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Question:

1D: $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$

vectors

transpose

k-DIM: $f_X(x_1,\dots,x_k) = \frac{1}{\sqrt{(2\pi)^k|\Sigma|}} \exp\left[-\frac{1}{2}(\bar{x}-\bar{\mu})^T \Sigma^{-1} (\bar{x}-\bar{\mu})\right]$, where

$\sqrt{(\bar{x}-\bar{\mu})\Sigma^{-1}(\bar{x}-\bar{\mu})}$ is the Mahalanobis distance

k=2

YOUR TASK

2-DIM Case:

$$(*) \quad f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right)\right],$$

where $\bar{\mu} = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}$

TASK: Prove that the k-DIM Gaussian reduces to (*) when $k=2$.

$\rho = \rho_{XY} = \frac{Cov(X,Y)}{\sigma_X\sigma_Y}$

NB! $\bar{x} = \vec{x} = \mathbf{x}$

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Solution:

$$f_x(x_1, x_2, \dots, x_k) = \frac{1}{\sqrt{(2\pi)^k |\Sigma|}} \exp\left[-\frac{1}{2}(\bar{x}-\bar{\mu})^T \Sigma^{-1} (\bar{x}-\bar{\mu})\right] \quad \text{— ①}$$

When $k = 2$ then the Covarince Matrix Σ is

$$\Sigma = \begin{bmatrix} \sigma_x^2 & \sigma_{x.y} \\ \sigma_{y.x} & \sigma_y^2 \end{bmatrix}$$

Note: $\rho = \frac{\operatorname{cov}(x,y)}{\sigma_x \sigma_y}$

$\Rightarrow \operatorname{cov}(x,y) = \rho \sigma_x \sigma_y$

ie, $\sigma_{xy} = \rho \sigma_x \sigma_y$

$$\Rightarrow \Sigma = \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}$$

Now taking the exp part from equation of ① for making solution simple in term of calculati-on.

Now, Given

$x = \begin{bmatrix} x \\ y \end{bmatrix}$ &

$\mu = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}$

$$-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)$$

$$\Rightarrow -\frac{1}{2}\begin{bmatrix} x-\mu_x & y-\mu_y \end{bmatrix} \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}^{-1} \begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix}$$

Note: putting the value of Σ

$$= \cancel{\begin{bmatrix} x-\mu_x \end{bmatrix}} \Rightarrow -\frac{1}{2}\begin{bmatrix} x-\mu_x & y-\mu_y \end{bmatrix} \frac{1}{\sigma_x^2\sigma_y^2 - \rho^2\sigma_x^2\sigma_y^2} \begin{bmatrix} \sigma_y^2 & -\rho\sigma_x\sigma_y \\ -\rho\sigma_x\sigma_y & \sigma_x^2 \end{bmatrix} \begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix}$$

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$$= -\frac{1}{2}\frac{1}{\sigma_x^2\sigma_y^2 - p^2\sigma_x^2\sigma_y^2}\Big[\sigma_y^2(x-\mu_x)^2 - 2p\sigma_x\sigma_y(x-\mu_x)(y-\mu_y) + \sigma_x^2(y-\mu_y)^2\Big]$$

$$= -\frac{1}{2}\left[\frac{\sigma_y^2(x-\mu_x)^2}{\sigma_x^2\sigma_y^2(1-p^2)} - \frac{2p\sigma_x\sigma_y(x-\mu_x)(y-\mu_y)}{\sigma_x^2\sigma_y^2(1-p^2)} + \frac{\sigma_x^2(y-\mu_y)^2}{\sigma_x^2\sigma_y^2(1-p^2)}\right]$$

$$= -\frac{1}{2}\left[\frac{(x-\mu_x)^2}{(1-p^2)\sigma_x} - \frac{2p(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y(1-p^2)} + \frac{(y-\mu_y)^2}{\sigma_y(1-p^2)}\right]$$

$$= -\frac{1}{2}\left[\frac{(x-\mu_x)^2}{(1-p^2)\sigma_x} - \frac{2p(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y(1-p^2)} + \frac{(y-\mu_y)^2}{\sigma_y(1-p^2)}\right] \quad (4)$$

Now, The Whole equation is

$$= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-p^2}}\exp\left[-\frac{1}{2(1-p^2)}\left(\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - \frac{2p(x-\mu_x)(y-\mu_y)}{\sigma_{xy}}\right)\right]$$