Intermediate Exam B 2020-12-10

1 a) The sampling frequency is chosen too low. The spectrum of the signal $x(t)$ shows that there are significant contributions up to at least $\omega \approx 75$ rad/s. Hence, the choice $\omega_s = 100\,\frac{rad}{s}$ violates the Nyquist-Shannon sampling theorem, which would require that $\omega_s > 2\omega_b$

$\Rightarrow \omega_s > 2 \cdot 75\,\text{rad/s} = 150\,\text{rad/s}$.

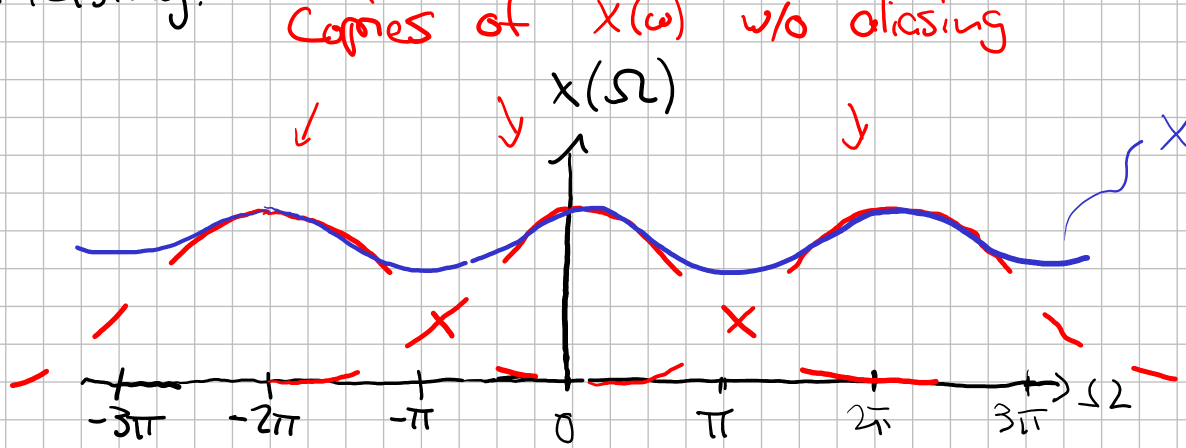
b) Filter 2 is to be preferred in this case. From $\omega_s = 100$ rad/s, it follows that $\omega_N = \frac{\omega_s}{2} = 50\,\frac{rad}{s}$, that is, the maximum allowable frequency at the ADC is 50 rad/s. Filter 1 lets a significant amount above 50 rad/s pass, whereas the stopband for filter 2 seems to start just around ω_N.

c) • We have the following frequency mapping:

$\omega_s = 100$ rad/s $\rightarrow 2\pi \pm m 2\pi$

$\omega_N = 50$ rad/s $\rightarrow \pi \pm m 2\pi$

• Since we have no prefilter, there will be aliasing.



2a) The system is stable since the pole-zero map shows that all poles are inside the unit circle.

b) Pole-zero form of the transfer function:

$$H(z) = K \frac{(z-z_1)(z-z_2)}{(z-p_1)(z-p_2)}.$$

From the pole-zero map:

- Zeros: $z_1 = \frac{1}{\sqrt{2}}(-1+j) = e^{j3\pi/4}$

  $z_2 = \frac{1}{\sqrt{2}}(-1-j) = e^{-j3\pi/4}$

- Poles: $p_1 = 0{,}5 + 0{,}5j = 0{,}5(1+j)$

  $p_2 = 0{,}5 - 0{,}5j = 0{,}5(1-j)$

Hence, the numerator polynomial is:

$$(z-z_1)(z-z_2) = (z-z_1)(z-z_1^*) = z^2 - 2\mathrm{Re}\{z_1\}z + |z_1|^2$$
$$= z^2 + \frac{2}{\sqrt{2}}z + 1 = z^2 + \sqrt{2} + 1.$$

Furthermore, the denominator is

$$(z-p_1)(z-p_2) = (z-p_1)(z-p_2^*) = z^2 - 2\mathrm{Re}\{p_1\} + |p_1|^2$$

$$= z^2 - 2 \cdot 0{,}5z + (0{,}5\sqrt{2})^2 = z^2 - 1z + 0{,}5$$

$$\Rightarrow H(z) = K \frac{z^2 + \sqrt{2}z + 1}{z^2 - z + 0{,}5}.$$

From the DC gain $H(\Omega=0) = H(z=1) = 1$ it follows that

$$1 = K \frac{1 + \sqrt{2} + 1}{1 - 1 + 0{,}5} = K \frac{2+\sqrt{2}}{1/2} \Rightarrow K = \frac{1}{2}(2+\sqrt{2})^{-1}$$

$$K = 0{,}15$$

Thus, the transfer function is

$$H(z) = 0{,}15 \frac{z^2 + \sqrt{2}\, z + 1}{z^2 - z + 0{,}5}$$

$$= \frac{0{,}15 + 0{,}15\sqrt{2}\, z^{-1} + 0{,}15\, z^{-2}}{1 - z^{-1} + 0{,}5 z^{-2}}$$

$$\underline{\underline{H(z) = \frac{0{,}15 + 0{,}21 z^{-1} + 0{,}15 z^{-2}}{1 - z^{-1} + 0{,}5 z^{-2}}}}$$

c) We have that

$$Y(z) = H(z) X(z)$$

$$= \frac{0{,}15 + 0{,}21 z^{-1} + 0{,}15 z^{-2}}{1 - z^{-1} + 0{,}5 z^{-2}} X(z)$$

$$Y(z)\left(1 - z^{-1} + 0{,}5 z^{-2}\right) = \left(0{,}15 + 0{,}21 z^{-1} + 0{,}15 z^{-2}\right) X(z)$$

$$\underline{\underline{y[k] - y[k-1] + 0{,}5 y[k-2] = 0{,}15 x[k] + 0{,}21 x[k-1] + 0{,}15 x[k-2]}}$$

d) We know that

$$H(\Omega) = H(z)\big|_{z = e^{j\Omega}}.$$

Furthermore, we see that there is a zero at $\Omega = \frac{3\pi}{4}$ and thus, the second component of the input is cancelled out. Thus, the output is

$$y[k] = |H(e^{j\pi/2})| \cos\left(\frac{\pi}{2}k + \angle H(e^{j\pi/2})\right).$$

Using a calculator, we find that

$$|H(e^{j\pi/2})| = 0{,}2 \quad \text{and} \quad \angle H(e^{j\pi/2}) = -153.4^\circ$$

Thus:

$$\underline{\underline{y[k] = 0{,}2 \cos\left(\frac{\pi}{2}k - 153{,}4°\right)}}$$

3a) 1. To obtain an error-free approximation, we have to make sure that we measure the whole interval of the time-limited signal, that is, from 0 s to 2 s. With a sampling frequency of $f_s = 100$ Hz, this means that

$$K T_s \geq 2s \Rightarrow \frac{K}{f_s} \geq 2s$$

$$\Rightarrow K \geq 2s \cdot f_s = 2s \cdot 100\,\text{Hz} = \underline{200 \text{ samples}}.$$

2. A spectral resolution of $\Delta f = 0{,}25$ Hz means that

$$\Delta f = \frac{f_s}{K} \Rightarrow K = \frac{f_s}{\Delta f} = \frac{100\,\text{Hz}}{0{,}25\,\text{Hz}} = \underline{400 \text{ samples}}$$

Hence, it would apper that we need $K = 400$ samples from the signal (4s). However, since we know that the signal is zero between 2s and 4s, we would just measure 200 samples of zeros. Hence, we only need to gather

$$\underline{K = 200 \text{ samples}}$$

and <u>use zero-padding to achieve the desired resolution</u>.

b) In this case, windowing with anything different from a rectangular window would change the signal. Hence, windowing introduces a source of error that is not present when using a rectangular window and it would thus worsen the sitution.

c) We know that the frequency response is the ratio betwen the output $Y(\omega)$ and the input $X(\omega)$,

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}.$$

Thus, we can use the DFT of the signals $x[k]$ and $y[k]$ to get approximations of the spectra $X(\omega)$ and $Y(\omega)$ in

$$\omega_l = l \frac{\omega_s}{2\pi},$$

through which we can approximate the frequency response in ω_l, that is,

$$H(\omega_l) = \frac{Y(\omega_l)}{X(\omega_l)}.$$