4/kv

Introduction to Computer Control Systems, 5 credits, 1RT485

Date: 2022-08-16

Teacher on duty: Dave Zachariah

Allowed aid:

- A basic calculator
- BETA mathematical handbook

Solutions have to be explained in detail and possible to reconstruct.

NB: Only one problem per sheet. Write your name and personal number if you do not have an anonymous code.

Best of luck!

Useful results

Laplace transform table

Table 1: Basic Laplace transforms

$f(t)$	$F(s)$	$f(t)$	$F(s)$
unit impulse $\delta(t)$	1	$\sinh(bt)$	$\frac{b}{s^2-b^2}$
unit step $1(t)$	$\frac{1}{s}$	$\cosh(bt)$	$\frac{s}{s^2-b^2}$
t	$\frac{1}{s^2}$	$\frac{1}{2b}t\sin(bt)$	$\frac{s}{(s^2+b^2)^2}$
t^n	$\frac{n!}{s^{n+1}}$	$t\cos(bt)$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
e^{-at}	$\frac{1}{s+a}$	$\frac{\cos(bt)-\cos(at)}{a^2-b^2}; (a^2 \neq b^2)$	$\frac{s}{(s^2+a^2)(s^2+b^2)}$
$\frac{1}{a}(1-e^{-at})$	$\frac{1}{s(s+a)}$	$\frac{\sin(at)+at\cos(at)}{2a}$	$\frac{s^2}{(s^2+a^2)^2}$
$\frac{1}{(n-1)!}t^{n-1}e^{-at}; (n=1,2,3...)$	$\frac{1}{(s+a)^n}$		
$\sin(bt)$	$\frac{b}{s^2+b^2}$		
$\cos(bt)$	$\frac{s}{s^2+b^2}$		
$e^{-at}\sin(bt)$	$\frac{b}{(s+a)^2+b^2}$		
$e^{-at}\cos(bt)$	$\frac{s+a}{(s+a)^2+b^2}$		

Table 2: Properties of Laplace Transforms

$\mathcal{L}[af(t)] = aF(s)$	$\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}$
$\mathcal{L}[f_1(t)+f_2(t)] = F_1(s)+F_2(s)$	$\mathcal{L}[t^2f(t)] = \frac{d^2}{ds^2}F(s)$
$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = sF(s)-f(0)$	$\mathcal{L}[t^nf(t)] = (-1)^n\frac{d^n}{ds^n}F(s), \quad n=1,2,3,\ldots$
$\mathcal{L}\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s)-sf(0)-f'(0)$	$\mathcal{L}\left[f\left(\frac{t}{a}\right)\right] = aF(as)$
$\mathcal{L}\left[\int f(t)\,dt\right] = \frac{F(s)}{s}+\frac{1}{s}\left[\int f(t)\,dt\right]_{t=0}$	$\mathcal{L}\left[\int_0^t f_1(t-\tau)f(t)d\tau\right] = F_1(s)F_2(s)$
$\mathcal{L}[f(t-a)] = e^{-as}F(s)$	$\mathcal{L}[e^{-at}f(t)] = F(s+a)$

Matrix exponential

$$\boxed{e^{At} \triangleq \mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\}}$$

Open-loop and sensitivity functions

$$G_o(s) = G(s)F_y(s), \qquad S(s) = \frac{1}{1+G_o(s)}, \qquad T(s) = 1-S(s)$$

State-space forms and transfer function relations

- State-space form and transfer function

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad \Rightarrow \quad \boxed{G(s) = C(sI - A)^{-1}B + D}$$

- Associated matrices

$$\mathcal{S} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \qquad \mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

- LTI system with transfer function

$$\boxed{G(s) = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}}$$

i) Observable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ -a_3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ b_3 - a_3 b_0 \\ \vdots \\ b_n - a_n b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

ii) Controllable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_1 - a_1 b_0 & b_2 - a_2 b_0 & \cdots & b_n - a_n b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

- Solution to state-space equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

can be written as

$$\boxed{x(t) = e^{At}x_0 + \int_0^t e^{A\tau} Bu(t - \tau) d\tau}$$

iii) Consider a system with transfer function

$$G(s) = \frac{s+1}{(s+2)(s+5)}.$$

Suppose the same system is described in state-space that is a minimal realization. Then the eigenvalues of A equal -2 and -5.

(3 p)

Problem 2 (6/30)

a) Consider a mechanical system with input $u(t)$ and output $y(t)$ that can be written as

$$Y(s) = G(s)U(s) = \frac{1}{s+a}U(s).$$

We apply feedback controller to output to follow a reference signal $r(t)$. Specifically a P-controller $F(s) = K$ is used with K set to 8.

We now are interested to know for which pole locations of $G(s)$ is the closed-loop system stable: For which a will $G_c(s)$ be stable?

(3 p)

b) Consider the following configuration of the system above

$$Y(s) = G(s)U(s) = \frac{1}{s+2}U(s),$$

which we control using a feedback controller $F(s) = 1 + \frac{K_i}{s}$.
Let the reference signal $r(t)$ be a step. That is,

$$r(t) = \begin{cases} r_0, & t \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Set $K_i = 10$ and calculate the stationary error of the controller: $e(t) = r(t) - y(t)$ as $t \to \infty$.

(3 p)

Problem 3 (6/30)

a) Consider a controlling a thermal process with an output modeled as

$$Y(s) = G(s)U(s),$$

where the process model can be written as

$$G(s) = \frac{s+3}{s^2+6s+11}.$$

Assuming the states can be obtained, design a stable closed-loop system from r to y with poles located at -2 and -5.

(4 p)

b) To prepare for the design of a state observer,

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x}),$$

write the system above on observable canonical form.

(2 p)

Problem 4 (6/30)

(a) When working with a control problem, Mr. Stu Dent generated the four Bode diagrams in Figure 1.

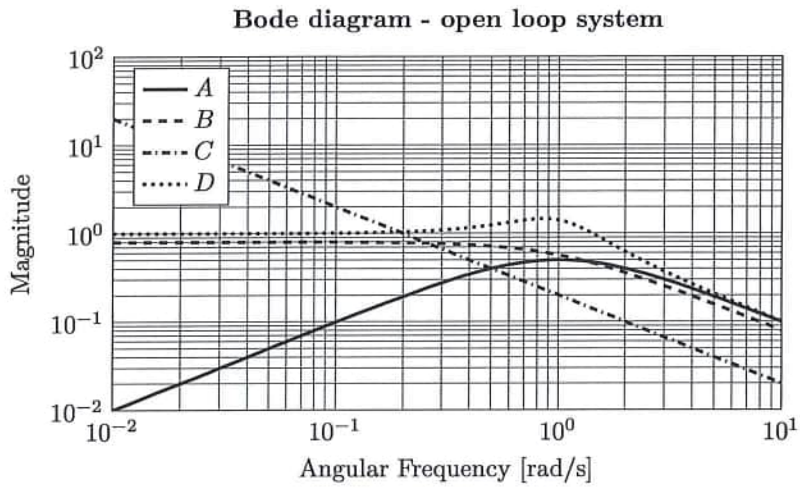

Figure 1: Bode diagrams of the open loop systems.

He then proceeded to generate step responses for both the open loop systems in Figure 2 and the closed loop systems in Figure 3. The closed loop systems used standard feedback with $F(s) = 1$. By accident, Stu mix up all the systems and used different labels in the legends. Help Stu match the Bode diagrams with the step responses.

(I) Pair the Bode diagrams A-D with the open loop step responses 1-4 in Figure 2.

(2 p)

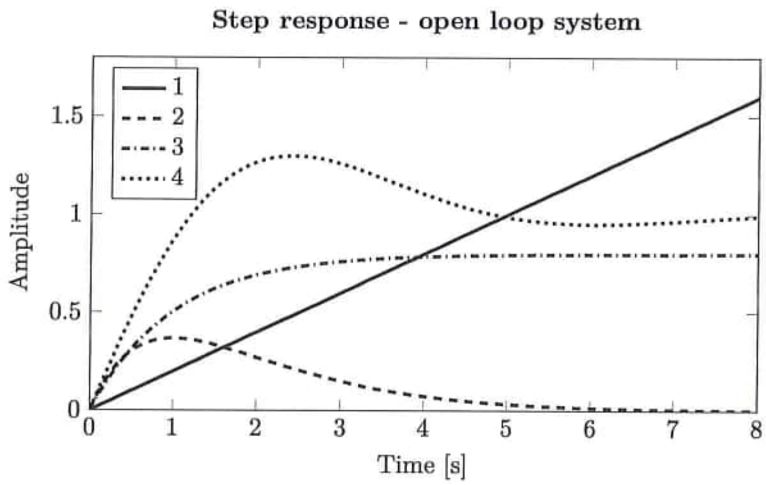

Figure 2: Step responses for the open loop systems.

(II) Pair the Bode diagrams A-D with the closed loop step responses I-IV in Figure 3.

(2 p)

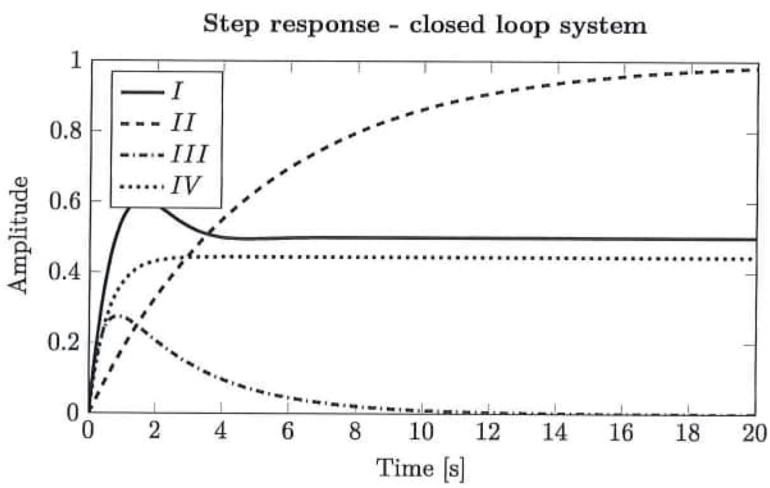

Figure 3: Step responses for the closed loop systems using standard feedback with $F(s) = 1$.

(b) The Bode diagram for an industrial process is shown in Figure 4.

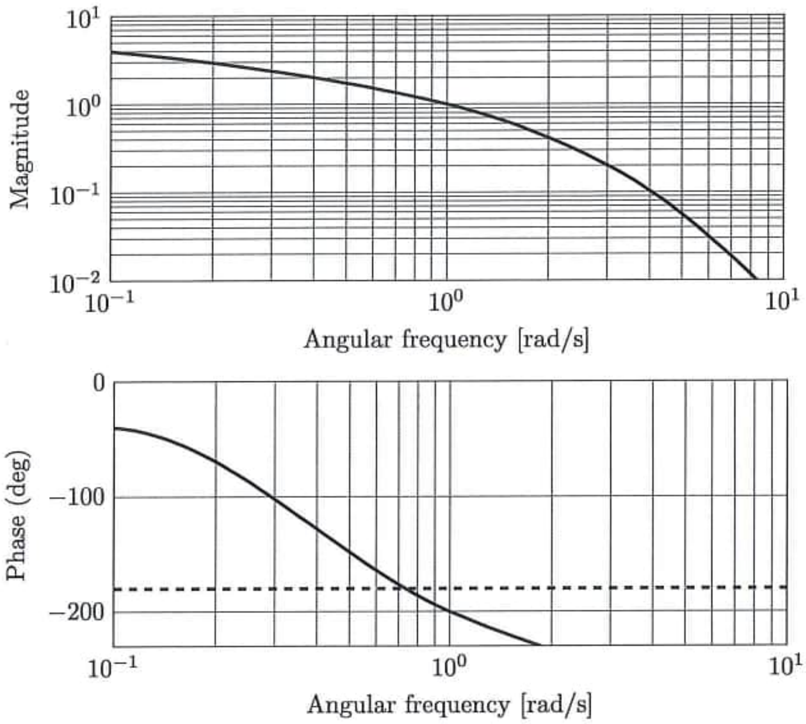

Figure 4: Bode diagram for an industrial process.

(I) What is the crossover frequency ω_c and the phase margin φ_m for the process?

(1 p)

(II) If a pure time delay of $T = 0.8$ s is added to the output of the process, that is we measure $\tilde{y}(t) = y(t - T)$ instead of $y(t)$, will the resulting closed loop system be stable when using standard feedback with $F(s) = 1$?

(1 p)

Problem 5 (6/30)

A certain physical model with the state vector

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

has the following state-space representation

$$\dot{x} = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

a) Discretise the state-space system (using zero order hold) with sampling time T.

(2 p)

b) For what values of T is the discretised system observable?

(2 p)

c) Pick $T = \frac{\pi}{2a}$ and construct an observer

$$K = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

such that the *discrete time* observer poles are located in ± 0.5.

(2 p)