Introduction to Computer Control Systems, 5 credits, 1RT485

Date: 2021-03-15

Place: Studium/Canvas

Teacher on duty: Dave Zachariah

Allowed aid:

- A basic calculator
- Beta mathematical handbook

Solutions have to be explained in detail and possible to reconstruct.

NB: **Only one problem per sheet.** Write your name and personal number if you do not have an anonymous code.

Best of luck!

Useful results

Laplace transform table

Table 1: Basic Laplace transforms

$f(t)$	$F(s)$	$f(t)$	$F(s)$
unit impulse $\delta(t)$	1	$\sinh(bt)$	$\frac{b}{s^2-b^2}$
unit step $1(t)$	$\frac{1}{s}$	$\cosh(bt)$	$\frac{s}{s^2-b^2}$
t	$\frac{1}{s^2}$	$\frac{1}{2b}t\sin(bt)$	$\frac{s}{(s^2+b^2)^2}$
t^n	$\frac{n!}{s^{n+1}}$	$t\cos(bt)$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
e^{-at}	$\frac{1}{s+a}$	$\frac{\cos(bt)-\cos(at)}{a^2-b^2}; (a^2 \neq b^2)$	$\frac{s}{(s^2+a^2)(s^2+b^2)}$
$\frac{1}{a}(1-e^{-at})$	$\frac{1}{s(s+a)}$	$\frac{\sin(at)+at\cos(at)}{2a}$	$\frac{s^2}{(s^2+a^2)^2}$
$\frac{1}{(n-1)!}t^{n-1}e^{-at}; (n = 1, 2, 3...)$	$\frac{1}{(s+a)^n}$		
$\sin(bt)$	$\frac{b}{s^2+b^2}$		
$\cos(bt)$	$\frac{s}{s^2+b^2}$		
$e^{-at}\sin(bt)$	$\frac{b}{(s+a)^2+b^2}$		
$e^{-at}\cos(bt)$	$\frac{s+a}{(s+a)^2+b^2}$		

Table 2: Properties of Laplace Transforms

$\mathcal{L}\left[af(t)\right] = aF(s)$	$\mathcal{L}\left[tf(t)\right] = -\frac{dF(s)}{ds}$
$\mathcal{L}\left[f_1(t)+f_2(t)\right] = F_1(s)+F_2(s)$	$\mathcal{L}\left[t^2 f(t)\right] = \frac{d^2}{ds^2}F(s)$
$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = sF(s)-f(0)$	$\mathcal{L}\left[t^n f(t)\right] = (-1)^n \frac{d^n}{ds^n}F(s), \quad n = 1, 2, 3, \ldots$
$\mathcal{L}\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s)-sf(0)-f'(0)$	$\mathcal{L}\left[f\left(\frac{t}{a}\right)\right] = aF(as)$
$\mathcal{L}\left[\int f(t)\,dt\right] = \frac{F(s)}{s}+\frac{1}{s}\left[\int f(t)\,dt\right]_{t=0}$	$\mathcal{L}\left[\int_0^t f_1(t-\tau)f(t)d\tau\right] = F_1(s)F_2(s)$
$\mathcal{L}\left[f(t-a)\right] = e^{-as}F(s)$	$\mathcal{L}\left[e^{-at}f(t)\right] = F(s+a)$

Matrix exponential

$$\boxed{e^{At} \triangleq \mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\}}$$

Open-loop and sensitivity functions

$$G_o(s) = G(s)F_y(s), \qquad S(s) = \frac{1}{1+G_o(s)}, \qquad T(s) = 1-S(s)$$

State-space forms and transfer function relations

- State-space form and transfer function

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad \Rightarrow \quad \boxed{G(s) = C(sI - A)^{-1}B + D}$$

- Associated matrices

$$\mathcal{S} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \qquad \mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

- LTI system with transfer function

$$\boxed{G(s) = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}}$$

i) Observable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ -a_3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ b_3 - a_3 b_0 \\ \vdots \\ b_n - a_n b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

ii) Controllable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_1 - a_1 b_0 & b_2 - a_2 b_0 & \cdots & b_n - a_n b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

- Solution to state-space equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

can be written as

$$\boxed{x(t) = e^{At} x_0 + \int_0^t e^{A\tau} Bu(t - \tau) d\tau}$$

- Observer system

$$\boxed{\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})}$$

Feedback control structures

General linear feedback in Laplace form:

$$\boxed{U(s) = F_r(s)R(s) - F_y(s)Y(s)}$$

Common control structures in this form.

- PID controller:

$$F_y(s) = F_r(s) = F(s) = K_p + \frac{K_i}{s} + K_d s,$$

  where $K_p, K_i, K_d \geq 0$

- Lead-lag controller:

$$F_y(s) = F_r(s) = F(s) = K\left(\frac{\tau_D s + 1}{\beta\tau_D s + 1}\right)\left(\frac{\tau_I s + 1}{\tau_I s + \gamma}\right),$$

  where $K, \tau_D, \tau_I > 0$ and $0 \leq \beta, \gamma < 1$

- State-feedback controller with observer:

$$\begin{aligned} F_r(s) &= \left(1 - L(sI - A + KC + BL)^{-1}B\right)\ell_0 \\ F_y(s) &= L(sI - A + KC + BL)^{-1}K \end{aligned}$$

Discrete-time state-space forms

A continuous time system with zero-order-hold input signal and sample period T can be written in discrete-time as:

$$\boxed{\begin{aligned} x(k+1) &= Fx(k) + Gu(k) \\ y(k) &= Hx(k) \end{aligned}}$$

where

$$\begin{aligned} F &= e^{AT} \\ G &= \int_{\tau=0}^{T} e^{A\tau} d\tau B = \lceil \text{if } A^{-1} \text{ exists} \rceil = A^{-1}(e^{AT} - I)B \\ H &= C \end{aligned}$$

Problem 1: basic questions (6/30)

Answer *'true'* or *'false'*. Each correct answer gives 1 point, each wrong answer gives -1 point. Minimum total points for Part A and B is 0 , respectively.

Part A

Note: Write 'skip' if your total home assignment score ≥ 8

i) We control the following system

$$G(s) = \frac{s}{s+4}$$

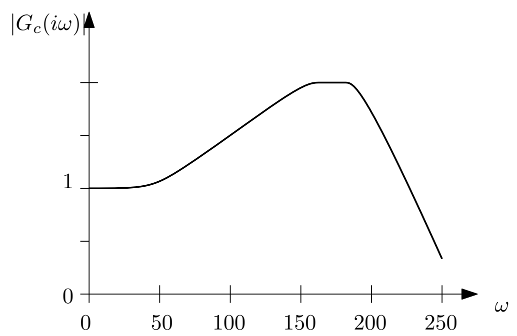
with a P-controller with constant K. The closed-loop system is stable for all $K \geq 4$.

ii) The following system description is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

controllable but not observable.

iii) Consider closed-loop system with a Bode-plot as shown below.



Suppose the reference signal is a sinusoid $r(t) = 100\sin(\omega t)$, then the output follows $y(t) = 150\sin(100\omega + \varphi)$ where φ is a phase-shift.

(3 p)

Part B

Note: Write 'skip' if your total home assignment score ≥ 12

i) Consider controlling a system

$$Y(s) = \frac{s}{s^2+2s+3} U(s)$$

with an open-loop controller $U(s) = F(s)R(s)$. Then an ideal controller is given by

$$F(s) = \frac{s^2 + 2s + 3}{s}$$

ii) The open-loop controller above is capable of attenuating disturbances on the output.

iii) Using feedback control it is possible to ensure stable closed-loop systems even if the system is not known exactly.

(3 p)

Problem 2 (6/30)

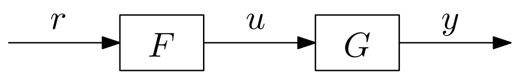
a) Consider a mechanical system with input $u(t)$ and output $y(t)$ that can be written as

$$Y(s) = G(s)U(s) = \frac{1}{s+3}U(s).$$

We first attempt to control the system using an open-loop controller, described as

$$F(s) = \frac{K}{s+K},$$

and illustrated in the block diagram.



Design K such that the resulting control system from reference to output is stable.

(2 p)

b) Next, we attempt to control the system using a simple feedback controller, described as

$$F(s) = K_p$$

Design K_p such that the resulting control system from reference to output is stable.

(2 p)

c) Assuming stabilized control systems, compare the stationary control error

$$e_f = \lim_{t\to\infty} e(t)$$

for each system

(2 p)

Problem 3 (6/30)

Consider controlling the rudder of an aircraft with an input signal u. The system has a state-space description given below:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

a) Show that the system is controllable.

(1 p)

b) The system is controlled using state-feedback controller (with an observer):

$$u = -\tilde{L}\hat{x} + \tilde{\ell}_0 r$$

where

$$\tilde{L} = \begin{bmatrix} 5 & -2 \end{bmatrix} \qquad \tilde{\ell}_0 = -2$$

Compute the poles of the resulting closed-loop system and motivate whether or not this controller is appropriate.

(2 p)

c) Design an alternative, better controller

$$u = -L\hat{x} + \ell_0 r$$

by specifying parameters L and ℓ_0.

(3 p)

Problem 4 (6/30)

(a) Pair each pole zero-plot in Figure 1 (1-4) with

(i) the correct step response plot in Figure 2 (I-IV).

(1.5 p)

(ii) the correct Bode plot in Figure 3 (A-D).

(1.5 p)

(b) The Bode plot for a system $G(s)$ is shown in Figure 4. The system is controlled with a PI-controller $F(s) = K_P + \frac{K_I}{s}$ with $K_P = K_I = \frac{1}{5}$. Let the reference signal be $r(t) = sin(3t)$. What is the output after a long time? The answer may be given in rounded decimals.

(3 p)

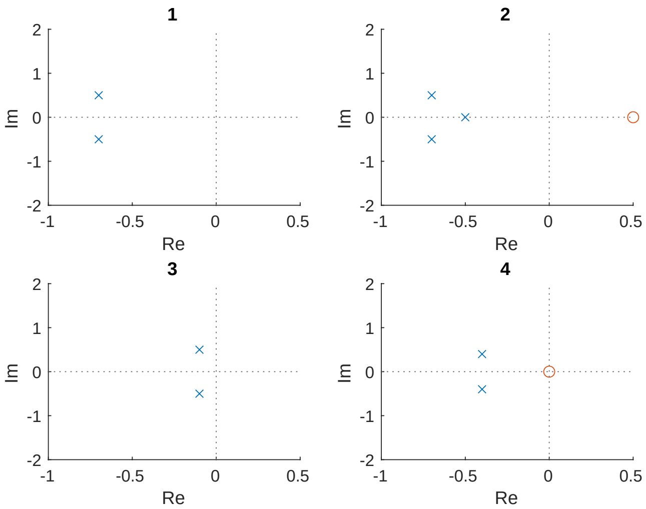


Figure 1

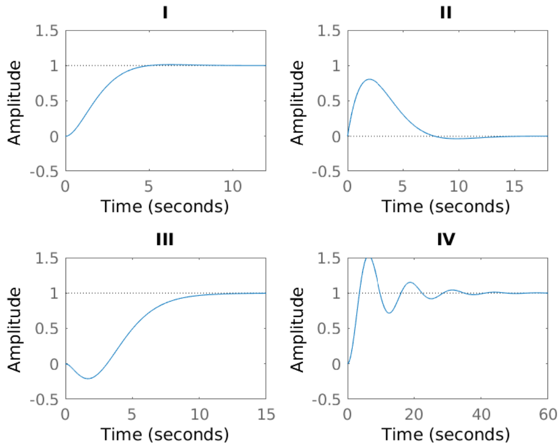


Figure 2

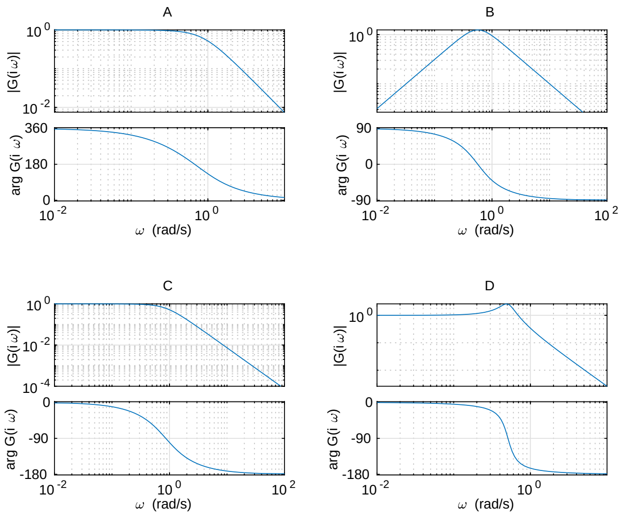


Figure 3

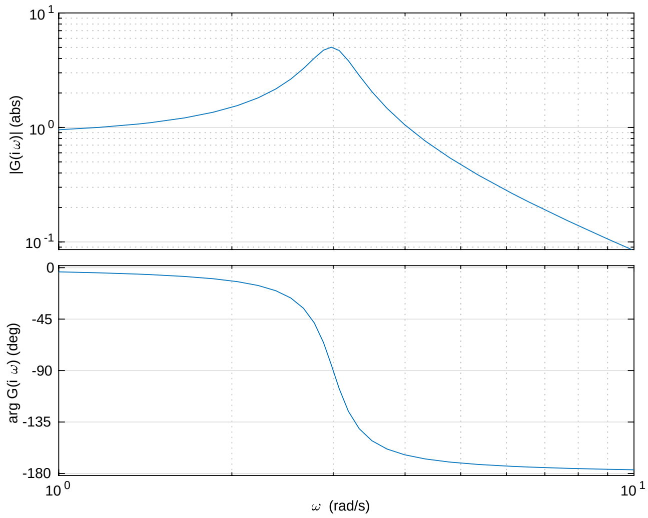
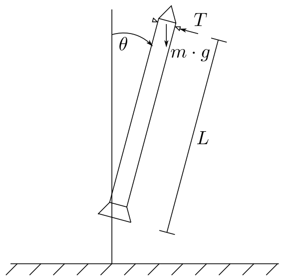


Figure 4: Bode plot for $G(s)$.

Problem 5 (6/30)

A startup is building a rocket which can land back on earth upright. To achieve this, the rocket has to be controlled shortly before touching down to be in an upright position in order to not fall over. To control the rotation we have small thrusters on top of the rocket which generate a momentum T.



The engineer is asked to analyze the system in detail. She derived the following equations to describe the rotations

$$\ddot{\theta}(t) - \frac{g}{L}\theta(t) = \frac{1}{ml}T(t)$$

The startup has a small scale model to test the rocket with $l = 1.09\ m$, $m = 0.8417\ kg$ and $g = 9.81\ \frac{m}{s^2}$. This gives the following transfer function and continuous state space model.

$$\frac{\theta(s)}{T(s)} = \frac{1}{s^2 - 9}$$

$$\dot{x} = \begin{bmatrix} 0 & 2.25 \\ 4 & 0 \end{bmatrix} x + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0.5 \end{bmatrix} x$$

a) Since the startup works with a real rocket, we need to use on-board control which means that we have to use discrete control. As a first step to analyzing the system we need to discretize the state space equations of the rocket. Use a sampling time T to discretize the state space model.

(3 p)

b) Now we want to analyze if the discretized rocket equations are stable and controllable. The first idea of the team is to use a high frequency control. We can approximate this for the beginning by $T \to 0\ s$. Use this sampling time and answer if the system is stable and controllable with high frequency control.

(1 p)

Later, the team wants to analyze the system for a possible second more realistic controller which runs at $T = \frac{1}{3}\ s$. Use this sampling time to again answer the question about stability and controllability. Which of the two systems ($T \to 0$ or $T = \frac{1}{3}$) do you suggest to the team for later usage?

(2 p)