Topological Hochschild homology of discrete valuation rings

MASTER'S THESIS

submitted by

Mirko STAPPERT

in partial fulfilment of the

requirements for the degree of

Bachelor of Science in the Department

of Mathematics at Saarland University

Supervisor:
Prof. Dr. Thomas NIKOLAUS

Examiner 1:
Prof. Dr. Gabriela WEITZE-SCHMITHÜSEN

Examiner 2:
Prof. Dr. Thomas NIKOLAUS

Statement in Lieu of an Oath

I hereby confirm that I have written this thesis on my own and that I have not used any other media or materials than the ones referred to in this thesis.

Ich versichere hiermit, dass ich die vorliegende Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Saarbrücken, August 25, 2021

Mirko Stappert

Contents

Todo list

Abstract

This thesis deals with the problem of the *functorial* identification of topological Hochschild homology on certain rings.
The included chapter contains an exposition of the work by Krause and Nikolaus which computes THH of complete mixed characteristic discrete valuation rings ([KN19]). This computation hinges on the choice of an uniformizer and is thus only functorial in maps that preserve the chosen uniformizers. In future chapters we will thus try to identify the action of THH on arbitrary maps of these rings.

1. Introduction

The algebraic K-theory groups are fundamental invariants of rings. They encapsulate deep knowledge about the ring via its category of modules. For rings of integers in number fields, the K-theory groups contain information about the class group and group of units of the ring, the Brauer group of the field and values of its Dedekind zeta function. Algebraic K-theory can also be applied to schemes. In fact, it was first defined in this context by Grothendieck during the work on his Riemann-Roch theorem, where he discovered a remarkable relationship to algebraic cycles. Later, Bloch pushed this relationship further to his newly defined higher Chow groups. But maybe its most spectacular applications have been found in geometric topology. Starting with the early work of Wall and the s-cobordism theorem by Barden-Mazur-Stalling it culminated in the incredible stable parametrized h-cobordism theorem by Waldhausen, Jahren and Rognes. It says, that the stable h-cobordism space of say a smooth manifold M can be obtained from the K-theory space of $\mathbb{S}[\Omega M]$, the spherical group ring of the loop space of M. This is remarkable, because this allows us - as does the classical h-cobordism theorem - to obtain geometric results for M purely using homotopy theory. One of its implications is that we can calculate many homotopy groups of the diffeomorphism group of M via the K-theory groups of $\mathbb{S}[\Omega M]$. For example the homotopy groups of the diffeomorphism group of a disc $\pi_*(\mathrm{Diff}(D^n))$ can (in a range depending on n) be obtained by computing the K-theory groups of the sphere spectrum. This boils down to the understanding of $K_*(\mathbb{Z})$ about which we know a lot by several major results in number theory.
These powerful results and theorems now present us with the challenge to actually compute K-theory groups. This problem is in general very hard! There are therefore several successful approaches towards these computations, which nicely complement each other. Among them are motivic methods, controlled algebra and trace methods. Let us only describe the last one, as it is the one of relevance for this thesis. The idea of trace methods is to consider other - ideally more computable - spectrum valued invariants of rings and compare K-theory to them via so-called trace maps. The hope is that while these invariants are more computable, the difference to K-theory is not too big, in other words the trace map is close to an isomorphism. The most successful of these comparisons is with the trace map $K \xrightarrow{\mathrm{tr}} \mathrm{TC}$ to the so-called *topological cyclic homology* introduced by Bökstedt-Hsiang-Madsen. By work of Dundas-Goodwillie-McCarthy and Hesselholt-Madsen, we now know that the trace map is a remarkably close approximation that has been used to calculate K-theory in many cases. Due to the recent work of Nikolaus-Scholze, topological cyclic homology itself can es-

sentially be computed via two spectral sequences out of yet another invariant: Topological Hochschild homology (THH). This is the main player of this thesis and the goal is to provide a new computation of THH, that has not been done before.

The rings, that we will consider are complete discrete valuation rings of mixed characteristic. To be concrete, all of them are certain extensions of the p-adic integers, like $\mathbb{Z}_p[\sqrt[n]{p}]$ or $\mathbb{Z}_p[\zeta_{p^n}]$. They arise for example as completions of rings of integers in number fields and are thus of fundamental importance in algebraic number theory. The topological Hochschild homology of these rings has already been computed in the work of Lindenstrauss-Madsen. Recently Krause and Nikolaus gave a more conceptual proof of the same result, which we present in the first chapter of this thesis. Unfortunately, both approaches do not give a functorial computation. They only identify THH of these rings but do not answer the question what it does to morphisms. This is unsatisfactory for at least two reasons: We might want to understand the action of the Galois group on K-theory. Because the trace map $K \xrightarrow{\mathrm{tr}} \mathrm{TC}$ is a natural transformation and TC is functorially obtained from THH, the first step to understand the functoriality of K-theory is to understand it for THH. Secondly, functorial descriptions are often needed for further computations. For example, in our computation, we crucially need that we not only know the topological Hochschild homology groups of $\mathbb{Z}_p[z]$ but *also* how endomorphisms of $\mathbb{Z}_p[z]$ act on the topological Hochschild homology groups of it.

Let us now give a short outline of our approach to understand the functoriality of $\mathrm{THH}_*(R)$ (more precisely we deal with the p-completions of these groups). Firstly we use, that by a general classification result all mixed characteristic discrete valuation rings are obtained from $\mathbb{Z}_p$ by adjoining a root of an Eisenstein polynomial, i.e. $R = \mathbb{Z}_p[z]/E(z)$ for $E(z) \in \mathbb{Z}_p[z]$ Eisenstein[1]. This allows us to write R as follows $R = \mathbb{Z}_p[z] \otimes_{\mathbb{Z}_p[z]} \mathbb{Z}_p$, where $\mathbb{Z}_p[z]$ acts on the left factor by the map $z \mapsto E(z)$ and on the right factor by $z \mapsto 0$. Now we can observe, that THH preserves tensor products of commutative rings, i.e. we get $\mathrm{THH}(R) = \mathrm{THH}(\mathbb{Z}_p[z]) \otimes_{\mathrm{THH}(\mathbb{Z}_p[z])} \mathrm{THH}(\mathbb{Z}_p)$. This is progress because we know what $\mathrm{THH}(\mathbb{Z}_p)$ is. Furthermore, by a Hochschild-Kostant-Rosenberg type result we also obtain $\mathrm{THH}(\mathbb{Z}_p[z]) \simeq \mathrm{THH}(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathrm{HH}(\mathbb{Z}[z]/\mathbb{Z})$. This is not a natural equivalence on the level of spectra, but it is natural after taking homotopy groups $\mathrm{THH}_*(\mathbb{Z}_p[z]) \simeq \mathrm{THH}_*(\mathbb{Z}_p) \otimes_{\mathbb{Z}} \Omega^*_{\mathbb{Z}[z]/\mathbb{Z}}$. We exactly know how Kähler differentials are functorial, thus we completely understand the functoriality on the level of homotopy groups. Now we want to go from these homotopy groups to $\mathrm{THH}_*(R)$ and for this we can employ the Tor-spectral sequence to the tensor product formula, we found above: $\mathrm{THH}(R) = \mathrm{THH}(\mathbb{Z}_p[z]) \otimes_{\mathrm{THH}(\mathbb{Z}_p[z])} \mathrm{THH}(\mathbb{Z}_p)$. This spectral sequences has E^2-page given by graded Tor-groups and converges to $\mathrm{THH}_*(R)$. A

[1] Actually we need to take the Witt vectors of the residue field of R, but for sake of exposition we stick to the easiest case that the residue field is $\mathbb{F}_p$ and the Witt vectors are $W(\mathbb{F}_p) = \mathbb{Z}_p$

map $R \to S$ induces a map between the respective spectral sequences. Therefore we need to understand the differentials and extension behaviour. This is in fact possible, because we know by the result of the first chapter, what $\mathrm{THH}(R)$ is and thus the target of the spectral sequence. In the last chapter we will then precisely analyze the spectral sequence, identify all differentials and extension problems and therefore give a full description of the functoriality of THH_* for all rings under consideration.

Outline the content of the chapters, check numbering/references, state main theorem

Acknowledgements

2. Preliminaries

We are going to use the language of higher category theory and higher algebra, mainly developed by Jacob Lurie in [Lur09] and [Lur17]. The online textbook project [Lur18b] by the same author contains a revised version of roughly the same content. For slightly different perspectives on higher category theory, we also recommend [RV21] and [Cis19]. Shorter summaries of most of the material we need can be found in [Gro20], [Cam13] and [Gep20].

2.1 Higher Category theory

Rewrite again later

Higher category theory is an extension of ordinary category theory, that is tailored to the needs of homotopy theory. In particular it follows the paradigm, that no equality should be taken literally and we should rather provide a specified homotopy, implemented as higher morphisms. The ∞ in ∞-category is supposed to signal, that we not only have objects and morphisms but also 2-morphisms between morphisms, 3-morphisms between 2-morphisms and so on. All diagrams should not commute strictly but only up to homotopy/2-morphism. A consequence of this is, that composition in ∞-categories is not strictly commutative and the notion of isomorphism of objects gets replaced by equivalence, which means that we have two maps that are inverse up to homotopy. Often times when we have more then one of these homotopies, they should in some sense be compatible, which is recorded via higher and higher homotopies. This train of thought goes under the name of **homotopy coherence**. We recommend [Rie18] and [Lur09, Section 1.2.6] to the unaquainted reader, who wants to become familiar with it. The benefit of this approach is that all concepts defined in this framework are naturally homtopy invariant, which is not the case for (co)limits in the category of topological spaces. A further advantage is that we can now consider spaces as ∞-categories themselves. Informally we do this by taking points as objects, paths as 1-morphisms, homotopies of paths as 2-morphism, homotopies between homotopies as 3-morphisms and so on. Note that this would not be possible in the realm of 1-categories, because the composition (concatenation) of paths is not strictly associative but only up to homotopy. The 1-categorical shadow of this is the fundamental groupoid. The reader can informally think about ∞-categories as some kind of a "greatest common" divisor of ordinary categories and topological spaces (or rather homotopy types).
The main thing to take away from this section is that most if not all concepts/constructions from ordinary category theory also exist in the world of ∞

categories. Precisely we will need the following higher categorical concepts:

- **∞-categories**, **functors** between them and **natural transformations**, all of which can be found in [Lur09, Chapter 1]. Every ordinary category can be considered as an infinity category via the nerve construction. There is also a way to go back[1] and to associate to every ∞-category $\mathcal{C}$ a 1-category, called its homotopy category $\mathrm{Ho}(\mathcal{C})$. The most important example of an ∞-category, that does not arise this way is the ∞-category of spaces $\mathcal{S}$ ([Lur09, Section 1.2.6]). Its role in higher category theory is analogous to the role, that the category of sets plays in ordinary category theory. An example of this paradigm is that in an ∞-category, we now no longer only have a set of maps between two objects but a whole mapping space.

- Most of the constructions and concepts from ordinary category theory have analogues in higher category theory. This includes **limits and colimits** ([Lur09, Section 1.2.3, Chapter 4]), **adjunctions**([Lur18b, Tag 02EJ]), opposite categories ([Lur09, Section 1.2.1]), slice categories ([Lur09, Section 1.2.9]), ... The only slice category that we will need is the ∞-category of pointed spaces, which is the slice category under the point $\mathcal{S}_* = \mathcal{S}_{*/}$. There is a free-forgetful adjunction $(-)_+ : \mathcal{S} \leftrightarrows \mathcal{S}_*$, whose left adjoint adds a disjoint basepoint. As usual left adjoints preserve colimits and right adjoints preserve limits ([Lur09, Proposition 5.2.3.5]). $\mathcal{S}$ and $\mathcal{S}_*$ are bicomplete and (co)limits agree with the classical notions of homotopy (co)limits. There are also notions of **filtered** and **sifted colimits** and the usual statements hold verbatim, i.e. filtered colimits commute with finite limits and sifted colimits commute with finite products.

- We will also crucially need **symmetric monoidal structures** on ∞-categories ([Lur17, Definition 2.0.0.7]) as well as **lax and strong symmetric monoidal functors**. The nerve of every ordinary symmetric monoidal category is a symmetric monoidal ∞-category. The ∞-category of spaces has all products, which equip $\mathcal{S}$ with a symmetric monoidal structure by [Lur17, Section 2.4.1]. The ∞-category of pointed spaces is also symmetric monoidal with the well known smash product $\wedge$ and the above functors in the adjunction between $\mathcal{S}$ and $\mathcal{S}_*$ have lax symmetric monoidal structures ([Lur17, Theorem 2.2.2.4]).

- Most ∞-categories[2], that we will consider in the next section and in this thesis in general, have an additional property. They are **stable ∞-categories**, a notion introduced in [Lur17, Section 1.1]. The definition is rather simple:

[1]This provides a (reflective) adjunction between the category of all categories and the ∞-category of all ∞-categories.

[2]Besides $\mathcal{S}$ and $\mathcal{S}_*$

An ∞-category is stable, if it has all finite limits and colimits, the thus provided initial and terminal object are equivalent [3] and a square is a pushout square if and only if it is a pullback square. Stable ∞-categories provide a natural place to do homological - or rather higher algebra. In particular they have a notion of exact sequences also called fiber sequences. We can add maps and thus also take kernels/cokernels of maps because we have a zero object. They are called (homotopy) **fiber/cofibers** and naturally fit into exact sequences $\mathrm{fib}(f) \to X \xrightarrow{f} Y$ and $X \xrightarrow{f} Y \to \mathrm{cofib}(f)$. Furthermore, by a result of Lurie [Lur17, Theorem 1.1.2.14] the homotopy category of a stable ∞-category has the structure of a triangulated category. Stable ∞-category are in some aspects preferable to triangulated categories. First of all a triangulation is extra structure, that has to be provided, while stability is a property. Secondly cones (or rather cofibers) are functorial in stable ∞-categories. They also "glue" together more naturally, which is relevant for descent questions in algebraic geometry. For example the functor, that sends a scheme to its derived ∞-category of quasi coherent sheaves is a sheaf, which is false on the level of triangulated categories. check details/reference

2.2 Spectra and higher algebra

The most important ∞-category for us will be the **∞-category of spectra**: Sp. Recall that it is defined as the following limit[4] in the (very large) ∞-category of large ∞-categories Cat_∞

$$\mathrm{Sp} \coloneqq \lim \left(\dots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \right).$$

The existence of such a functor $N(\mathbb{N})^{\mathrm{op}} \to \mathrm{Cat}_\infty$ is not obvious. A priori we have to supply further coherence data. That we can avoid doing this is the content of [Cis19, Corollary 7.3.17]. A further consequence of this corollary is that we can actually describe the objects of the resulting limit: An object of Sp - a **spectrum** - is a sequences of pointed spaces X_n together with equivalences $X_n \xrightarrow{\sim} \Omega X_{n+1}$. To differentiate them to other classically defined notions of spectra, they are also called Ω-spectra. Let us give the two main examples of interest. For an abelian group A, we can consider the sequence of Eilenberg-Maclane spaces $(K(A,0), K(A,1), K(A,2), \dots)$. With the usual equivalences $K(A,n) \xrightarrow{\sim} \Omega K(A,n+1)$ this defines a spectrum called the **Eilenberg-Maclane spectrum** of A. This construction assembles into a fully faithful functor $H : \mathrm{Ab} \to \mathrm{Sp}$ ([Lur17, Example 1.3.3.5]).Therefore, we will often abuse

[3] To be precise, the natural map from the initial to the terminal object should be an equivalence. An object is called initial if mapping spaces out of it are contractible and terminal if mapping spaces into it are contractible.

[4] That all limits exist in Cat_∞ is a result in [Lur09, Section 3.3.3]

notation and denote the Eilenberg-Maclane spectrum of R simply by R itself unless we want to stress, that we are talking about a discrete ring, in which case we will write HR. Spaces also give rise to spectra under the **suspension spectrum** functor $\Sigma^\infty : \mathcal{S}_* \to \mathrm{Sp}$. This allows us to define the most important spectrum, the **sphere spectrum** $\mathbb{S} := \Sigma^\infty(S^0)$. The suspension spectrum functor has a right adjoint $\Omega^\infty : \mathrm{Sp} \to \mathcal{S}_*$ simply given by sending a spectrum to its zeroth space. We can also compose this adjunction with the $(\mathcal{S}, \mathcal{S}_*)$-adunction to obtain an adjunction with unbased spaces $\Sigma^\infty_+ : \mathcal{S} \leftrightarrows \mathrm{Sp} : \Omega^\infty$. Similarly to spaces, spectra also have **homotopy groups**, which are defined via the ordinary homotopy groups of their spaces but now we can also define negative homotopy groups. For $X \in \mathrm{Sp}$ and $i \in \mathbb{Z}$ define $\pi_i(X) = \pi_{i+k}(X_k)$ as long as $i+k \geq 0$. This makes sense, because $X_k \simeq \Omega X_{k+1}$. They are all abelian because we can always write them as higher homotopy groups of some space. The homotopy groups of a suspension spectrum are the stable homotopy groups of the space. In particular $\pi_*(\mathbb{S})$ are the stable homotopy groups of spheres. A spectrum is called **connective**, if all its negative homotopy groups vanish. It is called **bounded below**, if they vanish below some degree (which is not necessarily 0). Suspension spectra and Eilenberg-Maclane spectra are connective.

Important structural properties of the ∞-category Sp are that it is a stable ∞-category with all limits and colimits. Furthermore it carries a closed symmetric monoidal structure with unit $\mathbb{S}$, called the tensor product of spectra $\otimes_{\mathbb{S}}$ ([Lur17, Corollary 4.8.2.19]). Being closed means exactly the same, as in ordinary category theory, i.e. for every object $X \in \mathrm{Sp}$, the functor $X \otimes_{\mathbb{S}} - : \mathrm{Sp} \to \mathrm{Sp}$ has a right adjoint. Closedness in particular implies, that tensoring with any spectrum preserves all colimits, as it is a left adjoint functor.

With these properties at hand, we are ready to do higher algebra. The monoidal structure lets us talk about monoids in Sp called **ring spectra**[5]. Here we have to point out two important differences between ordinary and higher algebra. First of all *commutativity and associativity are no longer properties but extra structure*. In the usual commutativity/associativity diagrams, we do not require that they strictly commute but only that they commute up to homotopy. Furthermore there is now a whole *hierarchy of commutativity*, starting with only associativity and no commutativity at all called $\mathbb{E}_1$-ring spectra[6], followed by $\mathbb{E}_2$, $\mathbb{E}_3$ and $\mathbb{E}_n$-ring spectra for every $n \in \mathbb{N}$ up to full homotopical commutativity $\mathbb{E}_\infty$. The proper way to precisely treat all this is via the theory of operads, developed by Peter May and transported into the world of ∞-categories by Jacob Lurie. He developed an extensive theory in [Lur17, Chapter 2, 3, 4]. The definition of the $\mathbb{E}_n$-operads is given in [Lur17, Definition 5.1.0.2], but we will not need any details.

[5]These monoid objects are called ring spectra, because spectra are already an homotopical incarnation of abelian groups, thus we only need the further multiplication. In particular the homotopy groups of a ring spectrum already form a graded ring.

[6]$\mathbb{E}_1$ is also called $\mathbb{A}_\infty$, where the 'A' stands for associativity. 'E' then stands for 'everything', meaning associative and commutative.

What is essential to us are the following facts:

- For every $n \in \mathbb{N} \cup \{\infty\}$ and every symmetric monoidal ∞-category $\mathcal{C}$ there is a notion of an $\mathbb{E}_n$**-algebra** in $\mathcal{C}$. These assemble into an ∞-category $\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C})$. In the most important special cases $n = 1, n = \infty$ we will denote them by $\mathrm{CAlg}(\mathcal{C}) \coloneqq \mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{C})$ and $\mathrm{Alg}(\mathcal{C}) \coloneqq \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C})$. If $\mathcal{C} = \mathrm{Sp}$ we will drop the dependence on $\mathcal{C}$ and simply write Alg and CAlg. The objects will be called ($\mathbb{E}_1$-)ring spectra and commutative/$\mathbb{E}_\infty$-ring spectra respectively. Every $\mathbb{E}_n$-algebra is also an $\mathbb{E}_{n-1}$-algebra and we get forgetful functors $\mathrm{CAlg}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})$. Lax symmetric monoidal functors $\mathcal{C} \to \mathcal{D}$ induce functors $\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{D})$. In particular since $H : \mathrm{Ab} \to \mathrm{Sp}$ is lax symmetric monoidal, we get functors $\mathrm{Alg}(\mathrm{Ab}) \to \mathrm{Alg}(\mathrm{Sp})$ and $\mathrm{CAlg}(\mathrm{Ab}) \to \mathrm{CAlg}(\mathrm{Sp})$. $\mathrm{Alg}(\mathrm{Ab})$ and $\mathrm{CAlg}(\mathrm{Ab})$ coincide with the ordinary categories of not-necessarily commutative and commutative rings respectively. Thus every ordinary ring R gives rise to an ring spectrum, which has an $\mathbb{E}_\infty$-structure if R is commutative. Another supply of algebras is given by **free algebras**. The forgetful functor $\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}) \to \mathcal{C}$ has a left adjoint [Lur17, Section 3.1], which provides us with several free spectra generated by spaces. Provided that $\mathcal{C}$ is bicomplete, the category $\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C})$ also has all limits and colimits by [Lur17, Section 3.2]. Limits and (at least in all of our situations also) sifted colimits are computed underlying. Finite coproducts in $\mathrm{CAlg}(\mathcal{C})$ are given by the tensor product of the underlying objects in $\mathcal{C}$.

- Given an algebra $A \in \mathrm{Alg}(\mathcal{C})$, we also able to talk about its ∞-category of left and right **modules** $\mathrm{LMod}_A, \mathrm{RMod}_A$ [Lur17, Definition 4.2.1.13]. In the case, that A is commutative, they are equivalent ([Lur17, Section 4.3.2]) and we will simply denote it by Mod_A. Given two algebras $R, S \in \mathrm{Alg}(\mathcal{C})$ there is the notion of a R-S-bimodule. Giving a R-S-bimodule structure is equivalent to giving a left module structure over $R \otimes_{\mathcal{C}} S^{\mathrm{op}}$. The categories LMod_A and RMod_A have all limits and colimits, provided that $\mathcal{C}$ does ([Lur17, Section 4.2.3]). For $\mathbb{S} \in \mathrm{Sp}$ we have $\mathrm{Mod}_{\mathbb{S}} = \mathrm{Sp}$ and for an ordinary ring R, we get its derived ∞-category $\mathrm{Mod}_{HR} = D(R)$.

- Given a right M and a left module N over a ring spectrum R, we can tensor them together and obtain a spectrum $M \otimes_R N$ [Lur17, Section 4.4.2]. For a commutative ring spectrum R its category of modules Mod_R obtains a symmetric monoidal structure this way ([Lur17, Theorem 4.5.2.1]). Thus we can talk about $\mathbb{E}_n$-algebras in there, which we will call $\mathbb{E}_n$**-R-algebras**.

- For a map $R \to S$ of ring spectra, we have the forgetful functor $\mathrm{LMod}_S \to \mathrm{LMod}_R$. This functor has both adjoints, in particular it preserves all limits and colimits. The left adjoint is given by a relative tensor product $M \mapsto M \otimes_R S$. For the tensor product we use the left R-module structure on

M and the right R-module structure on S. Because S is an S-S-bimodule, there remains a left S-module structure. If R and S are commutative, this left adjoint is symmetric monoidal [Lur17, Section 4.5.3]

- For a connective $\mathbb{E}_\infty$ ring spectrum R we have **Postnikov towers** in Mod_R as well as truncations/covers $\tau_{\geq n}, \tau_{\leq n} : \mathrm{Mod}_R \to \mathrm{Mod}_R$ ([Lur17, Proposition 7.1.1.13]). In particular they exist for spectra.

- **Filtered objects** (i.e. objects of $\mathrm{Fun}((N\mathbb{Z})^{\mathrm{op}}, \mathcal{C})$ in stable ∞-categories like the ones above provide **spectral sequences** with values in the abelian category of $\pi_0(R)$-modules ([Lur17, Section 1.2.2]). For a classical reference on spectral sequence, the reader can confer [Wei94, Chapter 5].

- Homotopy groups commute with arbitrary limits and filtered colimits. There is no general formula[7] for tensor products $\pi_*(X \otimes_{\mathbb{S}} Y)$. But in the case, that both X and Y are bounded below we can at least say something. Let $pi_n(X)$ and $pi_m(X)$ be the lowest non-zero homotopy groups of X, Y then $X \otimes_S Y$ is also bounded below with lowest homotopy group $\pi_{n+m}(X \otimes_{\mathbb{S}} Y) = \pi_n(X) \otimes_{\mathbb{Z}} \pi_m(Y)$. For connective module spectra M, N over a connective (commutative) ring spectrum R we similarly get that $M \otimes_R N$ is connective with $\pi_0(M \otimes_R N) = \pi_0(M) \otimes_{\pi_0(R)} \pi_0(N)$ ([Lur17, Corollary 7.2.1.23]).

The last construction from higher algebra, that we need is a generalization of p-completion of abelian groups. A textbook resource on p-completion of spectra is [BR20, Section 8.4.1]. Let us directly start with the definition, which is notationally identical to ordinary p-completion of abelian groups.

Definition 2.1. *Let X be a spectrum. We define its* p***-completion*** $X_p^\wedge$ *as the following limit*

$$X_p^\wedge := \lim(\ldots \to X/p^3 \to X/p^2 \to X/p).$$

This evidently provides an endofunctor of Sp. *X has a canonical map to $X_p^\wedge$ induced by the projections $X \to X/p^n$. We call X* p***-complete*** *if this map is an equivalence. A map of spectra is called a* p***-adic equivalence*** *if the induced map on p-completions is an equivalence.*

Let us be precise about the terms in the definition. For a spectrum X we of course always have the identity map $X \xrightarrow{1} X$. Since we can add maps between spectra, we can consider the n-fold addition of this map to itself to obtain $X \xrightarrow{n} X$. We denote the cofiber of this map as $X/n := \mathrm{cofib}(X \xrightarrow{n} X)$. The natural maps $X/p^n \to X/p^{n-1}$ are induced from taking horizontal cofibers in the relevant commutative diagram. An immediate observation is that p-completion commutes with arbitrary limits and finite colimits. From this we can also see, that $X/p \xrightarrow{\sim} (X_p^\wedge/p)$ is an equivalence. The crucial Lemma about p-complete spectra and the main reason why we have introduced the concept, is the following.

[7] But there is a spectral sequence of which we will make use of later.

Lemma 2.2. *Let $f : X \to Y$ be a map of spectra. Then f is a p-adic equivalence if and only if $f/p : X/p \to Y/p$ is an equivalence. In particular we can test whether a map of p-complete spectra is an equivalence by checking it mod p.*

Let us now collect a few further facts about p-completion, that we will need.

- The Eilenberg-Maclane spectrum of a p-complete abelian group is a p-complete spectrum. More precisely HA is p-complete if and only A is **derived p-complete** in the sense of Tag 091S.
- A spectrum is p-complete if and only if all of its homotopy groups are derived p-complete.
- p-completion is symmetric monoidal, in particular the p-completion of an $\mathbb{E}_n$-ring R spectrum is an $\mathbb{E}_n$-ring spectrum. Furthermore the map $R \to R_p^\wedge$ equips the p-completion with an $\mathbb{E}_n$-R-algebra structure.
- p-completion of tensor product, different ways to write it (use Lemma).

Finish p-completion

Given a commutative ring spectrum R and any element $z \in \pi_0(R)$, we can still talk about the z-completion of module spectra over R, which is defined completely analogously. The previous discussion was the case $R = \mathbb{S}$, $z = p \in \pi_0(\mathbb{S}) = \mathbb{Z}$ and $\mathrm{Mod}_{\mathbb{S}} = \mathrm{Sp}$. More or less everything goes through in this general setting aswell. For everything that we need (and much, much more) we refer to [Lur18c, Section 7.3].

Further things to cite:

- Nikolaus-Krause lecture notes [KN18]
- Dundas-Goodwillie-McCarthy (functoriality of THH for number rings is open problem, page 148, beginning of chapter 4) [DGM12, p. 148]

2.3 Topological Hochschild homology

Topological Hochschild homology is the higher algebra analogue of classical Hochschild homology, where one formally replaces the integers by the sphere spectrum. A classical reference on Hochschild homology which also discusses the relationship with algebraic K-theory is [Lod13].
In this section, we will define topological Hochschild homology and establish various properties of it.

Definition 2.3. *Let R be an $\mathbb{E}_1$-ring spectrum. We define the* ***topological Hochschild homology*** *of R as*

$$\mathrm{THH}(R) = R \otimes_{R \otimes_{\mathbb{S}} R^{op}} R.$$

If R is an $\mathbb{E}_1$-S-algebra, with $S \in \mathrm{CAlg}$ we can also define a relative version

$$\mathrm{THH}(R/S) = R \otimes_{R \otimes_S R^{op}} R.$$

In both cases, we abbreviate their p-completions:

$$\mathrm{THH}(R; \mathbb{Z}_p) \coloneqq \mathrm{THH}(R)^\wedge_p, \ \mathrm{THH}(R/S; \mathbb{Z}_p) \coloneqq \mathrm{THH}(R/S)^\wedge_p.$$

In the same way as for K-theory, we denote the homotopy groups by $\mathrm{THH}_*(R) \coloneqq \pi_*(\mathrm{THH}(R))$ and for the relative and p-completed version in the same way.

The left/right $R \otimes_{\mathbb{S}} R^{\mathrm{op}}$-module structure on R comes from the fact, that R is a R-R-bimodule which is equivalent to the data of an $R \otimes_{\mathbb{S}} R^{\mathrm{op}}$-module structure. In the case of an $\mathbb{E}_\infty$-ring spectrum we can drop the $(\ \ )^{op}$ by [Lur17, Section 4.6.3]. If S is an ordinary commutative ring[8], we will also use the notation $\mathrm{HH}(R/S) \coloneqq \mathrm{THH}(R/S)$ and $\mathrm{HH}(R/\mathbb{Z}) \coloneqq \mathrm{THH}(R/\mathbb{Z})$, because it agrees with (a derived version of) classical Hochschild homology. Using the Bar construction of the relative tensor product ([Lur17, Section 4.4.2] we get the following formula, which might be familiar from classical Hochschild homology.

$$\mathrm{THH}(R/S) = \mathrm{colim}_{\Delta^{\mathrm{op}}} \left(\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} R \otimes_S R \rightrightarrows R \right)$$

Example $\mathrm{THH}(\mathbb{S}) = \mathbb{S}$, THH of connective is connective. Lemma about p-completion? THH is sym mon. Module structure ...

finish

Proposition 2.4 (McClure-Schwänzl-Vogt). *Let R be an $\mathbb{E}_\infty$-ring spectrum, then the map $R \to \mathrm{THH}(R)$ is initial as a non-equivariant map of $\mathbb{E}_\infty$-ring spectra from R to an $\mathbb{E}_\infty$-ring spectrum with S^1-action (i.e. an object in $\mathrm{Fun}(BS^1, \mathrm{CAlg})$). In other words THH of an commutative ring spectrum is the colimit over the constant diagram $S^1 \to \mathrm{CAlg}$ with value R.*

Proof. (Idea) Use simplicial model of circle with n simplices in dimension n to obtain it from cyclic Bar construction [NS18, p. 114]. □

2.4 Cyclotomic spectra, TC and all that

The content of this section is not necessary for understanding the rest of the thesis. Nevertheless we want to include it, because it gives the main motivation, why one should be interested in THH in the first place. We can construct TC out of it which is extremely useful in calculations in K-theory. A cyclotomic structure on a spectrum is certain equivariant extra data, that are exactly what is necessary to define TC. To define it, we first need to discuss our equivariant setup. Let us stress that it is vastly different to genuine equivariant homotopy theory. In particular it is technically much simpler.

[8] From now on we will abuse notation and consider every ring as a ring spectrum without indication.

Definition 2.5. *Let G be a topological group like S^1 or C_p (the discrete cyclic group with p-elements) A* ***spectrum with G-action*** *is a functor $BG \to \mathrm{Sp}$ from the classifying space of G to spectra. A* ***G-equivariant map*** *is a natural transformation i.e. a 1-cell in the functor category $\mathrm{Fun}(BG, \mathrm{Sp})$. The* ***homotopy orbits*** *and* ***homotopy fixed points*** *of a spectrum with G-action $X : BG \to \mathrm{Sp}$ are defined as the colimit and limit of the functor.*

$$X^{hG} := \lim(F : BG \to \mathrm{Sp}), \quad X_{hG} := \mathrm{colim}(F : BG \to \mathrm{Sp})$$

There is also a third operation, that we need, called the ***Tate construction*** *X^{tG}, which comes equipped with a canonical map from the fixed points $X^{hG} \xrightarrow{\mathrm{can}} X^{tG}$.*

Do I need something else?

Definition 2.6. *[NS18, Chapter 2.1] A* ***cyclotomic spectrum*** *is a spectrum with S^1-action X together with S^1-equivariant maps $\varphi_p : X \to X^{tC_p}$ for every prime p.*

Our main examples of cyclotomic spectra will come from topological Hochschild homology. We will only define the cyclotomic structure on topological Hochschild homology in the setting of $\mathbb{E}_\infty$-ring spectra. It can also be constructed in the greater generality of $\mathbb{E}_1$-ring spectra but it is more complicated, see [NS18, Chapter 3.1]). The two main ingredients in the construction of the cyclotomic structure are the **Tate diagonal** and Proposition 2.4 from the last section. Since we already did not discuss the Tate construction, we will also blackbox the Tate diagonal and simply assert that there is a natural map of spectra $X \to (\underbrace{X \otimes_{\mathbb{S}} \cdots \otimes_{\mathbb{S}} X}_{p})^{tC_p}$. For details see [NS18, Section 3.1]. Let us now construct the cyclotomic structure on $\mathrm{THH}(R)$ for $R \in \mathrm{CAlg}$. We first note that we can construct an initial $\mathbb{E}_\infty$-map out of R, where the target is a spectrum with C_p-action: $R \to \underbrace{R \otimes \cdots \otimes R}_{p}$. Since $\mathrm{THH}(R)$ has a S^1-action and hence also a C_p-action we get a map $R \otimes_{\mathbb{S}} \cdots \otimes_{\mathbb{S}} R \to \mathrm{THH}(R)$. Applying the C_p-Tate construction to this map and precomposing it with the Tate diagonal of R we get an $\mathbb{E}_\infty$-map $R \xrightarrow{\Delta_p} (R \otimes \cdots \otimes R)^{tC_p} \to \mathrm{THH}(R)^{tC_p}$ whose target has $S^1/C_p \cong S^1$-action. Hence by Proposition 2.4 this factors through $\mathrm{THH}(R)$ and thus produces a map $\mathrm{THH}(R) \xrightarrow{\varphi_p} \mathrm{THH}(R)^{tC_p}$.

$$\begin{tikzcd}
R \arrow[r] \arrow[d, "\Delta_p"'] & \mathrm{THH}(R) \arrow[d, dashed, "\varphi_p"] \\
(R \otimes \cdots \otimes R)^{tC_p} \arrow[r] & \mathrm{THH}^{tC_p}
\end{tikzcd}$$

This map equips the topological Hochschild homology of any $\mathbb{E}_\infty$-ring spectrum with a cyclotomic structure. For every cyclotomic spectrum we can now define TC, which is our main object of interest for this section. First of all we define **negative topological cyclic homology** and **topological periodic homology**. For this we only need, that $\mathrm{THH}(R)$ has an S^1-action.

Definition 2.7. *Let R be a $\mathbb{E}_1$ ring spectrum. define*

$$\mathrm{TC}^{-}(R) \coloneqq \mathrm{THH}(R)^{hS^1}, \quad \mathrm{TP}(R) \coloneqq \mathrm{THH}(R)^{tS^1}$$

Definition 2.8. *Let X be a cyclotomic spectrum with structure maps $\varphi_p : X \to X^{tC_p}$, $p \in \mathbb{P}$. We define its* **topological cyclic homology** *as the following fiber:*

$$\mathrm{TC}(X) \to X^{hS^1} \xrightarrow{\prod_{p\in\mathbb{P}}\left((\varphi_p)^{hS^1} - can^{hS^1}\right)} \prod_{p\in\mathbb{P}} (X^{tC_p})^{hS^1}$$

We implicitely consider the map can^{hS^1} as being precomposed with an identication of its domain $can^{hS^1} : X^{hS^1} \simeq (X^{hC_p})^{hS^1} \to (X^{tC_p})^{hS^1}$. For $R \in \mathrm{Alg}$ we define $\mathrm{TC}(R) \coloneqq \mathrm{TC}(\mathrm{THH}(R))$.

If X is bounded below (e.g. $\mathrm{THH}(R)$ for R bounded below), we can rewrite the last term in an easier way using the result [NS18, Lemma II.4.2], that $(X^{tCp})^{hS^1} \simeq (X^{tS^1})^{\wedge}_p$. We thus get the formula

$$\mathrm{TC}(X) = \mathrm{fib}\left(\mathrm{TC}^{-}(X) \xrightarrow{\prod_{p\in\mathbb{P}}\left((\varphi_p)^{hS^1} - \mathrm{can}\right)} \prod_{p\in\mathbb{P}} \mathrm{TP}(X)^{\wedge}_p\right).$$

The crucial relationship with K-theory is provided via the **cyclotomic trace**, which is a natural transformation $K \xrightarrow{\mathrm{tr}} \mathrm{TC}$. A conceptual definition using a universal characterization of K-theory is given in [BGT13, Section 10.3]. The first crucial theorem establishing the intimate relationship between K-theory and TC was proven by Dundas-Goodwillie-McCarthy .

Theorem 2.9. *([DGM12, Theorem 7.2.2.1]) Let $R \to S$ be a map of $\mathbb{E}_1$-ring spectra. Assume that the induced map on π_0 is surjective with a nilpotent kernel. Then the induced maps on K-theory and* TC *together with the respective cyclotomic traces give a pullback square of spectra:*

$$\begin{tikzcd} K(R) \arrow[r] \arrow[d] & \mathrm{TC}(R) \arrow[d] \\ K(S) \arrow[r] & \mathrm{TC}(S) \end{tikzcd}$$

In fact, we can even get away with a weaker conditions on the map $R \to S$ by recent work of Clausen-Mathew-Morrow ([CMM21, Theorem A]).
The following result of Hesselholt and Madsen, proven using the Dundas-Goodwillie-McCarthy theorem, shows that for all rings of interest in this thesis, we can fully obtain their p-adic K-theory from knowledge of their topological cyclic homology.

.

Theorem 2.10. *[HM97, Theorem D] Let k be a perfect field of characteristic p and R a $W(k)$-algebra, that is finitely generated as a $W(k)$-module. Then the cyclotomic trace map induces the following isomorphism*

$$K(R)^{\wedge}_p \xrightarrow{\sim} \tau_{\geq 0}\,\mathrm{TC}(R)^{\wedge}_p.$$

For example we get $K(\mathbb{F}_p)^\wedge_p \simeq \tau_{\geq 0}\,\mathrm{TC}(\mathbb{F}_p)^\wedge_p = H\mathbb{Z}_p$. Together with the fact, that for all other primes q we have a q-adic equivalence with connective complex (topological) K-theory $K(\mathbb{F}_p)^\wedge_q \simeq (\mathrm{ku})^\wedge_q$, this allows us to fully determine the homotopy type of $K(\mathbb{F}_p)$. Furthermore, this result applies to all complete discrete valuation rings of mixed characteristic with complete residue field because they are finite $W(k)$-modules, i.e. all rings of interest in Chapter 3.
Another interesting application of a more geometric flavour is the following: The **A-theory** spectrum of a based space X is defined to be $K(\Sigma^\infty_+(\Omega X))$. Observe, that the $\mathbb{E}_1$-map $\Sigma^\infty_+(\Omega X) \to \Sigma^\infty_+(\Omega X) \otimes_{\mathbb{S}} \mathbb{Z} \to \mathbb{Z}[\pi_1(X)]$ is an isomorphism on π_0. Hence we can apply the Dundas-Goodwillie-McCarthy theorem and obtain the following pullback square.

$$\begin{tikzcd}
K(\Sigma^\infty_+(\Omega X)) \arrow[r] \arrow[d] & \mathrm{TC}(\Sigma^\infty_+(\Omega X)) \arrow[d] \\
K(\mathbb{Z}[\pi_1(X)]) \arrow[r] & \mathrm{TC}(\mathbb{Z}[\pi_1(X)])
\end{tikzcd}$$

This gives us significant insight into $A(X)$, because we know a lot about the three other constituents. $\mathrm{TC}(\Sigma^\infty_+(\Omega X))$ is to a large part understood by work of Bökstedt-Hsiang-Madsen [BHM93] (for a modern presentation see [NS18, Section IV.3]). $K(\mathbb{Z}[\pi_1(X)])$ can often be computed if the the Farrell-Jones conjecture for $\pi_1(X)$ is confirmed (see e.g. [BLR08]). Calculations of $A(X)$ are highly sought after because they contain a lot of geometric information about X due to the celebrated stable parametrized h-cobordism theorem by Waldhausen-Jahren-Rognes ([WJR13]). If X is a smooth manifold this allows us in particular to compute $\pi_*(\mathrm{Diff}(X))$ in a certain range of degrees. This connection (predating the invention of topological cyclic homology) was used by Farrell-Hsiang to compute $\mathbb{Q} \otimes \pi_*(\mathrm{Diff}(D^n))$ in [FH78] using a theorem of Waldhausen and Borel's work on the rational homotopy groups of $K(\mathbb{Z})$.

What is with $\mathrm{TC}(\mathbb{Z}[\pi_1(X)])$

3. Topological Hochschild Homology of CDVRs

In this chapter we will calculate topological Hochschild homology of (certain) complete discrete valuation rings.
Our main input for this is the following fundamental result of Bökstedt. We give two statements.

Theorem 3.1 (Bökstedt)**.** *On homotopy groups there is an isomorphism of graded rings* $\mathrm{THH}_*(\mathbb{F}_p) = \mathbb{F}_p[x]$ *for* $x \in \mathrm{THH}_2(\mathbb{F}_p)$ *a generator. We can also phrase this more conceptually as follows:* $\mathrm{THH}(\mathbb{F}_p)$ *is the free* $\mathbb{E}_1$*-*$\mathbb{F}_p$*-algebra on an element in degree 2. In other words* $\mathrm{THH}(\mathbb{F}_p) \cong \mathbb{F}_p \otimes_{\mathbb{S}} \Sigma^\infty_+ \Omega S^3$*.*

Note that the second statement implies the first[1], because the homotopy groups of $\mathbb{F}_p \otimes_{\mathbb{S}} \Sigma^\infty_+ \Omega S^3$ are the $\mathbb{F}_p$-homology groups of ΩS^3, which can be computed by the Serre spectral sequence associated to the path space fibration of S^3. The ring structure on $H_*(\Omega S^3; \mathbb{F}_p)$ is given by the Pontryagin product induced by the $\mathbb{E}_1$-structure on ΩS^3.
This periodicity in homotopy groups is called Bökstedt periodicity. In fact, the same phenomenon occurs in a much greater class of examples, namely for perfect $\mathbb{F}_p$-algebras and even all perfectoid rings. There is also a relative version for complete discrete valuation rings with perfect residue field of characteristic p. The proof for these results bootstraps the Bökstedt periodicity of $\mathbb{F}_p$ using two base change formulas that we will introduce now.

Proposition 3.2. *Let* $k' \to k$ *be a map of* $\mathbb{E}_\infty$*-ring spectra and* R *an* $\mathbb{E}_1$*-*k'*-algebra. Then we have*

$$\mathrm{THH}(R/k') \otimes_{k'} k = \mathrm{THH}(R \otimes_{k'} k/k).$$

Proof. The base change functor $- \otimes_{k'} k : \mathrm{Mod}_{k'} \to \mathrm{Mod}_k$ is left adjoint to the forgetful functor, so it preserves all colimits. It furthermore carries a symmetric monoidal structure. Since topological Hochschild homology in a symmetric monoidal category is built via the geometric realization of the cyclic bar construction it only uses the monoidal structures and geometric realization. Hence any

[1] Both statements are actually equivalent. For the other direction observe that a generator of $\mathrm{THH}_2(\mathbb{F}_p)$ gives a map of $\mathbb{E}_1$-$\mathbb{F}_p$-algebras $\mathbb{F}_p \otimes_{\mathbb{S}} \Sigma^\infty_+ \Omega S^3 \to \mathrm{THH}(\mathbb{F}_p)$ because the domain is the free $\mathbb{E}_1$-$\mathbb{F}_p$-algebra on an element in degree 2 and the target is even an $\mathbb{E}_\infty$-$\mathbb{F}_p$-algebra. By assumption this map induces an isomorphism on homotopy groups and since equivalences of $\mathbb{E}_n$-algebras can be detected on the underlying spectra, the map is an equivalence of $\mathbb{E}_1$-$\mathbb{F}_p$-algebras.

symmetric monoidal functor that also preserves geometric realizations will induce an equivalence on topological Hochschild homology. This applies in particular to the functor $- \otimes_{k'} k : \mathrm{Mod}'_k \to \mathrm{Mod}_k$, since it preserves all colimits. $\square$

An important special case is $k' = \mathbb{S}$, which gives us $\mathrm{THH}(R) \otimes_{\mathbb{S}} k = \mathrm{THH}(R \otimes_{\mathbb{S}} k/k)$.
The next base change formula will allow us to 'decompose' relative THH computations. We prepare it with a small Lemma.

Lemma 3.3. *Let $k' \to k$ be a map of $\mathbb{E}_\infty$-ring spectra and $M, N \in \mathrm{Mod}_k$ modules over k (and thus also over k'). Consider k as a module over $k \otimes_{k'} k$ via multiplication. We then have the following natural equivalence*

$$M \otimes_k N \simeq (M \otimes_{k'} N) \otimes_{k \otimes_{k'} k} k.$$

Proof. We have a canonical map of k-modules $M \otimes_{k'} N \to M \otimes_k N$ induced by the map $k' \to k$. Under the adjunction $- \otimes_{k \otimes_{k'} k} k : \mathrm{Mod}_{k \otimes_{k'} k} \rightleftarrows \mathrm{Mod}_k : \mathrm{Forget}$ this is adjoint to the map

$$(M \otimes_{k'} N) \otimes_{k \otimes_{k'} k} k \to M \otimes_k N.$$

We want to show that this is an equivalence. Because Mod_k is under colimits generated by k and $- \otimes_{k'} -$ preserves colimits in both variables separately, it suffices to check the statement in the case $M = N = k$. There we have

$$(k \otimes_{k'} k) \otimes_{k \otimes_{k'} k} k \to k \otimes_k k,$$

which is clearly an isomorphism. $\square$

Using exactly the same argument we also obtain the result

$$M_1 \otimes_k M_2 \otimes_k \cdots \otimes_k M_n \xleftarrow{\sim} (M_1 \otimes_{k'} M_2 \otimes_{k'} \cdots \otimes_{k'} M_n) \otimes_{k \otimes_{k'} \cdots \otimes_{k'} k} k,$$

for $M_1, \ldots M_n$ modules over k.

Proposition 3.4. *Let again $k' \to k$ be a map of $\mathbb{E}_\infty$-ring spectra and R an $\mathbb{E}_1$-k-algebra. Then k is a $\mathrm{THH}(k/k')$-module and we have*

$$\mathrm{THH}(R/k) = \mathrm{THH}(R/k') \otimes_{\mathrm{THH}(k/k')} k$$

Proof Sketch. Let us start by writing the left hand side in terms of its cyclic bar construction and applying the previous Lemma in the case where all k-modules are given by R.

$$\begin{aligned}
\mathrm{THH}(R/k) &= \mathrm{colim}_{\Delta^{\mathrm{op}}} (\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} R \otimes_k R \rightrightarrows R) \\
&\overset{3.3}{=} \mathrm{colim}_{\Delta^{\mathrm{op}}} \left(\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} (R \otimes_{k'} R) \otimes_{k \otimes_{k'} k} k \rightrightarrows R\right) \\
&= \mathrm{colim}_{\Delta^{\mathrm{op}}} \left((\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} R \otimes_{k'} R \rightrightarrows R) \otimes_{(\ldots k \otimes_{k'} k \rightrightarrows k)} (\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} k \rightrightarrows k)\right)
\end{aligned}$$

In the last line we consider the category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Sp})$ as a symmetric monoidal category equipped with the Day convolution tensor product. In this situation $(\cdots \rightrightarrows k \otimes_{k'} k \rightrightarrows k)$ is a commutative algebra object in $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Sp})$ and $(\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} R \otimes_{k'} R \rightrightarrows R)$, $(\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} k \rightrightarrows k)$ are modules over it. The tensor product over $(\cdots \rightrightarrows k \otimes_{k'} k \rightrightarrows k)$ is then to be understood in the sense of [Lur17, Section 4.5.2]. Now we can use that Δ^{op} is sifted, i.e. the diagonal $\Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$ is cofinal and thus also $\Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$ is cofinal (see [Lur09, Lemma 5.5.8.4]). Applying this cofinality (under the equivalent characterization of cofinality proven in [Lur09, Proposition 4.1.1.8]); that the tensor product commutes with colimits in both variables separately and that we can also pull geometric realizations out of the ring, we are tensoring over[2], we obtain:

$$\begin{aligned}
\mathrm{THH}(R/k) &= \mathrm{colim}_{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}} \big((\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} R \otimes_{k'} R \rightrightarrows R) \otimes_{(\ldots k \otimes_{k'} k \rightrightarrows k)} (\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} k \rightrightarrows k) \big) \\
&= \mathrm{colim}_{\Delta^{\mathrm{op}}} (\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} R \otimes_{k'} R \rightrightarrows R) \otimes_{\mathrm{colim}_{\Delta^{\mathrm{op}}} (\ldots k \otimes_{k'} k \rightrightarrows k)} \mathrm{colim}_{\Delta^{\mathrm{op}}} (\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} k \rightrightarrows k) \\
&= \mathrm{THH}(R/k) \otimes_{\mathrm{THH}(k/k')} k
\end{aligned}$$

This proves the desired result. $\square$

Again there is the important special case of $k' = \mathbb{S}$, which gives us $\mathrm{THH}(R/k) = \mathrm{THH}(R) \otimes_{\mathrm{THH}(k)} k$.
Note also that Proposition 3.4 implies Proposition 3.2. Indeed

$$\begin{aligned}
\mathrm{THH}(R \otimes_{k'} k/k) &\overset{3.4}{=} \mathrm{THH}(R \otimes_{k'} k/k') \otimes_{\mathrm{THH}(k/k')} k \\
&= \mathrm{THH}(R/k') \otimes_{k'} \mathrm{THH}(k/k') \otimes_{\mathrm{THH}(k/k')} k \\
&= \mathrm{THH}(R/k') \otimes_{k'} k.
\end{aligned}$$

3.1 Bökstedt periodicity for perfect $\mathbb{F}_p$- algebras

In this section we will prove Bökstedt periodicity for perfect[3] $\mathbb{F}_p$-algebras. Examples include perfect fields, in particular finite fields $\mathbb{F}_{p^n}$, as well as non-Noetherian examples like $\mathbb{F}_p[x^{1/p^\infty}]$. For the proof, we will rely on the following construction that establishes that we can not only deform perfect $\mathbb{F}_p$-algebras to characteristic zero via Witt vectors, but even to the sphere spectrum using a spherical analogue of the Witt vectors. (see [Lur18a], 5.2.5/5.2.7)

Proposition 3.5. *Let k be a perfect $\mathbb{F}_p$-algebra. Then there exists a p-complete $\mathbb{E}_\infty$-ring spectrum $\mathbb{S}_{W(k)}$ - the* ***spherical Witt vectors*** *of k - with base change*

[2] This itself uses the construction of the relative tensor product as the geometric realization of the bar construction ([Lur17, Section 4.4.2]) and once more the cofinality of $\Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$.

[3] For every $\mathbb{F}_p$-algebra R we have that the Frobenius $\varphi : R \to R,\ x \mapsto x^p$ is a ring homomorphism. We call R perfect if φ is an isomorphism.

$\mathbb{S}_{W(k)} \otimes_{\mathbb{S}} \mathbb{F}_p \simeq k$. *We thus get the following diagram of deformations of k:*

$$\begin{tikzcd}
\mathbb{S}_{W(k)} \arrow[r] & W(k) \arrow[r] & k \\
\mathbb{S} \arrow[r] \arrow[u] & \mathbb{Z} \arrow[r] \arrow[u] & \mathbb{F}_p \arrow[u]
\end{tikzcd}$$

In general for any $\mathbb{F}_p$-algebra k we have a map $\mathbb{F}_p \to k$, which induces a map of $\mathbb{E}_\infty$-ring spectra $\mathrm{THH}(\mathbb{F}_p) \to \mathrm{THH}(k)$. Since the target is a k-module spectrum, this further refines to $\mathrm{THH}(\mathbb{F}_p) \otimes_{\mathbb{F}_p} k \to \mathrm{THH}(k)$. The statement of Bökstedt periodicity for perfect $\mathbb{F}_p$-algebras is now the following.

Theorem 3.6. *For a perfect $\mathbb{F}_p$-algebra k the above map $\mathrm{THH}(\mathbb{F}_p)\otimes_{\mathbb{F}_p} k \to \mathrm{THH}(k)$ is an equivalence of $\mathbb{E}_\infty$-k-algebras. This in particular determines the homotopy groups as a graded ring:*

$$\mathrm{THH}_*(k) = k[x],\ |x| = 2.$$

Proof. Using our previous Proposition and that THH is symmetric monoidal, we see that:

$$\begin{aligned}
\mathrm{THH}(k) &= \mathrm{THH}(\mathbb{F}_p \otimes_{\mathbb{S}} \mathbb{S}_{W(k)}) \\
&= \mathrm{THH}(\mathbb{F}_p) \otimes_{\mathbb{S}} \mathrm{THH}(\mathbb{S}_{W(k)}) \\
&= \mathrm{THH}(\mathbb{F}_p) \otimes_{\mathbb{F}_p} (\mathbb{F}_p \otimes_{\mathbb{S}} \mathrm{THH}(\mathbb{S}_{W(k)}))
\end{aligned}$$

By Proposition 3.2 the term in brackets can be identified as $\mathbb{F}_p \otimes_{\mathbb{S}} \mathrm{THH}(\mathbb{S}_{W(k)}) = \mathrm{THH}(\mathbb{S}_{W(k)} \otimes_{\mathbb{S}} \mathbb{F}_p/\mathbb{F}_p) = \mathrm{THH}(k/\mathbb{F}_p)$. Furthermore since $\mathbb{F}_p$ is an ordinary ring $\mathrm{THH}(k/\mathbb{F}_p) = \mathrm{HH}(k/\mathbb{F}_p)$. To prove the desired result, it remains to show that $\mathrm{HH}(k/\mathbb{F}_p)$ is k concentrated in degree 0. Because k is commutative $\mathrm{HH}_0(k/\mathbb{F}_p) = k$, so we need to prove the vanishing of the higher homology groups.

We claim that for a perfect $\mathbb{F}_p$-algebra R the cotangent complex vanishes $L_{R/\mathbb{F}_p} = 0$. Together with the Hochschild-Kostant-Rosenberg filtration on Hochschild homology, this gives that $\mathrm{HH}_i(k/\mathbb{F}_p) = 0$ for $i > 0$, which then finishes the proof. For the vanishing of the cotangent complex, note that for any $\mathbb{F}_p$ algebra the Frobenius induces multiplication by p on the cotangent complex[4], i.e. the zero map. But since R is perfect, the Frobenius $\varphi : R \to R$ is an isomorphism and by functoriality still an isomorphism on the cotangent complex. The only way for the zero map to be an isomorphism is if already $L_{R/\mathbb{F}_p} = 0$. □

[4]To see this, recall that the cotangent complex is by definition the non-abelian derived functor of Kähler differentials. To compute $L_{R/\mathbb{F}_p}$ we thus have to simplicially resolve R by polynomial rings over $\mathbb{F}_p$ and then apply Kähler differentials level-wise. The Frobenius then acts via $\mathrm{d}x \mapsto \mathrm{d}(x^p) = p\mathrm{d}x$.

3.2 Relative Bökstedt periodicity for complete discrete valuation rings

Let R be a discrete valuation ring, i.e. a local principal ideal domain that is not a field. Denote its unique maximal ideal by $\mathfrak{m}$ and the residue field by $k = R/\mathfrak{m}$. In this section we assume that R is complete with respect to the ideal $\mathfrak{m}$, i.e. $R \xrightarrow{\sim} \lim(R/\mathfrak{m} \leftarrow R/\mathfrak{m}^2 \leftarrow \dots)$ and that the residue field k is perfect of characteristic $p > 0$. We will abreviate the term complete discrete valuation ring to CDVR. Any generator $\pi \in R$ of the maximal ideal will be called a *uniformizer*, completeness with respect to the ideal $\mathfrak{m}$ is then equivalent to completeness with respect to the element π.

Either R and k are both of characteristic p (then we say that R is of *equal characteristic*) or R is of characteristic 0 and k is of characteristic p (the *mixed characteristic* case). In the equal characteristic setting, it is known that $R = k[[x]]$ for k a perfect field of characteristic p (see [Ser95], Part One, Chapter 2.4, Theorem 2). In the mixed characteristic case there are more examples: We again get an example for every perfect field of characteristic p, namely $W(k)$. But there are also ramified examples[5] like $\mathbb{Z}_p[\sqrt[n]{p}]$. or $\mathbb{Z}_p[\zeta_p]$.

Make this explicit as we will use this again several times later.

To calculate $\mathrm{THH}(R)$ we first work relative to the maximal ideal which will allow us to use the result from the last section because the residue field is by assumption perfect. In the next section, we will then obtain the non-relative result using a descent spectral sequence.

We will work relative to the spectrum $\mathbb{S}[z] := \Sigma^\infty_+(\mathbb{N}_0)$, the spherical monoid ring of $\mathbb{N}_0$. Since $\mathbb{N}_0$ is a commutative monoid, hence an $\mathbb{E}_\infty$-monoid in spaces and $\Sigma^\infty_+ : \mathcal{S} \to \mathrm{Sp}$ has a strong symmetric monoidal structure, we get that $\mathbb{S}[z]$ is an $\mathbb{E}_\infty$-ring spectrum. To work relative to $\mathbb{S}[z]$ we need an $\mathbb{E}_1$-map $\mathbb{S}[z] \to HR$, which informally is just given by $z \mapsto \pi$ for π a uniformizer in R. To make this map precise, note that the $(\Sigma^\infty_+, \Omega^\infty)$-adjunction is compatible with the symmetric monoidal structures[6] on $\mathcal{S}$ and Sp, so we get an induced adjunction between $\mathrm{CAlg}(\mathcal{S})$ and $\mathrm{CAlg}(\mathrm{Sp})$. Thus giving a map $\mathbb{S}[z] \to HR$ of $\mathbb{E}_\infty$-ring spectra is equivalent to giving an map $\mathbb{N}_0 \to R$ of $\mathbb{E}_\infty$-monoids in spaces. This is just a morphism of ordinary (commutative) monoids and since $\mathbb{N}_0$ is the free monoid (in sets) on one generator, we get such a map for each element of R. In particular we have a map corresponding to $\pi \in R$ and this is what we mean by $\mathbb{S}[z] \to R$, $z \mapsto \pi$. We are now ready to state and prove a relative variant of Bökstedt's theorem for CDVRs.

Theorem 3.7. *Let R be a complete discrete valuation ring with perfect residue field of characteristic p. Choose a uniformizer $\pi \in R$ and give R a $\mathbb{S}[z]$-algebra*

[5] In general every mixed characteristic CDVR can be obtained by adjoining a root of an Eisenstein polynomial to W(k). See [Ser95] Part one, chapter 2.5.

[6] i.e. Σ^∞_+ is (strong) symmetric monoidal and Ω^∞ is (lax) symmetric monoidal with respect to the cartesian product on $\mathcal{S}$ and the tensor product on Sp.

structure by $z \mapsto \pi$. Then

$$\mathrm{THH}_*(R/\mathbb{S}[z];\mathbb{Z}_p) = R[x],\ |x| = 2.$$

Proof. Let us first treat the case that R is mixed characteristic. Since R is characteristic 0, p is not zero in R, so $p = u\pi^n$ for u a unit in R and $n \in \mathbb{N}$. This implies that an R-module is derived p-complete if and only if it is derived π-complete by [Stacks, Tag 091Q]. In particular we get that the homotopy groups of $\mathrm{THH}(R/\mathbb{S}[z];\mathbb{Z}_p)$ are derived π-complete because they are by definition derived p-complete. Using Proposition 3.2 and that the tensor product preserves colimits in both variables we can now calculate

$$\begin{aligned}\mathrm{THH}(R/\mathbb{S}[z];\mathbb{Z}_p)\otimes_{\mathbb{S}[z]}\mathbb{S} &\overset{3.2}{=} \mathrm{THH}(R\otimes_{\mathbb{S}[z]}\mathbb{S}/\mathbb{S};\mathbb{Z}_p)\\ &= \mathrm{THH}(R\otimes_{\mathbb{S}[z]}\mathrm{cofib}\left(\mathbb{S}[z]\xrightarrow{\cdot z}\mathbb{S}[z]\right);\mathbb{Z}_p)\\ &= \mathrm{THH}(R/\pi;\mathbb{Z}_p) = \mathrm{THH}(k;\mathbb{Z}_p) = \mathrm{THH}(k)\end{aligned}$$

In the last line we used that z acts by multiplication by π on R and that $\mathrm{THH}(k)$ is already p-complete since all its homotopy groups are already derived p-complete (even classically p-complete). On the other hand, again using that the tensor product preserves colimits, we can also see that $\mathrm{THH}(R/\mathbb{S}[z];\mathbb{Z}_p)\otimes_{\mathbb{S}[z]}\mathbb{S} = \mathrm{THH}(R/\mathbb{S}[z];\mathbb{Z}_p)/\pi$. Hence we get the cofiber sequence

$$\mathrm{THH}(R/\mathbb{S}[z];\mathbb{Z}_p)\xrightarrow{\cdot\pi}\mathrm{THH}(R/\mathbb{S}[z];\mathbb{Z}_p)\to\mathrm{THH}(k).$$

Since k is by assumption perfect of characteristic p it is in particular a perfect $\mathbb{F}_p$-algebra, so Theorem 3.6 applies and tells us the homotopy groups of $\mathrm{THH}(k)$. Thus the long exact sequence for the odd homotopy groups looks as follows

$$\ldots\to k\to\mathrm{THH}_{2i+1}(R/\mathbb{S}[z];\mathbb{Z}_p)\xrightarrow{\cdot\pi}\mathrm{THH}_{2i+1}(R/\mathbb{S}[z];\mathbb{Z}_p)\to 0\to\ldots$$

This tells us that the odd homotopy groups of $\mathrm{THH}(R/\mathbb{S}[z];\mathbb{Z}_p)$ are 0 mod π. But by our discussion at the beginning of the proof we also know that they are derived π-complete, so they must already be 0 before taking mod π reduction (see [Stacks, Tag 09B9]).
For the identification of the (non-negative) even homotopy groups, we proceed similarly. The long exact sequence takes the form

$$\ldots\to 0\to\mathrm{THH}_{2i}(R/\mathbb{S}[z];\mathbb{Z}_p)\xrightarrow{\cdot\pi}\mathrm{THH}_{2i}(R/\mathbb{S}[z];\mathbb{Z}_p)\to k\to\ldots,$$

which gives us that they are π-torsion free and their mod π reduction is k. They are also derived π-complete. The same holds for R, so once we have an R-module map (over k) we get an isomorphism by the derived Nakayama Lemma (π-torsion freeness guarantees that the derived mod π reduction agrees with the ordinary one). Let $M \coloneqq \mathrm{THH}_2(R/\mathbb{S}[z];\mathbb{Z}_p)$. By the long exact sequence we have a surjective map $M\to k$. Giving an R-module map $R\to M$ is the same as choosing an element $x \in M$. To also ensure that the map is an equivalence after reducing mod π, the reduction of x must generate k as an R-module. So we have to choose a preimage of any non-zero element of k under the map $M \to k$, which always exists, since the map is surjective by the long exact sequence. Thus choosing such a lift provides an isomorphism $\mathrm{THH}_2(R/\mathbb{S}[z]; \mathbb{Z}_p) \cong R$. The same applies to all positive even homotopy groups once we also choose lifts for all of them. A priori the choices of all these lifts might be unrelated. But since the map $\mathrm{THH}_*(R/\mathbb{S}[z]; \mathbb{Z}_p) \to \mathrm{THH}_*(k)$ is multiplicative, the elements $x^i \in \mathrm{THH}_{2i}(R/\mathbb{S}[z]; \mathbb{Z}_p)$ provide canonical choices of all of those. Hence the choice of x already suffices and gives us an isomorphism of graded rings $\mathrm{THH}_*(R/\mathbb{S}[z]; \mathbb{Z}_p) \xrightarrow{\sim} \mathrm{THH}_*(k)$.

If R is of equal characteristic, we have already remarked that necessarily $R = k[[z]]$ for k perfect of positive characteristic. We proceed similarly to the proof of Theorem 3.6. Defining $\mathbb{S}_{W(k)}[[z]]$ as the z-adic completion of $\mathbb{S}_{W(k)} \otimes_{\mathbb{S}} \mathbb{S}[z]$, gives us the following base change

$$\mathbb{S}_{W(k)}[[z]] \otimes_{\mathbb{S}} \mathbb{F}_p = k[[z]].$$

To see this, note that the right hand side is the z-completion of the left hand side, as in general we have $(X \otimes_{\mathbb{S}} Y)^\wedge_z = (X^\wedge_z \otimes_{\mathbb{S}} Y^\wedge_z)^\wedge_z$. To establish that the map to the completion is an isomorphism we thus have to show that the left hand side is already z-complete. Since $\mathbb{F}_p$ is finite type over the sphere, we can write it as a filtered colimit of finite spectra along strictly increasingly connective maps, i.e. $\mathbb{F}_p = \mathrm{colim}_i X_i$ with X_i finite and $X_i \to X_{i+1}$ is i-connective. Hence $\pi_n(\mathbb{F}_p \otimes_{\mathbb{S}} \mathbb{S}_{W(k)}[[z]]) = \pi_n(X_n \otimes_{\mathbb{S}} \mathbb{S}_{W(k)}[[z]])$. As X_n is finite and completion is preserved under *finite* colimits, $X_n \otimes_{\mathbb{S}} \mathbb{S}_{W(k)}[[z]]$ is z-complete, which implies that its homotopy groups are derived z-complete. Thus all homotopy groups of $\mathbb{S}_{W(k)}[[z]] \otimes_{\mathbb{S}} \mathbb{F}_p$ are derived z-complete, meaning that $\mathbb{S}_{W(k)}[[z]] \otimes_{\mathbb{S}} \mathbb{F}_p$ is z-complete[7].
After establishing this base change, we can use 3.2 again :

$$\begin{aligned}
\mathrm{THH}(k[[z]]/\mathbb{S}[z]) &= \mathrm{THH}(\mathbb{S}_{W(k)}[[z]] \otimes_{\mathbb{S}} \mathbb{F}_p) \\
&= \mathrm{THH}(\mathbb{S}_{W(k)}[[z]]/\mathbb{S}[z]) \otimes_{\mathbb{S}} \mathrm{THH}(\mathbb{F}_p/\mathbb{S}[z]) \\
&= \big(\mathrm{THH}(\mathbb{S}_{W(k)}[[z]]/\mathbb{S}[z]) \otimes_{\mathbb{S}} \mathbb{F}_p\big) \otimes_{\mathbb{F}_p} \mathrm{THH}(\mathbb{F}_p) \\
&\overset{3.2}{=} \mathrm{THH}(\mathbb{S}_{W(k)}[[z]] \otimes_{\mathbb{S}} \mathbb{F}_p/\mathbb{S}[z] \otimes_{\mathbb{S}} \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathrm{THH}(\mathbb{F}_p) \\
&= \mathrm{HH}(k[[z]]/\mathbb{F}_p[z]) \otimes_{\mathbb{F}_p} \mathrm{THH}(\mathbb{F}_p)
\end{aligned}$$

So we have to show that $\mathrm{HH}_i(k[[z]]/\mathbb{F}_p[z])$ vanishes for $i > 0$. For perfect $\mathbb{F}_p$-algebras we used perfectness to show this, which is not possible here, because $k[[z]]$ is clearly not perfect. But the map $\mathbb{F}_p[z] \to k[[z]]$ is relatively perfect[8],

[7]Note that this shows in general that although being complete is normally not closed under infinite colimits it is indeed preserved if we are taking a sequential colimit along strictly increasingly connective maps. We will use this again later.

[8]The naming comes from the fact, that an $\mathbb{F}_p$-algebra R is perfect if and only if $\mathbb{F}_p \to R$ is relatively perfect

which by definition means, that the following square is a pushout in CAlg.

$$\begin{tikzcd}
\mathbb{F}_p[z] \arrow[r] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\ulcorner"] & k[[z]] \arrow[d, "\varphi"'] \\
\mathbb{F}_p[z] \arrow[r] & k[[z]]
\end{tikzcd}$$

To see this, we first observe that the vertical maps in the diagram are on the level of ordinary rings simply the inclusions $\mathbb{F}_p[z^p] \hookrightarrow \mathbb{F}_p[z]$ and $k[[z^p]] \hookrightarrow k[[z]]$ (where we implicitely identify k with itself under the action of Frobenius using that k is perfect). Thus both maps exhibit the target as a free module over the domain on the basis $\{1, z, z^2, \dots, z^p\}$. Because equivalences in CAlg are detected on the underlying spectra, this also means that

$$\mathbb{F}_p[z] \otimes_{\mathbb{F}_p[z^p]} k[[z^p]] \cong k[[z]]$$

which implies that the square is a pushout. Using this isomorphism, we obtain an isomorphism in Hochschild homology aswell:

$$\begin{aligned}
\mathrm{HH}(k[[z]]/\mathbb{F}_p[z]) &\xleftarrow{\sim} \mathrm{HH}((\mathbb{F}_p[z] \otimes_{\mathbb{F}_p[z^p]} k[[z^p]])/\mathbb{F}_p[z]) \\
&= \mathrm{HH}(\mathbb{F}_p[z]/\mathbb{F}_p[z]) \otimes_{\mathbb{F}_p[z^p]} \mathrm{HH}(k[[z^p]]/\mathbb{F}_p[z]) \\
&= \mathbb{F}_p[z] \otimes_{\mathbb{F}_p[z^p]} \mathrm{HH}(k[[z^p]]/\mathbb{F}_p[z])
\end{aligned}$$

We now want to show, that this map induces the zero map on all $\pi_n,\ n > 0$ which will then finish the proof as it is also an isomorphism. Let us first observe, that since $\mathbb{F}_p[z]$ is free as an $\mathbb{F}_p[z^p]$-module we can pull the tensor product out of the homotopy/homology group:

$$\pi_n(\mathbb{F}_p[z] \otimes_{\mathbb{F}_p[z^p]} \mathrm{HH}(k[[z^p]]/\mathbb{F}_p[z])) = \mathbb{F}_p[z] \otimes_{\mathbb{F}_p[z^p]} \mathrm{HH}_n(k[[z^p]]/\mathbb{F}_p[z])$$

Under this identification the map in question

$$\mathbb{F}_p[z] \otimes_{\mathbb{F}_p[z^p]} \mathrm{HH}_n(k[[z^p]]/\mathbb{F}_p[z]) \to \mathrm{HH}_n(k[[z]]/\mathbb{F}_p[z])$$

is adjoined (under the left adjoint to the forgetful functor $\mathrm{Mod}_{\mathbb{F}_p[z]} \to \mathrm{Mod}_{\mathbb{F}_p[z^p]}$) to the map, which is itself obtained by applying $\mathrm{HH}_n(-/\mathbb{F}_p[z])$ to the inclusion $k[[z^p]] \to k[[z]]$. As the Hochschild homology groups are the homotopy groups of a simplicial commutative algebra and the latter always carry a divided power structure (see eg. [Ric09, Section 4], this inclusion necessarily induces the zero map because we are in characteristic p. $\square$

Using the same technique we can also compute relative THH of quotients of the CDVRs, that we have considered so far. In any discrete valuation ring all ideals are powers of the maximal ideal. So concretely we are dealing with rings of the form R/π^n, where R is a complete discrete valuation ring with perfect residue field of positive characteristic and π a uniformizer. For instance this covers the

cases $\mathbb{Z}_p/p^n = \mathbb{Z}/p^n$ or truncated polynomial algebras $k[[x]]/x^n = k[x]/x^n$ for k perfect of positive characteristic.
For these quotients we do not quite get polynomial homotopy groups as before but we need an additional divided power generator compensating for the quotient.

Theorem 3.8. *Let R be a complete discrete valuation ring with perfect residue field of positive characteristic, $n \geq 1$ and $R' = R/\pi^n$. Then we have:*

$$\mathrm{THH}_*(R'/\mathbb{S}[z]) = R'[x]\langle y\rangle, \ |x| = |y| = 2$$

Proof. Since π is not a zero divisor, the ordinary and derived quotient agree, so $R' = R \otimes_{\mathbb{S}[z]} (\mathbb{S}[z]/z^n)$. Here we write $\mathbb{S}[z]/z^n = \Sigma^\infty M$, where M is the (pointed) monoid $M = \{0, 1, x, \ldots, x^{n-1}\}$ with multiplicative monoid structure and $x^a \cdot x^b = 0$ if $a + b \geq n$. By the strong symmetric monoidality of $\Sigma^\infty : (\mathcal{S}_*, \wedge) \to (\mathrm{Sp}, \otimes_{\mathbb{S}})$ we obtain an $\mathbb{E}_\infty$-structure[9] on $\mathbb{S}[z]/z^n$.
Using this identification of R', we can calculate:

$$\begin{aligned}
\mathrm{THH}(R'/\mathbb{S}[z]) &= \mathrm{THH}(R/\mathbb{S}[z]) \otimes_{\mathbb{S}[z]} \mathrm{THH}(\mathbb{S}[z]/z^n/\mathbb{S}[z]) \\
&= \mathrm{THH}(R/\mathbb{S}[z]) \otimes_{\mathbb{Z}\otimes_{\mathbb{S}}\mathbb{S}[z]} (\mathbb{Z} \otimes_{\mathbb{S}} \mathrm{THH}((\mathbb{S}[z]/z^n)/\mathbb{S}[z]) \\
&\overset{3.2}{=} \mathrm{THH}(R/\mathbb{S}[z]) \otimes_{\mathbb{Z}[z]} \mathrm{HH}\left((\mathbb{Z}[z]/z^n)/\mathbb{Z}[z]\right)
\end{aligned}$$

As p is nilpotent in R', every R'-module spectrum is already p-complete, in particular $\mathrm{THH}(R'/\mathbb{S}[z])$ must be p-complete. This means that it does not matter whether we p-complete one (or even both) of the factors, i.e. we get

$$\mathrm{THH}(R'/\mathbb{S}[z]) = \mathrm{THH}(R/\mathbb{S}[z]; \mathbb{Z}_p) \otimes_{\mathbb{Z}[z]} \mathrm{HH}\left((\mathbb{Z}[z]/z^n)/\mathbb{Z}[z]\right)$$

We completely understand the homotopy groups of the first factor by the previous theorem, so again we need to do a calculation in Hochschild homology. The derived tensor product $(\mathbb{Z}[z]/z^n) \otimes^L_{\mathbb{Z}[z]} (\mathbb{Z}[z]/z^n) = \Lambda_{\mathbb{Z}[z]/z^n}(e)$ is given by an exterior algebra over $\mathbb{Z}[z]/z^n$ on a generator e in degree 1. Thus Hochschild homology is given by

$$\mathrm{HH}\left((\mathbb{Z}[z]/z^n)/\mathbb{Z}[z]\right) = (\mathbb{Z}[z]/z^n) \otimes^L_{\Lambda_{\mathbb{Z}[z]/z^n}(e)} (\mathbb{Z}[z]/z^n).$$

To calculate this derived tensor product, we can explicitely resolve $\mathbb{Z}[z]/z^n$ as a free divided power algebra over $\Lambda_{\mathbb{Z}[z]/z^n}(e)$ as follows

$$\Lambda_{\mathbb{Z}[z]/z^n}(e)\langle y\rangle := \frac{\Lambda_{\mathbb{Z}[z]/z^n}(e)[y_1, y_2, \ldots]}{y_i y_j = \binom{i+j}{i} y_{i+j}}, \ \partial y_i = e y_{i-1}, \ |y_i| = 2i.$$

Hence we get the following result for Hochschild homology

$$\mathrm{HH}\left((\mathbb{Z}[z]/z^n)/\mathbb{Z}[z]\right) = \Lambda_{\mathbb{Z}[z]/z^n}(e)\langle y\rangle \otimes_{\Lambda_{\mathbb{Z}[z]/z^n}(e)} (\mathbb{Z}[z]/z^n) = (\mathbb{Z}[z]/z^n)\langle y\rangle.$$

[9]The underlying spectrum of $\mathbb{S}[z]/z^n$ is simply given by taking the cofiber of $\mathbb{S}[z] \xrightarrow{\cdot z^n} \mathbb{S}[z]$ but this has a priori no ring structure anymore. That is why we give the alternative description which makes the existence of an $\mathbb{E}_\infty$-structure clear.

This finally enables us to compute the homotopy groups

$$\mathrm{THH}_*(R'/\mathbb{S}[z]) = R'[x]\langle y\rangle,\ |x| = |y| = 2.$$

$\square$

3.3 Absolute THH of CDVRs

In this section, we will calculate the absolute THH of (quotients of) complete discrete valuation rings with perfect residue field of positive characteristic. We have already computed it relative to $\mathbb{S}[z]$. To get from there to the absolute case we will use a descent spectral sequence that we construct now.

Construction 3.9. *Let R be an ordinary commutative ring equiped with a map of rings $\mathbb{Z}[z] \to R$ (i.e. the choice of an element of R). This also gives R the structure of an $\mathbb{E}_\infty$-$\mathbb{S}[z]$-algebra, via $\mathbb{S}[z] \to \mathbb{Z}[z]$. We can filter $\mathrm{HH}(\mathbb{Z}[z]$ by its (very short) Whitehead tower:*

$$0 = \tau_{\geq 2}\,\mathrm{HH}(\mathbb{Z}[z]) \to \tau_{\geq 1}\,\mathrm{HH}(\mathbb{Z}[z]) \to \tau_{\geq 0}\,\mathrm{HH}(\mathbb{Z}[z]) = \mathrm{HH}(\mathbb{Z}[z])$$

Observe that $\mathrm{HH}(\mathbb{Z}[z]) \overset{3.2}{=} \mathrm{THH}(\mathbb{S}[z]) \otimes_{\mathbb{S}} \mathbb{Z}$, which gives us an $\mathbb{E}_\infty$-$\mathrm{HH}(\mathbb{Z}[z])$-algebra structure on $\mathrm{THH}(R)$ via the map $\mathrm{THH}(\mathbb{S}[z]) \to \mathrm{THH}(R)$. We thus have the strong monoidal base change functor $\mathrm{Mod}_{\mathrm{HH}(\mathbb{Z}[z])} \to \mathrm{Mod}_{\mathrm{THH}(R)}$ and can level-wise apply it to the filtration to get a multiplicative filtration on $\mathrm{THH}(R)$

$$0 \to \tau_{\geq 1}\,\mathrm{HH}(\mathbb{Z}[z]) \otimes_{\mathrm{HH}(\mathbb{Z}[z])} \mathrm{THH}(R) \to \mathrm{HH}(\mathbb{Z}[z]) \otimes_{\mathrm{HH}(\mathbb{Z}[z])} \mathrm{THH}(R) = \mathrm{THH}(R)$$

Similar to filtered chain complex, filtered spectra also give rise to spectral sequence, whose E^2-page is given by the homotopy groups of the associated graded, see e.g. [Lur17, Chapter 1.2.2]. To state the spectral sequence, we thus have to identify the associated graded.
Since tensor products preserve cofiber sequences, the associated graded of the filtration on $\mathrm{THH}(R)$ can simply be obtained by tensoring the associated graded of the filtration on $\mathrm{HH}(\mathbb{Z}[z])$ with $\mathrm{THH}(R)$. The associated graded of the filtration on $\mathrm{HH}(\mathbb{Z}[z])$ is given by $\mathrm{HH}_i(\mathbb{Z}[z]) = \Omega^i_{\mathbb{Z}[z]/\mathbb{Z}}$. Using our base change formulas, we can relate this back to relative THH:

$$\begin{aligned}
\mathrm{gr}^j &= \mathrm{THH}(R) \otimes_{\mathrm{HH}(\mathbb{Z}[z])} \Omega^j_{\mathbb{Z}[z]/\mathbb{Z}} \\
&= \mathrm{THH}(R) \otimes_{\mathrm{HH}(\mathbb{Z}[z])} \mathbb{Z}[z] \otimes_{\mathbb{Z}[z]} \Omega^j_{\mathbb{Z}[z]/\mathbb{Z}} \\
&\overset{3.2}{=} \big(\mathrm{THH}(R) \otimes_{\mathrm{THH}(\mathbb{S}[z])\otimes_{\mathbb{S}}\mathbb{Z}} (\mathbb{S}[z] \otimes_{\mathbb{S}} \mathbb{Z})\big) \otimes_{\mathbb{Z}[z]} \Omega^j_{\mathbb{Z}[z]/\mathbb{Z}} \\
&= \big(\mathrm{THH}(R) \otimes_{\mathrm{THH}(\mathbb{S}[z])} \mathbb{S}[z]\big) \otimes_{\mathbb{Z}[z]} \Omega^j_{\mathbb{Z}[z]/\mathbb{Z}} \\
&\overset{3.4}{=} \mathrm{THH}(R/\mathbb{S}[z]) \otimes_{\mathbb{Z}[z]} \Omega^j_{\mathbb{Z}[z]/\mathbb{Z}}
\end{aligned}$$

Therefore we get a homological (Serre graded) spectral sequence

$$E^2_{i,j} = \mathrm{THH}_i(R/\mathbb{S}[z]) \otimes_{\mathbb{Z}[z]} \Omega^j_{\mathbb{Z}[z]/\mathbb{Z}} \Rightarrow \mathrm{THH}_{i+j}(R).$$

The spectral sequence is concentrated in degrees $(i,j) \in [0,\infty) \times [0,1]$, in particular it is first quadrant and converges. Furthermore the spectral sequence is multiplicative since the filtration is multiplicative[10].
By p-completing everywhere and using that p-completion is exact we also get the following spectral sequence with the same formal properties:

$$E^2_{i,j} = \mathrm{THH}_i(R/\mathbb{S}[z];\mathbb{Z}_p) \otimes_{\mathbb{Z}_p[z]} \Omega^j_{\mathbb{Z}_p[z]/\mathbb{Z}_p} \Rightarrow \mathrm{THH}_{i+j}(R;\mathbb{Z}_p)$$

In our case of interest R is again a CDVR with perfect residue field of positive characteristic. We completely know the E^2-page which by Theorem 3.7 takes the form $E^2 = R[x] \otimes_Z \Lambda_{\mathbb{Z}}(dz)$ for elements $|x| = (0,2)$ and $|dz| = (1,0)$ as pictured below.

$$\begin{tikzcd}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots \\
R\{dz\} & 0 & R\{xdz\} & 0 & R\{x^2dz\} & \cdots \\
R & 0 & R\{x\} \arrow[ull] & 0 & R\{x^2\} \arrow[ull] & \cdots
\end{tikzcd}$$

Since the E^2-page is already concentrated in the first two rows, there is only room for the d^2-differential (recall in Serre grading d^r goes r to the left and $r-1$ up.) Furthermore because multiplicatively everything with potential for non-vanishing differential is generated by x and the spectral sequence is multiplicative, it suffices to identify

$$d^2 : R\{x\} \to R\{dz\}.$$

If R has equal characteristic, this differential actually has to vanish. The reason for this is that by the construction of the spectral sequence, we know that the edge homomorphism is given by $\mathrm{THH}_*(R;\mathbb{Z}_p) \to \mathrm{THH}_*(R/\mathbb{S}[z];\mathbb{Z}_p)$ and everything that lies in the image of this map, has to be a permanent cycle. Because R has equal characteristic, we can precompose the map by $\mathrm{THH}(k) \to \mathrm{THH}(R;\mathbb{Z}_p)$ and choose $x \in \mathrm{THH}_2(R/\mathbb{S}[z];\mathbb{Z}_p)$ to be the image of the Bökstedt element under the composition of the two maps. Thus x has to be a permanent cycle and d^2 vanishes.

[10] i.e. it is an algebra in the category of filtered spectra $\mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathrm{Sp})$ equipped with Day convolution

For mixed characteristic R we will identify it in the following Lemma. Recall from footnote 5 that every mixed characteristic CDVR R with perfect residue field k has the form $R = W(k)[\pi]$, where π satisfies $\Phi(\pi) = 0$ for an Eisenstein polynomial $\Phi \in W(k)[z]$.[11]

Lemma 3.10. *For a mixed characteristic CDVR R, there is a generator $x \in \mathrm{THH}_2(R;\mathbb{Z}_p)$ with differential $d^2(x) = \Phi'(\pi)dz$.*

Proof. We want to work relative to $\mathbb{S}_{W(k)}$ because this will enable us to make use of the description $R = W(k)[z]/\Phi(z)$. To do this we again employ the base change for relative THH:

$$\mathrm{THH}(R/\mathbb{S}_{W(k)};\mathbb{Z}_p) \overset{3.4}{=} \mathrm{THH}(R;\mathbb{Z}_p) \otimes_{\mathrm{THH}(\mathbb{S}_{W(k)};\mathbb{Z}_p)} \mathbb{S}_{W(k)}$$

This does not look helpful so far but we can simplify it further. In the proof of Theorem 3.6 we already saw that $\mathbb{F}_p \otimes_{\mathbb{S}} \mathrm{THH}(\mathbb{S}_{W(k)};\mathbb{Z}_p) = k$. We also have a map $\mathbb{S}_{W(k)} \to \mathrm{THH}(\mathbb{S}_{W(k)};\mathbb{Z}_p)$. Both spectra are connective and p-complete, so we can check equivalence on $\mathbb{F}_p$-homology. There it is an isomorphism by the above computation. Thus we get

$$\mathrm{THH}(R/\mathbb{S}_{W(k)};\mathbb{Z}_p) = \mathrm{THH}(R;\mathbb{Z}_p).$$

As R is p-complete and finite type over $\mathbb{S}_{W(k)}$, we can even drop the p-completion, i.e. $\mathrm{THH}(R/\mathbb{S}_{W(k)}) = \mathrm{THH}(R/\mathbb{S}_{W(k)};\mathbb{Z}_p)$. This holds because we can write $\mathrm{THH}(R/\mathbb{S}_{W(k)})$ as a colimit along the skeletal filtration of its cyclic Bar construction. These maps get increasingly connective and all terms are by induction p-complete, which allows us to apply the observation from footnote 7 and conclude that $\mathrm{THH}(R/\mathbb{S}_{W(k)})$ is already p-complete.
We can now identify the differential using that Hochschild homology and topological Hochschild homology agree in low degrees:

$$\mathrm{THH}_1(R;\mathbb{Z}_p) = \mathrm{THH}_1(R/\mathbb{S}_{W(k)}) = \mathrm{HH}_1(R/W(k)).$$

Furthermore Hochschild homology and Kähler differentials always agree in degree 1 and Kähler differentials are easy to compute, since $R = W(k)[z]/\Phi(z)$.

$$\mathrm{HH}_1(R/W(k)) = \Omega^1_{R/W(k)} = R\{dz\}/\Phi'(\pi)dz.$$

By the spectral sequence we also know

$$\mathrm{THH}_1(R;\mathbb{Z}_p) = R\{dz\}/\operatorname{im}(d^2 : E^2_{2,0} \to E^2_{0,1}),$$

which finally implies that $\operatorname{im}(d^2 : E^2_{2,0} \to E^2_{0,1})$ is equal to the submodule of $R\{dz\}$ generated by $\Phi'(\pi)dz$. This only pinpoints the generator of this submodule up to a unit, which we can choose. $\square$

[11] In this case π is a uniformizer of R. For example $\mathbb{Z}_p[\sqrt[n]{p}] = W(\mathbb{F}_p)[\pi]$, $\Phi(z) = z^n - p$.

This result will easily allow us to compute $\mathrm{THH}_*(R;\mathbb{Z}_p)$, which we will do momentarily. But let us first issue the following warning.

Warning 3.11. *The result of the Lemma implicitely depends on the choice of a uniformizer. While this is not a problem if we are only interested in* THH_* *of one particular CDVR, it is a problem if we are dealing with a map* $R \to S$ *of two of those. We are only able to determine the effect on* THH_* *of those maps, which preserve the chosen uniformizers. Only in this case is the map* $R \to S$ *actually a map of* $\mathbb{S}[z]$*-modules with respect to the module structures induced by the choices of the respective uniformizers. Consider for example the inclusion* $\mathbb{Z}_p \hookrightarrow \mathbb{Z}_p[\sqrt{p}]$*. Any uniformizer in* $\mathbb{Z}_p$ *has the form up for* $u \in \mathbb{Z}_p^\times$*, but of course none of them generate the maximal ideal of* $\mathbb{Z}_p[\sqrt{p}]$*. There is therefore no choice of uniformizers that is compatible with this map and our method is not able to calculate the induced map* $\mathrm{THH}(\mathbb{Z}_p;\mathbb{Z}_p) \to \mathrm{THH}(\mathbb{Z}_p[\sqrt{p}];\mathbb{Z}_p)$*. Exactly the same problem occurs for* $\mathbb{Z}_p \hookrightarrow \mathbb{Z}_p[\sqrt[n]{p}],\ n \geq 2$*.*
But even if we can choose the uniformizers such that one maps onto the other as for example in the case $\mathbb{Z}_p[\sqrt{p}] \to \mathbb{Z}_p[\sqrt{p}],\ \sqrt{p} \mapsto -\sqrt{p}$*, this is of not much use. For a meaningful calculation, we would need to choose the same uniformizer in domain and target. Compare this to the analogous situation for vector spaces. If we want to find a matrix representation of a vector space endomorphism, we should of course choose the same basis in domain and target. Otherwise every automorphism can be represented by the identity matrix when taking any basis in the domain and the image of this basis in the target.*

Let us now state the evaluation of the spectral sequence for mixed characteristic CDVRs. This result was first obtained in [LM00, Theorem 5.1] using an entirely different spectral sequence due to Brun.

Proposition 3.12. *Let* R *be a mixed characteristic complete discrete valuation ring with perfect residue field and uniformizer* π*. The topological Hochschild homology groups are then given by*

$$\mathrm{THH}_i(R;\mathbb{Z}_p) = \begin{cases} R & \textit{for } n = 0 \\ R/n\Phi'(\pi) & \textit{for } i = 2n-1, n \geq 1 \\ 0 & \textit{else} \end{cases}$$

In the case $R = \mathbb{Z}_p$, we can make this more explicit using the Chinese remainder theorem[12]: $\mathrm{THH}_{2n-1}(\mathbb{Z}_p;\mathbb{Z}_p) = \mathbb{Z}_p/n\mathbb{Z}_p = \mathbb{Z}/p^{\nu_p(n)}$, where ν_p denotes the p-adic valuation. Emplyoing a fracture square, this can also be used to recover Bökstedt's calculation of $\mathrm{THH}_*(\mathbb{Z})$.

Proof of Proposition 3.12. By Lemma 3.10 and the multiplicativity of the spectral sequence we know that $d^2(x^n) = nd^2(x)x^{n-1} = n\Phi'(\pi)x^{n-1}dz$. All higher

[12]together with knowledge of the quotients: $\mathbb{Z}_p/n = 0$ for $n \neq p^k$ and $\mathbb{Z}_p/p^k = \mathbb{Z}/p^k$.

differentials vanish for degree reasons, so the E^3-page is already the E^∞-page, which takes the form

$$
\begin{array}{c|cccccc}
 & 0 & 0 & 0 & 0 & 0 & \dots \\
 & R/\Phi'(\pi) & 0 & R/2\Phi'(\pi) & 0 & R/3\Phi'(\pi) & \dots \\
 & R & 0 & 0 & 0 & 0 & \dots \\
\hline
\end{array}
$$

There is no room for extensions and we can directly read of the result. $\square$

For the case $R' = R/\pi^n$, we can initially proceed similarly: Construction 3.9 gives us a spectral sequence which by Proposition 3.8 takes the form

$$E^2 = R'[x]\langle y\rangle \otimes_Z \Lambda_{\mathbb{Z}}(dz) \Rightarrow \mathrm{THH}_*(R').$$

We can again identify the d^2 differential, which is the only one that can occur

$$d^2(x) = \Phi'(\pi)dz,\ d^2(y) = k\pi^{k-1}dz.$$

Thus we get an explicit differential graded algebra, whose homology computes $\mathrm{THH}_*(R')$. But depending on n there is no longer a closed form expression for its homology. For explicit examples see [KN19, Chapter 6].

4. A Functorial identification of THH of CDVRs

In this chapter, we will use the Tor-spectral sequence to understand the effect of THH on maps between complete discrete valuation rings. See [Lur17, Proposition 7.2.1.19] for a statement of the Tor spectral sequence for module spectra over a ring spectrum or [Stacks, Tag 061Y] for the more classical result in the case of chain complexes.
Let R be a mixed characteristic complete discrete valuation ring with perfect residue field. The following are the main steps in our proof.

1. Write R as a pushout $R \cong \mathbb{Z}_p[z] \otimes_{\mathbb{Z}_p[z]} \mathbb{Z}_p$ via action through the Eisenstein polynomial (see footnote 5)

2. THH preserves this pushout because the involved rings are commutative, and for $\mathbb{E}_\infty$-rings, THH is given by the colimit over S^1 in CAlg which means that it commutes with arbitrary colimits.

3. Use the fact that $\mathrm{THH}(\mathbb{Z}_p[z]) \simeq \mathrm{THH}(\mathbb{Z}_p) \otimes_{\mathbb{Z}} \mathrm{HH}(\mathbb{Z}[z]/\mathbb{Z})$ but not naturally. But have natural isomorphism on homotopy groups $\mathrm{THH}*(\mathbb{Z}_p[z]) \simeq \mathrm{THH}_*(\mathbb{Z}_p) \otimes_{\mathbb{Z}} \Omega_{\mathbb{Z}[z]/\mathbb{Z})}$.

4. Now employ Tor-spectral sequence:
$E^2_{i,j} = \mathrm{Tor}^i_{\pi_*(R)}\left(\pi_*(M), \pi_*(M)\right)_{(j)} \Rightarrow \pi_{i+j}\left(M \otimes_R N\right)$, i.e. in our case:
$\mathrm{Tor}^i_{\mathrm{THH}_*(\mathbb{Z}_p[z])}\left(\mathrm{THH}_*(\mathbb{Z}_p[z]), \mathrm{THH}_*(\mathbb{Z}_p)\right)_{(j)} \Rightarrow \mathrm{THH}_{i+j}(R)$ (or rather the p-completed version)

5. Analyze the differentials and extension problems in the spectral sequence.

Bibliography

[BR20] David Barnes and Constanze Roitzheim. *Foundations of Stable Homotopy Theory*. Vol. 185. Cambridge University Press, 2020.

[BLR08] Arthur Bartels, Wolfgang Lück, and Holger Reich. "On the Farrell–Jones conjecture and its applications". In: *Journal of Topology* 1.1 (2008), pp. 57–86.

[BGT13] Andrew J Blumberg, David Gepner, and Gonçalo Tabuada. "A universal characterization of higher algebraic K-theory". In: *Geometry & Topology* 17.2 (2013), pp. 733–838.

[BHM93] Marcel Bökstedt, Wu Chung Hsiang, and Ib Madsen. "The cyclotomic trace and algebraic K-theory of spaces". In: *Inventiones mathematicae* 111.1 (1993), pp. 465–539.

[Cam13] Omar Antolın Camarena. "A whirlwind tour of the world of $(\infty, 1)$-categories". In: *Mexican mathematicians abroad: recent contributions* (2013), pp. 15–61.

[Cis19] Denis-Charles Cisinski. *Higher categories and homotopical algebra*. Vol. 180. Cambridge University Press, 2019.

[CMM21] Dustin Clausen, Akhil Mathew, and Matthew Morrow. "K-theory and topological cyclic homology of henselian pairs". In: *Journal of the American Mathematical Society* 34.2 (2021), pp. 411–473.

[DGM12] Bjørn Ian Dundas, Thomas G Goodwillie, and Randy McCarthy. *The local structure of algebraic K-theory*. Vol. 18. Springer Science & Business Media, 2012.

[FH78] F. Thomas Farrell and Wu-Chung Hsiang. "On the rational homotopy groups of the diffeomorphism groups of discs, spheres and aspherical manifolds". In: *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part*. Vol. 1. 1978, pp. 325–337.

[Gep20] David Gepner. "An introduction to higher categorical algebra". In: *Handbook of Homotopy Theory* (2020), p. 487.

[Gro20] Moritz Groth. "A short course on ∞-categories". In: *Handbook of Homotopy Theory*. Chapman and Hall/CRC, 2020, pp. 549–617.

[HM97] Lars Hesselholt and Ib Madsen. "On the K-theory of finite algebras over Witt vectors of perfect fields". In: *Topology* 36.1 (1997), pp. 29–101. ISSN: 0040-9383. DOI: `10.1016/0040-9383(96)00003-1`. URL: `https://doi.org/10.1016/0040-9383(96)00003-1`.

[KN18] Achim Krause and Thomas Nikolaus. *Lectures on topological Hochschild homology and cyclotomic spectra*. 2018. eprint: `https://www.uni-muenster.de/IVV5WS/WebHop/user/nikolaus/papers.html`.

[KN19] Achim Krause and Thomas Nikolaus. *Bökstedt periodicity and quotients of DVRs*. 2019. arXiv: `1907.03477 [math.AT]`.

[LM00] Ayelet Lindenstrauss and Ib Madsen. "Topological Hochschild homology of number rings". In: *Trans. Am. Math. Soc.* 352.5 (2000), pp. 2179–2204. ISSN: 0002-9947; 1088-6850/e.

[Lod13] Jean-Louis Loday. *Cyclic homology*. Vol. 301. Springer Science & Business Media, 2013.

[Lur09] Jacob Lurie. *Higher topos theory*. Vol. 170. Princeton, NJ: Princeton University Press, 2009, pp. xv + 925. ISBN: 978-0-691-14049-0.

[Lur17] Jacob Lurie. "Higher algebra." In: *preprint available from the author's website* (2017). URL: `https://www.math.ias.edu/~lurie/papers/HA.pdf`.

[Lur18a] Jacob Lurie. "Elliptic cohomology II: orientations". In: *preprint available from the author's website* (2018). URL: `https://www.math.ias.edu/~lurie/papers/Elliptic-II.pdf`.

[Lur18b] Jacob Lurie. *Kerodon*. `https://kerodon.net`. 2018.

[Lur18c] Jacob Lurie. "Spectral Algebraic Geometry." In: *preprint available from the author's website* (2018). URL: `https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf`.

[NS18] Thomas Nikolaus and Peter Scholze. *On topological cyclic homology*. 2018. arXiv: `1707.01799 [math.AT]`.

[Ric09] Birgit Richter. "Divided power structures and chain complexes". In: *Alpine perspectives on algebraic topology. Third Arolla conference on algebraic topology, Arolla, Switzerland, August 18–24, 2008*. Providence, RI: American Mathematical Society (AMS), 2009, pp. 237–254. ISBN: 978-0-8218-4839-5.

[Rie18] Emily Riehl. "Homotopy coherent structures". In: *arXiv preprint 1801.07404* (2018).

[RV21] Emily Riehl and Dominic Verity. "Elements of ∞-category theory". In: *Preprint available at www.math.jhu.edu/~eriehl/elements.pdf* (2021).

[Ser95] Jean Pierre Serre. *Local Fields*. Graduate Texts in Mathematics. Springer New York, 1995. ISBN: 9780387904245.

[Stacks] The Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu`. 2018.

[WJR13] Friedhelm Waldhausen, Bjørn Jahren, and John Rognes. *Spaces of PL manifolds and categories of simple maps*. 186. Princeton University Press, 2013.

[Wei94] Charles A Weibel. *An introduction to homological algebra*. 38. Cambridge university press, 1994.