Envariabel. Del 2

$\boxed{2015-05-28}$

Lösningar.

① a) $$\int_0^1 x e^{2x}\,dx = \left(\begin{matrix}\text{Partiell}\\ \text{Integration}\end{matrix}\right) = \Big/_0^1 x\cdot\frac{e^{2x}}{2} - \frac{1}{2}\int_0^1 e^{2x}\,dx$$

$$= \frac{e^2}{2} - \frac{1}{2}\Big/_0^1 \frac{e^{2x}}{2} = \frac{e^2}{2} - \frac{1}{2}\left(\frac{e^2}{2} - \frac{e^0}{2}\right) =$$

$$= \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} = \boxed{\frac{e^2+1}{4}}$$

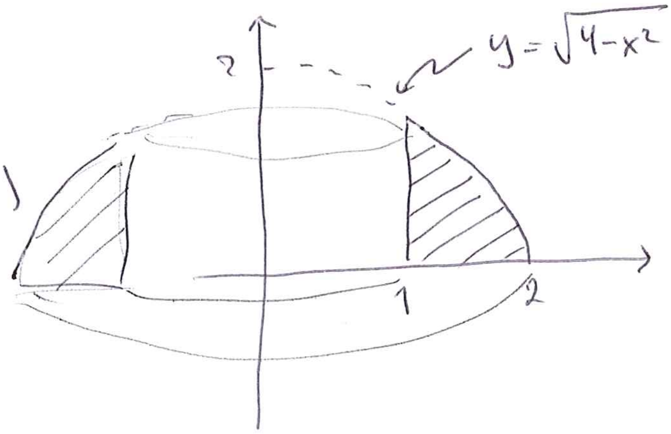
b) $$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = \left\{\begin{matrix}\text{SUBSTITUTION} & \\ \sqrt{x} = t & \frac{dx}{\sqrt{x}} = 2\,dt \\ \frac{dt}{dx} = \frac{1}{2\sqrt{x}} & \end{matrix}\right\} =$$

$$= \int \cos t \cdot 2\,dt = 2\sin t + C = \boxed{2\sin\sqrt{x} + C}$$

② $y = \sqrt{4-x^2}$

$y^2 = 4-x^2$

$x^2+y^2 = 2^2$ (Cirkel)



Rör formeln:

$$V = 2\pi\int_1^2 x\sqrt{4-x^2}\,dx$$

Så $V = 2\pi \int_1^2 x\sqrt{4-x^2}\,dx = \left\{ \begin{array}{ll} \text{SUBST.} & x\,dx = -\frac{1}{2}dt \\ 4-x^2 = t & x=1 \Rightarrow t=3 \\ -2x\,dx = dt & x=2 \Rightarrow t=0 \end{array} \right\} =$

$$= 2\pi \int_3^0 \sqrt{t}\left(-\frac{1}{2}\right)dt = \pi \int_0^3 \sqrt{t}\,dt = \pi \int_0^3 t^{1/2}\,dt =$$

$$= \pi \left[\frac{t^{3/2}}{3/2} \right]_0^3 = \frac{2\pi}{3} \cdot 3^{3/2} = \frac{2\pi}{3} \cdot 3\sqrt{3} = \boxed{2\pi\sqrt{3}}$$

③ Partialbråksuppdela:

$$\frac{1}{x^3-x} = \frac{1}{x(x^2-1)} = \frac{1}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1} =$$

$$= \frac{A(x+1)(x-1) + Bx(x-1) + Cx(x+1)}{x(x+1)(x-1)} =$$

$$= \frac{\boxed{(A+B+C)x^2 + (C-B)x + (-A)}}{x(x+1)(x-1)}$$

Identifiera: $\begin{cases} A+B+C = 0 \\ C-B = 0 \\ -A = 1 \end{cases} \Rightarrow \begin{array}{l} B+C = 1 \\ C-B = 0 \end{array} \quad \begin{array}{l} C = 1/2 \\ B = 1/2 \end{array}$

Så integralen blir:

$$\int\left(\frac{-1}{x}+\frac{1/2}{x+1}+\frac{1/2}{x-1}\right)dx=$$

$$=-\ln|x|+\frac{1}{2}\ln|x+1|+\frac{1}{2}\ln|x-1|+C=$$

$$=\frac{1}{2}\ln\underbrace{|x+1|\cdot|x-1|}_{|x^2-1|}-\ln|x|+C=\ln\sqrt{|x^2-1|}-\ln|x|+C=$$

$$=\boxed{\ln\frac{\sqrt{|1-x^2|}}{|x|}+C}$$

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④ a) $\displaystyle\int_e^{X}\frac{dx}{x(\ln x)^2}=\left\{\begin{array}{ll}\text{Sätt } t=\ln x & x=e\Rightarrow t=1\\ \frac{dt}{dx}=\frac{1}{x}\quad \frac{dx}{x}=dt & x=X\Rightarrow t=\ln X\end{array}\right\}$

$$=\int_1^{\ln X}\frac{1}{t^2}\,dt=\Big[-\frac{1}{t}\Big]_{t=1}^{t=\ln X}=1-\frac{1}{\ln X}\xrightarrow[X\to\infty]{}1$$

<u>Svar:</u> Integralens värde är 1.

b) Integralen är generaliserad i $x=0$

(eftersom $\ln 0$ inte existerar)

$$\int_{\varepsilon}^{1} \ln x \, dx = \left(\begin{matrix}\text{partiell}\\ \text{int.}\end{matrix}\right) = \Big/ x \cdot \ln x \Big/_{\varepsilon}^{1} - \int_{\varepsilon}^{1} x \cdot \frac{1}{x} \, dx =$$

$$= (\underbrace{1 \cdot \ln 1}_{=0} - \varepsilon \cdot \ln \varepsilon) - \int_{\varepsilon}^{1} 1 \, dx =$$

$$= -\varepsilon \cdot \ln \varepsilon - \Big/ x \Big/_{\varepsilon}^{1} = -\varepsilon \cdot \ln \varepsilon - (1 - \varepsilon)$$

Men $\lim\limits_{\varepsilon \to 0^+} \varepsilon \cdot \ln \varepsilon = \left\{\begin{matrix} \varepsilon = 1/t \\ \varepsilon \to 0^+ \Rightarrow t \to \infty \end{matrix}\right\} =$

$$= \lim_{t \to \infty} \frac{1}{t} \ln \frac{1}{t} = \lim_{t \to \infty} \frac{-\ln t}{t} = \underline{\underline{0}}.$$

Och $-1 + \varepsilon \to -1$

Så $\lim\limits_{\varepsilon \to 0^+} (-\varepsilon \cdot \ln \varepsilon - (1 - \varepsilon)) = -1$

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(5) Ekvationen är <u>separabel</u>:

$$y^2 \frac{dy}{dx} = \frac{1}{x^2 + 2x + 5} \quad \text{eller} \quad y^2 dy = \frac{dx}{x^2 + 2x + 5}$$

som ger: $$\int y^2 \, dy = \int \frac{dx}{x^2 + 2x + 5}$$

Integreras:

$$\int y^2\,dy = y^3/3 + C$$

$$\int \frac{dx}{x^2+2x+5} = \int \frac{dx}{(x^2+2x+1)+4} = \int \frac{dx}{(x+1)^2+2^2} =$$

$$= \frac{1}{2^2}\int \frac{dx}{\left(\frac{x+1}{2}\right)^2+1} = \left\{ \begin{array}{c} \frac{x+1}{2} = t \\ dx = 2\,dt \end{array} \right\} =$$

$$= \frac{1}{2^2}\int \frac{2\,dt}{t^2+1} = \frac{1}{2^2}\cdot 2\arctan t + C = \frac{1}{2}\arctan\left(\frac{x+1}{2}\right) + C$$

Så lösningen ges implicit av

$$\boxed{\frac{y^3}{3} = \frac{1}{2}\arctan\left(\frac{x+1}{2}\right) + C}$$

$y(-1) = 1$ ger $\frac{1}{3} = \frac{1}{2}\arctan 0 + C \Rightarrow \underline{\underline{C = 1/3}}$

Svar: $\boxed{y = \sqrt[3]{\frac{3}{2}\arctan\frac{x+1}{2} + 1}}$

(6) $\begin{cases} y'' + 2y' + 5y = e^{-2x} \\ y(0) = 0, \quad y'(0) = 1 \end{cases}$

a) Homogen lösning.

Karakteristisk ekvation: $r^2 + 2r + 5 = 0$

$r = -1 \pm \sqrt{1-5}$

$\boxed{r = -1 \pm 2i}$

$$\boxed{y_H = e^{-x}(A\cos 2x + B\sin 2x)}$$

b) Ansatz: $y_p = Ae^{-2x}$

$y_p' = -2Ae^{-2x}$

$y_p'' = 4Ae^{-2x}$

Så $y_p'' + 2y_p' + 5y_p = 4Ae^{-2x} + 2(-2Ae^{-2x}) + 5Ae^{-2x} =$

$= 5Ae^{-2x}$ och detta blir precis e^{-2x}

om $\underline{A = 1/5}$.

Så allmän lösning är:

$$\boxed{y = e^{-x}(A\cos 2x + B\sin 2x) + \frac{1}{5}e^{-2x}} \quad (1)$$

Derivera:

$$y' = -e^{-x}(A\cos 2x + B\sin 2x) + e^{-x}(-2A\sin 2x + 2B\cos 2x) - \frac{2}{5}e^{-2x} \quad (1)$$

Sätt in $y(0)=0$, $y'(0)=1$: (1) och (2)

$$\begin{cases} A + \frac{1}{5} = 0 \qquad\qquad A = -\frac{1}{5} \\ -A + 2B - \frac{2}{5} = 1 \end{cases}$$

$$2B = 1 - \frac{1}{5} + \frac{2}{5} = \frac{6}{5}; \quad B = \frac{3}{5}$$

Svar: $\boxed{y = e^{-x}\left(\frac{3}{5}\sin 2x - \frac{1}{5}\cos 2x\right) + \frac{1}{5}e^{-2x}}$

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(7) a) Vi undersöker termernas storlek:

$$a_n = \frac{n+1}{\underbrace{2n^2\sqrt{n}}_{n^{2+\frac{1}{2}} = n^{5/2}} - n} = \frac{n(1 + 1/n)}{n^{5/2}\left(2 - 1/n^{3/2}\right)} = \frac{\left(1 + \frac{1}{n}\right)}{n^{3/2}\left(2 - \frac{1}{n^{3/2}}\right)}$$

jämför (kvotform) med $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

Vi har:

$$\frac{a_n}{1/n^{3/2}} = \frac{\frac{(1+\frac{1}{n})}{n^{3/2}(2-1/n^{3/2})}}{\frac{1}{n^{3/2}}} = \frac{(1+\frac{1}{n})}{(2-\frac{1}{n^{3/2}})} \xrightarrow[n\to\infty]{} \underline{\underline{\frac{1}{2}}}$$

och $\sum \frac{1}{n^{3/2}}$ är konvergent (p serie med $p = 3/2$)

Så den givna serien är också konvergent.

b) Har är $a_n = \frac{n^{10}}{2^n}$ och vi använder

$\boxed{\text{KVOTKRITERIET}}$ (som ej är jämförelsesatsen i kvotform).

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^{10}}{2^{n+1}}}{\frac{n^{10}}{2^n}} = \frac{1}{2} \cdot \frac{(n+1)^{10}}{n^{10}} = \frac{1}{2}\left(\frac{n+1}{n}\right)^{10} =$$

$$= \frac{1}{2}\left(1+\frac{1}{n}\right)^{10} \to \frac{1}{2} \cdot 1 = \frac{1}{2} \text{ då } n \to \infty$$

Så eftersom $\frac{a_{n+1}}{a_n}$ går mot ett tal som är < 1

så är serien konvergent även i detta fall.

Problem 8

Löst i F 28. Del 1