UPPSALA UNIVERSITET
Matematiska institutionen
Maksim Maydanskiy

Midterm
1MA016/1MA183
Several Variable Calculus
2017-03-10

Time: 14.00 - 16.00. Tools allowed: only materials for writing. Please provide ***full explanations and calculations*** *in order to get full credit. The midterm consists of 3 problems worth 5 points each, for a total of 15 points. You may write answers in English or Swedish. Your total score can be used as a backup to substitute for the total score on the first 3 problems on the final combined.*

1. (5 points) Find the length of the curve parametrised by

$$\vec{r}(t) = (3t, 3t^2, 2t^3), \quad t \in [-1, 1]$$

We find $\vec{v}(t) = (3, 6t, 6t^2) = 3(1, 2t, 2t^2)$, so

$$\begin{aligned}|\vec{v}(t)| &= 3\sqrt{1^2 + (2t)^2 + (2t^2)^2} = \\ &= 3\sqrt{1 + 4t^2 + 4t^4} = \\ &= 3\sqrt{(1 + 2t^2)^2} \quad = 3|1 + 2t^2| = 3(1 + 2t^2)\end{aligned}$$

Then the length $s = \int_{-1}^{1} 3(1 + 2t^2) = 6 + 3 \times 2 \times \frac{2}{3} = 10$.

2. (5 points) Let

$$f(x, y) = \begin{cases} \frac{x^5 + y^5}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{otherwise} \end{cases}$$

(a) Is $f(x, y)$ continuous on $\mathbb{R}^2$?

Everywhere outside of $(0, 0)$ function f is continuous by limit theorems (it is a ratio of non-zero continuous functions $x^5 + y^5$ and $x^2 + y^2$).
At $(0, 0)$ we can compute

$$\lim_{(x,y)\to(0,0)} f(x,y) - f(0,0) = \lim_{(x,y)\to(0,0)} \frac{x^5+y^5}{x^2+y^2} - 0$$
$$= \lim_{r\to 0} \frac{r^5\cos^5(\theta) + r^5\sin^5(\theta)}{r^2} = \lim_{r\to 0} r^3(\cos^5(\theta) + \sin^5(\theta))$$

This is a product of a function r^3 which goes to zero, and function $(\cos^5(\theta) + \sin^5(\theta))$ whose values are bounded between -2 and 2 so by squeeze lemma it goes to zero. Hence f is continuous at $(0,0)$.

(b) Is $f(x,y)$ differentiable on $\mathbb{R}^2$?

Method 1: Everywhere outside of $(0,0)$ function f is differentiable by limit theorems (it is a ratio of non-zero differentiable functions x^5+y^5 and x^2+y^2). At $(0,0)$ we suspect that it is differentiable with zero derivative (because the top vanishes much faster than the bottom), and to confirm we can compute from definition

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - 0(x-0) - 0(y-0)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{x^5+y^5}{(x^2+y^2)^{3/2}} =$$
$$= \lim_{r\to 0} \frac{r^5\cos^5(\theta) + r^5\sin^5(\theta)}{r^3} = \lim_{r\to 0} r^2(\cos^5(\theta) + \sin^5(\theta))$$

This is a product of a function r^2 which goes to zero, and function $(\cos^5(\theta) + \sin^5(\theta))$ whose values are bounded between -2 and 2 so by squeeze lemma it goes to zero. Hence f is differentiable at $(0,0)$ (with derivative $Df = [0,0]$).

Method 2: We know that if a function has continuous partial derivatives on $\mathbb{R}^2$ then it is differentiable on $\mathbb{R}^2$.
We compute at any point $(x,y) \neq (0,0)$

$$f_1(x,y) = \frac{5x^4(x^2+y^2) - (x^5+y^5)(2x)}{(x^2+y^2)^2} = \frac{3x^6 + 5x^4y^2 - 2xy^5}{(x^2+y^2)^2} =$$
$$\frac{r^6(3\cos^6 t + 5\cos^4 t\sin^2 t - 2\cos t\sin^5 t}{r^4} =$$
$$r^2(3\cos^6 t + 5\cos^4 t\sin^2 t - 2\cos t\sin^5 t)$$

This is continuous everywhere outside $(0,0)$. Near $(0,0)$, this is a product of a function r^2 which goes to zero, and function $(3\cos^6 t + 5\cos^4 t\sin^2 t -$

$2\cos t \sin^5 t)$ whose values are bounded between -10 and 10 so by squeeze lemma it goes to zero near $(0,0)$.
On the other hand $f_1(0,0) = \lim x \to 0 \frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0} \frac{x^5}{x^2 x} = 0$.
So indeed, $f_1(x,y)$ exists everywhere and is continuous everywhere.
Since the function is symmetric in (x,y), the same argument applies to $f_2(x,y)$.
We conclude that f has continuous partials, and so is differentiable everywhere in $\mathbb{R}^2$.

3. (5 points)

Let

$$f(x,y) = x^3 - 3xy + y^3$$

(a) Find critical points of $f(x,y)$ on $\mathbb{R}^2$ and classify them into local maxima, local minima, or saddle points.

$\nabla f = (3x^2 - 3y, 3y^2 - 3x)$, so for (x,y) to be critical we must have $y = x^2, x = y^2$. This implies $x^4 = x$, so $x = 0$ or $x^3 = 1$ i.e. $x = 1$. The critical points are $P_1 = (0,0)$ and $P_2 = (1,1)$.
The Hessian of f is given by $f_{11}(x,y) = 6x, f_{22}(x,y) = 6y, f_{1,2}(x,y) = -3$
At P_1 this is $\begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$ representing $-6xy$ which is indefinite and we have a saddle point.
At P_2 this is $\begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$ representing $6x^2 + 6y^2 - 6xy = 6(x^2 - xy + y^2) = 6((x^2 - \frac{y}{2})^2 + \frac{3}{4}y^2$ which is positive definite, and we have a local minimum.

(b) Find absolute maximum and minimum values of $f(x,y)$ on the square $D = \{x,y|0 \le x \le 4, 0 \le y \le 4\}$.

D is closed and bounded, so f attains both maximum and minimum, which are also local maximum and minimum, and so can be found among:
1) Singular points of f - none.
2) Critical points of f on D. P_2 is a local minimum, with value $f(P_2) = 1 - 3 + 1 = -1$.
3) Boundary points of D.
There are 4 pieces:
a) $x = 0$; then $f(0,y) = y^3$ min 0 at $(0,0)$, max $4^3 = 64$ at $(0,4)$.
b) $y = 0$; symmetrically, then $f(x,0) = x^3$ min 0 at $(0,0)$, max $4^3 = 64$ at $(4,0)$.

c) $x = 4$, then $f(4, y) = y^3 - 12y + 64$. Critical points: $3y^2 - 12 = 0$, $y^2 = \pm 2$, minus outside of D, so $(x, y) = (4, 2)$. Value $f(4, 2) = 8 - 24 + 64 = 48$. Boundary: $(4, 0)$ and $(4, 4)$. Values 64 as before and $64 - 48 + 64 = 80$.

d) $y = 4$ symmetrically - one critical pint $(2, 4)$ with value 48 and boundary points $(0, 4)$ with value 64 and $(4, 4)$ with value 80.

So, the minimum is at $(1, 1)$ and is equal -1, maximum is at $(4, 4)$ and is equal to 80.