Introduction to Computer Control Systems, 5 credits, 1RT485

Date: 2023-06-14

Teacher on duty: Carl Andersson

Number of problems: 5

Allowed aid: A calculator and mathematical handbooks (e.g. BETA)

Preliminary grades:	grade 3	15 points
	grade 4	21 points
	grade 5	26 points

Some general instructions and information:

- Your solutions can be given in Swedish or in English.
- Write only on one side of the paper.
- Write your exam code and page number on all pages.
- Do not use a red pen.
- Use separate sheets of paper for the different problems (i.e. the numbered problems, 1–5).

With the exception of Problem 1, ***all your answers must be clearly motivated!*** *A correct answer without a proper motivation will score zero points!*

Best of luck!

Useful results

Laplace transform table

Table 1: Basic Laplace transforms

$f(t)$	$F(s)$	$f(t)$	$F(s)$
unit impulse $\delta(t)$	1	$\sinh(bt)$	$\frac{b}{s^2-b^2}$
unit step $1(t)$	$\frac{1}{s}$	$\cosh(bt)$	$\frac{s}{s^2-b^2}$
t	$\frac{1}{s^2}$	$\frac{1}{2b} t \sin(bt)$	$\frac{s}{(s^2+b^2)^2}$
t^n	$\frac{n!}{s^{n+1}}$	$t \cos(bt)$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
e^{-at}	$\frac{1}{s+a}$	$\frac{\cos(bt)-\cos(at)}{a^2-b^2}; (a^2 \neq b^2)$	$\frac{s}{(s^2+a^2)(s^2+b^2)}$
$\frac{1}{a}(1-e^{-at})$	$\frac{1}{s(s+a)}$	$\frac{\sin(at)+at\cos(at)}{2a}$	$\frac{s^2}{(s^2+a^2)^2}$
$\frac{1}{(n-1)!} t^{n-1} e^{-at}; (n = 1, 2, 3...)$	$\frac{1}{(s+a)^n}$		
$\sin(bt)$	$\frac{b}{s^2+b^2}$		
$\cos(bt)$	$\frac{s}{s^2+b^2}$		
$e^{-at}\sin(bt)$	$\frac{b}{(s+a)^2+b^2}$		
$e^{-at}\cos(bt)$	$\frac{s+a}{(s+a)^2+b^2}$		

Table 2: Properties of Laplace Transforms

$\mathcal{L}\left[af(t)\right] = aF(s)$	$\mathcal{L}\left[tf(t)\right] = -\frac{dF(s)}{ds}$
$\mathcal{L}\left[f_1(t) + f_2(t)\right] = F_1(s) + F_2(s)$	$\mathcal{L}\left[t^2 f(t)\right] = \frac{d^2}{ds^2} F(s)$
$\mathcal{L}\left[\frac{d}{dt} f(t)\right] = sF(s) - f(0)$	$\mathcal{L}\left[t^n f(t)\right] = (-1)^n \frac{d^n}{ds^n} F(s), \quad n = 1, 2, 3, \ldots$
$\mathcal{L}\left[\frac{d^2}{dt^2} f(t)\right] = s^2 F(s) - sf(0) - f'(0)$	$\mathcal{L}\left[f\left(\frac{t}{a}\right)\right] = aF(as)$
$\mathcal{L}\left[\int f(t)\, dt\right] = \frac{F(s)}{s} + \frac{1}{s}\left[\int f(t)\, dt\right]_{t=0}$	$\mathcal{L}\left[\int_0^t f_1(t-\tau) f(t) d\tau\right] = F_1(s) F_2(s)$
$\mathcal{L}\left[f(t-a)\right] = e^{-as} F(s)$	$\mathcal{L}\left[e^{-at} f(t)\right] = F(s+a)$

Matrix exponential

$$\boxed{e^{At} \triangleq \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}}$$

Open-loop and sensitivity functions

$$G_o(s) = G(s) F_y(s), \qquad S(s) = \frac{1}{1 + G_o(s)}, \qquad T(s) = 1 - S(s)$$

State-space forms and transfer function relations

- State-space form and transfer function

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad \Rightarrow \quad \boxed{G(s) = C(sI - A)^{-1}B + D}$$

- Associated matrices

$$\mathcal{S} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \qquad \mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

- LTI system with transfer function

$$\boxed{G(s) = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}}$$

i) Observable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ -a_3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ b_3 - a_3 b_0 \\ \vdots \\ b_n - a_n b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

ii) Controllable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_1 - a_1 b_0 & b_2 - a_2 b_0 & \cdots & b_n - a_n b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

- Solution to state-space equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

can be written as

$$\boxed{x(t) = e^{At}x_0 + \int_0^t e^{A\tau} Bu(t - \tau) d\tau}$$

- Observer system

$$\boxed{\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})}$$

Feedback control structures

General linear feedback in Laplace form:

$$\boxed{U(s) = F_r(s)R(s) - F_y(s)Y(s)}$$

Common control structures in this form.

- PID controller:

$$F_y(s) = F_r(s) = F(s) = K_p + \frac{K_i}{s} + K_d s,$$

  where $K_p, K_i, K_d \geq 0$

- Lead-lag controller:

$$F_y(s) = F_r(s) = F(s) = K\left(\frac{\tau_D s + 1}{\beta\tau_D s + 1}\right)\left(\frac{\tau_I s + 1}{\tau_I s + \gamma}\right),$$

  where $K, \tau_D, \tau_I > 0$ and $0 \leq \beta, \gamma < 1$

- State-feedback controller with observer:

$$\begin{aligned} F_r(s) &= \big(1 - L(sI - A + KC + BL)^{-1}B\big)\,\ell_0 \\ F_y(s) &= L(sI - A + KC + BL)^{-1}K \end{aligned}$$

Discrete-time state-space forms

A continuous time system with zero-order-hold input signal and sample period T can be written in discrete-time as:

$$\boxed{\begin{aligned} x(k+1) &= Fx(k) + Gu(k) \\ y(k) &= Hx(k) \end{aligned}}$$

where

$$\begin{aligned} F &= e^{AT} \\ G &= \int_{\tau=0}^{T} e^{A\tau} d\tau B = \lceil \text{if } A^{-1} \text{ exists} \rceil = A^{-1}(e^{AT} - I)B \\ H &= C \end{aligned}$$

Problem 1: basic questions (6/30)

Answer only *'true'* or *'false'*. Each correct answer gives 1 point, each wrong answer gives -1 point. Minimum total points for Part A and B is 0 , respectively.

Part A

Note: Write 'skip' if your total home assignment score ≥ 8

i) P-controllers are always able to bring the stationary control error to 0.

ii) We control the following system

$$G(s) = \frac{s}{s+4}$$

with a P-controller with constant K. The closed-loop system is unstable for all $K \geq 4$.

iii) The system

$$G(s) = \frac{(s+2)(s-1)}{(s^2+2s-3)(s+5)}$$

is input-output stable.

(3 p)

Part B

Note: Write 'skip' if your total home assignment score ≥ 12

i) All state space models are either observable or controllable.

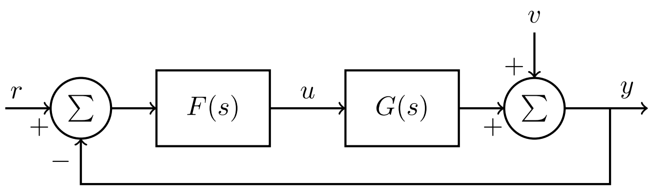
ii) With a sufficiently well-designed regulator, one can simultaneously achieve $|T(i\omega_B)| \leq 0.34$ and $|S(i\omega_B)| \leq 0.34$ for both sensitivity functions, where ω_B is the bandwidth.

iii) Consider a closed-loop system using a state-feedback controller. If we introduce an observer to the closed-loop system, its poles change.

(3 p)

Problem 2 (6/30)

The block diagram below shows a closed-looped system



where

$$
\begin{aligned}
G(s) &= \frac{1}{s(s-1.2)} \\
F(s) &= 1 + \tau s, \quad \tau \in \mathbb{R}
\end{aligned}
\tag{1}
$$

a) The regulator in (1), $F(s)$, is of a commonly used type. State what this regulator type is called.

(1p)

b) The feedback system in the block diagram and (1) can be written $Y(s) = G_c(s)R(s) + S(s)V(s)$. Provide the transfer function $G_c(s)$ and the *sensitivity function*, $S(s)$.

(3p)

c) For what values of $\tau \in \mathbb{R}$ does the feedback system in the block diagram and (1) become stable?

(2p)

Problem 3 (6/30)

a) Consider controlling a thermal process with an output modeled as

$$Y(s) = G(s)U(s),$$

where the process model can be written as

$$G(s) = \frac{s+3}{s^2+6s+11}. \tag{2}$$

Assuming the states can be obtained, design a controller so that the closed-loop system from reference r to output y has the poles located at -2 and -5.

(4 p)

b) To prepare for the design of a state observer,

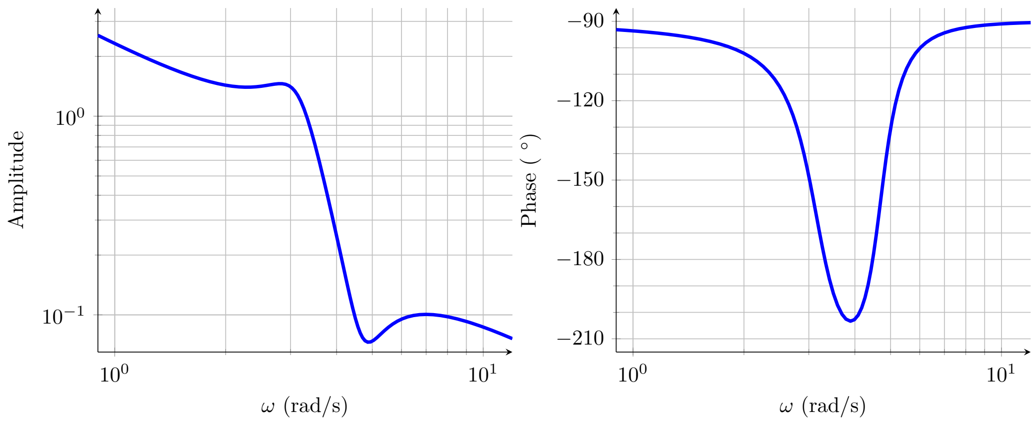
$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x}),$$

write the system (2) on observable canonical form.

(2 p)

Problem 4 (6/30)

a) Below is the Bode diagram for the stable system $Y(s) = G(s)U(s)$.

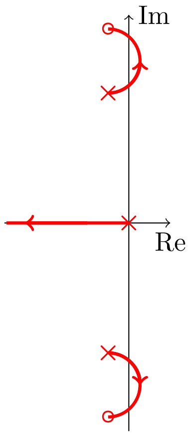


One controls the system with proportional feedback, $U(s) = K(R(s) - Y(s))$. Use the Nyquist criterion to determine for which $K > 0$ the closed-loop system is stable.

Hint: You may want to sketch the Nyquist curve of $G(s)$.

(3 p)

b)



A feedback system has the pole polynomial

$$s(s^2 + s + 10) + K(s^2 + s + 22), \tag{3}$$

where $K \geq 0$ is a regulator parameter. On the left is shown the root locus of the poles of the closed-loop system (i.e., the zeros of (3)) with respect to $K \geq 0$. As can be seen from the root locus, the imaginary axis intersects for two different values of $K > 0$. As we know, poles on the imaginary axis give rise to self-oscillation. State the two values of $K > 0$ for which self-oscillation occurs, and also what angular frequency ω the self-oscillation has in each case.

(3 p)

Problem 5 (6/30)

A system can be described by the following continuous-time state space model:

$$
\begin{aligned}
\begin{bmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \end{bmatrix} &= \begin{bmatrix} -3 & 2 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
\end{aligned} \tag{4}
$$

a) Is the continuous-time system a minimal realization?

(1 p)

b) Your friend has designed a state-feedback controller with the continuous-time poles placed in -6 for the closed-loop system. Now, we can only measure the output y of the system, so we also need an observer in order to be able to estimate the states of the continuous-time system. Choose suitable values for the observer poles and compute the observer gain K. Remember to *motivate* your choice of observer poles!

(3 p)

c) Discretize the continuous-time system (4) with sampling time T and find the system matrix F. Note, in this exercise you don't need to compute the matrices H and G of the discrete-time system.

(2 p)