# Introduction to Computer Control Systems, 5 credits, 1RT485

**Date:** 2023-08-22, 14:00-19:00

**Teacher on duty:** Niklas Wahlström

**Teacher visit:** Ca 16:00

**Number of problems:** 5

**Allowed aid**: A calculator and mathematical handbooks (e.g. Beta)

| | | |
|---|---|---|
| **Preliminary grades:** | grade 3 | 15 points |
| | grade 4 | 21 points |
| | grade 5 | 26 points |

Some general instructions and information:

- Your solutions can be given in Swedish or in English.
- Write only on one side of the paper.
- Write your exam code and page number on all pages.
- Do not use a red pen.
- Use separate sheets of paper for the different problems (i.e. the numbered problems, 1–5).

*With the exception of Problem 1, **all your answers must be clearly motivated!** A correct answer without a proper motivation will score zero points!*

Best of luck!

# Useful results

## Laplace transform table

Table 1: Basic Laplace transforms

| $f(t)$ | $F(s)$ | $f(t)$ | $F(s)$ |
|---|---|---|---|
| unit impulse $\delta(t)$ | $1$ | $\sinh(bt)$ | $\frac{b}{s^2-b^2}$ |
| unit step $1(t)$ | $\frac{1}{s}$ | $\cosh(bt)$ | $\frac{s}{s^2-b^2}$ |
| $t$ | $\frac{1}{s^2}$ | $\frac{1}{2b}t\sin(bt)$ | $\frac{s}{(s^2+b^2)^2}$ |
| $t^n$ | $\frac{n!}{s^{n+1}}$ | $t\cos(bt)$ | $\frac{s^2-b^2}{(s^2+b^2)^2}$ |
| $e^{-at}$ | $\frac{1}{s+a}$ | $\frac{\cos(bt)-\cos(at)}{a^2-b^2}; (a^2 \neq b^2)$ | $\frac{s}{(s^2+a^2)(s^2+b^2)}$ |
| $\frac{1}{a}(1-e^{-at})$ | $\frac{1}{s(s+a)}$ | $\frac{\sin(at)+at\cos(at)}{2a}$ | $\frac{s^2}{(s^2+a^2)^2}$ |
| $\frac{1}{(n-1)!}t^{n-1}e^{-at}; (n=1,2,3...)$ | $\frac{1}{(s+a)^n}$ | | |
| $\sin(bt)$ | $\frac{b}{s^2+b^2}$ | | |
| $\cos(bt)$ | $\frac{s}{s^2+b^2}$ | | |
| $e^{-at}\sin(bt)$ | $\frac{b}{(s+a)^2+b^2}$ | | |
| $e^{-at}\cos(bt)$ | $\frac{s+a}{(s+a)^2+b^2}$ | | |

Table 2: Properties of Laplace Transforms

| | |
|---|---|
| $\mathcal{L}[af(t)] = aF(s)$ | $\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}$ |
| $\mathcal{L}[f_1(t)+f_2(t)] = F_1(s)+F_2(s)$ | $\mathcal{L}\left[t^2 f(t)\right] = \frac{d^2}{ds^2}F(s)$ |
| $\mathcal{L}\left[\frac{d}{dt}f(t)\right] = sF(s)-f(0)$ | $\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s), \quad n=1,2,3,\ldots$ |
| $\mathcal{L}\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s)-sf(0)-f'(0)$ | $\mathcal{L}\left[f\left(\frac{t}{a}\right)\right] = aF(as)$ |
| $\mathcal{L}\left[\int f(t)\,dt\right] = \frac{F(s)}{s}+\frac{1}{s}\left[\int f(t)\,dt\right]_{t=0}$ | $\mathcal{L}\left[\int_0^t f_1(t-\tau)f(t)d\tau\right] = F_1(s)F_2(s)$ |
| $\mathcal{L}[f(t-a)] = e^{-as}F(s)$ | $\mathcal{L}\left[e^{-at}f(t)\right] = F(s+a)$ |

## Matrix exponential

$$\boxed{e^{At} \triangleq \mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\}}$$

## Open-loop and sensitivity functions

$$G_o(s) = G(s)F_y(s), \qquad S(s) = \frac{1}{1+G_o(s)}, \qquad T(s) = 1-S(s)$$

## State-space forms and transfer function relations

- State-space form and transfer function

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad \Rightarrow \quad \boxed{G(s) = C(sI - A)^{-1}B + D}$$

- Associated matrices

$$\mathcal{S} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \qquad \mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

- LTI system with transfer function

$$\boxed{G(s) = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}}$$

i) Observable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ -a_3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ b_3 - a_3 b_0 \\ \vdots \\ b_n - a_n b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

ii) Controllable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_1 - a_1 b_0 & b_2 - a_2 b_0 & \cdots & b_n - a_n b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

- Solution to state-space equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

can be written as

$$\boxed{x(t) = e^{At} x_0 + \int_0^t e^{A\tau} B u(t - \tau) d\tau}$$

- Observer system

$$\boxed{\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})}$$

## Feedback control structures

General linear feedback in Laplace form:

$$\boxed{U(s) = F_r(s)R(s) - F_y(s)Y(s)}$$

Common control structures in this form.

- PID controller:

$$F_y(s) = F_r(s) = F(s) = K_p + \frac{K_i}{s} + K_d s,$$

  where $K_p, K_i, K_d \geq 0$

- Lead-lag controller:

$$F_y(s) = F_r(s) = F(s) = K\left(\frac{\tau_D s + 1}{\beta\tau_D s + 1}\right)\left(\frac{\tau_I s + 1}{\tau_I s + \gamma}\right),$$

  where $K, \tau_D, \tau_I > 0$ and $0 \leq \beta, \gamma < 1$

- State-feedback controller with observer:

$$\begin{aligned} F_r(s) &= \left(1 - L(sI - A + KC + BL)^{-1}B\right)\ell_0 \\ F_y(s) &= L(sI - A + KC + BL)^{-1}K \end{aligned}$$

## Discrete-time state-space forms

A continuous time system with zero-order-hold input signal and sample period $T$ can be written in discrete-time as:

$$\boxed{\begin{aligned} x(k+1) &= Fx(k) + Gu(k) \\ y(k) &= Hx(k) \end{aligned}}$$

where

$$\begin{aligned} F &= e^{AT} \\ G &= \int_{\tau=0}^{T} e^{A\tau} d\tau B = \lceil \text{if } A^{-1} \text{ exists} \rceil = A^{-1}(e^{AT} - I)B \\ H &= C \end{aligned}$$

# Problem 1: basic questions (6/30)

Answer only *'true'* or *'false'*. Each correct answer gives 1 point, each wrong answer gives −1 point, each unanswered answer gives 0 points. Minimum total points for Part A and B is 0 , respectively.

## Part A

*Note:* Write 'skip' if your total home assignment score $\geq 8$

i) The following system is controllable

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

ii) For a system with measurable disturbances, a control structure using feed forward can be applied.

iii) If you control the system $G(s) = \frac{1}{s}$ with the controller $U(s) = K(Y(s) - R(s))$, the closed-loop system will be stable for all $K > 0$.

**(3 p)**

## Part B

*Note:* Write 'skip' if your total home assignment score $\geq 12$

i) All controllable systems are input-output stable.

ii) The rise time and bandwidth both relate to the quickness of a system.

iii) Consider a system with transfer function

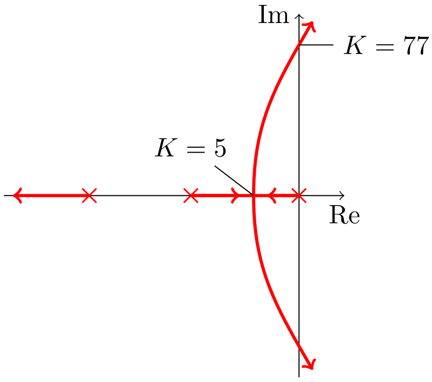
$$G(s) = \frac{s+1}{(s+2)(s+5)}.$$

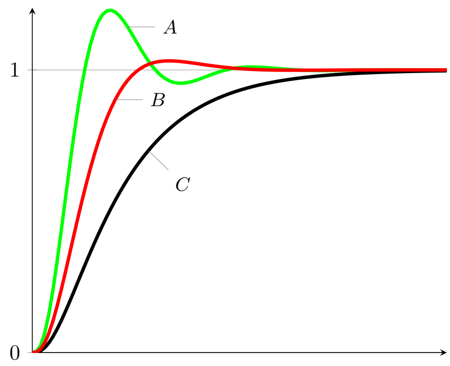
Suppose the same system is described as a state-space model that is a minimal realization. Then the eigenvalues of the system matrix $A$ of the state-space model are equal to −2 and −5.

**(3 p)**

## Problem 2 (6/30)

The system $Y(s) = G(s)U(s)$ is controlled with proportional feedback, $u = K(r - y)$. Below left is the root locus of the poles of the closed-loop system with respect to $K \geq 0$. On the right, the output signal of the closed-loop system is shown for three different values of $K$ when the reference signal is a unit step.




**a)** For what $K > 0$ is the closed-loop system stable?

**(1 p)**

**b)** For all step responses above, the error $e(t) = y(t) - r(t) \to 0$ as $t \to \infty$, i.e. $G_c(0) = 1$, applies. What characteristic must the *system* $G(s)$ have in order for this to occur?[1]

**(1 p)**

**c)** Four different regulators are evaluated with $K = 4$, $K = 8$, $K = 16$, $K = 80$. Which step response, $A$, $B$, and $C$, goes with which value of $K$ above? Note that for one of the four regulators, the corresponding step response is not displayed above.

**(4 p)**

[1]That the system has this characteristic in this case can be easily read out in the root locus.

## Problem 3 (6/30)

Consider the system

$$Y(s) = G(s)U(s), \qquad G(s) = \frac{1}{(s+1)^3} = \frac{1}{s^3 + 3s^2 + 3s + 1}. \tag{1}$$

**a)** Set up the state-space representation of (1) on *controllable canonical form.*

**(1 p)**

**b)** Is the state-space model in **a)** *asymptotically stable*?

**(1 p)**

**c)** One controls the system (1) with proportional feedback, $u = K(r - y)$. For what $K \in \mathbb{R}$ is the closed-loop system input-output stable?

**(4 p)**

*Hint:*
For a polynomial

$$a_0 s^n + b_0 s^{n-1} + a_1 s^{n-2} + b_1 s^{n-3} + \cdots = 0$$

Routh's table looks like

$$\begin{array}{ccccc} a_0 & a_1 & a_2 & a_3 & \ldots \\ b_0 & b_1 & b_2 & b_3 & \ldots \\ c_0 & c_1 & c_2 & \ldots & \\ d_0 & d_1 & d_2 & \ldots & \end{array}$$

where

$$\begin{aligned} c_k &= \frac{b_0 a_{k+1} - a_0 b_{k+1}}{b_0} \\ d_k &= \frac{c_0 b_{k+1} - b_0 c_{k+1}}{c_0} \\ &\vdots \end{aligned}$$

## Problem 4 (6/30)

A classic method of setting the controller parameters for a PID-controller by experimental means is the Ziegler-Nichols method. It is based on first using proportional feedback, $u = K(r - y)$. During the experiment, one increases carefully the gain $K$ until the closed-loop system starts to self-oscillate with a constant amplitude. One notes the value of $K$ and the self-oscillation frequency, and these values are then used to determine the PID-controller parameters.

Now suppose such an experiment is done for the system

$$Y(s) = \frac{1}{s(s+1)^3}U(s). \tag{2}$$

This means, the control law $u = K(r-y)$ is used for this system, and for $K = K_1$ the closed-loop system starts to self-oscillate with the angular frequency $\omega_1$.

**a)** Determine $K_1$.

**(2p)**

**b)** What will $\omega_1$ be?

**(1p)**

**c)** Suppose that you have done the experiment above, and that you control the system (2) with the feedback

$$U(s) = F_{PD}(s)(R(s) - Y(s)), \quad F_{PD}(s) = K_p + K_d s, \tag{3}$$
$$\text{with} \quad K_p = K_d = \omega_1(\omega_1^2 + 1).$$

Determine the crossover frequency $\omega_c$ and the phase margin $\varphi_m$ when the controller (3) is used on system (2).
(It should be pointed out here that the regulator (3) *does not* correspond to any Ziegler-Nichols setting.)

**(3p)**

## Problem 5 (6/30)

A harmonic oscillator can be described by the following continuous-time, second order system

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix} x + \begin{bmatrix} 8 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 0.5 \end{bmatrix} x\end{aligned}$$

**a)** Determine if the continuous-time system is observable.

**(1 p)**

**b)** Using zero-order hold, discretize the system with sampling time $T$.

**(2 p)**

**c)** We can only measure the output, so we need an observer in order to be able to estimate the states of the discrete-time system. Compute the observer gain $K = [k_1, k_2]^T$ for the discrete-time system derived in **b)** so that the observer has disctere-time poles in $e^{-T}$ (corresponds to continuous-time poles in -1). Consider a sampling time of $T = \pi/8$.

**(3 p)**