HW1

P45-1.

记$(-1,3),(2,5),(3,7)$分别为$(x_0,f(x_0)),(x_1,f(x_1)),(x_2,f(x_2))$,

$$
\begin{aligned}
L_2(x) &= f(x_0)l_0(x) + f(x_1)l_1(x) + f(x_2)l_2(x) \\
&= 3\frac{(x-2)(x-3)}{(-1-2)(-1-3)} + 5\frac{(x+1)(x-3)}{(2+1)(2-3)} + 7\frac{(x+1)(x-2)}{(3+1)(3-2)} \\
&= \frac{1}{4}(x-2)(x-3) - \frac{5}{3}(x+1)(x-3) + \frac{7}{4}(x+1)(x-2)
\end{aligned}
$$

计算$L_2(0)=3$.

P45-3(2). 构造4点三次Lagrange多项式$L_3(x)$.

$$
\begin{aligned}
L_3(x) &= f(x_0)l_0(x) + f(x_1)l_1(x) + f_(x_2)l_2(x) + f_(x_3)l_3(x) \\
&= 2\frac{x(x-2)(x-3)}{-1(-1-2)(-1-3)} + 0 + 1\frac{(x+1)x(x-3)}{(2+1)2(2-3)} + 3\frac{(x+1)x(x-2)}{(3+1)3(3-2)} \\
&= -\frac{1}{6}x(x-2)(x-3) - \frac{1}{6}x(x+1)(x-3) + \frac{1}{4}x(x+1)(x-2)
\end{aligned}
$$

P45-5

$$
\begin{aligned}
L(x) &= 9\frac{(x-100)(x-121)}{(81-100)(81-121)} + 10\frac{(x-81)(x-121)}{(100-81)(100-121)} + 11\frac{(x-81)(x-100)}{(121-81)(121-100)} \\
&= \frac{-x^2+601x+29700}{7980}
\end{aligned}
$$

故

$$
L(105) = \frac{1363}{133} \approx 10.2481203
$$

Lagrange插值余项

$$
R(x) = f(x) - L(x) = \frac{f^{(3)}(\xi_x)}{3!}(x-81)(x-100)(x-121), \quad 81 < \xi_x < 121,
$$

其中

$$
|f^{(3)}(\xi_x)| = \frac{3}{8}\xi_x^{-\frac{5}{2}} \le \frac{3}{8}\cdot\frac{1}{81^{\frac{5}{2}}} = M
$$

代入$x=105$,

$$
|R(x)| \le \frac{1}{6}\cdot\frac{3}{8}\cdot\frac{1}{81^{\frac{5}{2}}}|105-81||105-100||105-121| \approx 0.00203221
$$

实际误差为$|\sqrt{105}-L(105)| = 0.00116953$小于所求的上界.

HW2

P45-6 差商表如下

x_i	$f(x_i)$	$f[x_{i-1},x_i]$	$f[x_{i-2},x_{i-1},x_i]$	$f[x_{i-3},x_{i-2},x_{i-1},x_i]$
-1	3			
2	5	2/3		
3	7	2	1/3	
4	5	-2	-2	-7/15

牛顿插值多项式

$$N_3(x) = 3 + \frac{2}{3}(x+1) + \frac{1}{3}(x+1)(x-2) - \frac{7}{15}(x+1)(x-2)(x-3)$$

所以$f(1.2)$的近似值$N_3(1.2) = 2.4016$.

P45-7

(1)差商的大小与点排列的顺序无关. Newton插值多项式

$$\begin{aligned} N(x) &= f(4) + f[4,1](x-4) + f[4,1,3](x-4)(x-1) + f[4,1,3,2](x-4)(x-1)(x-3) \\ &= 1 + 2(x-4) + (x-4)(x-1) - (x-4)(x-1)(x-3) \end{aligned}$$

(2)$N(x)$于$x=1,2,3,4$插值$f(x)$, $f(2) = N(2) = -7$.

$$f[2,3,4,1] = \frac{f[1,3,4] - f[2,3,4]}{-1} = -1 \Rightarrow 1 - f[2,3,4] = 1$$

所以$f[2,3,4] = 0$.

P45-9

差商的性质: $f[t_1, t_2, \cdots, t_{r+1}] = \frac{f^{(r)}(\zeta)}{r!} (t_1 \le \zeta \le t_{r+1})$.

$$\begin{aligned} f[2^0, 2^1] &= f(2) - f(1) = -2089; \\ f[2^0, 2^1, \cdots, 2^7] &= \frac{f^{(7)}(\zeta)}{7!} = \frac{7!}{7!} = 1; \\ f[2^0, 2^1, \cdots, 2^8] &= \frac{f^{(8)}(\zeta)}{8!} = 0. \end{aligned}$$

P45-12. 已知$f(3) = 5, f(5) = 15, f^{'}(5) = 7$, 二次插值设$L_2(x) = f(3)l_0(x) + f(5)l_2(x) + f^{'}(5)l_3(x)$, 其中$l_i(x), i = 1,2,3$满足

$$\begin{aligned} &l_0(3) = 1,\ l_0(5) = 0,\ l_0^{'}(5) = 0, \Rightarrow l_0(x) = a_1(x-5)^2, \\ &l_1(3) = 0,\ l_1(5) = 1,\ l_1^{'}(5) = 0, \Rightarrow l_1(x) = (x-3)(a_2x + b_2), \\ &l_2(3) = 0,\ l_2(5) = 0,\ l_2^{'}(5) = 1, \Rightarrow l_2(x) = a_3(x-3)(x-5). \end{aligned} \tag{0.1}$$

用待定系数法可以解得

$$l_0(x) = \frac{1}{4}(x-5)^2,\ l_1(x) = \frac{1}{4}(x-3)(7-x),\ l_2(x) = \frac{1}{2}(x-3)(x-5).$$

故

$$L_2(x) = \frac{5}{4}(x-5)^2 - \frac{15}{4}(x-3)(x-7) + \frac{7}{2}(x-3)(x-5).$$

Lagrange型插值余项$R(x) = \frac{f^{(3)}(\xi)}{3!}(x-3)(x-5)^2$. $f(3.7)$的近似值$L_2(3.7) = 7.59$.

P45-15. 已知$f(1)=0.5,\ f(2)=1,\ f^{'}(1)=0.5,\ f^{'}(2)=-1,\ f^{''}(2)=1$列差商表构造四次Newton插值多项式$N_4(x)$

x_i	$f(x_i)$	$f[x_{i-1},x_i]$	$f[x_{i-2},x_{i-1},x_i]$	$f[x_{i-3},x_{i-2},x_{i-1},x_i]$	$f[x_{i-4},x_{i-3},x_{i-2},x_{i-1},x_i]$
1	0.5				
1	0.5	0.5			
2	1	0.5	0		
2	1	-1	-1.5	-1.5	
2	1	-1	0.5	2	3.5

其中$f[2,2,2]=\frac{f^{''}(2)}{2!}$ 容易忘记除掉阶乘. 故Newton插值多项式

$$N_4(x)=0.5+0.5(x-1)+0(x-1)^2-1.5(x-1)^2(x-2)+3.5(x-1)^2(x-2)^2.$$

Newton型插值余项$R(x)=f[x,1,1,2,2,2](x-1)^2(x-3)^3$.

补充题1: hermite插值误差估计

设$f\in C^{2n+2}[a,b]$, $H_{2n+1}(x)$为$f(x)$关于$[a,b]$上互不相同的节点$\{x_i,i=0,1,\cdot,n\}$的hermite插值多项式, 即

$$H_{2n+1}(x_i)=f(x_i),\quad H_{2n+1}^{'}(x_i)=f^{'}(x_i),\ i=0,1,\cdots,n.$$

记$R(x)=f(x)-H_{2n+1}(x)$, 则由插值条件可知$\{x_0,x_1,\cdots,x_n\}$为$R(x)$的二重根, 所以可设

$$R(x)=k(x)(x-x_0)^2(x-x_1)^2\cdots(x-x_n)^2$$

以下计算$k(x)$, 引入变量为t的函数$\phi(t)$

$$\phi(t)=f(t)-H_{2n+1}(t)-k(x)(t-x_0)^2(t-x_1)^2\cdots(t-x_n)^2$$

则有$\phi(x_i)=0,\ \phi^{'}(x_i)=0,\ i=0,1,\cdots,n.$且$\phi(x)=0$. 若$x$为插值节点中的某个点, 则插值余项为0, 现考虑$x$非插值节点. 则$\phi(t)$有$n+2$个互不相同的根. 由Rolle定理, $\phi^{'}(t)$在每两个相邻的插值节点之间都至少有一个零点, 而这些插值节点也为$\phi^{'}(t)$的零点, 所以$\phi^{'}(t)$至少有$2n+2$个互不相同的零点. 反复使用Rolle定理可得$\phi^{(2n+2)}$至少有一个零点, 记为ξ.

对$\phi(t)$求$2n+2$次导, 并取$t=\xi$

$$\phi^{(2n+2)}(\xi)=f^{(2n+2)}(\xi)-0-k(x)/(2n+2)!=0$$

所以$k(x)=\frac{f^{(2n+2)}(\xi)}{(2n+2)!},\xi\in[a,b]$, 所以插值余项为

$$R(x)=\frac{f^{(2n+2)}(\xi)}{(2n+2)!}(x-x_0)^2(x-x_1)^2\cdots(x-x_n)^2,\ \xi\in[a,b]$$

P46-16

利用课本41-43页的方法, 计算得到

$$h_0=1,h_1=2,h_2=1$$

$$\lambda_1=\frac{2}{3},\mu_1=\frac{1}{3},d_1=-12$$

$$\lambda_2 = \frac{1}{3}, \mu_2 = \frac{2}{3}, d_2 = 12$$

关于M_1, M_2的方程组

$$\begin{pmatrix} 2 & \frac{2}{3} \\ \frac{2}{3} & 2 \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \begin{pmatrix} -12 \\ 12 \end{pmatrix}$$

解得$M_1 = -9, M_2 = 9$, 故

$$S(x) = \begin{cases} -1.5x^3 - 9x^2 - 9.5x + 1 \\ 1.5x^3 - 0.5x + 4 \\ -1.5x^3 + 9x^2 - 9.5x + 7 \end{cases}$$

$S(0) = 4$.

P46-17

计算得到

$$h_0 = 1, h_1 = 1, h_2 = 2$$
$$\lambda_1 = \frac{1}{2}, \mu_1 = \frac{1}{2}, d_1 = 0$$
$$\lambda_2 = \frac{2}{3}, \mu_2 = \frac{1}{3}, d_2 = 23$$

通过$S^{'}(-1) = 5, S^{'}(3) = 29$得到的方程为

$$2M_0 + M_1 = -24$$
$$M_2 + 2M_3 = \frac{99}{2}$$

故关于M_0, M_1, M_2, M_3的方程组为

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ \frac{1}{2} & 2 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & 2 & \frac{2}{3} \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \end{pmatrix} = \begin{pmatrix} -24 \\ 0 \\ 23 \\ \frac{99}{2} \end{pmatrix}$$

解得

$$M_0 = -13.2273, M_1 = 2.45455, M_2 = 3.40909, M_3 = 23.0455$$

故

$$S(x) = \begin{cases} 2.61364x^3 + 1.22728x^2 - 0.386367x + 3 & x \in [-1, 0] \\ 0.15909x^3 + 1.22728x^2 - 0.386365x + 3 & x \in [0, 1] \\ 1.63637x^3 - 3.20456x^2 + 4.04545x + 1.52274 & x \in [1, 3] \end{cases}$$

$S(2) = 9.88636$.

补充题2

分段二次插值的截断误差$R(x) = \frac{f^{(3)}(\xi)}{3!}(x - x_{i-1})(x - x_i)(x - x_{i+1}),\ x \in [x_{i-1}, x_{i+1}]$. 代入$x = x_i + ht$,

$$R(t) = \frac{f^{(3)}(\xi)}{3!} h^3 (t^3 - t),\ t \in [-1, 1]$$

$t^3 - t$在$t = \pm\frac{\sqrt{3}}{3}$处取得极值. 所以令

$$R(t) = \frac{e^{\xi}}{6} h^3 (t^3 - t) \le \frac{e^{\xi}}{6} h^3 \frac{2\sqrt{3}}{9} \le 1e - 6 \Rightarrow h^3 \le \frac{9\sqrt{3}}{e^{\xi}} \cdot 10^{-6},\ \forall \xi \in [-4, 4]$$

所以$h^3 \le \frac{9\sqrt{3}}{e^4} \cdot 10^{-6}$, 解得$h \le 6.58 \times 10^{-3}$.

HW3

P58-3

设该组点为$\{(x_i,y_i),i=1,\cdots,5\}$, 考虑$n$次多项式拟合. 设拟合函数$\phi(x)=a_0+a_1x+\cdots+a_nx^n$, 对应的误差平方和函数

$$Q(a_0,a_1,\cdots,a_n)=\sum_{i=1}^{5}(a_0+a_1x_i+\cdots+a_nx_i^n-y_i)^2$$

当Q达到最小值时满足

$$\frac{\partial Q}{\partial a_j}=2\sum_{i=1}^{5}(a_0+a_1x_i+\cdots+a_nx_i^n-y_i)x_i^j=0,\quad j=0,1,\cdots,n.$$

当$n=1$, 对应线性拟合的方程组为

$$\begin{pmatrix}5 & \sum_{i=1}^5 x_i\\ \sum_{i=1}^5 x_i & \sum_{i=1}^5 x_i^2\end{pmatrix}\begin{pmatrix}a_0\\ a_1\end{pmatrix}=\begin{pmatrix}\sum_{i=1}^5 y_i\\ \sum_{i=1}^5 x_iy_i\end{pmatrix}$$

即

$$\begin{pmatrix}5 & -0.5\\ -0.5 & 1.875\end{pmatrix}\begin{pmatrix}a_0\\ a_1\end{pmatrix}=\begin{pmatrix}9.32\\ 2.855\end{pmatrix}$$

解得$a_0=2.0715,a_1=2.0751$, 误差平方和0.1476.

当$n=2$, 对应线性拟合的方程组为

$$\begin{pmatrix}5 & \sum_{i=1}^5 x_i & \sum_{i=1}^5 x_i^2\\ \sum_{i=1}^5 x_i & \sum_{i=1}^5 x_i^2 & \sum_{i=1}^5 x_i^3\\ \sum_{i=1}^5 x_i^2 & \sum_{i=1}^5 x_i^3 & \sum_{i=1}^5 x_i^4\end{pmatrix}\begin{pmatrix}a_0\\ a_1\\ a_2\end{pmatrix}=\begin{pmatrix}\sum_{i=1}^5 y_i\\ \sum_{i=1}^5 x_iy_i\\ \sum_{i=1}^5 x_i^2y_i\end{pmatrix}$$

即

$$\begin{pmatrix}5 & -0.5 & 1.875\\ -0.5 & 1.875 & -0.6875\\ 1.875 & -0.6875 & 1.3828\end{pmatrix}\begin{pmatrix}a_0\\ a_1\\ a_2\end{pmatrix}=\begin{pmatrix}9.32\\ 2.855\\ 2.7137\end{pmatrix}$$

解得$a_0=1.9074,a_1=2.2044,a_2=0.4722$, 误差平方和0.0266.

P59-7 令$g=\frac{1}{f},y=\frac{1}{x}$, 则

$$g=ay+b$$

求解

y	0.4762	0.4	0.3571	0.3125
g	1.6428	1.4601	1.3572	1.2329

$$\begin{pmatrix}0.4762 & 1\\ 0.4 & 1\\ 0.3571 & 1\\ 0.3125 & 1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}1.6428\\ 1.4601\\ 1.3572\\ 1.2329\end{pmatrix}$$

法方程组为

$$\begin{pmatrix} 0.6119 & 1.5458 \\ 1.5458 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2.2363 \\ 5.6930 \end{pmatrix}$$

解得$a = 2.4821,\ b = 0.4625.$

P59-8(1)

解矛盾方程组

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 4 \end{pmatrix}$$

对应的法方程为

$$\begin{pmatrix} 6 & 5 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 21 \\ 20 \end{pmatrix}$$

解得$x_1 = \frac{26}{11} = 2.3636, x_2 = \frac{15}{11} = 1.3636.$

P137-3

代入$f(x) = 1, x, x^2$得到关于a_{-1}, a_0, a_1的方程组

$$\begin{cases} 3h = a_{-1} + a_0 + a_1 \\ \frac{3}{2}h = -a_{-1} + 2a_1 \\ 3h = a_{-1} + 4a_1 \end{cases}$$

解得

$$\begin{cases} a_{-1} = 0 \\ a_0 = \frac{9}{4}h \\ a_1 = \frac{3}{4}h \end{cases}$$

所以$I(f) = \frac{9}{4}hf(0) + \frac{3}{4}hf(2h)$. 而$I(x^3) = 6h^4 \neq \int_{-h}^{2h} x^3 dx = \frac{15}{4}h^4$, 所以代数精度为2.