Maxwell-Elasto-Brittle rheology: Implementation in MITgcm

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November 3, 2021

- momentum equations

$$m\frac{D\vec{\boldsymbol{u}}}{Dt} = -mf\vec{\boldsymbol{k}}\times\vec{\boldsymbol{u}} + \vec{\boldsymbol{\tau}}_{air} + \vec{\boldsymbol{\tau}}_{ocean} - m\vec{\nabla}\phi(0) + \vec{\nabla}\cdot\boldsymbol{\sigma}, \tag{1}$$

  with $m = \rho\, h\, c$, ρ=density, h=thickness, c=concentration (sea ice fraction), etc., $H = h\, c$

- VP-rheology (Reiner-Rivlin relation):

$$\begin{aligned}\sigma_{ij} &= 2\eta\dot{\epsilon}_{ij} + \left[(\zeta-\eta)\dot{\epsilon}_{kk} - \frac{P}{2}\right]\delta_{ij} && (2)\\ &= \left[\boldsymbol{K}(\zeta-\eta,\eta) : \dot{\boldsymbol{\epsilon}}\right]_{ij} - \frac{P}{2}\delta_{ij} && (3)\end{aligned}$$

  with

$$\begin{aligned}\zeta &= \frac{P}{2\Delta} && (4)\\ \eta &= \frac{\zeta}{e^2} = \frac{P}{2\Delta e^2} && (5)\end{aligned}$$

  MEB-equations

$$\begin{aligned}&\frac{1}{E}\frac{D\boldsymbol{\sigma}}{Dt} + \frac{\boldsymbol{\sigma}}{\eta_v} = \left[\boldsymbol{K}(\Lambda, M) : \dot{\boldsymbol{\epsilon}}\right] && (6)\\ \leftrightarrow\quad &\frac{D\boldsymbol{\sigma}}{Dt} + \frac{E}{\eta_v}\boldsymbol{\sigma} = \frac{D\boldsymbol{\sigma}}{Dt} + \lambda(t)\boldsymbol{\sigma} = E\left[\boldsymbol{K}(\Lambda, M) : \dot{\boldsymbol{\epsilon}}\right] && (7)\end{aligned}$$

with the Elasticity $E = E(t)$, viscosity $\eta_v = \eta_v(t)$, reciprocal of the relaxation time scale $\lambda(t) = \frac{E(t)}{\eta_v(t)} = \tau_r^{-1}(t)$, and

$$(\boldsymbol{K}(\Lambda, M) : \dot{\boldsymbol{\epsilon}})_{ij} = \frac{\nu}{(1+\nu)(1-\nu)}\dot{\epsilon}_{kk}\delta_{ij} + 2\,\frac{1}{2(1+\nu)}\dot{\epsilon}_{ij} \tag{8}$$

$$= \Lambda\dot{\epsilon}_{kk}\delta_{ij} + 2M\dot{\epsilon}_{ij} \tag{9}$$

$$\tag{10}$$

with the Poisson ratio $\nu = 0.3$, and the Lamé coefficients Λ and M for planar 2D stress [reference]. By comparing Eq. (**??**) to (**??**), we can see the following anaologies:

$$P \rightarrow 0 \quad \text{(PRESS)} \tag{11}$$

$$E\Lambda \rightarrow \zeta - \eta \tag{12}$$

$$EM = \frac{E}{2(1+\nu)} \rightarrow \eta \quad \text{(ETA)} \tag{13}$$

so that

$$\text{(ZETA:)} \qquad \zeta \leftarrow E(\Lambda + M) \tag{14}$$

$$= E\left\{\frac{\nu}{(1+\nu)(1-\nu)} + \frac{1}{2(1+\nu)}\right\}$$

$$= \frac{E\,(3\nu+1)}{2(1+\nu)(1-\nu)}$$

$$\text{(PRESS0:)} \qquad E = E_0\,d\,H\,\exp\left[-C^*(1-c)\right] \tag{15}$$

$$E_0 = \text{SEAICE\_strength (runtime parameter)} \tag{16}$$

- we introduce the following abbreviations and conventions:

$$\dot{\epsilon}_+ = \dot{\epsilon}_{11} + \dot{\epsilon}_{22} \tag{17}$$

$$\dot{\epsilon}_- = \dot{\epsilon}_{11} - \dot{\epsilon}_{22} \tag{18}$$

$$\sigma_+ = \sigma_{11} + \sigma_{22} \tag{19}$$

$$\sigma_- = \sigma_{11} - \sigma_{22} \tag{20}$$

with the abbreviations we have for the VP equations

$$\sigma_+ = 2\,\zeta\,\dot{\epsilon}_+ - P \tag{21}$$

$$\sigma_- = 2\,\eta\,\dot{\epsilon}_- = 2\,(\zeta/e^2)\,\dot{\epsilon}_- \tag{22}$$

and for the MEB equations

$$E\,(\boldsymbol{K} : \dot{\boldsymbol{\epsilon}})_{+} = 2\,\zeta\,\dot{\epsilon}_{+} = 2\,(\Lambda + M)\,\dot{\epsilon}_{+} \tag{23}$$

$$E\,(\boldsymbol{K} : \dot{\boldsymbol{\epsilon}})_{-} = 2\,\eta\,\dot{\epsilon}_{-} = 2\,M\,\dot{\epsilon}_{-} \tag{24}$$

principle stress components:

$$\sigma_1 = \frac{\sigma_{11} + \sigma_{22}}{2} + \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2} \tag{25}$$

$$\sigma_2 = \frac{\sigma_{11} + \sigma_{22}}{2} - \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2} \tag{26}$$

stress invariants:

divergent stress: $$\sigma_I = \frac{1}{2}\,(\sigma_1 + \sigma_2) = \frac{1}{2}\,(\sigma_{11} + \sigma_{22}) \tag{27}$$

shear stress: $$\sigma_{II} = \frac{1}{2}\,(\sigma_1 - \sigma_2) = \frac{1}{2}\sqrt{(\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2} \tag{28}$$

and similarly for $\dot{\epsilon}_{ij}$.

- some parameters:

$$
\begin{aligned}
E_0 &\approx 10^9\,\mathrm{N\,m^{-2}} \quad \text{elasticity modulus of undamaged ice} && (29)\\
\lambda_0 &\approx 10^{-7}\,\mathrm{s} \quad \text{relaxation time scale of undamaged ice} && (30)\\
\eta_0 &= E_0\lambda_0^{-1} \quad \text{apparent viscosity of undamaged ice in } \mathrm{N\,m^{-2}\,s} && (31)\\
C &\approx 40,000\ \mathrm{N\,m^{-2}} \quad \text{cohesion} && (32)\\
\nu &= 0.3 \quad \text{Poisson ratio} && (33)\\
\mu &= 0.7 \quad \text{internal friction coefficient} && (34)\\
q &= \left[\sqrt{\mu^2+1}+\mu\right]^2 \quad \text{slope of M-C yield curve in principle stress space} && (35)\\
s &= \frac{q-1}{q+1} \quad \text{slope of M-C yield curve in stress invariant space} && (36)\\
\sigma_c &= \frac{2C\,H}{\sqrt{\mu^2+1}-\mu} \quad \text{uniaxial (unconfined) compressive strength} && (37)\\
&= 2C\,H\left[\sqrt{\mu^2+1}+\mu\right] = 2C\,H\sqrt{q}\\
\sigma_t &= -\frac{\sigma_c}{q} = -\frac{2C\,H}{\sqrt{q}} \quad \text{tensile strength cut-off} && (38)\\
d_\mathrm{crit} &= \min\left[1, \frac{\sigma_t}{\sigma_1}, \frac{\sigma_c}{\sigma_2 - q\,\sigma_1}\right] \quad \text{critical damage parameter} && (39)\\
\eta_v(t) &= \eta_0\, d^{-\alpha}\, H\, \exp\left[-C^*(1-c)\right] && (40)\\
\lambda^{-1}(t) &= \tau_r(t) = \frac{\eta_0}{E_0}\, d^{\alpha-1}, \quad \lambda(t) = \frac{E_0}{\eta_0}\, d^{1-\alpha} && (41)
\end{aligned}
$$

- We exploit the analogy (**??**)-(**??**) to use existing VP-code and turn it into MEB-code;

- S/R `seaice_calc_stress`: calculate VP stress from coefficients and strain rates $\dot\epsilon_{ij}$

- S/R `seaice_meb_update_sigma`: update global variables sigPlus, sigMinus, sigma12, based on

- S/R `seaice_meb_calc_stress`: compute new stress at time level n from σ_{ij}^{n-1} and $\dot\epsilon_{ij}^{n}$, neglecting advection of stress, based on ($\beta = 1$: implicit; $= 0$: explicit time stepping):

$$
\frac{\boldsymbol{\sigma}^n - \boldsymbol{\sigma}^{n-1}}{\Delta t} + \lambda^n\left\{\beta\boldsymbol{\sigma}^n + (1-\beta)\boldsymbol{\sigma}^{n-1}\right\} = E^n(\boldsymbol{K} : \dot{\boldsymbol{\epsilon}}^n) - \text{advection} = \dot{\boldsymbol{\sigma}}^n \quad (42)
$$

$$\leftrightarrow \boldsymbol{\sigma}^n = \frac{\boldsymbol{\sigma}^{n-1}\left\{1-(1-\beta)\Delta t\lambda^n\right\} + \Delta t\,\dot{\boldsymbol{\sigma}}^n}{1+\beta\Delta t\lambda^n}, \quad \left(\lambda^n = \frac{E^n}{\eta_v^n}\right) \tag{43}$$

drop explicit/implicit factor β

$$\boldsymbol{\sigma}^n = \frac{1}{1+\lambda^n}\left[E^n\Delta t\,\boldsymbol{K}:\dot{\boldsymbol{\epsilon}}^n + \boldsymbol{\sigma}^{n-1}\right] \tag{44}$$

from the previous timestep:

$$\boldsymbol{\sigma}^{n-1} = \frac{1}{1+\lambda^{n-1}}\left[E^{n-1}\Delta t\,\boldsymbol{K}:\dot{\boldsymbol{\epsilon}}^{n-1} + \boldsymbol{\sigma}^{n-2}\right] \tag{45}$$

so

$$\boldsymbol{\sigma}^n = \frac{1}{1+\lambda^n}\left[E^n\Delta t\,\boldsymbol{K}:\dot{\boldsymbol{\epsilon}}^n + \frac{1}{1+\lambda^{n-1}}\left[E^{n-1}\Delta t\,\boldsymbol{K}:\dot{\boldsymbol{\epsilon}}^{n-1} + \boldsymbol{\sigma}^{n-2}\right]\right] \tag{46}$$

$$= \frac{1}{1+\lambda^n}\left[E^n\Delta t\,\boldsymbol{K}:\dot{\boldsymbol{\epsilon}}^n + \frac{1}{1+\lambda^{n-1}}\left[E^{n-1}\Delta t\,\boldsymbol{K}:\dot{\boldsymbol{\epsilon}}^{n-1}\ldots\right.\right. \tag{47}$$

$$\left.\left.+\frac{1}{1+\lambda^{n-2}}\left[E^{n-2}\Delta t\,\boldsymbol{K}:\dot{\boldsymbol{\epsilon}}^{n-2}+\ldots\right]\right]\right]$$

so that (**??**) and (**??**) become:

$$\frac{E\lambda}{1+\beta\Delta t\lambda(t)} \to \zeta - \eta \tag{48}$$

$$\frac{EM}{1+\beta\Delta t\lambda(t)} \to \eta \tag{49}$$

$$\zeta \leftarrow \frac{E(\Lambda+M)}{1+\beta\Delta t\lambda(t)} \tag{50}$$

and (**??**) and (**??**)

$$\frac{E\left(\boldsymbol{K}:\dot{\boldsymbol{\epsilon}}\right)_+}{1+\beta\Delta t\lambda(t)} = 2\,\zeta\,\dot{\epsilon}_+ \tag{51}$$

$$\frac{E\left(\boldsymbol{K}:\dot{\boldsymbol{\epsilon}}\right)_-}{1+\beta\Delta t\lambda(t)} = 2\,\eta\,\dot{\epsilon}_- \tag{52}$$

The first part of eq. (**??**)

$$\frac{\boldsymbol{\sigma}^{n-1}\left\{1-(1-\beta)\Delta t\lambda(t)\right\}}{1+\beta\Delta t\lambda(t)}$$

is added to the rhs of the momentum equations

- S/R `seaice_recip_relaxtime`: compute $\lambda(t) = \frac{E(t)}{\eta_v(t)}$.
- S/R `seaice_meb_update_rhs`: update right hand side of momentum equations with stress divergence of previous time level $\nabla \frac{\boldsymbol{\sigma}^{n-1}[1-(1-\beta)\Delta t\lambda(t)]}{1+\beta\Delta t\lambda(t)}$
- S/R `seaice_calc_viscosity`: replace definition of ζ, η, and P with the appropriate forms (??)-(??); include time stepping part $\frac{1}{1+\Delta t\lambda(t)}$ as in (??) and (??)
- S/R `seaice_update_damage`: determine failure criteria and step damage equation in time
- averaging: η needs to be averaged to Z-points ($\overline{\eta}^Z$) to compute σ_{12}, σ_{12} and $\dot{\epsilon}_{12}$ need to be averaged to C-points ($\overline{\sigma}_{12}^C$ and $\overline{\dot{\epsilon}}_{12}^C$) to compute stress criteria and diagnostics

$$\overline{\sigma}_{12,ij}^C = \frac{\sigma_{12,ij} + \sigma_{12,i+1,j} + \sigma_{12,i,j+1} + \sigma_{12,i+1,j+1}}{4} \tag{53}$$

$$= \frac{2\,\overline{\eta}_{ij}^Z\dot{\epsilon}_{12,ij} + 2\,\overline{\eta}_{ij}^Z\dot{\epsilon}_{12,i+1,j} + 2\,\overline{\eta}_{ij}^Z\dot{\epsilon}_{12,i,j+1} + 2\,\overline{\eta}_{ij}^Z\dot{\epsilon}_{12,i+1,j+1}}{4} \tag{54}$$

or

$$\overline{\sigma}_{12,ij}^C = \sqrt{\frac{\sigma_{12,ij}^2 + \sigma_{12,i+1,j}^2 + \sigma_{12,i,j+1}^2 + \sigma_{12,i+1,j+1}^2}{4}} \tag{55}$$

or

$$\overline{\sigma}_{12,ij}^C = 2\,\eta\,\frac{\dot{\epsilon}_{12,ij} + \dot{\epsilon}_{12,i+1,j} + \dot{\epsilon}_{12,i,j+1} + \dot{\epsilon}_{12,i+1,j+1}}{4} \tag{56}$$

or

$$\overline{\sigma}_{12,ij}^C = 2\,\eta\sqrt{\frac{\dot{\epsilon}_{12,ij}^2 + \dot{\epsilon}_{12,i+1,j}^2 + \dot{\epsilon}_{12,i,j+1}^2 + \dot{\epsilon}_{12,i+1,j+1}^2}{4}} \tag{57}$$

the latter two (??) and (??) are preferred for stress evaluation when computing damage because it involves less averaging, but it is not clear if it is better or more consistent than (??). Average of the squares is energy consistent?

Timestepping This is how it is currently done in the MITgcm sea ice model:

1. solve momentum equations for ice velocitis $\boldsymbol{u}^n$ with stress computed from eq.(**??**):

$$\boldsymbol{\sigma}^n = \frac{\boldsymbol{\sigma}_M^{n-1} + \Delta t\, E^{n-1}(\boldsymbol{K} : \dot{\boldsymbol{\epsilon}}^n)}{1+\Delta t \lambda^{n-1}} = \frac{1}{1+\Delta t\lambda^{n-1}} \left\{ \Delta t\, E^{n-1}(\boldsymbol{K} : \dot{\boldsymbol{\epsilon}}^n) + \boldsymbol{\sigma}_M^{n-1} \right\}$$

   (S/Rs `seaice_calc_viscosities`, `seaice_meb_update_rhs`) Note, that $\dot{\boldsymbol{\epsilon}}^n$ is from the current time level, implying an implicit treatment of the momentum equations; σ_M^{n-1} is stored in the variables `seaice_sigma1/2/12`;

2. S/R `seaice_meb_calc_stress`: diagnose stress from new ice velocities (strain rates) using the same equation;

3. S/R `seaice_update_damage`: compute critical damage $\Psi = \frac{\Delta t}{\tau_d}\left\{\frac{\tau_d}{\Delta t} + d_{\text{crit}}^n - 1\right\} \approx d_{\text{crit}}^n$ if the stress is over-critical and step damage equations (explicit time stepping):

$$d^n = d^{n-1}\left\{1 + \frac{\Delta t}{\tau_d}(d_{\text{crit}} - 1)\right\} + \frac{\Delta t}{\tau_h} \tag{58}$$

   without healing ($\tau_h = \infty$), this is the same as:

$$d^n = d^{n-1}\Psi$$

   and with $\tau_d = \Delta t$

$$= d^{n-1}\, d_{\text{crit}}$$

4. adjust/recompute stress to satisfy yield criteria and store stress:

$$\boldsymbol{\sigma}_M^n = \boldsymbol{\sigma}^n\, \Psi \tag{59}$$

$$= \frac{\Psi \boldsymbol{\sigma}_M^{n-1} + \Delta t\, E^n(\boldsymbol{K} : \dot{\boldsymbol{\epsilon}}^n)}{1+\Delta t\lambda^{n-1}} \tag{60}$$

   because $E^n = d_{\text{crit}} E^{n-1} \approx \Psi E^{n-1}$.

5. (possibly) iterate 1.-5. until convergence à la **?**.

6. step advection of thickness, concentration, etc., including damage d

7. step thermodynamic equations (S/R `seaice_growth`)

Problems:

- σ_{12} is not co-located with the remaining variables and averaging is required. This can lead to a mismatch between the stress that is used to evaluate the critical damage (σ_{12} averaged to C-points) and the stress that is used in computing the ice velocities (Ψ averaged to Z-points). This increases the numerical stencil and also leads to excessive smoothing, if not done properly.

- defined at C-points: Ψ, λ, E, σ_{11}, σ_{22}, σ_1, σ_2

- defined at Z-points: σ_{12}, $\dot\epsilon_{12}$

- averaging between σ_{12}^C (needed for computing Ψ) and σ_{12}^Z (needed for computing $\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}$):

$$\sigma_{12}^{n,C} = \frac{1}{1+\Delta t\lambda^{n-1}}\left\{\frac{\Delta t E^{n-1}}{1+\nu}\,\overline{\dot\epsilon_{12}^{n}}^{C} + \overline{\sigma_{M,12}^{n-1}}^{C}\right\} \tag{61}$$

$$\sigma_{M,12}^{n,C} = \sigma_{12}^{n,C}\,\Psi \tag{62}$$

$$\sigma_{M,12}^{n,Z} = \frac{1}{1+\Delta t\overline{\lambda^{n-1}}^{Z}}\left\{\frac{\Delta t\overline{E^{n-1}}^{Z}}{1+\nu}\,\dot\epsilon_{12}^{n} + \overline{\sigma_{M,12}^{n,C}}^{Z}\right\}; \tag{63}$$

If we store the strain rates $\dot{\boldsymbol{\epsilon}}_M$ instead of the stresses, we can write:

$$\overline{\sigma_{12}^{n}}^{C} = \frac{1}{1+\Delta t\lambda^{n-1}}\left\{\frac{\Delta t E^{n-1}}{1+\nu}\,\overline{\dot\epsilon_{12}^{n}}^{C} + \frac{1}{1+\Delta t\lambda^{n-1}}\left\{\frac{\Delta t E^{n-1}}{1+\nu}\,\overline{\dot\epsilon_{M,12}^{n-1,Z}}^{C}\right\}\right\} \tag{64}$$

$$\sigma_{12}^{n} = \frac{1}{1+\Delta t\overline{\lambda^{n-1}}^{Z}}\left\{\frac{\Delta t\overline{E^{n-1}}^{Z}}{1+\nu}\,\dot\epsilon_{12}^{n} + \frac{\overline{\Psi}^{Z}}{1+\Delta t\overline{\lambda^{n-1}}^{Z}}\left\{\frac{\Delta t\overline{E^{n-1}}^{Z}}{1+\nu}\,\dot\epsilon_{M,12}^{n-1}\right\}\right\} \tag{65}$$

$$\dot\epsilon_{M,12}^{n} = \dot\epsilon_{12}^{n} + \frac{\dot\epsilon_{M,12}^{n-1}}{1+\Delta t\overline{\lambda^{n-1}}^{Z}} \tag{66}$$

and we may get away with less averaging. Eq. (**??**) is used to evaluate the principle stresses at C-points and eq. (**??**) is used to compute the new velocity field from $\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}$.