Introduction to Computer Control Systems, 5 credits, 1RT485

Date and Time: 2018-03-09

Place: Fyrislundsgatan 80, sal 2.

Teacher on duty: Hans Rosth.

Allowed aid:

- A basic calculator
- BETA mathematical handbook

Solutions have to be explained in detail and possible to reconstruct.

<u>NB:</u> **Only one problem per sheet.** Write your anonymous exam code on each sheet. Write your name if you do not have an anonymous code.

Best of luck!

Useful results

Laplace transform table

Table 1: Basic Laplace transforms

$f(t)$	$F(s)$	$f(t)$	$F(s)$
unit impulse $\delta(t)$	1	$\sinh(bt)$	$\frac{b}{s^2-b^2}$
unit step $1(t)$	$\frac{1}{s}$	$\cosh(bt)$	$\frac{s}{s^2-b^2}$
t	$\frac{1}{s^2}$	$\frac{1}{2b}t\sin(bt)$	$\frac{s}{(s^2+b^2)^2}$
t^n	$\frac{n!}{s^{n+1}}$	$t\cos(bt)$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
e^{-at}	$\frac{1}{s+a}$	$\frac{\cos(bt)-\cos(at)}{a^2-b^2}; (a^2 \neq b^2)$	$\frac{s}{(s^2+a^2)(s^2+b^2)}$
$\frac{1}{a}(1-e^{-at})$	$\frac{1}{s(s+a)}$	$\frac{\sin(at)+at\cos(at)}{2a}$	$\frac{s^2}{(s^2+a^2)^2}$
$\frac{1}{(n-1)!}t^{n-1}e^{-at}; (n=1,2,3...)$	$\frac{1}{(s+a)^n}$		
$\sin(bt)$	$\frac{b}{s^2+b^2}$		
$\cos(bt)$	$\frac{s}{s^2+b^2}$		
$e^{-at}\sin(bt)$	$\frac{b}{(s+a)^2+b^2}$		
$e^{-at}\cos(bt)$	$\frac{s+a}{(s+a)^2+b^2}$		

Table 2: Properties of Laplace Transforms

$\mathcal{L}[af(t)] = aF(s)$	$\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}$
$\mathcal{L}[f_1(t)+f_2(t)] = F_1(s)+F_2(s)$	$\mathcal{L}\left[t^2f(t)\right] = \frac{d^2}{ds^2}F(s)$
$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = sF(s)-f(0)$	$\mathcal{L}[t^nf(t)] = (-1)^n\frac{d^n}{ds^n}F(s), \quad n=1,2,3,\ldots$
$\mathcal{L}\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s)-sf(0)-f'(0)$	$\mathcal{L}\left[f\left(\frac{t}{a}\right)\right] = aF(as)$
$\mathcal{L}\left[\int f(t)\,dt\right] = \frac{F(s)}{s}+\frac{1}{s}\left[\int f(t)\,dt\right]_{t=0}$	$\mathcal{L}\left[\int_0^t f_1(t-\tau)f(t)d\tau\right] = F_1(s)F_2(s)$
$\mathcal{L}[f(t-a)] = e^{-as}F(s)$	$\mathcal{L}[e^{-at}f(t)] = F(s+a)$

Matrix exponential

$$\boxed{e^{At} \triangleq \mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\}}$$

Open-loop and sensitivity functions

$$G_o(s) = G(s)F_y(s), \qquad S(s) = \frac{1}{1+G_o(s)}, \qquad T(s) = 1-S(s)$$

State-space forms and transfer function relations

- State-space form and transfer function

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad \Rightarrow \quad \boxed{G(s) = C(sI - A)^{-1}B + D}$$

- Associated matrices

$$\mathcal{S} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \qquad \mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

- LTI system with transfer function

$$\boxed{G(s) = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}}$$

i) Observable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ -a_3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ b_3 - a_3 b_0 \\ \vdots \\ b_n - a_n b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

ii) Controllable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_1 - a_1 b_0 & b_2 - a_2 b_0 & \cdots & b_n - a_n b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

- Solution to state-space equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

can be written as

$$\boxed{x(t) = e^{At} x_0 + \int_0^t e^{A\tau} B u(t - \tau) d\tau}$$

- Observer system

$$\boxed{\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})}$$

Feedback control structures

General linear feedback in Laplace form:

$$\boxed{U(s) = F_r(s)R(s) - F_y(s)Y(s)}$$

Common control structures in this form.

- PID controller:

$$F_y(s) = F_r(s) = F(s) = K_p + \frac{K_i}{s} + K_d s,$$

  where $K_p, K_i, K_d \geq 0$

- Lead-lag controller:

$$F_y(s) = F_r(s) = F(s) = K\left(\frac{\tau_D s + 1}{\beta\tau_D s + 1}\right)\left(\frac{\tau_I s + 1}{\tau_I s + \gamma}\right),$$

  where $K, \tau_D, \tau_I > 0$ and $0 \leq \beta, \gamma < 1$

- State-feedback controller with observer:

$$\begin{aligned} F_r(s) &= \left(1 - L(sI - A + KC + BL)^{-1}B\right)\ell_0 \\ F_y(s) &= L(sI - A + KC + BL)^{-1}K \end{aligned}$$

Discrete-time state-space forms

A continuous time system with zero-order-hold input signal and sample period T can be written in discrete-time as:

$$\boxed{\begin{aligned} x(k+1) &= Fx(k) + Gu(k) \\ y(k) &= Hx(k) \end{aligned}}$$

where

$$\begin{aligned} F &= e^{AT} \\ G &= \int_{\tau=0}^{T} e^{A\tau} d\tau B = [\text{if } A^{-1} \text{ exists}] = A^{-1}(e^{AT} - I)B \\ H &= C \end{aligned}$$

Problem 1: basic questions (6/30)

Answer only *'true'* or *'false'*. Each correct answer gives 1 point, each wrong answer gives -1 point. Minimum total points for Part A and B is 0 , respectively.

Part A

Note: Write 'skip' if your total home assignment score ≥ 8

i) We want to control the following system

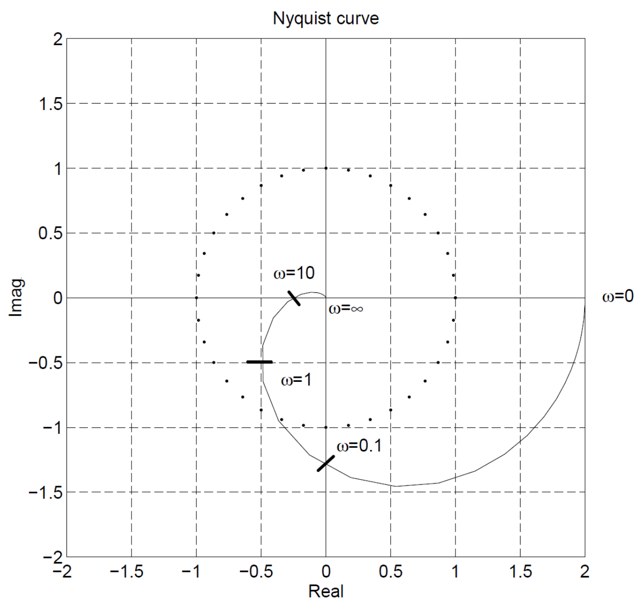
$$G(s) = \frac{5s}{s+3}$$

with a P-controller with constant K. The closed-loop system is stable for all $K > 0$.

ii) The following system is controllable

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

iii) Consider a system $G(s)$ with a P controller of gain K. The Nyquist diagram of $G(s)$ is depicted below. Knowing that $G(s)$ does not have any poles in



the right hand side of the plane, the closed-loop system is stable for $K = 6$

(3 p)

Part B

Note: Write ‘skip’ if your total home assignment score ≥ 12

i) If $|\Delta_G(i\omega)| < \frac{1}{|T(i\omega)|}$ does not hold for all ω, then we know that the closed-loop system is unstable.

ii) A dynamical system is completely described by its impulse response.

iii) The system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is input-output stable.

(3 p)

Proposed solution to problem 1

Part A

i) True. The poles are located at $-\frac{3}{1+5K}$

ii) True. Check the determinant of the controllability matrix

iii) False. It will circle -1 if K=6.

Part B

i) False. The robustness criterion is sufficient, not necessary.

ii) True. The impulse response $g(t) = \mathcal{L}^{-1}[G(s)]$ contains all information about the system.

iii) True. It is not a minimal realisation and $G(s) = \frac{1}{s+1}$.

Problem 2 (6/30)

Consider a system with a liquid in two communicated compartments. A chemical agent is injected into the first compartment. Its mixing can be described by a two-compartment model and has been identified with the following transfer function:

$$\frac{Y_2(s)}{U(s)} = \frac{4}{s^2 + 7s + 4}$$

where Y_2 is the concentration of the agent in the second compartment.

a) In order to make it faster, we want to control the input (i.e. how we inject the agent) with a proportional (P) controller with gain K. For which values of K is the system stable?

(3 p)

b) Sketch the root locus of the system and briefly comment on the system's response: fastness, oscillations and static gain.

(3 p)

Proposed solution to problem 2

a) The closed loop system is given by:

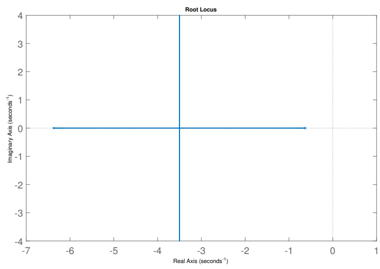
$$G_{cl}(s) = \frac{4K}{s^2 + 7s + 4 + 4K}$$

Its poles are located at:

$$s = \frac{-7 \pm \sqrt{49 - 16 - 16K}}{2}$$

The real part of the poles is always negative for all $K > 0$. Hence, the system is stable for all $K > 0$.

b) Going for different values of K, it can be seen that for $K = 0$, the poles are located at -6.37 and -0.63. As K increases, the poles will go towards each other and will coincide at -3.5, at which point they will branch out, being complex conjugate.



Regarding the behavior of the system, it can be seen that for low values of K the system will become faster (i.e. less rise time) as K increases. When the poles branch out, oscillations will appear. Regarding steady state error, it will be large for low values of K (this can be seen using the final value theorem). As $K \to \infty$, the error will converge to 0.

Problem 3 (6/30)

Consider the system

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

a) Show that the system is observable.

(1 p)

b) Assume that x_2 can not be measured, so that you have to make an estimation $\hat{x}$ of x. Construct an observer such that the observer poles are placed in -3.

(2 p)

c) Given the observer in **b)**, let the initial error in $\hat{x}_2$ be 1, that is $\tilde{x}_2(0) = \hat{x}_2(0) - x_2(0) = 1$. Determine the value of this error at the time $t = 1/3$, that is $\tilde{x}_2(1/3)$. Since x_1 is measured, use that $\tilde{x}_1(0) = 0$.

Hint: Use that $\dot{\tilde{x}} = (A - KC)\tilde{x}$, $\tilde{x}_2 = [0 \;\; 1]\tilde{x}$ and note that the initial values are important here. Also, recall partial fraction decomposition.

(3 p)

Proposed solution to problem 3

a) The observability matrix is

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1. \end{bmatrix}$$

Since $\det \mathcal{O} = 1 \neq 0$, the system is observable.

b) The characteristic polynomial of $A - KC$ is

$$\det(sI - A + KC) = \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} k_1 & 0 \\ k_2 & 0 \end{bmatrix}\right) = \det\left(\begin{bmatrix} s+2+k_1 & -1 \\ 1+k_2 & s \end{bmatrix}\right) =$$
$$= s^2 + (k_1+2)s + k_2 + 1.$$

The desired polynomial is

$$(s+3)^2 = s^2 + 6s + 9,$$

so we get

$$\begin{cases} k_1 + 2 = 6 \\ k_2 + 1 = 9 \end{cases} \Rightarrow \begin{cases} k_1 = 4 \\ k_2 = 8 \end{cases}$$

Answer: $k_1 = 4$ and $k_2 = 8$

c) Laplace transform $\dot{\tilde{x}} = (A - KC)\tilde{x}$ and include the initial values. This gives

$$s\tilde{X}(s) - \tilde{x}(0) = (A-KC)\tilde{X}(s) \Leftrightarrow (sI - A + KC)\tilde{X}(s) = \tilde{x}(0),$$

so

$$\tilde{X}(s) = (sI - A + KC)^{-1}\tilde{x}(0),$$

and

$$\tilde{X}_2(s) = [0 \;\; 1]\tilde{X}(s) = [0 \;\; 1]\begin{bmatrix} s+6 & -1 \\ 9 & s \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{(s+3)^2}[0 \;\; 1]\begin{bmatrix} s & 1 \\ -9 & s+6 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$= \frac{s+6}{(s+3)^2} = \text{/partial fraction decomposition/} = \left[\frac{1}{s+3} + \frac{3}{(s+3)^2}\right].$$

Hence,

$$\tilde{x}_2(t) = \mathcal{L}^{-1}\left[\frac{1}{s+3} + \frac{3}{(s+3)^2}\right] = e^{-3t} + 3te^{-3t} = (1+3t)e^{-3t},$$

and

$$\tilde{x}_2(1/3) = (1+1)e^{-1} = \frac{2}{e}.$$

Answer: $\tilde{x}_2(1/3) = 2/e$

***Alternatively*:** From the solution to the state-space equation (bottom of page 3), it is seen that

$$\tilde{x}(t) = e^{(A-KC)t}\tilde{x}(0),$$

so

$$\begin{aligned}\tilde{x}_2(t) &= [0\ \ 1]\tilde{x}(t) = [0\ \ 1]e^{(A-KC)t}\tilde{x}(0) = [0\ \ 1]\mathcal{L}^{-1}\left[(sI-A+KC)^{-1}\right]\begin{bmatrix}0\\1\end{bmatrix}\\ &= \mathcal{L}^{-1}\left[[0\ \ 1](sI-A+KC)^{-1}\begin{bmatrix}0\\1\end{bmatrix}\right] = \mathcal{L}^{-1}\left[\frac{s+6}{(s+3)^2}\right] = (1+3t)e^{-3t},\end{aligned}$$

where the details are as above.

Problem 4 (6/30)

a) Consider the second order system

$$G(s) = \frac{b}{s^2 + a_1 s + a_2}.$$

A plot of $|G(i\omega)|$ versus ω is shown in Figure 1. For a low-damped system like this one, the approximation $a_2 \approx \omega_\mathrm{p}^2$ is valid. Determine the positive coefficients b and a_1 (as functions of ω_p, K and M_p).

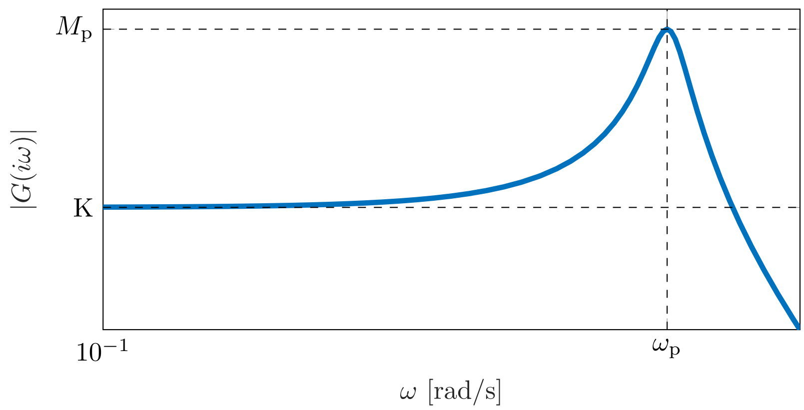


Figure 1: Magnitude plot of the system G.

(1.5 p)

b) The bode diagram for the stable system $G(s)$ is shown in Figure 2. The system is controlled using a PD-controller $F(s) = K_\mathrm{P} + K_\mathrm{D}s$.

(i) For what values of $K_\mathrm{P}, K_\mathrm{D} > 0$ is the closed-loop system stable?

(1.5 p)

(ii) Let $K_\mathrm{P} = K_\mathrm{D} = 1$ and the reference signal $r(t) = \sin(2t)$. What will the output $y(t)$ be when the transients have vanished? Answer in rounded decimals is fine.

(3 p)

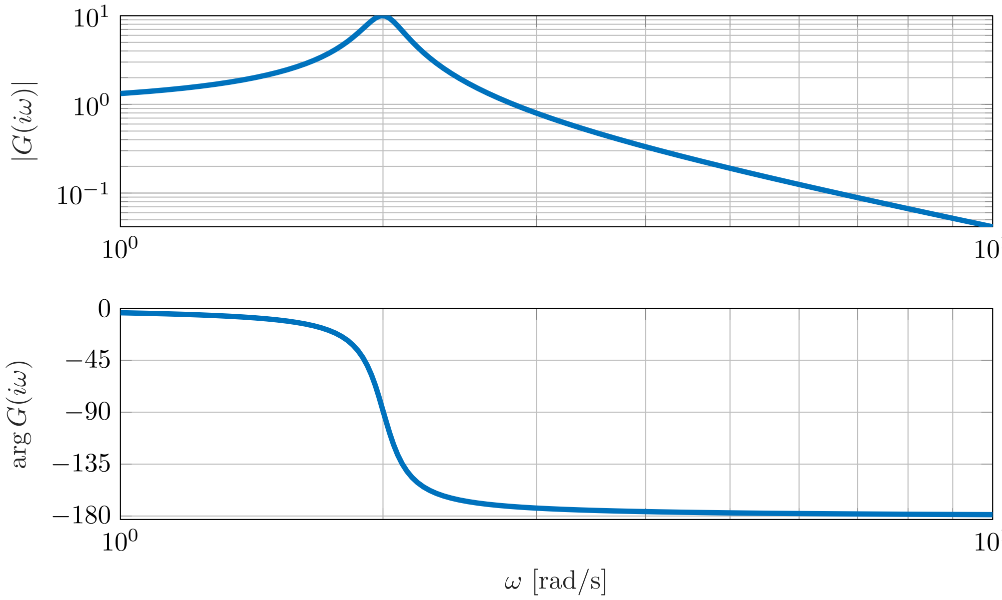


Figure 2: Bode diagram of the system G.

Proposed solution to problem 4

a) We have that

$$|G(i\omega)| = \frac{|b|}{|i^2\omega^2 + a_1 i\omega + a_2|} = \frac{b}{|a_2 - \omega^2 + ia_1\omega|} = \frac{b}{\sqrt{(a_2-\omega^2)^2 + a_1^2\omega^2}}.$$

We are given that $\omega_{\mathrm{p}}^2 \approx a_2$, hence

$$M_{\mathrm{p}} = |G(i\omega_{\mathrm{p}})| = \frac{b}{\sqrt{(a_2-\omega_{\mathrm{p}}^2)^2 + a_1^2\omega_{\mathrm{p}}^2}} \approx \frac{b}{a_1\omega_{\mathrm{p}}}.$$

Furthermore

$$K = |G(0)| = \frac{b}{a_2} \approx \frac{b}{\omega_{\mathrm{p}}^2}.$$

Hence, we have that $b \approx K\omega_{\mathrm{p}}^2$ and $a_1 \approx \frac{b}{M_{\mathrm{p}}\omega_{\mathrm{p}}} \approx \frac{K\omega_{\mathrm{p}}}{M_{\mathrm{p}}}$.

Answer: $b = K\omega_{\mathrm{p}}^2$ and $a_1 = \frac{K\omega_{\mathrm{p}}}{M_{\mathrm{p}}}$.

b) (i) The open-loop system will be given by $G_o(s) = F(s)G(s)$. We have that

$$\arg G_o(i\omega) = \arg F(i\omega) + \arg G(i\omega),$$

where

$$\arg F(i\omega) = \arg(K_{\mathrm{P}} + iK_{\mathrm{D}}\omega) = \arctan\frac{K_{\mathrm{D}}\omega}{K_{\mathrm{P}}} \geq 0,$$

so $\arg G_o(i\omega) \geq \arg G(iw) > -180°$. Hence, $G_o(i\omega)$ can not enclose -1 and the closed-loop system is stable for all $K_{\mathrm{P}}, K_{\mathrm{D}} > 0$.

Answer: The closed-loop system is stable for all $K_{\mathrm{P}}, K_{\mathrm{D}} > 0$.

(ii) According to 'sine-in sine-out' the output will be given by

$$y(t) = |G_c(i2)| \sin\left(2t + \arg G_c(i2)\right),$$

where the closed-loop system

$$G_c(s) = \frac{G_o(s)}{1 + G_o(s)}.$$

From the bode diagram, we have that $|G(i2)| = 10$ and $\arg G(i2) = -90° = -\frac{\pi}{2}$ rad. Furthermore, $|F(i\omega)| = \sqrt{1+\omega^2}$, so $|F(i2)| = \sqrt{5}$ and $\arg F(i2) = \arctan 2$. Thus,

$$G_o(i2) = |G(i2)||F(i2)|e^{i(\arg G(i2) + \arg F(i2))} = 10\sqrt{5}e^{i(-\frac{\pi}{2}+\arctan 2)} \approx 22.36e^{-0.46i}.$$

Moreover,

$$\begin{aligned}
1 + G_o(i2) &= 1 + 10\sqrt{5}\cos(-\frac{\pi}{2} + \arctan 2) + i10\sqrt{5}\sin(-\frac{\pi}{2} + \arctan 2) \\
&\approx 1 + 22.36\cos(-0.46) + i22.36\sin(-0.46) \\
&= 20.93 - i9.93,
\end{aligned}$$

so

$$
\begin{aligned}
|1 + G_o(i2)| &= \sqrt{20.93^2 + 9.93^2} \approx 23.17, \\
\arg(1 + G_o(i2)) &= -\arctan \frac{9.93}{20.93} \approx -0.44.
\end{aligned}
$$

Hence

$$
\begin{aligned}
|G_c(i2)| &= \frac{22.36}{23.17} \approx 0.97, \\
\arg(G_c(i2)) &= -0.46 - (-0.44) = -0.02.
\end{aligned}
$$

Answer: $y(t) = 0.97 \sin(2t - 0.02)$

Problem 5 (6/30)

A minesweeper vessel uses hydrophones in order to detect underwater mines. In particular, the hydrophones are arranged in an array in order to cover a larger terrain. The transfer function from the drive shaft which is connected to the vessel to the hydrophone array can be given as a second order system:

$$G(s) = \frac{1}{Js^2 + k_d s}$$

where J is the moment of inertia of the array and k_d represents the viscous force of the water. The transfer function above can be rewritten as a state-space representation by using e.g. the controller canonical form:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} -\frac{k_d}{J} & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & \frac{1}{J} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where u is the input from the drive shaft and y is the output to the array.

We want to control the array with a computer.

a) Discretize the state-space system above (using zero order hold) with a sampling time T.

(3 p)

b) Assume that $J = 4$ Nm s^2 rad^{-1}, $k_d = 2$ Nm s rad^{-1} and $T = 2 \log 2$ s, where log denotes natural logarithm. By using state feedback on the discrete-time system, compute a suitable matrix $\mathbf{L}$ so that the system error will decay as e^{-t}, i.e. the poles in the continuous-time system are at -1.

Note: Answers in rounded decimals are fine.

(3 p)

Proposed solution to problem 5

a) To do so, we use that for a ZOH the discretized system is given by:

$$F = e^{AT} \quad G = \int_0^T e^{A\tau} B \, \mathrm{d}\tau$$

and the matrix exponential can be computed as:

$$e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}]$$

The inverse of $(sI - A)$ is given by:

$$(sI - A)^{-1} = \begin{bmatrix} s + \frac{k_d}{J} & 0 \\ -1 & s \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s + \frac{k_d}{J}} & 0 \\ \frac{1}{s\left(s + \frac{k_d}{J}\right)} & \frac{1}{s} \end{bmatrix}$$

Taking the inverse Laplace transform:

$$F = e^{AT} = \begin{bmatrix} e^{-\frac{k_d}{J}T} & 0 \\ \frac{J}{k_d}\left(1 - e^{-\frac{k_d}{J}T}\right) & 1 \end{bmatrix}$$

Once this is known, computing G is straightforward taking into account that the integral operator can go inside each component of the resulting matrix:

$$\begin{aligned} G &= \int_0^T \begin{bmatrix} e^{-\frac{k_d}{J}\tau} & 0 \\ \frac{J}{k_d}\left(1 - e^{-\frac{k_d}{J}\tau}\right) & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathrm{d}\tau \\ &= \int_0^T \begin{bmatrix} e^{-\frac{k_d}{J}\tau} \\ \frac{J}{k_d}\left(1 - e^{-\frac{k_d}{J}\tau}\right) \end{bmatrix} \mathrm{d}\tau \\ &= \begin{bmatrix} -\frac{J}{k_d}\left(e^{-\frac{k_d}{J}T} - 1\right) \\ \frac{J}{k_d}\left(T + \frac{J}{k_d} e^{-\frac{k_d}{J}T} - \frac{J}{k_d}\right) \end{bmatrix} \end{aligned}$$

and $H = C$.

b) Substituting:

$$F = \begin{bmatrix} 0.5 & 0 \\ 1 & 1 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 \\ 4\log 2 - 2 \end{bmatrix}$$

Using state feedback control ($u = -Lx$), the matrix which gives the system dynamics in closed loop is given by $F - GL$:

$$F - GL = \begin{bmatrix} 0.5 - \ell_1 & -\ell_2 \\ 1 - 0.773\ell_1 & 1 - 0.773\ell_2 \end{bmatrix}$$

In order to perform pole placement, the characteristic polynomial of $F - GL$ (which will give the eigenvalues λ) must be computed and then made equal to the desired characteristic polynomial:

$$\det(\lambda I - (F - GL)) = \left\| \begin{array}{cc} \lambda - 0.5 + \ell_1 & \ell_2 \\ -1 + 0.773\ell_1 & \lambda - 1 + 0.773\ell_2 \end{array} \right\| =$$

$$= (\lambda - 0.5 + \ell_1)(\lambda - 1 + 0.773\ell_2) - \ell_2(-1 + 0.773\ell_1)$$

$$= \lambda^2 + (-1.5 + 0.773\ell_2 + \ell_1)\lambda + 0.5 + 0.614\ell_2 - \ell_1 = 0$$

The desired poles are at -1 in continuous time. They must be translated into discrete time with the expression $p_d = e^{p_c T}$, where p_d is the pole in discrete time and p_c is the pole in continuous time. Doing so, the desired characteristic polynomial is $(\lambda - e^{-2\log(2)})^2 = (\lambda - 0.25)^2 = \lambda^2 - 0.5\lambda + 0.0625 = 0$. Since we want this polynomial to be the same as the one obtained above for the system, the coefficients of both polynomials must be equal, obtaining the following system of equations:

$$\begin{array}{lllll} \lambda: & \ell_1 + 0.773\ell_2 - 1.5 & & & = -0.5 \\ 1: & & -\ell_1 & +0.614\ell_2 + 0.5 & = 0.0625 \end{array}$$

which once solved yield the values $\ell_1 = 0.687$ and $\ell_2 = 0.406$.