Minimizing Chan-Vese type energies using Finsler minimal paths.

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1 Introduction

To each domain $\Omega \subset \mathbb{R}^2$ attach the energy

$$\mathcal{E}(\Omega) := \int_\Omega \rho + \int_{\partial\Omega} \sigma, \tag{1}$$ eqdef:EOmega

where $\rho : \mathbb{R}^2 \to \mathbb{R}$ has arbitrary sign, and $\sigma : \mathbb{R}^2 \to \mathbb{R}_+^*$ is strictly positive. If ρ has negative sign, then the integral over Ω acts as a balloon force. The objective is to (locally) minimize $\mathcal{E}$ among domains $\Omega \subset \mathbb{R}^2$.

Definition 1. *Let $\Omega \subset \mathbb{R}^2$ be open, simply connected, with Lipschitz boundary, and let $x_0, x_1 \in \partial\Omega$. Then one can parametrize $\partial\Omega$ as the union of two Lipschitz curves γ_+, γ_-, with common endpoints $\gamma_+(0) = \gamma_-(0) = x_0$ and $\gamma_+(1) = \gamma_-(1) = x_1$, going respectively counter-clockwise and clockwise. We write*

$$\Omega = [\gamma_+, \gamma_-].$$

Conversely, let γ_+, γ_- be two curves which are Lipschitz, without self intersections, and which do not intersect each other except at their common endpoints. Then perhaps exchanging the roles of γ_+ and γ_- for orientation one has $\Omega = [\gamma_+, \gamma_-]$.

Let $\Omega = [\gamma_+, \gamma_-]$, and let $x_0 = \gamma_+(0) = \gamma_-(0)$, $x_1 = \gamma_+(1) = \gamma_-(1)$. We design neighborhoods V_+, V_- of γ_+, γ_-, path lengths $\mathrm{length}_+, \mathrm{length}_-$ with respect to adequate Finsler metrics on $V+, V_-$, and a potential function $p : V_+ \cap V_- \to \mathbb{R}$. Then for any curves Γ_+, Γ_- within V_+, V_-, originating from $x_0 = \Gamma_+(0) = \Gamma_-(0)$, and with common endpoint X_1 one has

$$\mathcal{E}([\Gamma_+, \Gamma_-]) = \mathrm{length}_+(\Gamma_+) + \mathrm{length}_-(\Gamma_-) - p(X_1) + p(x_0).$$

Denote by d_+, d_- the geodesic distance from x_0 with respect to $\mathrm{length}_+, \mathrm{length}_-$. Then for improving (1) it suffices to minimize

$$d_+(X_1) + d_-(X_1) - p(X_1)$$

and then extract shortest paths Γ_+, Γ_-. The paths are then reversed and exchanged $[-\Gamma_-, -\Gamma_+] = [\Gamma_+, \Gamma_-]$, in order to exchange the roles of x_0, X_1, and the procedure is repeated.

2 Defining the metrics

Lemma 2. *(*Which are the correct spaces ? What estimates on the growth of Ω ?*) Let $V \subset \mathbb{R}^2$ be a bounded domain, let $\gamma : [0,1] \to \mathbb{R}^2$ be a Lipschitz curve without self-intersections, and let $\rho : V \to \mathbb{R}^2$. Then there exists $\omega : V \to \mathbb{R}^2$ such that (*?? Probably not, but what is the right estimate ??*) $|\omega(x)| \leq \|\rho\|_\infty d(x, \gamma)$.*

Let V_+ (resp. V_-) be a neighborhood of γ_+ (resp. γ_-), and $\omega_+ : V_+ \to \mathbb{R}^2$ (resp. ω_-) a solution to

$$\operatorname{rot} \omega_+ = \rho, \qquad\qquad \operatorname{rot} \omega_- = -\rho.$$

We choose these vector fields

We denoted $\operatorname{rot} \omega := \partial_1 \omega_2 - \partial_2 \omega_1$. By construction one has $\operatorname{rot}(\omega_+ + \omega_-) = 0$ on $V_+ \cap V_-$, hence there exists $p : V_+ \cap V_- \to \mathbb{R}$ such that

$$\omega_+ + \omega_- = \nabla p.$$

The solution ω_+ to $\operatorname{rot} \omega_+ = \rho$ can be extended globally on $\mathbb{R}^2$, hence denoting by τ the counter-clockwise tangent vector to $\partial\Omega$ we can write

$$\begin{aligned}
\int_\Omega \rho &= \int_\Omega \operatorname{rot} \omega_+ \\
&= \int_{\partial\Omega} \langle \omega_+, \tau \rangle \\
&= -\int_0^1 \langle \omega_+, \gamma_+' \rangle + \int_0^1 \langle \omega_+, \gamma_-' \rangle \\
&= -\int_0^1 \langle \omega_+, \gamma_+' \rangle + \int_0^1 \langle \nabla p - \omega_-, \gamma_-' \rangle \\
&= -\int_0^1 \langle \omega_+, \gamma_+' \rangle - \int_0^1 \langle \omega_-, \gamma_-' \rangle + p(y) - p(x).
\end{aligned}$$

For any path (*...*)

$$\mathcal{E}([\gamma_+, \gamma_-]) = \operatorname{length}_+(\gamma_+) + \operatorname{length}_-(\gamma_-).$$

Remark 3 (Divergence theorem)**.** *Denoting by R the counter-clockwise rotation by $\pi/2$, one has $\operatorname{rot} \omega = \nabla \cdot R\omega$ and $\tau = Rn$ where n is the exterior normal. Thus using the identity $R^T = -R$*

$$\int_\Omega \operatorname{rot} \omega = \int_\Omega \nabla \cdot R\omega = \int_{\partial\Omega} \langle R\omega, n \rangle = \int_{\partial\Omega} \langle \omega, R^T n \rangle = -\int_{\partial\Omega} \langle \omega, \tau \rangle.$$

3 Implementation