积分变换笔记

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目录

第一章　Fourier 变换

1.1   Fourier 积分公式

Fourier 积分公式

当在 t 处连续时，有

$$f\left(t\right) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}f\left(\tau\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\mathrm{d}\tau\right]\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega$$

当在 t 处间断时，有

$$f\left(t\right) = \frac{f\left(t+0\right)+f\left(t-0\right)}{2}$$

注意：当在 t 处间断时，应取该点左右极限计算

Fourier 积分公式的复数形式

$$f\left(t\right) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}f\left(\tau\right)\mathrm{e}^{-\mathrm{i}\omega\tau}\mathrm{d}\tau\right]\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega$$

Dirichlet 积分

$$\int_{0}^{+\infty}\frac{\sin x}{x}\mathrm{d}x = \frac{\pi}{2}$$

1.2   Fourier 变换与其逆变换

Fourier 变换式

$$\mathscr{F}\left[f\left(t\right)\right] = \int_{-\infty}^{+\infty}f\left(t\right)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t = F\left(\omega\right)$$

$F\left(\omega\right)$ 为 $f\left(t\right)$ 的象函数

Fourier 逆变换式

$$\mathscr{F}^{-1}\left[F\left(\omega\right)\right] = \frac{1}{2\pi}\int_{-\infty}^{+\infty}F\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega = f\left(t\right)$$

$f\left(t\right)$ 为 $F\left(\omega\right)$ 的象原函数

1.3　单位脉冲函数及其 Fourier 变换

Dirac 函数（δ 函数）

若 $\delta(t)$ 满足

$$\delta(t) = 0, t \neq 0$$

以及

$$\int_{-\infty}^{+\infty} \delta(t)\,\mathrm{d}t = 1$$

则 $\delta(t)$ 为 δ 函数

性质

对性质良好的 $f(t)$，δ 函数有如下性质

性质一

$$\int_{-\infty}^{+\infty} \delta(t) f(t)\,\mathrm{d}t = f(0)$$

$$\int_{-\infty}^{+\infty} \delta(t - t_0) f(t)\,\mathrm{d}t = f(t_0)$$

性质二

$$\int_{-\infty}^{+\infty} \delta'(t) f(t)\,\mathrm{d}t = -f'(0)$$

$$\int_{-\infty}^{+\infty} \delta^{(n)}(t) f(t)\,\mathrm{d}t = (-1)^n f^{(n)}(0)$$

性质三

$$\delta(t) = \delta(-t)$$

性质四

$$\int_{-\infty}^{t} \delta(\tau)\,\mathrm{d}\tau = u(t)$$

$$u(t) = \begin{cases} 1, t > 0 \\ 0, t < 0 \end{cases}$$

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = \delta(t)$$

性质五

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

$$\delta(at - b) = \frac{1}{|a|}\delta\left(t - \frac{b}{a}\right)$$

常见函数的 Fourier 变换

表 1.1: 常见函数的 Fourier 变换

函数	Fourier 变换
$\delta\left(t\right)$	1
$\delta\left(t-t_0\right)$	$\mathrm{e}^{-\mathrm{i}\omega t_0}$
$u\left(t\right)$	$\frac{1}{\mathrm{i}\omega}+\pi\delta\left(\omega\right)$
$\sin\left(\omega_0 t\right)$	$\pi\mathrm{i}\left[\delta\left(\omega+\omega_0\right)-\delta\left(\omega-\omega_0\right)\right]$
$\cos\left(\omega_0 t\right)$	$\pi\left[\delta\left(\omega+\omega_0\right)+\delta\left(\omega-\omega_0\right)\right]$
1	$2\pi\delta\left(\omega\right)$
$\delta'\left(t\right)$	$\mathrm{i}\omega$
$\delta^{(n)}\left(t\right)$	$\left(\mathrm{i}\omega\right)^n$

1.4   Fourier 变换的性质

A、**线性性质**

若 $\mathscr{F}\left[f_1\left(t\right)\right]=F_1\left(\omega\right)$，$\mathscr{F}\left[f_2\left(t\right)\right]=F_2\left(\omega\right)$，则

$$\mathscr{F}\left[\alpha f_1\left(t\right)+\beta f_2\left(t\right)\right]=\alpha F_1\left(\omega\right)+\beta F_2\left(\omega\right)$$

$$\mathscr{F}^{-1}\left[\alpha F_1\left(\omega\right)+\beta F_2\left(\omega\right)\right]=\alpha f_1\left(t\right)+\beta f_2\left(t\right)$$

B、**位移性质**

若 $\mathscr{F}\left[f\left(t\right)\right]=F\left(\omega\right)$，则

$$\mathscr{F}\left[f\left(t+t_0\right)\right]=\mathrm{e}^{\mathrm{i}\omega t_0}F\left(\omega\right)$$

$$\mathscr{F}\left[\mathrm{e}^{\mathrm{i}\omega_0 t}f\left(t\right)\right]=F\left(\omega-\omega_0\right)$$

C、微分性质

设 $\mathscr{F}[f(t)] = F(\omega)$，若 $f(t)$ 在 $(-\infty, +\infty)$ 上连续或只有有限个可去间断点，且当 $|t| \to +\infty$，$f(t) \to 0$，则

$$\mathscr{F}[f'(t)] = \mathrm{i}\omega F(\omega)$$

$$\mathscr{F}\left[f^{(n)}(t)\right] = (\mathrm{i}\omega)^n F(\omega)$$

$$\mathscr{F}[tf(t)] = \mathrm{i}F'(\omega)$$

$$\mathscr{F}[t^n f(t)] = \mathrm{i}^n F^{(n)}(\omega)$$

D、积分性质

设 $\mathscr{F}[f(t)] = F(\omega)$，

当 $t \to +\infty$ 时，$g(t) = \int_{-\infty}^{t} f(t)\,\mathrm{d}t \to 0$，则

$$\mathscr{F}\left[\int_{-\infty}^{t} f(t)\,\mathrm{d}t\right] = \frac{1}{\mathrm{i}\omega} F(\omega)$$

当 $\lim_{t \to +\infty} g(x) \neq 0$ 时，则

$$\mathscr{F}\left[\int_{-\infty}^{t} f(t)\,\mathrm{d}t\right] = \frac{1}{\mathrm{i}\omega} F(\omega) + \pi F(0)\delta(\omega)$$

E、相似性质

设 $\mathscr{F}[f(t)] = F(\omega)$，且 $a \neq 0$，则

$$\mathscr{F}[f(at)] = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

补充

在求 Fourier 变换时，我们或许会遇到被变换式中有 $\sin\omega_0 t$ 或是 $\cos\omega_0 t$，此时我们需要用定义展开，同时用欧拉公式表示出 $\sin\omega_0 t$ 或 $\cos\omega_0 t$，根据 Euler 公式，我们有

$$\sin\omega_0 t = \frac{\mathrm{e}^{\mathrm{i}\omega_0 t} - \mathrm{e}^{-\mathrm{i}\omega_0 t}}{2\mathrm{i}}$$

以及

$$\cos\omega_0 t = \frac{\mathrm{e}^{\mathrm{i}\omega_0 t} + \mathrm{e}^{-\mathrm{i}\omega_0 t}}{2}$$

1.5 卷积

概念

$$f_1\left(t\right) * f_2\left(t\right) = \int_{-\infty}^{\infty} f_1\left(\tau\right) f_2\left(t - \tau\right) \mathrm{d}\tau$$

满足的运算律

（1）交换律

$$f_1\left(t\right) * f_2\left(t\right) = f_2\left(t\right) * f_1\left(t\right)$$

（2）结合律

$$f_1\left(t\right) * \left[f_2\left(t\right) * f_3\left(t\right)\right] = \left[f_1\left(t\right) * f_2\left(t\right)\right] * f_3\left(t\right)$$

（3）分配律

$$f_1\left(t\right) * \left[f_2\left(t\right) + f_3\left(t\right)\right] = f_1\left(t\right) * f_2\left(t\right) + f_1\left(t\right) * f_3\left(t\right)$$

（4）设 α 为常数，则

$$\alpha\left[f_1\left(t\right) * f_2\left(t\right)\right] = \left[\alpha f_1\left(t\right)\right] * f_2\left(t\right) = f_1\left(t\right) * \left[\alpha f_2\left(t\right)\right]$$

（5）函数卷积的绝对值小于等于函数绝对值的卷积，即

$$\left|f_1\left(t\right) * f_2\left(t\right)\right| \leqslant \left|f_1\left(t\right)\right| * \left|f_2\left(t\right)\right|$$

卷积定理

设 $\mathscr{F}\left[f_1\left(t\right)\right] = F_1\left(\omega\right)$，$\mathscr{F}\left[f_2\left(t\right)\right] = F_2\left(\omega\right)$，则

$$\mathscr{F}\left[f_1\left(t\right) * f_2\left(t\right)\right] = F_1\left(\omega\right) \cdot F_2\left(\omega\right)$$

$$\mathscr{F}^{-1}\left[F_1\left(\omega\right) \cdot F_2\left(\omega\right)\right] = f_1\left(t\right) * f_2\left(t\right)$$

$$\mathscr{F}\left[f_1\left(t\right) \cdot f_2\left(t\right)\right] = \frac{1}{2\pi} F_1\left(\omega\right) * F_2\left(\omega\right)$$

$$\mathscr{F}\left[f_1\left(t\right) f_2\left(t\right) \cdots f_n\left(t\right)\right] = \frac{1}{\left(2\pi\right)^{n-1}} F_1\left(\omega\right) * F_2\left(\omega\right) * \cdots * F_n\left(\omega\right)$$

第二章　Laplace 变换

2.1   Laplace 变换的概念

定义

Laplace 变换

$$F\left(s\right) = \mathscr{L}\left[f\left(t\right)\right] = \int_0^{+\infty} f\left(t\right) \mathrm{e}^{-st} \mathrm{d}t$$

Laplace 逆变换

$$f\left(t\right) = \mathscr{L}^{-1}\left[F\left(s\right)\right]$$

常见函数的 Laplace 变换

表 2.1: 常见函数的 Laplace 变换

函数	Laplace 变换
1	$\frac{1}{s}$
$u\left(t\right)$	$\frac{1}{s}$
e^{kt}	$\frac{1}{s-k}$
t^m	$\frac{m!}{s^{m+1}}$
$\sin\left(kt\right)$	$\frac{k}{s^2+k^2}$
$\cos\left(kt\right)$	$\frac{s}{s^2+k^2}$
$\delta\left(t\right)$	1
$\delta^{(n)}\left(t\right)$	s^n

几种单位阶跃函数变形的 Laplace 变换

例 2.1.1.

$$\mathscr{L}\left[u\left(t-1\right)\right] = \mathscr{L}\left[u\left(t-1\right)u\left(t-1\right)\right] = \mathrm{e}^{-s}\mathscr{L}\left[u\left(t\right)\right] = \frac{\mathrm{e}^{-s}}{s}$$

例 2.1.2.

$$\mathscr{L}\left[u\left(1-t\right)\right] = \mathscr{L}\left[1 - u\left(t-1\right)\right] = \frac{1}{s} - \frac{\mathrm{e}^{-s}}{s}$$

例 2.1.3.

$$\mathscr{L}\left[u\left(2t-2\right)\right] = \mathscr{L}\left[u\left(t-1\right)\right] = \mathscr{L}\left[u\left(t-1\right)u\left(t-1\right)\right] = \mathrm{e}^{-s}\mathscr{L}\left[u\left(t\right)\right] = \frac{\mathrm{e}^{-s}}{s}$$

例 2.1.4.

$$\mathscr{L}\left[u\left(t+1\right)\right] = \frac{1}{s}$$

2.2 Laplace 变换的性质

A、**线性性质**

若 $\mathscr{L}[f_1(t)] = F_1(s)$，$\mathscr{L}[f_2(t)] = F_2(s)$，则

$$\mathscr{L}[\alpha f_1(t) + \beta f_2(t)] = \alpha F_1(s) + \beta F_2(s)$$

$$\mathscr{L}^{-1}[\alpha F_1(s) + \beta F_2(s)] = \alpha \mathscr{L}^{-1}[F_1(s)] + \beta \mathscr{L}[F_2(s)]$$

B、**微分性质**

若 $\mathscr{L}[f(t)] = F(s)$，则

$$\mathscr{L}[f'(t)] = sF(s) - f(0)$$

$$\mathscr{L}[f''(t)] = s^2 F(s) - sf(0) - f'(0)$$

$$\mathscr{L}[tf(t)] = -F'(s)$$

$$\mathscr{L}[t^n f(t)] = (-1)^n F^{(n)}(s)$$

C、**积分性质**

若 $\mathscr{L}[f(t)] = F(s)$，则

$$\mathscr{L}\left[\int_0^t f(\tau)\,\mathrm{d}\tau\right] = \frac{1}{s}F(s)$$

$$\mathscr{L}\left[\frac{f(t)}{t}\right] = \int_s^{+\infty} F(s)\,\mathrm{d}s$$

$$\int_0^{+\infty} \frac{f(t)}{t}\mathrm{d}t = \int_0^{+\infty} F(s)\,\mathrm{d}s$$

D、**位移性质**

若 $\mathscr{L}[f(t)] = F(s)$，则

$$\mathscr{L}[\mathrm{e}^{at} f(t)] = F(s - a)$$

E、**延迟性质**

若 $\mathscr{L}[f(t)] = F(s)$，则对于 $\forall \tau > 0$，有

$$\mathscr{L}[f(t - \tau)u(t - \tau)] = \mathrm{e}^{-s\tau} F(s)$$

Fourier 与 Laplace 变换性质

<div align="center">表 2.2: Fourier 与 Laplace 变换性质</div>

性质	Fourier 变换	Laplace 变换
微分性质	$\mathscr{F}\left[f'\left(t\right)\right]=\mathrm{i}\omega F\left(\omega\right)$	$\mathscr{L}\left[f'\left(t\right)\right]=sF\left(s\right)-f\left(0\right)$
	$\mathscr{F}\left[f^{(n)}\left(t\right)\right]=\left(\mathrm{i}\omega\right)^{n}F\left(\omega\right)$	$\mathscr{L}\left[f''\left(t\right)\right]=s^{2}F\left(s\right)-sf\left(0\right)-f'\left(0\right)$
	$\mathscr{F}\left[tf\left(t\right)\right]=\mathrm{i}F'\left(\omega\right)$	$\mathscr{L}\left[tf\left(t\right)\right]=-F'\left(s\right)$
	$\mathscr{F}\left[t^{n}f\left(t\right)\right]=\mathrm{i}^{n}F^{(n)}\left(\omega\right)$	$\mathscr{L}\left[t^{n}f\left(t\right)\right]=\left(-1\right)^{n}F^{(n)}\left(s\right)$
积分性质	$\mathscr{F}\left[\int_{-\infty}^{t}f\left(t\right)\mathrm{d}t\right]=\frac{1}{\mathrm{i}\omega}F\left(\omega\right)$	$\mathscr{L}\left[\int_{0}^{t}f\left(\tau\right)\mathrm{d}\tau\right]=\frac{1}{s}F\left(s\right)$
		$\mathscr{L}\left[\frac{f(t)}{t}\right]=\int_{s}^{+\infty}F\left(s\right)\mathrm{d}s$
		$\int_{0}^{+\infty}\frac{f(t)}{t}\mathrm{d}t=\int_{0}^{+\infty}F\left(s\right)\mathrm{d}s$
位移性质	$\mathscr{F}\left[f\left(t+t_{0}\right)\right]=\mathrm{e}^{\mathrm{i}\omega t_{0}}F\left(\omega\right)$	$\mathscr{L}\left[\mathrm{e}^{at}f\left(t\right)\right]=F\left(s-a\right)$
	$\mathscr{F}\left[\mathrm{e}^{\mathrm{i}\omega_{0}t}f\left(t\right)\right]=F\left(\omega-\omega_{0}\right)$	
Fourier 变换相似性质 Laplace 变换延迟性质	$\mathscr{F}\left[f\left(at\right)\right]=\frac{1}{|a|}F\left(\frac{\omega}{a}\right)$	$\mathscr{L}\left[f\left(t-\tau\right)u\left(t-\tau\right)\right]=\mathrm{e}^{-s\tau}F\left(s\right)$

2.3   Laplace 逆变换

部分分式法

有些题目可将有理分式化为多个真分式之和，真分式形如

$$\frac{b}{\left(x+a\right)^{m}}$$

$$\frac{dx+e}{\left(ax^{2}+bx+c\right)^{n}},a\neq0,b^{2}-4ac<0$$

配方法

有些题目可将分母进行配方，配凑出 $\sin\left(kt\right)$ 或者 $\cos\left(kt\right)$ 的 Laplace 变换形式

例 2.3.1.

$$
\begin{aligned}
\mathscr{L}^{-1}\left[\frac{2}{\left(s+1\right)\left(s+2\right)}\right] &= \mathscr{L}^{-1}\left[2\cdot\left(\frac{1}{s+1}-\frac{1}{s+2}\right)\right]\\
&= 2\mathscr{L}^{-1}\left[\frac{1}{s+1}-\frac{1}{s+2}\right]\\
&= 2\left(\mathrm{e}^{-t}-\mathrm{e}^{-2t}\right)
\end{aligned}
$$

例 2.3.2.

$$\mathscr{L}^{-1}\left[\frac{s-1}{s\,(s+1)\,(s+2)}\right]$$

不妨设

$$\frac{s-1}{s\,(s+1)\,(s+2)}=\frac{a}{s}+\frac{b}{s+1}+\frac{c}{s+2}$$

易得

$$s-1=a\,(s+1)\,(s+2)+bs\,(s+2)+cs\,(s+1)$$

因此有

$$\frac{s-1}{s\,(s+1)\,(s+2)}=\frac{-\frac{1}{2}}{s}+\frac{2}{s+1}+\frac{-\frac{3}{2}}{s+2}$$

从而

$$
\begin{aligned}
\mathscr{L}^{-1}\left[\frac{s-1}{s\,(s+1)\,(s+2)}\right] &= \mathscr{L}^{-1}\left[\frac{-\frac{1}{2}}{s}+\frac{2}{s+1}+\frac{-\frac{3}{2}}{s+2}\right]\\
&= -\frac{1}{2}\mathscr{L}^{-1}\left[\frac{1}{s}\right]+2\mathscr{L}^{-1}\left[\frac{1}{s+1}\right]-\frac{3}{2}\mathscr{L}^{-1}\left[\frac{1}{s+2}\right]\\
&= -\frac{1}{2}+2\mathrm{e}^{-t}-\frac{3}{2}\mathrm{e}^{-2t}
\end{aligned}
$$

例 2.3.3.

$$\mathscr{L}^{-1}\left[\frac{s+2}{s\,(s+1)^{2}}\right]$$

不妨设

$$\frac{s+2}{s\,(s+1)^{2}}=\frac{a}{s}+\frac{b}{s+1}+\frac{c}{(s+1)^{2}}$$

则我们有

$$s+2=a\,(s+1)^{2}+bs\,(s+1)+cs$$

易得

$$\frac{s+2}{s\,(s+1)^{2}}=\frac{2}{s}+\frac{-2}{s+1}+\frac{-1}{(s+1)^{2}}$$

而

$$\frac{-1}{(s+1)^{2}}=\left(\frac{1}{s+1}\right)'$$

且

$$\mathscr{L}\left[tf\,(t)\right]=-F'\,(s)$$

因此

$$\mathscr{L}^{-1}\left[\frac{-1}{(s+1)^{2}}\right]=t\mathrm{e}^{-t}$$

从而

$$
\begin{aligned}
\mathscr{L}^{-1}\left[\frac{s+2}{s\,(s+1)^2}\right] &= \mathscr{L}^{-1}\left[\frac{2}{s}+\frac{-2}{s+1}+\frac{-1}{(s+1)^2}\right] \\
&= 2\mathscr{L}^{-1}\left[\frac{1}{s}\right]-2\mathscr{L}^{-1}\left[\frac{1}{s+1}\right]+\mathscr{L}^{-1}\left[\frac{-1}{(s+1)^2}\right] \\
&= 2-2\mathrm{e}^{-t}+t\mathrm{e}^{-t}
\end{aligned}
$$

例 2.3.4.

$$
\mathscr{L}^{-1}\left[\frac{s}{(s^2+1)^2}\right]
$$

注意到

$$
\left(\frac{1}{s^2+1}\right)'=-\frac{2s}{(s^2+1)^2}
$$

从而

$$
\begin{aligned}
\mathscr{L}^{-1}\left[\frac{s}{(s^2+1)^2}\right] &= \mathscr{L}^{-1}\left[\frac{-2s}{(s^2+1)^2}\right]\times\left(-\frac{1}{2}\right) \\
&= -\frac{1}{2}\mathscr{L}^{-1}\left[\left(\frac{1}{s^2+1}\right)'\right] \\
&= \frac{1}{2}\mathscr{L}^{-1}\left[-\left(\frac{1}{s^2+1}\right)'\right] \\
&= \frac{1}{2}t\sin t
\end{aligned}
$$

2.4　卷积

概念

$$
f_1\left(t\right)*f_2\left(t\right)=\int_0^t f_1\left(\tau\right)f_2\left(t-\tau\right)\mathrm{d}\tau
$$

满足的运算律

对于 Laplace 卷积，其满足的运算律与 Fourier 卷积满足的运算律一致

卷积定理

设 $\mathscr{L}\left[f_1\left(t\right)\right]=F_1\left(s\right)$，$\mathscr{L}\left[f_2\left(t\right)\right]=F_2\left(s\right)$，则

$$\mathscr{L}\left[f_1\left(t\right)*f_2\left(t\right)\right]=F_1\left(s\right)\cdot F_2\left(s\right)$$

$$\mathscr{L}^{-1}\left[F_1\left(s\right)\cdot F_2\left(s\right)\right]=f_1\left(t\right)*f_2\left(t\right)$$

例 2.4.1. 求

$$f_1\left(t\right)=\begin{cases}0,t\leqslant 0\\t,t>0\end{cases}$$

与

$$f_2\left(t\right)=\begin{cases}0,t\leqslant 0\\\sin t,t>0\end{cases}$$

的卷积

解

1) 当 $t\leqslant 0$ 时，$f_1\left(t\right)*f_2\left(t\right)=0$

2) 当 $t>0$ 时，

$$\mathscr{L}\left[f_1\left(t\right)*f_2\left(t\right)\right]=\mathscr{L}\left[f_1\left(t\right)\right]\cdot\mathscr{L}\left[f_2\left(t\right)\right]$$

由于

$$\mathscr{L}\left[f_1\left(t\right)\right]=\frac{1}{s^2}$$

$$\mathscr{L}\left[f_2\left(t\right)\right]=\frac{1}{s^2+1}$$

则

$$\mathscr{L}\left[f_1\left(t\right)*f_2\left(t\right)\right]=\frac{1}{s^2}\cdot\frac{1}{s^2+1}=\frac{1}{s^2}-\frac{1}{s^2+1}$$

因此

$$f_1\left(t\right)*f_2\left(t\right)=\mathscr{L}^{-1}\left[\frac{1}{s^2}-\frac{1}{s^2+1}\right]=t-\sin t$$

2.5　Laplace 变换的应用

解线性微分方程

例 2.5.1. 解线性微分方程:

$$y'' + 2y' - 3y = \mathrm{e}^{-t}$$

且满足初值条件:

$$y(0) = 0, y'(0) = 1$$

解

令 $y = y(t)$ 是该微分方程的解, 且 $\mathscr{L}[y(t)] = Y(s)$

则我们有

$$\mathscr{L}[y''(t)] + 2\mathscr{L}[y'(t)] - 3\mathscr{L}[y(t)] = \mathscr{L}[\mathrm{e}^{-t}]$$

而又由于

$$\mathscr{L}[y''(t)] = s^2 Y(s) - sy(0) - y'(0)$$

$$\mathscr{L}[y'(t)] = sY(s) - y(0)$$

则

$$s^2 Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) - 3Y(s) = \frac{1}{s+1}$$

整理得

$$\left(s^2 + 2s - 3\right) Y(s) = \frac{s+2}{s+1}$$

即

$$Y(s) = \frac{s+2}{(s+1)(s-1)(s+3)}$$

而又由于

$$\frac{s+2}{(s+1)(s-1)(s+3)} = \frac{-\frac{1}{4}}{s+1} + \frac{\frac{3}{8}}{s-1} + \frac{-\frac{1}{8}}{s+3}$$

则

$$
\begin{aligned}
y(t) &= \mathscr{L}^{-1}\left[\frac{-\frac{1}{4}}{s+1} + \frac{\frac{3}{8}}{s-1} + \frac{-\frac{1}{8}}{s+3}\right] \\
&= -\frac{1}{4}\mathrm{e}^{-t} + \frac{3}{8}\mathrm{e}^{t} - \frac{1}{8}\mathrm{e}^{-3t} \\
&= \frac{1}{8}\left(3\mathrm{e}^{t} - 2\mathrm{e}^{-t} - \mathrm{e}^{-3t}\right)
\end{aligned}
$$