Baskurs i matematik – tenta 2015-01-17 (lösningar)

1. $|x-5| > 2 \iff x-5>2$ eller $x-5<-2 \iff x>8$ eller $x<3$ (alt. $x \in (-\infty,3] \cup [8,\infty)$)

2. $$\frac{-4+7i}{3-2i} = \frac{(-4+7i)(3+2i)}{(3-2i)(3+2i)} = \frac{-26+13i}{13} = -2+i$$

3. $$\frac{1}{x-3} - \frac{6}{x^2-9} = \frac{x+3-6}{x^2-9} = \frac{x-3}{x^2-9} = \frac{x-3}{(x+3)(x-3)} = \frac{1}{x+3}$$

4. $$\ln 18x^3 - 2\ln 3x = \ln 18x^3 - \ln 9x^2 = \ln\left(\frac{18x^3}{9x^2}\right) = \ln 2x$$

5. $$\sum_{i=1}^{5} 2^i - 3 = \left(\sum_{i=1}^{5} 2^i\right) - 5\cdot 3 = 2+4+8+16+32-15 = 47$$

6. $$4^x \cdot 2^x = 64 \iff 2^{2x}\cdot 2^x = 64 \iff 2^{3x} = 64 \iff 3x = 6 \iff x = 2$$

7. $$\sin\frac{13\pi}{3} = \sin\left(\frac{12\pi}{3} + \frac{\pi}{3}\right) = \sin(4\pi + \pi/3) = \sin(\pi/3) = \frac{\sqrt{3}}{2}$$

8. $$\operatorname{Im}\big((4+3i)(-2+7i)\big) = \operatorname{Im}(-29+22i) = 22$$

9. $$8x^2 - 4x + y^2 = \frac{3}{2}$$

$$8\left(x^2 - \frac{x}{2}\right) + y^2 = \frac{3}{2}$$

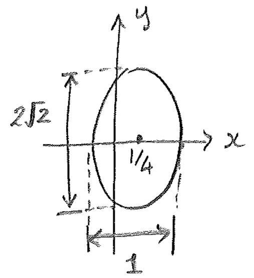
$$8\left(\left(x-\frac{1}{4}\right)^2 - \frac{1}{16}\right) + y^2 = 3/2$$

$$8\left(x-\frac{1}{4}\right)^2 + y^2 = 2$$

$$4\left(x-\frac{1}{4}\right)^2 + \frac{y^2}{2} = 1$$

$$\frac{\left(x-\frac{1}{4}\right)^2}{(1/2)^2} + \frac{y^2}{(\sqrt{2})^2} = 1$$

mittpunkt : $(1/4, 0)$
x-längd : $2\cdot 1/2 = 1$
y-längd : $2\sqrt{2}$



10. Binomial koefficienten $\binom{n}{5} = 21$ ger svaret.
Pascals Δ ger $n = 7$, dvs. $|A| = 7$.

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

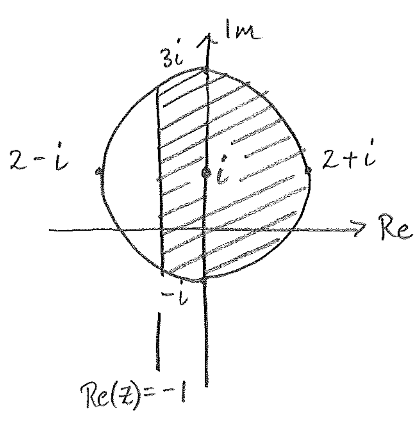
$\binom{7}{5} = 21$

11. $$\sin(2x - \pi) = \frac{1}{2} \iff \left.\begin{aligned} 2x_1 - \pi &= \pi/6 + 2k\pi \\ 2x_2 - \pi &= 5\pi/6 + 2k\pi \end{aligned}\right\}$$

$$\left.\begin{aligned} 2x_1 &= 7\pi/6 + 2k\pi \\ 2x_2 &= 11\pi/6 + 2k\pi \end{aligned}\right\}$$

$$\left.\begin{aligned} x_1 &= 7\pi/12 + k\pi \\ x_2 &= 11\pi/12 + k\pi \end{aligned}\right\} \quad k \in \mathbb{Z}$$

12.



//// $\{z : |z-i| \leq 2 \text{ och } \operatorname{Re}(z) \geq -1\}$

13. $5-5i = 5(1-i) = 5\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)$

$= 5\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$

Av de Moivres sats

$(5-5i)^n = 5^n\left(\cos\frac{7n\pi}{4} + i\sin\frac{7n\pi}{4}\right)$

Så vi söker det minsta $n \in \mathbb{Z}_{>0}$ så att $\frac{7n\pi}{4} = k\pi$ där $k \in \mathbb{Z}$.

<u>Svar</u>: $n = 4$ $\quad (5-5i)^4 = -2500$

14. $\left(x^2 - \frac{1}{x}\right)^9 = \sum_{k=0}^{9}\binom{9}{k} x^{2k}\left(\frac{-1}{x}\right)^{9-k}$

vi behöver $\frac{x^{2k}}{x^{9-k}} = x^{3k-9} = x^0$ för att få konstanttermen

Dvs. $3k = 9 \Rightarrow k = 3$

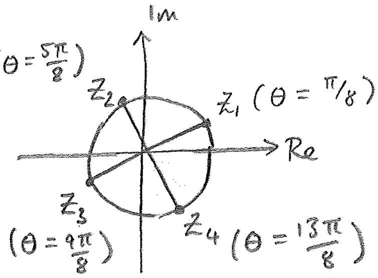
Koefficienten är $\binom{9}{3}(-1)^{9-3} = \binom{9}{3} = 84$

15. $z^4 = 81i$

$\Leftrightarrow r^4 e^{4i\theta} = 81 e^{\pi i/2}$ $\quad$ (om $z = re^{i\theta}$)

$\underbrace{r^4 = 81}_{r=3}$ och $4\theta = \pi/2 + 2k\pi$, $k \in \mathbb{Z}$

$\theta = \pi/8 + \frac{k\pi}{2}$



$z = 3e^{i\theta}$ där $\theta \in \left\{\pi/8, \frac{5\pi}{8}, \frac{9\pi}{8}, 13\pi/8\right\}$

16. $f(z) = z^4 + 3z^3 - 6z^2 + 12z - 40$ har lösningen $z = 2i$ och därför $z = -2i$ för f har reella koefficienter så komplexa rötter sker i konjugatpar.

$\Rightarrow (z-2i)(z+2i)$ delar f exakt

$\Leftrightarrow z^2 + 4$ delar f exakt

$$
\begin{array}{r|l}
 & z^2 + 3z - 10 \\
\hline
z^2 + 0z + 4 & z^4 + 3z^3 - 6z^2 + 12z - 40 \\
 & z^4 + 0z^3 + 4z^2 \\
\hline
 & 0 \quad 3z^3 - 10z^2 + 12z \\
 & 3z^3 + 0z^2 + 12z \\
\hline
 & 0 \quad -10z^2 + 0z - 40 \\
 & -10z^2 + 0z - 40 \\
\hline
 & 0 \quad 0 \quad 0
\end{array}
$$

$\Rightarrow \frac{f}{z^2+4} = z^2 + 3z - 10$ $\quad$ Lös $z^2 + 3z - 10 = 0$

$(z+5)(z-2) = 0$

$z = 5$ eller $z = 2$

<u>Svar</u>: $z_1 = 2i$, $z_2 = -2i$, $z_3 = 5$, $z_4 = 2$

17. $P(n): \sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}$

Så $P(1)$ LHS $= \frac{1}{1 \cdot 2} = \frac{1}{2}$

$P(1)$ RHS $= 1 - \frac{1}{2} = \frac{1}{2}$

Antag $P(m)$ är sant och betrakta

$$\sum_{k=1}^{m+1} \frac{1}{k(k+1)} = \sum_{k=1}^{m} \frac{1}{k(k+1)} + \frac{1}{(m+1)(m+2)}$$

$$= 1 - \frac{1}{m+1} + \frac{1}{(m+1)(m+2)} \quad \text{av } P(m)$$

$$= 1 + \frac{-(m+2)+1}{(m+1)(m+2)}$$

$$= 1 - \frac{\cancel{m+1}}{\cancel{(m+1)}(m+2)}$$

$$= 1 - \frac{1}{m+2}$$

Alltså $P(m) \Rightarrow P(m+1)$

Eftersom $P(1)$ är sant och $P(m) \Rightarrow P(m+1)$ då måste $P(n)$ gäller för alla $n \in \mathbb{N}$.

18. $\log_3(x-3) + \log_9(x+5)^2 = 2 \quad (*)$

Ändra basen: $\log_9(x+5)^2 = \log_3 y$

$\Rightarrow 2\log_9(x+5) = \log_3 y$

$\Rightarrow 3^{2\log_9(x+5)} = 3^{\log_3 y}$

$\Rightarrow 9^{\log_9(x+5)} = 3^{\log_3 y}$

$\Rightarrow y = x+5$

Så $(*)$ blir $\log_3(x-3) + \log_3(x+5) = 2$

$\Leftrightarrow \log_3 (x-3)(x+5) = 2$

$(x-3)(x+5) = 9$

$x^2 + 2x - 15 = 9$

$x^2 + 2x - 24 = 0$

$(x+6)(x-4) = 0$

$x = -6$ eller $x = 4$

men endast $x = 4$ ger ett acceptabelt svar i $(*)$

<u>Svar</u>: $x = 4$