Uppsala University
Department of Information Technology
Division of Scientific Computing

Scientific Computing for Partial Differential Equations, 5.0 credits, 2023-03-15

Exam time: $8^{00} - 13^{00}$ (5 hrs)
Aids: Attached formulae, calculator, Physics Handbook, Mathematics Handbook

Unless otherwise specified, all solutions must include detailed reasoning and complete calculations.

Grade requirements:
Grade 3: At least 12/24 points.
Grade 4: At least 17/24 points.
Grade 5: At least 21/24 points.

1. Consider the following PDE and initial condition:

$$\begin{aligned} C\mathbf{u}_t &= B\mathbf{u}_x + \mathbf{F}, \quad x \in (0, L), \quad t > 0, \\ \mathbf{u} &= \mathbf{f}, \quad x \in [0, L], \quad t = 0, \end{aligned} \tag{1}$$

where $\mathbf{F} = \mathbf{F}(x, t)$ is a forcing function, $\mathbf{f} = \mathbf{f}(x)$ is initial data, and

$$C(x) = \begin{bmatrix} c_1(x) & \\ & c_2(x) \end{bmatrix}, \quad B = \begin{bmatrix} & i \\ -i & \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

where $c_{1,2}$ are real-valued, strictly positive functions.

(a) Use the energy method to prove that (1) is well-posed with *periodic* boundary conditions. That is, you may disregard any boundary terms. (2p)

(b) For the non-periodic initial-boundary-value problem (IBVP) to be well posed, how many boundary conditions are required at $x = 0$ and at $x = L$? Motivate your answer. (1p)

(c) Derive a set of well-posed boundary conditions. (1p)

(d) State an SBP-SAT discretization of the PDE combined with the set of boundary conditions that you derived in the previous problem. You may here assume that $\mathbf{F} = 0$ and c_1 and c_2 are *constant*. You do *not* need to prove stability. You may use up to 4 unspecified *scalar* penalty parameters in your SATs. (2p)

2. The second-order wave equation in a bounded 2D domain $\Omega \subset \mathbb{R}^2$ is given by

$$\begin{aligned} &\phi_{tt} = c^2 \nabla \cdot \nabla \phi, && \vec{x} \in \Omega, && t > 0, \\ &\frac{\partial \phi}{\partial \hat{\mathbf{n}}} + \alpha \phi_t = 0, && \vec{x} \in \partial\Omega, && t > 0, \\ &\phi = \phi_0, && \vec{x} \in \Omega, && t = 0, \\ &\phi_t = \varphi_0, && \vec{x} \in \Omega, && t = 0, \end{aligned} \tag{2}$$

where $c = c(\vec{x}) > 0$, $\alpha \in \mathbb{R}$, $\partial\Omega$ denotes the boundary of Ω, and $\hat{\mathbf{n}}$ denotes the outward unit normal. We assume that the solution is real.

(a) Consider the case $\alpha = 0$. Prove that the IBVP (2) conserves energy. (2p)

(b) For $\alpha > 0$, prove that the IBVP (2) dissipates energy. (1p)

3. A 1D version of the second-order wave equation with constant wave speed is

$$
\begin{array}{lll}
\phi_{tt} = c^2 \phi_{xx}, & x \in (0, L) & t > 0, \\
\alpha\phi_t - \phi_x = 0, & x = 0, & t > 0, \\
\alpha\phi_t + \phi_x = 0, & x = L, & t > 0, \\
\phi = \phi_0, & x \in [0, L], & t = 0, \\
\phi_t = \varphi_0, & x \in [0, L], & t = 0,
\end{array}
\tag{3}
$$

where $c > 0$ is constant, $L > 0$, and $\alpha \in \mathbb{R}$. We assume that the solution is real.

(a) State an SBP-SAT discretization of the PDE and boundary conditions. You do not need to prove stability. You may use one unspecified scalar penalty parameter per boundary. You may solve the problem with $\alpha = 0$ for 1 point out of 2. (2p)

(b) Select appropriate penalty parameters and prove that your SBP-SAT scheme is stable for $\alpha \geq 0$. You may solve the problem with $\alpha = 0$ for 2 points out of 3. (3p)

4. Consider the heat equation with constant coefficients in the interval $\mathcal{I} = (0, L)$:

$$
\begin{array}{rll}
u_t = au_{xx}, & x \in \mathcal{I}, & t > 0, \\
u = g, & x = 0, & t > 0, \\
u_x = 0, & x = L, & t > 0, \\
u = f, & x \in \mathcal{I}, & t = 0,
\end{array}
\tag{4}
$$

where g, $a > 0$ and $L > 0$ are real constants. We assume that the solution is real.

(a) Derive the weak form of (4) with appropriate spaces. (2p)

(b) State the finite element approximation of the weak form, using appropriate spaces of piecewise linear functions on a uniform mesh of n intervals. (2p)

(c) Derive the system of ODE corresponding to the finite element approximation. You do not need to evaluate any nonzero integrals, but you should indicate the structure of all matrices and vectors and show where they have nonzero entries. (3p)

5. Consider the linear system $Ax = b$, where

$$
A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & -32 & 4 \\ 1 & 0 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 8 \\ 0 \\ 4 \end{bmatrix}.
$$

(a) Perform one iteration using the Gauss-Seidel method. Use the starting guess $x_0 = [1, 1, 8]^T$. (1p)

(b) Do you expect the Gauss-Seidel method to converge for this linear system? Motivate your answer. (2p)

Collection of formulae

Summation-by-parts operators

Consider a uniform grid of $m+1$ points, with grid spacing $h = \frac{L}{m}$. Let $\mathbf{e}_\ell$ and $\mathbf{e}_r$ denote the following vectors in $\mathbb{R}^{m+1}$:

$$\mathbf{e}_\ell = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_r = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Definition of D_1

A difference operator D_1 approximating $\partial/\partial x$ is a first-derivative SBP operator with quadrature matrix H if $H = H^T > 0$ and

$$HD_1 = \mathbf{e}_r\mathbf{e}_r^T - \mathbf{e}_\ell\mathbf{e}_\ell^T - D_1^T H.$$

Definition of D_2

A difference operator D_2 approximating $\partial^2/\partial x^2$ is a second-derivative SBP operator with quadrature matrix H if $H = H^T > 0$ and

$$HD_2 = \mathbf{e}_r\mathbf{d}_r^T - \mathbf{e}_\ell\mathbf{d}_\ell^T - M,$$

where $M = M^T \geq 0$, and $\mathbf{d}_\ell^T v \simeq u_x|_{x=0}$, $\mathbf{d}_r^T v \simeq u_x|_{x=L}$ are finite difference approximations of the first derivatives at the left and right boundary points.

Discrete inner product

Let $(\cdot,\cdot)_H$ denote the discrete inner product, defined by

$$(\mathbf{u},\mathbf{v})_H = \mathbf{u}^* H\mathbf{v}.$$

In the discrete inner product, the SBP operators satisfy

$$(\mathbf{u}, D_1\mathbf{v})_H = (\mathbf{e}_r^T\mathbf{u})^*(\mathbf{e}_r^T\mathbf{v}) - (\mathbf{e}_\ell^T\mathbf{u})^*(\mathbf{e}_\ell^T\mathbf{v}) - (D_1\mathbf{u},\mathbf{v})_H$$

and

$$(\mathbf{u}, D_2\mathbf{v})_H = (\mathbf{e}_r^T\mathbf{u})^*(\mathbf{d}_r^T\mathbf{v}) - (\mathbf{e}_\ell^T\mathbf{u})^*(\mathbf{d}_\ell^T\mathbf{v}) - \mathbf{u}^* M\mathbf{v}.$$

Integration by parts in multiple dimensions

Let Ω denote a bounded domain in $\mathbb{R}^n$, with boundary $\partial\Omega$ and outward unit normal $\hat{\mathbf{n}}$. For (sufficiently smooth) scalar functions $u, v, \alpha \in L^2(\Omega)$,

$$\int_{\Omega} u\nabla \cdot \alpha\nabla v \,\mathrm{d}\Omega = \oint_{\partial\Omega} u\alpha\frac{\partial v}{\partial \hat{\mathbf{n}}}\,\mathrm{d}S - \int_{\Omega} \nabla u \cdot \alpha\nabla v \,\mathrm{d}\Omega.$$

This relation follows from the divergence theorem.

Finite element methods

Given a mesh of n intervals

$$x_0 < x_1 < \ldots < x_n,$$

the usual basis functions (the hat functions) in the corresponding space of piecewise linear functions satisfy, for $i = 1, \ldots, n-1$,

$$(\varphi_i', \varphi_j') = \begin{cases} \dfrac{1}{h_i} + \dfrac{1}{h_{i+1}}, & i = j \\ -\dfrac{1}{h_{i+1}}, & j = i+1 \\ -\dfrac{1}{h_i}, & j = i-1 \\ 0, & |i-j| > 1 \end{cases}$$

where $h_i = x_i - x_{i-1}$. Further, for $i = 1, \ldots, n-1$:

$$(\varphi_i, \varphi_j) = \begin{cases} \dfrac{h_i}{3} + \dfrac{h_{i+1}}{3}, & i = j \\ \dfrac{h_{i+1}}{6}, & j = i+1 \\ \dfrac{h_i}{6}, & j = i-1 \\ 0, & |i-j| > 1 \end{cases}$$