Linear Regression Inference

Statistics on a linear model

Download the section 16 .Rmd handout to `STAT240/lectures/16-regression-inference`.

Material in this section is covered by Chapter 14 on the notes website.

Download the files `lake-monona-winters-2024.csv`, `riley.txt` and `lions.csv` to `STAT240/data`.

We have not done any statistics on our model yet.

$$\hat{y}_i \;=\; \hat{\beta}_0 + \hat{\beta}_1 x_i$$

Is there an actual linear relationship between x and y? There is when the slope β_1 is nonzero.

Let's build an interval estimate for $\hat{\beta}_1$.

point estimate ± critical value × standard error

Here, the point estimate is $\hat{\beta}_1$.

What is the estimation error of $\hat{\beta}_1$?

$$se(\hat{\beta}_1) \;=\; \frac{\sigma}{\sqrt{(n-1)s_X^2}}$$

- Numerator: spread of points around line
- Denominator: variability of X

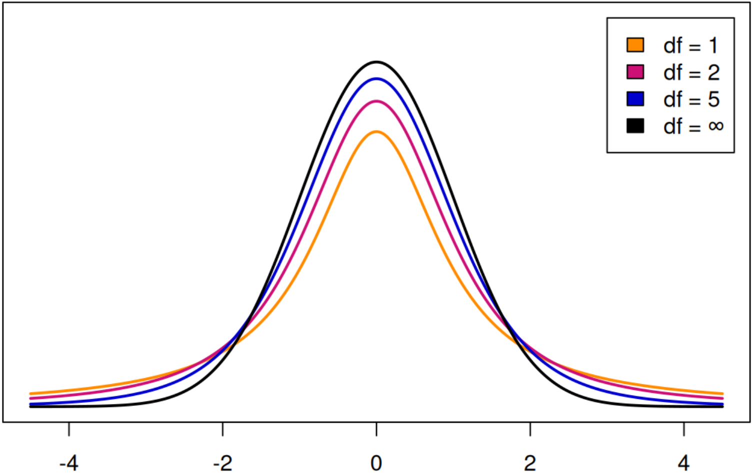
To guarantee $1 - \alpha$ coverage, we use a quantile from the T distribution.

Why? Need to estimate σ with S.

- "Residual standard error"

In simple linear regression, df $= n - 2$.

T distributions



$$\hat{\beta}_1 \;\pm\; t_{\alpha/2,n-2} \times \frac{S}{\sqrt{(n-1)s_X^2}}$$

Find these values with the `lm` summary.

A 95% CI for slope in the height model is:

$$(0.244, 0.256)$$

$$\hat{\beta}_1 \pm t_{\alpha/2,n-2} \times \frac{S}{\sqrt{(n-1)s_X^2}}$$

Build and interpret a 98% CI for the slope of the Lake Monona linear model.

- Try to calculate SE by hand

Are we confident that year and duration are related?

We can also perform a T test for the slope. The test statistic looks like:

$$\frac{\text{Esimated slope} \; - \; \text{Slope from } H_0}{\text{Estimation error}}$$

If the null is true, the test statistic is close to 0.

Let’s test whether the slope of the Lake Monona model is negative, with $\alpha = 0.05$.

This is a one-sided test of

$$H_0 : \beta_1 = 0 \quad \text{versus} \quad H_A : \beta_1 < 0$$

For a slope test, our test statistic is

$$T = \frac{\hat{\beta}_1 - \beta_{null}}{\frac{S}{\sqrt{(n-1)s_X^2}}}$$

If H_0 is true and $\beta_1 = 0$, then T follows a T distribution with $n-2$ degrees of freedom.

We have

$$t_{obs} = \frac{-0.224 - 0}{0.02636} = -8.5$$

Is this value consistent with a t_{n-2} distribution?
No - very small p-value in “less than” direction

What if we just want to know if x and y are related?

$$H_0 : \beta_1 = 0 \quad \text{versus} \quad H_A : \beta_1 \neq 0$$

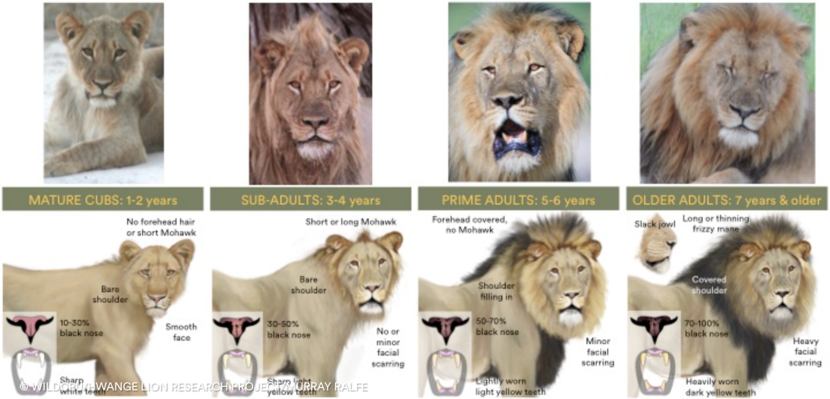
We would have the same test statistic, but a different p-value.

This is also given in the `lm()` output.

We've seen how to predict a value with a linear model. Let's turn that into an interval.

predicted value ± critical value × prediction error

In the lion ages data, we want to relate a lion's age to the % of its nose that is black.



© WILDCRU/HWANGE LION RESEARCH PROJECT/MURRAY RALFE

Six traits can be used to accurately estimate a lion's age: nose darkness, mane growth (in males), facial scarring, teeth color and wear, and jowl slackness. Due to variance between individuals, age should be estimated based on multiple characteristics.

We predict a 5-year-old lion to have a 36% black nose. Formally,

$$\left(\hat{y} \mid x^* \ = \ 5\right) \ = \ 0.36$$

$$\text{(Fitted value given } x^*\text{)} \ = \ \hat{\beta}_0 + \hat{\beta}_1 x^*$$

The uncertainty in this point estimate depends on what exactly we are predicting.

- Predicting the position of the line itself. This is the *average* nose % for all 5-year-old lions.
- Predicting the nose % for a *single* 5 year old lion.

The first type of prediction, the position of the line, is $E(\hat{y} \mid x^*)$.

Let's investigate this with simulation.

- Generate *n* random points from $\beta_0 + \beta_1 x + \epsilon$
- Calculate $\hat{\beta}_1$ and $\hat{\beta}_0$ and plot the line

The estimated standard error of the position of the line is

$$\hat{se}(E(\hat{y} \mid x^*)) = S\sqrt{\frac{1}{n} + \frac{(x^* - \bar{x})^2}{(n-1)s_X^2}}$$

The critical value for our CI is the same as before. It is a $\alpha/2$ critical value from the T with $n-2$ degrees of freedom.

$$\hat{y}|x^* \ \pm \ t_{\alpha/2,n-2} \times S\sqrt{\frac{1}{n} \ + \ \frac{(x^* - \bar{x})^2}{\sum(x_i - \bar{x})^2}}$$

This is what `geom_smooth()` is doing!

Use `predict()` to calculate the CI for us.

Set `interval = "confidence"`. We can also plot this against the data.

The uncertainty in this point estimate depends on what exactly we are predicting.

- Predicting the position of the line itself. This is the *average* nose % for all 5-year-old lions.
- Predicting the nose % for a *single* 5 year old lion.

The second type of prediction is $\hat{y} \mid x^*$.

Again, the point estimate is just found by plugging x^* into the model.

This type of prediction has more error than predicting the position of the line.

The estimated standard error of a new prediction is

$$\hat{se}(\hat{y} \mid x^*) \;=\; S\sqrt{1+\frac{1}{n} \;+\; \frac{(x^*-\bar{x})^2}{\sum(x_i-\bar{x})^2}}$$

We have an extra $+1$ term for applying our model to a new data point.

This gives us a **prediction** interval.

$$\hat{y}|x^* \ \pm \ t_{\alpha/2,n-2} \times S\sqrt{1 + \frac{1}{n} \ + \ \frac{(x^* - \bar{x})^2}{\sum(x_i - \bar{x})^2}}$$

Again, use `predict()`.

Confidence interval: position of the line
Prediction interval: new y value

- PIs are wider than CIs
- Both intervals are wider when we are further from $\bar{x}$