UPPSALA UNIVERSITY
Department of Mathematics
Laertis Vaso, Johan Andersson, Olof Bergvall

Linear Algebra II
Written exam
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Time: 08:00–13:00. No aids except writing instruments. Solutions shall be accompanied with explanatory text. Each problem can give a maximum of 5 points. The exam can be awarded at most 40 points. A total of 18, 25 resp. 32 points will yield grade 3, 4 resp. 5. En version av tentamen på svenska finns på det andra bladet.

1. For each of the following sets, determine whether they are a subspace of $\mathbb{R}^3$. Motivate your answer!
   (a) $U_1 = \big\{(x, y, z) \in \mathbb{R}^3 \mid y = 1\big\}$.
   (b) $U_2 = \big\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 0\big\}$.
   (c) $U_3 = \big\{(x, y, z) \in \mathbb{R}^3 \mid x = y\big\}$.
   (d) $U_4 = \big\{(x, y, z) \in \mathbb{R}^3 \mid xyz = 0\big\}$.

   Solution

   (a) U_1 is not a subspace since $(0, 0, 0) \notin U_1$.
   (b) The equation $x^2 + y^2 + z^2 = 0$ implies $x^2 = y^2 = z^2 = 0$ and so $x = y = z = 0$. Hence $U_2 = \{(0, 0, 0)\}$, which is a subspace of $\mathbb{R}^3$.
   (c) We have
   $$\begin{aligned} U_3 &= \big\{(x, x, z) \in \mathbb{R}^3\big\} \\ &= \{x(1, 1, 0) + z(0, 0, 1) \mid x, z \in \mathbb{R}\} \\ &= \operatorname{Span}\{(1, 1, 0), (0, 0, 1)\} \end{aligned}$$
   and so U_3 is a subspace.
   (d) We have $(1, 0, 0) \in U_4$ and $(0, 1, 1) \in U_4$ but $(1, 0, 0) + (0, 1, 1) = (1, 1, 1) \notin U_4$. Since U_4 is not closed under addition, it is not a subspace of $\mathbb{R}^3$.

2. Let $f : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}^3$ be the linear map given by
   $$f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a + b, a + c, a + d).$$
   (a) Find the matrix of f relative to the standard bases.
   (b) Determine whether f is surjective.
   (c) Determine whether f is injective. Is f bijective?

   Solution

   (a) Let $\mathcal{E} = \{e_1, e_2, e_3\}$ denote the standard basis of $\mathbb{R}^3$ where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$. We have
   $$\begin{aligned} f\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} &= (1, 1, 1) = (1, 0, 0) + (0, 1, 0) + (0, 0, 1) = 1e_1 + 1e_2 + 1e_3, \\ f\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} &= (1, 0, 0) = 1e_1 + 0e_2 + 0e_3, \\ f\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} &= (0, 1, 0) = 0e_1 + 1e_2 + 0e_3, \\ f\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} &= (0, 0, 1) = 0e_1 + 0e_2 + 1e_3, \end{aligned}$$
   and so the matrix of f relative to the standard bases is
   $$[f] = \left(\left[f\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]_{\mathcal{E}} \middle| \left[f\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right]_{\mathcal{E}} \middle| \left[f\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right]_{\mathcal{E}} \middle| \left[f\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]_{\mathcal{E}} \right) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

(b) We need to check whether for any $(x, y, z) \in \mathbb{R}^3$ there exists an $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$ such that

$$f \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (x, y, z), \tag{1}$$

or equivalently

$$(a + b, a + c, a + d) = (x, y, z).$$

This gives the system of equations

$$\begin{cases} a + b = x, \\ a + c = y, \\ a + d = z, \end{cases}$$

or equivalently

$$\begin{cases} b = x + a, \\ c = y + a, \\ d = z + a, \\ a \in \mathbb{R}. \end{cases}$$

Since the equation (1) has a solution, the linear map f is surjective.

(c) Since f is surjective we have $\operatorname{im}(f) = \mathbb{R}^3$ and so $\dim(\operatorname{im}(f)) = \dim\left(\mathbb{R}^3\right) = 3$. Then the dimension formula for linear maps

$$\dim(\ker(f)) + \dim(\operatorname{im}(f)) = \dim\left(M_{2\times 2}(\mathbb{R})\right.$$

becomes

$$\dim(\ker(f)) + 3 = 4$$

and so $\dim(\ker(f)) = 1$. It follows that $\ker(f) \neq \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}$ and so f is not injective. Since f is not injective, we conclude that f is not bijective.

3. Find bases for the row space, column space and the null space of the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 & 1 \\ 1 & -2 & -1 & 10 & 2 \\ 1 & -1 & 0 & 7 & 1 \\ 1 & 2 & 3 & 0 & 0 \end{pmatrix},$$

and verify the rank-nullity theorem for the matrix.

Solution We first compute the reduced row echelon form of A.

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 & 1 \\ 1 & -2 & -1 & 10 & 2 \\ 1 & -1 & 0 & 7 & 1 \\ 1 & 2 & 3 & 0 & 0 \end{pmatrix} \xrightarrow[\substack{R_3 - R_1 \\ R_4 - R_1}]{R_2 - R_1} \begin{pmatrix} 1 & 1 & 2 & 3 & 1 \\ 0 & -3 & -3 & 7 & 1 \\ 0 & -2 & -2 & 4 & 0 \\ 0 & 1 & 1 & -3 & -1 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_4} \begin{pmatrix} 1 & 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & -2 & -2 & 4 & 0 \\ 0 & -3 & -3 & 7 & 1 \end{pmatrix}$$

$$\xrightarrow[R_4 + 3R_2]{R_3 + 2R_2} \begin{pmatrix} 1 & 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & -2 & -2 \end{pmatrix} \xrightarrow{-\frac{1}{2}R_3} \begin{pmatrix} 1 & 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -2 & -2 \end{pmatrix} \xrightarrow{R_4 + 2R_3} \begin{pmatrix} 1 & 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & -3 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[R_2 + 3R_3]{R_1 - 3R_3} \begin{pmatrix} 1 & 1 & 2 & 0 & -2 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 - R_2} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = T.$$

Since T is in reduced row echelon form, a basis of T is given by its nonzero rows. Hence a basis of $R(T)$ is $\{(1, 0, 1, 0, 0), (0, 1, 1, 0, -2), (0, 0, 0, 1, 1)\}$. Since $R(T) = R(A)$, the also form a basis of the row space of A.

Next, the first, second and fourth columns of T are the columns with the leading 1's. Hence the corresponding columns of A form a basis of $K(A)$. It follows that a basis of the column space of A is the set

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 10 \\ 7 \\ 0 \end{pmatrix} \right\}.$$

For the null space of A we solve the system $Ax = 0$ or equivalently $Tx = 0$. That is we have the system of linear equations

$$\begin{cases} x_1 + x_3 & = 0, \\ x_2 + x_3 - 2x_5 & = 0, \\ x_4 + x_5 & = 0, \end{cases}$$

This is a system of three equations with five unkowns hence we need $5 - 3 = 2$ parameters. Then we get

$$\begin{cases} x_1 & = -x_3, \\ x_2 & = -x_3 + 2x_5, \\ x_4 & = -x_5, \\ x_3 & = t \in \mathbb{R}, \\ x_5 & = s \in \mathbb{R}, \end{cases}$$

and so

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -x_3 \\ -x_3 + 2x_5 \\ x_3 \\ -x_5 \\ x_5 \end{pmatrix} = \begin{pmatrix} -t \\ -t + 2s \\ t \\ -s \\ s \end{pmatrix} = t \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 2 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

Since the vectors $\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$ span the null space $N(A)$ and are linearly independent (because they are not a scalar multiple of each other), they form a basis of $N(A)$.

To verify the rank nullity theorem for A we need to show that

$$\dim(N(A)) + \operatorname{rank}(A) = n,$$

where n is the number of columns of A. We have $\dim(N(A)) = 2$, $\operatorname{rank}(A) = \dim(R(A)) = 3$ and $n = 5$ and so

$$2 + 3 = 5$$

as required.

4. The linear operator $f : \mathbb{R}^3 \to \mathbb{R}^3$ acts geometrically as orthogonal projection in the plane $P : x + 2y + z = 0$. Let A be the matrix of f relative the standard basis ($\mathbb{R}^3$ is equipped with the euclidean inner product).

   (a) Find the eigenvalues of the operator f as well as ON-bases for the corresponding eigenspaces.

   (b) Find an orthogonal matrix P and a diagonal matrix D such that $P^T AP = D$.

   Solution

   (a) Since $f : \mathbb{R}^3 \to \mathbb{R}^3$, it follows that A is a 3×3 matrix and so the characteristic polynomial of A is of degree 3. Hence it has at most 3 distinct eigenvalues. Since f is the orthogonal projection onto the plane P, we know that vectors $v \in P$ satisfy $f(v) = v$. Hence $\lambda_1 = 1$ is an eigenvalue of f. Let us pick two linearly independent vectors in P. For example, let us pick first $b_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. Since we are looking for an orthonormal basis of $E(1)$, let us pick the second vector to be orthogonal to b_1. If we let $b_2 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, we need for b_2 to satisfy both conditions

$$b_1 \cdot b_2 = 0 \text{ and } b_2 \in P,$$

or equivalently

$$a - c = 0 \text{ and } a + 2b + c = 0.$$

One solution of this system is $b_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. Hence the vectors b_1, b_2 belong to $E(1)$ and $E(1)$ has dimension at least 2. Therefore the geometric multiplicity of $\lambda_1 = 1$ is at least 2, and hence the algebraic multiplicity of λ_1 is also at least 2.

Next, vectors u parallel to the line $L = \text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$ are orthogonal to P and hence $f(u) = 0 = 0u$. Therefore, $\lambda_2 = 0$ is another eigenvalue. Similarly, $b_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ belongs to $E(0)$ and hence $E(0)$ has dimension at least 1. Therefore the geometric multiplicity of $\lambda_2 = 0$ is at least 1 and hence the algebraic multiplicity of λ_2 is also at least 1.

Since the sum of all the algebraic multiplicities should be exactly 3, we conclude that there are no other eigenvalues. Moreover, we find that λ_1 has algebraic multiplicity exactly 2 and λ_1 has algebraic multiplicity exactly 1, and similar for their geometric multiplicities. It follows that $\{b_1, b_2\}$ is a basis of $E(1)$ and $\{b_3\}$ is a basis of $E(0)$.

To find an orthonormal basis of $E(1)$, it is enough to normalize b_1 and b_2 since they are orthogonal. Hence an orthonormal basis of $E(1)$ is

$$\left\{ \frac{b_1}{\|b_1\|}, \frac{b_2}{\|b_2\|} \right\} = \left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \right\}.$$

Similarly, an orthonormal basis of $E(0)$ is

$$\left\{ \frac{b_3}{\|b_3\|} \right\} = \left\{ \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} \right\}$$

(b) Note that by (a) the matrix A is diagonalizable, and it satisfies $P^{-1}AP = D$, where $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}$. In particular, the columns of P are orthonormal vectors and so P is an orthogonal matrix. Hence $P^{-1} = P^T$ and so $P^TAP = D$ as required.

5. Let

$$U = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

be a subspace of $\mathbb{R}^4$. Find an ON-basis, relative the euclidean inner product, for U and determine the distance between the vector $\mathbf{v} = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}^T$ and the subspace U.

Solution We apply the Gram-Schmidt algorithm to find an orthogonal basis of U first. Let us name the vectors in the given basis of U first:

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

We now set

$$v_1 = u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

and then

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle}{\left\| \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\|^2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ 1 \end{pmatrix}.$$

Finally, to make the basis $\{v_1, v_2\}$ orthonormal, we set

$$w_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

and

$$w_2 = \frac{v_2}{\|v_2\|} = \frac{3}{\sqrt{15}} \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ 1 \end{pmatrix} = \frac{1}{\sqrt{15}} \begin{pmatrix} -2 \\ 1 \\ 1 \\ 3 \end{pmatrix}.$$

Then, $\{w_1, w_2\}$ is an orthonormal basis of U.

To find the distance of v from U we have that

$$\mathrm{d}(v, U) = \|v - \mathrm{proj}_U(v)\|.$$

We first compute $\mathrm{proj}_U(v)$ using the orthonormal basis $\{w_1, w_2\}$. We have

$$\begin{aligned}
\mathrm{proj}_U(v) &= \langle v, w_1 \rangle w_1 + \langle v, w_2 \rangle w_2 \\
&= \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{15}} \begin{pmatrix} -2 \\ 1 \\ 1 \\ 3 \end{pmatrix} \right\rangle \frac{1}{\sqrt{15}} \begin{pmatrix} -2 \\ 1 \\ 1 \\ 3 \end{pmatrix} \\
&= \frac{3}{\sqrt{3}} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \frac{3}{\sqrt{15}} \frac{1}{\sqrt{15}} \begin{pmatrix} -2 \\ 1 \\ 1 \\ 3 \end{pmatrix} \\
&= \frac{1}{5} \begin{pmatrix} 3 \\ 6 \\ 6 \\ 3 \end{pmatrix}
\end{aligned}$$

Now we have

$$\begin{aligned}
\mathrm{d}(v,U) &= \|v - \mathrm{proj}_U(v)\| \\
&= \left\| \begin{pmatrix}1\\1\\1\\1\end{pmatrix} - \frac{1}{5}\begin{pmatrix}3\\6\\6\\3\end{pmatrix} \right\| \\
&= \left\| \frac{1}{5}\begin{pmatrix}2\\-1\\-1\\2\end{pmatrix} \right\| \\
&= \frac{1}{5}\sqrt{4+1+1+4} \\
&= \frac{\sqrt{10}}{5}.
\end{aligned}$$

6. Solve the following system of differential equations

$$\begin{cases} y_1' = y_1 + 4y_2, \\ y_2' = y_1 + y_2, \end{cases}$$

where $y_1(0) = 1$ and $y_2(0) = 2$.

Solution We can write the given system as

$$\begin{pmatrix}y_1'\\y_2'\end{pmatrix} = \begin{pmatrix}1 & 4\\1 & 1\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}.$$

Set $A = \begin{pmatrix}1 & 4\\1 & 1\end{pmatrix}$. We will find a matrix P that diagonalizes A. First we find the eigenvalues of A. We have

$$\det(\lambda I - A) = \begin{vmatrix}\lambda - 1 & -4\\ -1 & \lambda - 1\end{vmatrix} = (\lambda-1)^2 - 4 = (\lambda - 1 - 2)(\lambda - 1 + 2) = (\lambda-3)(\lambda+1),$$

and so the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -1$. To find basis for the eigenspaces we have

$$E(3) = N(3I - A) = \mathrm{Span}\left\{\begin{pmatrix}2\\1\end{pmatrix}\right\}, \text{ since } \begin{pmatrix}2 & -4\\-1 & 2\end{pmatrix} \sim \begin{pmatrix}1 & -2\\0 & 0\end{pmatrix},$$

$$E(-1) = N(-I - A) = \mathrm{Span}\left\{\begin{pmatrix}2\\-1\end{pmatrix}\right\}, \text{ since } \begin{pmatrix}-2 & -4\\-1 & -2\end{pmatrix} \sim \begin{pmatrix}1 & 2\\0 & 0\end{pmatrix}.$$

Therefore for $P = \begin{pmatrix}2 & 2\\1 & -1\end{pmatrix}$ we have $P^{-1}AP = D = \begin{pmatrix}3 & 0\\0 & -1\end{pmatrix}$ and so the change of variables $\begin{pmatrix}y_1\\y_2\end{pmatrix} = P\begin{pmatrix}z_1\\z_2\end{pmatrix}$ gives the equivalent system

$$\begin{pmatrix}z_1'\\z_2'\end{pmatrix} = \begin{pmatrix}3 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}z_1\\z_2\end{pmatrix}$$

which we can write as

$$\begin{cases} z_1' &= 3z_1, \\ z_2' &= -z_2 \end{cases}.$$

This system has the general solution

$$\begin{cases} z_1 &= c_1 e^{3x}, \\ z_2 &= c_2 e^{-x} \end{cases}.$$

Now we can find the general solutions y_1 and y_2:

$$\begin{pmatrix}y_1\\y_2\end{pmatrix} = \begin{pmatrix}2 & 2\\1 & -1\end{pmatrix}\begin{pmatrix}z_1\\z_2\end{pmatrix} = \begin{pmatrix}2 & 2\\1 & -1\end{pmatrix}\begin{pmatrix}c_1e^{3x}\\c_2e^{-x}\end{pmatrix} = \begin{pmatrix}2c_1e^{3x} + 2c_2e^{-x}\\ c_1e^{3x} - c_2e^{-x}\end{pmatrix}.$$

For the specific solution, we use the initial conditions $y_1(0) = 1$ and $y_2(0) = 2$. By plugging in these equations to the general solution we obtain the system

$$\begin{cases} 2c_1 + 2c_2 = 1, \\ c_1 - c_2 = 2 \end{cases},$$

which we can easily solve to find $c_1 = \frac{5}{4}$ and $c_2 = -\frac{3}{4}$. Hence the solution to the initial value problem is

$$\begin{cases} y_1 &= \frac{5}{2}e^{3x} - \frac{3}{2}e^{-x}, \\ y_2 &= \frac{5}{4}e^{3x} + \frac{3}{4}e^{-x}. \end{cases}$$

7. Recall that the *trace*, $\mathrm{Tr}(A)$, of an $n \times n$-matrix A is the sum of its diagonal elements. Let $M_{2\times 2}(\mathbb{R})$ be the vector space of 2×2-matrices with real coefficients. Let A and B be 2×2-matrices and define $\langle A, B\rangle$ as the trace of $A^T B$, i.e.

$$\langle A, B\rangle = \mathrm{Tr}(A^T B) = \sum_{i=1}^{2} (A^T B)_{ii}$$

(a) Compute $\langle A, B\rangle$ for

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

(b) Show that $\langle A, B\rangle$ gives an inner product on $M_{2\times 2}(\mathbb{R})$.

(c) Show that if A and B are symmetric 2×2-matrices, then

$$|\mathrm{Tr}(A^T B)|^2 \leq \mathrm{Tr}(A^2)\mathrm{Tr}(B^2)$$

Solution

(a) We have

$$\langle A, B\rangle = \mathrm{Tr}(A^T B) = \mathrm{Tr}\left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = \mathrm{Tr}\begin{pmatrix} 1 & 3 \\ 3 & 7 \end{pmatrix} = 1 + 7 = 8.$$

(b) We have to show that $\langle A, B\rangle$ satisfies the four axioms of inner product.

(IP1) We need to show that $\langle A, B\rangle = \langle B, A\rangle$. We use two facts: first that for two matrices $X, Y \in M_{2\times 2}(\mathbb{R})$ we have $(XY)^T = X^T Y^T$ and second that since the trace is the sum of all diagonal elements of a matrix, we have $\mathrm{Tr}(X) = \mathrm{X^T}$. With this in mind, we have:

$$\langle A, B\rangle = \mathrm{Tr}(A^T B) = \mathrm{Tr}\left(\left(B^T A\right)^T\right) = \mathrm{Tr}\left(B^T A\right) = \langle B, A\rangle.$$

(IP2) We need to show that $\langle A + C, B\rangle = \langle A, B\rangle + \langle C, B\rangle$. We use two facts: first that for two matrices $X, Y \in M_{2\times 2}(\mathbb{R})$ we have $(X + Y)^T = X^T + Y^T$ and second that $\mathrm{Tr}(X + Y) = \mathrm{Tr}(X) + \mathrm{Tr}(Y)$. With this in mind, we have:

$$\begin{aligned} \langle A + B, C\rangle &= \mathrm{Tr}\left((A + C)^T B\right) = \mathrm{Tr}\left(\left(A^T + C^T\right) B\right) \\ &= \mathrm{Tr}\left(A^T B + C^T B\right) = \mathrm{Tr}(A^T B) + \mathrm{C^T B} \\ &= \langle A, B\rangle + \langle C, B\rangle. \end{aligned}$$

(IP3) We need to show that for any $c \in \mathbb{R}$ we have $\langle cA, B\rangle = c\langle A, B\rangle$. We use two facts: first that for a matrix $X \in M_{2\times 2}(\mathbb{R})$ we have $(cX)^T = cX^T$ and second that $\mathrm{Tr}(cX) = c\mathrm{X}$. With this in mind, we have:

$$\langle cA, B\rangle = \mathrm{Tr}\left((cA)^T B\right) = \mathrm{Tr}\left(cA^T B\right) = \mathrm{Tr}\left(c\left(A^T B\right)\right) = c\mathrm{Tr}\left(A^T B\right) = c\langle A, B\rangle.$$

(IP4) We need to show that for $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$ we have $\langle A, A\rangle \geq 0$ and $\langle A, A\rangle = 0$ if and only if $A = 0$. We have

$$\begin{aligned}
\langle A, A\rangle = \mathrm{Tr}(A^T A) &= \mathrm{Tr}\left(\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) \\
&= \mathrm{Tr}\begin{pmatrix} a_{11}^2 + a_{21}^2 & a_{11}a_{12} + a_{21}a_{22} \\ a_{12}a_{11} + a_{22}a_{21} & a_{12}^2 + a_{22}^2 \end{pmatrix} \\
&= a_{11}^2 + a_{21}^2 + a_{12}^2 + a_{22}^2 \geq 0.
\end{aligned}$$

Hence $\langle A, A\rangle \geq 0$ and $\langle A, A\rangle = 0$ if and only if $a_{11}^2 + a_{21}^2 + a_{12}^2 + a_{22}^2 = 0$, which is true if and only if $a_{11} = a_{21} = a_{12} = a_{22} = 0$, as required.

(c) By the Cauchy-Schwarz inequality we have

$$|\langle A, B\rangle| \leq \|A\| \, \|B\| \tag{2}$$

If we compute $\|A\|$ we have

$$\|A\| = \sqrt{\langle A, A\rangle} = \sqrt{\mathrm{Tr}(A^T A)} \overset{A \text{ is symmetric}}{=} \sqrt{\mathrm{Tr}(AA)} = \sqrt{\mathrm{Tr}(A^2)}.$$

Similarly, we have $\|B\| = \sqrt{\mathrm{Tr}(B^2)}$. By replacing these and the definition of $\langle A, B\rangle$ in (2), we get

$$|\langle A, B\rangle| \leq \sqrt{\mathrm{Tr}(A^2)}\sqrt{\mathrm{Tr}(B^2)},$$

and by squaring both sides we have

$$|\langle A, B\rangle|^2 \leq \mathrm{Tr}(A^2)\mathrm{Tr}(B^2),$$

as required.

8. Let the quadratic form $q : \mathbb{R}^3 \to \mathbb{R}$ be given by

$$q(x) = q(x_1, x_2, x_3) = x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2^2 + 2x_2x_3 + 2x_3^2$$

(a) Find a symmetric matrix A such that $q(x) = x^T Ax$.

(b) Determine if q is positive definite.

(c) Let $v = (1, 1, 1)^T$ and define $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x) = \frac{1}{2}(q(x + v) - q(x) - q(v))$$

Show that f is linear.

Solution

(a) The matrix of the quadratic form q is $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

(b) We need to find the eigenvalues of A. To do this we first find the characteristic polynomial of A.

$$\begin{aligned}
\mathrm{Det}(\lambda I - A) &= \begin{vmatrix} \lambda - 1 & -1 & -1 \\ -1 & \lambda - 2 & -1 \\ -1 & -1 & \lambda - 2 \end{vmatrix} \overset{R_2 \to R_2 - R_3}{=} \begin{vmatrix} \lambda - 1 & -1 & -1 \\ 0 & \lambda - 1 & -\lambda + 1 \\ -1 & -1 & \lambda - 2 \end{vmatrix} \\
&= (\lambda - 1)\begin{vmatrix} \lambda - 1 & -1 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & \lambda - 2 \end{vmatrix} \overset{C_3 \to C_3 + C_2}{=} (\lambda - 1)\begin{vmatrix} \lambda - 1 & -1 & -2 \\ 0 & 1 & 0 \\ -1 & -1 & \lambda - 3 \end{vmatrix} \\
&\overset{\text{expand along } R_2}{=} (\lambda - 1)\begin{vmatrix} \lambda - 1 & -2 \\ -1 & \lambda - 3 \end{vmatrix} = (\lambda - 1)((\lambda - 1)(\lambda - 3) - (-1)(-2)) \\
&= (\lambda - 1)(\lambda^2 - 4\lambda + 1)
\end{aligned}$$

From the characteristic polynomial of A we immediately find the root $\lambda_1 = 1 > 0$. For the other roots we need to find the roots of $\lambda^2 - 4\lambda + 1$. The discriminant is

$$D = (-4)^2 - 4 = 12$$

and so the roots are

$$\lambda_2 = \frac{-(-4) + \sqrt{12}}{2} = \frac{4 + 2\sqrt{3}}{2} = 2 + \sqrt{3} > 0,$$

$$\lambda_3 = \frac{-(-4) - \sqrt{12}}{2} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3} > 0.$$

Since all eigenvalues of A are positive, the quadratic form is positive definite.

(c) First let us rewrite $f(x)$ using the matrix A. We have

$$\begin{aligned} f(x) &= \frac{1}{2}\left(q(x+v) - q(x) - q(v)\right) \\ &= \frac{1}{2}\left((x+v)^T A(x+v) - x^T Ax - v^T Av\right) \\ &= \frac{1}{2}\left(x^T Ax + x^T Av + v^T Ax + v^T Av - x^T Ax - v^T Av\right) \\ &= \frac{1}{2}\left(x^T Av + v^T Ax\right). \end{aligned}$$

Using this expression we can show that f is linear, but we can simplify this even more. Notice that $x^T Av$ is just a number, that is an 1×1 matrix. Hence $(x^T Av)^T = x^T Av$. Then, since A is symmetric, we have

$$x^T Av = (x^T Av)^T = v^T A^T x = v^T Ax.$$

Hence we can rewrite

$$f(x) = \frac{1}{2}\left(x^T Av + v^T Ax\right) = \frac{1}{2}\left(x^T Av + x^T Av\right) = x^T Av,$$

and so we have shown that $f(x) = x^T Av$. To show that f is linear then we have to show the two axioms for linear maps. Let $x, y \in \mathbb{R}^3$ and $c \in \mathbb{R}$. Then

(L1) $f(x+y) = (x+y)^T Av = (x^T + y^T)Av = x^T Av + y^T Av = f(x) + f(y)$,

(L2) $f(cx) = (cx)^T Av = (c(x^T))Av = c(x^T Av) = cf(x)$,

and so f is linear as required.