Baskurs i matematik 2015-03-12 Lösningar

1. $\dfrac{1-9x^2}{2+6x} = \dfrac{(1+3x)(1-3x)}{2(1+3x)} = \dfrac{1-3x}{2}$ Svar: $\dfrac{1-3x}{2}$

2. $\cos\left(\frac{7\pi}{3}\right) = \cos\left(\frac{6\pi}{3}+\frac{\pi}{3}\right) = \cos(2\pi + \pi/3) = \cos(\pi/3) = \frac{1}{2}$ Svar: $1/2$

3. $|x-\frac{1}{2}| < 3 \iff -3 < x - 1/2 < 3$
$\iff -\frac{5}{2} < x < \frac{7}{2}$ Svar: $-5/2 < x < 7/2$

4. $\dfrac{14+5i}{3+2i} = \dfrac{(14+5i)(3-2i)}{(3+2i)(3-2i)} = \dfrac{52-13i}{13} = 4-i$ Svar: $4-i$

5. $\displaystyle\sum_{k=2}^{5} \frac{k}{2^k} = \frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32} = \frac{16+12+8+5}{32} = \frac{41}{32}$ Svar: $\frac{41}{32}$

6. $\log_2 88 - 2\log_2\sqrt{11} = \log_2 88 - \log_2 11 = \log_2 \frac{88}{11} = \log_2 8 = 3$ Svar: 3

7.
$$\begin{array}{c} 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\ 1 \quad 6 \quad \underline{15} \quad 20 \quad \underline{15} \quad 6 \quad 1 \end{array}$$
$\binom{6}{2}$ $\binom{6}{4}$

Svar: $n=6$, $k=2$ (or $k=4$)

8. $\left(\frac{27}{8}\right)^{-2/3} = \left(\left(\frac{8}{27}\right)^{1/3}\right)^2 = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$ Svar: $4/9$

9. $\sin(3x - \pi/3) = \frac{\sqrt{3}}{2}$ p.v. $\arcsin\frac{\sqrt{3}}{2} = \pi/3$

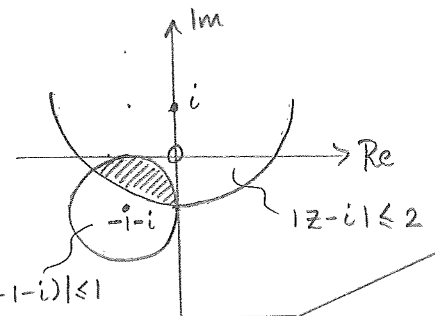
$$\left.\begin{aligned} 3x_1 - \tfrac{\pi}{3} &= \pi/3 + 2k\pi \\ 3x_2 - \pi/3 &= \pi - \pi/3 + 2k\pi \end{aligned}\right\} k \in \mathbb{Z}$$

$$\left.\begin{aligned} 3x_1 &= 2\pi/3 + 2k\pi \\ 3x_2 &= \pi + 2k\pi \end{aligned}\right\} k \in \mathbb{Z}$$

$$\left.\begin{aligned} x_1 &= 2\pi/9 + \frac{2k\pi}{3} \\ x_2 &= \pi/3 + \frac{2k\pi}{3} \end{aligned}\right\} k \in \mathbb{Z}$$

Svar: $\left.\begin{aligned} x_1 &= \frac{2\pi}{9} + \frac{2k\pi}{3} \\ x_2 &= \frac{\pi}{3} + \frac{2k\pi}{3} \end{aligned}\right\} k \in \mathbb{Z}$

10. Cirklar: $|z-i| = 2$, $|z-(-1-i)| = 1$



//// $|z-i| \leq 2$ och $|z+1+i| \leq 1$

11.

Fall	$\lvert x+6\rvert$	$\lvert x\rvert$	
$x<-6$	$-(x+6)$	$-x$	(1)
$-6 \leq x < 0$	$x+6$	$-x$	(2)
$x \geq 0$	$x+6$	x	(3)

(1) ger $-x-6+3x = 1$
$2x = 7$
$x = 7/2$ ※ stämmer inte

(2) ger $x+6+3x = 1$
$4x = -5$
$x = -5/4$

(3) ger $x+6-3x = 1$
$-2x = -5$
$x = 5/2$

Svar: $x_1 = -5/4$, $x_2 = 5/2$

12.

$$
\begin{array}{r l}
 & x^2 - 6x + 5 \\
x - 4 & \overline{\big) \, x^3 - 10x^2 + 29x - 20} \\
 & \underline{x^3 - 4x^2} \\
 & 0 \quad -6x^2 + 29x \\
 & \quad \underline{-6x^2 + 24x} \\
 & \qquad\qquad 5x - 20 \\
 & \qquad\qquad \underline{5x - 20} \\
 & \qquad\qquad\qquad 0
\end{array}
$$

Svar : $x^2 - 6x + 5$

13.

$$\log_2(x-1) + \log_2(x+6) = 3 \qquad (*)$$

$$\log_2\big((x-1)(x+6)\big) = 3$$

$$(x-1)(x+6) = 2^3$$

$$x^2 + 5x - 6 = 8$$

$$x^2 + 5x - 14 = 0$$

$$(x-2)(x+7) = 0$$

$$x_1 = 2 \quad , \quad \underbrace{x_2 = -7}_{\text{ger ingen lösning i } (*)}$$

Svar : $x = 2$

14.

$$(x^2+3)^7 = \sum_{k=0}^{7} \binom{7}{k} x^{2k} (3)^{7-k}$$

$k = 6$ ger x^{12} termen :

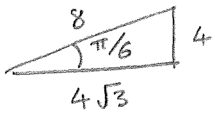
$$\binom{7}{6} \cdot 3^{7-6} = 7 \cdot 3 = 21$$

Svar : $x = 21$

15. Euler form :

$$|4\sqrt{3} + 4i| = |4||\sqrt{3} + i| = 4\sqrt{3+1} = 4 \cdot 2 = 8$$

$$\arg(4\sqrt{3} + 4i) = \pi/6$$



$$z^3 = r^3 e^{3i\theta} = 8e^{\pi i/6}$$

$$\Rightarrow \ r^3 = 8 \ \text{och} \ 3\theta = \pi/6 + 2k\pi$$

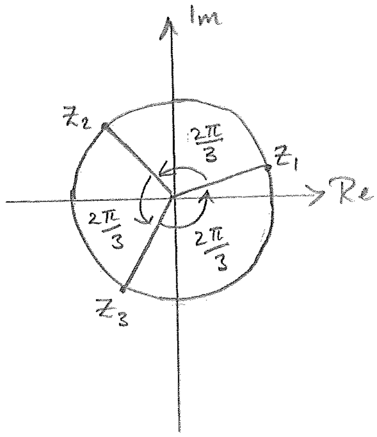
$$\Rightarrow \ r = 2 \ \text{och} \ \theta = \pi/18 + \frac{2k\pi}{3}$$

k	0	1	2
θ	$\pi/18$	$13\pi/18$	$25\pi/18$

Svar : $z_1 = 2e^{\pi i/18}$

$z_2 = 2e^{13\pi i/18}$

$z_3 = 2e^{25\pi i/18}$



16. (1) $16x^2 + y^2 - 4y - 12 = 0$
(2) $y = x - 2$

$\Rightarrow 16x^2 + (x-2)^2 - 4(x-2) - 12 = 0$

$\Rightarrow 17x^2 - 8x = 0$

$x(17x - 8) = 0$

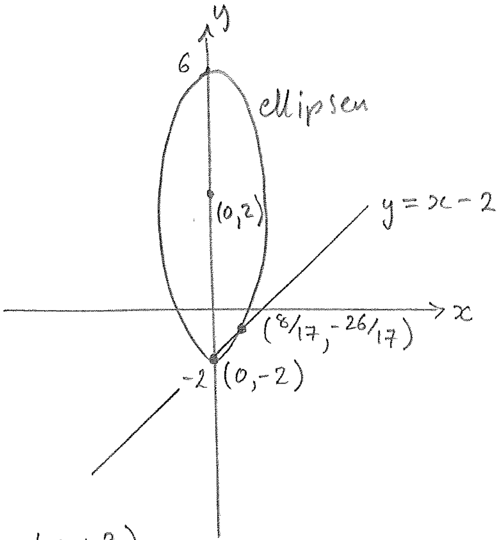
$x_1 = 0 \quad x_2 = 8/17$

$y_1 = -2 \quad y_2 = 8/17 - 2 = -26/17$ enligt (2)

$16x^2 + y^2 - 4y - 12 = 0$ har en medelpunkt på y-axeln $(x = 0)$

Om $x = 0$, $y^2 - 4y - 12 = 0$

$(y+2)(y-6) = 0 \Rightarrow y_1 = -2 \quad y_2 = 6$ medelpunkt $(0,2)$



Svar: $(0,-2)$
$(8/17, -26/17)$

17. $P(n): \sum_{k=1}^{n} \frac{k+1}{2} = \frac{n(n+3)}{4}$

1. $P(1)$: $VL = \frac{1+1}{2} = 1 \qquad HL = \frac{1(1+3)}{4} = 1$

$P(1)$ är sant

2. Antag $P(m)$ är sant.

Mål
$$\sum_{k=1}^{m+1} \frac{k+1}{2} = \frac{(m+1)(m+4)}{4}$$

Betrakta
$$\sum_{k=1}^{m+1} \frac{k+1}{2} = \left(\sum_{k=1}^{m} \frac{k+1}{2}\right) + \frac{m+1+1}{2}$$

$$= \frac{m(m+3)}{4} + \frac{m+2}{2} \quad \text{av induktionsantagandet}$$

$$= \frac{m^2 + 3m + 2m + 4}{4}$$

$$= \frac{m^2 + 5m + 4}{4}$$

$$= \frac{(m+1)(m+4)}{4} \Rightarrow P(m+1) \text{ gäller}$$

Så $P(m) \Rightarrow P(m+1)$.

3. Eftersom $P(1)$ är sant och $P(m) \Rightarrow P(m+1)$ då är $P(n)$ sant för alla $n \in \mathbb{N}, n \geq 1$.

18. $z_1 = 2i$ är en lösning $\Rightarrow$ $z_2 = -2i$ är en lösning (konjugatet)

$\Rightarrow$ $(z-2i)(z+2i) = z^2+4$ är en faktor av $f(z) = z^4+2z^3+2z^2+8z-8$

Polynomdivision

$$
\begin{array}{r|l}
 & z^2 + 2z - 2 \\
\hline
z^2+0z+4 & z^4+2z^3+2z^2+8z-8 \\
 & \underline{z^4+0z^3+4z^2} \quad \downarrow \\
 & 0 \quad 2z^3-2z^2+8z \\
 & \quad \underline{2z^3+0z^2+8z} \\
 & \quad 0 \quad -2z^2+0z-8 \\
 & \quad\quad \underline{-2z^2+0z-8} \\
 & \quad\quad 0 \quad 0 \quad 0
\end{array}
$$

Lös $z^2+2z-2=0$

$(z+1)^2-1-2=0$

$z=-1\pm\sqrt{3}$

Svar : $z_1 = 2i$, $z_2=-2i$, $z_3 = -1-\sqrt{3}$, $z_4=-1+\sqrt{3}$