Guide.

DISCRETE MATHEMATICS, TEST 1, TRINITY 2024

Instructions to Candidates:

(i) Answer all questions.

(ii) Write your name and registration number on top of the paper.

(iii) All questions MUST be answered in the spaces provided on the question paper.

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1. Use the truth table to prove the validity of the following argument. If there is a chance of rain or her red headband is missing, then Lois will not mow her lawn. Whenever the temperature is over 20°C there is no chance for rain. Today the temperature is 22°C and Lois is wearing her red headband. Therefore Lois will mow her lawn. (6 marks)

let p be it rains
q be her red headband is missing.
r be Lois will mow her lawn
s temperature is over 20°C.

$\therefore (p \vee q) \to \neg r$
$s \to \neg p$
$s \wedge \neg q$
r

p	q	r	s	$p \vee q$	$\neg r$	$\neg p$	$(p \vee q) \to \neg r$	$s \to \neg p$	$s \wedge \neg q$	r
T	T	T	T	T	F	F	F	F	F	T
T	T	T	F	T	F	F	F	T	F	T
T	T	F	T	T	T	F	T	F	F	F
T	T	F	F	T	T	F	T	T	F	F
T	F	T	T	T	F	F	F	F	T	T
T	F	T	F	T	F	F	F	T	F	T
T	F	F	T	T	T	F	T	F	T	F
T	F	F	F	T	T	F	T	T	F	F
F	T	T	T	T	F	T	F	T	F	T
F	T	T	F	T	F	T	F	T	F	T
F	T	F	T	T	T	T	T	T	F	F
F	T	F	F	T	T	T	T	T	F	F
F	F	T	T	F	F	T	T	F	T	T
F	F	T	F	F	F	T	T	T	F	T
F	F	F	T	F	T	T	T	F	F	F
F	F	F	F	F	T	T	T	T	F	F

∴ The Argument is a fallacy which is an invalid argument.

2. Re-write $A \wedge B \vee C$ using only formal connectives. (3 marks)

$A \wedge B \vee C \equiv (A \wedge B) \vee C \equiv (\neg(A \to \neg B)) \vee C$

$\equiv \neg(\neg(A \to \neg B)) \to C$

3. Write down all the subformulas for $\rho = ((\neg A_1 \to A_7) \to (A_8 \to A_1))$. (3 marks)

$S(\rho) = \{\neg A_1, A_1, A_7, A_8, (\neg A_1 \to A_7), (A_8 \to A_1), ((\neg A_1 \to A_7) \to (A_8 \to A_1))\}$.

4. Define the following terms.

(a) A language. A language is a set of symbols and rules for assembling symbols into formulas of a language (1 mark)

(b) A proposition. A proposition is a statement that expresses a single idea which is either true or false but not both. (1 mark)

5. Given α and β are atomic formulas, find the length of $\neg\alpha \to \beta$. (2 marks)

$L\{\neg\alpha \to \beta\} = 1 + L\{\neg\alpha\} + L\{\beta\} = 1 + 1 + L\{\alpha\} + L\{\beta\}$

$= 1 + 1 + 1 + 1 = 4$.

6. Prove that $(p \to q), (q \to r) \models (p \to r)$. (4 marks)

p	q	r	p→q	q→r	p→q
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

$\therefore (p \to q), (q \to r) \not\models (p \to q)$ since whenever both premises are true, the conclusion is also true.

7. Construct a truth table for each of the following and determine which of them is a tautology or a contradiction.

(a) $\neg A \to B \wedge C$. (4 marks)

$\equiv \neg A \to (B \wedge C)$

A	B	C	$\neg A$	$B \wedge C$	$\neg A \to (B \wedge C)$
T	T	T	F	T	T
T	T	F	F	F	T
T	F	T	F	F	T
T	F	F	F	F	T
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	F	F
F	F	F	T	F	F

$\neg A \to (B \wedge C)$ is neither a tautology nor a Contradiction

(b) $P \leftrightarrow R \vee S$. (4 marks)

P	R	S	$R \vee S$	$P \leftrightarrow (R \vee S)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	T

$P \leftrightarrow (R \vee S)$ is a fallacy.

8. Prove that $\sqrt{3}$ is irrational. (4 marks)

Suppose $\sqrt{3}$ is a rational number.

Then, $\sqrt{3} = \frac{a}{b}$ $-①$ where $a, b \in \mathbb{Z}$, $\gcd(a,b) = 1$

Squaring both side of ① gives $(\sqrt{3})^2 = \left(\frac{a}{b}\right)^2 \Rightarrow 3 = \frac{a^2}{b^2}$

$\Rightarrow a^2 = 3b^2$ $-②$ ② implies that 3 divides a^2 and therefore 3 divides a. Since 3 divides a, $a = 3l$, $l \in \mathbb{Z}$.

Substitute $a = 3l$ into ② to get, $(3l)^2 = 3b^2 \Rightarrow b^2 = 3l^2$

This implies that 3 divides b^2 and therefore 3 divides b.

$\therefore \gcd(a,b) = 3$ which contradicts that $\gcd(a,b) = 1$ $\therefore$ by contradiction, $\sqrt{3}$ is irrational.

9. With explanation, determine whether the following statements are true or false.

(a) The set of natural numbers is close under subtraction. (2 marks)

False

Since $2, 3 \in \mathbb{N}$ but $2 - 3 = -1 \notin \mathbb{N}$ (counter example).

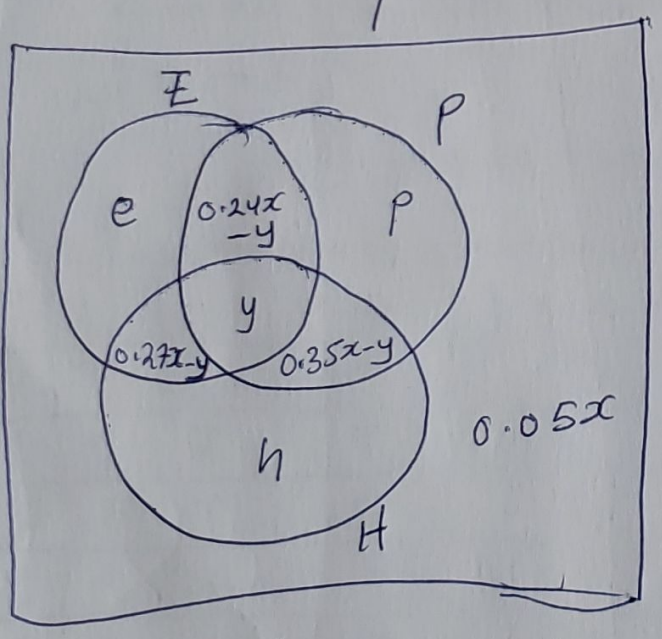
(b) The set of integers is closed under division. (2 marks)

False since $2, 3 \in \mathbb{Z}$ but $3/2 = 1.5 \notin \mathbb{Z}$ (counter example).

10. In a certain examination, 53% of students passed Economics, 61% passed Politics, 60% passed history, 24% passed both Economics and Politics, 35% passed Politics and History, 27% passed Economics and history, and 5% of the students passed none of these subjects.

(a) What percentage of the students passed all the three subjects? (6 marks)

Let the number of students be x.



$0.53x = e + 0.24x - y + 0.27x - y + y$

$e = 0.02x + y$

$0.6x = h + 0.27x - y + y + 0.35x - y$

$h = -0.02x + y$

$0.61x = p + 0.24x - y + y + 0.35x - y$

$p = 0.02x + y$.

$x = e + h + p + 0.35x - y + y + 0.27x - y + 0.24x - y + 0.05x$

$x = 0.02x + y + -0.02x + y + 0.02x + y + 0.35x + 0.27x - y + 0.24x - y + 0.05x$

$x = 0.93x + y \Rightarrow y = 0.07x$.

7% passed all the three subjects.

(b) What percentage of the students passed only one subject? (2 marks)

Percentage of students that passes only one subject = e + p + h

$= 0.02x + y - 0.02x + y + 0.02x + y$

$= 0.07x + 0.07x + 0.02x + 0.07x = 0.23x$

∴ 23% passes only one subject

11. Given that $n(\xi) = 18$, $n(A) = 7$, $n(B) = 8$ and $n(A \cap B') = 4$. Find the following

(a) $n(A \cap B)$ (2 marks)

$n(B) = n(A \cap B) + n(A' \cap B)$.

also $n(A) = n(A \cap B) + n(A \cap B')$

$7 = n(A \cap B) + 4$.

$n(A \cap B) = 7 - 4 = 3$.

(b) $n(A \cup B)$ (2 marks)

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$= 7 + 8 - 3 = 12$.

(c) $n(A' \cap B)$ (2 marks)

$n(A' \cap B) = n(B) - n(A \cap B) = 8 - 3 = 5$

12. Given two sets A and B. Define a cartesian product of two sets by $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$. If $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$. Find $A \times B$. (2 marks)

$A \times B = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c), (3,a), (3,b), (3,c)\}$.

13. Write the set $A = \{x : x^2 + 3x + 2 = 0\}$ in tabular notation. (3 marks)

$x^2 + 3x + 2 = 0 \Rightarrow x^2 + x + 2x + 2 = 0$

$x(x+1) + 2(x+1) = 0$

$(x+2)(x+1) = 0 \Rightarrow x = -1$ or $x = -2$.

$A = \{-1, -2\}$.

14. Given a set $A = \{\{1\}, \{2, 3\}, \{a\}\}$ Write down the power set of A. (2 marks)

$P(A) = \{\{\{1\}\}, \{\{2,3\}\}, \{\{a\}\}, \{\{1\}, \{2,3\}\}, \{\{1\}, \{a\}\}, \{\{2,3\}, \{a\}\}, \{\{1\}, \{2,3\}, \{a\}\}, \{\{\}\}\}$.

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15. Given sets

$U = \{1, 2, 3, \ldots, 15\}$

$E = \{\text{even numbers in } U\}$

$F = \{\text{numbers in } U \text{ divisible by } 5\}$

List down the following sets:

(a) F^c = $\{1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14\}$. (1 mark)

(b) $E \cap F$ = $\{10\}$. (1 mark)

(c) $E \cup F^c$ = $\{1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$. (1 mark)