Uppsala Universitet
Institutionen för Informationsteknologi
Avdelningen för Beräkningsvetenskap

Exam – Scientific Computing for Partial Differential Equations, 2022-10-24

Time: $8^{00} - 13^{00}$
Allowed resources: No aids of any kind are allowed.

Start a new solution on a new sheet of paper. To get full credit for your solution good arguments for your method of solution and detailed calculations are required. The total credit sum determines the grade.

To pass the exam: You need to get at least **3 points** from Finite Difference Method; **3 points** from Finite Element Methods; **3 points** from Iterative Methods. Then the following grade limits will be applied: grade 3 at least 9 points, grade 4 at least 14 points, grade 5 at least 19 points.

Part I: Finite Difference Method

1. Consider the following problem,

$$\begin{array}{ll} i\,\mathbf{u}_t = \mathbf{A}\mathbf{u}_x + \mathbf{V}\mathbf{u} & -1 \leq x \leq 1,\ t \geq 0, \\ L_l\mathbf{u} = 0, & x = -1,\ t \geq 0, \\ L_r\mathbf{u} = 0, & x = 1,\ t \geq 0, \\ \mathbf{u} = \mathbf{f}(x), & -1 \leq x \leq 1,\ t = 0, \end{array} \tag{1}$$

where L_l and L_r are the boundary operators, $\mathbf{V} > 0$, $\mathbf{f} = \mathbf{f}(x)$ is the initial data and

$$\mathbf{u} = \begin{bmatrix} u^{(1)} \\ u^{(2)} \end{bmatrix}, \mathbf{A} = \begin{bmatrix} \alpha & 1 \\ -1 & 0 \end{bmatrix},$$

i is the imaginary number, and α possibly a complex constant. Below we present the definitions of the first-derivate SBP operator:

$$D_1 = H^{-1}(Q + B), \quad B = -\tfrac{1}{2}e_1e_1^T + \tfrac{1}{2}e_me_m^T, \quad H = H^T > 0, \quad (Q + Q^T) = 0, \quad e_m^T = [0, \ldots, 0, 1]$$

(a) What are the requirements for (1) to be well-posed, disregarding the boundary conditions, i.e. here assume $BT \leq 0$. (1p)

(b) Let $\alpha = 0$, and derive a set of well-posed boundary conditions for (1), thats leads to damping of energy. This means finding L_l and L_r. (1p)

(c) Let $\alpha = 0$, and derive two sets of well-posed boundary conditions for (1), thats leads to conservation of energy. This means finding L_l and L_r. (1p)

(d) Derive an SBP-Projection approximation of (1), with any set of the well-posed boundary conditions derived in (c), where $\alpha = 0$. (1p)

(e) Show stability for the SBP-Projection approximation in (d). (2p)

(f) Explain why Euler forward (RK1) is not a suitable time-integrator for the SBP-Projection approximation derived in (e). Propose a more suitable time-integrator and give a rough estimate how to chose the time-step to obtain stability for an arbitrary grid-spacing h. (2p)

Part II: Finite Element Method

2. Consider the following problem in $\Omega = (0,1)$:

$$\begin{aligned} u_t - u_{xx} + \alpha u_x &= 0, \quad x \in \Omega,\ t > 0, \\ u(0,t) &= 1, \quad t > 0, \\ u'(1,t) &= 1, \quad t > 0. \\ u(x,t) &= u_0(x), \quad x \in \Omega, \end{aligned} \tag{2}$$

where $\alpha > 0$, and $u_0(x)$ is a given initial condition.

(a) Write down a weak and finite element formulations for (2) with appropriate spaces. (2p)

(b) Now split the interval into N equally spaced sub-intervals: $0 = x_0 < x_1 < \ldots < x_N = 1$. Construct the corresponding system of ordinary differential equations. Compute the elements of the resulting matrices.

Hint: you can use the following Simpson's rule:

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{b-a}{6}\Big(f(a) + 4f\Big(\frac{a+b}{2}\Big) + f(b)\Big),$$

to compute the elements of the mass matrix $\int_0^1 \varphi_j \varphi_i \,\mathrm{d}x$. (4p)

(c) Discretize the ODE system in (b) using the explicit Euler method. (2p)

Part III: Iterative Methods

3. Consider the linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, where

$$\mathbf{A} = \begin{pmatrix} y & 0 & 1 \\ 0 & -8 & 1 \\ z & 0 & 1 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

where y and z are some parameters.

(a) Write down the Gauss-Seidel method to solve the system and do one iteration using the starting vector $\boldsymbol{x}^0 = (1,1,1)^\top$. (1p)

(b) For which values of y and z the Gauss-Seidel method can be expected to converge for this problem? Motivate your answer.

Hint: note that the first element of the last row of the inverse of the lower triangular matrix would be $-\frac{b}{a}$. (3p)

(c) For which values of y and z can one use the Conjugate-Gradient method to solve the above problem? Motivate your answer. (1p)

(d) For some iterative method $\boldsymbol{x}^{k+1} = \mathbf{M}\boldsymbol{x}^k + \boldsymbol{c}$, $k = 0, 1, \ldots$, we know that the matrix $\mathbf{M}$ is strictly diagonally dominant. Prove that the iterative method converges. (3p)

Good Luck!