Suggested Solutions for Exam Aug19

David Lundberg[1]

[1]Uppsala University, david.lundberg.math@uu.se

3 juni 2020

We suggest solutions for the Exam in Multivariable Calculus from Augst 2019. Observe that there may be more ways of solving the problems.

In case there are typos or errors found, please contact the Author or the Course Responsible.

1 Problem 1

The function is

$$f(x, y) = y^2 - xy - 3y + 2x \tag{1}$$

and we want to find maximum and minimum on the triangle T in $\mathbb{R}^2$ spanned by the points $(4, 0)$, $(0, 0)$ and $(0, 4)$. Since f is continuous we know that maxima and minima are attained.

To look for maxima/minima in the interior, we solve for $\nabla f = \vec{0}$. We have

$$\begin{aligned} \frac{\partial f}{\partial x} &= -y + 2, \\ \frac{\partial f}{\partial y} &= 2y - x - 3. \end{aligned} \tag{2}$$

It is easily seen that the point $(1, 2)$ is the only solution and $f(1, 2) = -2$. We need not analyze the type of stationary point in $(1, 2)$, (it will be a saddle point for the very interested reader), but we rather compare the value -2 with how f behaves on the boundary. We have $f(4, 0) = 8$, $f(0, 0) = 0$ and $f(0, 4) = 4$.

We investigate how f behaves on the boundary of T along the x-axis, i.e. the straight line between $(0, 0)$ and $(4, 0)$. If $g(t) = f(t, 0) = 2t$ we see that we have no interior extremal point. If $g(t) = f(0, t) = t^2 - 3t$, then $g'(t) = 0$ implies $t = 3/2$ and furthermore $g''(t) = 2 > 0$, so that this point is a local minimum. We have $g(3/2) = -9/4$. Finally on the line between $(4, 0)$ and $(0, 4)$ we let

$$\begin{aligned} g(t) &= f(t, 4 - t) \\ &= (4 - t)^2 - t(4 - t) - 3(4 - t) + 2t \\ &= 2t^2 - 7t + 4, \end{aligned} \tag{3}$$

so that $g'(t) = 4t - 7 = 0$ implies $t = 7/4$ and $g''(t) = 4 > 0$ so that $t = 7/4$ is a local minimum. We have $f(7/4) = -17/8$.

We compare all extremal points and conclude that

$$\max_T(f) = 8, \qquad \min_T(f) = -\frac{9}{4}. \tag{4}$$

2 Problem 2

The PDE is

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2y^2 \tag{5}$$

on $\{x, y > 0\}$. We do the change of variables

$$u = xy, \qquad v = \frac{y}{x} \tag{6}$$

in the standard way:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \\ &= \frac{\partial f}{\partial u} y + \frac{\partial f}{\partial v}\Big(- \frac{y}{x^2}\Big) \end{aligned} \tag{7}$$

and

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} \\ &= \frac{\partial f}{\partial u} x + \frac{\partial f}{\partial v}\Big(\frac{1}{x}\Big). \end{aligned} \tag{8}$$

We plug this into the left hand side of Equation 5:

$$\begin{aligned} x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} &= x\Big(\frac{\partial f}{\partial u} y + \frac{\partial f}{\partial v}\Big(- \frac{y}{x^2}\Big)\Big) \\ &\quad + y\Big(\frac{\partial f}{\partial u} x + \frac{\partial f}{\partial v}\Big(\frac{1}{x}\Big)\Big) \\ &= 2xy\frac{\partial f}{\partial u} \\ &= 2u\frac{\partial f}{\partial u}. \end{aligned} \tag{9}$$

The right hand side of Equation 5 is $2y^2 = 2uv$. Hence

$$\frac{\partial f}{\partial u} = v. \tag{10}$$

We integrate this equation with respect to u and get

$$f(u, v) = vu + g(v), \tag{11}$$

where $g \in C^1(\mathbb{R}^+)$, where $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$. Plugging in the initial variables x, y we get

$$f(x, y) = y^2 + g\Big(\frac{y}{x}\Big). \tag{12}$$

It is not difficult to see that f satisfies Equation 5, which on the exam you should check.

3 Problem 3

(a)

We see two ways of solving this. On the one hand we may use the fact that the gradient is ortogonal to the level curves at all points. Hence

$$\begin{aligned} \nabla g &= \Big(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\Big) \\ &= \Big(3x^2 - 2y, -2x - 3y^2\Big), \end{aligned} \tag{13}$$

which at $P = (1, 1)$ evaluates to $\nabla g = (1, -5)$. We see that the vector $v = (5, 1)$ is orthogonal and so the straight line given by

$$\gamma(t) = p + tv \tag{14}$$

solves the problem. To the get this on the form $y = kx + m$, we observe that the vector v gives

$$\begin{aligned} k &= \frac{\Delta y}{\Delta x} \\ &= \frac{1}{5} \end{aligned} \tag{15}$$

and then we solve for m by plugging in P into the straight line:

$$\begin{aligned} m &= y - kx \\ &= 1 - \frac{1}{5} \cdot 1 \\ &= \frac{4}{5}. \end{aligned} \tag{16}$$

Alternatively, we may use the Implicit function theorem. Since again

$$\begin{aligned} \frac{\partial g}{\partial y}\bigg|_P &= (-2x - 3y^2)|_P \\ &= -5 \\ &\neq 0, \end{aligned} \tag{17}$$

the Implicit Function Theorem says that the level curve $g(x, y) = 0$ can be written as a graph of $y(x)$ "near" $(1, 1)$. From the calculation

$$\begin{aligned} 0 &= \frac{d}{dx} g(x, y(x)) \\ &= \frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}\frac{dy(x)}{dx} \\ &= \Big(3x^2 - 2y\Big) + \Big(-2x - 3y^2\Big)\frac{dy(x)}{dx}. \end{aligned} \tag{18}$$

Evaluating at $P(1, 1)$ gives

$$\begin{aligned} \frac{dy}{dx}\bigg|_{(1,1)} &= -\frac{1}{-5} \\ &= \frac{1}{5} \end{aligned} \tag{19}$$

and we would have solved for m the same way as above.

(b)

We comment that this Problem was prone to misinterpretation. We provide solutions to both interpretations.

We to approximate the graph of the level curve $g(x, y) = 0$ by its Taylor polynomial near $P = (1, 1)$. To do this we find the second derivative, which in turn we find by deriving Equation 18 once more:

$$0 = \Big(6x - 2\frac{dy}{dx}\Big) + \Big(-2 - 6y\frac{dy}{dx}\Big)\frac{dy(x)}{dx} + \Big(-2x - 3y^2\Big)\frac{d^2y(x)}{dx^2}, \tag{20}$$

which evaluated at $P = (1, 1)$ gives

$$\begin{aligned} -5\frac{d^2y}{dx^2} &= \Big(6 - \frac{2}{5}\Big) + \Big(-2 - \frac{6}{5}\Big)\frac{1}{5} \\ &= \frac{156}{25} \end{aligned} \tag{21}$$

so that

$$\frac{d^2y}{dx^2} = -\frac{156}{125} \tag{22}$$

at $P = (1,1)$. Hence, the second order curve

$$\begin{aligned} y(x) &= y(1) + y'(1)(x-1) + y''(1)\frac{(x-1)^2}{2} \\ &= 1 + \frac{1}{5}(x-1) - \frac{156}{125}\frac{(x-1)^2}{2} \end{aligned} \tag{23}$$

approximates $g(x,y) = 0$ near P.

We present the alternative (and by the Examiner intended) interpretation, that we want a second order polynomial in both x, y that approximates the level curve $g(x,y) = 0$ near P. We have

$$\begin{aligned} \frac{\partial^2 g}{\partial x^2} &= 6x, \\ \frac{\partial^2 g}{\partial x \partial y} &= -2 \\ \frac{\partial^2 g}{\partial y^2} &= -6y \end{aligned} \tag{24}$$

so that

$$\begin{aligned} g(x,y) &\approx g(1,1) + \left.\frac{\partial g}{\partial x}\right|_{(1,1)}(x-1) + \left.\frac{\partial g}{\partial y}\right|_{(1,1)}(y-1) \\ &\quad + \frac{1}{2}\left(\left.\frac{\partial^2 g}{\partial x^2}\right|_{(1,1)}(x-1)^2 + 2\left.\frac{\partial^2 g}{\partial x \partial y}\right|_{(1,1)}(x-1)(y-1) + \left.\frac{\partial^2 g}{\partial y^2}\right|_{(1,1)}(y-1)^2\right) \\ &= (x-1) - 5(y-1) + 3(x-1)^2 - 2(x-1)(y-1) - 3(y-1)^2 \\ &= 2 - 3x + 3y + 3x^2 - 2xy - 3y^2. \end{aligned} \tag{25}$$

4 Problem 4

(a)

We calculate the rotation $\nabla \times \vec{F}$:

$$\begin{aligned} \nabla \times \vec{F} &= \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \partial_x & \partial_x & \partial_x \\ F_x & F_y & F_z \end{vmatrix} \\ &= \left(e^{yz} + yze^{yz} - (e^{yz} + yze^{yz}), 0 - 0, \frac{-2y}{(1+y^2)^2} - \frac{-2y}{(1+y^2)^2}\right) \\ &= (0,0,0). \end{aligned} \tag{26}$$

Look up e.g. Sarrus' rule for how to calculate the determinant, in case this is not clear.

(b)

We now observe that since the vector field $\vec{F}$ is irrotational and $\mathbb{R}^3$ is simply connected, $\vec{F}$ is **conservative**, i.e. there exits $\varphi(x,y,z)$ such that $\nabla\varphi = \vec{F}$. By inspection we find

$$\varphi = \frac{x}{1+y^2} + e^{yz} + z^2 \tag{27}$$

and since the curve integral of a conservative vector field is independent of the path, we have

$$\begin{aligned} \int_\gamma \vec{F} \cdot d\vec{r} &= \varphi(1,0,2\pi) - \varphi(1,0,0) \\ &= 1 + (2\pi)^2 - 1 \\ &= 4\pi^2. \end{aligned} \tag{28}$$

As an alternative route, we may use Stokes Theorem, which in our case, since $\vec{F}$ was irrotational, says that the curve integral along any piecewise smooth curve must vanish. We let γ_1 denote the curve in question and add the two curves γ_2 and γ_3, where γ_2 passes from $(1,0,2\pi)$ to $(1,0,0)$ and γ_3 passes from $(1,0,0)$ to itself clock-wise seen from the positive z-axis. By Stoke's Theorem we then have

$$\int_{\gamma_1} \vec{F}\cdot d\vec{r} + \int_{\gamma_2} \vec{F}\cdot d\vec{r} + \int_{\gamma_3} \vec{F}\cdot d\vec{r} = 0. \tag{29}$$

To calculate the γ_2-integral, we parametrize $\gamma_2(t) = (1,0,2\pi - t)$ so that $\dot{\gamma_2}(t) = (0,0,-1)$ and $t \in (0,2\pi)$. We get

$$\begin{aligned}\int_{\gamma_2} \vec{F}\cdot d\vec{r} &= \int_0^{2\pi} \vec{F}(\gamma_2)\cdot \dot{\gamma_2}(t)dt \\ &= \int_0^{2\pi} -2(2\pi - t)dt \\ &= \dots \\ &= -4\pi^2.\end{aligned} \tag{30}$$

To do the γ_3-integral, we parametrize $\gamma_3(t) = (\cos(t), -\sin(t), 0)$ with $t \in (0,2\pi)$ so that $\dot{\gamma_3}(t) = (-\sin(t), -\cos(t), 0)$. We get

$$\begin{aligned}\int_{\gamma_3} \vec{F}\cdot d\vec{r} &= \int_0^{2\pi} \vec{F}(\gamma_3)\cdot \dot{\gamma_3} dt \\ &= \int_0^{2\pi} \left(\frac{1}{1+\sin^2(t)}, -2\frac{\cos(t)(-\sin(t))}{(1+\sin^2(t))}, 0\right)\cdot(-\sin(t), -\cos(t), 0)dt \\ &= \int_0^{2\pi} (-\sin(t))\frac{(1+\sin^2(t)+2\cos^2(t))}{(1+\sin^2(t))^2}dt \\ &= 0.\end{aligned} \tag{31}$$

That the last step follows, i.e. that the integral vanishes, follows as the trigonometric functions are 2π-periodic, so that the integral equals the same integral over the interval $[-\pi,\pi]$ and then the $\sin(t)$-term is odd and the terms in the quotient is even over this interval.

We have shown that

$$\int_{\gamma_1} \vec{F}\cdot d\vec{r} - 4\pi^2 + 0 = 0. \tag{32}$$

So try to find a scalar potential of you can!

5 Problem 5

We apply the Divergence Theorem to solve for the flow. This is allowed since the vector field has smooth components and the boundary ∂T consists of piecewise smooth surfaces. It is straightforward to calculate the divergence of $\vec{F}$:

$$\begin{aligned}\mathrm{div}\vec{F} &= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_y}{\partial y} \\ &= z^2 - 1 + 1 \\ &= z^2.\end{aligned} \tag{33}$$

From the Divergence Theorem and iterated integration we get

$$\begin{aligned}
\oiint_{\partial T} \vec{F} \cdot d\vec{S} &= \iiint_T \mathrm{div}\vec{F} dV \\
&= \iiint z^2 dxdydz \\
&= \int_{z=0}^{1} \left(\iint_{A_z} z^2 dxdy \right) dz \\
&= \int_{z=0}^{1} z^2 \mathrm{Vol}(A_z) dz
\end{aligned} \tag{34}$$

where A_z is the half rectangle at height z. It is not difficult to see from the equation of the plane (e.g. fix $y = 0$ so that $x = 1 - z$ on the equilateral triangle seen from above) that

$$\mathrm{Vol}(A_z) = \frac{(1-z)^2}{2}. \tag{35}$$

Continuing the iterated integration yields

$$\begin{aligned}
\int_{z=0}^{1} z^2 \mathrm{Vol}(A_z) dz &= \int_{z=0}^{1} z^2 \frac{(1-z)^2}{2} dz \\
&= \dots \\
&= \frac{1}{60}.
\end{aligned} \tag{36}$$

6 Problem 6

We let I denote the integral and

$$K = \{x^2 + y^2 + (z-2)^2 \leq 4\} \cap \{z \geq 2\}, \tag{37}$$

which is the half-sphere centered at $(0, 0, 2)$ with radius 2. We start by observing that ("easy integrals"!) we have both

$$\iiint_K x dV = 0 \tag{38}$$

and similarly for y by the symmetry and

$$\begin{aligned}
\iint_K 5 dV &= 5\mathrm{Vol}(K) \\
&= 5 \frac{4\pi 2^3}{2 \cdot 3} \\
&= \frac{4\pi}{3} \cdot 20.
\end{aligned} \tag{39}$$

The z-integral is calculated with iterated integration:

$$\begin{aligned}\iiint_K z dV &= \int_{z=2}^{4} \left(\iint_{C_z} z dx dy \right) dz \\ &= \int_{z=2}^{4} z \mathrm{Vol}(C_z) dz \\ &= \int_{2}^{4} z\pi \left(4 - (z-2)^2 \right) dz \\ &= \pi \left[\frac{4}{3} z^3 - \frac{z^4}{4} \right]_2^4 \\ &= \pi \left(\frac{4}{3} 4^3 - \frac{4^4}{4} \right) - \pi \left(\frac{4}{3} 2^3 - \frac{2^4}{4} \right) \\ &= \dots \\ &= \frac{4\pi}{3} \cdot 11. \end{aligned} \tag{40}$$

Hence

$$\begin{aligned} I &= \frac{4\pi}{3} \cdot 11 + \frac{4\pi}{3} \cdot 20 \\ &= \frac{4\pi}{3} \cdot 31. \end{aligned} \tag{41}$$

Alternatively, we can calculate the z-integral with polar coordinates. Firstly, we shift the z-variable:

$$\begin{aligned} x &\to u, \\ y &\to v, \\ z - 2 &\to w, \end{aligned} \tag{42}$$

so that

$$K = \{u^2 + v^2 + w^2 \leq 4\} \cap \{w \geq 2\}. \tag{43}$$

We claim that this change of variables does not change the volume measure:

$$\iiint_K z\, dx dy dz = \iiint_K (w+2) du dv dw, \tag{44}$$

which you should check to be true (Just look at the Jacobian). The 2-integral is just the volume of the half-sphere:

$$\begin{aligned} \iiint_K 2 du dv dw &= \frac{2}{2} \frac{4\pi 2^3}{3} \\ &= \frac{4\pi}{3} \cdot 8. \end{aligned} \tag{45}$$

The w-integral is calculated with spherical polar coordinates:

$$\begin{aligned} u &= r\cos(\varphi)\cos(\theta), \\ v &= r\sin(\varphi)\cos(\theta), \\ w &= r\sin(\theta). \end{aligned} \tag{46}$$

It is not difficult to see that

$$\frac{\partial(u, v, w)}{\partial(r, \varphi, \theta)} = r^2 \cos(\theta) \tag{47}$$

and you should do the calculation so as to agree. We note that here $\theta \in (-\pi/2, \pi/2)$ (which may disagree with the convention sometimes used that θ runs from 0 to π and you should work out that this parametrization makes no difference) and on K $\theta \in (0, \pi/2)$. It follows that

$$\begin{aligned}
\iiint_K z\, dxdydz &= \iiint_K w\, dudvdw + \frac{4\pi}{3}8 \\
&= \int_{r=0}^{2} \int_{\theta=0}^{\pi/2} \int_{\varphi=0}^{2\pi} \big(r\sin(\theta)\big) r^2 \cos(\theta) dr d\theta d\varphi + \frac{4\pi}{3}8 \\
&= \int_{r=0}^{2} r^3 dr \cdot \int_{\theta=-\pi/2}^{\pi/2} \sin(\theta)\cos(\theta) d\theta \cdot \int_{\varphi=0}^{2\pi} d\varphi + \frac{4\pi}{3}8 \\
&= \frac{2^4}{4} \cdot \frac{1}{2} \cdot 2\pi + \frac{4\pi}{3}8 \\
&= \frac{4\pi}{3} \cdot 11,
\end{aligned} \tag{48}$$

since the θ-integral can be computed with, for instance, some smart change of variables, or some boring trigonometric identity. The rest follows.

7 Problem 7

We have the vector field

$$\vec{F} = \left(e^{\sin(x)} - x^2 y, e^{y^2}\right) \tag{49}$$

and so we may apply Green's formula:

$$\begin{aligned}
\oint_\gamma \vec{F} \cdot d\vec{r} &= \iint_D \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dxdy \\
&= \iint_D \left(0 - (-x^2)\right) dxdy \\
&= \frac{1}{2} \iint_D (x^2 + y^2) dxdy \\
&= \frac{1}{2} \int_{r=0}^{1} \int_{\varphi=0}^{2\pi} r^2 r dr d\varphi \\
&= \frac{1}{2} 2\pi \int_0^1 r^3 dr \\
&= \pi \left[\frac{r^4}{4}\right]_0^1 \\
&= \frac{\pi}{4}.
\end{aligned} \tag{50}$$

We used the symmetry of x and y over the circle in the third equality and then changed to polar coordinates.

8 Problem 8

The differential form is on the form $M(x, y)dx + N(x, y)dy = 0$, and we check that it is exact:

$$\begin{aligned}
\frac{\partial M}{\partial y} &= 3x\cos(y) - 2y\sin(x), \\
\frac{\partial N}{\partial x} &= 3x\cos(y) - 2y\sin(x)
\end{aligned} \tag{51}$$

so that the form is exact. We solve for a scalar potential φ and we see directly that

$$\varphi(x, y) = x^2\sin(y) + y^2\sin(x) + y \tag{52}$$

(modulo constants) solves for the form. The solutions are the equations $\varphi = C$, for any real-valued constant C.

Since

$$\varphi(0,\pi) = \pi^2 + \pi \tag{53}$$

this solves the equation passing through $(0,\pi)$.

Referenser

[1] *Calculus: A Complete Course,* 10^{th} *Edition*, R Adams, C. Essex.