Exam in Automatic Control II
Reglerteknik II 5hp (1RT495)

Date: August 16, 2021
Time: 8:00 – 13:00

Responsible teacher: Hans Rosth
Aiding material: Calculator, mathematical handbooks, textbooks by Glad & Ljung (Reglerteori/Control theory & Reglerteknik). Additional notes in the textbooks are allowed.

Preliminary grades: 23p for grade 3, 33p for grade 4, 43p for grade 5.

Your solutions should be submitted as a pdf file — scan your solutions with best possible quality!

Important: Your solutions should be well motivated unless else is stated in the problem formulation! Vague or lacking motivations may lead to a reduced number of points.

Use Swedish or English in your solutions.

Problem 6 is an alternative to the homework assignments from the spring semester 2021. (In case you choose to hand in a solution to Problem 6 you will be accounted for the best performance of the homework assignments and Problem 6.)

Good luck!

Problem 1 Consider the continuous-time stochastic process below:

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & -2 \\ 3 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v_1(t), & Ev_1(t) = 0, \quad \Phi_1(\omega) = 15, \\ y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) + v_2(t), & Ev_2(t) = 0, \quad \Phi_2(\omega) = 1. \end{cases} \tag{1}$$

The noise processes v_1 and v_2 are uncorrelated.

(a) Determine the covariance matrix of the state vector in (1). **(2p)**

(b) What is the spectral density of y in (1)? **(3p)**

(c) Determine the Kalman filter for (1). **(4p)**

(d) What is the spectral density of the output innovations, $\nu = y - C\hat{x}$, for the Kalman filter in (c)? **(2p)**

Problem 2 When discretizing a continuous-time system with zero-order-hold (ZOH) sampling, as in Theorem 4.1, we have the following mapping:

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx, \end{cases} \quad \overset{\text{ZOH}}{\mapsto} \quad \begin{cases} qx = F_h x + G_h u, \\ y = Cx, \end{cases} \tag{2}$$

where h denotes the sampling interval, indicating the dependence thereof.

(a) Assume that the continuous-time system in (2) is *stable*. What can be said about the stability properties of the ZOH sampled model in (2)? A precise answer and a thorough motivation is required. **(2p)**

(b) Assume that the continuous-time system in (2) is a *minimal realization*. Then, is the ZOH sampled model in (2) also a minimal realization? A precise answer and a thorough motivation is required. **(2p)**

(c) In the demo lab a double tank system was controlled by use of MPC. A linearized model of the double tank is

$$Y(s) = \frac{1}{(s+1)^2} U(s). \tag{3}$$

Give the transfer operator for the ZOH sampled model, $y(k) = G_{ZOH}(q)u(k)$, of (3). Your answer should be expressed in the sampling interval h.[1] **(4p)**

[1]... and in the shift operator q.

Problem 3 The difference equation

$$y(k) = ay(k-1) + u(k-2) - w(k-1), \qquad 0 < a < 1, \tag{4}$$

represents a "leaky" inventory model. Here $y(k)$ is the level of the inventory, $u(k)$ is the ordered amount and $w(k)$ is the amount sold, all at day k. The positive number a is slightly less than one, and indicates that the fraction $1-a$ of the inventory level disappears from one day to the next.

The inventory model can also be represented by the state space model

$$\begin{cases} x(k+1) = \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} x(k) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ -a \end{bmatrix} w(k), \\ y(k) = \begin{bmatrix} \frac{1}{a} & \frac{1}{a} \end{bmatrix} x(k). \end{cases} \tag{5}$$

(a) Show that the state space model (5) indeed is equivalent to the difference equation (4). **(3p)**

(b) Assume that the inventory is controlled by proportional control, that is, by use of the control law $u(k) = K(r(k) - y(k))$. For which gains $K \in \mathbb{R}$ is the closed loop system stable? **(3p)**

(c) Consider the system (5) with $w(k)$ as input and $y(k)$ as output (that is, disregard from $u(k)$). Is that system a *minimal realization*? **(2p)**

Problem 4 In this problem we will (again) consider the state space model (5) in Problem 3. The system (inventory) should be controlled by state feedback control,

$$u(k) = -Lx(k) + L_r r(k), \qquad \text{with} \quad L = \begin{bmatrix} l_1 & l_2 \end{bmatrix}, \tag{6}$$

such that the criterion

$$V = E\left[a^2 y(k)^2 + \rho u(k)^2\right], \qquad \rho \geq 0,$$

is minimized. We assume here that $w(k)$ is zero mean white noise.

(a) How should the gain $L_r \in \mathbb{R}$ in (6) be chosen in order to have *unit static gain* from r to y? Your answer should be expressed in l_1 and l_2. **(2p)**

(b) Show that the solution of the associated Riccati equation can be written as

$$S = \sigma \begin{bmatrix} 0 & 0 \\ 0 & a^2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

for some scalar σ (that depends on ρ). (You need not solve for σ.)[2] **(4p)**

(c) Given S in (b), what are the poles of the closed loop system? Specifically, what are the closed loop poles *(i)* when $\rho \to 0$, and *(ii)* when $\rho \to \infty$? **(3p)**

(d) Explain/motivate why the results in (c) are reasonable, given the role of ρ in the criterion V above. **(2p)**

[2] It is possible to show that $1 \leq \sigma < \frac{1}{1-a^2}$ for all $\rho \geq 0$.

Problem 5 Specify for each of the following statements whether it is true or false. No motivations required — only answers "true" / "false" are considered!

(a) In MPC the *control horizon* is normally much longer than the *prediction horizon.*

(b) In MPC the computational load increases with the *control horizon.*

(c) The Nyquist frequency is always 50% of the sampling frequency.

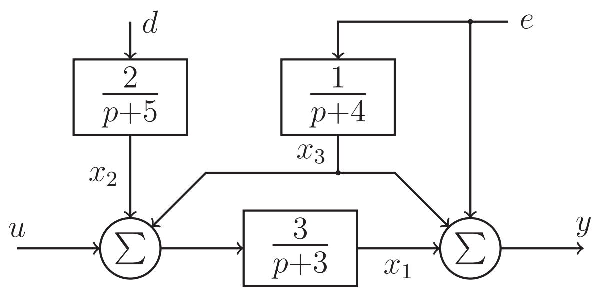
(d) For white noise the spectral density is $\Phi(\omega) = 0$ for $\omega \neq 0$.

(e) For white noise the covariance function is $r(\tau) = 0$ for $\tau \neq 0$.

Each correct answer scores +1, each incorrect answer scores −1, and omitted answers score 0 points. (Minimal total score is 0 points.) **(5p)**

Problem 6 *The HW bonus points (from the spring 2021) are exchangeable for this problem.*

(a) The block diagram below shows a continuous-time system.



Here d and e are uncorrelated white noise processes, with intensities R_d and R_e respectively. Give a state space representation of the system in the "standard form", that is, find the matrices and vectors A, B, N and C in

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Nv_1(t), & Ev_1(t) = 0, \quad \Phi_1(\omega) = R_1, \\ y(t) = Cx(t) + v_2(t), & Ev_2(t) = 0, \quad \Phi_2(\omega) = R_2. \end{cases}$$

Use the state vector $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$. Also, specify v_1 and v_2, and give the noise intensities R_1 and R_2, as well as the cross-intensity R_{12}. **(4p)**

(b) The spectral density of the (short term) wind fluctuations was measured at a potential future wind farm site. It was found that

$$\Phi_w(\omega) = \frac{9\omega^2 + 81}{\omega^4 + 29\omega^2 + 100} \tag{7}$$

was a good approximation of the spectral density. Find a stable, minimum phase transfer operator $G_w(p)$ such that the stochastic process

$$w(t) = G_w(p)v(t), \qquad Ev(t) = 0, \quad \Phi_v(\omega) = 1,$$

as a model of the fluctuations, has the spectral density $\Phi_w(\omega)$ in (7). **(3p)**