Intermediate Exam 1 Signals & Transforms

1 a) The Fourier transform can be seen as a decomposition of the aperiodic signal $x(t)$ into components with all possible ω. The transform gives a measure of how strong each component is.

b) 1. The signal $x(t)$ must be causal.
2. The Fourier transform for $x(t)$ must exist (i.e., the integral must converge).
3. The Laplace transform must be evaluated in $s=j\omega$.

c) The signal $z(t)$ is a product in the time domain. This will result in a convolution in the frequency domain:

$$Z(\omega) = \mathcal{F}\left\{\frac{2}{\pi}\operatorname{sinc}\left(\frac{2t}{\pi}\right)x(t)\right\}$$

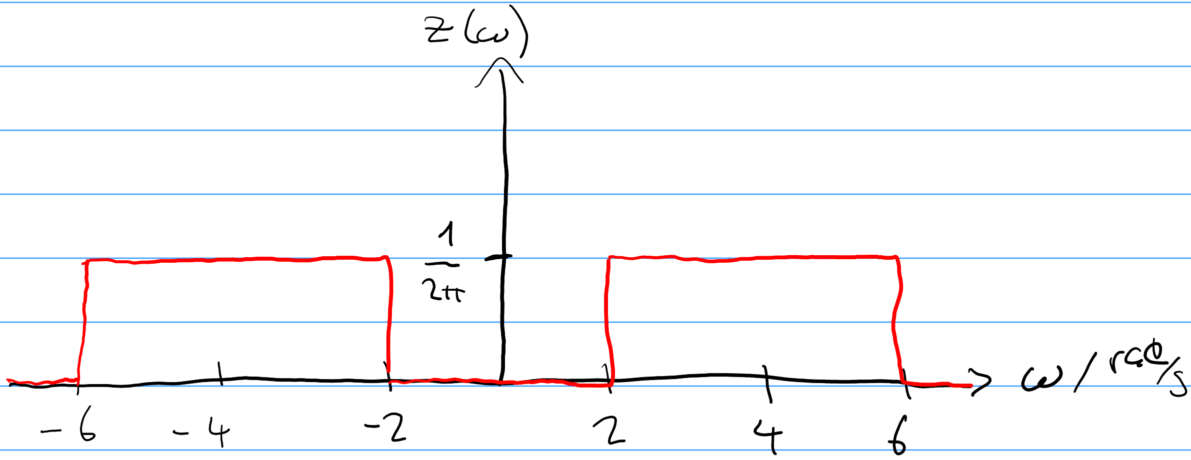
$$= \frac{1}{2\pi}\mathcal{F}\left\{\frac{2}{\pi}\operatorname{sinc}\left(\frac{2t}{\pi}\right)\right\} * \mathcal{F}\left\{x(t)\right\}$$

$$= \frac{1}{2\pi}\mathcal{F}\left\{\frac{2}{\pi}\operatorname{sinc}\left(\frac{2t}{\pi}\right)\right\} * X(\omega)$$

$$= \frac{1}{2\pi}\underbrace{\operatorname{rect}\left(\frac{\omega}{2\cdot 2}\right)}_{\text{A rect of width 4, centered around 0.}} * \underbrace{X(\omega)}_{\text{Two Dirac delta fcns. at }\pm 4\text{, weight 1.}}$$

$$= \frac{1}{2\pi}\operatorname{rect}\left(\frac{\omega}{4}\right) * \left(\delta(\omega-4)+\delta(\omega+4)\right)$$

$$= \frac{1}{2\pi}\Big(\underbrace{\mathrm{rect}\left(\frac{\omega-4}{4}\right)+\mathrm{rect}\left(\frac{\omega+4}{4}\right)}_{\text{Two rects of width 4 centered at } \pm 4}\Big).$$



2 a) Yes! We see from the figure that

$$\underline{\underline{x_3(t) = x_1(t-2) + x_2(t-2) \mapsto y_1(t-2) + y_2(t-2)}}$$

b) There are several ways of determining $h(t)$ from the figure. One way is to look at $x_2(t)$, which is $x_2(t) = -\delta(t)$. Hence, $-x_2(t) \mapsto h(t)$ or $x_2(t) \mapsto -h(t)$, that is, inverting the output observed for $x_2(t)$ yields the impulse response.

This yields:

$$\underline{\underline{h(t) = \mathrm{rect}\left(\frac{t-1}{2}\right)}}$$

c) The input is a pure cosine. Hence, we can use the sine in, sine out principle:

$$x(t) = 2\cos\left(\frac{\pi}{2}t\right) \mapsto y(t) = 2|H(\tfrac{\pi}{2})|\cos\left(\frac{\pi}{2}t + \angle H(\tfrac{\pi}{2})\right)$$

where $H(\frac{\pi}{2}) = |H(\frac{\pi}{2})|e^{j\angle H(\frac{\pi}{2})}$ is the frequency response evaluated in $\omega = \frac{\pi}{2}$.

$$H(\omega) = \mathcal{F}\{h(t)\} = \mathcal{F}\left\{\text{rect}\left(\frac{t-1}{2}\right)\right\}$$

Time shifting, rect $\circ\!\!-\!\!\bullet$ sinc (from table)

$$= 2e^{-j\omega}\,\text{sinc}\left(\frac{2\omega}{2\pi}\right)$$

$$\Rightarrow H(\tfrac{\pi}{2}) = 2e^{-j\frac{\pi}{2}}\,\text{sinc}\left(\frac{2\,\pi/2}{2\pi}\right) = 2e^{-j\frac{\pi}{2}}\,\text{sinc}\left(\frac{1}{2}\right) \approx 1{,}3e^{-j\frac{\pi}{2}}$$

$$\Rightarrow |H(\tfrac{\pi}{2})| = 1{,}3\,,\quad \angle H(\tfrac{\pi}{2}) = -\pi/2$$

Thus, $y(t) = 2\cdot 1{,}3\cos\left(\frac{\pi}{2}t - \pi/2\right)$

$$\underline{\underline{y(t) = 2{,}6\cos\left(\frac{\pi}{2}t - \frac{\pi}{2}\right)}}$$

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3 a) $H(s) = \dfrac{1}{(s+2)(s+3)}$

The system has two real-valued poles. Hence, we can use partial fraction expansion to calculate the inverse transform (and thus the impulse response):

$$h(t) \circ\!\!-\!\!\bullet H(s).$$

Hence: $H(s) = \frac{c_1}{s+2} + \frac{c_2}{s+3} = \frac{c_1(s+3)}{s+2} + \frac{c_2(s+2)}{s+3}$

$$= \frac{c_1 s + 3c_1 + c_2 s + 2c_2}{(s+2)(s+3)}$$

Hence, we have that

$$s(c_1 + c_2) + 3c_1 + 2c_2 = 1$$

$$\begin{cases} c_1 + c_2 = 0 \\ 3c_1 + 2c_2 = 1 \end{cases} \Rightarrow c_1 = -c_2$$

$$3c_1 + 2c_2 = 1$$
$$-3c_2 + 2c_2 = 1$$
$$-c_2 = 1$$
$$c_2 = -1$$

$\rightarrow c_1 = 1, \quad c_2 = -1$

Hence:

$$H(s) = \frac{1}{s+2} - \frac{1}{s+3}$$

(Transform table)

$$\underline{\underline{h(t) = e^{-2t} u(t) - e^{-3t} u(t)}}$$

b) The frequency response is related to the transfer function as

$$H(\omega) = H(s)\big|_{s=j\omega}$$

Hence,

$$H(\omega) = H(s)\big|_{s=j\omega} = \frac{1}{(s+2)(s+3)}\Bigg|_{s=j\omega}$$

$$= \frac{1}{s^2+5s+6}\Bigg|_{s=j\omega}$$

$$\underline{\underline{H(\omega) = \frac{1}{(j\omega)^2+5j\omega+6}}}$$

c) To sketch the Bode plot, we first rewrite the transfer function to Bode form:

$$H(s) = \frac{1}{(s+2)(s+3)} = \frac{1}{2\cdot 3\cdot (s/2+1)(s/3+1)}$$

$$= \frac{1}{6}\cdot\frac{1}{(s/2+1)(s/3+1)}.$$

Thus, we have three terms:

1. A constant $k_0 = 1/6$
2. A first order term $\frac{1}{s/2+1}$
3. A first order term $\frac{1}{s/3+1}$

Term	Frequency	Magnitude	Phase
k_0	All	$20\log_{10}(1/6) = -15{,}6\,dB$	0
$\frac{1}{s/2+1}$	$\omega \ll 2$	$0\,dB$	0°
	$\omega \approx 2$	$-3\,dB$	-45°
	$\omega \gg 2$	$-20\,dB/decade$	-90°
$\frac{1}{s/3+1}$	$\omega \ll 3$	$0\,dB$	0°
	$\omega \approx 3$	$-3\,dB$	-45°
	$\omega \gg 3$	$-20\,dB/decade$	-90°

This results in the attached Bode plots.

4a) $\ddot{y}(t)+6\dot{y}(t)+13y(t)=\dot{x}(t)+2x(t)$

$\circ\!\!-\!\!\bullet \ \mathcal{L}\{\cdot\}$ (zero initial conditions)

$s^2Y(s)+6sY(s)+13Y(s)=sX(s)+2X(s)$

$Y(s)(s^2+6s+13)=(s+2)X(s)$

$$\frac{Y(s)}{X(s)}=H(s)=\frac{s+2}{s^2+6s+13}$$

b) Zeros: Roots of the numerator

$s+2=0$

$s=-2 \Rightarrow z_1=-2$

Poles: Roots of the denominator

$s^2+6s+13=0$

Using the quadratic formula:

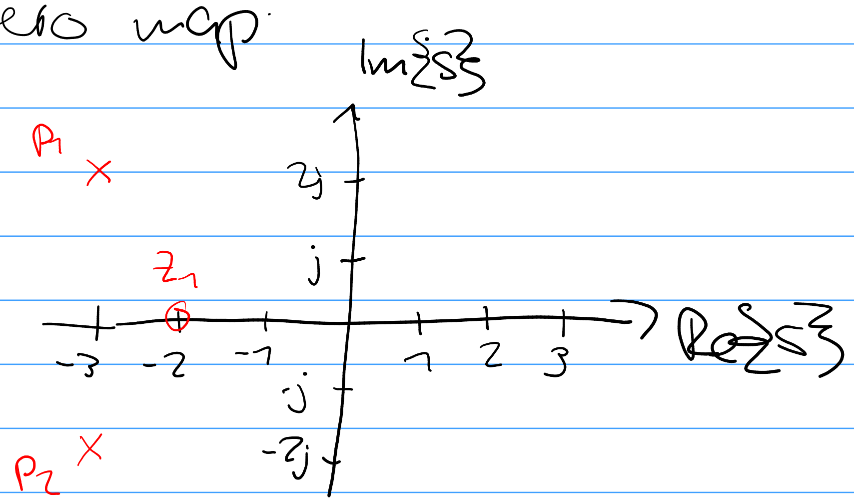
$$p_{1,2}=-\frac{6}{2}\pm\frac{\sqrt{36-4\cdot13}}{2}=-3\pm\frac{\sqrt{36-52}}{2}$$

$$=-3\pm\frac{\sqrt{-16}}{2}=-3\pm j\frac{4}{2}$$

$$=-3\pm j2$$

Hence: Zeros $z_1=-2$, poles $p_1=-3+j2$, $p_2=-3-j2$

Pole-zero map:



The system is stable since all (both) poles have negative real part.

c) The input-output relationship is

$$Y(s) = H(s)X(s).$$

$H(s)$ is as given above, and $x(t) = u(t)$, which yields $X(s) = \frac{1}{s}$.

Hence,

$$Y(s) = \frac{s+2}{s^2+6s+13} \cdot \frac{1}{s}$$

$$= \frac{\cancel{s}}{\cancel{s}(s^2+6s+13)} + \frac{2}{s(s^2+6s+13)}$$

The denominator can be written as

$$s^2+6s+13 = (s+3)^2+4$$

such that

$$Y(s) = \frac{1}{(s+3)^2+4} + \frac{2}{s[(s+3)^2+4]}$$

- For the first term, we can use the transform pair

$$\frac{\omega_0}{(s+a)^2+\omega_0^2} \circ\!-\!\!\!-\!\!\circ\; e^{-at}\sin(\omega_0 t)u(t)$$

with $a=3$ and $\omega_0=2$:

$$\frac{1}{2}\,\frac{2}{(s+3)^2+4} \circ\!-\!\!\!-\!\!\circ\; \underline{\frac{1}{2}e^{-3t}\sin(2t)u(t)}$$

- For the second term, we have to use partial fraction expansion:

$$\frac{2}{s[(s+3)^2+4]} = \frac{c_1}{s} + \frac{c_2 s + c_3}{s^2+6s+13}$$

$$= \frac{c_1(s^2+6s+13)+s(c_2 s+c_3)}{s(s^2+6s+13)}$$

$$= \frac{c_1 s^2+6sc_1+13c_1+c_2 s^2+c_3 s}{s(s^2+6s+13)}$$

Hence:

$$s^2(c_1+c_2)+s(6c_1+c_3)+13c_1=2$$

$$c_1+c_2=0 \Rightarrow c_2=-c_1$$

$$6c_1+c_3=0 \Rightarrow c_3=-6c_1$$

$$13c_1=2 \Rightarrow c_1=2/13$$

$$c_1 = \frac{2}{13}, \quad c_2 = -\frac{2}{13}, \quad c_3 = -\frac{12}{13}$$

Hence,

$$\frac{2}{s([s+3]^2+4)} = \frac{2/13}{s} - \frac{2}{13}\,\frac{s+6}{(s+3)^2+4}$$

$$= \frac{2}{13}\left(\frac{1}{s} - \frac{s+3}{(s+3)^2+4} - \frac{3}{(s+3)^2+4}\right)$$

$$\frac{2}{13}\left(u(t) - e^{-3t}\cos(2t)u(t) - \frac{3}{2}e^{-3t}\sin(3t)\,u(t)\right)$$

Finally, we have:

$$y(t) = \frac{1}{2}e^{-3t}\sin(3t)u(t) + \frac{2}{13}u(t) - \frac{2}{13}e^{-3t}\cos(3t)u(t)$$
$$-\frac{3}{13}e^{-3t}\sin(3t)u(t)$$

$$y(t) = \left[\frac{1}{2}\sin(3t) - \frac{2}{13}\cos(3t) - \frac{3}{13}\sin(3t)\right]e^{-3t}u(t) + \frac{2}{13}u(t)$$

Bode Diagram