Decremental Connectivity in Trees

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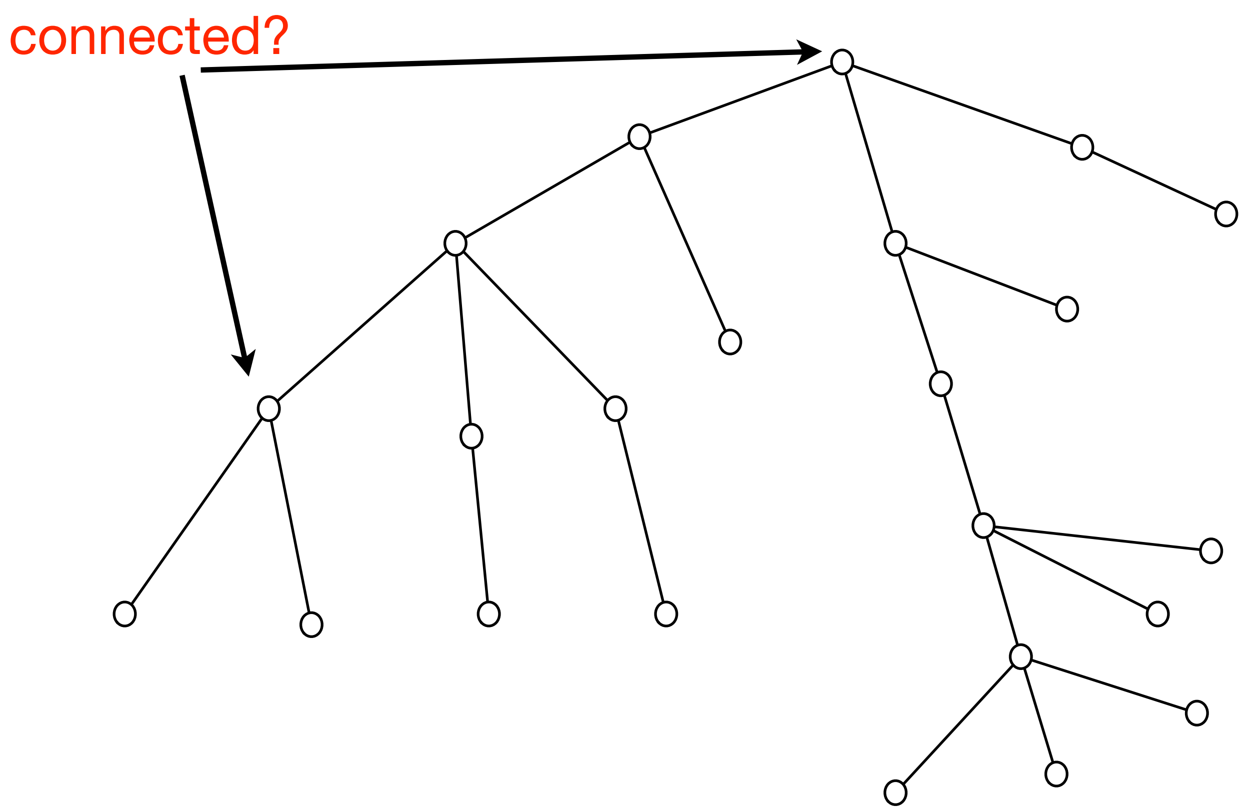
Outline

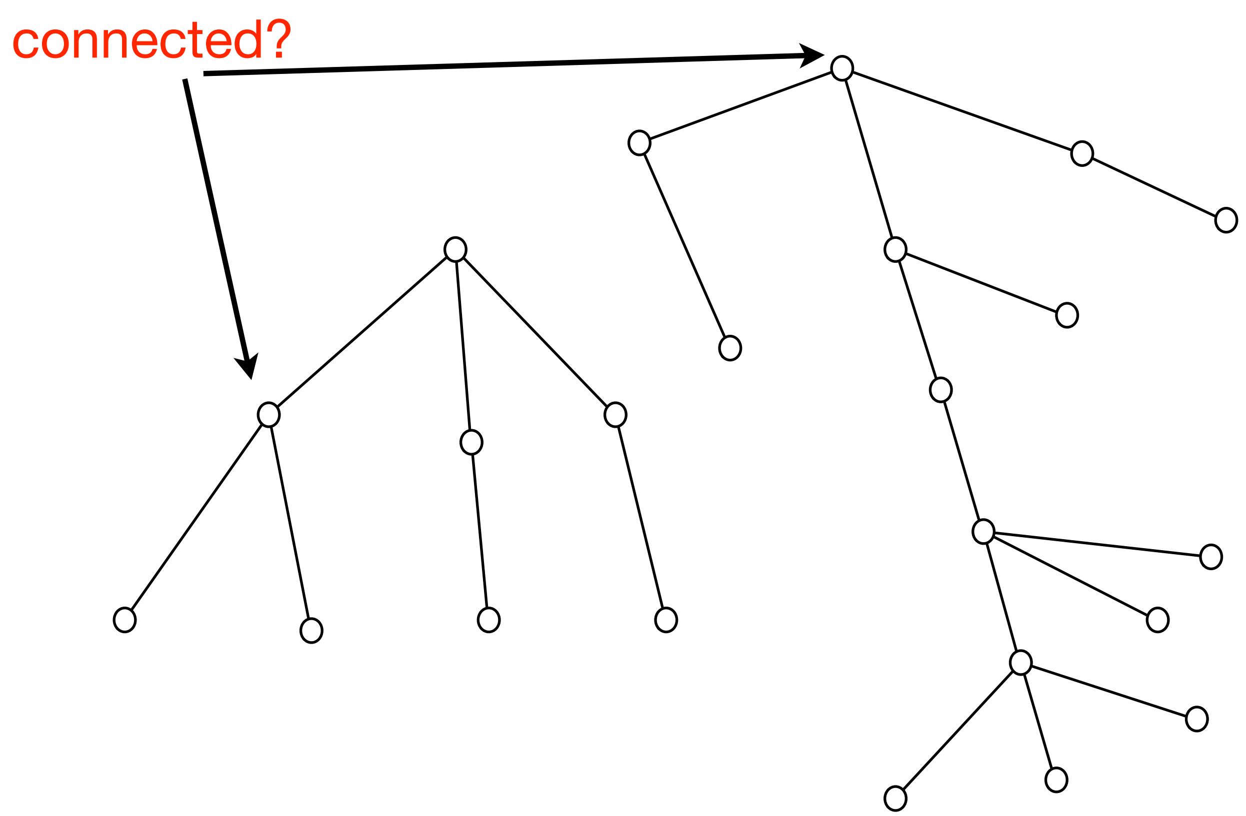
- Decremental Connectivity in Trees Problem
- Decremental Connectivity in Paths
    - First Tradeoffs
    - Two-Level Solution
- Decremental Connectivity in Trees
    - Two-Level Solution

Decremental Connectivity Problem

Decremental Connectivity Problem

- The *decremental connectivity problem*: Starting with a tree T with n nodes support the following operations:
- connected(v,u): return true if v and w are connected.
- delete(e): delete the edge e.

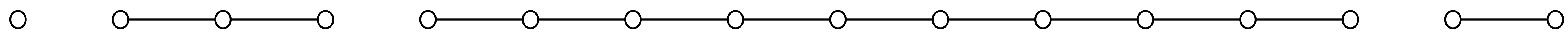

connected?


connected?

Applications

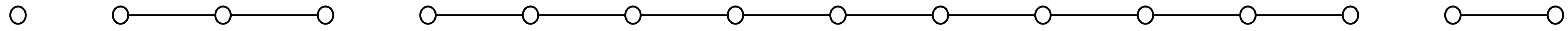
- Special case of dynamic graph algorithm (inserting and deleting edges/nodes) while supporting some query/queries.
- Nice illustration of techniques for trees and word-level parallelism.
- Nice illustration of algorithmic theory useful in practice.

Overview



- First consider the simple case of *paths.* Later generalize to trees.
- Goal:
  - O(1) for connected.
  - O(n) total for executing all n-1 delete operations.

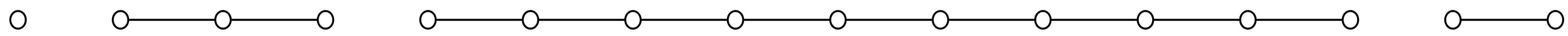
Overview



- Solution in 3 steps:
  - First tradeoffs: queries vs. updates
  - Balanced relabeling
  - Clustering with word-level parallelism

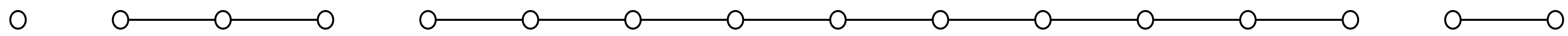
First Tradeoffs

First Tradeoffs



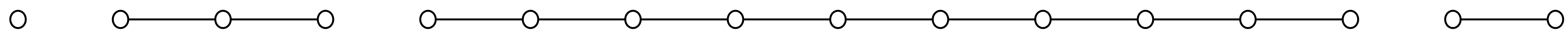
- What tradeoffs can we get for connected queries vs. deletions?

Fast Deletions



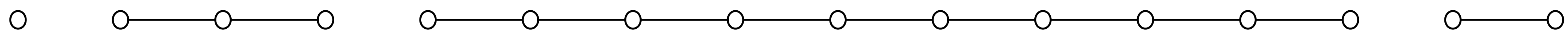
- How fast deletions can we get if we ignore the time for connected?

Fast Deletions



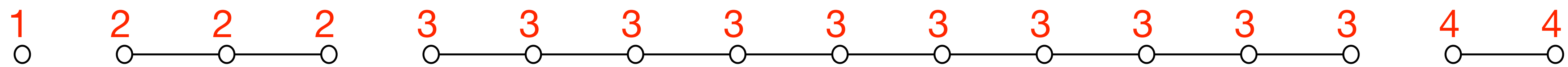
- Solution:
    - delete(e): remove e from path.
    - connected(v,u): traverse component from v to see if u is in component containing v.
- O(1) for deletion => O(n) for n-1 deletions
- O(n) for connected

Fast Connected Queries



- How fast connected queries can we get if we ignore the time for deletions?

Fast Connected Queries

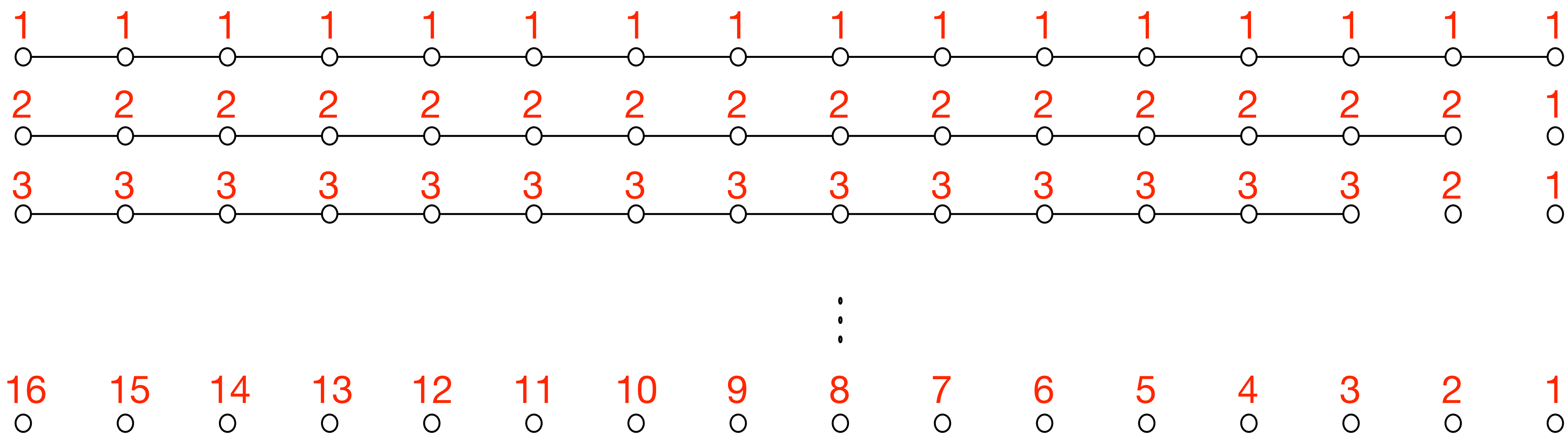


- Solution: Maintain a component ID for each node.
- v and u are connected iff ID(v) = ID(u).
  - connected(v,u): return true iff ID(v) = ID(u)
  - delete(e): relabel node IDs of left endpoint of e (or right endpoint of e).
- O(1) for connected
- O(n) for single delete => O(n^2) for n-1 deletions

Better Tradeoffs?

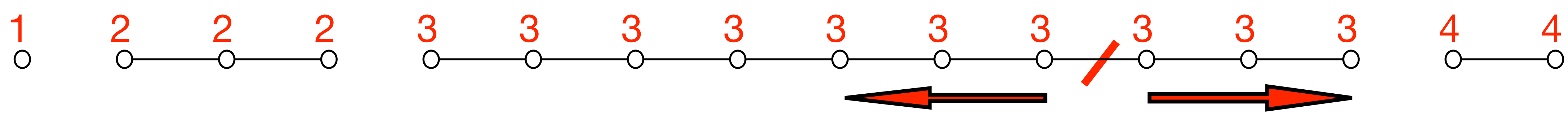
- We have two extreme tradeoffs:
  - connected $O(n)$ and $n-1$ deletes $O(n)$.
  - connected $O(1)$ and $n-1$ deletes $O(n^2)$.
- How can we get better tradeoffs?
- Consider the bad case for relabeling algorithm.

A Bad Delete Sequence



- Main problem: n-1 delete operations take
  - $O((n-1) + (n-2) + \cdots + 1) = O(n^2)$
- How can we do better?
- Idea: After a delete update ID for the *smaller component.*

Balanced Relabeling



- New delete(e):
  - Traverse components for endpoints of e *in parallel*.
  - Stop when we find the *smallest* component.
  - Update ID for smallest component.
- ID(v) is only updated when v is in the smaller component.
- => After first update to ID(v) the size of v's component is ≤ n/2, next ≤ n/4, next ≤ n/8,...
- => At most log n updates for ID(v).
- Total time for n-1 deletions = O(Total number of ID updates) = O(n log n).

Summary

- Theorem: We can solve decremental connectivity in paths in
  - O(1) time for connected.
  - O(n log n) time for n-1 deletes.
- How can we *shave* off the log-factor?

Shaving a Log-Factor

LOG FACTORS

"A goal is a dream with a deadline"

— Napoleon Hill

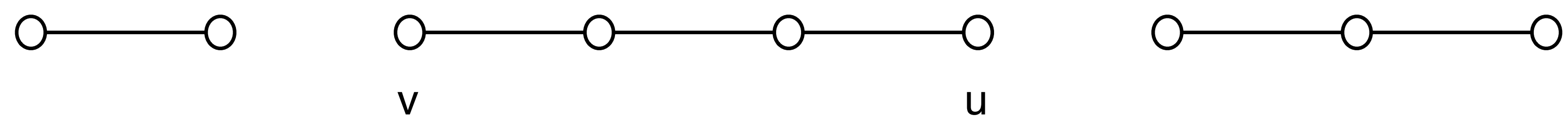
Overview

- Goal:
  - $O(1)$ time connected
  - $O(n)$ time for $n-1$ deletes.
- Solution by two-level data structure:
  - Divide path into n/w subpath of w nodes.
  - Level 1: Balanced relabeling data structure over path of n/w nodes.
  - Level 2: New data structure for paths of length $\leq w$ supporting delete and connected in $O(1)$ time.

Data Structure for Short Paths

Data Structure for Shorts Paths

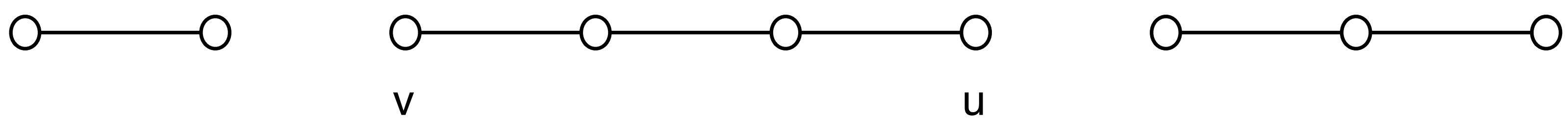
R(v)	0	0	1	1	1	1	1	1
L(u)	1	1	1	1	1	0	0	0
C	1	0	1	1	1	0	1	1



- Paths of length ≤ w.
- Data structure consists of bitmasks:
- C: C[i] = 1 iff edge i exists.
- For each node v:
    - R(v): R(v)[i] = 1 iff i is to the right of v.
    - L(v): L(v)[i] = 1 iff i is to the left of v.
- O(w) bitmasks of length ≤ w => O(w) space.

Delete

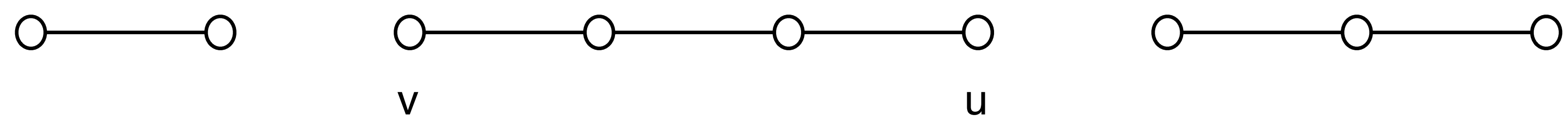
R(v)	0	0	1	1	1	1	1	1
L(u)	1	1	1	1	1	0	0	0
C	1	0	1	1	1	0	1	1



- How can we implement delete in $O(1)$ time?
- delete(edge i): set C[i] = 0.

Connected

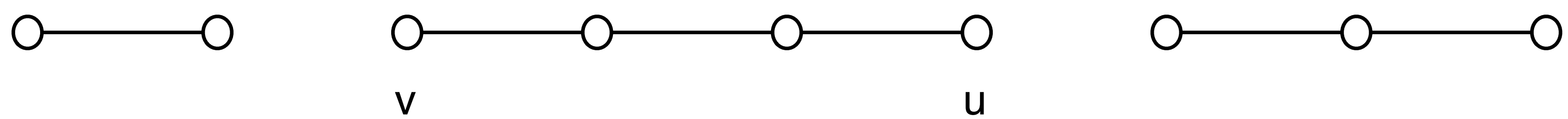
R(v)	0	0	1	1	1	1	1	1
L(u)	1	1	1	1	1	0	0	0
C	1	0	1	1	1	0	1	1



- How can we implement connected in O(1) time?
- connect(v,u) = Return true iff R(v) & L(u) & ¬C = 0

Summary

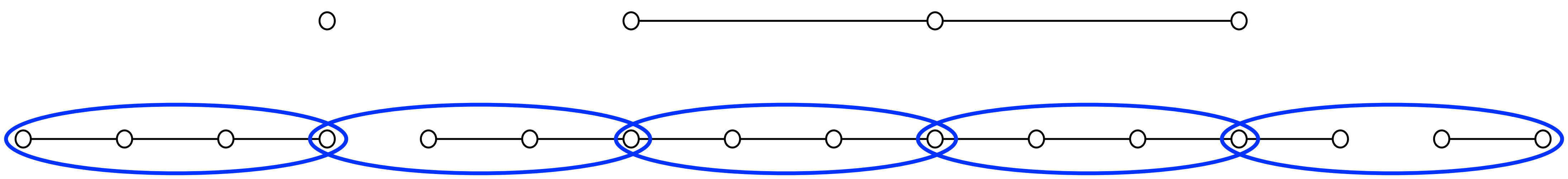
R(v)	0	0	1	1	1	1	1	1
L(u)	1	1	1	1	1	0	0	0
C	1	0	1	1	1	0	1	1



- Theorem: We can solve decremental connectivity in paths of *length* ≤ *w* in
  - O(1) time for connected.
  - O(1) time for delete.

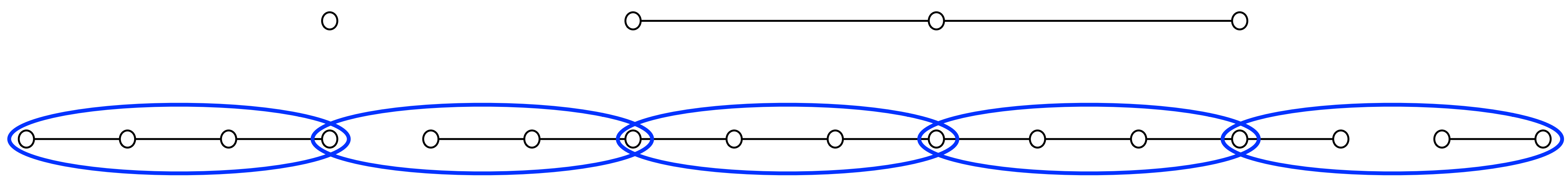
Two-Level Data Structure

A Two-Level Solution



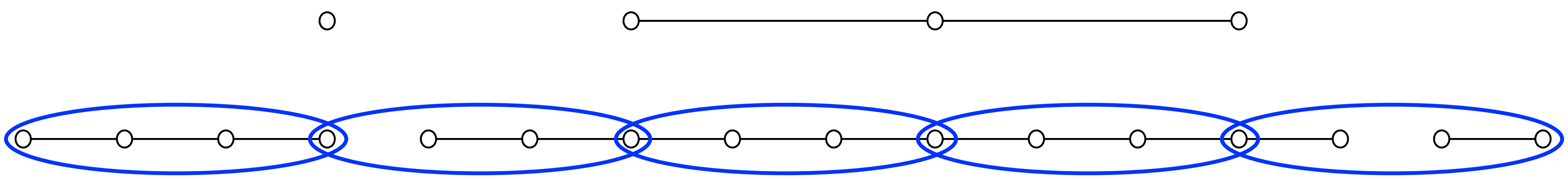
- Divide path into n/w subpath of w nodes.
- Level 1: The O(n/w) boundary nodes of the subpaths. Edge between two boundary nodes iff connected by subpath. Maintain using balanced relabeling data structure.
- Level 2: Short path data structure for each subpath.

A Two-Level Solution



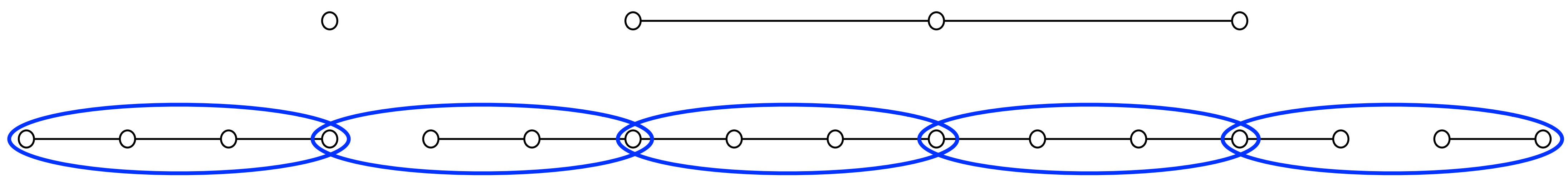
- delete(e): delete edge in level 2. If first deletion in subpath also delete in level 1 using balanced relabeling strategy.
- connected(v,u):
  - Case 1: v and u in same subpath. Use level 2 data structure.
  - Case 2: v and u in different subpaths. Return true iff
    - connected(v, right-boundary(v)) &
    - connected(u, left-boundary(u)) &
    - connected(right-boundary(v), left-boundary(u))

A Two-Level Solution



- O(1) time for connected (at most 1 connected query in level 1 + 2 connected queries in level 2)
- n-1 delete operations:
  - Level 2: O(n)
  - Level 1: $O(n/w \cdot \log n) = O(n)$
- => O(n) in total.

Summary



- Theorem: We can solve decremental connectivity in paths in
  - O(1) time for connected
  - O(n) time for n-1 deletions

Decremental Connectivity in Trees

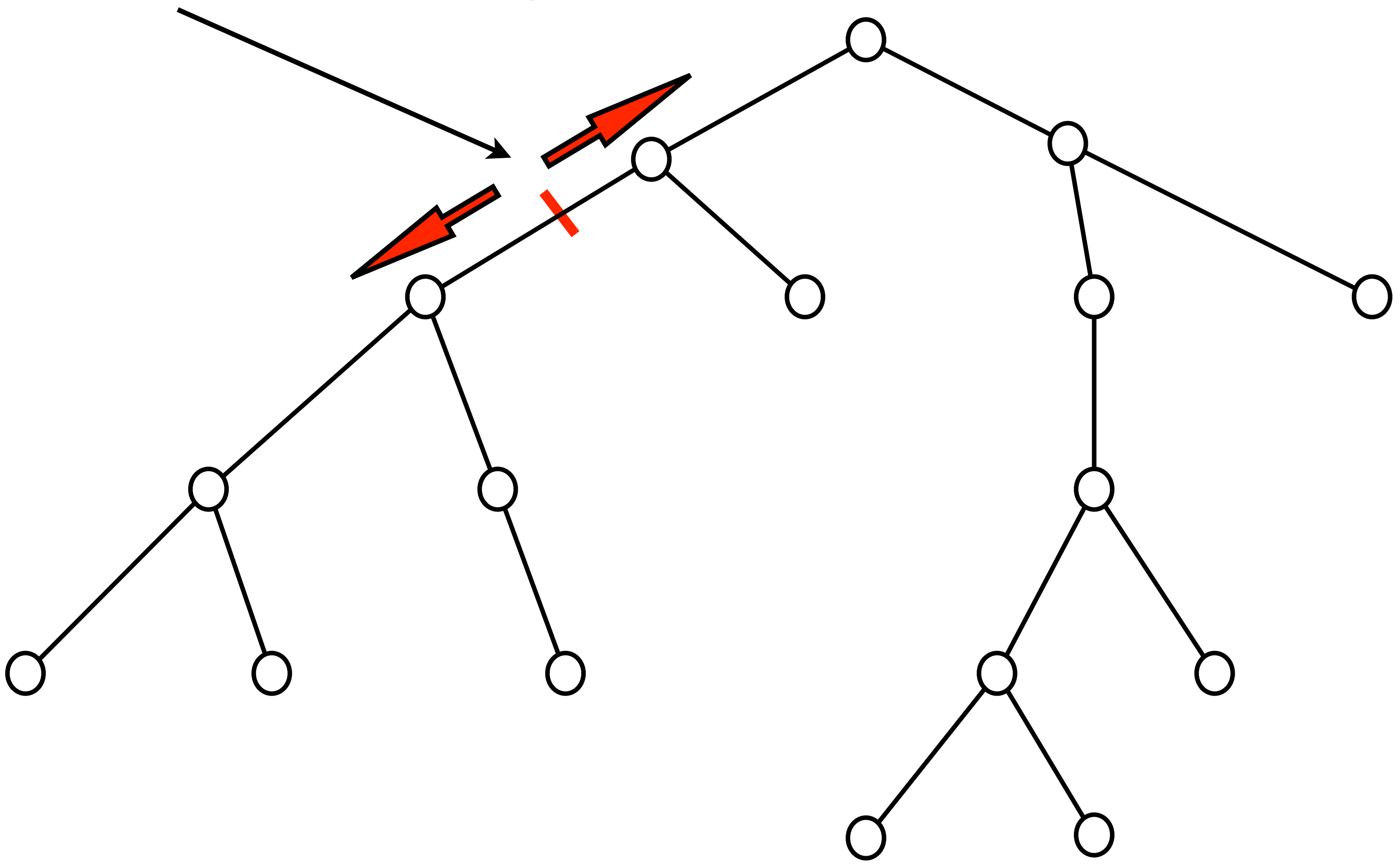
Decremental Connectivity in Trees

- Goal: Generalize two-level data structure from paths to trees.
- Simplifying assumption: Maximum degree of nodes in tree is 3.
- We need:
  - A balanced relabeling algorithm for trees.
  - An algorithm to divide trees into $O(n/w)$ subtrees of $\leq w$ nodes that only overlap in boundary nodes (does such division even exist?)
  - A fast data structure for decremental connectivity on subtrees with $\leq w$ nodes.

Balanced Relabeling for Trees

Balanced Relabeling for Trees

Replace left and right search with
Breadth-first search away from edge



Balanced Relabeling for Trees

- Breadth-first search uses time linear in size of component.
- => Same analysis as with paths
- => $O(n \log n)$ time for $n-1$ deletes.

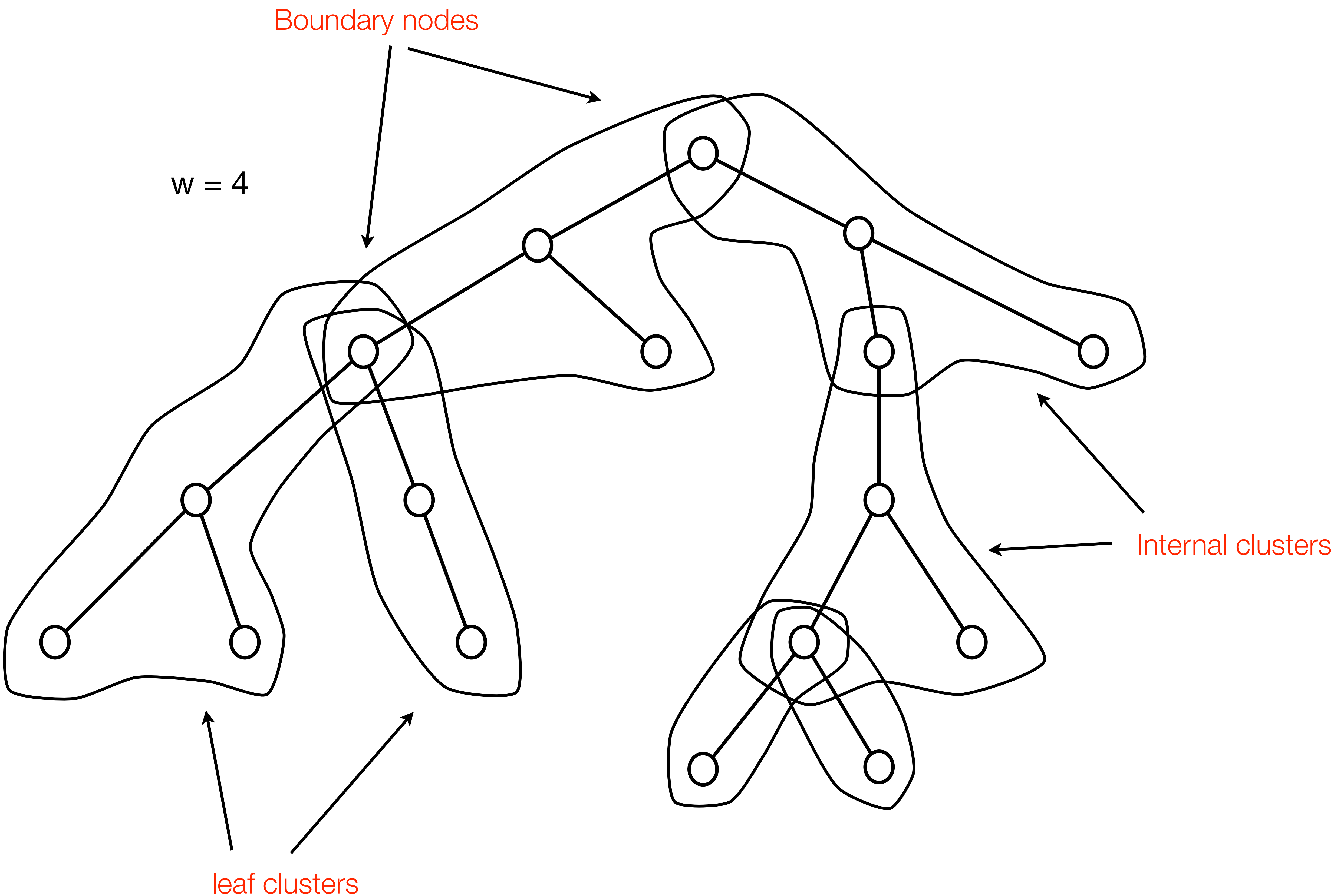
Summary

- Theorem: We can solve decremental connectivity in *trees* in
  - O(1) time for connected.
  - O(n log n) time for n-1 deletes.

Tree Clustering

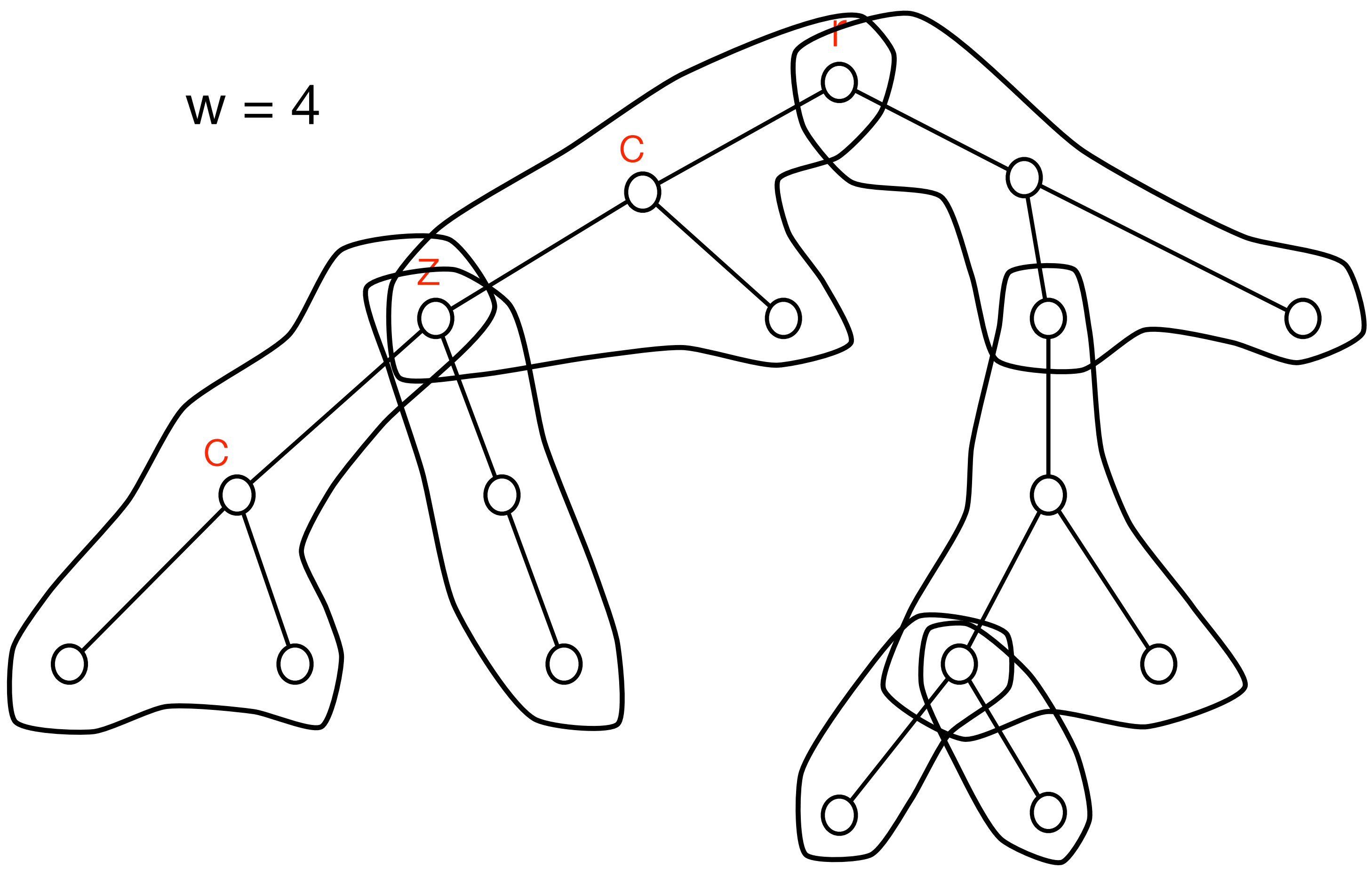
Tree Clustering

- Goal: Given a tree T with maximum degree ≤ 3 compute a *cluster decomposition* of T:
  - Divide T into O(n/w) connected subtrees (*clusters)* of ≤ w nodes.
  - Each cluster overlaps with other clusters in at most 2 *boundary nodes.*


Boundary nodes
w = 4
Internal clusters
leaf clusters

Tree Clustering

- Lemma: A cluster decomposition exists and we can compute it in O(n) time.
- Main idea:
  - Root T at arbitrary node r.
  - Construct clusters greedily top-down.



- For each child c of root r:
  - If c has ≤ w - 1 descendants. Form leaf cluster from r and descendants of c with r as boundary node.
  - If c has ≥ w descendants. Pick descendant z of max depth to form internal cluster of ≤ w nodes with r and z as boundary nodes. Recurse on z.

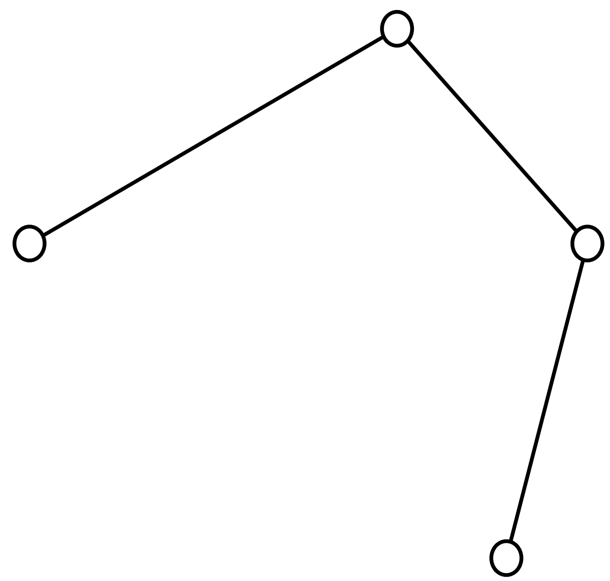
Tree Clustering

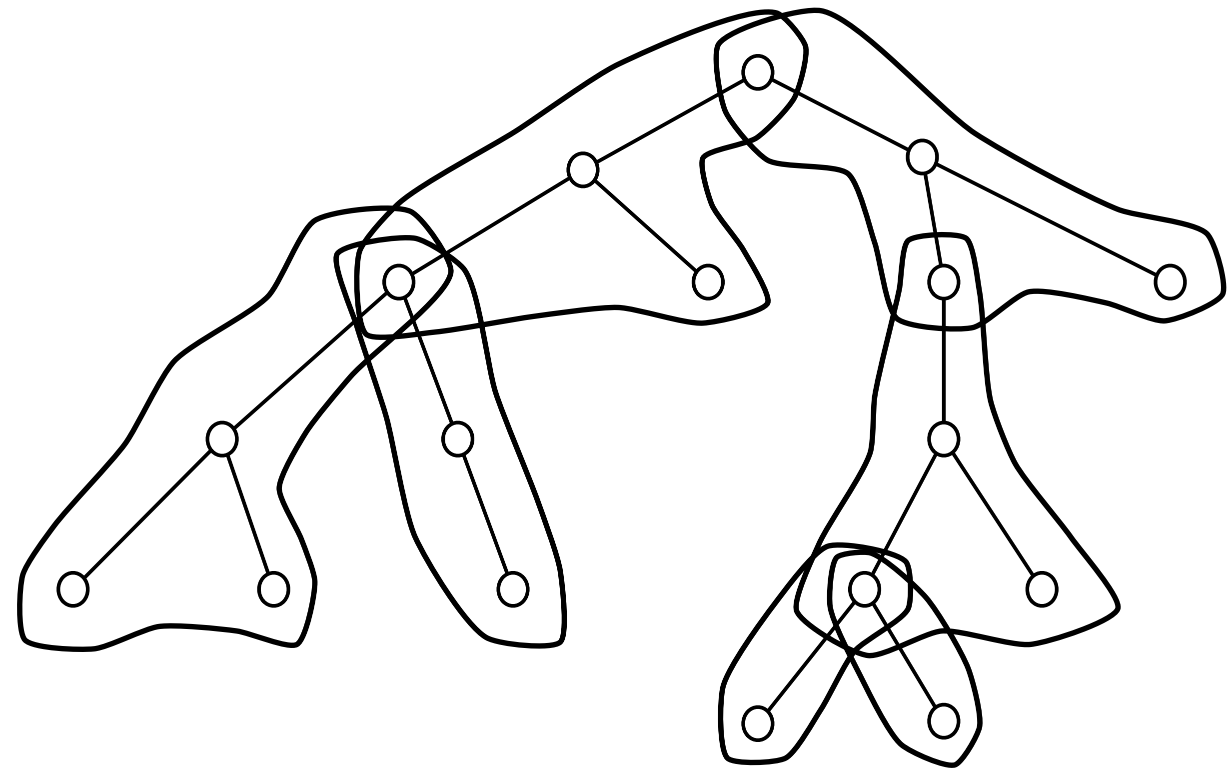
- Why does the clustering algorithm produce $O(n/w)$ clusters each with $\leq w$ nodes?
- Each cluster has $\leq w$ nodes.
- How many clusters do we get?
- Intuition:
  - Greed => A constant fraction of cluster will have $\Omega(w)$ nodes.
  - => There will be a most $O(n/w)$ clusters.

Data Structure for Small Trees

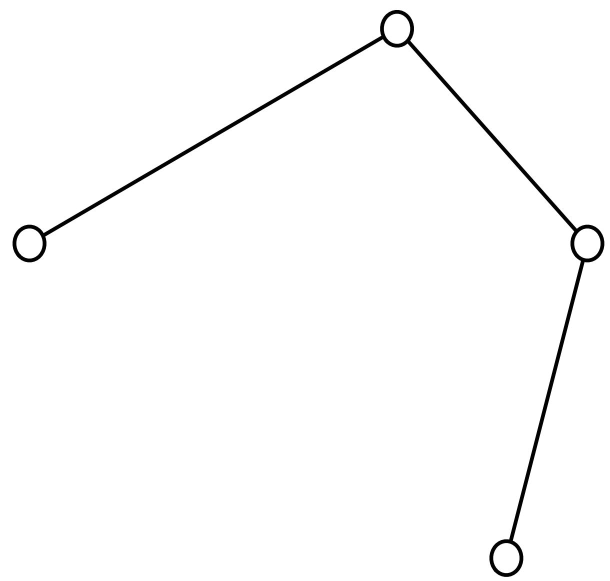
Data Structure for Small Trees

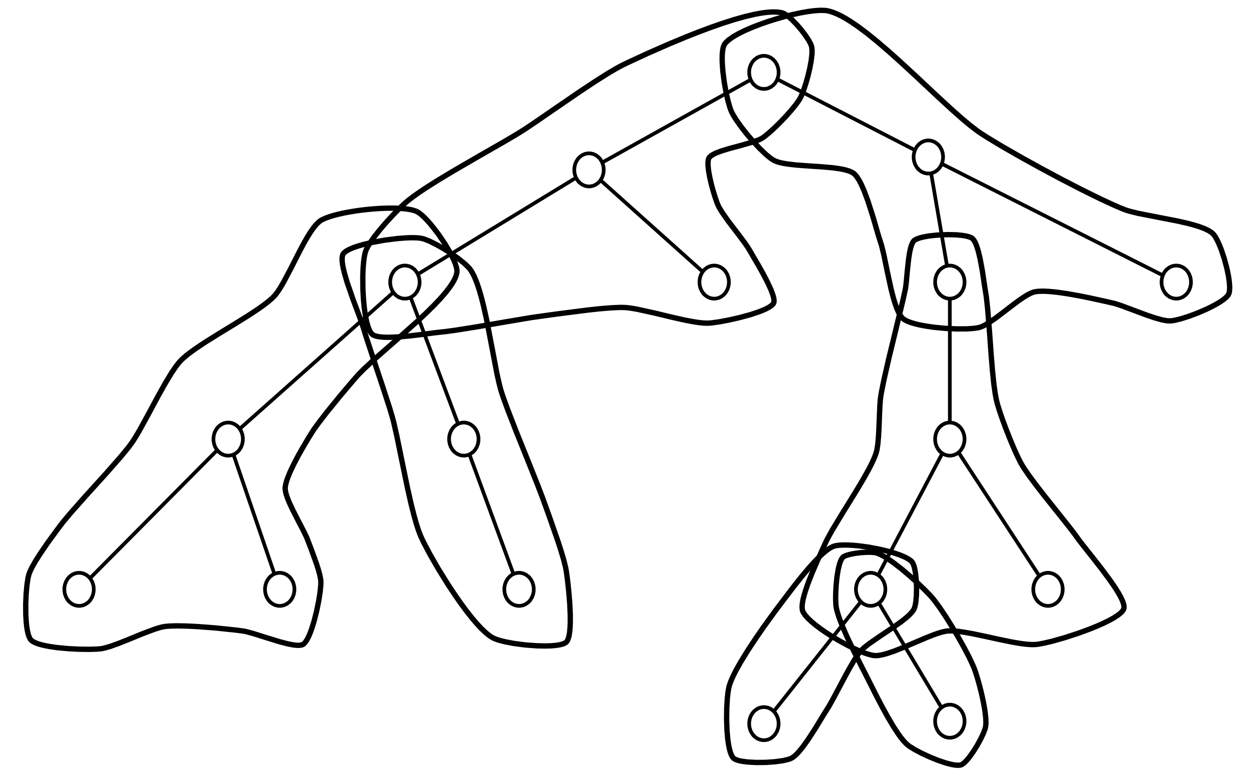
- Theorem: We can solve decremental connectivity in trees with at most ≤ w nodes in
  - O(1) time for connected.
  - O(1) time for delete





- Two-level data structure.
- Level 1: The O(n/w) boundary nodes of the clusters. Edge between two boundary nodes iff connected in cluster. Maintain using balanced relabeling data structure.
- Level 2: Small tree data structure for each cluster.





- Analysis: Exactly as in the path case
- Theorem: We can solve decremental connectivity in trees in
  - O(1) time for connected
  - O(n) time for n-1 deletions

Summary

- Decremental Connectivity in Trees Problem
- Decremental Connectivity in Paths
    - First Tradeoffs
    - Two-Level Solution
- Decremental Connectivity in Trees
    - Two-Level Solution

References

- S. Alstrup, J. P. Secher, M. Spork: Optimal On-Line Decremental Connectivity in Trees, Inf. Process. Lett., 1997
- S. Alstrup, J. P. Secher, M. Thorup: Word encoding tree connectivity works. SODA, 2000