Division of Computing Science
Department of Information Technology
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Written examination in 1TD184 Optimization

- Date: 2022-06-10, 14.00-19.00
- Allowed tools: Pocket calculator, one A4 paper with notes (computer typed, font size minimum 10 pt).
- Maximum number of points: 36 (18 to pass).
- All assumptions and answers *must* be motivated for full points.

(1) Describe the following concepts: 1) Convex optimization problem for minimization as well as for maximization, and 2) Gauss-Newton method. (4p)

(2) Consider a circle with center at origin $(0,0)^T$ and radius r. We would like to place a rectangle inside the circle such that the rectangle's area is maximum possible. Formulate this task as an optimization problem and examine if it is a convex optimization problem. Next, solve the problem and argue why the solution you obtained is the optimum. (5p)

(3) True or false (answers *must* be motivated):

a) A search direction p is a descent direction of function $f(x)$ if and only if condition $p^T \nabla f(x) < 0$ holds. (1p)

b) For a quadratic problem $\min \frac{1}{2}x^T Qx - b^T x$, Newton's direction will be a descent direction. (1p)

c) If the computation in each iteration returns the optimum, a sequential quadratic programming (SQP) method will monotonically improve the objective function. (1p)

d) Degeneration in LP occurs if there are multiple optima. (1p)

(4) Consider minimizing a strictly convex function $f(x)$ over a polytope defined by a set of linear inequalities $\{a_i^T x \leq b_i, i = 1, \ldots, m\}$. We have point $\bar{x}$ with $\nabla f(\bar{x}) = 0$.

a) Suppose $\bar{x}$ is feasible but the last constraint is active, i.e., $a_i^T \bar{x} < b_i, i = 1, \ldots, m-1$, and $a_m^T \bar{x} = b_m$. Is $\bar{x}$ optimal? (1p)

b) Suppose $a_i^T \bar{x} < b_i, i = 1, \ldots, m-1$, remain to be true, but $\bar{x}$ violates the last constraint. Prove that at optimum x^*, the last constraint must be active, i.e., $a_m^T x^* = b_m$. Is x^* the point on $a_m^T x = b_m$ being closest to $\bar{x}$? Why or why not? (2p)

(5) Consider minimizing the one-dimensional function $f(x) = \frac{1}{4}x^4 + x^3 - 2x$.

- Give one point for which the steepest descent direction and Newton's direction coincide, and one point for which these two directions differ. What is the underlying reason for the latter case? (3p)
- Explain the underlying rationale of Armijo line search. Then, for the function above, and point $x = 0$ and search direction $p = 1$, formulate the Armijo's condition. Is there a step size satisfying the condition for Armijo parameter $\mu = 0.5$? (3p)

(6) Consider Quasi-Newton methods, and denote by B_k and B_{k+1} the matrices used for obtaining the search directions of iterations k and $k+1$, respectively, Moreover, let x_k and x_{k+1} be the solutions of the two iterations, and define $s_k = x_{k+1} - x_k$, and $y_k = \nabla f(x_{k+1}) - \nabla f(x_k)$.

- Define the secant condition and explain the reason of imposing it. (2p)
- In the rank-one update, $B_{k+1} = B_k + \frac{(y_k - B_k s_k)(y_k - B_k s_k)^T}{(y_k - B_k s_k)^T s_k}$. Why is this a rank-one update? Does B_{k+1} satisfy the secant condition by this update? (2p)

(7) We have the following constrained optimization problem.

$$
\begin{aligned}
\min \quad & (x_1 - 3)^2 + (x_2 - \frac{1}{2})^2 \\
& x_1 + x_2 \leq 3 \\
& x_1 - x_2 \leq 2 \\
& x_1 \geq 0 \\
& x_1 \leq 2
\end{aligned}
$$

a) Consider point $(2,0)^T$. Use the first-order necessary condition to show this is not a local minimum. (*Hint: Convert constraints of* $\leq$ *to* $\geq$*, if you are more used to dealing with the latter.*) (1p)

b) For the above point, what is the search direction that is feasible and closest to the steepest descent direction? What is the optimal (and feasible) step size along this direction? (2p)

c) Suppose we know (or conjecture) that the only active constraint at minimum is $x_1 = 2$. Then we can deploy a penalty function $(x_1-3)^2 + (x_2 - \frac{1}{2})^2 + \frac{1}{2}\rho(x_1 - 2)^2$ and omit all the constraints. What point is the minimum to this penalty function when $\rho \to \infty$? Is this the point reached by taking the optimal step size in b)? (2p)

(8) We have the following linear program (LP) and its dual, where c is an unknown parameter.

LP	Dual
$\min \quad -2x_1 - x_2 - 2x_3 + x_4$	$\max \quad 4y_1 + cy_2$
$x_1 + 2x_2 + x_3 = 4$	$y_1 + y_2 \leq -2$
$x_1 - x_2 + x_4 = c$	$2y_1 - y_2 \leq -1$
$x_1, x_2, x_3, x_4 \geq 0$	$y_1 \leq -2$
	$y_2 \leq 1$

- Use a geometric argument to find out when the LP dual has unbounded optimum because of the value of c. (1p)
- Let $c = 1$ and consider the extreme point $y_1 = -2$ and $y_2 = -3$ in the dual. What would be the direction taken by the simplex method at this point for solving the dual? What is the optimal solution of the dual? (2p)
- Use complementary slackness to derive the optimum to the primal LP for $c = 1$, and show that the solution satisfies the optimality criterion using reduced cost. (2p)

Good Luck!