HW7、8、9

P74 T2

(i)

$$x_{k+1} = \left(\frac{5x_k^2 + 19x_k - 42}{2}\right)^{\frac{1}{3}}$$

令

$$\phi(x) = \left(\frac{5x^2 + 19x - 42}{2}\right)^{\frac{1}{3}}$$

在 $x = 3$ 处

$$|\phi'(3)| < 1$$

故该迭代格式收敛。

(ii)

$$x_{k+1} = \sqrt{\frac{2x_k^3 - 5x_k^2 + 42}{5}}$$

因

$$|\phi'(3)| > 1$$

故在 $x = 3$ 处不收敛。

(ii)

$$x_{k+1} = \frac{2x_k^3 - 5x_k^2 + 42}{19}$$

因

$$|\phi'(3)| > 1$$

故在 $x = 3$ 处不收敛。

T3

```
I=[0.000000,1.000000],x*=0.500000,f(x*)=-0.625000
I=[0.500000,1.000000],x*=0.750000,f(x*)=-0.015625
I=[0.750000,1.000000],x*=0.875000,f(x*)= 0.435547
I=[0.750000,0.875000],x*=0.812500,f(x*)= 0.196533
I=[0.750000,0.812500],x*=0.781250,f(x*)= 0.087189
I=[0.750000,0.781250],x*=0.765625,f(x*)= 0.034977
I=[0.750000,0.765625],x*=0.757812,f(x*)= 0.009476
I=[0.750000,0.757812],x*=0.753906,f(x*)=-0.003124
I=[0.753906,0.757812],x*=0.755859,f(x*)= 0.003164
I=[0.753906,0.755859],x*=0.754883,f(x*)= 0.000017
请按任意键继续. . .
```

根为 $x^* = 0.754883$。

T5 令

$$F(x) = x^n - a$$

牛顿迭代格式为

$$x_{k+1} = x_k - \frac{x_k^n - a}{n x_k^{n-1}}$$

取 $n = 5, a = 9, x_0 = 2$, 迭代

```
   2.00000000000000
   1.71250000000000          0.287500000000000
   1.57929082235757          0.133209177642430
   1.55278303901599          2.650778334158344E-002
   1.55184670518788          9.363338281074274E-004
请按任意键继续. . .
```

近似值为 1.5518467。

T7 迭代格式为

$$x_{k+1} = x_k - \frac{(x_k^3 - 3x_k - 2)(x_k - x_{k-1})}{(x_k^3 - 3x_k - 2) - (x_{k-1}^3 - 3x_{k-1} - 2)}$$

取 $x_0 = 1, x_1 = 3$

```
   1.00000000000000
   3.00000000000000
   1.40000000000000          1.60000000000000         -3.45600000000000
   1.68421052631579         0.284210526315789         -2.27525878407931
   2.23187710033605         0.547666574020265          2.42196317464224
   1.94949140812989         0.282385692206160        -0.439399473085182
   1.99285540620987         4.336399807998093E-002 -6.399543748593484E-002
   2.00024770294914         7.392296739266113E-003  2.229694697973628E-003
   1.99999881609453         2.488868546099976E-004 -1.065514081055596E-005
请按任意键继续. . .
```

根为 $x^* = 1.999998816$。

T8

$$J(x,y)=\begin{bmatrix}2x & 2y\\ 3x^2 & -1\end{bmatrix}$$

迭代格式为

$$\begin{bmatrix}x_{k+1}\\ y_{k+1}\end{bmatrix}=\begin{bmatrix}x_k\\ y_k\end{bmatrix}-J^{-1}(x_k,y_k)\begin{bmatrix}f(x_k,y_k)\\ g(x_k,y_k)\end{bmatrix}$$

得

```
  0.600000000000000       0.800000000000000
  0.919125683060109       0.560655737704918       0.319125683060109
  0.833417265720939       0.559252682215940       8.570841733916998E-002
  0.826087834352686       0.563605855734621       7.329431368253368E-003
  0.826031360794393       0.563624160685555       5.647355829263629E-005
请按任意键继续. . .
```

解为

$$x^*=0.82603\quad y^*=0.563624$$

补充：证明 Newton 迭代是二次收敛的。

课本 P65,66

P94 T1(1)
结果均为

$$x_1 = 10.00 \quad x_2 = 1.000$$

T5
(1) 由 Doolittle 分解

$$\begin{bmatrix} 2 & 1 & 2 \\ -2 & 2 & -1 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

求解

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 18 \\ -39 \\ 24 \end{bmatrix}$$

得

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 18 \\ -21 \\ 27 \end{bmatrix}$$

求解

$$\begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 18 \\ -21 \\ 27 \end{bmatrix}$$

得

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \\ 9 \end{bmatrix}$$

T6
(1) 有 Crout 分解

$$\begin{bmatrix} 5 & 1 & 2 \\ 1 & 3 & -1 \\ 2 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 1 & \frac{14}{5} & 0 \\ 2 & \frac{13}{5} & \frac{11}{2} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{5} & \frac{2}{5} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} = LU$$

求解 $Ly=b$，得

$$y=\begin{bmatrix}2\\0\\2\end{bmatrix}$$

求解 $Ux=y$，得

$$x=\begin{bmatrix}1\\1\\2\end{bmatrix}$$

T7

(1) 由 LDL^T 分解

$$\begin{bmatrix}-6&3&2\\3&5&1\\2&1&6\end{bmatrix}=\begin{bmatrix}1&0&0\\-\frac{1}{2}&1&0\\-\frac{1}{3}&\frac{4}{13}&1\end{bmatrix}\begin{bmatrix}-6&0&0\\0&\frac{13}{2}&0\\0&0&\frac{236}{39}\end{bmatrix}\begin{bmatrix}1&-\frac{1}{2}&-\frac{1}{3}\\0&1&\frac{4}{13}\\0&0&1\end{bmatrix}=LDL^T$$

求解 $Lz=b$，得

$$z=\begin{bmatrix}-4\\9\\-\frac{472}{39}\end{bmatrix}$$

求解 $Dy=z$，得

$$y=\begin{bmatrix}\frac{2}{3}\\\frac{18}{13}\\-2\end{bmatrix}$$

求解 $L^Tx=y$，得

$$y=\begin{bmatrix}1\\2\\-2\end{bmatrix}$$

T8

(1) 有分解

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 2 & 7 & 3 & 0 \\ 0 & 2 & 7 & 3 \\ 0 & 0 & 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ & 2 & 1 & \\ & & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & & \\ & 1 & 3 & \\ & & 1 & 3 \\ & & & 1 \end{bmatrix} = LU$$

求解 $Ly = b$，得

$$y = \begin{bmatrix} -2 \\ -4 \\ 2 \\ 1 \end{bmatrix}$$

求解 $Ux = y$，得

$$x = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

补充：Gauss 消元法，消元部分的运算量为

$$\sum_{i=1}^{n-1}(n-i)(2n-2i+3) = \frac{2}{3}n^3 + \frac{1}{2}n^2 - \frac{7}{6}n = \frac{2}{3}n^3 + O\left(n^2\right)$$

回代过程的运算量为 n^2，故 Gauss 消元法的运算量为

$$\frac{2}{3}n^3 + O(n^2)$$

与 Gauss 消元法相比，列主元法多了 $\sum_{i=1}^{n-1}(n-i) = \frac{n^2-n}{2}$ 次查找，全主元法多了 $\sum_{i=1}^{n-1}(n-i)^2 = \frac{2n^3-3n^2+n}{6}$ 次查找，回代过程运算量相同，总的运算量仍是 $O(n^3)$ 次浮点运算，一般情形下，列主元法比全主元法有时间上的优势。

P107 T1(2)

$$A = \begin{bmatrix} 5 & 1 & 1 \\ 0 & 3 & 0 \\ -1 & 1 & 6 \end{bmatrix}$$

$$||A||_1 = 7 \quad ||A||_\infty = 8$$

$$A^TA = \begin{bmatrix} 26 & 4 & -1 \\ 4 & 11 & 7 \\ -1 & 7 & 37 \end{bmatrix}$$

特征值为 $(\lambda_1, \lambda_2, \lambda_3) = (38.7648, 26.9591, 8.27603)$, 所以

$$||A||_2 = \sqrt{38.7648} = 6.22614$$

计算知 A 的特征值

$$(\lambda_1, \lambda_2, \lambda_3) = (\frac{11+\sqrt{3}i}{2}, \frac{11-\sqrt{3}i}{2}, 3)$$

故 $\rho(A) = 5.56776$

T3

(1) Jacobi 迭代格式为

$$\begin{aligned} x_1^{k+1} &= 0.1x_2^k + 0.1 \\ x_2^{k+1} &= 0.1x_1^k + 0.1x_3^k \\ x_3^{k+1} &= 0.1x_2^k + 0.1x_4^k + 0.1 \\ x_4^{k+1} &= 0.1x_3^k + 0.2 \end{aligned}$$

计算得

$$X_1 = \begin{bmatrix} 0.1 \\ 0 \\ 0.1 \\ 0.2 \end{bmatrix} \quad X_2 = \begin{bmatrix} 0.1 \\ 0.02 \\ 0.12 \\ 0.21 \end{bmatrix} \quad X_3 = \begin{bmatrix} 0.102 \\ 0.022 \\ 0.123 \\ 0.212 \end{bmatrix}$$

(2) Gauss-Seidel 迭代格式为

$$
\begin{aligned}
x_1^{k+1} &= 0.1x_2^k + 0.1 \\
x_2^{k+1} &= 0.1x_1^{k+1} + 0.1x_3^k \\
x_3^{k+1} &= 0.1x_2^{k+1} + 0.1x_4^k + 0.1 \\
x_4^{k+1} &= 0.1x_3^{k+1} + 0.2
\end{aligned}
$$

计算得

$$
X_1 = \begin{bmatrix} 0.1 \\ 0.01 \\ 0.101 \\ 0.2101 \end{bmatrix} \quad X_2 = \begin{bmatrix} 0.101 \\ 0.0202 \\ 0.12303 \\ 0.212303 \end{bmatrix} \quad X_3 = \begin{bmatrix} 0.10202 \\ 0.022505 \\ 0.123481 \\ 0.212348 \end{bmatrix}
$$

(3) Jacobi 迭代矩阵:

$$
R = \begin{bmatrix} 0 & 0.1 & 0 & 0 \\ 0.1 & 0 & 0.1 & 0 \\ 0 & 0.1 & 0 & 0.1 \\ 0 & 0 & 0.1 & 0 \end{bmatrix}
$$

Gauss-Seidel 迭代矩阵:

$$
S = \begin{bmatrix} 0 & 0.1 & 0 & 0 \\ 0 & 0.01 & 0.1 & 0 \\ 0 & 0.001 & 0.01 & 0.1 \\ 0 & 0.0001 & 0.001 & 0.01 \end{bmatrix}
$$

计算得

$$
\rho(R) = 0.402248 \quad \rho(S) = 0.161803
$$

故 Jacobi 和 Gasuu-Seidel 迭代都收敛。

T6

(1) Jacobi 迭代矩阵：

$$
R = \begin{bmatrix} 0 & -t \\ -\frac{t}{2} & 0 \end{bmatrix}
$$

$$
|\lambda I - R| = \lambda^2 - \frac{t^2}{2} = 0
$$

故特征值为

$$\lambda_{1,2} = \pm\frac{|t|}{\sqrt{2}}$$

谱半径为

$$\rho(R) = \frac{|t|}{\sqrt{2}} < 1$$

得

$$-\sqrt{2} < t < \sqrt{2}$$

(2) Gauss-Seidel 迭代矩阵为:

$$S = \begin{bmatrix} 0 & -t \\ 0 & \frac{t^2}{2} \end{bmatrix}$$

$$|\lambda I - S| = \lambda(\lambda - \frac{t^2}{2}) = 0$$

故特征值为

$$\lambda_1 = 0 \quad \lambda_2 = \frac{t^2}{2}$$

谱半径为

$$\rho(S) = \frac{t^2}{2} < 1$$

得

$$-\sqrt{2} < t < \sqrt{2}$$

T7

(1) Jacobi:

$$R = \begin{bmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{bmatrix}$$

$$|\lambda I - R| = \lambda^3 = 0$$

故特征值为

$$\lambda_{1,2,3} = 0$$

谱半径为

$$\rho(R) = 0 < 1$$

故 Jacobi 迭代收敛

Gauss-Seidel:

$$S = \begin{bmatrix} 0 & -2 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

$$|\lambda I - S| = \lambda(\lambda - 2)^2 = 0$$

故特征值为

$$\lambda_1 = 0 \quad \lambda_{2,3} = 2$$

谱半径为

$$\rho(S) = 2 > 1$$

故 Gauss-Seidel 迭代不收敛

(2) Jacobi:

$$R = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & -1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

$$|\lambda I - R| = \lambda(\lambda^2 + \frac{5}{4}) = 0$$

故特征值为

$$\lambda_1 = 0 \quad \lambda_2 = \frac{\sqrt{5}i}{2} \quad \lambda_3 = -\frac{\sqrt{5}i}{2}$$

谱半径为

$$\rho(R) = \frac{\sqrt{5}}{2} > 1$$

故 Jacobi 迭代不收敛

Gauss-Seidel:

$$S = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

$$|\lambda I - S| = \frac{\lambda}{4}(2\lambda + 1)^2 = 0$$

故特征值为

$$\lambda_1 = 0 \quad \lambda_{2,3} = -\frac{1}{2}$$

谱半径为

$$\rho(S) = \frac{1}{2} < 1$$

故 Gauss-Seidel 迭代收敛

补充:

$$A = \begin{bmatrix} 0.2161 & 0.1441 \\ 1.2968 & 0.8648 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} 60055.6 & -10006.9 \\ -90055.6 & 15006.9 \end{bmatrix}$$

则

$$||A||_\infty = 2.1616 \quad ||A^{-1}||_\infty = 105062.5$$

故

$$cond(A) = ||A||_\infty ||A^{-1}||_\infty = 227103.1$$

所以 A 是病态的。具体来说，

$$b = \begin{bmatrix} 0.1440 \\ 0.8640 \end{bmatrix} \quad x = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$\delta b = \begin{bmatrix} 10^{-8} \\ -10^{-8} \end{bmatrix} \quad \delta x = \begin{bmatrix} 0.000700625 \\ -0.00105063 \end{bmatrix}$$

则

$$||b||_\infty = 0.8640, ||x||_\infty = 2$$

$$||\delta b||_\infty = 10^{-8}, ||\delta x||_\infty = 0.00105063$$

故

$$\frac{||\delta x||_\infty}{||x||_\infty} = 45387 \frac{||\delta b||_\infty}{||b||_\infty}$$

在方程组右端项给予一个微小扰动时，解的变化较大，因此该方程组确为病态的。