Division of Computing Science
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Written examination in 1TD184 Optimization

- Date: 2024-01-05, 14.00-19.00
- Allowed tools: Pocket calculator, one A4 paper with notes (computer typed, font size minimum 10 pt).
- Maximum number of points: 36. Thresholds for 3/4/5 = 18/24/30.
- All assumptions and answers *must* be motivated for full points.

(1) Explain the following concepts: 1) Quasi-Newton methods, 2) Sequential quadratic programming. (4p)

(2) Consider a finite set of points with given coordinates (on the two-dimensional space): $C = \{(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)\}$. The centroid of C is the location where the sum of squared Euclidean distances between this location and those in C is minimized. Formulate the task of finding the centroid as an optimization problem. Next, utilize the knowledge from the optimization course to determine the coordinates of the centroid. (3p)

(3) True or false (answers *must* be motivated):

a) If function $f(x)$ is convex, then the region defined by $f(x) \leq 0$ is also convex. (1p)

b) For a quadratic function, the basic Newton method converges in one iteration. (1p)

c) For minimization, if the search direction is a descent direction, then Armijo line search with backtracking is guaranteed to make improvement in the function value. (1p)

d) In the simplex method for linear programming (LP) with minimization, selecting the variable with most negative reduced cost gives the largest improvement. (1p)

(4) Consider the following two problems. Note: All the answers *must* be motivated.

I	II
minimize $f(x)$	minimize $f(x)$
	subject to
	$Ax \leq b$

a) Is a local minimum of Problem I also a local minimum of Problem II? Conversely, is a local minimum of Problem II also a local minimum of Problem I? (2p)

b) Suppose you are given a software for solving Problem II. The software expects you to provide with function f together with a linear inequality system $Ax \leq b$. Also, the software does not accept A to be an all-zero matrix. Can you use the software to solve Problem I? Next, suppose you are given a software for solving Problem I. Can you use the software to solve Problem II? (2p)

(5) Consider the following quadratic problem with two variables: min $2x_1^2 + x_2^2 + 2x_1x_2 - 2x_1$.

a) Consider starting point $(0, 0)^T$. Derive the steepest descent direction and the Newton direction[1] at this point. (2p)

b) For the steepest descent direction, apply exact line search (i.e., formulate line search in form of a one-dimensional minimization problem with step size α as the variable, and find the optimal α). For the Newton direction, apply the default step size. What are the new points obtained by taking the respective step sizes? (2p)

c) Is any of the two points obtained above (locally) optimal? Justify your answer. (2p)

[1]For a 2×2 matrix $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, its inverse matrix is given by $\frac{1}{\alpha\delta - \beta\gamma} \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix}$.

(6) Consider the linear least-squares data fitting problem. For a linear function $b = x_1 t + x_2$, x_1 and x_2 are our optimization variables defining the function. There are m data points $(t_1, b_1), (t_2, b_2), \ldots, (t_m, b_m)$. If there is a perfect fit, then there exist x_1 and x_2 such that $b_i = x_1 t_i + x_2, i = 1, \ldots, m$. This is however very unlikely, hence there will be some errors. Given x_1 and x_2, the error for data point i is the absolute difference between $x_1 t_i + x_2$ and b_i.

- Suppose the objective function is to minimize the sum of squares of errors. Formulate the problem, and show that a local optimum is also a global optimum. (2p)
- Now consider instead the maximum error (i.e., the largest error among all data points), and we would like to minimize the maximum error. Formulate the problem with this new objective function. Is the problem convex? (Hint: You need auxiliary variables and constraints). (2p)

(7) Consider the following constrained problem.

$$
\begin{aligned}
\min \quad & (x_1 - 1)^2 + (x_2 + 2)^2 \\
& -x_1 + x_2 \geq -2 \\
& -3x_1 - 2x_2 \geq -6 \\
& x_1 \geq 0
\end{aligned}
$$

a) Consider point $(2, 0)^T$. Use first-order optimality condition based on Lagrangian function to show this point cannot be optimal. Also, specify all descent and feasible directions at $(2, 0)^T$. (3p)

b) There are two active constraints at $(2, 0)^T$. Consider an active-set method that keeps one of the constraints active. Which constraint is kept active? What would be the search direction? What is the optimal (and feasible) step size for this direction? (2p)

c) For the new point obtained by taking the step size, show it does satisfy the first-order necessary condition. Is this point a minimum of the constrained problem? (2p)

(8) We have the following LP and its dual LP.

Primal LP

$$
\begin{aligned}
\max \quad & 4y_1 + 3y_2 + 2y_3 \\
& y_1 + y_2 \leq -2 \\
& y_1 + y_3 \leq -1 \\
& y_1 \leq 0, y_2 \leq 0, y_3 \leq 0
\end{aligned}
$$

Dual LP

$$
\begin{aligned}
\min \quad & -2x_1 - x_2 \\
& x_1 + x_2 \leq 4 \\
& x_1 \leq 3 \\
& x_2 \leq 2 \\
& x_1 \geq 0, x_2 \geq 0
\end{aligned}
$$

a) How many extreme points exist for the dual LP? Which of them is optimal? (2p)

b) Derive and prove the optimum of the primal LP[2]. (2p)

Good Luck!

[2] In case you use the simplex method, note that 1) the primal is maximization, and 2) you need to consider the standard form.