Topological Hochschild homology of discrete valuation rings

Master's Thesis

submitted by

Mirko Stappert

in partial fulfilment of the

requirements for the degree of

Bachelor of Science in the Department

of Mathematics at Saarland University

Supervisor:
Prof. Dr. Thomas Nikolaus

Examiner 1:
Prof. Dr. Gabriela Weitze-Schmithüsen

Examiner 2:
Prof. Dr. Thomas Nikolaus

Statement in Lieu of an Oath

I hereby confirm that I have written this thesis on my own and that I have not used any other media or materials than the ones referred to in this thesis.

Ich versichere hiermit, dass ich die vorliegende Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Saarbrücken, August 25, 2021

Mirko Stappert

Contents

1. Introduction

The algebraic K-theory groups are fundamental invariants of rings. They encapsulate deep knowledge about the ring via its category of modules. For rings of integers in number fields, the K-theory groups contain information about the class group and group of units of the ring, the Brauer group of the field and values of its Dedekind zeta function. Algebraic K-theory can also be applied to schemes. In fact, it was first defined in this context by Grothendieck during the work on his Riemann-Roch theorem, where he discovered a remarkable relationship to algebraic cycles. Later, Bloch pushed this relationship further to his newly defined higher Chow groups. There are also spectacular applications towards geometric topology. Starting with the early work of Wall and the s-cobordism theorem by Barden-Mazur-Stalling it culminated in the stable parametrized h-cobordism theorem by Waldhausen, Jahren and Rognes. It says that the stable h-cobordism space of say a smooth manifold M can be obtained from the K-theory space of $\mathbb{S}[\Omega M]$, the spherical group ring of the loop space of M. This is remarkable because this allows us - as does the classical h-cobordism theorem - to obtain geometric results for M purely using homotopy theory. One of its implications is that we can calculate many homotopy groups of the diffeomorphism group of M via the K-theory groups of $\mathbb{S}[\Omega M]$. For example, the homotopy groups of the diffeomorphism group of a disc $\pi_*(\mathrm{Diff}(D^n))$ can (in a range depending on n) be obtained by computing the K-theory groups of the sphere spectrum. This boils down to the understanding of $K_*(\mathbb{Z})$ about which we know a lot using motivic homotopy theory and several major results in number theory that allow us to compute certain étale cohomology groups.

These powerful results and theorems now present us with the challenge to actually compute K-theory groups. This problem is in general very hard! There are several successful approaches towards these computations, which nicely complement each other. Among them are motivic methods, controlled algebra and trace methods. Let us only describe the last one, as it is the one of relevance for this thesis. The idea of trace methods is to consider other - ideally more computable - spectrum valued invariants of rings and compare K-theory to them via so-called trace maps. The hope is that while these invariants are more computable, the difference to K-theory is not too big. In other words, the trace map is close to an isomorphism. The most successful of these comparisons is with the trace map $K \xrightarrow{\mathrm{tr}} \mathrm{TC}$ to the so-called *topological cyclic homology* introduced by Bökstedt-Hsiang-Madsen. By work of Dundas-Goodwillie-McCarthy and Hesselholt-Madsen, we now know that the trace map is a remarkably close approximation that has been used to calculate K-theory in many cases. Due to the recent work of Nikolaus-Scholze, topological cyclic homology itself can essentially be computed via two spectral sequences out of yet another invariant: Topological Hochschild homology THH, which is the main player of this thesis.

Let us now describe the problem we want to solve. We will specifically study (topological) Hochschild homology of complete discrete valuation rings of mixed characteristic with perfect residue field of characteristic p. To be concrete, all of these rings are certain extensions of the p-adic integers, like $\mathbb{Z}_p[\sqrt[n]{p}]$ or $\mathbb{Z}_p[\zeta_{p^n}]$. They arise for example as completions of rings of integers in number fields and are thus of fundamental importance in algebraic number theory. The topological Hochschild homology of them has already been computed in the work of Lindenstrauss-Madsen. Recently Krause and Nikolaus gave a more conceptual proof of the same result, which we present in Chapter 3 of this thesis. Unfortunately, both approaches do not give a *functorial* computation. They only identify THH of these rings but do not answer the question what it does to morphisms. This is unsatisfactory for at least two reasons: We might want to understand the action of the Galois group on K-theory. Because the trace map $K \xrightarrow{\text{tr}} \mathrm{TC}$ is a natural transformation and TC is functorially obtained from THH, the first step to understand the functoriality of K-theory is to understand it for THH. Secondly, functorial descriptions are often needed for further computations. For example, in our computation, we crucially need that we not only know the topological Hochschild homology groups of $\mathbb{Z}_p[z]$ but *also* how endomorphisms of $\mathbb{Z}_p[z]$ act on the topological Hochschild homology groups of it.

We do not succeed to describe the functoriality of THH for this class of rings in complete generality. Instead we obtain three partial results, which we explain in the three section of Chapter 4

In Section 4.1, we give a fully functorial description of (p-completed) Hochschild homology for CDVRs. We achieve this using the Hochschild-Kostant-Rosenberg filtration and by carefully keeping track of the functoriality in the setting of a relative computation. Since the homotopy groups of $\mathrm{THH}(R)_p^\wedge$ and $\mathrm{HH}(R)_p^\wedge$ are naturally isomorphic in degrees less than $2p-1$, we thus also give a functorial description of the low homotopy groups of THH.

In the second section, we describe an approach to understand the functoriality using the Tor spectral sequence. For this, we use that by a general classification result all mixed characteristic discrete valuation rings are obtained from $\mathbb{Z}_p$ by adjoining a root of an Eisenstein polynomial, i.e. $R = \mathbb{Z}_p[z]/E(z)$ for $E(z) \in \mathbb{Z}_p[z]$ Eisenstein[1]. This allows us to write R as $R = \mathbb{Z}_p[z] \otimes_{\mathbb{Z}_p[z]} \mathbb{Z}_p$, which gives us $\mathrm{THH}(R) = \mathrm{THH}(\mathbb{Z}_p[z]) \otimes_{\mathrm{THH}(\mathbb{Z}_p[z])} \mathrm{THH}(\mathbb{Z}_p)$ We can then use the Tor spectral sequence, associated to this tensor product, to try to understand $\mathrm{THH}_*(R)$.

Unfortunately, this approach has two problems. First of all, we are only able

[1]Actually we need to take the Witt vectors of the (perfect) residue field of R, but for sake of exposition we stick to the easiest case that the residue field is $\mathbb{F}_p$ and the Witt vectors are $W(\mathbb{F}_p) = \mathbb{Z}_p$

to understand the effect of maps between CDVRs, which we can 'lift' to the level of presentations $\mathbb{Z}_p[r] \xrightarrow{r \mapsto E(z)} \mathbb{Z}_p[z] \to R$. Secondly, while we can compute the E^2-page and understand several things about the differentials and extension problems, there are still many indeterminacies left which make it hard to give a concrete result. But we can for example say something in the tamely ramified setting .

In the last section we give another partial result which works for all complete discrete valuation rings with perfect residue field, but again only for very special maps. Specifically we can only deal with 'monomial' maps, i.e. those which send one uniformizer to a power of the other uniformizer.

2. Preliminaries

We are going to use the language of higher category theory and higher algebra, mainly developed by Jacob Lurie in [Lur09] and [Lur17]. The online textbook project [Lur18b] by the same author contains a revised version of roughly the same content. For slightly different perspectives on higher category theory, we also recommend [RV21] and [Cis19]. Shorter summaries of most of the material we need can be found in [Gro20], [Cam13] and [Gep20].

2.1 Higher Category theory

Higher category theory is an extension of ordinary category theory that is tailored to the needs of homotopy theory. In particular it follows the paradigm that no equality should be taken literally and we should rather provide a specified homotopy, implemented as *higher morphisms*. The ∞ in ∞-category is supposed to signal that we not only have objects and morphisms but also 2-morphisms between morphisms, 3-morphisms between 2-morphisms and so on. All diagrams should not commute strictly but only up to homotopy/2-morphism. A consequence of this is that composition in ∞-categories is not strictly commutative and the notion of isomorphism of objects gets replaced by equivalence, which means that we have two maps that are inverse up to homotopy. Often times when we have more then one of these homotopies they should in some sense be compatible which is recorded via higher and higher homotopies. This train of thought goes under the name of **homotopy coherence**. We recommend [Rie18] and [Lur09, Section 1.2.6] to the unaquainted reader who wants to become familiar with it. The benefit of this approach is that all concepts defined in this framework are naturally homotopy invariant. A further advantage is that we can now consider spaces as ∞-categories themselves. Informally we do this by taking points as objects, paths as 1-morphisms, homotopies of paths as 2-morphism, homotopies between homotopies as 3-morphisms and so on. Note that this would not be possible in the realm of 1-categories, because the composition (concatenation) of paths is not strictly associative but only up to homotopy. The 1-categorical shadow of this is the fundamental groupoid. The reader can informally think about ∞-categories as some kind of a "least common multiple" of ordinary categories and topological spaces (or rather homotopy types).
The main thing to take away from this section is that most if not all concepts/constructions from ordinary category theory also exist in the world of ∞-categories. Precisely we will need the following higher categorical concepts:

- **∞-categories**, **functors** between them and **natural transformations**, all of which can be found in [Lur09, Chapter 1]. Every ordinary category can be considered as an infinity category via the nerve construction. There is also a way to go back[1] and to associate to every ∞-category $\mathcal{C}$ a 1-category, called its homotopy category $\mathrm{Ho}(\mathcal{C})$. The most important example of an ∞-category that does not arise this way is the ∞-category of spaces $\mathcal{S}$ ([Lur09, Section 1.2.16]). Its role in higher category theory is analogous to the role that the category of sets plays in ordinary category theory. An example of this paradigm is that in an ∞-category we now no longer only have a set of maps between two objects but a whole mapping space.

- Most of the constructions and concepts from ordinary category theory have analogues in higher category theory. This includes **limits and colimits** ([Lur09, Section 1.2.3, Chapter 4]), **adjunctions**([Lur18b, Tag 02EJ]), opposite categories ([Lur09, Section 1.2.1]), slice categories ([Lur09, Section 1.2.9]), ... The only slice category that we will need is the ∞-category of pointed spaces which is the slice category under the point $\mathcal{S}_* = \mathcal{S}_{*/}$. There is a free-forgetful adjunction $(-)_+ : \mathcal{S} \leftrightarrows \mathcal{S}_*$ whose left adjoint adds a disjoint basepoint. As usual left adjoints preserve colimits and right adjoints preserve limits ([Lur09, Proposition 5.2.3.5]). $\mathcal{S}$ and $\mathcal{S}_*$ are bicomplete and (co)limits agree with the classical notions of homotopy (co)limits. There are also notions of **filtered** and **sifted colimits** and the usual statements hold verbatim, i.e. filtered colimits commute with finite limits and sifted colimits commute with finite products.

- We will also crucially need **symmetric monoidal structures** on ∞-categories ([Lur17, Definition 2.0.0.7]) as well as **lax and strong symmetric monoidal functors**. The nerve of every ordinary symmetric monoidal category is a symmetric monoidal ∞-category. The ∞-category of spaces has all products, which equip $\mathcal{S}$ with a symmetric monoidal structure by [Lur17, Section 2.4.1]. The ∞-category of pointed spaces is also symmetric monoidal with the well known smash product $\wedge$ and the above functors in the adjunction between $\mathcal{S}$ and $\mathcal{S}_*$ have lax symmetric monoidal structures ([Lur17, Theorem 2.2.2.4]). Similarly if an ∞-category has all finite coproducts it also admits a symmetric monoidal structure by [Lur17, Section 2.4.3].

- Most ∞-categories[2] that we will consider in the next section and in this thesis in general, have an additional property. They are **stable ∞-categories**, a notion introduced in [Lur17, Section 1.1]. The definition is rather simple:

[1] This provides a (reflective) adjunction between the (2,1)-category of all categories and the ∞-category of all ∞-categories.

[2] Besides $\mathcal{S}$ and $\mathcal{S}_*$

An ∞-category is stable, if it has all finite limits and colimits, the thus provided initial and terminal object are equivalent[3] and a square is a pushout square if and only if it is a pullback square. Stable ∞-categories provide a natural place to do homological - or rather higher algebra. In particular they have a notion of exact sequences also called fiber sequences. We can add maps and thus also take kernels/cokernels of maps because we have a zero object. They are called (homotopy) **fiber/cofibers** and naturally fit into exact sequences $\mathrm{fib}(f) \to X \xrightarrow{f} Y$ and $X \xrightarrow{f} Y \to \mathrm{cofib}(f)$. Furthermore, the homotopy category of a stable ∞-category has the structure of a triangulated category (see e.g. [Lur17, Theorem 1.1.2.14]). Stable ∞-category are in some aspects preferable to triangulated categories. First of all a triangulation is extra structure that has to be provided, while stability is a property. Secondly cones (or rather cofibers) are functorial in stable ∞-categories. They also "glue" together more naturally which is relevant for descent questions in algebraic geometry. For example the functor that sends a scheme to its derived ∞-category of quasi-coherent sheaves is a sheaf of ∞-categories, which is false on the level of triangulated categories (see [GR19, Chapter 3]).

2.2 Spectra and higher algebra

The most important ∞-category for us will be the **∞-category of spectra**: Sp. Recall that it is defined as the following limit[4] in the (very large) ∞-category of large ∞-categories Cat_∞

$$\mathrm{Sp} := \lim \left(\dots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \right).$$

The existence of such a functor $N(\mathbb{N})^{\mathrm{op}} \to \mathrm{Cat}_\infty$ is not obvious. A priori we have to supply further coherence data. That we can avoid doing this is the content of [Cis19, Corollary 7.3.17]. A further consequence of this corollary is that we can actually describe the objects of the resulting limit: An object of Sp - a **spectrum** - is a sequence of pointed spaces $\{X_n,\ n = 0, 1, 2, \dots\}$ together with chosen equivalences $X_n \xrightarrow{\sim} \Omega X_{n+1}$ for all $n = 0, 1, 2, \dots$. To differentiate them to other classically defined notions of spectra, they are also called Ω-spectra. Let us give the two main examples of interest. For an abelian group A, we can consider the sequence of Eilenberg-Maclane spaces $(K(A, 0),\ K(A, 1),\ K(A, 2), \dots)$. With the usual equivalences $K(A, n) \xrightarrow{\sim} \Omega K(A, n + 1)$ this defines a spectrum called the **Eilenberg-Maclane spectrum** of A. This construction assembles into a fully faithful functor $H : \mathrm{Ab} \to \mathrm{Sp}$ ([Lur17, Example 1.3.3.5]). Therefore, we will

[3] An object is called initial if mapping spaces out of it are contractible and terminal if mapping spaces into it are contractible.

[4] That all limits exist in Cat_∞ is a result in [Lur09, Section 3.3.3]

often abuse notation and denote the Eilenberg-Maclane spectrum of R simply by R itself unless we want to stress that we are talking about a discrete ring, in which case we will write HR. Spaces also give rise to spectra under the **suspension spectrum** functor $\Sigma^\infty : \mathcal{S}_* \to \mathrm{Sp}$. This allows us to define the important **sphere spectrum** $\mathbb{S} \coloneqq \Sigma^\infty(S^0)$. The suspension spectrum functor has a right adjoint $\Omega^\infty : \mathrm{Sp} \to \mathcal{S}_*$ given by sending a spectrum to its zeroth space. We can also compose this adjunction with the $(\mathcal{S}, \mathcal{S}_*)$-adunction to obtain an adjunction with unbased spaces $\Sigma^\infty_+ : \mathcal{S} \leftrightarrows \mathrm{Sp} : \Omega^\infty$. Similarly to spaces, spectra also have **homotopy groups**, which are defined via the ordinary homotopy groups of their spaces but now we can also define negative homotopy groups. For $X \in \mathrm{Sp}$ and $i \in \mathbb{Z}$ define $\pi_i(X) = \pi_{i+k}(X_k)$ as long as $i + k \geq 0$. This makes sense because $X_k \simeq \Omega X_{k+1}$. They are all abelian because we can always write them as higher homotopy groups of some space. The homotopy groups of a suspension spectrum are the stable homotopy groups of the space. In particular $\pi_*(\mathbb{S})$ are the stable homotopy groups of spheres. A spectrum is called **connective**, if all its negative homotopy groups vanish. It is called **bounded below**, if they vanish below some degree (which is not necessarily 0). Suspension spectra and Eilenberg-Maclane spectra are connective.
Important structural properties of the ∞-category Sp are that it is a stable ∞-category with all limits and colimits. Furthermore it carries a closed symmetric monoidal structure with unit $\mathbb{S}$, called the tensor product of spectra $\otimes_{\mathbb{S}}$ ([Lur17, Corollary 4.8.2.19]). Being closed means exactly the same, as in ordinary category theory, i.e. for every object $X \in \mathrm{Sp}$, the functor $X \otimes_{\mathbb{S}} - : \mathrm{Sp} \to \mathrm{Sp}$ has a right adjoint. Closedness in particular implies that tensoring with any spectrum preserves all colimits, as it is a left adjoint functor.
With these properties at hand we are ready to do higher algebra. The monoidal structure lets us talk about monoids in Sp called **ring spectra**[5]. Here we have to point out two important differences between ordinary and higher algebra. First of all *commutativity and associativity are no longer properties but extra structure*. In the usual commutativity/associativity diagrams, we do not require that they strictly commute but only that they commute up to homotopy. Furthermore there is now a whole *hierarchy of commutativity*, starting with only associativity and no commutativity at all called $\mathbb{E}_1$-ring spectra[6], followed by $\mathbb{E}_2$, $\mathbb{E}_3$ and $\mathbb{E}_n$-ring spectra for every $n \in \mathbb{N}$ up to full homotopical commutativity $\mathbb{E}_\infty$. The proper way to precisely treat all this is via the theory of operads, pioneered by Boardman-Vogt ([BV06],[BV68]) and May ([May06]). Lurie transported the theory into the world of ∞-categories, where he developed an extensive theory in [Lur17, Chapter 2, 3, 4]. The definition of the $\mathbb{E}_n$-operads is given in [Lur17,

[5]These algebra objects are called ring spectra because spectra are already a homotopical incarnation of abelian groups, thus we only need the further multiplication. In particular, the homotopy groups of a ring spectrum already form a graded ring.

[6]$\mathbb{E}_1$ is also called $\mathbb{A}_\infty$, where the 'A' stands for associativity. 'E' then stands for 'everything', meaning associative and commutative.

Definition 5.1.0.2], but we will not need precise details.
What is essential to us are the following facts:

- For every $n \in \mathbb{N} \cup \{\infty\}$ and every symmetric monoidal ∞-category $\mathcal{C}$ there is a notion of an **$\mathbb{E}_n$-algebra** in $\mathcal{C}$. These assemble into an ∞-category $\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C})$, which is itself again symmetric monoidal. In the most important special cases $n = 1, n = \infty$ we will denote them by $\mathrm{CAlg}(\mathcal{C}) := \mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{C})$ and $\mathrm{Alg}(\mathcal{C}) := \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C})$. If $\mathcal{C} = \mathrm{Sp}$ we will drop the dependence on $\mathcal{C}$ and simply write $\mathrm{Alg} := \mathrm{Alg}(\mathrm{Sp})$ and $\mathrm{CAlg} := \mathrm{CAlg}(\mathrm{Sp})$. The objects will be called ($\mathbb{E}_1$-)ring spectra and commutative/$\mathbb{E}_\infty$-ring spectra respectively. Every $\mathbb{E}_n$-algebra is also an $\mathbb{E}_{n-1}$-algebra and we get forgetful functors $\mathrm{CAlg}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_{n-1}}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C}) \to \mathcal{C}$.

- Lax symmetric monoidal functors $\mathcal{C} \to \mathcal{D}$ induce functors $\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{D})$. In particular since $H : \mathrm{Ab} \to \mathrm{Sp}$ is lax symmetric monoidal, we get functors $\mathrm{Alg}(\mathrm{Ab}) \to \mathrm{Alg}(\mathrm{Sp})$ and $\mathrm{CAlg}(\mathrm{Ab}) \to \mathrm{CAlg}(\mathrm{Sp})$. The categories $\mathrm{Alg}(\mathrm{Ab})$ and $\mathrm{CAlg}(\mathrm{Ab})$ coincide[7] with the (nerves of the) ordinary categories of not-necessarily commutative and commutative rings respectively. Thus every ordinary ring R gives rise to a ring spectrum, which has an $\mathbb{E}_\infty$-structure if R is commutative.

- $\mathbb{E}_n$-algebras in the symmetric monoidal ∞-category of $\mathbb{E}_m$-algebras are $\mathbb{E}_{n+m}$-algebras, i.e. $\mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Alg}_{\mathbb{E}_m}(\mathcal{C})) = \mathrm{Alg}_{\mathbb{E}_{n+m}}(\mathcal{C})$ for any symmetric monoidal ∞-category $\mathcal{C}$. In particular, we can interpret an $\mathbb{E}_n$-structure as n commuting associative structures. This equivalence goes under the name of Dunn additivity, see [Lur17, Theorem 5.1.2.2].

- $\mathbb{E}_n$-algebras in the category of spaces equipped with the cartesian tensor product are called $\mathbb{E}_n$-spaces. The suspension spectrum functor is strong symmetric monoidal and hence suspension spectra of $\mathbb{E}_n$-spaces are $\mathbb{E}_n$-ring spectra. An n-fold space i.e. a space of the form $\Omega^n X$ for some space X is an $\mathbb{E}_n$-space. In fact a space is an n-fold loop space if and only if it is an $\mathbb{E}_n$-space and π_0 is a group under the monoid structure induced by the $\mathbb{E}_1$-structure (this is called the recognition theorem, see e.g. [Lur17, Section 5.2.6] for a statement and proof in our language).

[7]In general in a 1-category, this hierarchy of commutativity collapses in the sense that $\mathbb{E}_1$-algebras are ordinary algebras, while $\mathbb{E}_n$-algebras for $n \geq 2$ are already ordinary commutative algebras. In a general n-category, the collapse happens at step $n+1$, i.e. $\mathbb{E}_1, \mathbb{E}_2, \ldots \mathbb{E}_n$-algebra structures are all different, while from $\mathbb{E}_{n+1}$ on they are already the same data as an $\mathbb{E}_\infty$-structure. For example in the 2-category of all ordinary categories, we therefore have $\mathbb{E}_1, \mathbb{E}_2$ and $\mathbb{E}_\infty$-algebras, which are given by (ordinary) monoidal, braided monoidal and symmetric monoidal categories respectively. In a general ∞-category as opposed to n-categories for finite n, the notions all differ.

- Provided that $\mathcal{C}$ is bicomplete and the tensor product commutes with colimits in both variables separately[8] we have two further statements. The category $\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C})$ also has all limits and colimits by [Lur17, Section 3.2]. Limits and sifted colimits are computed underlying. Without any restictions finite coproducts in $\mathrm{CAlg}(\mathcal{C})$ are given by the tensor product of the underlying objects in $\mathcal{C}$. We also get another supply of algebras called **free algebras**, because the forgetful functor $\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}) \to \mathcal{C}$ has a left adjoint [Lur17, Corollary 3.1.3.5]. In particular, this provides us with several free spectra generated by spaces.

- Given an algebra $A \in \mathrm{Alg}(\mathcal{C})$, we also able to talk about its ∞-category of left and right **modules** $\mathrm{LMod}_A, \mathrm{RMod}_A$ [Lur17, Definition 4.2.1.13]. In the case that A is commutative they are equivalent ([Lur17, Section 4.3.2]) and we will simply denote it by Mod_A. Given two algebras $R, S \in \mathrm{Alg}(\mathcal{C})$ there is the notion of a R-S-bimodule. Giving a R-S-bimodule structure is equivalent to giving a left module structure over $R \otimes_{\mathcal{C}} S^{\mathrm{op}}$. The categories LMod_A and RMod_A have all limits and colimits, provided that $\mathcal{C}$ does ([Lur17, Section 4.2.3]). For $\mathbb{S} \in \mathrm{Sp}$ we have $\mathrm{Mod}_{\mathbb{S}} = \mathrm{Sp}$ and for an ordinary ring R, we get its derived ∞-category $\mathrm{LMod}_{HR} \simeq D(R)$ (see [Lur17, Proposition 7.1.1.15] and the Remark following the Proposition).

- Given a right M and a left module N over a ring spectrum R, we can tensor them together and obtain a spectrum $M \otimes_R N$. More generally for $R, S, T \in \mathrm{Alg}$, M an R-S-bimodule and N an S-T-bimodule, then $M \otimes_S N$ carries the structure of an R-T-bimodule ([Lur17, Section 4.4.2]). For a commutative ring spectrum R every element in its category of modules Mod_R has an R-R-bimodule structure. The tensor product of two of these thus carries an R-R-bimodule structure, in particular it determines a left R-module, i.e. an element of $\mathrm{LMod}_R = \mathrm{Mod}_R$. Through this Mod_R obtains a symmetric monoidal structure ([Lur17, Theorem 4.5.2.1]). Thus we can talk about $\mathbb{E}_n$-algebras in there, which we will call **$\mathbb{E}_n$-R-algebras**.

- For a map $R \to S$ of ring spectra, we have the forgetful functor $\mathrm{LMod}_S \to \mathrm{LMod}_R$. This functor has both adjoints, in particular it preserves all limits and colimits. The left adjoint is given by a relative tensor product $M \mapsto M \otimes_R S$. For the tensor product we use the left R-module structure on M and the right R-module structure on S. Because S is an S-S-bimodule, there remains a left S-module structure. If R and S are commutative, this left adjoint is symmetric monoidal [Lur17, Section 4.5.3]

- For a connective ring spectrum R we have **Postnikov towers** in Mod_R as well as truncations/covers $\tau_{\geq n}, \tau_{\leq n} : \mathrm{Mod}_R \to \mathrm{Mod}_R$ ([Lur17, Proposition 7.1.1.13]). In particular they exist for spectra.

[8] This is satisfied in all of our cases of interest.

- **Filtered objects** (i.e. objects of $\mathrm{Fun}((N\mathbb{Z})^{\mathrm{op}}, \mathcal{C})$ in stable ∞-categories like Mod_R for R connective, provide **spectral sequences** with values in the abelian category of $\pi_0(R)$-modules ([Lur17, Section 1.2.2]). We will use these spectral sequences repeatedly. For a classical reference on spectral sequence, the reader can confer [Wei94, Chapter 5].

- Homotopy groups commute with arbitrary products and filtered colimits. There is no general formula for the homotopy groups of tensor products[9] $\pi_*(X \otimes_{\mathbb{S}} Y)$. But in the case that both X and Y are bounded below we can at least say something. Let $\pi_n(X)$ and $\pi_m(X)$ be the lowest non-zero homotopy groups of X, Y then $X \otimes_S Y$ is also bounded below with lowest homotopy group $\pi_{n+m}(X \otimes_{\mathbb{S}} Y) = \pi_n(X) \otimes_{\mathbb{Z}} \pi_m(Y)$. For connective (left and right) module spectra M, N over a connective ring spectrum R we similarly get that $M \otimes_R N$ is connective with $\pi_0(M \otimes_R N) = \pi_0(M) \otimes_{\pi_0(R)} \pi_0(N)$ ([Lur17, Corollary 7.2.1.23]). Another fact about homotopy groups, that will often help us compute them, is that fiber/cofiber sequences of spectra give us long exact sequences on the level of homotopy groups.

The last construction from higher algebra that we need is a generalization of p-completion of abelian groups. A textbook resource on p-completion of spectra is [BR20, Section 8.4.1]. Let us directly start with the definition, which is notationally identical to ordinary p-completion of abelian groups.

Definition 2.1. *Let X be a spectrum. We define its p-**completion** $X_p^\wedge$ as the following limit*

$$X_p^\wedge := \lim(\ldots \to X/p^3 \to X/p^2 \to X/p).$$

This provides an endofunctor of Sp*. X has a canonical map to $X_p^\wedge$ induced by the projections $X \to X/p^n$. We call X p-**complete** if this map is an equivalence. A map of spectra is called a p-**adic equivalence** if the induced map on p-completions is an equivalence.*

Let us be precise about the terms in the definition. For a spectrum X we always have the identity map $X \xrightarrow{1} X$. Since we can add maps between spectra, we can consider the n-fold addition of this map to itself to obtain $X \xrightarrow{n} X$. We denote the cofiber of this map as $X/n := \mathrm{cofib}(X \xrightarrow{n} X)$. The natural maps $X/p^n \to X/p^{n-1}$ are induced from taking horizontal cofibers in the relevant commutative diagram. An immediate observation is that p-completion commutes with arbitrary limits and finite colimits. From this we can also see that $X/p \xrightarrow{\sim} (X_p^\wedge/p)$ is an equivalence. The crucial Lemma about p-complete spectra and the main reason why we have introduced the concept, is the following.

Lemma 2.2. *Let $f : X \to Y$ be a map of spectra. Then f is a p-adic equivalence if and only if $f/p : X/p \to Y/p$ is an equivalence. In particular*[10] *we can test*

[9] But there is a spectral sequence of which we will make use of later.

[10] Here we use that p-completion is idempotent, i.e. $(X_p^\wedge)_p^\wedge \simeq X_p^\wedge$.

whether a map of p-complete spectra is an equivalence by checking it mod p. If furthermore X and Y are connective, the map f is already a p-adic equivalence provided that $X \otimes_{\mathbb{S}} H\mathbb{F}_p \to Y \otimes_{\mathbb{S}} H\mathbb{F}_p$ is an equivalence (i.e. we can check it on $\mathbb{F}_p$-homology).

Let us now collect a few further facts about p-completion that we will need.

- The Eilenberg-Maclane spectrum of a p-complete abelian group is a p-complete spectrum. More precisely HA is p-complete if and only A is **derived p-complete** in the sense of Tag 091S.
- A spectrum is p-complete if and only if all of its homotopy groups are derived p-complete.
- For every spectrum X, the spectrum X/p is p-complete.
- p-completion $(\ \ )^\wedge_p : \mathrm{Sp} \to \mathrm{Sp}$ is a lax symmetric monoidal functor (but of course not strongly). In particular, we have that the p-completion of an $\mathbb{E}_n$-ring R spectrum is an $\mathbb{E}_n$-ring spectrum again. Furthermore the map $R \to R^\wedge_p$ equips the p-completion with an $\mathbb{E}_n$-R-algebra structure.
- Let X, Y be module spectra over a connective ring spectrum R. Then we can write the p-completion of their tensor product in several different (but equivalent) ways:
$$(X \otimes_R Y)^\wedge_p \simeq \left(X^\wedge_p \otimes_R Y\right)^\wedge_p \simeq \left(X^\wedge_p \otimes_R Y^\wedge_p\right)^\wedge_p \simeq \left(X^\wedge_p \otimes_{R^\wedge_p} Y^\wedge_p\right)^\wedge_p$$
These equivalences hold because they hold after mod p-reduction and we can use Lemma 2.2.
- In the case that all R, X, Y are all connective, X is of finite type[11] over R and Y is p-complete, then $X \otimes_R Y$ is also p-complete.
This can be seen as follows: Write $X = \operatorname{colim} X_i$ with X_i finite. Tensoring with a finite spectrum preserves p-completeness, so $X_i \otimes_R Y$ is p-complete and hence all homotopy groups of $X_i \otimes_R Y$ are derived p-complete. But because the colimit is filtered and along increasingly connective maps we have
$$\pi_n(X \otimes_R Y) = \operatorname{colim}_i \pi_n(X_i \otimes_R Y) = \pi_n(X_n \otimes_R Y).$$
Therefore all these groups are derived p-complete and hence by the second point in this list, $X \otimes_R Y$ is p-complete.

Given a commutative ring spectrum R and any element $z \in \pi_0(R)$, we can still talk about the z-adic completion of module spectra over R, which is defined completely analogously. The previous discussion was the case $R = \mathbb{S}$, $z = p \in$

[11]This means that X can be written as a sequential colimit of finite R-module spectra along increasingly connective maps, i.e. $X = \operatorname{colim}_i X_i$ with X_i finite and $X_i \to X_{i+1}$ is i-connective.

$\pi_0(\mathbb{S}) = \mathbb{Z}$ and $\mathrm{Mod}_{\mathbb{S}} = \mathrm{Sp}$. What we discussed so far goes through in this more general setting as well. For everything that we need (and much, much more) we refer to [Lur18c, Section 7.3].

2.3 Topological Hochschild homology

Topological Hochschild homology is the higher algebra analogue of classical Hochschild homology, where one formally replaces the integers by the sphere spectrum. A classical reference on Hochschild homology which also discusses the relationship with algebraic K-theory is [Lod13].

In this section, we will define topological Hochschild homology and establish various properties of it.

Definition 2.3. *Let R be an $\mathbb{E}_1$-ring spectrum. We define the* ***topological Hochschild homology*** *of R as*

$$\mathrm{THH}(R) = R \otimes_{R \otimes_{\mathbb{S}} R^{op}} R.$$

If R is an $\mathbb{E}_1$-S-algebra, with $S \in \mathrm{CAlg}$ we can also define a relative version

$$\mathrm{THH}(R/S) = R \otimes_{R \otimes_S R^{op}} R.$$

In both cases, we abbreviate their p-completions:

$$\mathrm{THH}(R; \mathbb{Z}_p) \coloneqq \mathrm{THH}(R)^\wedge_p, \ \mathrm{THH}(R/S; \mathbb{Z}_p) \coloneqq \mathrm{THH}(R/S)^\wedge_p.$$

In the same way as for K-theory, we denote the homotopy groups by $\mathrm{THH}_*(R) \coloneqq \pi_*(\mathrm{THH}(R))$ and for the relative and p-completed version in the same way.

The left/right $R \otimes_{\mathbb{S}} R^{\mathrm{op}}$-module structure on R comes from the fact that R is a R-R-bimodule which is equivalent to the data of an $R \otimes_{\mathbb{S}} R^{\mathrm{op}}$-module structure. In the case of an $\mathbb{E}_\infty$-ring spectrum we can drop the $(\ \ )^{op}$ by [Lur17, Section 4.6.3]. If S is an ordinary commutative ring[12], we will also use the notation $\mathrm{HH}(R/S) \coloneqq \mathrm{THH}(R/S)$ and $\mathrm{HH}(R/\mathbb{Z}) \coloneqq \mathrm{THH}(R/\mathbb{Z})$, because it agrees with (a derived version of) classical Hochschild homology.

Using the bar construction for the relative tensor product ([Lur17, Section 4.4.2] we get the following formula for THH, which might be familiar from classical Hochschild homology.

$$\mathrm{THH}(R/S) = \mathrm{colim}_{\Delta^{\mathrm{op}}} \left(\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} R \otimes_S R \rightrightarrows R\right)$$

Evidently $\mathrm{THH}(R/S)$ is functorial in R, but we note that it is also functorial in S and even better it is functorial in squares, i.e. a commutative square of ring spectra induces a map on the relative topological Hochschild homology spectra, because it induces a map on the bar construction.

Let us record a few facts, that that we will need, in the following Lemma.

[12] From now on we will abuse notation and consider every ring as a ring spectrum without indication.

Lemma 2.4. *1.)* $\mathrm{THH}(\mathbb{S}) = \mathbb{S}$
2.) $\mathrm{THH}(R)$ *is connective if* R *is connective.*
3.) For any ordinary ring R*, the natural map* $\mathrm{THH}(R) \to \mathrm{HH}(R)$ *induces an isomorphism on* π_0, π_1, π_2 *and a surjection on* π_3*. More generally, for a commutative, connective ring spectrum* S *and an ordinary* $\pi_0(S)$*-algebra* R*, the same hold for the map* $\mathrm{THH}(R/S) \to \mathrm{HH}(R/\pi_0(S))$.
4.) If S *is a connective* $\mathbb{E}_\infty$*-ring spectrum and* R *a connective,* p*-complete* S*-algebra spectrum, which is of finite type over* S*, then* $\mathrm{THH}(R/S)$ *is also* p*-complete.*
5.) If S *is a connective* $\mathbb{E}_\infty$*-ring spectrum and* R *a connective,* p*-complete* S*-algebra spectrum, which is of finite type over* $S^\wedge_p$*, then*

$$\mathrm{THH}(R/S; \mathbb{Z}_p) = \mathrm{THH}(R/S^\wedge_p)$$

In particular, it is p*-complete by the part 4.).*

Proof. 1.) This is clear, because $\mathbb{S}$ is the unit of the tensor product on Sp.
2.) By induction, we see that $R \otimes_{\mathbb{S}} \cdots \otimes_{\mathbb{S}} R$ is connective, because the tensor product of two connective spectra is again connective. Any colimit of connective spectra is again connective and thus $\mathrm{THH}(R)$ is connective because of the cyclic bar construction.
3.) We show the statement for the map $\mathrm{THH}(R) \to \mathrm{HH}(R)$. For the proof of the general case, replace every $\mathbb{S}$ by S and $\mathbb{Z}$ by $\pi_0(S)$.
We need to show that the fiber $F \coloneqq \mathrm{fib}(\mathrm{THH}(R) \to \mathrm{HH}(R))$ is 3-connective, i.e. $\pi_0(F) = \pi_1(F) = \pi_2(F) = 0$. Because fibers and colimits commute, we can use the cyclic bar construction model to write F as the geometric realization of the levelwise fibers, i.e.

$$F = \mathrm{colim}_{\Delta^{\mathrm{op}}} \left(\cdots \substack{\rightarrow\\[-1ex]\rightarrow\\[-1ex]\rightarrow} \mathrm{fib}(R \otimes_{\mathbb{S}} R \to R \otimes_{\mathbb{Z}} R) \rightrightarrows \mathrm{fib}(R \to R) \right)$$

F is filtered by its skeletal filtration with associated graded given by $\mathrm{gr}^n = \Sigma^n F_n$. The spectral sequence for this filtered spectrum has its first page given by $E^1_{p,q} = \pi_q(F_p)$. Because every F_n is connective and $\mathrm{fib}(R \to R) = 0$, it remains to show that $\pi_0(F_1) = \pi_1(F_1) = \pi_0(F_2) = 0$. We can write every abelian group as a colimit of copies of $\mathbb{Z}$ and colimits only increase connectivity, so it suffices to show the statement in the case $R = \mathbb{Z}$. Let us thus consider the map $\mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z} \to \mathbb{Z}$. This map admits a retraction

$$\tau_{\leq 0}\mathbb{S} \otimes_{\mathbb{S}} \mathbb{Z} \simeq \mathbb{Z} \to \mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z} \to \mathbb{Z}.$$

The retraction exhibits $\mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z}$ as a direct sum of $\mathbb{Z}$ and the cofiber of the map $\tau_{\leq 0}\mathbb{S} \otimes_{\mathbb{S}} \mathbb{Z} \to \mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z}$. The fiber is given by $\tau_{\geq 1}\mathbb{S} \otimes_{\mathbb{S}} \mathbb{Z}$, thus the cofiber is $\Sigma\,(\tau_{\geq 1}\mathbb{S} \otimes_{\mathbb{S}} \mathbb{Z})$. By the direct sum decomposition this is then also the fiber, which we are after, i.e. $F_1 = \Sigma\,(\tau_{\geq 1}\mathbb{S} \otimes_{\mathbb{S}} \mathbb{Z})$. Because $\tau_{\geq 1}\mathbb{S}$ is by definition 1-connective and we tensor it with a connective spectrum, the resulting spectrum

is also 1-connective. Shifting this gives that F_1 is 2-connective.
The case of F_2 is easier. The long exact sequence for the fiber sequence $F_2 \to \mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z} \to \mathbb{Z}$ looks like

$$0 = \pi_1(\mathbb{Z}) \to \pi_0(F_2) \to \pi_0(\mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z}) \xrightarrow{\sim} \pi_0(\mathbb{Z}).$$

Because $\pi_1(\mathbb{Z}) = 0$ and the last map is an isomorphism, we necessarily have $\pi_0(F_2) = 0$.
4.) Recall from the last section that the tensor product of two connective modules over a connective base ring is p-complete provided one factor is of finite type over the base ring and the other module is p-complete. By our assumption we can see that $R \otimes_S R$ is p-complete and by induction that all the term of the form $R \otimes_S \cdots \otimes_S R$ are p-complete. Now we want to argue that by the bar construction $\mathrm{THH}(R/S)$ is also p-complete. This does not work directly because infinite colimits do not preserve p-completeness in general. But since finite colimits preserve p-completeness, we know that the colimit over $\Delta^{\mathrm{op}}_{\leq n}$ is p-complete. The full colimit over Δ^{op} is then given by a filtered colimit of these finite stage colimits along increasingly connective maps. This then implies that $\mathrm{THH}(R/S)$ is p-complete.
5.) We have a map $\mathrm{THH}(R/S) \to \mathrm{THH}(R/S^\wedge_p)$ and want to show that it is a p-adic equivalence. This suffices to show the statement, since the target is already p-complete by part 4.). By Lemma 2.2 we thus need to show that the map on mod p-reductions $\mathrm{THH}(R/S)/p \to \mathrm{THH}(R/S^\wedge_p)/p$ is an equivalence. For this we can for example use Proposition 3.2:

$$\begin{aligned} \mathrm{THH}(R/S)/p = \mathrm{THH}(R/S) \otimes_S (S/p) &\overset{3.2}{=} \mathrm{THH}((R/p)/(S/p)) \\ \to \mathrm{THH}(R/S^\wedge_p)/p = \mathrm{THH}(R/S^\wedge_p) \otimes_{\mathbb{S}^\wedge_p} (S^\wedge_p/p) &\overset{3.2}{=} \mathrm{THH}((R/p)/(S^\wedge_p/p)) \end{aligned}$$

And this map is an equivalence, because $S/p \to S^\wedge_p/p$ is. □

Part 3.) is especially useful when combined with the fact that the explicit map $\Omega^1_{R/S} \cong \mathrm{HH}_1(R/S), adb \mapsto a \otimes b$ is also a natural isomorphism for all (ordinary) commutative rings R and S. By the universal property of the de-Rham complex this induces a natural map from the (underlying graded-commutative algebra of the) de-Rham complex to the homology of the Hochschild complex $\Omega^*_{R/S} \cong \mathrm{HH}_*(R/S)$. In the case that R/S is smooth, this map is an isomorphism due to the **Hochschild-Kostant-Rosenberg** (HKR) theorem (see [Lod13, Section 3.4]). This result is also useful for the study of non-smooth algebras. Using the machinery of non-abelian derived functors it provides us, for *any* ring, with a natural filtration on its Hochschild complex. For a precise statement see [NS18, Proposition IV.4.1]. This filtration is called the **HKR-filtration** and we will crucially make use of it in Section 4.1.
The following proposition will only be used in the next section, where it is an important ingredient in the cyclotomic structure on THH.

Proposition 2.5. *[MSV97] Let R be an $\mathbb{E}_\infty$-ring spectrum, then the map $R \to \mathrm{THH}(R)$ is initial as a non-equivariant map of $\mathbb{E}_\infty$-ring spectra from R to an $\mathbb{E}_\infty$-ring spectrum with S^1-action (i.e. an object in $\mathrm{Fun}(BS^1, \mathrm{CAlg})$). In other words THH of a commutative ring spectrum is the colimit over the constant diagram $S^1 \to \mathrm{CAlg}$ with value R.*

Proof. We will use the simplicial model of the circle S^1 whose set of n-vertices S^1_n has exactly $n+1$ elements. This gives the presentation $S^1 = \operatorname{colim}_{\Delta^{\mathrm{op}}} S^1_n$ and thus allows us to decompose the colimit over S^1.

$$\operatorname{colim}_{S^1} R = \operatorname{colim}_{\Delta^{\mathrm{op}}}(\operatorname{colim}_{S^1_n} R) = \operatorname{colim}_{\Delta^{\mathrm{op}}} R^{\otimes(n+1)}$$

Which is precisely the cyclic bar construction from above, i.e. we get the desired result $\operatorname{colim}_{S^1} R = \mathrm{THH}(R)$. Here we used two facts mentioned in section 2.2: 1.) Finite coproducts in CAlg are given by tensor products. 2.) Filtered colimits in CAlg are computed on the level of the underlying spectra. □

The same is also true for relative THH. We have that $\mathrm{THH}(R/S)$ is the colimit of the functor $S^1 \to \mathrm{CAlg}_S$ with constant value R.
We also need to address the monoidality THH.

Proposition 2.6. *For any commutative ring spectrum S, the functor $\mathrm{THH}(-/S) : \mathrm{Alg}_S \to \mathrm{Mod}_S$ is symmetric monoidal. In particular, $\mathrm{THH} : \mathrm{Alg} \to \mathrm{Sp}$ is symmetric monoidal and thus THH of an $\mathbb{E}_n$-ring spectrum is $\mathbb{E}_{n-1}$. Most importantly THH of a commutative ring spectrum is again a commutative ring spectrum.*

For the case of commutative ring spectra this is clear by Proposition 2.5, because colimits commute with colimits and THH as well as the tensor product are given by colimits in the case of commutative ring spectra. The general case is more complicated.

2.4 Topological cyclic homology and the cyclotomic trace

The content of this section is not necessary for understanding the rest of the thesis. Nevertheless, we want to include it, because it gives an important motivation for being interest in THH in the first place. We can construct TC out of it which is extremely useful in calculations in K-theory[13]. A cyclotomic structure on a spectrum is certain additional equivariant data that are exactly what is necessary to define TC. To define it, we first need to discuss our equivariant setup. Let us stress that it is vastly different to genuine equivariant homotopy theory. In particular it is technically much simpler.

[13]See the appendix for a recollection on the definition of K-theory.

Definition 2.7. *Let G be a topological group like S^1 or C_p (the discrete cyclic group with p-elements) A **spectrum with G-action** is a functor $BG \to \mathrm{Sp}$ from the classifying space of G to spectra. A G-**equivariant map** is a natural transformation i.e. a 1-cell in the functor category $\mathrm{Fun}(BG, \mathrm{Sp})$. The **homotopy orbits** and **homotopy fixed points** of a spectrum with G-action $X : BG \to \mathrm{Sp}$ are defined as the colimit and limit of the functor.*

$$X^{hG} \coloneqq \lim(F : BG \to \mathrm{Sp}), \quad X_{hG} \coloneqq \mathrm{colim}(F : BG \to \mathrm{Sp})$$

*There is also a third operation that we need, called the **Tate construction** X^{tG}, which comes equipped with a canonical map from the fixed points $X^{hG} \xrightarrow{\mathrm{can}} X^{tG}$.*

Definition 2.8. *[NS18, Chapter 2.1] A **cyclotomic spectrum** is a spectrum with S^1-action X together with S^1-equivariant maps $\varphi_p : X \to X^{tC_p}$ for every prime p.*

Our main examples of cyclotomic spectra will come from topological Hochschild homology. We will only define the cyclotomic structure on topological Hochschild homology in the setting of $\mathbb{E}_\infty$-ring spectra. It can also be constructed in the greater generality of $\mathbb{E}_1$-ring spectra but it is more complicated, see [NS18, Chapter 3.1]). The two main ingredients in the construction of the cyclotomic structure are the **Tate diagonal** and Proposition 2.5 from the last section. Since we already did not discuss the Tate construction, we will also blackbox the Tate diagonal and simply assert that there is a natural map of spectra $X \to (\underbrace{X \otimes_{\mathbb{S}} \cdots \otimes_{\mathbb{S}} X}_{p})^{tC_p}$. For details see [NS18, Section 3.1]. Let us now construct the cyclotomic structure on $\mathrm{THH}(R)$ for $R \in \mathrm{CAlg}$. We first note that we can construct an initial $\mathbb{E}_\infty$-map out of R, where the target is a spectrum with C_p-action: $R \to \underbrace{R \otimes \cdots \otimes R}_{p}$. Since $\mathrm{THH}(R)$ has a S^1-action and hence also a C_p-action we get a map $R \otimes_{\mathbb{S}} \cdots \otimes_{\mathbb{S}} R \to \mathrm{THH}(R)$. Applying the C_p-Tate construction to this map and precomposing it with the Tate diagonal of R we get an $\mathbb{E}_\infty$-map $R \xrightarrow{\Delta_p} (R \otimes \cdots \otimes R)^{tC_p} \to \mathrm{THH}(R)^{tC_p}$ whose target has $S^1/C_p \cong S^1$-action.
Hence by Proposition 2.5 this factors through $\mathrm{THH}(R)$ and thus produces a map $\mathrm{THH}(R) \xrightarrow{\varphi_p} \mathrm{THH}(R)^{tC_p}$.

$$\begin{tikzcd}
R \arrow[r] \arrow[d, "\Delta_p"'] & \mathrm{THH}(R) \arrow[d, dashed, "\varphi_p"] \\
(R \otimes \cdots \otimes R)^{tC_p} \arrow[r] & \mathrm{THH}^{tC_p}
\end{tikzcd}$$

This map equips the topological Hochschild homology of any $\mathbb{E}_\infty$-ring spectrum with a cyclotomic structure. For every cyclotomic spectrum we can now define TC, which is our main object of interest for this section. First of all we define **negative topological cyclic homology** and **topological periodic homology**. For this we only need that $\mathrm{THH}(R)$ has an S^1-action.

Definition 2.9. *Let R be a $\mathbb{E}_1$-ring spectrum. define*

$$\mathrm{TC}^-(R) \coloneqq \mathrm{THH}(R)^{hS^1}, \quad \mathrm{TP}(R) \coloneqq \mathrm{THH}(R)^{tS^1}$$

Definition 2.10. *Let X be a cyclotomic spectrum with structure maps $\varphi_p : X \to X^{tC_p}$, $p \in \mathbb{P}$. We define its* ***topological cyclic homology*** *as the following fiber:*

$$\mathrm{TC}(X) \to X^{hS^1} \xrightarrow{\prod_{p\in\mathbb{P}}\left((\varphi_p)^{hS^1} - can^{hS^1}\right)} \prod_{p\in\mathbb{P}} (X^{tC_p})^{hS^1}$$

We implicitly consider the map can^{hS^1} as being precomposed with an identification of its domain $can^{hS^1} : X^{hS^1} \simeq (X^{hC_p})^{hS^1} \to (X^{tC_p})^{hS^1}$. For $R \in \mathrm{Alg}$ we define $\mathrm{TC}(R) \coloneqq \mathrm{TC}(\mathrm{THH}(R))$.

If X is bounded below (e.g. $\mathrm{THH}(R)$ for R connective) we can rewrite the last term in an easier way using the result [NS18, Lemma II.4.2] that $(X^{tCp})^{hS^1} \simeq (X^{tS^1})^\wedge_p$. We thus get the formula

$$\mathrm{TC}(X) = \mathrm{fib}\left(\mathrm{TC}^-(X) \xrightarrow{\prod_{p\in\mathbb{P}}\left((\varphi_p)^{hS^1} - \mathrm{can}\right)} \prod_{p\in\mathbb{P}} \mathrm{TP}(X)^\wedge_p\right).$$

The crucial relationship with K-theory is provided via the **cyclotomic trace**, which is a natural transformation $K \xrightarrow{\mathrm{tr}} \mathrm{TC}$. A conceptual definition using a universal characterization of K-theory is given in [BGT13, Section 10.3]. The first crucial theorem establishing the intimate relationship between K-theory and TC was proven by Dundas-Goodwillie-McCarthy .

Theorem 2.11. *([DGM12, Theorem 7.2.2.1]) Let $R \to S$ be a map of connective ring spectra. Assume that the induced map on π_0 is surjective with kernel a nilpotent ideal. Then the induced maps on K-theory and* TC *together with the respective cyclotomic traces give a pullback square of spectra:*

$$\begin{array}{ccc} K(R) & \xrightarrow{\mathrm{tr}} & \mathrm{TC}(R) \\ \downarrow & & \downarrow \\ K(S) & \xrightarrow{\mathrm{tr}} & \mathrm{TC}(S) \end{array}$$

In fact, if we are only interested in the p-completed case (for any prime p) we can even get away with a weaker condition on the map $R \to S$ by recent work of Clausen-Mathew-Morrow ([CMM21, Theorem A]).
The following result of Hesselholt and Madsen, proven using the Dundas-Goodwillie-McCarthy theorem, shows that for all rings of interest in this thesis, we can fully obtain their p-adic K-theory from knowledge of their topological cyclic homology.
.

Theorem 2.12. *[HM97, Theorem D] Let k be a perfect field of characteristic p and R a $W(k)$-algebra that is finitely generated as a $W(k)$-module. Then the cyclotomic trace map induces the following isomorphism*

$$K(R)^{\wedge}_p \xrightarrow{\simeq} \tau_{\geq 0} \operatorname{TC}(R)^{\wedge}_p.$$

For example we get $K(\mathbb{F}_p)^{\wedge}_p \simeq \tau_{\geq 0} \operatorname{TC}(\mathbb{F}_p)^{\wedge}_p = H\mathbb{Z}_p$. Together with the fact that for all other primes q we have a q-adic equivalence with connective complex (topological) K-theory $K(\mathbb{F}_p)^{\wedge}_q \simeq (\mathrm{ku})^{\wedge}_q$, this allows us to fully determine the homotopy type of $K(\mathbb{F}_p)$. Furthermore, this result applies to all complete discrete valuation rings of mixed characteristic with perfect residue field k because they are finite $W(k)$-modules, i.e. all rings of interest in Chapters 3 and 4.
Another interesting application of a more geometric flavour is the following: The **A-theory** spectrum of a based space X is defined to be $K(\Sigma^{\infty}_{+}(\Omega X))$. Observe that the $\mathbb{E}_1$-map $\Sigma^{\infty}_{+}(\Omega X) \to \Sigma^{\infty}_{+}(\Omega X) \otimes_{\mathbb{S}} \mathbb{Z} \to \mathbb{Z}[\pi_1(X)]$ is an isomorphism on π_0. Hence we can apply the Dundas-Goodwillie-McCarthy theorem and obtain the following pullback square.

$$\begin{array}{ccc} K(\Sigma^{\infty}_{+}(\Omega X)) & \longrightarrow & \operatorname{TC}(\Sigma^{\infty}_{+}(\Omega X)) \\ \downarrow & & \downarrow \\ K(\mathbb{Z}[\pi_1(X)]) & \longrightarrow & \operatorname{TC}(\mathbb{Z}[\pi_1(X)]) \end{array}$$

This gives us significant insight into $A(X)$, because we know a lot about the other constituents. $\operatorname{TC}(\Sigma^{\infty}_{+}(\Omega X))$ is to a large part understood by work of Bökstedt-Hsiang-Madsen [BHM93] (for a modern presentation see [NS18, Section IV.3]). $K(\mathbb{Z}[\pi_1(X)])$ can often be computed if the Farrell-Jones conjecture for $\pi_1(X)$ is confirmed (see e.g. [BLR08]). Calculations of $A(X)$ are highly sought after because they contain a lot of geometric information about X due to the celebrated stable parametrized h-cobordism theorem by Waldhausen-Jahren-Rognes ([WJR13]). If X is a smooth manifold this allows us in particular to compute $\pi_*(\mathrm{Diff}(X))$ in a certain range of degrees. This connection (predating the invention of topological cyclic homology) was used by Farrell-Hsiang to compute $\mathbb{Q} \otimes \pi_*(\mathrm{Diff}(D^n))$ in [FH78] using a theorem of Waldhausen ([Wal85, Corollary 2.3.8]) and Borel's work on the rational homotopy groups of $K(\mathbb{Z})$ ([Bor74, Chapter 12]).

3. Topological Hochschild Homology of CDVRs

In this chapter we will calculate topological Hochschild homology of (certain) complete discrete valuation rings.
Our main input for this is the following fundamental result of Bökstedt.

Theorem 3.1 (Bökstedt)**.** *On homotopy groups there is an isomorphism of graded rings* $\mathrm{THH}_*(\mathbb{F}_p) = \mathbb{F}_p[x]$ *for* $x \in \mathrm{THH}_2(\mathbb{F}_p)$ *a generator. We can also phrase this more conceptually as follows:* $\mathrm{THH}(\mathbb{F}_p)$ *is the free* $\mathbb{E}_1$*-*$\mathbb{F}_p$*-algebra on an element in degree 2, i.e. on the two sphere* S^2*. In other words* $\mathrm{THH}(\mathbb{F}_p) \cong \mathbb{F}_p \otimes_{\mathbb{S}} \Sigma^\infty_+ \Omega S^3$*.*

Note that the second statement implies the first[1], because the homotopy groups of $\mathbb{F}_p \otimes_{\mathbb{S}} \Sigma^\infty_+ \Omega S^3$ are the $\mathbb{F}_p$-homology groups of ΩS^3, which can be computed by the Serre spectral sequence associated to the path space fibration of S^3. The ring structure on $H_*(\Omega S^3; \mathbb{F}_p)$ is given by the Pontryagin product induced by the $\mathbb{E}_1$-structure on ΩS^3.
This periodicity in homotopy groups is called *Bökstedt periodicity.* In fact, the same phenomenon occurs in a much greater class of examples, namely for perfect $\mathbb{F}_p$-algebras and even all perfectoid rings ([BMS19, Chapter 6]). There is also a relative version for complete discrete valuation rings with perfect residue field of characteristic p. The proof for these results bootstraps the Bökstedt periodicity of $\mathbb{F}_p$ using two base change formulas that we will introduce now.

3.1 Base change formulas

Proposition 3.2. *Let* $k' \to k$ *be a map of* $\mathbb{E}_\infty$*-ring spectra and* R *an* $\mathbb{E}_1$*-*k'*-algebra. Then we have*

$$\mathrm{THH}(R/k') \otimes_{k'} k = \mathrm{THH}(R \otimes_{k'} k/k).$$

Proof. The base change functor $- \otimes_{k'} k : \mathrm{Mod}_{k'} \to \mathrm{Mod}_k$ is left adjoint to the forgetful functor, so it preserves all colimits. It furthermore carries a symmetric

[1] Both statements are actually equivalent. For the other direction observe that a generator of $\mathrm{THH}_2(\mathbb{F}_p)$ gives a map of $\mathbb{E}_1$-$\mathbb{F}_p$-algebras $\mathbb{F}_p \otimes_{\mathbb{S}} \Sigma^\infty_+ \Omega S^3 \to \mathrm{THH}(\mathbb{F}_p)$ because the domain is the free $\mathbb{E}_1$-$\mathbb{F}_p$-algebra on an element in degree 2 and the target is even an $\mathbb{E}_\infty$-$\mathbb{F}_p$-algebra. By assumption this map induces an isomorphism on homotopy groups and since equivalences of $\mathbb{E}_n$-algebras can be detected on the underlying spectra, the map is an equivalence of $\mathbb{E}_1$-$\mathbb{F}_p$-algebras.

monoidal structure. Since topological Hochschild homology is built via the geometric realization of the cyclic bar construction it only uses the monoidal structure and geometric realization. Hence any symmetric monoidal functor that also preserves geometric realizations will induce an equivalence on topological Hochschild homology. This applies in particular to the functor $-\otimes_{k'} k : \mathrm{Mod}'_k \to \mathrm{Mod}_k$, since it preserves all colimits. □

An important special case is $k' = \mathbb{S}$, which gives us $\mathrm{THH}(R) \otimes_{\mathbb{S}} k = \mathrm{THH}(R \otimes_{\mathbb{S}} k/k)$.

The next base change formula will allow us to 'decompose' relative THH computations. We prepare it with a small Lemma.

Lemma 3.3. *Let $k' \to k$ be a map of $\mathbb{E}_\infty$-ring spectra and $M, N \in \mathrm{Mod}_k$ modules over k (and thus also over k'). Consider k as a module over $k \otimes_{k'} k$ via multiplication. We then have the following natural equivalence*

$$M \otimes_k N \simeq (M \otimes_{k'} N) \otimes_{k \otimes_{k'} k} k.$$

Proof. We have a canonical map of k-modules $M \otimes_{k'} N \to M \otimes_k N$ induced by the map $k' \to k$. Under the adjunction $- \otimes_{k \otimes_{k'} k} k : \mathrm{Mod}_{k \otimes_{k'} k} \rightleftarrows \mathrm{Mod}_k : \mathrm{Forget}$ this is adjoint to the map

$$(M \otimes_{k'} N) \otimes_{k \otimes_{k'} k} k \to M \otimes_k N.$$

We want to show that this is an equivalence. Because Mod_k is under colimits generated by k and $- \otimes_{k'} -$ preserves colimits in both variables separately, it suffices to check the statement in the case $M = N = k$. There we have

$$(k \otimes_{k'} k) \otimes_{k \otimes_{k'} k} k \to k \otimes_k k,$$

which is clearly an isomorphism. □

Using exactly the same argument we also obtain the result

$$M_1 \otimes_k M_2 \otimes_k \cdots \otimes_k M_n \xleftarrow{\sim} (M_1 \otimes_{k'} M_2 \otimes_{k'} \cdots \otimes_{k'} M_n) \otimes_{k \otimes_{k'} \cdots \otimes_{k'} k} k,$$

for $M_1, \ldots M_n$ modules over k.

Proposition 3.4. *Let again $k' \to k$ be a map of $\mathbb{E}_\infty$-ring spectra and R an $\mathbb{E}_1$-k-algebra. Then k is a $\mathrm{THH}(k/k')$-module and we have*

$$\mathrm{THH}(R/k) = \mathrm{THH}(R/k') \otimes_{\mathrm{THH}(k/k')} k$$

Proof. Let us start by writing the left hand side in terms of its cyclic bar construction and applying the previous Lemma in the case where all k-modules are given by R.

$$\begin{aligned}
\mathrm{THH}(R/k) &= \mathrm{colim}_{\Delta^{\mathrm{op}}} (\cdots \Rrightarrow R \otimes_k R \rightrightarrows R) \\
&\overset{3.4}{=} \mathrm{colim}_{\Delta^{\mathrm{op}}} \left(\cdots \Rrightarrow (R \otimes_{k'} R) \otimes_{k \otimes_{k'} k} k \rightrightarrows R\right) \\
&= \mathrm{colim}_{\Delta^{\mathrm{op}}} \left((\cdots \Rrightarrow R \otimes_{k'} R \rightrightarrows R) \otimes_{(\ldots k \otimes_{k'} k \rightrightarrows k)} (\cdots \Rrightarrow k \rightrightarrows k)\right)
\end{aligned}$$

In the last line we consider the category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Sp})$ as a symmetric monoidal category equipped with the Day convolution tensor product (see [Lur17, Section 2.2.6]). In this situation $(\cdots \rightrightarrows k \otimes_{k'} k \rightrightarrows k)$ is a commutative algebra object in $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Sp})$ and $(\cdots \substack{\to\\\to\\\to} R \otimes_{k'} R \rightrightarrows R)$, $(\cdots \substack{\to\\\to\\\to} k \rightrightarrows k)$ are modules over it. The tensor product over $(\cdots \rightrightarrows k \otimes_{k'} k \rightrightarrows k)$ is then to be understood in the sense of [Lur17, Section 4.5.2].
Now we can use that Δ^{op} is sifted, i.e. the diagonal $\Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$ is cofinal and thus also $\Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$ is cofinal (see [Lur09, Lemma 5.5.8.4]). Applying this cofinality (under the equivalent characterization of cofinality proven in [Lur09, Proposition 4.1.1.8]); that the tensor product commutes with colimits in both variables separately and that we can also pull geometric realizations out of the ring, we are tensoring over[2], we obtain:

$$\begin{aligned}
\mathrm{THH}(R/k) &= \mathrm{colim}_{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}} \left((\cdots \substack{\to\\\to\\\to} R \otimes_{k'} R \rightrightarrows R) \otimes_{(\ldots k \otimes_{k'} k \rightrightarrows k)} (\cdots \substack{\to\\\to\\\to} k \rightrightarrows k) \right) \\
&= \mathrm{colim}_{\Delta^{\mathrm{op}}} (\cdots \substack{\to\\\to\\\to} R \otimes_{k'} R \rightrightarrows R) \otimes_{\mathrm{colim}_{\Delta^{\mathrm{op}}} (\ldots k \otimes_{k'} k \rightrightarrows k)} \mathrm{colim}_{\Delta^{\mathrm{op}}} (\cdots \substack{\to\\\to\\\to} k \rightrightarrows k) \\
&= \mathrm{THH}(R/k) \otimes_{\mathrm{THH}(k/k')} k
\end{aligned}$$

□

Again there is the important special case of $k' = \mathbb{S}$, which gives us $\mathrm{THH}(R/k) = \mathrm{THH}(R) \otimes_{\mathrm{THH}(k)} k$.
Note also that Proposition 3.4 implies Proposition 3.2. Indeed

$$\begin{aligned}
\mathrm{THH}(R \otimes_{k'} k/k) &\overset{3.4}{=} \mathrm{THH}(R \otimes_{k'} k/k') \otimes_{\mathrm{THH}(k/k')} k \\
&= \mathrm{THH}(R/k') \otimes_{k'} \mathrm{THH}(k/k') \otimes_{\mathrm{THH}(k/k')} k \\
&= \mathrm{THH}(R/k') \otimes_{k'} k.
\end{aligned}$$

3.2 Bökstedt periodicity for perfect $\mathbb{F}_p$-algebras

In this section we will prove Bökstedt periodicity for perfect[3] $\mathbb{F}_p$-algebras. Examples include perfect fields, in particular finite fields $\mathbb{F}_{p^n}$, as well as non-Noetherian examples like $\mathbb{F}_p[x^{1/p^\infty}]$. For the proof, we will rely on the following construction that establishes that we can not only deform perfect $\mathbb{F}_p$-algebras to characteristic zero via Witt vectors, but even to the sphere spectrum using a spherical analogue of the Witt vectors. (see [Lur18a], 5.2.5/5.2.7)

Proposition 3.5. *Let k be a perfect $\mathbb{F}_p$-algebra. Then there exists a p-complete $\mathbb{E}_\infty$-ring spectrum $\mathbb{S}_{W(k)}$ - the **spherical Witt vectors** of k - with base change*

[2] This itself uses the construction of the relative tensor product as the geometric realization of the bar construction ([Lur17, Section 4.4.2]) and once more the cofinality of $\Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$.

[3] For every $\mathbb{F}_p$-algebra R we have that the Frobenius $\varphi : R \to R,\ x \mapsto x^p$ is a ring homomorphism. We call R perfect if φ is an isomorphism.

$\mathbb{S}_{W(k)} \otimes_{\mathbb{S}} \mathbb{F}_p \simeq k$. *We thus get the following diagram of deformations of k:*

$$\begin{tikzcd} \mathbb{S}_{W(k)} \arrow[r] & W(k) \arrow[r] & k \\ \mathbb{S} \arrow[r] \arrow[u] & \mathbb{Z} \arrow[r] \arrow[u] & \mathbb{F}_p \arrow[u] \end{tikzcd}$$

In general for any $\mathbb{F}_p$-algebra k we have a map $\mathbb{F}_p \to k$, which induces a map of $\mathbb{E}_\infty$-ring spectra $\mathrm{THH}(\mathbb{F}_p) \to \mathrm{THH}(k)$. Since the target is a k-module spectrum, this further refines to $\mathrm{THH}(\mathbb{F}_p) \otimes_{\mathbb{F}_p} k \to \mathrm{THH}(k)$. The statement of Bökstedt periodicity for perfect $\mathbb{F}_p$-algebras is now the following.

Theorem 3.6. *For a perfect $\mathbb{F}_p$-algebra k the above map $\mathrm{THH}(\mathbb{F}_p) \otimes_{\mathbb{F}_p} k \to \mathrm{THH}(k)$ is an equivalence of $\mathbb{E}_\infty$-k-algebras. This in particular determines the homotopy groups as a graded ring:*

$$\mathrm{THH}_*(k) = k[x], \ |x| = 2.$$

Proof. Using our previous Proposition and that THH is symmetric monoidal, we see that:

$$\begin{aligned} \mathrm{THH}(k) &= \mathrm{THH}(\mathbb{F}_p \otimes_{\mathbb{S}} \mathbb{S}_{W(k)}) \\ &= \mathrm{THH}(\mathbb{F}_p) \otimes_{\mathbb{S}} \mathrm{THH}(\mathbb{S}_{W(k)}) \\ &= \mathrm{THH}(\mathbb{F}_p) \otimes_{\mathbb{F}_p} (\mathbb{F}_p \otimes_{\mathbb{S}} \mathrm{THH}(\mathbb{S}_{W(k)})) \end{aligned}$$

By Proposition 3.2 the term in brackets can be identified as
$\mathbb{F}_p \otimes_{\mathbb{S}} \mathrm{THH}(\mathbb{S}_{W(k)}) = \mathrm{THH}(\mathbb{S}_{W(k)} \otimes_{\mathbb{S}} \mathbb{F}_p/\mathbb{F}_p) = \mathrm{THH}(k/\mathbb{F}_p)$. Furthermore since $\mathbb{F}_p$ is an ordinary ring $\mathrm{THH}(k/\mathbb{F}_p) = \mathrm{HH}(k/\mathbb{F}_p)$. To prove the desired result, it remains to show that $\mathrm{HH}(k/\mathbb{F}_p)$ is k concentrated in degree 0. Because k is commutative $\mathrm{HH}_0(k/\mathbb{F}_p) = k$, so we need to prove the vanishing of the higher homology groups.
We claim that for a perfect $\mathbb{F}_p$-algebra R the cotangent complex vanishes $\mathbb{L}_{R/\mathbb{F}_p} = 0$. Together with the Hochschild-Kostant-Rosenberg filtration on Hochschild homology, this gives that $\mathrm{HH}_i(k/\mathbb{F}_p) = 0$ for $i > 0$, which then finishes the proof. For the vanishing of the cotangent complex, note that for any $\mathbb{F}_p$ algebra the Frobenius induces multiplication by p on the cotangent complex[4], i.e. the zero map. But since R is perfect, the Frobenius $\varphi : R \to R$ is an isomorphism and by functoriality still an isomorphism on the cotangent complex. The only way for the zero map to be an isomorphism is if already $\mathbb{L}_{R/\mathbb{F}_p} = 0$. □

[4]To see this, recall that the cotangent complex is by definition the non-abelian derived functor of Kähler differentials. To compute $\mathbb{L}_{R/\mathbb{F}_p}$ we thus have to simplicially resolve R by polynomial rings over $\mathbb{F}_p$ and then apply Kähler differentials level-wise. The Frobenius then acts via $\mathrm{d}x \mapsto \mathrm{d}(x^p) = p\mathrm{d}x$.

3.3 Relative Bökstedt periodicity for CDVRs

Let R be a discrete valuation ring, i.e. a local principal ideal domain that is not a field. Denote its unique maximal ideal by $\mathfrak{m}$ and the residue field by $k = R/\mathfrak{m}$. In this section we assume that R is complete with respect to the ideal $\mathfrak{m}$, i.e. $R \xrightarrow{\sim} \lim(R/\mathfrak{m} \leftarrow R/\mathfrak{m}^2 \leftarrow \dots)$ and that the residue field k is perfect of characteristic $p > 0$. We will abbreviate the term complete discrete valuation ring to CDVR. Any generator $\pi \in R$ of the maximal ideal will be called a *uniformizer*. Completeness with respect to the ideal $\mathfrak{m}$ is then equivalent to completeness with respect to the element π.

Either R and k are both of characteristic p (then we say that R is of *equal characteristic*) or R is of characteristic 0 and k is of characteristic p (the *mixed characteristic* case). In the equal characteristic setting, it is known that $R = k[[x]]$ for k a perfect field of characteristic p . In the mixed characteristic case there are more examples: We again get an example for every perfect field of characteristic p, namely $W(k)$. But there are also ramified examples like $\mathbb{Z}_p[\sqrt[n]{p}]$. or $\mathbb{Z}_p[\zeta_p]$. We can fully characterize them, as certain extensions of the unramified examples. This will be extremely useful later on.

Proposition 3.7. *Let R be a complete discrete valuation ring of mixed characteristic with perfect residue field k. Then R is of the form $R \cong W(k)[z]/E(z)$ for an Eisenstein polynomial*[5] *$E(z) \in W(k)[z]$. Under the map $W(k)[z] \to R$, the element z goes to a uniformizer in R and E is a minimal polynomial of π.*

Proof. See [Ser95, Section 2.5]. □

To calculate $\mathrm{THH}(R)$ we first work relative to the maximal ideal which allows us to use the result from the previous section because the residue field is by assumption perfect. In the next section, we will then obtain the non-relative result using a descent spectral sequence.

We will work relative to the spectrum $\mathbb{S}[z] := \Sigma^\infty_+(\mathbb{N}_0)$, the **spherical monoid ring** of $\mathbb{N}_0$. Since $\mathbb{N}_0$ is a commutative monoid, hence an $\mathbb{E}_\infty$-algebra in spaces and $\Sigma^\infty_+ : \mathcal{S} \to \mathrm{Sp}$ has a strong symmetric monoidal structure, we get that $\mathbb{S}[z]$ is an $\mathbb{E}_\infty$-ring spectrum. To work relative to $\mathbb{S}[z]$, we need an $\mathbb{E}_1$-map $\mathbb{S}[z] \to HR$, which informally is just given by $z \mapsto \pi$ for π a uniformizer in R. To make this map precise, note that the $(\Sigma^\infty_+, \Omega^\infty)$-adjunction is compatible with the symmetric monoidal structures[6] on $\mathcal{S}$ and Sp, so we get an induced adjunction between $\mathrm{CAlg}(\mathcal{S})$ and $\mathrm{CAlg}(\mathrm{Sp})$. Thus giving a map $\mathbb{S}[z] \to HR$ of $\mathbb{E}_\infty$-ring spectra is equivalent to giving a map $\mathbb{N}_0 \to R$ of $\mathbb{E}_\infty$-monoids in spaces. This is just a morphism of ordinary commutative monoids and since $\mathbb{N}_0$ is the free monoid (in

[5] A polynomial is called Eisenstein if it is of the form $z^n + a_{n-1}z^{n-1} + \dots a_1 z + a_0$ such that all a_i are divisible by p and a_0 is not divisible by p^2.

[6] i.e. Σ^∞_+ is (strong) symmetric monoidal and Ω^∞ is (lax) symmetric monoidal with respect to the cartesian product on $\mathcal{S}$ and the tensor product on Sp.

sets) on one generator, we get such a map for each element of R. In particular we have a map corresponding to $\pi \in R$ and this is what we mean by $\mathbb{S}[z] \to R,\ z \mapsto \pi$. We are now ready to state and prove a relative variant of Bökstedt's theorem for CDVRs.

Theorem 3.8. *Let R be a complete discrete valuation ring with perfect residue field of characteristic p. Choose a uniformizer $\pi \in R$ and give R a $\mathbb{S}[z]$-algebra structure by $z \mapsto \pi$. Then*

$$\mathrm{THH}_*(R/\mathbb{S}[z]; \mathbb{Z}_p) = R[x],\ |x| = 2.$$

Proof. Let us first treat the case that R is mixed characteristic. Since R is characteristic 0, p is not zero in R, so $p = u\pi^n$ for u a unit in R and $n \in \mathbb{N}$. This implies that an R-module is derived p-complete if and only if it is derived π-complete by [Stacks, Tag 091Q]. In particular we get that the homotopy groups of $\mathrm{THH}(R/\mathbb{S}[z]; \mathbb{Z}_p)$ are derived π-complete because they are by definition derived p-complete. Using Proposition 3.2 and that the tensor product preserves colimits in both variables we can now calculate

$$\begin{aligned}
\mathrm{THH}(R/\mathbb{S}[z]; \mathbb{Z}_p) \otimes_{\mathbb{S}[z]} \mathbb{S} &\overset{3.2}{=} \mathrm{THH}(R \otimes_{\mathbb{S}[z]} \mathbb{S}/\mathbb{S}; \mathbb{Z}_p) \\
&= \mathrm{THH}(R \otimes_{\mathbb{S}[z]} \mathrm{cofib}\left(\mathbb{S}[z] \xrightarrow{\cdot z} \mathbb{S}[z]\right); \mathbb{Z}_p) \\
&= \mathrm{THH}(R/\pi; \mathbb{Z}_p) = \mathrm{THH}(k; \mathbb{Z}_p) = \mathrm{THH}(k)
\end{aligned}$$

In the last line we used that z acts by multiplication by π on R and that $\mathrm{THH}(k)$ is already p-complete since all its homotopy groups are already derived p-complete (even classically p-complete). On the other hand, again using that the tensor product preserves colimits, we can also see that $\mathrm{THH}(R/\mathbb{S}[z]; \mathbb{Z}_p) \otimes_{\mathbb{S}[z]} \mathbb{S} = \mathrm{THH}(R/\mathbb{S}[z]; \mathbb{Z}_p)/\pi$. Hence we get the cofiber sequence

$$\mathrm{THH}(R/\mathbb{S}[z]; \mathbb{Z}_p) \xrightarrow{\cdot \pi} \mathrm{THH}(R/\mathbb{S}[z]; \mathbb{Z}_p) \to \mathrm{THH}(k).$$

Since k is by assumption perfect of characteristic p, it is in particular a perfect $\mathbb{F}_p$-algebra, so Theorem 3.6 applies and tells us the homotopy groups of $\mathrm{THH}(k)$. Thus the long exact sequence for the odd homotopy groups looks as follows

$$\ldots \to k \to \mathrm{THH}_{2i+1}(R/\mathbb{S}[z]; \mathbb{Z}_p) \xrightarrow{\cdot \pi} \mathrm{THH}_{2i+1}(R/\mathbb{S}[z]; \mathbb{Z}_p) \to 0 \to \ldots$$

This tells us that the odd homotopy groups of $\mathrm{THH}(R/\mathbb{S}[z]; \mathbb{Z}_p)$ are 0 mod π. But by our discussion at the beginning of the proof we also know that they are derived π-complete, so they must already be 0 before taking mod π reduction (see [Stacks, Tag 09B9]).
For the identification of the (non-negative) even homotopy groups, we proceed similarly. The long exact sequence takes the form

$$\ldots \to 0 \to \mathrm{THH}_{2i}(R/\mathbb{S}[z]; \mathbb{Z}_p) \xrightarrow{\cdot \pi} \mathrm{THH}_{2i}(R/\mathbb{S}[z]; \mathbb{Z}_p) \to k \to \ldots,$$

which gives us that they are π-torsion free and their mod π reduction is k. They are also derived π-complete. The same holds for R, so once we have an R-module map (over k) we get an isomorphism by the derived Nakayama Lemma (π-torsion freeness guarantees that the derived mod π reduction agrees with the ordinary one). Let $M := \mathrm{THH}_2(R/\mathbb{S}[z];\mathbb{Z}_p)$. By the long exact sequence we have a surjective map $M \to k$. Giving an R-module map $R \to M$ is the same as choosing an element $x \in M$. To also ensure that the map is an equivalence after reducing mod π, the reduction of x must generate k as an R-module. So we have to choose a preimage of any non-zero element of k under the map $M \to k$, which always exists, since the map is surjective by the long exact sequence. Thus choosing such a lift provides an isomorphism $\mathrm{THH}_2(R/\mathbb{S}[z];\mathbb{Z}_p) \cong R$. The same applies to all positive even homotopy groups once we also choose lifts for all of them. A priori the choices of all these lifts might be unrelated. But since the map $\mathrm{THH}_*(R/\mathbb{S}[z];\mathbb{Z}_p) \to \mathrm{THH}_*(k)$ is multiplicative, the elements $x^i \in \mathrm{THH}_{2i}(R/\mathbb{S}[z];\mathbb{Z}_p)$ provide canonical choices of all of those. Hence the choice of x already suffices and gives us an isomorphism of graded rings $\mathrm{THH}_*(R/\mathbb{S}[z];\mathbb{Z}_p) \xrightarrow{\sim} \mathrm{THH}_*(k)$.

If R is of equal characteristic, we have already remarked that necessarily $R = k[[z]]$ for k perfect of positive characteristic. We proceed similarly to the proof of Theorem 3.6. Defining $\mathbb{S}_{W(k)}[[z]]$ as the z-adic completion of $\mathbb{S}_{W(k)} \otimes_{\mathbb{S}} \mathbb{S}[z]$, gives us the following base change

$$\mathbb{S}_{W(k)}[[z]] \otimes_{\mathbb{S}} \mathbb{F}_p = k[[z]].$$

To see this, note that the right hand side is the z-adic completion of the left hand side, as in general we have $(X \otimes_{\mathbb{S}} Y)^\wedge_z = (X^\wedge_z \otimes_{\mathbb{S}} Y^\wedge_z)^\wedge_z$. To establish that the map to the completion is an isomorphism we thus have to see that the left hand side is already z-complete. This holds because $\mathbb{F}_p$ is finite type over the sphere and $\mathbb{S}_{W(k)}[[z]]$ is z-complete.
After establishing this base change, we can use 3.2 again :

$$\begin{aligned}
\mathrm{THH}(k[[z]]/\mathbb{S}[z]) &= \mathrm{THH}(\mathbb{S}_{W(k)}[[z]] \otimes_{\mathbb{S}} \mathbb{F}_p) \\
&= \mathrm{THH}(\mathbb{S}_{W(k)}[[z]]/\mathbb{S}[z]) \otimes_{\mathbb{S}} \mathrm{THH}(\mathbb{F}_p/\mathbb{S}[z]) \\
&= \big(\mathrm{THH}(\mathbb{S}_{W(k)}[[z]]/\mathbb{S}[z]) \otimes_{\mathbb{S}} \mathbb{F}_p\big) \otimes_{\mathbb{F}_p} \mathrm{THH}(\mathbb{F}_p) \\
&\overset{3.2}{=} \mathrm{THH}(\mathbb{S}_{W(k)}[[z]] \otimes_{\mathbb{S}} \mathbb{F}_p/\mathbb{S}[z] \otimes_{\mathbb{S}} \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathrm{THH}(\mathbb{F}_p) \\
&= \mathrm{HH}(k[[z]]/\mathbb{F}_p[z]) \otimes_{\mathbb{F}_p} \mathrm{THH}(\mathbb{F}_p)
\end{aligned}$$

So we have to show that $\mathrm{HH}_i(k[[z]]/\mathbb{F}_p[z])$ vanishes for $i > 0$. For perfect $\mathbb{F}_p$-algebras we used perfectness to show this, which is not possible here, because $k[[z]]$ is clearly not perfect. But the map $\mathbb{F}_p[z] \to k[[z]]$ is relatively perfect[7],

[7]The naming comes from the fact that an $\mathbb{F}_p$-algebra R is perfect if and only if $\mathbb{F}_p \to R$ is relatively perfect

which by definition means that the following square is a pushout in CAlg.

$$\begin{tikzcd} \mathbb{F}_p[z] \arrow[r] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\ulcorner"] & k[[z]] \arrow[d, "\varphi"'] \\ \mathbb{F}_p[z] \arrow[r] & k[[z]] \end{tikzcd}$$

To see this, we first observe that the vertical maps in the diagram are on the level of ordinary rings simply the inclusions $\mathbb{F}_p[z^p] \hookrightarrow \mathbb{F}_p[z]$ and $k[[z^p]] \hookrightarrow k[[z]]$ (where we implicitly identify k with itself under the action of Frobenius using that k is perfect). Thus both maps exhibit the target as a free module over the domain on the basis $\{1, z, z^2, \ldots, z^p\}$. Because equivalences in CAlg are detected on the underlying spectra, this also means that

$$\mathbb{F}_p[z] \otimes_{\mathbb{F}_p[z^p]} k[[z^p]] \cong k[[z]]$$

which implies that the square is a pushout. Using this isomorphism, we obtain an isomorphism in Hochschild homology as well:

$$\begin{aligned} \mathrm{HH}(k[[z]]/\mathbb{F}_p[z]) &\xleftarrow{\sim} \mathrm{HH}((\mathbb{F}_p[z] \otimes_{\mathbb{F}_p[z^p]} k[[z^p]])/\mathbb{F}_p[z]) \\ &= \mathrm{HH}(\mathbb{F}_p[z]/\mathbb{F}_p[z]) \otimes_{\mathbb{F}_p[z^p]} \mathrm{HH}(k[[z^p]]/\mathbb{F}_p[z]) \\ &= \mathbb{F}_p[z] \otimes_{\mathbb{F}_p[z^p]} \mathrm{HH}(k[[z^p]]/\mathbb{F}_p[z]) \end{aligned}$$

We now want to show that this map induces the zero map on all $\pi_n,\ n > 0$ which will then finish the proof as it is also an isomorphism. Let us first observe that since $\mathbb{F}_p[z]$ is free as an $\mathbb{F}_p[z^p]$-module we can pull the tensor product out of the homotopy/homology group:

$$\pi_n(\mathbb{F}_p[z] \otimes_{\mathbb{F}_p[z^p]} \mathrm{HH}(k[[z^p]]/\mathbb{F}_p[z])) = \mathbb{F}_p[z] \otimes_{\mathbb{F}_p[z^p]} \mathrm{HH}_n(k[[z^p]]/\mathbb{F}_p[z])$$

Under this identification the map in question

$$\mathbb{F}_p[z] \otimes_{\mathbb{F}_p[z^p]} \mathrm{HH}_n(k[[z^p]]/\mathbb{F}_p[z]) \to \mathrm{HH}_n(k[[z]]/\mathbb{F}_p[z])$$

is adjoined (under the left adjoint to the forgetful functor $\mathrm{Mod}_{\mathbb{F}_p[z]} \to \mathrm{Mod}_{\mathbb{F}_p[z^p]}$) to the map, which is itself obtained by applying $\mathrm{HH}_n(-/\mathbb{F}_p[z])$ to the inclusion $k[[z^p]] \to k[[z]]$. As the Hochschild homology groups are the homotopy groups of a simplicial commutative algebra and the latter always carry a divided power structure (see eg. [Ric09, Section 4], this inclusion necessarily induces the zero map because we are in characteristic p. $\square$

Using the same technique we can also compute relative THH of quotients of the CDVRs that we have considered so far. In any discrete valuation ring all ideals are powers of the maximal ideal. So concretely we are dealing with rings of the form R/π^n, where R is a complete discrete valuation ring with perfect residue field of positive characteristic and π a uniformizer. For instance this covers the

cases $\mathbb{Z}_p/p^n = \mathbb{Z}/p^n$ or truncated polynomial algebras $k[[x]]/x^n = k[x]/x^n$ for k perfect of positive characteristic.
For these quotients we do not quite get polynomial homotopy groups as before but we need an additional divided power generator compensating for the quotient.

Theorem 3.9. *Let R be a complete discrete valuation ring with perfect residue field of positive characteristic, $n \geq 1$ and $R' = R/\pi^n$. Then we have:*

$$\mathrm{THH}_*(R'/\mathbb{S}[z]) = R'[x]\langle y\rangle,\ |x| = |y| = 2$$

Proof. Since π is not a zero divisor, the ordinary and derived quotient agree, so $R' = R \otimes_{\mathbb{S}[z]} (\mathbb{S}[z]/z^n)$. Here we write $\mathbb{S}[z]/z^n = \Sigma^\infty M$, where M is the (pointed) monoid $M = \{0, 1, x, \dots, x^{n-1}\}$ with multiplicative monoid structure and $x^a \cdot x^b = 0$ if $a + b \geq n$. By the strong symmetric monoidality of $\Sigma^\infty : (\mathcal{S}_*, \wedge) \to (\mathrm{Sp}, \otimes_{\mathbb{S}})$ we obtain an $\mathbb{E}_\infty$-structure[8] on $\mathbb{S}[z]/z^n$.
Using this identification of R', we can calculate:

$$\begin{aligned}
\mathrm{THH}(R'/\mathbb{S}[z]) &= \mathrm{THH}(R/\mathbb{S}[z]) \otimes_{\mathbb{S}[z]} \mathrm{THH}(\mathbb{S}[z]/z^n/\mathbb{S}[z]) \\
&= \mathrm{THH}(R/\mathbb{S}[z]) \otimes_{\mathbb{Z}\otimes_{\mathbb{S}}\mathbb{S}[z]} (\mathbb{Z} \otimes_{\mathbb{S}} \mathrm{THH}((\mathbb{S}[z]/z^n)/\mathbb{S}[z]) \\
&\overset{3.2}{=} \mathrm{THH}(R/\mathbb{S}[z]) \otimes_{\mathbb{Z}[z]} \mathrm{HH}\left((\mathbb{Z}[z]/z^n)/\mathbb{Z}[z]\right)
\end{aligned}$$

As p is nilpotent in R', every R'-module spectrum is already p-complete, in particular $\mathrm{THH}(R'/\mathbb{S}[z])$ must be p-complete. This means that it does not matter whether we p-complete one (or even both) of the factors, i.e. we get

$$\mathrm{THH}(R'/\mathbb{S}[z]) = \mathrm{THH}(R/\mathbb{S}[z]; \mathbb{Z}_p) \otimes_{\mathbb{Z}[z]} \mathrm{HH}\left((\mathbb{Z}[z]/z^n)/\mathbb{Z}[z]\right)$$

We completely understand the homotopy groups of the first factor by the previous theorem, so again we need to do a calculation in Hochschild homology. The derived tensor product $(\mathbb{Z}[z]/z^n) \otimes^L_{\mathbb{Z}[z]} (\mathbb{Z}[z]/z^n) = \Lambda_{\mathbb{Z}[z]/z^n}(e)$ is given by an exterior algebra over $\mathbb{Z}[z]/z^n$ on a generator e in degree 1. Thus Hochschild homology is given by

$$\mathrm{HH}\left((\mathbb{Z}[z]/z^n)/\mathbb{Z}[z]\right) = (\mathbb{Z}[z]/z^n) \otimes^L_{\Lambda_{\mathbb{Z}[z]/z^n}(e)} (\mathbb{Z}[z]/z^n).$$

To calculate this derived tensor product, we can explicitely resolve $\mathbb{Z}[z]/z^n$ as a free divided power algebra over $\Lambda_{\mathbb{Z}[z]/z^n}(e)$ as follows

$$\Lambda_{\mathbb{Z}[z]/z^n}(e)\langle y\rangle := \frac{\Lambda_{\mathbb{Z}[z]/z^n}(e)[y_1, y_2, \dots]}{y_i y_j = \binom{i+j}{i} y_{i+j}},\ \partial y_i = e y_{i-1},\ |y_i| = 2i.$$

Hence we get the following result for Hochschild homology

$$\mathrm{HH}\left((\mathbb{Z}[z]/z^n)/\mathbb{Z}[z]\right) = \Lambda_{\mathbb{Z}[z]/z^n}(e)\langle y\rangle \otimes_{\Lambda_{\mathbb{Z}[z]/z^n}(e)} (\mathbb{Z}[z]/z^n) = (\mathbb{Z}[z]/z^n)\langle y\rangle.$$

This finally enables us to compute the homotopy groups:

$$\mathrm{THH}_*(R'/\mathbb{S}[z]) = R'[x]\langle y\rangle,\ |x| = |y| = 2. \qquad \square$$

[8]The underlying spectrum of $\mathbb{S}[z]/z^n$ is simply given by taking the cofiber of $\mathbb{S}[z] \xrightarrow{\cdot z^n} \mathbb{S}[z]$ but this has a priori no ring structure anymore. That is why we give the alternative description which makes the existence of an $\mathbb{E}_\infty$-structure clear.

3.4 Absolute THH of CDVRs

In this section, we calculate the absolute THH of (quotients of) complete discrete valuation rings with perfect residue field of positive characteristic. We have already computed it relative to $\mathbb{S}[z]$. To get from there to the absolute case we use a descent spectral sequence that we construct now.

Construction 3.10. *Let R be an ordinary commutative ring equipped with a map of rings $\mathbb{Z}[z] \to R$ (i.e. the choice of an element of R). This also gives R the structure of an $\mathbb{E}_\infty$-$\mathbb{S}[z]$-algebra, via $\mathbb{S}[z] \to \mathbb{Z}[z]$. We can filter $\mathrm{HH}(\mathbb{Z}[z]$ by its (very short) Whitehead tower:*

$$0 = \tau_{\geq 2} \mathrm{HH}(\mathbb{Z}[z]) \to \tau_{\geq 1} \mathrm{HH}(\mathbb{Z}[z]) \to \tau_{\geq 0} \mathrm{HH}(\mathbb{Z}[z]) = \mathrm{HH}(\mathbb{Z}[z])$$

Observe that $\mathrm{HH}(\mathbb{Z}[z]) \overset{3.2}{=} \mathrm{THH}(\mathbb{S}[z]) \otimes_{\mathbb{S}} \mathbb{Z}$, which gives us an $\mathbb{E}_\infty$-$\mathrm{HH}(\mathbb{Z}[z])$-algebra structure on $\mathrm{THH}(R)$ via the map $\mathrm{THH}(\mathbb{S}[z]) \to \mathrm{THH}(R)$. We thus have the strong monoidal base change functor $\mathrm{Mod}_{\mathrm{HH}(\mathbb{Z}[z])} \to \mathrm{Mod}_{\mathrm{THH}(R)}$ and can levelwise apply it to the filtration to get a multiplicative filtration on $\mathrm{THH}(R)$

$$0 \to \tau_{\geq 1} \mathrm{HH}(\mathbb{Z}[z]) \otimes_{\mathrm{HH}(\mathbb{Z}[z])} \mathrm{THH}(R) \to \mathrm{HH}(\mathbb{Z}[z]) \otimes_{\mathrm{HH}(\mathbb{Z}[z])} \mathrm{THH}(R) = \mathrm{THH}(R)$$

This filtered spectrum gives rise to a spectral sequence, whose E^2-page is given by the homotopy groups of the associated graded $E^2_{i,j} = \pi_{i+j} \mathrm{gr}^j$, see[9] *e.g. [Lur17, Chapter 1.2.2]. To compute the E^2-page, we thus have to identify the associated graded.*

Since tensor products preserve cofiber sequences, the associated graded of the filtration on $\mathrm{THH}(R)$ can simply be obtained by tensoring the associated graded of the filtration on $\mathrm{HH}(\mathbb{Z}[z])$ with $\mathrm{THH}(R)$. The associated graded of the filtration on $\mathrm{HH}(\mathbb{Z}[z])$ is given by $\mathrm{HH}_i(\mathbb{Z}[z]) = \Omega^i_{\mathbb{Z}[z]/\mathbb{Z}}$. Using our base change formulas, we can relate this to relative THH:

$$\begin{aligned}
\mathrm{gr}^j &= \mathrm{THH}(R) \otimes_{\mathrm{HH}(\mathbb{Z}[z])} \Omega^j_{\mathbb{Z}[z]/\mathbb{Z}}[j] \\
&= \mathrm{THH}(R) \otimes_{\mathrm{HH}(\mathbb{Z}[z])} \mathbb{Z}[z] \otimes_{\mathbb{Z}[z]} \Omega^j_{\mathbb{Z}[z]/\mathbb{Z}}[j] \\
&\overset{3.2}{=} \big(\mathrm{THH}(R) \otimes_{\mathrm{THH}(\mathbb{S}[z]) \otimes_{\mathbb{S}} \mathbb{Z}} (\mathbb{S}[z] \otimes_{\mathbb{S}} \mathbb{Z})\big) \otimes_{\mathbb{Z}[z]} \Omega^j_{\mathbb{Z}[z]/\mathbb{Z}}[j] \\
&= \big(\mathrm{THH}(R) \otimes_{\mathrm{THH}(\mathbb{S}[z])} \mathbb{S}[z]\big) \otimes_{\mathbb{Z}[z]} \Omega^j_{\mathbb{Z}[z]/\mathbb{Z}}[j] \\
&\overset{3.4}{=} \mathrm{THH}(R/\mathbb{S}[z]) \otimes_{\mathbb{Z}[z]} \Omega^j_{\mathbb{Z}[z]/\mathbb{Z}}[j]
\end{aligned}$$

Therefore we get a homological (Serre graded) spectral sequence

$$E^2_{i,j} = \mathrm{THH}_i(R/\mathbb{S}[z]) \otimes_{\mathbb{Z}[z]} \Omega^j_{\mathbb{Z}[z]/\mathbb{Z}} \Rightarrow \mathrm{THH}_{i+j}(R).$$

[9] Note that Lurie works with accending filtrations, while the Whitehead filtration is descending. The differentials and indexing therefore work differently, which is why we reindex to be consistent in our usage of homological Serre grading. As a result of this reindexing, the differentials work in such a way that we start with the E^2-page instead of an E^1-page.

The spectral sequence is concentrated in degrees $(i, j) \in [0, \infty) \times [0, 1]$, in particular it is first quadrant and converges. Furthermore the spectral sequence is multiplicative since the Whitehead filtration is multiplicative[10].
By p-completing everywhere and using that p-completion is exact we also get the following spectral sequence with the same formal properties:

$$E^2_{i,j} = \mathrm{THH}_i(R/\mathbb{S}[z]; \mathbb{Z}_p) \otimes_{\mathbb{Z}_p[z]} \Omega^j_{\mathbb{Z}_p[z]/\mathbb{Z}_p} \Rightarrow \mathrm{THH}_{i+j}(R; \mathbb{Z}_p)$$

In our case of interest R is again a CDVR with perfect residue field of positive characteristic. We completely know the E^2-page which by Theorem 3.8 takes the form $E^2 = R[x] \otimes_{\mathbb{Z}_p} \Lambda_{\mathbb{Z}_p}(dz)$ for elements $|x| = (0, 2)$ and $|dz| = (1, 0)$ as pictured below.

$$\begin{tikzcd}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots \\
R\{dz\} & 0 & R\{xdz\} & 0 & R\{x^2dz\} & \cdots \\
R & 0 & R\{x\} \arrow[ull] & 0 & R\{x^2\} \arrow[ull] & \cdots
\end{tikzcd}$$

Since the E^2-page is already concentrated in the first two rows, there is only room for the d^2-differential (recall in Serre grading d^r goes r to the left and $r - 1$ up.) Furthermore because multiplicatively everything with potential for non-vanishing differential is generated by x and the spectral sequence is multiplicative, it suffices to identify

$$d^2 : R\{x\} \to R\{dz\}.$$

If R has equal characteristic, this differential actually has to vanish. The reason for this is that by the construction of the spectral sequence, we know that the edge homomorphism is given by $\mathrm{THH}_*(R; \mathbb{Z}_p) \to \mathrm{THH}_*(R/\mathbb{S}[z]; \mathbb{Z}_p)$ and everything that lies in the image of this map, has to be a permanent cycle. Because R has equal characteristic, we can precompose the map by $\mathrm{THH}(k) \to \mathrm{THH}(R; \mathbb{Z}_p)$ and choose $x \in \mathrm{THH}_2(R/\mathbb{S}[z]; \mathbb{Z}_p)$ to be the image of the Bökstedt element under the composition of the two maps. Thus x has to be a permanent cycle and d^2 vanishes. We hence get that $\mathrm{THH}_*(R; \mathbb{Z}_p) = R[x] \otimes_{\mathbb{Z}_p} \Lambda_{\mathbb{Z}_p}(dz)$, in particular $\mathrm{THH}_i(R; \mathbb{Z}_p) = R$ for all $i \geq 0$.
For mixed characteristic R we will identify it in the following Lemma. Recall from Proposition 3.7 that every mixed characteristic CDVR R with perfect residue

[10] i.e. it is an algebra in the ∞-category of filtered spectra $\mathrm{Fun}((N\mathbb{Z})^{\mathrm{op}}, \mathrm{Sp})$ equipped with the Day convolution tensor product

field k has the form $R = W(k)[z]/E(z)$, where $E(z) \in W(k)[z]$ is an Eisenstein polynomial.

Lemma 3.11. *For a mixed characteristic CDVR of the form $R = W(k)[z]/E(z)$, there is a generator $x \in \mathrm{THH}_2(R; \mathbb{Z}_p)$ with differential $d^2(x) = E'(\pi)dz$.*

Proof. We want to work relative to $\mathbb{S}_{W(k)}$ because this will enable us to make use of the description $R = W(k)[z]/E(z)$. To do this we again employ the base change for relative THH:

$$\mathrm{THH}(R/\mathbb{S}_{W(k)}; \mathbb{Z}_p) \overset{3.4}{=} \mathrm{THH}(R; \mathbb{Z}_p) \otimes_{\mathrm{THH}(\mathbb{S}_{W(k)}; \mathbb{Z}_p)} \mathbb{S}_{W(k)}$$

This does not look helpful so far but we can simplify it further. In the proof of Theorem 3.6 we already saw that $\mathbb{F}_p \otimes_{\mathbb{S}} \mathrm{THH}(\mathbb{S}_{W(k)}; \mathbb{Z}_p) = k$. We also have a map $\mathbb{S}_{W(k)} \to \mathrm{THH}(\mathbb{S}_{W(k)}; \mathbb{Z}_p)$. Both spectra are connective and p-complete, so we can check equivalence on $\mathbb{F}_p$-homology. There it is an isomorphism by the above computation. Thus we get

$$\mathrm{THH}(R/\mathbb{S}_{W(k)}; \mathbb{Z}_p) = \mathrm{THH}(R; \mathbb{Z}_p).$$

As R is p-complete and finite type over $\mathbb{S}_{W(k)}$, we can even drop the p-completion by part 4 of Proposition 2.4. We can now identify the differential using that Hochschild homology and topological Hochschild homology agree in low degrees (Lemma 2.4.part 3):

$$\mathrm{THH}_1(R; \mathbb{Z}_p) = \mathrm{THH}_1(R/\mathbb{S}_{W(k)}) = \mathrm{HH}_1(R/W(k)).$$

Furthermore Hochschild homology and Kähler differentials always agree in degree 1 and Kähler differentials are easy to compute, since $R = W(k)[z]/E(z)$.

$$\mathrm{HH}_1(R/W(k)) = \Omega^1_{R/W(k)} = R\{dz\}/E'(\pi)dz.$$

By the spectral sequence we also know

$$\mathrm{THH}_1(R; \mathbb{Z}_p) = R\{dz\}/\operatorname{im}(d^2 : E^2_{2,0} \to E^2_{0,1}),$$

which finally implies that $\operatorname{im}(d^2 : E^2_{2,0} \to E^2_{0,1})$ is equal to the submodule of $R\{dz\}$ generated by $E'(\pi)dz$. This only pinpoints the generator of this submodule up to a unit, which we can choose. $\square$

This result will easily allow us to compute $\mathrm{THH}_*(R; \mathbb{Z}_p)$, which we will do momentarily. But let us first issue the following warning.

Warning 3.12. *The result of the Lemma implicitly depends on the choice of a uniformizer. While this is not a problem if we are only interested in* THH_* *of one particular CDVR, it is a problem if we are dealing with a map $R \to S$ of two of those. We can only determine the effect on* THH_* *of those maps, which preserve*

the chosen uniformizers. Only in this case is the map $R \to S$ actually a map of $\mathbb{S}[z]$-modules with respect to the module structures induced by the choices of the respective uniformizers. Consider for example the inclusion $\mathbb{Z}_p \hookrightarrow \mathbb{Z}_p[\sqrt{p}]$. Any uniformizer in $\mathbb{Z}_p$ has the form up for $u \in \mathbb{Z}_p^\times$, but of course none of them generate the maximal ideal of $\mathbb{Z}_p[\sqrt{p}]$. There is therefore no choice of uniformizers that is compatible with this map and our method is not able to calculate the induced map $\mathrm{THH}(\mathbb{Z}_p;\mathbb{Z}_p) \to \mathrm{THH}(\mathbb{Z}_p[\sqrt{p}];\mathbb{Z}_p)$. The same problem occurs for $\mathbb{Z}_p \hookrightarrow \mathbb{Z}_p[\sqrt[n]{p}]$, $n \geq 2$.
But even if we can choose the uniformizers such that one maps onto the other as for example in the case $\mathbb{Z}_p[\sqrt{p}] \to \mathbb{Z}_p[\sqrt{p}]$, $\sqrt{p} \mapsto -\sqrt{p}$, this is of not much use. For a meaningful calculation, we would need to choose the same uniformizer in domain and target. Compare this to the analogous situation for vector spaces. If we want to find a matrix representation of a vector space endomorphism, we should of course choose the same basis in domain and target. Otherwise every automorphism can be represented by the identity matrix when taking any basis in the domain and the image of this basis in the target.

Let us now state the evaluation of the spectral sequence for mixed characteristic CDVRs.

Proposition 3.13. *Let R be a mixed characteristic complete discrete valuation ring with perfect residue field and uniformizer π with minimal polynomial E. The topological Hochschild homology groups are then given by*

$$\mathrm{THH}_i(R;\mathbb{Z}_p) = \begin{cases} R & \text{for } n = 0 \\ R/nE'(\pi) & \text{for } i = 2n-1, n \geq 1 \\ 0 & \text{else} \end{cases}$$

In the case $R = \mathbb{Z}_p$, we can make this more explicit using the Chinese remainder theorem[11]: $\mathrm{THH}_{2n-1}(\mathbb{Z}_p;\mathbb{Z}_p) = \mathbb{Z}_p/n\mathbb{Z}_p = \mathbb{Z}/p^{\nu_p(n)}$, where ν_p denotes the p-adic valuation. Emplyoing a fracture square, this can also be used to recover Bökstedt's calculation of $\mathrm{THH}_*(\mathbb{Z})$.

Proof of Proposition 3.13. By Lemma 3.11 and the multiplicativity of the spectral sequence we know that $d^2(x^n) = nd^2(x)x^{n-1} = nE'(\pi)x^{n-1}dz$. All higher differentials vanish for degree reasons, so the E^3-page is already the E^∞-page,

[11]together with knowledge of the quotients: $\mathbb{Z}_p/n = 0$ for $p \nmid n$ and $\mathbb{Z}_p/p^k = \mathbb{Z}/p^k$.

which takes the form

$$
\begin{array}{|cccccc}
0 & 0 & 0 & 0 & 0 & \cdots \\
R/E'(\pi) & 0 & R/2E'(\pi) & 0 & R/3E'(\pi) & \cdots \\
R & 0 & 0 & 0 & 0 & \cdots \\
\hline
\end{array}
$$

There is no room for extensions and we can directly read of the result. ☐

Remark 3.14. *The above result was first obtained in [LM00, Theorem 5.1] using an entirely different spectral sequence due to Brun. They describe it in terms of the* ***inverse different***

$$\mathrm{THH}_{2n-1}(R;\mathbb{Z}_p) = \mathcal{D}_R^{-1}/n.$$

The inverse different $\mathcal{D}_R^{-1}$ of R (or more precisely, the inverse different relative to the Witt vectors of the residue field $\mathcal{D}_{R/W(k)}^{-1}$) is a fractional ideal of R. As the name suggests it is the inverse fractional ideal of the different $\mathcal{D}_R$, which is an ideal of R. The different can be defined as the annihilator ideal of the module of Kähler differentials (see [Ser95, Chapter 3.7])

$$\mathcal{D}_R \coloneqq \mathrm{Ann}_R(\Omega^1_{R/\mathbb{Z}_p})$$

We can rewrite the annihilator ideal to make its functoriality more clear

$$\mathrm{Ann}_R(\Omega^1_{R/\mathbb{Z}_p}) = \ker\left(R \to \mathrm{End}(\Omega^1_{R/\mathbb{Z}_p})\right).$$

Here we consider elements of R as endomorphisms via their multiplication action. Since $M \mapsto \mathrm{End}(M) = \mathrm{Hom}(M,M)$ is a combination of a covariant and a contravariant functor, it is only functorial in automorphisms. Thus the description $\mathrm{THH}_{2n-1}(R;\mathbb{Z}_p) = \mathcal{D}_R^{-1}/n$ only has the chance to be natural for automorphisms $R \xrightarrow{\sim} R$. But even in this case, both sides behave completely differently. In degree 1, we have $\mathrm{THH}_1(R;\mathbb{Z}_p) = \mathrm{HH}_1(R;\mathbb{Z}_p) = \Omega^1_{R/\mathbb{Z}_p}$. Thus for an automorphism $\varphi : R \to R$ the induced map on Kähler differentials is given by the derivative $\mathrm{d}\varphi$, while the map $\mathcal{D}_R \to \mathcal{D}_R$ is simply the restriction of φ to $\mathcal{D}_R$. For a concrete counterexample, consider the automorphism $\varphi : \mathbb{Z}_p[\zeta_p] \to \mathbb{Z}_p[\zeta_p],\ \zeta_p \mapsto \zeta_p^2$ (for $p \neq 2$). This shows that the isomorphism $\mathrm{THH}_1(R;\mathbb{Z}_p) \cong \mathcal{D}_R^{-1}/R$ cannot be a natural isomorphism.
Nevertheless the description using the inverse different, while not functorial, is still useful because it allows us to deduce statements for THH *of rings of integers*

in number fields. For these rings, the definition of the different as the annihilator ideal of the Kähler differentials still makes sense if we take the Kähler differentials relative[12] *to $\mathbb{Z}$ instead of $\mathbb{Z}_p$. It is then possible to express this different ideal in terms of the different ideals of all the completions of the ring of integers. This allowed Lindenstrauss and Madsen to give the following expression: For A the ring of integers in a number field, we have*

$$\mathrm{THH}_{2n-1}(A) = \mathcal{D}_A^{-1}/nA.$$

For the case of quotients $R' = R/\pi^n$, we can initially proceed similarly: Construction 3.10 gives us a spectral sequence which by Proposition 3.9 takes the form

$$E^2 = R'[x]\langle y\rangle \otimes_{\mathbb{Z}_p} \Lambda_{\mathbb{Z}_p}(dz) \Rightarrow \mathrm{THH}_*(R').$$

We can again identify the d^2 differential, which is the only one that can occur

$$d^2(x) = E'(\pi)dz,\ d^2(y) = k\pi^{k-1}dz.$$

Thus we get an explicit differential graded algebra whose homology computes $\mathrm{THH}_*(R')$. But depending on n there is no longer a closed-form expression for its homology. For explicit examples see [KN19, Chapter 6].

[12] This should then more appropriately be called the diffent relative to $\mathbb{Z}$, while the different we discussed before is the different relative to $\mathbb{Z}_p$ or the Witt vectors of the residue field.

4. Functoriality of THH for CDVRs

In this chapter, we will investigate functorial computations of (topological) Hochschild homology for CDVRs.
In the first section we give a fully functorial description of the Hochschild homology groups.
In the second section we employ the Tor spectral sequence. We can compute the E^2-page and understand some aspects of how the differentials and extension problems have to work out. Several indeterminacies and restrictions remain, but we can for example say something in the case of tamely ramified CDVRs.
In the third and last section we discuss a 'partial' functoriality of the descent spectral sequence from section 3.4, which allows us to compute THH for 'monomial' maps of CDVRs.

4.1 Functoriality of HH for CDVRs

In this section we will deal with the functoriality of Hochschild homology for CDVRs. In the same vein as Chapter 3, the computation of absolute Hochschild homology will first pass through an auxiliary relative result. While achieving this, we need to be attentive to the functoriality. So what is the correct domain category for $\mathrm{HH}(-/\mathbb{Z}_p[z])$? The first answer one might come up with is probably the category of $\mathbb{Z}_p[z]$-algebras. But as we have previously seen, this is not suitable for our purposes, because the morphism sets are too restrictive. Given two CDVRs, we want to talk about arbitrary maps between them, but only maps that preserve choices of uniformizers (which give the $\mathbb{Z}_p[z]$-algebra structures) are $\mathbb{Z}_p[z]$-algebra homomorphisms. We will therefore now introduce a more refined category, which allows us to also consider these maps.
For ease of notation, we first only deal with CDVRs with residue field $\mathbb{F}_p$, i.e. purely ramified extensions of $\mathbb{Z}_p$. We discuss the general case in Remark 4.5.
Let $\mathrm{CDVR}^{\mathrm{Lift}}$ be the category, whose *objects* are given by pairs (R, π), where R is a complete discrete valuation rings of mixed characteristic with residue field $\mathbb{F}_p$ and π is a uniformizer of R. The element π determines a surjective map $ev_\pi : \mathbb{Z}_p[z] \to R,\ z \mapsto \pi$. A *morphism* between two objects (R, π) and (S, ε) is now given by a pair (φ, f), where $\varphi : R \to S$ and $f : \mathbb{Z}_p[z] \to \mathbb{Z}_p[w]$ are $\mathbb{Z}_p$-algebra

homomorphisms that make the following diagram commute.

$$\begin{tikzcd} \mathbb{Z}_p[z] \arrow[r, "f"] \arrow[d, "ev_\pi"] & \mathbb{Z}_p[w] \arrow[d, "ev_\varepsilon"] \\ R \arrow[r, "\varphi"] & S \end{tikzcd}$$

The commutativity expresses that f is a *lift* of φ, hence the name for the category. Giving the map $f : \mathbb{Z}_p[z] \to \mathbb{Z}_p[w]$ is equivalent to providing a polynomial in $\mathbb{Z}_p[w]$, which we abusively also call f. The map is then given by $g(z) \mapsto g(f(w))$, i.e. by substitution.
Abstractly, this category is the full subcategory of the arrow category of $\mathbb{Z}_p$-algebras on objects of the form $\mathbb{Z}_p[z] \to R$, where R is a mixed characteristic CDVR with residue field $\mathbb{F}_p$ and the map sends z to a uniformizer of R. Several constructions/computations that we will now make will only be functorial in this input category. In particular $\mathrm{HH}_*(-/\mathbb{Z}_p[z])$ canonically provides a functor from $\mathrm{CDVR}^{\mathrm{Lift}}$ to graded $\mathbb{Z}_p$-algebras. There is an evident forgetful functor U from $\mathrm{CDVR}^{\mathrm{Lift}}$ to CDVR, the full subcategory of $\mathbb{Z}_p$-algebras with objects mixed characteristic complete discrete valuation rings with residue field $\mathbb{F}_p$. (p-completed) Hochschild homology can be considered as a functor $\mathrm{HH}_*(-; \mathbb{Z}_p) : \mathrm{CDVR} \to \mathrm{gr}\,\mathbb{Z}_p\text{-Alg}$ with values in the category of graded $\mathbb{Z}_p$-algebras. Via precomposition with the forgetful functor we obtain a functor with domain $\mathrm{CDVR}^{\mathrm{Lift}}$. Our main result is a natural computation of $U \circ \mathrm{HH}_*(-; \mathbb{Z}_p)$. We do this by giving a natural isomorphism from $\mathrm{HH}_*(R; \mathbb{Z}_p)$ to the homology of a very easy dga. This natural isomorphism is an equivalence of functors $\mathrm{CDVR}^{\mathrm{Lift}} \to \mathrm{gr}\,\mathbb{Z}_p\text{-Alg}$. Since Hochschild homology is independent of the choice of lift, the homology of the dga is necessarily also independent and hence descends to a functor $\mathrm{CDVR} \to \mathrm{gr}\,\mathbb{Z}_p\text{-Alg}$. The upshot is that our dga description can be used to functorially compute (p-completed) Hochschild homology by first choosing uniformizers and then choosing a lift of a given map, but the such computed map on Hochschild homology does not depend of the choice of uniformizers or lift.
The proof of the main result consists of the following steps:

- Functorially compute the second relative Hochschild homology $\mathrm{HH}_2(R/\mathbb{Z}[z]; \mathbb{Z}_p)$.
- Establish that the (full) relative Hochschild homology is given by the divided power algebra of the degree 2 part: $\mathrm{HH}_*(R/\mathbb{Z}[z]; \mathbb{Z}_p) = \Gamma_R(\mathrm{HH}_2(R/\mathbb{Z}[z]; \mathbb{Z}_p))$.
- Use a descent spectral sequence as in Section 3.4 to obtain the absolute Hochschild homology from the relative term.

For the first two steps in the program, we will use the HKR-filtration, which is expressed in terms of **nonabelian derived functors**. They are a generalization of ordinary derived functors, which have the advantage that they also work in

non-additive settings, i.e. for non-additive categories or non-additive functors. This originally goes back to [DP61], for a modern formulation see [Lur09, Section 5.5.8] or [CS19, Section 5.1.4], where the terminology *animation* is used. The main idea is to use simplicial resolutions, which make sense in every category, instead of resolutions by chain complexes, which are only sensible in an additive setting[1].
Two functors that we want to derive are the non-additive functor $\Lambda^n_R : \mathrm{Mod}_R \to \mathrm{Mod}_R$ (here we really mean an ordinary ring and the 1-category of ordinary modules over it) and Kähler differentials $\Omega^1_{-} : \mathrm{CRing} \to \mathrm{Ab}$ (note that Ω^1_R has a R-module structure). Deriving these provides (∞-) functors $L\Lambda^n_R : D(R)_{\geq 0} \to D(R)_{\geq 0}$ and $\mathbb{L}_{-} : s\,\mathrm{CRing} \to D(\mathbb{Z})_{\geq 0}$.
Now recall that the HKR-filtration is a natural, multiplicative filtration on relative Hochschild homology $\mathrm{HH}(R/S)$, with associated graded given by $\mathrm{gr}^n_{\mathrm{HKR}} = (L\Lambda^n_R \mathbb{L}_{R/S})[n]$ (see eg. [NS18, Theorem IV.4.1]).
Using this, we can now give a proof of the following functorial computation of relative Hochschild homology.

Proposition 4.1. *Let $(R,\pi) \in \mathrm{CDVR}^{\mathrm{Lift}}$ and $I = \ker(\mathbb{Z}_p[z] \to R)$ the kernel of the π-substitution morphism. Then we have a natural isomorphism of functors from $\mathrm{CDVR}^{\mathrm{Lift}}$ to $\mathrm{gr}\,\mathbb{Z}_p$-Alg*

$$\mathrm{HH}_*(R/\mathbb{Z}[z];\mathbb{Z}_p) = \mathrm{HH}_*(R/\mathbb{Z}_p[z]) = \Gamma_R(I/I^2).$$

Let us prepare this result, by first establishing the computation in the second degree.

Lemma 4.2. *Let $(R,\pi) \in \mathrm{CDVR}^{\mathrm{Lift}}$ and $I = \ker(\mathbb{Z}_p[z] \to R)$. For the second Hochschild homology, we have the following natural isomorphism of functors from $\mathrm{CDVR}^{\mathrm{Lift}}$ to $\mathbb{Z}_p$-Mod.*

$$\mathrm{HH}_2(R/\mathbb{Z}_p[z]) = I/I^2$$

Proof. Because of the presentation $R = \mathbb{Z}_p[z]/I$ we naturally have $\mathbb{L}_{R/\mathbb{Z}_p[z]} = I/I^2[1]$ (see e.g. [Stacks, Tag 08SJ]). Now the HKR-filtration provides us with a natural isomorphism $\mathrm{HH}_2(R/\mathbb{Z}_p[z]) = \mathbb{L}_{R/\mathbb{Z}_p[z]}$, thus yielding the result. □

The module I/I^2 is isomorphic to R. To describe the isomorphism, let $E \in \mathbb{Z}_p[z]$ be the minimal polynomial of π. Then E generates the ideal $I = (E)$, so we also have $\mathrm{HH}_2(R/\mathbb{Z}_p[z]) = (E)/(E^2)$. The isomorphism is now given by $R \xrightarrow{\sim} I/I^2, 1 \mapsto [E]$, with inverse $I/I^2 \xrightarrow{\sim} R, [g] \mapsto (g/E)(z = \pi)$.
The reason that we write I/I^2 instead of R, is to indicate the correct functoriality. Let us be completely explicit about this: Assume $(\varphi, f) : (R,\pi) \to (S,\varepsilon)$ is a morphism in $\mathrm{CDVR}^{\mathrm{Lift}}$. Let $E \in \mathbb{Z}_p[z]$ and $F \in \mathbb{Z}_p[w]$ be minimal polynomials of

[1] In the case of non-additive functors between additive categories, we can obtain a simplicial resolution by applying the Dold-Kan correspondence to a projective resolution.

π and ε respectively. Under the identifications $R \simeq (E)/(E^2)$ and $S \simeq (F)/(F)^2$, the induced map on the relative second Hochschild homology

$$R \xrightarrow{\sim} (E)/(E)^2 \simeq \mathrm{HH}_2(R/\mathbb{Z}_p[z]) \to \mathrm{HH}_2(S/\mathbb{Z}_p[w]) \simeq (F)/(F)^2 \xrightarrow{\sim} S$$

is given by sending $1 \mapsto [E(z)] \mapsto [E(f(w))] \mapsto \frac{E(f(w))}{F(w)}(w = \varepsilon)$. This uniquely determines the map by requiring that it is φ-semilinear[2], i.e. we send $R \ni r \mapsto \varphi(r)\frac{E(f(w))}{F(w)}(w = \varepsilon)$.

Proof of Proposition 4.1. The first isomorphism follows from Lemma 2.4.5) since R is p-complete and of finite type over $\mathbb{Z}_p[z]$.
For the second isomorphism: By [Ill06, §5.4.3], we have a natural isomorphism $L\Lambda^n_R(I/I^2[1]) = \Gamma^n_R(I/I^2)[n]$. Thus the spectral sequence induced from the HKR-filtration looks as follows

$$\begin{array}{ccccc}
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \Gamma^2_R(I/I^2) & \cdots \\
0 & 0 & \Gamma_R(I/I^2) & 0 & \cdots \\
0 & I/I^2 & 0 & 0 & \cdots \\
R & 0 & 0 & 0 & \cdots
\end{array}$$

There is no room for differentials and it degenerates on the E^1-page. We also do not have extension problems. The isomorphisms $L\Lambda^n_R(I/I^2[1]) = \Gamma^n_R(I/I^2)[n]$ for varying n are multiplicatively compatible, i.e. we have $L\Lambda^*_R(I/I^2[1]) = \Gamma_R(I/I^2)$. This determines the multiplicative structure on the $E^1 = E^\infty$-page and hence gives the result. □

Let us again unravel, what this means concretely. The R-module I/I^2 is free of rank 1 with generator provided by the class of E, the minimal polynomial of the uniformizer. Let us instead denote this generator by $x := [E] \in (E)/(E)^2$. We can then identify Hochschild homology with the free divided power algebra on x: $\mathrm{HH}_*(R/\mathbb{Z}_p[z]) \simeq \Gamma_R\{x\}$. Let now $(\varphi, f) : (R, \pi) \to (S, \varepsilon)$ be a morphism

[2]Recall: Let R, S be rings, M a module over R, N a module over S and $\varphi : R \to S$ a map of rings. Then an additive map $f : M \to N$ is called φ-semilinear if $f(rm) = \varphi(r)f(m)$ for all $m \in M$, and $r \in R$.

in $\mathrm{CDVR}^{\mathrm{Lift}}$ and $E \in \mathbb{Z}_p[z], F \in \mathbb{Z}_p[w]$ minimal polynomials of π, ε. The induced map on relative Hochschild homology

$$\Gamma_R\{x\} \simeq \mathrm{HH}_*(R/\mathbb{Z}_p[z]) \to \mathrm{HH}_*(S/\mathbb{Z}_p[w]) \simeq \Gamma_S\{y\}$$

is then uniquely determined by sending $x^{[n]}$, the n-the divided power of x, to $\left(\frac{E(f(w))}{F(w)}(w = \varepsilon)\right)^n y^{[n]}$ and being φ-semilinear.

Now we want to obtain the absolute Hochschild homology from the relative term. For this we proceed in the same way as in Section 3.4. Again, we have a descent spectral sequence for obtaining $\mathrm{HH}(R;\mathbb{Z}_p)$ out of $\mathrm{HH}(R/\mathbb{Z}_p[z])$. This works out as follows: We use the trivial isomorphism $\mathrm{HH}(R;\mathbb{Z}_p) = \mathrm{HH}(R;\mathbb{Z}_p) \otimes_{\mathrm{HH}(\mathbb{Z}_p[z]/\mathbb{Z}_p)} \mathrm{HH}(\mathbb{Z}_p[z]/\mathbb{Z}_p)$. Then we filter $\mathrm{HH}(\mathbb{Z}_p[z]/\mathbb{Z}_p)$ by its Whitehead tower $\tau_{\geq \bullet} \mathrm{HH}(\mathbb{Z}_p[z]/\mathbb{Z}_p)$ and tensor this up to obtain the filtration $\mathrm{HH}(R;\mathbb{Z}_p) \otimes_{\mathrm{HH}(\mathbb{Z}_p[z]/\mathbb{Z}_p)} \tau_{\geq \bullet} \mathrm{HH}(\mathbb{Z}_p[z]/\mathbb{Z}_p)$ on $\mathrm{HH}(R;\mathbb{Z}_p)$. The associated graded of this is given by:

$$\begin{aligned}
\mathrm{gr}^j &= \mathrm{HH}(R;\mathbb{Z}_p) \otimes_{\mathrm{HH}(\mathbb{Z}_p[z]/\mathbb{Z}_p)} \Omega^j_{\mathbb{Z}_p[z]/\mathbb{Z}_p}[j] \\
&= \big(\mathrm{HH}(R;\mathbb{Z}_p) \otimes_{\mathrm{HH}(\mathbb{Z}_p[z]/\mathbb{Z}_p)} \mathbb{Z}_p[z]\big) \otimes_{\mathbb{Z}_p[z]} \Omega^j_{\mathbb{Z}_p[z]/\mathbb{Z}_p}[j] \\
&\overset{3.4}{=} \mathrm{HH}(R/\mathbb{Z}_p[z]) \otimes_{\mathbb{Z}_p[z]} \Omega^j_{\mathbb{Z}_p[z]/\mathbb{Z}_p}[j]
\end{aligned}$$

The descent spectral sequence associated to this filtration is natural in $\mathrm{CDVR}^{\mathrm{Lift}}$. It takes the form

$$E^2_{i,j} = \mathrm{HH}_i(R/\mathbb{Z}_p[z]) \otimes_{\mathbb{Z}_p[z]} \Omega^j_{\mathbb{Z}_p[z]/\mathbb{Z}_p} \Rightarrow \mathrm{HH}_{i+j}(R;\mathbb{Z}_p).$$

Due to degree reasons, the spectral sequence degenerates at the E^3-page. Thus we can identify the absolute (p-completed) Hochschild homology with the homology of the dga given by the E^2-page of the spectral sequence equipped with the d^2-differential. I.e. we get a natural isomorphism of functors $\mathrm{CDVR}^{\mathrm{Lift}} \to \mathrm{gr}\,\mathbb{Z}_p$-Alg

$$\mathrm{HH}_*(R/\mathbb{Z}_p) = H_*(\Gamma_R(I/I^2) \otimes_{\mathbb{Z}_p[z]} \Omega^*_{\mathbb{Z}_p[z]/\mathbb{Z}_p}, \partial = d^2)$$

Because the left hand side is even a functor on CDVR, the right hand side also descends to one. Since the forgetful functor $\mathrm{CDVR}^{\mathrm{Lift}} \to \mathrm{CDVR}$ is full[3] this allows us to compute the effect of Hochschild homology by choosing an arbitrary lift and then compute the functoriality via the homology of the dga.

The final thing, that remains to be done, is identifying the homology of the dga.

Theorem 4.3. *Let $(R, \pi) \in \mathrm{CDVR}^{\mathrm{Lift}}$, $E \in \mathbb{Z}_p[z]$ the minimal polynomial of π and let $x := [E] \in (E)/(E)^2 = \mathrm{HH}_2(R/\mathbb{Z}_p[z])$. The differential of the descent spectral sequence $d^2 : R\{x^{[n+1]}\} \to R\{x^{[n]}dz\}$ is injective and has image given by the principal ideal generated by $E'(\pi)x^{[n]}dz$. The nonzero Hochschild homology groups are therefore $\mathrm{HH}_0(R;\mathbb{Z}_p) = R$ and $\mathrm{HH}_{2n+1} \simeq R/(E'(\pi))$*

[3]This holds because, $\mathbb{Z}_p[z] \to R$ is surjective, so we can lift every map $\varphi : R \to S$ to a map $\mathbb{Z}_p[z] \to \mathbb{Z}_p[w]$.

Regarding the functoriality: Suppose, we are given another (S, ε)*, a map* $(\varphi, f) : (R, \pi) \to (S, \varepsilon)$ *in* $\mathrm{CDVR}^{\mathrm{Lift}}$*,* $F \in \mathbb{Z}_p[w]$ *the minimal polynomial of* ε *and denote* $y := [F] \in \mathrm{HH}_2(S; \mathbb{Z}_p[w])$*. The functoriality is then completely encoded in the commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & R\{x^{[n+1]}\} \arrow[r, "d^2"] \arrow[d] & R\{x^{[n]}dz\} \arrow[r, two heads] \arrow[d] & \mathrm{HH}_{2n+1}(R; \mathbb{Z}_p) \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & S\{y^{[n+1]}\} \arrow[r, "d^2"] & S\{y^{[n]}dw\} \arrow[r, two heads] & \mathrm{HH}_{2n+1}(S; \mathbb{Z}_p) \arrow[r] & 0
\end{tikzcd}$$

The map $R\{x^{[n]}dz\} \to S\{y^{[n]}dw\}$ *is given as the unique* φ*-semilinear map that acts on* $x^{[n]}dz$ *as*

$$x^{[n]}dz \mapsto \left(\frac{E(f(w))}{F(w)} (w = \varepsilon) \right)^n f'(\varepsilon) y^{[n]} dw.$$

Proof. Let us start with a picture of the spectral sequence.

$$\begin{tikzcd}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots \\
R\{dz\} & 0 & R\{xdz\} & 0 & R\{x^{[2]}dz\} & \cdots \\
R & 0 & R\{x\} \arrow[ull, "d^2"] & 0 & R\{x^{[2]}\} \arrow[ull, "d^2"] & \cdots
\end{tikzcd}$$

We see that it suffices to understand the effect of d^2 on $x, x^{[2]}, x^{[3]}, \ldots$. Knowing that $\mathrm{HH}_1(R; \mathbb{Z}_p) \simeq R/E'(\pi)$, we obtain that d^2 must send x to a generator of the ideal generated by $E'(\pi)$, i.e. $d^2(x) = uE'(\pi)$ for some $u \in R^\times$. By the Leibniz rule, we then also get $d^2(x^n) = nd^2(x)x^{n-1}$. But we need the differential on divided powers of x and not of the actual powers. We claim that d^2 is a PD-derivation, i.e. $d^2(x^{[n]}) = x^{[n-1]}d^2(x)$. In fact, this follows from multiplicativity:

$$\begin{aligned}
0 &= d^2(x^{[n]}) - x^{[n-1]}d^2(x) \\
\iff 0 &= n!(d^2(x^{[n]}) - x^{[n-1]}d^2(x)) \\
&= d^2(n!x^{[n]}) - n((n-1)!x^{[n-1]})d^2(x) \\
&= d^2(x^n) - nx^{n-1}d^2(x)
\end{aligned}$$

The equivalence holds, because all involved modules are torsion-free. This proves the first part of the theorem.

For the functoriality: We get the map of short exact sequences from the functoriality of the descent spectral sequence. The map $R\{x^{[n]}dz\} \to S\{y^{[n]}dw\}$

is the tensor product of the map $\mathrm{HH}_{2n}(R/\mathbb{Z}_p[z]) \to \mathrm{HH}_{2n}(S/\mathbb{Z}_p[w])$ and the map $\Omega^1_{\mathbb{Z}_p[z]} \xrightarrow{df} \Omega^1_{\mathbb{Z}_p[w]/\mathbb{Z}_p}$. We know the effect of the first map from Proposition 4.1 and the map on Kähler differentials works as the formal derivative, i.e. $dz \mapsto d(f(w)) = f'(w)dw$. The tensor product of these two maps is exactly the map we claimed in the theorem. □

Let us also observe that, by the result of the last section, this already allows us to naturally understand the first few homotopy groups of $\mathrm{THH}(R;\mathbb{Z}_p)$.

Lemma 4.4. *For all (ordinary) rings R, the map $\mathrm{THH}(R;\mathbb{Z}_p) \to \mathrm{HH}(R;\mathbb{Z}_p)$ induces a natural isomorphism on homotopy groups in degrees $< 2p-1$*

$$\mathrm{THH}_i(R;\mathbb{Z}_p) \xrightarrow{\sim} \mathrm{HH}_i(R;\mathbb{Z}_p), \text{ for } i = 0, 1, \dots 2p-2$$

Proof. Proposition 3.4 allows us to write the map as

$$\mathrm{THH}(R;\mathbb{Z}_p)\otimes_{\mathrm{THH}(\mathbb{Z}_p;\mathbb{Z}_p)}\mathrm{THH}(\mathbb{Z}_p;\mathbb{Z}_p) \to \mathrm{THH}(R;\mathbb{Z}_p)\otimes_{\mathrm{THH}(\mathbb{Z}_p;\mathbb{Z}_p)}\mathbb{Z}_p \overset{3.4}{=} \mathrm{HH}(R;\mathbb{Z}_p).$$

We know by Proposition 3.13 that the map $\mathrm{THH}(\mathbb{Z}_p;\mathbb{Z}_p) \to \mathbb{Z}_p$ is an isomorphism on π_i for $i = 0, \dots 2p-2$ and hence also $\mathrm{THH}(R;\mathbb{Z}_p) \to \mathrm{HH}(R;\mathbb{Z}_p)$ is because tensoring can only increase connectivity. □

Together with the content of the last section this functorially identifies the first $2p-1$ homotopy groups of $\mathrm{THH}(R;\mathbb{Z}_p)$. This is the best that we can hope for. In the next degree we only get a natural surjection but not an isomorphism, we e.g. have $\mathrm{THH}_{2p+1}(\mathbb{Z}_p;\mathbb{Z}_p) = \mathbb{F}_p$ but $\mathrm{HH}_{2p+1}(\mathbb{Z}_p;\mathbb{Z}_p) = 0$.

Remark 4.5. *Let us now indicate, which modification we need to make in the general case of an arbitrary perfect residue field of positive charactersitic. We redefine $\mathrm{CDVR}^{\mathrm{Lift}}$ to be the full subcategory of the arrow category of $\mathbb{Z}_p$-algebras on objects of the form $W(k)[z] \to R$. Here R is a mixed characteristic complete discrete valuation ring with perfect residue field k and we require that the restriction $W(k)[z] \supset W(k) \to R$ is the unique map over k (with respect to the canonical projections $W(k) \to k \leftarrow R$) and that the map sends z to a uniformizer of R. Again, this data is equivalent to the choice of an R and a uniformizer. Maps from (R,π) to (S,ε) are pairs (φ, f), where $\varphi : R \to S$ and $f : W(R/\pi)[z] \to W(S/\varepsilon)[w]$ are $\mathbb{Z}_p$-algebra homomorphisms, such that the evident diagram commutes.*
The functor $\mathrm{HH}(-/\mathbb{Z}[z];\mathbb{Z}_p) : \mathrm{CDVR}^{\mathrm{Lift}} \to \mathrm{gr}\,\mathbb{Z}_p\text{-Alg}$ is now no longer naturally isomorphic to $\mathrm{HH}(-/\mathbb{Z}_p[z])$ but instead to the functor $(R,\pi) \mapsto \mathrm{HH}(R/W(R/\pi)[z])$. To see this, we use the p-completed isomorphism from Proposition 3.4:

$$\mathrm{HH}(R/W(R/\pi)[z]) = \mathrm{HH}(R/\mathbb{Z}[z];\mathbb{Z}_p) \otimes_{\mathrm{HH}(W(R/\pi)[z]/\mathbb{Z}[z];\mathbb{Z}_p)} W(R/\pi)[z]$$

Now we need to show that $\mathrm{HH}(W(R/\pi)[z]/\mathbb{Z}[z];\mathbb{Z}_p) = W(R/\pi)[z][0]$. By Lemma 2.2, we can check this on mod p-reduction. This holds using the base change formula again and that $k := R/\pi$ is perfect, hence $\mathrm{HH}(k/\mathbb{F}_p) = k[0]$ (see the proof

of Theorem 3.6).
We can then prove Proposition 4.1 in exactly the same way using the presentation $R = W(R/\pi)[z]/I$, which gives us the cotangent complex $\mathbb{L}_{R/W(R/\pi)[z]} = I/I^2$. The descent spectral sequence construction associates to $(R, \pi) \in \mathrm{CDVR}^{\mathrm{Lift}}$ with $R/\pi = k$ the spectral sequence

$$\mathrm{HH}_i(R/W(k)[z]) \otimes_{W(k)[z]} \Omega^j_{W(k)[z]/W(k)} \Rightarrow \mathrm{HH}_{i+j}(R; \mathbb{Z}_p).$$

We identify the differential in the same way and thus obtain the natural dga-description. This again gives us that the positive odd homotopy groups of $\mathrm{HH}(R; \mathbb{Z}_p)$ are given by $R/E'(\pi)$ for E the minimal polynomial of π. Regarding the functoriality: For a map $(\varphi, f) : (R, \pi) \to (S, \varepsilon)$ with $k = R/\pi$ and $l = S/\varepsilon$ the following diagram encodes the functoriality of p-completed Hochschild homology.

$$\begin{tikzcd}
{} \arrow[r, "d^2"] & \mathrm{HH}_{2n}(R/W(k)[z]) \otimes_{W(k)[z]} \Omega^1_{W(k)[z]/W(k)} = R\{x^{[n]}dz\} \arrow[d] \arrow[r, two heads] & \mathrm{HH}_{2n+1}(R; \mathbb{Z}_p) \arrow[d, dashed] \\
{} \arrow[r, "d^2"] & \mathrm{HH}_{2n}(S/W(l)[w]) \otimes_{W(l)[w]} \Omega^1_{W(l)[w]/W(l)} = S\{y^{[n]}dw\} \arrow[r, two heads] & \mathrm{HH}_{2n+1}(S; \mathbb{Z}_p)
\end{tikzcd}$$

Here it is maybe worth pointing out that in the case of a general residue field, there are multiple maps $W(k) \to W(l)$ (which are all necessarily inclusions), while there is only one ring map $\mathbb{Z}_p \to \mathbb{Z}_p$ in the case of the fixed residue field $\mathbb{F}_p$. The inclusion $W(k) \to W(l)$ that is relevant for us, is induced by the map $k \to l$ which is itself induced by the map $\varphi : R \to S$.

Remark 4.6. *In our discussion above we actually nowhere used that in the presentation $R = W(k)/E(z)$, the polynomial E is Eisenstein. All results hold verbatim for arbitrary quotients of $W(k)[z]$ by principal ideals. We do not know whether this encompasses any interesting cases that cannot be reduced to CDVRs.*

4.2 Functoriality of THH via the Tor spectral sequence

In this section we try to employ the Tor spectral sequence to understand the functoriality of THH for CDVRs. See [Lur17, Proposition 7.2.1.19] for a statement of the Tor spectral sequence for module spectra over a ring spectrum. Let R be a mixed characteristic complete discrete valuation ring with perfect residue field of characteristic p. The following are the main steps in the computation.

1. Write R as a pushout $R = \mathbb{Z}_p[z] \otimes_{\mathbb{Z}_p[z]} \mathbb{Z}_p$ via the action through the Eisenstein polynomial.

2. THH preserves this pushout, i.e. we have $\mathrm{THH}(R) = \mathrm{THH}(\mathbb{Z}_p[z]) \otimes_{\mathrm{THH}(\mathbb{Z}_p[z])} \mathrm{THH}(\mathbb{Z}_p)$.

3. Use the natural isomorphism $\mathrm{THH}_*(\mathbb{Z}_p[z]) \simeq \mathrm{THH}_*(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \Omega^*_{\mathbb{Z}_p[z]/\mathbb{Z}_p}$.

4. Now employ the Tor spectral sequence for this tensor product $\mathrm{Tor}^i_{\mathrm{THH}_*(\mathbb{Z}_p[z])}(\mathrm{THH}_*(\mathbb{Z}_p[z]), \mathrm{THH}_*(\mathbb{Z}_p))_{(j)} \Rightarrow \mathrm{THH}_{i+j}(R)$
(or rather the p-completed version)

5. Analyze the differentials and extension problems in the spectral sequence.

Using Lemma 3.7 to write all our rings as quotients of $\mathbb{Z}_p[z]$ allows us to deduce computations for R from the computation of $\mathrm{THH}(\mathbb{Z}_p[z])$ and the understanding of the the functoriality of $\mathrm{THH}(\mathbb{Z}_p[z])$ under self maps of $\mathbb{Z}_p[z]$. We do not understand the functoriality of $\mathrm{THH}(\mathbb{Z}_p[z])$ on the level of spectra, but for homotopy groups we have a relative Hochschild-Kostant-Rosenberg style result.

Lemma 4.7. *We have an equivalence*

$$\mathrm{THH}(\mathbb{Z}_p[z]) \simeq \mathrm{THH}(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathrm{HH}(\mathbb{Z}_p[z]).$$

This is not a natural equivalence of spectra. But the resulting isomorphism on homotopy groups $\mathrm{THH}_*(\mathbb{Z}_p[z]) \simeq \mathrm{THH}_*(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \Omega^*_{\mathbb{Z}_p[z]/\mathbb{Z}_p}$ *is a natural isomorphism (in ring maps* $\mathbb{Z}_p[z] \to \mathbb{Z}_p[z]$*).*

Proof. Using the basic properties of THH and the equivalence $\mathbb{Z}_p[z] \simeq \mathbb{Z}_p \otimes_{\mathbb{S}} \mathbb{S}[z]$ we obtain

$$\begin{aligned}
\mathrm{THH}(\mathbb{Z}_p[z]) \simeq \mathrm{THH}(\mathbb{Z}_p \otimes_{\mathbb{S}} \mathbb{S}[z]) &\simeq \mathrm{THH}(\mathbb{Z}_p) \otimes_{\mathbb{S}} \mathrm{THH}(\mathbb{S}[z]) \\
&\simeq \mathrm{THH}(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} (\mathbb{Z}_p \otimes \mathrm{THH}(\mathbb{S}[z])) \\
&\simeq \mathrm{THH}(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} (\mathrm{THH}(\mathbb{Z}_p \otimes \mathbb{S}[z]/\mathbb{Z})) \\
&\simeq \mathrm{THH}(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathrm{HH}(\mathbb{Z}_p[x])
\end{aligned}$$

Regarding the naturality: By [HN20, Section 15.1.3] and because $\mathrm{THH}(\mathbb{Z}_p[z])$ is $H\mathbb{Z}$-linear we have a natural differential $d : \mathrm{THH}_*(\mathbb{Z}_p[z]) \to \mathrm{THH}_*(\mathbb{Z}_p[z])$. Composing this differential with the map $\mathbb{Z}_p[z] \to \mathrm{THH}_*(\mathbb{Z}_p[z])$, we obtain a derivation with domain $\mathbb{Z}_p[z]$ and target a strictly graded-commutative algebra. By the universal property of Kähler differentials, we thus get a natural map $\Omega^*_{\mathbb{Z}_p[z]/\mathbb{Z}_p} \to \mathrm{THH}_*(\mathbb{Z}_p[z])$. We also have a natural map $\mathrm{THH}_*(\mathbb{Z}_p) \to \mathrm{THH}_*(\mathbb{Z}_p[z])$ induced by $\mathbb{Z}_p \to \mathbb{Z}_p[z]$. Combining both of them yields a natural map

$$\mathrm{THH}_*(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \Omega^*_{\mathbb{Z}_p[z]/\mathbb{Z}_p} \to \mathrm{THH}_*(\mathbb{Z}_p[z]).$$

And we know by the above that it is an isomorphism. $\square$

Taking the p-completion yields

$$\mathrm{THH}_*(\mathbb{Z}_p[z]; \mathbb{Z}_p) \simeq \mathrm{THH}_*(\mathbb{Z}_p; \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \Omega_{\mathbb{Z}_p[z]/\mathbb{Z}_p}.$$

Let us now apply this result. For this and for the rest of the section let R be a mixed-characteristic complete discrete discrete valuation ring with residue field[4] $\mathbb{F}_p$. By Lemma 3.7 we can find an Eisenstein polynomial $E(z)$ such that

$$R = \mathbb{Z}_p[z]/E(z) = \mathbb{Z}_p[z] \otimes_{\mathbb{Z}_p[z]} \mathbb{Z}_p,$$

with actions prescribed by the maps $\mathbb{Z}_p[z] \to \mathbb{Z}_p[z], z \mapsto E(z)$ and $\mathbb{Z}_p[z] \to \mathbb{Z}_p, z \mapsto 0$.
Using the monoidality of THH (Proposition 2.6) we therefore get

$$\mathrm{THH}(R) = \mathrm{THH}(\mathbb{Z}_p[z]) \otimes_{\mathrm{THH}(\mathbb{Z}_p[z])} \mathrm{THH}(\mathbb{Z}_p).$$

To p-complete this, we again only have to p-complete all the involved terms and not on the outside. We then know all the homotopy groups of the terms involved on the right side, but we actually want to know the homotopy groups of $\mathrm{THH}(R; \mathbb{Z}_p)$. For this we can use the Tor spectral sequence. It takes the form

$$E^2_{i,j} = \mathrm{Tor}^i_{\mathrm{THH}_*(\mathbb{Z}_p[z];\mathbb{Z}_p)} (\mathrm{THH}_*(\mathbb{Z}_p[z]; \mathbb{Z}_p), \mathrm{THH}_*(\mathbb{Z}_p; \mathbb{Z}_p))_{(j)} \Rightarrow \mathrm{THH}_{i+j}(R; \mathbb{Z}_p)$$

Importantly, we see that **the E^2-page only contains terms that we functorially know**. To evaluate the graded Tor groups, we have to calculate the homology of the following derived tensor product of graded algebras

$$\begin{aligned}
&\mathrm{THH}_*(\mathbb{Z}_p[z]; \mathbb{Z}_p) \otimes^L_{\mathrm{THH}_*(\mathbb{Z}_p[z];\mathbb{Z}_p)} \mathrm{THH}_*(\mathbb{Z}_p; \mathbb{Z}_p) \\
\overset{4.7}{=} &(\mathrm{THH}_*(\mathbb{Z}_p; \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \Omega^*_{\mathbb{Z}_p[z]/\mathbb{Z}_p}) \otimes^L_{(\mathrm{THH}_*(\mathbb{Z}_p;\mathbb{Z}_p)\otimes_{\mathbb{Z}_p}\Omega^*_{\mathbb{Z}_p[z]/\mathbb{Z}_p})} \mathrm{THH}_*(\mathbb{Z}_p; \mathbb{Z}_p) \\
&= \Omega^*_{\mathbb{Z}_p[z]/\mathbb{Z}_p} \otimes^L_{\Omega^*_{\mathbb{Z}_p[z]/\mathbb{Z}_p}} \mathrm{THH}_*(\mathbb{Z}_p; \mathbb{Z}_p) \\
&= (\Omega^*_{\mathbb{Z}_p[z]/\mathbb{Z}_p} \otimes^L_{\Omega^*_{\mathbb{Z}_p[z]/\mathbb{Z}_p}} \mathbb{Z}_p) \otimes^L_{\mathbb{Z}_p} \mathrm{THH}_*(\mathbb{Z}_p; \mathbb{Z}_p)
\end{aligned}$$

Thus we need to resolve $\mathbb{Z}_p$ as a $\Omega^*_{\mathbb{Z}_p[z]/\mathbb{Z}_p}$-module. A resolution can be provided by a free divided power algebra on an exterior algebra over $\Omega^*_{\mathbb{Z}_p[z]/\mathbb{Z}_p}$, i.e. we have a quasi-isomorphism of graded $\Omega^*_{\mathbb{Z}_p[z]/\mathbb{Z}_p}$-modules

$$\mathbb{Z}_p \xleftarrow{\sim} \Gamma_{\Lambda_{\Omega^*_{\mathbb{Z}_p[z]/\mathbb{Z}_p}}(b)}\{a\},\ |a| = (1, 0),\ |b| = (1, 1),\ \partial(a) = dz,\ \partial(b) = z.$$

Explicitly the resolution looks as follows:

$$\mathbb{Z}_p \leftarrow \Omega^*_{\mathbb{Z}_p[z]/\mathbb{Z}_p} \leftarrow \Omega^*_{\mathbb{Z}_p[z]/\mathbb{Z}_p}\{a, b\} \leftarrow \Omega^*_{\mathbb{Z}_p[z]/\mathbb{Z}_p}\{a^{[2]}, ba\} \leftarrow \dots$$

with the maps (from left to right) given by $[0 \leftarrow\!\shortmid z, dz]$, $[z \leftarrow\!\shortmid b, dz \leftarrow\!\shortmid a]$ and $[az - bdz \leftarrow\!\shortmid ba, adz \leftarrow\!\shortmid a^{[2]}]$. Let us abbreviate $\Gamma := \Gamma_{\Lambda_{\Omega^*_{\mathbb{Z}_p[z]/\mathbb{Z}_p}}(b)}\{a\}$. The E^2-page is then the homology of

$$\mathrm{THH}(\mathbb{Z}_p; \mathbb{Z}_p) \otimes^L_{\mathbb{Z}_p} \left(\Omega^*_{\mathbb{Z}_p[z]/\mathbb{Z}_p} \otimes_{\Omega^*_{\mathbb{Z}_p[z]/\mathbb{Z}_p}} \Gamma\right).$$

[4] Again, this can be generalized to perfect residue fields of characteristic p.

Taking the tensor product with $\Omega^*_{\mathbb{Z}_p[z]/\mathbb{Z}_p} \otimes_{\Omega^*_{\mathbb{Z}_p[z]/\mathbb{Z}_p}} -$ only has the effect that it changes the differential of Γ to $\partial(a) = E'(z)dz,\ \partial(b) = E(z)$. Let us denote this (modified) dga with Γ'. By Theorem 3.13, we know that $\mathrm{THH}_i(\mathbb{Z}_p; \mathbb{Z}_p)$ is only nonzero in degrees 0 and $i = 2p^k l - 1$ (for $k, l \geq 1$ and $p \nmid l$), where it is given by $\mathbb{Z}/p^k$, i.e. we have

$$\mathrm{THH}_*(\mathbb{Z}_p; \mathbb{Z}_p) = \bigoplus_{k,l \geq 1,\ p \nmid l} \mathbb{Z}/p^k[2p^k l - 1].$$

Hence to fully compute the E^2-page, we must first identify the homology of Γ' and then take the direct sum of this homology with infinitely many copies of shifts of its derived mod p^n reduction, i.e. for each $k, l \geq 1$ and $p \nmid l$ we add another copy of the mod p^k reduction of the homology of Γ' shifted by $(2p^k l - 1)$.
Evaluating the homology of Γ' yields[5]:

$$\begin{array}{|cccc}
 & & & \cdots \\
0 & 0 & R/E'(\pi)\{a^{[2]}dz\} & \\
0 & R/E'(\pi)\{adz\} & 0 & \\
R/E'(\pi)\{dz\} & 0 & 0 & \\
R & 0 & 0 & \\
\hline
\end{array}$$

Since R is a DVR it is also rather easy to calculate the derived mod p^n reduction of $R/E'(\pi)$. This is because in any valuation ring the set of all elements is totally ordered by divisibility, so we have $E'(\pi)|p^n$ or $p^n|E'(\pi)$. In the first case p^n is zero in $R/E'(\pi)$ and in the second case $E'(\pi)$ is zero in R/p^n. Thus we obtain:

$$(R/E'(\pi))//p^n = \begin{cases} R/E'(\pi)[0] \oplus R/E'(\pi)[1], & \text{if } E'(\pi)|p^n \\ R/p^n[0] \oplus R/p^n[1], & \text{if } p^n|E'(\pi) \end{cases}$$

This gives a completely explicit description of the E^2-page of the Tor spectral sequence.
Let us for example consider that case that R is tamely ramified[6] over $\mathbb{Z}_p$. We

[5] If in everything that we have done so far in this section, we would have replaced THH by HH, then this part of the spectral sequence would already be everything.

[6] i.e. its Eisenstein polynomial has a degree which is not divisibly by p

then have $E'(\pi) = a + pb$ for $a, b \in R$ and $p \nmid a$, in particular $p \nmid E'(\pi)$ and hence $E'(\pi) | p^n$ for all $n \in \mathbb{N}$. The non-zero entries of the E^2-page look as follow (or rather a finite subset of the infinitely many terms):

$$
\begin{array}{r|ccccc}
\cdots & & & R/E'(\pi)\{adz\} & R/E'(\pi)\{adz\} & \\
 & & R/E'(\pi)\{dz\} & R/E'(\pi)\{dz\} & & \\
2p^2-1 & & R/p^2 & & & \\
 & & & & \cdots & \cdots \\
\cdots & & & R/E'(\pi)\{adz\} & R/E'(\pi)\{adz\} & \\
 & & R/E'(\pi)\{dz\} & R/E'(\pi)\{dz\} & & \\
4p-1 & & R/p & & & \\
 & & & & \cdots & \cdots \\
 & & & R/E'(\pi)\{adz\} & R/E'(\pi)\{adz\} & \\
 & & R/E'(\pi)\{dz\} & R/E'(\pi)\{dz\} & & \\
2p-1 & & R/p & & & \cdots \\
 & & & & R/E'(\pi)\{a^{[2]}dz\} & \\
 & & & R/E'(\pi)\{adz\} & & \\
 & & R/E'(\pi)\{dz\} & & & \\
 & & R & & & \\
\hline
\end{array}
$$

Because we know, by Theorem 3.13, to what the spectral sequence converges, we can obtain knowledge about the differentials and extensions. In the tamely ramified case, the d^{p+1}-differential is necessarily an isomorphism (if domain and target are non-zero), because the even homotopy groups of $\mathrm{THH}(R;\mathbb{Z}_p)$ vanish. Furthermore the only degrees in which extension problem exist, are $2p^kl-1$. There we have an extension of the form

$$0 \to R/p^k \to \mathrm{THH}_{2p^kl-1}(R;\mathbb{Z}_p) \to R/E'(\pi) \to 0.$$

In the wildly ramified case[7], the band starting at $2p^kl-1$ (with $p \nmid l$) can either contain $R/E'(\pi)$'s or R/p^k's depending on k and the coefficients of the Eisenstein polynomial. The analysis of the differentials and extensions is not as easy as in the tamely ramified case, in particular the differentials do not have to be isomorphisms. We can only conclude that they have to be surjective, because the target has to vanish on the E^∞-page. Since domain and target are potentially different, there will remain a kernel in general, which necessarily participates in an extension to yield the known result $\mathrm{THH}_i(R;\mathbb{Z}_p) = R/E'(\pi)$ (for $i \neq 2pk-1$). For the differentials connecting the first and second band and R wildly ramified with $E'(\pi) = pr$, we have the following short exact sequence for the d^{p+1}-differential

$$0 \to R/r = \ker(d^{p+1}) \to R/E'(\pi) \xrightarrow{d^{p+1}} R/p \to 0$$

and the next one for the extension problem ($i \neq 2pk-1$)

$$0 \to R/p \to \mathrm{THH}_i(R;\mathbb{Z}_p) \to R/r \to 0.$$

These extensions prohibt us from concretely understanding the functoriality, because we do not know the two non-zero maps in it.
If we are able to understand differentials and extension, we can then calculate the effects of maps $\varphi : R \to S$ on THH via this spectral sequence *provided* that we can find lifts f, g in the following diagram (π and ε are chosen uniformizers of R and S and E, F are their minimal polynomials):

$$\begin{tikzcd}
\mathbb{Z}_p[r] \arrow[r, dashed, "g"] \arrow[d, "r\mapsto E_R(z)"'] & \mathbb{Z}_p[s] \arrow[d, "s\mapsto E_S(w)"] \\
\mathbb{Z}_p[z] \arrow[r, dashed, "f"] \arrow[d, "z\mapsto \pi_R"'] & \mathbb{Z}_p[w] \arrow[d, "w\mapsto \pi_S"] \\
R \arrow[r, "\varphi"] & S
\end{tikzcd}$$

While the first lift can always be found, finding the second one is unfortunately not always possible[8]. But sometimes it is, for example in the following cases:

[7] i.e. the degree of the Eisenstein polynomial is divisible by p

[8] For example the interesting case $\mathbb{Z}_p[\zeta_p] \to \mathbb{Z}_p[\zeta_p],\ \zeta_p \to \zeta_p^2$ is not covered.

- The inclusion $\mathbb{Z}_p \hookrightarrow \mathbb{Z}_p[\sqrt{p}]$ or more generally $\mathbb{Z}_p[\sqrt[k]{p}] \hookrightarrow \mathbb{Z}_p[\sqrt[kn]{p}]$
- $\mathbb{Z}_p[\sqrt{p}] \to \mathbb{Z}_p[\sqrt{p}], \sqrt{p} \mapsto -\sqrt{p}$
  or more generally $\mathbb{Z}_p[\sqrt[2n]{p}] \to \mathbb{Z}_p[\sqrt[2n]{p}],\ \sqrt[2n]{p} \mapsto -\sqrt[2n]{p}$
- $\mathbb{Z}_p[\sqrt[p-1]{p}] \to \mathbb{Z}_p[\sqrt[p-1]{p}],\ \sqrt[p-1]{p} \mapsto \zeta_{p-1} \sqrt[p-1]{p}$, for $p > 2$ (Recall that $\mathbb{Z}_p$ has all $(p-1)$-th roots of unity by Hensel's Lemma)

In these cases, the dashed map in the following diagram induces the map on E^2-pages. Here, the left and right square of the cube are derived pushout squares.

$$\begin{tikzcd}
& \mathbb{Z}_p \arrow[rr, "\mathrm{id}"] \arrow[dd] & & \mathbb{Z}_p \arrow[dd] \\
\Omega^*_{\mathbb{Z}_p[r]/\mathbb{Z}_p} \arrow[ur] \arrow[rr, "dg"] \arrow[dd] & & \Omega^*_{\mathbb{Z}_p[s]/\mathbb{Z}_p} \arrow[ur] \arrow[dd] & \\
& \Omega^*_{\mathbb{Z}_p[z]/\mathbb{Z}_p} \otimes^L_{\Omega^*_{\mathbb{Z}_p[r]/\mathbb{Z}_p}} \mathbb{Z}_p \arrow[rr, dashed] & & \Omega^*_{\mathbb{Z}_p[w]/\mathbb{Z}_p} \otimes^L_{\Omega^*_{\mathbb{Z}_p[s]/\mathbb{Z}_p}} \mathbb{Z}_p \\
\Omega^*_{\mathbb{Z}_p[z]/\mathbb{Z}_p} \arrow[ur] \arrow[rr, "df"] & & \Omega^*_{\mathbb{Z}_p[w]/\mathbb{Z}_p} \arrow[ur] &
\end{tikzcd}$$

To summarize: If we are dealing with a map of CDVRs that we can lift twice *and* we sufficiently understand differentials and extension, then the Tor spectral sequence allows us to compute the effect on $\mathrm{THH}_*(-;\mathbb{Z}_p)$. For example, by our discussion for tamely ramified CDVRs, we can understand what the maps $\mathbb{Z}_p[\sqrt[k]{p}] \hookrightarrow \mathbb{Z}_p[\sqrt[nk]{p}]$ (for $p \nmid k, n$), $\mathbb{Z}_p[\sqrt[2n]{p}] \to \mathbb{Z}_p[\sqrt[2n]{p}],\ \sqrt[2n]{p} \mapsto -\sqrt[2n]{p}$ (for $p \nmid n$) and $\mathbb{Z}_p[\sqrt[p-1]{p}] \to \mathbb{Z}_p[\sqrt[p-1]{p}],\ \sqrt[p-1]{p} \mapsto \zeta_{p-1} \sqrt[p-1]{p}$ (for $p > 2$) do on $\mathrm{THH}_i(-;\mathbb{Z}_p)$ for $i \neq 2pk-1,\ k \geq 1$ (in degrees of the form $2pk-1$, we only understand the map up to an extension).

4.3 Functoriality of THH via the descent spectral sequence

Let us try to mimic the procedure of Section 4.1 for *topological* Hochschild homology. We can define the ∞-category of **CDVRs with spherical lifts** $\mathrm{CDVR}^{\mathbb{S}-\mathrm{Lift}}$ as the full subcategory of the arrow category of CAlg on objects of the form $\mathbb{S}[z] \to R$, where R is a mixed characteristic CDVR with perfect residue field and z maps to a uniformizer of R. Again objects are equivalently given by pairs (R, π), where $\pi \in R$ is a uniformizer. This is because, every map $\mathbb{S}[z] \to R$ factors over $\mathbb{Z}[z] \to R$ and $\mathbb{Z}[z]$ is the free ordinary commutative ring on one generator. This category is the natural home for the relative THH computations that we want to do. We have forgetful functors $\mathrm{CDVR}^{\mathbb{S}-\mathrm{Lift}} \to \mathrm{CDVR}^{\mathrm{Lift}} \to \mathrm{CDVR}$ (the first one sends the pair $\mathbb{S}[z] \to R$ to the induced map $W(R/\pi)[z] \to R$). The

problem is that the first functor is *not full*. We cannot lift an arbitrary polynomial map $\mathbb{Z}_p[z] \to \mathbb{Z}_p[w]$ to an $\mathbb{E}_\infty$-map $\mathbb{S}[z] \to \mathbb{S}[w]$. This is only possible for monomial maps, i.e. maps of the form $\mathbb{Z}_p[z] \to \mathbb{Z}_p[w],\ z \to w^n$. In this specific case, we can define the lift $\mathbb{S}[z] \to \mathbb{S}[w]$ via applying Σ^∞_+ to the map of monoids $\mathbb{N} \xrightarrow{\cdot n} \mathbb{N}$. That we cannot lift general maps $\mathbb{Z}_p[z] \to \mathbb{Z}_p[w]$ is a recent observation due to Thomas Nikolaus which uses the existence of the *Tate-valued Frobenius* ([NS18, Section IV.1])). This is a certain additional structure, which only exists in the world of (commutative) ring spectra and not in ordinary algebra (i.e. even in the case that ordinary rings are involved, the Tate valued Frobenius is not $H\mathbb{Z}$-linear).
Relative THH and the descent spectral sequence from Construction 3.10 are functorial in $\mathrm{CDVR}^{\mathbb{S}-\mathrm{Lift}}$. We can use this for maps between CDVRs that lie in the image of the forgetful functor. For example in the case of the inclusion $\mathbb{Z}_p \hookrightarrow \mathbb{Z}_p[\sqrt[n]{p}]$ or $\mathbb{Z}_p[\sqrt[k]{p}] \hookrightarrow \mathbb{Z}_p[\sqrt[kn]{p}]$.
To understand the functoriality we can then proceed very similarly to Section 4.1. We first observe that by Lemma 2.4.3) we have a natural isomorphism $\mathrm{THH}_2(-/\mathbb{S}[z];\mathbb{Z}_p) \xrightarrow{\sim} \mathrm{HH}_2(-/\mathbb{Z}[z];\mathbb{Z}_p)$ of functors $\mathrm{CDVR}^{\mathbb{S}-\mathrm{Lift}} \to \mathbb{Z}_p$-Mod. Using Lemma 4.2 we thus get a natural description of $\mathrm{THH}_2(-\mathbb{S}[z];\mathbb{Z}_p)$. We now want to functorially understand the full homotopy ring. Theorem 3.8 already gives us the (non-functorial) description $\mathrm{THH}_*(R/\mathbb{S}[z];\mathbb{Z}_p) = R[x]$. From this, we get a natural map $\mathrm{Sym}(\mathrm{THH}_2(R/\mathbb{S}[z];\mathbb{Z}_p)) \to \mathrm{THH}_*(R/\mathbb{S}[z];\mathbb{Z}_p)$ via the universal property of the symmetric algebra and because we know that the target is a commutative algebra. This is an isomorphism, because the higher homotopy groups are generated by powers of a generator of the second homotopy group. We thus have the natural identification

$$\mathrm{Sym}(I/I^2) = \mathrm{THH}_*(R/\mathbb{S}[z];\mathbb{Z}_p) \tag{4.3.1}$$

for $(\mathbb{S}[z] \to R) \in \mathrm{CDVR}^{\mathbb{S}-\mathrm{Lift}}$ and I the kernel of the associated map $\mathbb{Z}[z] \to R$. We can now formulate the main result in the following theorem. This is the THH analogue of Theorem 4.3.

Theorem 4.8. *We have a natural isomorphism of functors from* $\mathrm{CDVR}^{\mathbb{S}-\mathrm{Lift}}$ *to graded* $\mathbb{Z}_p$*-algebras.*

$$\mathrm{THH}_*(R;\mathbb{Z}_p) = H_*(\mathrm{Sym}(I/I^2) \otimes_{W(k)[z]} \Omega^*_{W(k)[z]/W(k)},\ d^2) \tag{4.3.2}$$

Here k is the residue field of R. This explicitly means that we get a short exact sequence for the odd homotopy groups of $\mathrm{THH}(R;\mathbb{Z}_p)$*:*

$$0 \to R\{x^{n+1}\} \xrightarrow{d^2} R\{x^n dz\} \twoheadrightarrow \mathrm{THH}_{2n+1}(R;\mathbb{Z}_p) \to 0$$

Given a map $(\varphi, f) : (R, \pi) \to (S, \varepsilon)$ in $\mathrm{CDVR}^{\mathbb{S}-\mathrm{Lift}}$ *the following diagram com-*

pletely encodes the functoriality.

$$\begin{tikzcd}
0 \arrow[r] & R\{x^{n+1}\} \arrow[r, "d^2"] \arrow[d] & R\{x^n dz\} \arrow[r, two heads] \arrow[d] & \mathrm{THH}_{2n+1}(R;\mathbb{Z}_p) \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & S\{x^{n+1}\} \arrow[r, "d^2"] & S\{x^n dw\} \arrow[r, two heads] & \mathrm{THH}_{2n+1}(S;\mathbb{Z}_p) \arrow[r] & 0
\end{tikzcd}$$

Here, the map $R\{x^n dz\} \to S\{y^n dw\}$ has the form $x^n dz \mapsto \left(\frac{E(f(w))}{F(w)}|_{w=\varepsilon}\right)^n f'(\varepsilon) y^n dw$, where E, F are Eisenstein polynomials of π, ε.

Proof. Statement 4.3.2 follows from the natural identification of relative THH in 4.3.1, the naturality of the descent spectral sequence (Construction 3.10) and the identification of the d^2-differential (Lemma 3.11).
The map of short exact sequences also comes from the naturality of the descent spectral sequence. Finally, the claimed map $R\{x^n dz\} \to S\{y^n dw\}$ is the tensor product of the maps $\mathrm{Sym}^n_R((E)/(E)^2) \to \mathrm{Sym}^n_S((F)/(F)^2)$ and the map on Kähler differentials. □

Remark 4.9. *We again want to stress that we have only proven this dga-description for maps of CDVRs that send one uniformizer to a power of the other. Nevertheless, formally 4.3.2 makes sense for completely arbitrary maps and it would be a natural question to wonder whether it could be a correct description in full generality.*

A. K-Theory

In this appendix, we give a brief review of the definition of algebraic K-theory for ring spectra. The construction can be performed as the composition of four intermediate steps, which we will outline now.
For this, let R be a connective ring spectrum.
1) An R-module spectrum M is called **finitely generated projective**, if it is a retract of a finitely generated, free R-module, i.e. there is another R-module N and $n \in \mathbb{N}$ such that $M \oplus N = R^n$. Denote by $\mathrm{Proj}_R \subset \mathrm{Mod}_R$ the (full) subcategory of finitely generated projective R-module spectra. This is a symmetric monoidal ∞-category under the direct sum of R-modules.
2) For any ∞-category $\mathcal{C}$, we can take its **groupoid core** $\mathcal{C}^{\simeq}$, which is the largest subcategory of $\mathcal{C}$ containing only equivalences. This is by definition an ∞-groupoid or equivalently a space. If $\mathcal{C}$ has a symmetric monoidal structure, it induces[1] an $\mathbb{E}_\infty$-structure on $\mathcal{C}^{\simeq}$. In particular, $(\mathrm{Proj}_R)^{\simeq}$ is an $\mathbb{E}_\infty$-space.
3) Next we need the concept of an **grouplike $\mathbb{E}_\infty$-space**, which is simply an $\mathbb{E}_\infty$-space X, such that $\pi_0(X)$ is a group under the monoid operation induced by the $\mathbb{E}_\infty$-structure. We define the category of grouplike $\mathbb{E}_\infty$-spaces $\mathrm{CAlg}_{\mathrm{gp}}(\mathcal{S})$ as the full subcategory of $\mathrm{CAlg}(\mathcal{S})$ containing those $\mathbb{E}_\infty$-spaces that are grouplike. The crucial fact is that the thus provided inclusion $\mathrm{CAlg}_{\mathrm{gp}}(\mathcal{S}) \hookrightarrow \mathrm{CAlg}(\mathcal{S})$ admits a left adjoint $X \to X^{\mathrm{gp}}$, called **group completion**. On the π_0 level, it actually induces the ordinary group completion which provides an initial abelian group for any commutative monoid. The existence of the adjoint can either be proved using the adjoint functor theorem or by providing a concrete construction, like $X^{\mathrm{gp}} = \Omega B X$.
4) The last step entails the equivalence of ∞-categories between connective spectra and grouplike $\mathbb{E}_\infty$-spaces: $\mathrm{Sp}_{\geq 0} \simeq \mathrm{CAlg}_{\mathrm{gp}}(\mathcal{S})$. One functor is (on objects) given by $\mathrm{Sp}_{\geq 0} \ni X \mapsto \Omega^\infty X$. Let us denote its inverse by $B^\infty : \mathrm{CAlg}_{\mathrm{gp}}(\mathcal{S}) \xrightarrow{\sim} \mathrm{Sp}_{\geq 0}$.
These four construction now allow us to define the K-theory spectrum $K(R)$ of a connective ring spectrum R as the composition of all of them. I.e. the **K-theory spectrum** of R is the spectrum corresponding to the group completion of the groupoid core of the symmetric monoidal ∞-category of projective R-modules.

$$K : \mathrm{Alg}(\mathrm{Sp})_{\geq 0} \ni R \mapsto \mathrm{Proj}_R \mapsto \mathrm{Proj}_R^{\simeq} \mapsto (\mathrm{Proj}_R^{\simeq})^{\mathrm{gp}} \mapsto B^\infty(\mathrm{Proj}_R^{\simeq})^{\mathrm{gp}} \in \mathrm{Sp}_{\geq 0}$$

Remark A.1. *If R is a commutative ring spectrum, $K(R)$ also is, more precisely*

[1] This is because the functor $(\ )^{\simeq} : \mathrm{Cat}_\infty \to \mathcal{S}$ is symmetric monoidal, where both domain and target are symmetric monoidal under product. Commutative algebras in Cat_∞ are exactly symmetric monoidal categories.

for an $\mathbb{E}_n$-ring spectrum R, $K(R)$ is a connective $\mathbb{E}_{n-1}$-ring spectrum. In the language, used here, this is proved in [GGN16]. The last chapter in there also contains more details on the above construction of the K-theory spectrum and proofs of the claimed facts.

Remark A.2. *In the case of a non-connective ring spectrum R, this definition is no longer suitable and we need to proceed differently. We should then not consider* Proj_R *but* Perf_R *instead. This is the smallest stable subcategory of* Mod_R *containing R, which is closed under retracts. The ∞-category* Perf_R *has the structure of a Waldhausen ∞-category in the sense of Barwick and we can define its K-theory, see [Bar16, Chapter 11]. This construction is also suitably monoidal by [Bar13, Proposition 3.8]. In particular we again have the statement that $K(R)$ is an $\mathbb{E}_{n-1}$-ring spectrum for an $\mathbb{E}_n$-ring spectrum R,*

Bibliography

[BR20] David Barnes and Constanze Roitzheim. *Foundations of Stable Homotopy Theory*. Vol. 185. Cambridge University Press, 2020.

[BLR08] Arthur Bartels, Wolfgang Lück, and Holger Reich. “On the Farrell–Jones conjecture and its applications”. In: *Journal of Topology* 1.1 (2008), pp. 57–86.

[Bar13] Clark Barwick. “Multiplicative structures on algebraic K-theory”. In: *arXiv preprint arXiv:1304.4867* (2013).

[Bar16] Clark Barwick. “On the algebraic K-theory of higher categories”. In: *Journal of Topology* 9.1 (2016), pp. 245–347.

[BMS19] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. “Topological Hochschild homology and integral p-adic Hodge theory”. In: *Publ. Math. Inst. Hautes Études Sci.* 129 (2019), pp. 199–310. ISSN: 0073-8301. DOI: 10.1007/s10240-019-00106-9. URL: https://doi.org/10.1007/s10240-019-00106-9.

[BGT13] Andrew J Blumberg, David Gepner, and Gonçalo Tabuada. “A universal characterization of higher algebraic K-theory”. In: *Geometry & Topology* 17.2 (2013), pp. 733–838.

[BV68] John Michael Boardman and Rainer M Vogt. “Homotopy-everything H-spaces”. In: *Bulletin of the American mathematical society* 74.6 (1968), pp. 1117–1122.

[BV06] John Michael Boardman and Rainer M Vogt. *Homotopy invariant algebraic structures on topological spaces*. Vol. 347. Springer, 2006.

[BHM93] Marcel Bökstedt, Wu Chung Hsiang, and Ib Madsen. “The cyclotomic trace and algebraic K-theory of spaces”. In: *Inventiones mathematicae* 111.1 (1993), pp. 465–539.

[Bor74] Armand Borel. “Stable real cohomology of arithmetic groups”. In: *Annales scientifiques de l’École Normale Supérieure*. Vol. 7. 2. 1974, pp. 235–272.

[Cam13] Omar Antolın Camarena. “A whirlwind tour of the world of $(\infty, 1)$-categories”. In: *Mexican mathematicians abroad: recent contributions* (2013), pp. 15–61.

[CS19] Kestutis Cesnavicius and Peter Scholze. “Purity for flat cohomology”. In: *arXiv preprint arXiv:1912.10932* (2019).

[Cis19] Denis-Charles Cisinski. *Higher categories and homotopical algebra*. Vol. 180. Cambridge University Press, 2019.

[CMM21] Dustin Clausen, Akhil Mathew, and Matthew Morrow. “K-theory and topological cyclic homology of henselian pairs”. In: *Journal of the American Mathematical Society* 34.2 (2021), pp. 411–473.

[DP61] Albrecht Dold and Dieter Puppe. “Homologie nicht-additiver funktoren. anwendungen”. In: *Annales de l'institut Fourier*. Vol. 11. 1961, pp. 201–312.

[DGM12] Bjørn Ian Dundas, Thomas G Goodwillie, and Randy McCarthy. *The local structure of algebraic K-theory*. Vol. 18. Springer Science & Business Media, 2012.

[FH78] F. Thomas Farrell and Wu-Chung Hsiang. “On the rational homotopy groups of the diffeomorphism groups of discs, spheres and aspherical manifolds”. In: *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part*. Vol. 1. 1978, pp. 325–337.

[GR19] Dennis Gaitsgory and Nick Rozenblyum. *A study in derived algebraic geometry: Volume I: correspondences and duality*. Vol. 221. American Mathematical Society, 2019.

[Gep20] David Gepner. “An introduction to higher categorical algebra”. In: *Handbook of Homotopy Theory* (2020), p. 487.

[GGN16] David Gepner, Moritz Groth, and Thomas Nikolaus. “Universality of multiplicative infinite loop space machines”. In: *Algebraic & Geometric Topology* 15.6 (2016), pp. 3107–3153.

[Gro20] Moritz Groth. “A short course on ∞-categories”. In: *Handbook of Homotopy Theory*. Chapman and Hall/CRC, 2020, pp. 549–617.

[HM97] Lars Hesselholt and Ib Madsen. “On the K-theory of finite algebras over Witt vectors of perfect fields”. In: *Topology* 36.1 (1997), pp. 29–101. ISSN: 0040-9383. DOI: `10.1016/0040-9383(96)00003-1`. URL: `https://doi.org/10.1016/0040-9383(96)00003-1`.

[HN20] Lars Hesselholt and Thomas Nikolaus. “Topological cyclic homology”. English. In: *Handbook of homotopy theory*. Boca Raton, FL: CRC Press, 2020, pp. 619–656. ISBN: 978-0-8153-6970-7; 978-1-351-25162-4. DOI: `10.1201/9781351251624-15`.

[Ill06] Luc Illusie. *Complexe cotangent et déformations I*. Vol. 239. Springer, 2006.

[KN18] Achim Krause and Thomas Nikolaus. *Lectures on topological Hochschild homology and cyclotomic spectra*. 2018. eprint: `https://www.uni-muenster.de/IVV5WS/WebHop/user/nikolaus/papers.html`.

[KN19] Achim Krause and Thomas Nikolaus. *Bökstedt periodicity and quotients of DVRs*. 2019. arXiv: `1907.03477 [math.AT]`.

[LM00] Ayelet Lindenstrauss and Ib Madsen. "Topological Hochschild homology of number rings". In: *Trans. Am. Math. Soc.* 352.5 (2000), pp. 2179–2204. ISSN: 0002-9947; 1088-6850/e.

[Lod13] Jean-Louis Loday. *Cyclic homology*. Vol. 301. Springer Science & Business Media, 2013.

[Lur09] Jacob Lurie. *Higher topos theory*. Vol. 170. Princeton, NJ: Princeton University Press, 2009, pp. xv + 925. ISBN: 978-0-691-14049-0.

[Lur17] Jacob Lurie. "Higher algebra." In: *preprint available from the author's website* (2017). URL: `https://www.math.ias.edu/~lurie/papers/HA.pdf`.

[Lur18a] Jacob Lurie. "Elliptic cohomology II: orientations". In: *preprint available from the author's website* (2018). URL: `https://www.math.ias.edu/~lurie/papers/Elliptic-II.pdf`.

[Lur18b] Jacob Lurie. *Kerodon*. `https://kerodon.net`. 2018.

[Lur18c] Jacob Lurie. "Spectral Algebraic Geometry." In: *preprint available from the author's website* (2018). URL: `https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf`.

[May06] J Peter May. *The geometry of iterated loop spaces*. Vol. 271. Springer, 2006.

[MSV97] James McClure, Roland Schwänzl, and Rainer Vogt. "THH$(R) \cong R \otimes S^1$ for $\mathbb{E}_\infty$ ring spectra". In: *Journal of Pure and Applied Algebra* 121.2 (1997), pp. 137–159.

[NS18] Thomas Nikolaus and Peter Scholze. *On topological cyclic homology*. 2018. arXiv: `1707.01799 [math.AT]`.

[Ric09] Birgit Richter. "Divided power structures and chain complexes". In: *Alpine perspectives on algebraic topology. Third Arolla conference on algebraic topology, Arolla, Switzerland, August 18–24, 2008*. Providence, RI: American Mathematical Society (AMS), 2009, pp. 237–254. ISBN: 978-0-8218-4839-5.

[Rie18] Emily Riehl. "Homotopy coherent structures". In: *arXiv preprint 1801.07404* (2018).

[RV21] Emily Riehl and Dominic Verity. "Elements of ∞-category theory". In: *Preprint available at www.math.jhu.edu/~eriehl/elements.pdf* (2021).

[Ser95] Jean Pierre Serre. *Local Fields*. Graduate Texts in Mathematics. Springer New York, 1995. ISBN: 9780387904245.

[Stacks] The Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu`. 2018.

[Wal85] Friedhelm Waldhausen. "Algebraic K-theory of spaces". In: *Algebraic and geometric topology*. Springer, 1985, pp. 318–419.

[WJR13] Friedhelm Waldhausen, Bjørn Jahren, and John Rognes. *Spaces of PL manifolds and categories of simple maps*. 186. Princeton University Press, 2013.

[Wei94] Charles A Weibel. *An introduction to homological algebra*. 38. Cambridge university press, 1994.