Abstract

This thesis deals with the problem of the *functorial* identification of topological Hochschild homology on certain rings.
The included chapter contains an exposition of the work by Krause and Nikolaus which computes THH of complete mixed characteristic discrete valuation rings ([KN19]). This computation hinges on the choice of an uniformizer and is thus only functorial in maps that preserve the chosen uniformizers. In future chapters we will thus try to identify the action of THH on arbitrary maps of these rings.

Extended Abstract

The algebraic K-theory groups are fundamental invariants of rings. They encapsulate deep knowledge about the ring via its module category. For rings of integers in number fields, the K-theory groups contain information about the class group and group of units of the ring, the Brauer group of the field and values of the Dedekind zeta function. Algebraic K-theory can also be applied to schemes where it is deeply related to algebraic cycles in the form of (higher) Chow groups. But maybe its most spectacular applications have been found in geometry. Starting with the early work of Wall, the s-cobordism theorem by Barden-Mazur-Stalling it culminated in the stable parametrized h-cobordism by Waldhausen, Jahren and Rognes. One of its implications is that we can calculate many homotopy groups of diffeomorphism groups of spaces as the K-theory groups of certain rings (more precisely ring spectra), which are more less the integral group ring of the fundamental group. For example the homotopy groups of the diffeomorphism group of a disc $\pi_*(\mathrm{Diff}(D^n))$ can (in a range depending on n) be obtained by computing the K-theory groups of $\mathbb{Z}$.
These powerful results and theorems now present us with the challenge to actually compute K-theory groups. The problem is that this is in general very hard! There are therefore several successful approaches towards these computations, which nicely complement each other. Among them are motivic methods, controlled algebra conjecture and trace methods. Let us only describe the last one, as it is the only one of relevance for this thesis. The idea of trace methods is to consider other - ideally more computable - invariants of rings and compare K-theory to them via so called trace maps. The hope is that while these invariants are more computable, the difference to K-theory is not too big, in other words the trace map is close to an isomorphism. The most successful of these comparisons is with the trace map $K \xrightarrow{\mathrm{tr}} \mathrm{TC}$ to the so called *topological cyclic homology* introduced by Bökstedt-Hsiang-Madsen. By work of Dundas-Goodwillie-McCarthy and Hesselholt-Madsen, we now know, that the trace map is a remarkably close approximation which has been used to calculate K-theory in many cases. Due to the recent work of Nikolaus-Scholze, topological cyclic homology itself can essentially be computed via two spectral sequences out of yet another invariant: Topological Hochschild homology (THH). This is the main player of this thesis and the goal is to provide a new computation of THH, that has not been done before.

The rings, that we will consider are complete discrete valuation rings of mixed characteristic. To be concrete, all of them are certain extensions of the p-adic integers, like $\mathbb{Z}_p[\sqrt[n]{p}]$ or $\mathbb{Z}_p[\zeta_{p^n}]$. They arise for example as completions of rings of integers in number fields and are thus of fundamental importance in algebraic number theory. The topological Hochschild homology of these kind of rings has already been computed by work of Lindenstrauss-Madsen. Recently Krause and Nikolaus gave a more conceptual proof of the same result, which we present in the second chapter. Unfortunately both approaches do not give a functorial computation. They only identify THH of these rings, but do not answer the question what it does to morphisms. This is unsatisfactory for at least two reasons: We might want to understand the action of the Galois group on K-theory. Because the trace map $K \xrightarrow{\mathrm{tr}} \mathrm{TC}$ is a natural transformationa and TC is functorially obtained from THH, the first step to understand the functoriality of K-theory is to understand it for THH. Secondly, functorial description are actually often needed for further computations. For example, in our computation we crucially need that we not know only the topological Hochschild homology groups of $\mathbb{Z}_p[z]$ but *also* how endomorphisms of $\mathbb{Z}_p[z]$ act on the topological Hochschild homology groups of it.

Let us now give a short outline of our approach to understand the functoriality of $\mathrm{THH}_*(R)$ (more precisely we deal with the p-completions of these groups). Firstly we use, that by a general classification result all mixed characteristic discrete valuation rings are obtained from $\mathbb{Z}_p$ by adjoining a root of an Eisenstein polynomial, i.e. $R = \mathbb{Z}_p[z]/E(z)$ for $E(z) \in \mathbb{Z}_p[z]$ Eisenstein[1]. This allows us to write R as follows $R = \mathbb{Z}_p[z] \otimes_{Z_p[z]} \mathbb{Z}_p$, where $Z_p[z]$ acts on the left factor by the map $z \mapsto E(z)$ and on the right factor by $z \mapsto 0$. Now we can observe, that THH preserves tensor products of commutative rings, i.e. we get $\mathrm{THH}(R) = \mathrm{THH}(\mathbb{Z}_p[z]) \otimes_{\mathrm{THH}(\mathbb{Z}_p[z])} \mathrm{THH}(\mathbb{Z}_p)$. This is progress, because we know what $\mathrm{THH}(\mathbb{Z}_p)$ is. Furthermore, by an Hochschild-Kostant-Rosenberg type result we also obtain $\mathrm{THH}(\mathbb{Z}_p[z]) \simeq \mathrm{THH}(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathrm{HH}(\mathbb{Z}[z]/\mathbb{Z})$. This is not a natural equivalence on the level of spectra, but it is natural after taking homotopy groups $\mathrm{THH}_*(\mathbb{Z}_p[z]) \simeq \mathrm{THH}_*(\mathbb{Z}_p) \otimes_{\mathbb{Z}} \Omega^*_{\mathbb{Z}[z]/\mathbb{Z}}$. We exactly know how Kähler differentials are functorial, thus we now completely understand the functoriality on the level of homotopy groups.

[1]Actually we need to take the Witt vectors of the residue field of R, but for sake of exposition we stick to the easiest case that the residue field is $\mathbb{F}_p$ and the Witt vectors are $W(\mathbb{F}_p) = \mathbb{Z}_p$

1. Preliminaries

Topics that need to be explained (or at least give reference):

- spectra,
- constructions like $\Sigma^\infty, \Omega^\infty$
- Eilenberg-Maclane spectra.
- (co)fiber sequences
- $\otimes$ of spectra uniquely determined by being closed symmetric monoidal with unit $\mathbb{S}$ and preserves colimits in each variable (separately)
- p-completeness (maybe also of ab. groups)
- Basics of infinity categories
- (sym) monoidal structures,
- (co)limits
- adjunctions
- Ring spectra ($\mathbb{E}_1$,... $\mathbb{E}_\infty$)
- Postnikov towers?
- Definition of THH, relative THH aswell as Hochschild homology
- maybe also shortly cyclotomic structure and definition of TC

Further things to cite:

- Barnes-Roitzheim for p-completeness
- SAG: z-adic completeness.
- Groth, Camarena and Gepner for quick intros to higher algebra and higher category theory.
- Nikolaus-Krause lecture notes
- Dundas-Goodwillie-McCarthy (functoriality of THH for number rings is open problem, page 148, beginning of chapter 4)

1.1 ∞-categories

1.2 Spectra

Definition 1.1. *The ∞-category of spectra is defined as the following limit in the (very large) category* Cat_∞

$$\mathrm{Sp} \coloneqq \lim \left(\dots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \right)$$

Objects of this category are thus ...

By definition of this limit we have a functor $\mathrm{Sp} \to \mathcal{S}_*$, which we call Ω^∞. Using the adjoint functor theorem [Lur17, p. 5.5.2.9] we obtain a right adjoint $\Sigma^\infty : \mathcal{S}_* \to \mathrm{Sp}$. Composing with the free-forgetful adjunction $\mathcal{S}_* \leftrightarrows \mathcal{S}$ we obtain a further adjunction $\mathrm{Sp} \underset{\Omega^\infty}{\overset{\Sigma^\infty_+}{\leftrightarrows}} \mathcal{S}$.

2. Functorial identifications of THH of CDVRs

In this chapter, we will use the Tor-spectral sequence to understand the effect of THH on maps between complete discrete valuation rings. See [Lur17, Proposition 7.2.1.19] for a statement of the Tor spectral sequence for modules spectra over a ring spectrum or [Stacks, Tag 061Y] for the more classical result in the case of chain complexes.
Let R be a mixed characteristic complete discrete valuation ring with perfect residue field.

1. Write R as a pushout $R \cong \mathbb{Z}_p[z] \otimes_{\mathbb{Z}_p[z]} \mathbb{Z}_p$ via Eisenstein polynomial (see footnote 5)

2. THH preserves this pushout because the involved rings are commutative, and for $\mathbb{E}_\infty$-rings, THH is given by the colimit over S^1 in CAlg which means that it commutes with arbitrary colimits.

3. Fact that $\mathrm{THH}(\mathbb{Z}_p[z]) \simeq \mathrm{THH}(\mathbb{Z}_p) \otimes_{\mathbb{Z}} \mathrm{HH}(\mathbb{Z}[z]/\mathbb{Z})$ but not functorially, but have functoriality on homotopy groups $\mathrm{THH}*(\mathbb{Z}_p[z]) \simeq \mathrm{THH}_*(\mathbb{Z}_p) \otimes_{\mathbb{Z}} \Omega_{\mathbb{Z}[z]/\mathbb{Z})}$ functorialy!

4. Now employ Tor-spectral sequence:
   $E^2_{i,j} = \mathrm{Tor}^i_{\pi_*(R)}\left(\pi_*(M), \pi_*(M)\right)_{(j)} \Rightarrow \pi_{i+j}\left(M \otimes_R N\right)$, i.e. in our case:
   $\mathrm{Tor}^i_{\mathrm{THH}_*(\mathbb{Z}_p[z])}\left(\mathrm{THH}_*(\mathbb{Z}_p[z]), \mathrm{THH}_*(\mathbb{Z}_p)\right)_{(j)} \Rightarrow \mathrm{THH}_{i+j}(R)$ (or rather the p-completed version)

Bibliography

[KN19] Achim Krause and Thomas Nikolaus. *Bökstedt periodicity and quotients of DVRs*. 2019. arXiv: `1907.03477 [math.AT]`.

[Lur17] Jacob Lurie. "Higher algebra." In: *preprint available from the author's website* (2017).

[Stacks] The Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu`. 2018.