Introduction to Computer Control Systems, 5 credits, 1RT485

Date: 2020-03-17

Place: Bergsbrunnagatan 15, Sal 2.

Teacher on duty: Dave Zachariah.

Allowed aid:

- A basic calculator
- BETA mathematical handbook

Solutions have to be explained in detail and possible to reconstruct.

NB: Only one problem per sheet. Write your anonymous exam code on each sheet. Write your name if you do not have an anonymous code.

Best of luck!

Useful results

Laplace transform table

Table 1: Basic Laplace transforms

$f(t)$	$F(s)$	$f(t)$	$F(s)$
unit impulse $\delta(t)$	1	$\sinh(bt)$	$\frac{b}{s^2-b^2}$
unit step $1(t)$	$\frac{1}{s}$	$\cosh(bt)$	$\frac{s}{s^2-b^2}$
t	$\frac{1}{s^2}$	$\frac{1}{2b}t\sin(bt)$	$\frac{s}{(s^2+b^2)^2}$
t^n	$\frac{n!}{s^{n+1}}$	$t\cos(bt)$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
e^{-at}	$\frac{1}{s+a}$	$\frac{\cos(bt)-\cos(at)}{a^2-b^2}; (a^2 \neq b^2)$	$\frac{s}{(s^2+a^2)(s^2+b^2)}$
$\frac{1}{a}(1-e^{-at})$	$\frac{1}{s(s+a)}$	$\frac{\sin(at)+at\cos(at)}{2a}$	$\frac{s^2}{(s^2+a^2)^2}$
$\frac{1}{(n-1)!}t^{n-1}e^{-at}; (n=1,2,3...)$	$\frac{1}{(s+a)^n}$		
$\sin(bt)$	$\frac{b}{s^2+b^2}$		
$\cos(bt)$	$\frac{s}{s^2+b^2}$		
$e^{-at}\sin(bt)$	$\frac{b}{(s+a)^2+b^2}$		
$e^{-at}\cos(bt)$	$\frac{s+a}{(s+a)^2+b^2}$		

Table 2: Properties of Laplace Transforms

$\mathcal{L}[af(t)] = aF(s)$	$\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}$
$\mathcal{L}[f_1(t)+f_2(t)] = F_1(s)+F_2(s)$	$\mathcal{L}\left[t^2f(t)\right] = \frac{d^2}{ds^2}F(s)$
$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = sF(s)-f(0)$	$\mathcal{L}[t^nf(t)] = (-1)^n\frac{d^n}{ds^n}F(s), \quad n=1,2,3,\ldots$
$\mathcal{L}\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s)-sf(0)-f'(0)$	$\mathcal{L}\left[f\left(\frac{t}{a}\right)\right] = aF(as)$
$\mathcal{L}\left[\int f(t)\,dt\right] = \frac{F(s)}{s}+\frac{1}{s}\left[\int f(t)\,dt\right]_{t=0}$	$\mathcal{L}\left[\int_0^t f_1(t-\tau)f(t)d\tau\right] = F_1(s)F_2(s)$
$\mathcal{L}[f(t-a)] = e^{-as}F(s)$	$\mathcal{L}[e^{-at}f(t)] = F(s+a)$

Matrix exponential

$$\boxed{e^{At} \triangleq \mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\}}$$

Open-loop and sensitivity functions

$$G_o(s) = G(s)F_y(s), \qquad S(s) = \frac{1}{1+G_o(s)}, \qquad T(s) = 1-S(s)$$

State-space forms and transfer function relations

- State-space form and transfer function

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad \Rightarrow \quad \boxed{G(s) = C(sI - A)^{-1}B + D}$$

- Associated matrices

$$\mathcal{S} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \qquad \mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

- LTI system with transfer function

$$\boxed{G(s) = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}}$$

i) Observable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ -a_3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ b_3 - a_3 b_0 \\ \vdots \\ b_n - a_n b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

ii) Controllable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_1 - a_1 b_0 & b_2 - a_2 b_0 & \cdots & b_n - a_n b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

- Solution to state-space equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

can be written as

$$\boxed{x(t) = e^{At}x_0 + \int_0^t e^{A\tau} B u(t - \tau) d\tau}$$

- Observer system

$$\boxed{\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})}$$

Feedback control structures

General linear feedback in Laplace form:

$$\boxed{U(s) = F_r(s)R(s) - F_y(s)Y(s)}$$

Common control structures in this form.

- PID controller:

$$F_y(s) = F_r(s) = F(s) = K_p + \frac{K_i}{s} + K_d s,$$

  where $K_p, K_i, K_d \geq 0$

- Lead-lag controller:

$$F_y(s) = F_r(s) = F(s) = K\left(\frac{\tau_D s + 1}{\beta\tau_D s + 1}\right)\left(\frac{\tau_I s + 1}{\tau_I s + \gamma}\right),$$

  where $K, \tau_D, \tau_I > 0$ and $0 \leq \beta, \gamma < 1$

- State-feedback controller with observer:

$$\begin{aligned} F_r(s) &= \big(1 - L(sI - A + KC + BL)^{-1}B\big)\,\ell_0 \\ F_y(s) &= L(sI - A + KC + BL)^{-1}K \end{aligned}$$

Discrete-time state-space forms

A continuous time system with zero-order-hold input signal and sample period T can be written in discrete-time as:

$$\boxed{\begin{aligned} x(k+1) &= Fx(k) + Gu(k) \\ y(k) &= Hx(k) \end{aligned}}$$

where

$$\begin{aligned} F &= e^{AT} \\ G &= \int_{\tau=0}^{T} e^{A\tau}d\tau B = \lceil \text{if } A^{-1} \text{ exists} \rceil = A^{-1}(e^{AT} - I)B \\ H &= C \end{aligned}$$

Problem 1: basic questions (6/30)

Answer only *'true'* or *'false'*. Each correct answer gives 1 point, each wrong answer gives -1 point. Minimum total points for Part A and B is 0 , respectively.

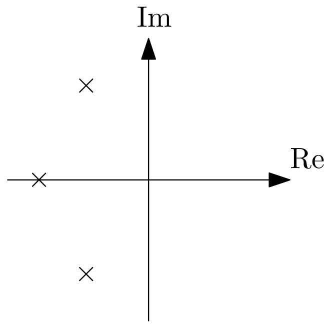
Part A

Note: Write 'skip' if your total home assignment score ≥ 8

i) The following system
$$G(s) = \frac{s^2 - 3s + 2}{s^2 + s - 6}$$
is input-output stable.

ii) The system $G(s)$ above is a 'minimum phase' system.

iii) Consider a system with a state-space description, where the matrix A has eigenvalues as shown in the figure below.



Then this system $G(s)$ from $u(t)$ to $y(t)$ will exhibit oscillations.

(3 p)

Part B

Note: Write 'skip' if your total home assignment score ≥ 12

i) The damping of a closed-loop system can be studied by observing its overshoot of $y(t)$ when $r(t)$ is a step function.

ii) When we design the observer poles by the matrix K, then we are determining how fast its errors will decay.

iii) The static gain $G_c(0)$ of the closed-loop system determines its stationary control error $r(t) - y(t)$ as $t \to \infty$

(3 p)

Proposed solution to problem 1

Part A

i) True. After factorization, we observe one pole in the left-half plane

ii) False. After factorization, we observe one zero in the right-half plane

iii) False. In general the poles of the system $G(s)$ are a subset of the eigenvalues of the system matrix A

Part B

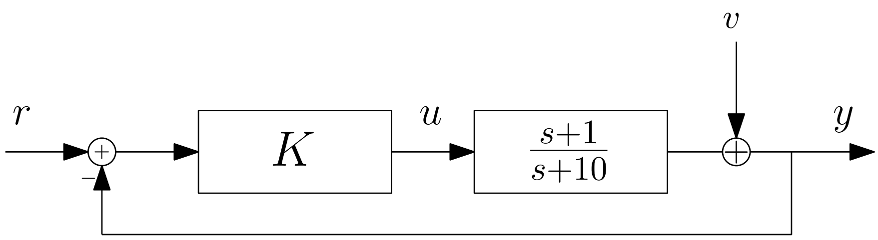
i) True.

ii) True.

iii) True.

Problem 2 (6/30)

a) As a control engineer, your task is to control a mechanical system subject to a high-frequency disturbance on the output. You are using a P-controller, as illustrated in the figure below.



Provide the sensitivity function of the system, that is, the transfer function from disturbance $v(t)$ to the output $y(t)$.

(3 p)

b) The disturbance $v(t)$ has most of its energy at high frequencies $\omega \gg 100$. Show that increasing K makes the controlled system less sensitive to disturbances.

(3 p)

Proposed solution to problem 2

a) The sensitivity function w.r.t. the disturbance $v(t)$ is

$$S(s) = \frac{1}{1 + G_o(s)} = \frac{1}{1 + \frac{s+1}{s+10}K} = \frac{s + 10}{s + 10 + (s + 1)K} = \frac{s + 10}{(K + 1)s + K + 10}$$

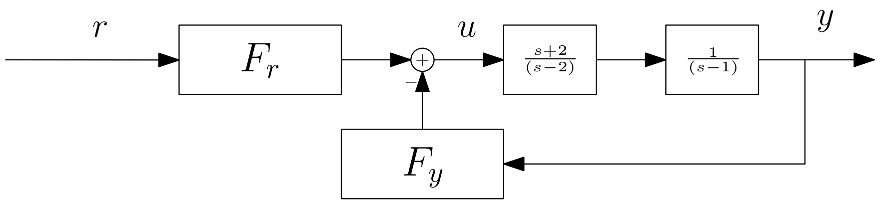
b) The frequency response can be expressed as

$$S(i\omega) = \frac{i\omega + 10}{(K + 1)i\omega + K + 10} = \frac{1 + \frac{10}{i\omega}}{(K + 1) + \frac{K+10}{i\omega}} \approx \frac{1}{(K + 1)}$$

for large ω. Thus increasing K, decreases the magnitude $|S(i\omega)| \approx \frac{1}{(K+1)}$ at these frequencies, which therefore attenuates the high-freqeuency disturbance.

Problem 3 (6/30)

a) You have been assigned to designed a general linear feedback controller to a highly unstable system, illustrated in the figure below.



Derive a state-space description of the system in controllable canonical form.

(2 p)

b) Design the parameters L of a state feedback controller such that the closed-loop system is stable.

(4 p)

Proposed solution to problem 3

a) Using the block diagram we see that

$$G(s) = \frac{s+2}{(s-2)(s-1)} = \frac{s+2}{s^2-3s+2}$$

Thus the state-space description of the system is

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + 0u \end{aligned}$$

where

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

in controllable canonical form.

b) Using a state-feedback controller, the poles of the closed-loop system is given by

$$\det(sI - A + BL) = 0$$

which we want to be stable. That is, we want to match it to two poles in the left-half plane, e.g. $(s+a)(s+b)$ for postive constants. We may for instance choose $(s+1)^2 = s^2 + 2s + 1$. Then

$$\det(sI - A + BL) = \det\left(\begin{bmatrix} s + \ell_1 - 3 & \ell_2 + 2 \\ -1 & s \end{bmatrix}\right) = s^2 + (\ell_1 - 3)s + (\ell_2 + 2)$$

matches the poles when

$$\begin{cases} \ell_1 - 3 = 2 \\ \ell_2 + 2 = 1 \end{cases}$$

so that the parameters of the state-feedback controller equals:

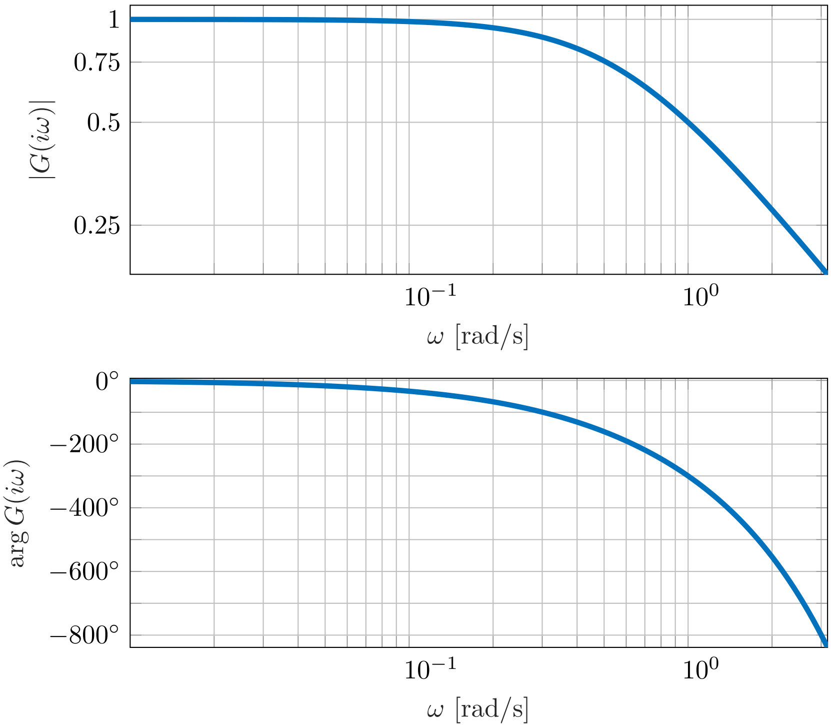
$$L = \begin{bmatrix} 5 & -1 \end{bmatrix}$$

Problem 4 (6/30)

Since you are considered a control expert, you've been asked to investigate the first order system

$$G(s) = e^{-\tau s}\frac{a}{s+b},$$

where $a, b, \tau > 0$ and the exponential factor is due to a time delay of τ seconds from the input to the output. The bode diagram of $G(s)$ is shown below[1].



a) Determine a, b and τ.

(2 p)

b) You want to control the system using a P-controller $F(s) = K > 0$. One way of tuning the gain K is through the so-called *Ziegler-Nichols* method. The idea is to slowly increase the gain until the value K_0, a point at which the closed loop system oscillates with constant amplitude. We then choose $K = 0.5K_0$ for the controller.

Determine K according to this approach, and compute the final value of the output $y(t)$ when the reference signal $r(t)$ is a unit step.

(2 p)

[1] Readings in the diagram do not need to be exact, but should be reasonable.

c) In order to eliminate the error, you know that the controller needs integral action. Hence, you consider the PI-controller

$$F(s) = K\left(1 + \frac{1}{s}\right),$$

using the same K that you found in **b)**. Does this controller result in a stable closed loop system?

(2 p)

Proposed solution to problem 4

a) We have that

$$
\begin{aligned}
|G(i\omega)| &= \underbrace{|e^{-\tau i\omega}|}_{=1} \frac{|a|}{|iw + b|} = \frac{a}{\sqrt{\omega^2 + b^2}}, \\
\arg G(iw) &= \arg e^{-\tau i\omega} + \underbrace{\arg a}_{=0} - \arg(i\omega + b) = -\tau\omega - \arctan \frac{w}{b}.
\end{aligned}
$$

From the bode diagram we see that $|G(0)| = 1$, so

$$
1 = \frac{a}{b} \Leftrightarrow a = b.
$$

Now pick any frequency in the magnitude plot, e.g. $\omega = 1$, from which we find that $|G(i1)| = 0.5$ and so

$$
\begin{aligned}
0.5 &= \frac{b}{\sqrt{1 + b^2}} \\
\Leftrightarrow 0.25(1 + b^2) &= b^2 \\
\Leftrightarrow b &= \frac{1}{\sqrt{3}} \approx 0.58.
\end{aligned}
$$

Next, pick a frequency in the phase plot, again e.g. $\omega = 1$, and note that

$$
\arg G(i1) = -300^\circ = -\frac{5\pi}{3} \text{ rad},
$$

so

$$
\begin{aligned}
-\frac{5\pi}{3} &= -\tau - \arctan \frac{1}{b} \\
\Leftrightarrow \tau &= \frac{5\pi}{3} - \arctan \frac{1}{b} \approx 4.19.
\end{aligned}
$$

Answer: $a = b = 0.58$ and $\tau = 4.19$.

b) Oscillations with constant amplitude implies that the closed loop system is marginally stable, and that the Nyquist curve passes through the point -1, i.e. $G_o(i\omega_0) = -1$. To identify ω_0, note that

$$
\begin{aligned}
G_o(i\omega) &= F(i\omega)G(i\omega) = KG(i\omega), \\
|G_o(i\omega)| &= K|G(i\omega)|, \\
\arg G_o(i\omega) &= \underbrace{\arg K}_{=0} + \arg G(i\omega) = \arg G(i\omega).
\end{aligned}
$$

As only the magnitude is affected, we have that

$$
-180^\circ = \arg G(i\omega_0),
$$

and from the phase plot we find $\omega_0 \approx 0.57$. Then note that

$$
|G(i\omega_0)| = \frac{b}{\sqrt{\omega_0^2 + b^2}} \approx 0.71.
$$

(We could also read $|G(i\omega_0)|$ from the magnitude plot.) Thus we have

$$\begin{aligned} 1 &= K_0|G(i\omega_0)| \\ \Leftrightarrow\ K_0 &= \frac{1}{|G(i\omega_0)|} \approx 1.40. \end{aligned}$$

Therefore we choose $K = 0.5K_0 \approx 0.70$. The final value of the output becomes

$$\lim_{t\to\infty} y(t) = G_c(0) = \frac{KG(0)}{1+KG(0)} = \frac{K}{1+K} \approx 0.41.$$

Answer: $K = 0.70$ and $\lim_{t\to\infty} y(t) = 0.41$.

c) For this controller we have

$$\begin{aligned} F(i\omega) &= K\left(1+\frac{1}{i\omega}\right) = K\left(1-i\frac{1}{\omega}\right), \\ |F(i\omega)| &= K\sqrt{1+1/\omega^2}, \\ \arg F(i\omega) &= -\arctan(1/\omega). \end{aligned}$$

Now we find

$$\begin{aligned} 1 &= |G_o(i\omega_c)| = |Fi\omega_c||G(i\omega_c)| = K\sqrt{1+1/\omega_c^2}\frac{b}{\sqrt{\omega_c^2+b^2}} \\ \Leftrightarrow\ 0 &= \omega_c^4 + b^2(1-K^2)\omega_c^2 - b^2K \\ \Leftrightarrow\ \omega_c^2 &= -\frac{1}{2}b^2(1-K^2) + b\sqrt{\frac{1}{4}b^2(1-K^2)^2+K} \approx 0.33 \\ \Leftrightarrow\ \omega_c &\approx 0.57, \end{aligned}$$

since we must have $\omega_c > 0$. Furthermore

$$\arg F(i\omega_c) = -\arctan(1/\omega_c) \approx -60^\circ.$$

From the phase plot we have $\arg G(i\omega_c) \approx -180^\circ$, and so we get $\arg G_o(i\omega_c) \approx -180^\circ - 60^\circ = -240^\circ$. The Nyquist criterion is clearly not fulfilled ...

Answer: No, the closed loop system is not stable with this controller.

Problem 5 (6/30)

Consider the following second order, continuous time system, which approximate the dynamics of an inherently unstable system.

$$\dot{x} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

a) The continuous time system is analyzed in a simulation tool and we want to design a controller for it. Assume that we have full access to the state vector. Show that the system is controllable and design a state-feedback controller such that the closed loop poles are at -0.5.

(2 p)

b) In order to apply on-board digital control, we need to discretize the system dynamics. A sampling time of $T = 1$ is chosen. Compute the discrete time system matrices.

(2 p)

c) The same controller as designed for the continuous time system (see part a)) is considered out of simplicity for the discrete time dynamics. Give the stability criteria for discrete time poles and compute the closed-loop poles in order to check if the closed-loop system stable still stable.

(2 p)

Proposed solution to problem 5

a) Compute the controllability matrix gives

$$S = \begin{bmatrix} 0 & 3 \\ 1 & 1 \end{bmatrix}$$

which has rank of 2 and therefore the system is controllable.

The controller is given by $u = -Lx$.

The characteristic polynomial yields

$$\det(sI - A + BL) = s^2 + s(\ell_2 - 2) + 1 - \ell_2 + 3\ell_1$$

comparing this with the desired characteristic polynomial $s^2 + s + 1/4$ gives

$$\begin{aligned} \ell_2 - 2 &= 1 \\ 1 - \ell_2 + 3\ell_1 &= 1/4 \end{aligned}$$

which gives $L = \begin{bmatrix} 3/4 & 3 \end{bmatrix}$ as solution.

b) $H = C$. For $F = e^{AT}$ we need

$$(sI - A)^{-1} = \begin{bmatrix} s-1 & -3 \\ 0 & s-1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s-1} & \frac{3}{(s-1)^2} \\ 0 & \frac{1}{s-1} \end{bmatrix}$$

Using the inverse Laplace transformation gives the matrix exponential as

$$F = \begin{bmatrix} e^T & 3Te^T \\ 0 & e^T \end{bmatrix} = \{\text{using } T = 1\} = \begin{bmatrix} e & 3e \\ 0 & e \end{bmatrix}$$

For the input matrix we can make use of $G = A^{-1}(F - I)B$ since A is invertible. Hence,

$$G = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e-1 & 3e \\ 0 & e-1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ e-1 \end{bmatrix}$$

c) The characteristic polynomial is given by

$$\begin{aligned} 0 &= \det(\lambda I - F + GL) \\ &= \lambda^2 + \lambda(e - 3/4) + \frac{e^2}{4} - \frac{3e}{2} \\ &\approx \lambda^2 - 1.9683\lambda - 2.2302 \end{aligned}$$

with roots at

$$\lambda_{1,2} \approx \{0.8043, -2.7726\}$$

For stability we need $|\lambda| < 1$ which is not given here. So the system is not stable since the second pole is smaller than -1.