Introduction to Computer Control Systems, 5 credits, 1RT485

Date: 2023-03-16

Teacher on duty: Niklas Wahlström

Teacher visit: around 10:15

Number of problems: 5

Allowed aid: A calculator and mathematical handbooks

Preliminary grades:	grade 3	15 points
	grade 4	21 points
	grade 5	26 points

Some general instructions and information:

- Your solutions can be given in Swedish or in English.
- Write only on one side of the paper.
- Write your exam code and page number on all pages.
- Do not use a red pen.
- Use separate sheets of paper for the different problems (i.e. the numbered problems, 1–5).

*With the exception of Problem 1, **all your answers must be clearly motivated!** A correct answer without a proper motivation will score zero points!*

Best of luck!

Useful results

Laplace transform table

Table 1: Basic Laplace transforms

$f(t)$	$F(s)$	$f(t)$	$F(s)$
unit impulse $\delta(t)$	1	$\sinh(bt)$	$\frac{b}{s^2-b^2}$
unit step $1(t)$	$\frac{1}{s}$	$\cosh(bt)$	$\frac{s}{s^2-b^2}$
t	$\frac{1}{s^2}$	$\frac{1}{2b}t\sin(bt)$	$\frac{s}{(s^2+b^2)^2}$
t^n	$\frac{n!}{s^{n+1}}$	$t\cos(bt)$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
e^{-at}	$\frac{1}{s+a}$	$\frac{\cos(bt)-\cos(at)}{a^2-b^2}; (a^2 \neq b^2)$	$\frac{s}{(s^2+a^2)(s^2+b^2)}$
$\frac{1}{a}(1-e^{-at})$	$\frac{1}{s(s+a)}$	$\frac{\sin(at)+at\cos(at)}{2a}$	$\frac{s^2}{(s^2+a^2)^2}$
$\frac{1}{(n-1)!}t^{n-1}e^{-at}; (n=1,2,3...)$	$\frac{1}{(s+a)^n}$		
$\sin(bt)$	$\frac{b}{s^2+b^2}$		
$\cos(bt)$	$\frac{s}{s^2+b^2}$		
$e^{-at}\sin(bt)$	$\frac{b}{(s+a)^2+b^2}$		
$e^{-at}\cos(bt)$	$\frac{s+a}{(s+a)^2+b^2}$		

Table 2: Properties of Laplace Transforms

$\mathcal{L}[af(t)] = aF(s)$	$\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}$
$\mathcal{L}[f_1(t)+f_2(t)] = F_1(s)+F_2(s)$	$\mathcal{L}\left[t^2 f(t)\right] = \frac{d^2}{ds^2}F(s)$
$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = sF(s)-f(0)$	$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s), \quad n=1,2,3,\ldots$
$\mathcal{L}\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s)-sf(0)-f'(0)$	$\mathcal{L}\left[f\left(\frac{t}{a}\right)\right] = aF(as)$
$\mathcal{L}\left[\int f(t)\,dt\right] = \frac{F(s)}{s}+\frac{1}{s}\left[\int f(t)\,dt\right]_{t=0}$	$\mathcal{L}\left[\int_0^t f_1(t-\tau)f(t)d\tau\right] = F_1(s)F_2(s)$
$\mathcal{L}[f(t-a)] = e^{-as}F(s)$	$\mathcal{L}[e^{-at}f(t)] = F(s+a)$

Matrix exponential

$$\boxed{e^{At} \triangleq \mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\}}$$

Open-loop and sensitivity functions

$$G_o(s) = G(s)F_y(s), \qquad S(s) = \frac{1}{1+G_o(s)}, \qquad T(s) = 1-S(s)$$

State-space forms and transfer function relations

- State-space form and transfer function

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad \Rightarrow \quad \boxed{G(s) = C(sI - A)^{-1}B + D}$$

- Associated matrices

$$\mathcal{S} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \qquad \mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

- LTI system with transfer function

$$\boxed{G(s) = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}}$$

i) Observable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ -a_3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ b_3 - a_3 b_0 \\ \vdots \\ b_n - a_n b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

ii) Controllable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_1 - a_1 b_0 & b_2 - a_2 b_0 & \cdots & b_n - a_n b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

- Solution to state-space equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

can be written as

$$\boxed{x(t) = e^{At}x_0 + \int_0^t e^{A\tau} Bu(t - \tau) d\tau}$$

- Observer system

$$\boxed{\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})}$$

Feedback control structures

General linear feedback in Laplace form:

$$\boxed{U(s) = F_r(s)R(s) - F_y(s)Y(s)}$$

Common control structures in this form.

- PID controller:

$$F_y(s) = F_r(s) = F(s) = K_p + \frac{K_i}{s} + K_d s,$$

  where $K_p, K_i, K_d \geq 0$

- Lead-lag controller:

$$F_y(s) = F_r(s) = F(s) = K\left(\frac{\tau_D s + 1}{\beta\tau_D s + 1}\right)\left(\frac{\tau_I s + 1}{\tau_I s + \gamma}\right),$$

  where $K, \tau_D, \tau_I > 0$ and $0 \leq \beta, \gamma < 1$

- State-feedback controller with observer:

$$F_r(s) = \left(1 - L(sI - A + KC + BL)^{-1}B\right)\ell_0$$
$$F_y(s) = L(sI - A + KC + BL)^{-1}K$$

Discrete-time state-space forms

A continuous time system with zero-order-hold input signal and sample period T can be written in discrete-time as:

$$\boxed{\begin{aligned} x(k+1) &= Fx(k) + Gu(k) \\ y(k) &= Hx(k) \end{aligned}}$$

where

$$\begin{aligned} F &= e^{AT} \\ G &= \int_{\tau=0}^{T} e^{A\tau}d\tau B = \lceil\text{if } A^{-1} \text{ exists}\rceil = A^{-1}(e^{AT} - I)B \\ H &= C \end{aligned}$$

Problem 1: basic questions (6/30)

Answer only *‘true’* or *’false’*. Each correct answer gives 1 point, each wrong answer gives -1 point. Minimum total points for Part A and B is 0 , respectively.

Part A

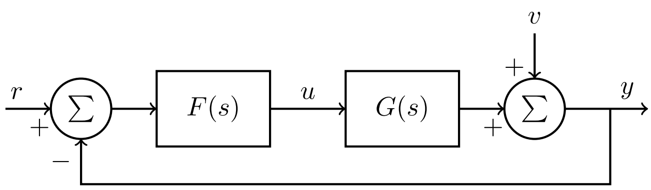
Note: Write ‘skip’ if your total home assignment score ≥ 8

i) The following system is observable

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

ii) Consider the control system



where we measure y and where v is a disturbance. If we also could measure the disturbance v, cascade control could be used to improve the performance of the control system.

iii) A state-space description is a ’minimal realization’ if it is either observable or controllable.

(3 p)

Part B

Note: Write ‘skip’ if your total home assignment score ≥ 12

i) A system with poles in -1 and -10 is faster than a system with poles in -2 and -5, i.e. the rise time of a step response lower for the first system.

ii) The system

$$G(s) = 3\frac{s-1}{(s-1)(s+3)}$$

is input-output stable.

iii) Figure 1 on next page depicts the Nyquist diagram of a stable open loop system, $G_o(i\omega)$. With this information we can guarantee that the closed loop system $G_c(s) = \frac{G_o(s)}{1+G_o(s)}$ is stable.

(3 p)

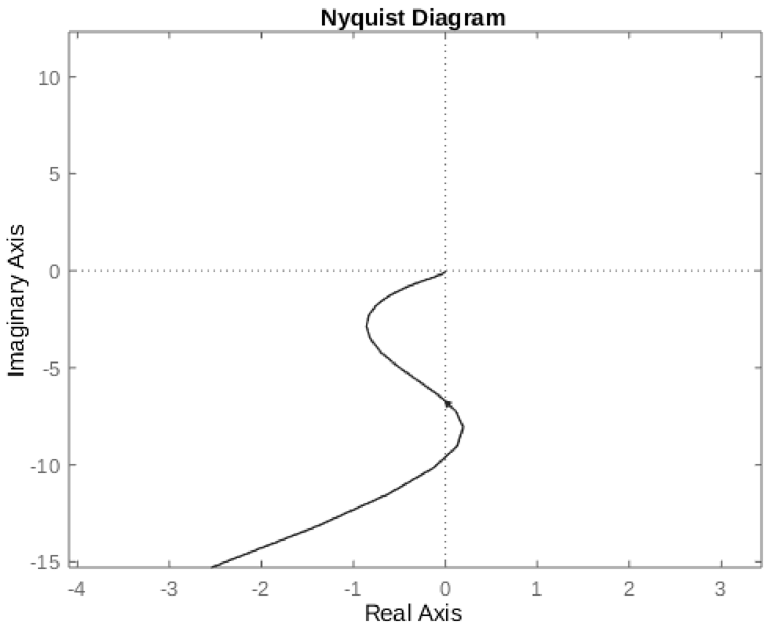
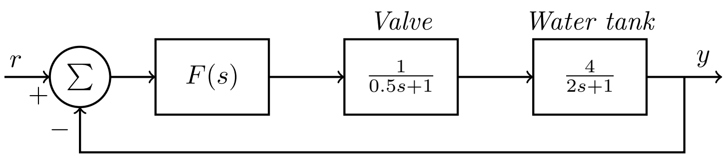


Figure 1: A Nyquist diagram for an open system in Problem 1 Part B iii) where ω goes from 0 to ∞

Problem 2 (6/30)

A description for controlling the water level in a tank is provided by the figure below:



a) Suppose we use a P-controller

$$F(s) = K_P$$

What are the poles of the closed loop system $G_c(s)$ from r to y as a function of K_p? For which values of $K_p > 0$ is the closed-loop system stable?

(2 p)

b) Assume now that we instead use a PI-controller

$$F(s) = K_P + K_I \frac{1}{s}$$

The control system is evaluated by using a step as reference signal for some different combinations, see below, of the coefficients in PI-controller. Match the step responses in Figure 2 on next page with the coefficients. Motivate!

- 1: $K_P = 0.5, K_I = 0$ 2: $K_P = 1, K_I = 0$ 3: $K_P = 4, K_I = 0$
- 4: $K_P = 1, K_I = 1$ 5: $K_P = 1, K_I = 2$

Note: One of the regulators do not have a corresponding step response.

(2 p)

c) Provide a state-space description of the *closed loop system* $G_c(s)$ from r to y where $F(s) = 1$.

(2 p)

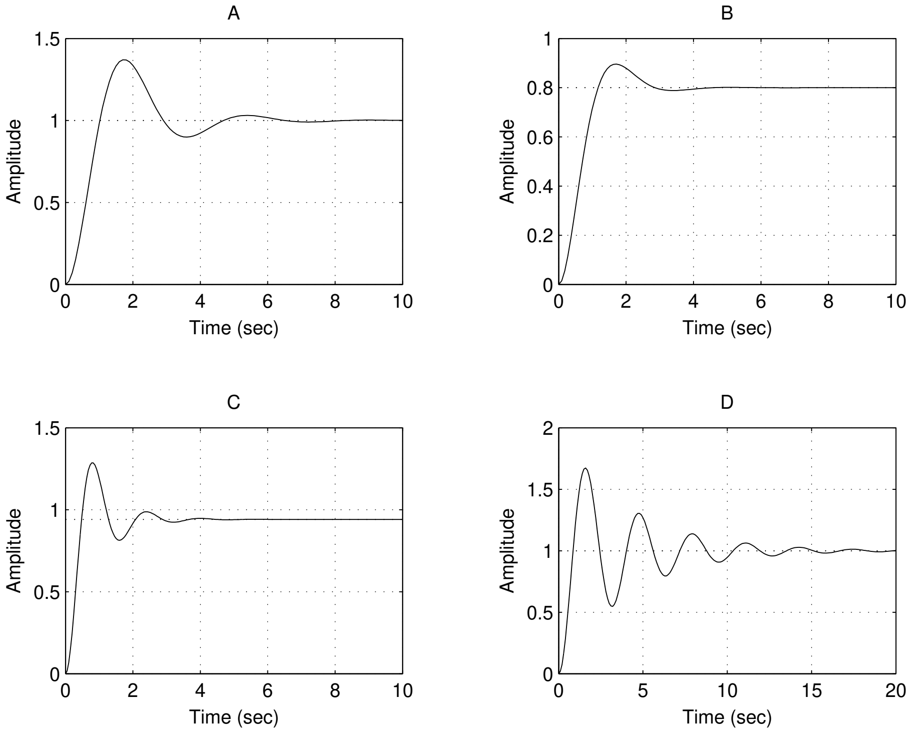
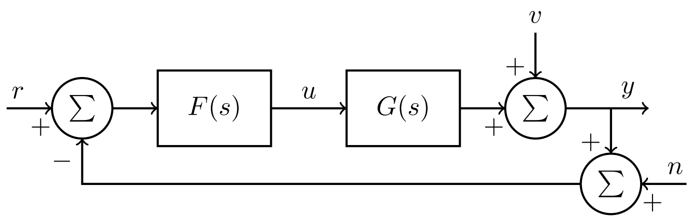


Figure 2: Step responses in Problem 2b

Problem 3 (6/30)

An antenna is used to communicate with several different satellites. The antenna must be able to be aimed at the different satellites, and switch from one satellite to another. It must also be able to follow a satellite that is not geostationary, i.e. the satellite is moving relative to the earth. We want to control the direction of the antenna $y(t)$ and the control signal is applied voltage $u(t)$. The antenna is affected by an additive disturbance $v(t)$, and the angle is measured with a measurement error $n(t)$. The system $G(s)$ does not have any poles in the right half plane. The block diagram below shows the system.



In problem a-c below a P-controller with $F(s) = 1$ is considered, which results in a stable closed-loop system $G_c(s)$. With this controller the Bode diagram for $G_o(s) = F(s)G(s)$ is displayed in Figure 3, and Bode diagram for $S(s) = \frac{1}{1+G_o(s)}$ and $G_c(s) = T(s) = \frac{G_o(s)}{1+G_o(s)}$ are displayed in Figure 4 on the following two pages.

a) What is the cross-over frequency and phase margin of the open-loop system $G_o(s)$ if we use the suggested P-controller $F(s) = 1$?

(1 p)

b) Assume that $r(t) = 0$ and $n(t) = 0$ and that the disturbance $v(t) = \sin(t)$. What will $y(t)$ be when all transients have died out?

(2 p)

c) Which frequencies of the measurement noise n will be damped by the control system and which frequencies will be amplified by the control system, respectively?

(1 p)

d) Assume we want to try two other P-controllers, one with $F(s) = 5$ and one with $F(s) = 10$. Are the closed-loop systems still stable with these controllers? If so, what is the performance in respect to damping and quickness for these controllers in relation to the P-controller $F(s) = 1$. Remember to motivate your answer properly.

(2 p)

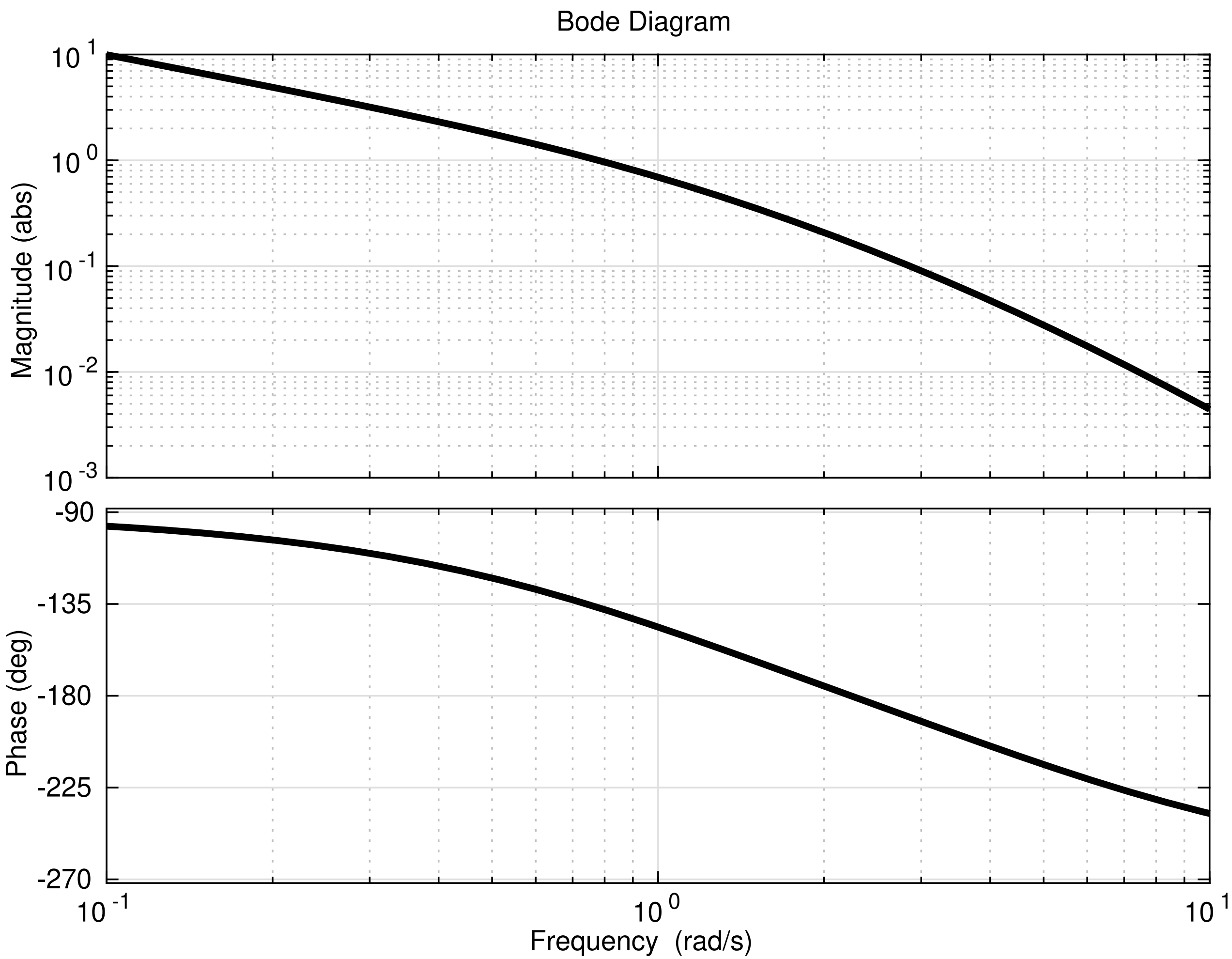


Figure 3: Bode diagram for open-loop system $G_o(s) = F(s)G(s)$ with $F(s) = 1$ in problem 3

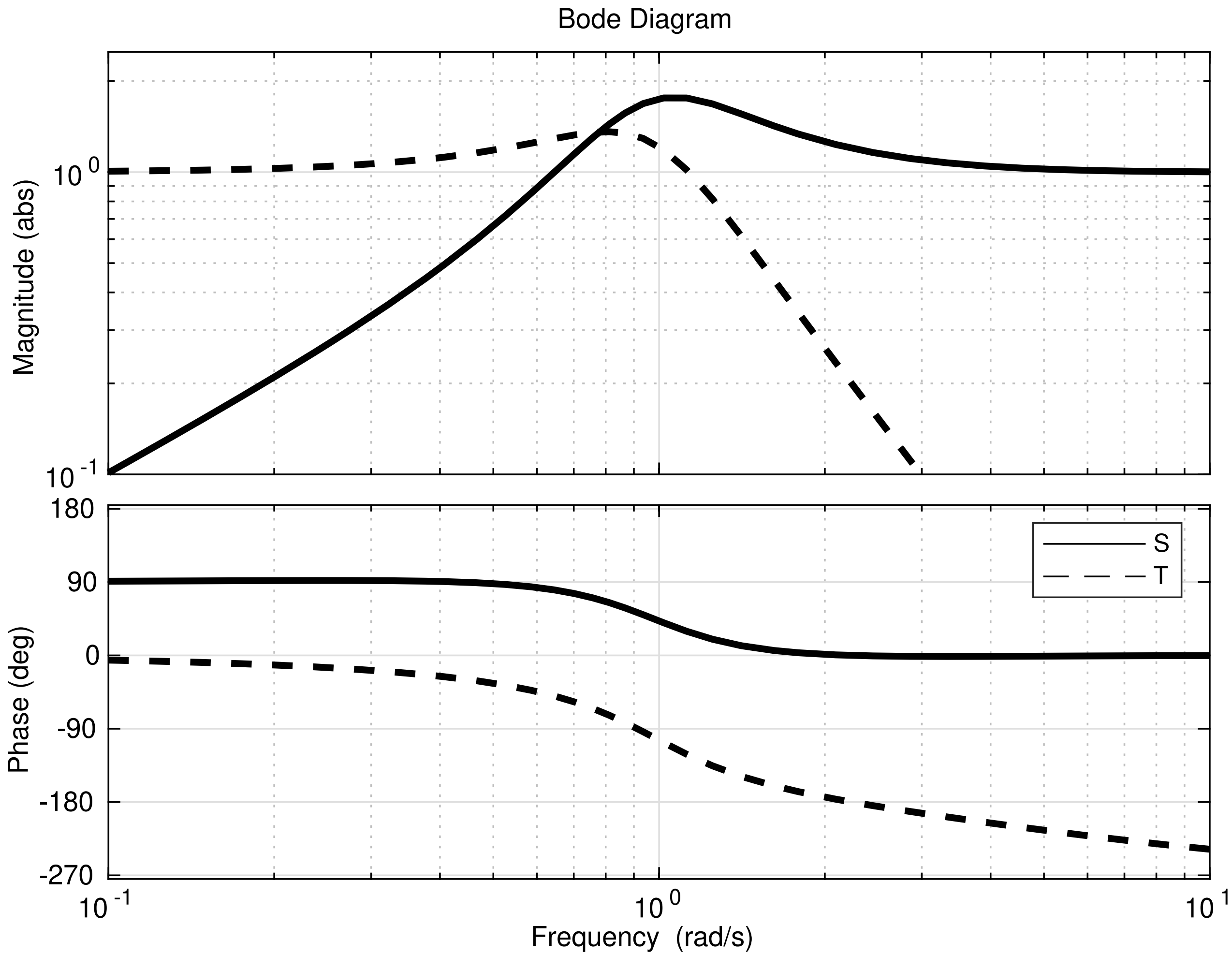


Figure 4: Bode diagram for $S(s)$ (solid line) and $T(s) = G_c(s)$ (dashed line) with $F(s) = 1$ in problem 3

Problem 4 (6/30)

Your colleagues are interested in an unstable system,

$$\frac{Y(s)}{U(s)} = \frac{s+11}{(s+6)(s+4)(s-1)} = G(s).$$

They wish to control it and fortunately this system can be stabilized with proportional feedback, i.e with the control law,

$$U(s) = K(R(s) - Y(s)).$$

a) To investigate the systems behavior your colleagues produced a root locus of the poles of the closed loop system (see Figure 5 on next page), comment on the qualitative behavior (stability, quickness and oscillations) of the closed loop system for increasing values of the proportionality constant, K in the whole range $0 < K < \infty$. In this exercise, you don't need to compute value of K when the qualitative behavior changes.

(2 p)

b) For which values of the proportionality constant, K, is the closed loop system stable?

(2 p)

c) With your help your colleagues choose K so that the closed loop system is stable and presented you with a bode diagram of the closed loop system (see Figure 6 on next page). However, they noticed that they made an error while identifying the system and the unstable pole could be up to 10% off. That is, the true system, $G^*(s)$, looks like

$$G^*(s) = \frac{s+11}{(s+6)(s+4)(s-a)}$$

where $0.9 < a < 1.1$. Can you provided them with any insurance that the real closed loop system is stable anyway?

(2 p)

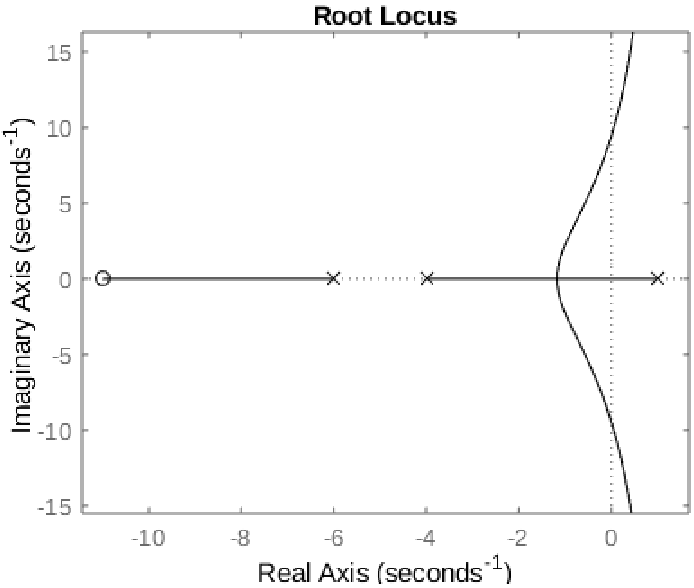


Figure 5: Root locus in Problem 4 a). $\times$ denotes start points and $\circ$ denotes end points.

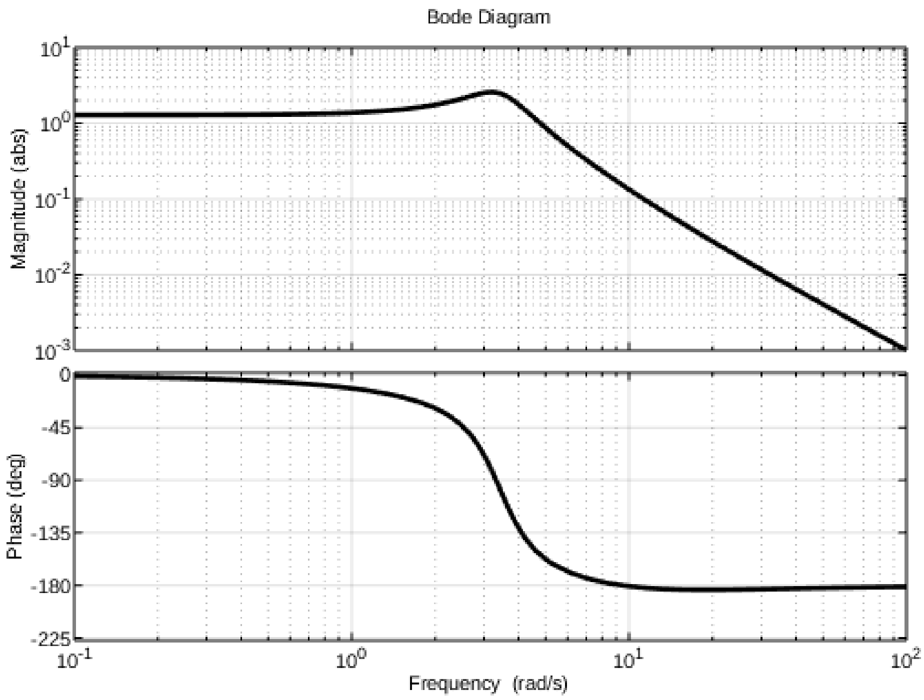


Figure 6: Bode diagram of the closed loop system in Problem 4 c)

Problem 5 (6/30)

A toy car is driving on a track and we want to control its position, z, and velocity, v. The only force acting on the car comes from its motor and is denoted F.

$$F = m\ddot{z}$$

For this system we can write down the state-space model as,

$$\begin{aligned}\begin{bmatrix} \dot{z} \\ \dot{v} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} z \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} F \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} z \\ v \end{bmatrix}.\end{aligned}$$

a) Is it possible to control the car from $z(0) = v(0) = 0$ to any position $z(t_{\text{end}})$ and velocity $v(t_{\text{end}})$ given this model and assuming there is no limitation on how much force F can be applied?

(1 p)

b) Discretize the state-space model given above with sampling time T.

(2 p)

c) Suppose that we want to control the car's position and velocity, and assume that we have access to the states. Design a state-feedback controller $F = -\ell_1 z - \ell_2 v + r$ for the discrete-time system so that the closed loop system has discrete poles corresponding to $-10\ln(2)$ in continuous time. Use the values $T = 0.1\ s$ and $m = 0.01\ kg$.

Note: In this exercise you do not need to determine the static gain ℓ_0.

(3 p)