DUGGA WITH SOLUTIONS - LINEAR ALGEBRA II 2018/11/21

JULIAN KÜLSHAMMER

Time: 14.00–16.00. No aids allowed except a pen. All solutions should be accompanied with justifications.

Exercise 1. Let $A = \begin{pmatrix} 1 & 2 & 0 & 1 & 2 \\ 1 & 2 & 0 & 1 & 2 \\ 2 & 4 & 1 & 1 & 5 \end{pmatrix}$. Give bases for each of the following spaces.

(i) the null space of A,
(ii) the row space of A,
(iii) the column space of A.

Possible solution 1a: Applying Gaussian elimination (with elementary row operations) we obtain:

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 2 \\ 1 & 2 & 0 & 1 & 2 \\ 2 & 4 & 1 & 1 & 5 \end{pmatrix} \overset{II-I, III-2I}{\rightsquigarrow} \begin{pmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix} \overset{II\leftrightarrow II}{\rightsquigarrow} \begin{pmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(i) A vector $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$ is in the null space of A if and only if it is a solution to the linear system of equations corresponding to A. Since elementary row operations don't change the solution space to a linear system of equations $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$ is in the null space of A if and only if

$$\begin{aligned} x_1 + 2x_2 + x_4 + 2x_5 &= 0 \\ x_3 - x_4 + x_5 &= 0 \end{aligned}$$

A parametric solution of the linear system of equations is therefore given by

$$N(A) = \left\{ \begin{pmatrix} -2u - t - 2s \\ u \\ t - s \\ t \\ s \end{pmatrix} \middle| \, s, t, u \in \mathbb{R} \right\}$$

Setting the parameters equal to $u = 1, s = 0, t = 0$; $u = 0, t = 1, s = 0$; $u = 0, t = 0, s = 1$ we obtain the three vectors

$$\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix},\begin{pmatrix}-1\\0\\1\\1\\0\end{pmatrix},\begin{pmatrix}-2\\0\\-1\\0\\1\end{pmatrix}$$

which obviously form a spanning set of $N(A)$ and are linearly independent. Therefore they form a basis for $N(A)$.

(ii) Elementary row operations don't change the row space of A. Therefore the row space of A is equal to the row space of the row echelon form of A as obtained above. It is easy to see that the first two rows of the row echelon form are linearly independent since they are not multiples of each other. Furthermore $\operatorname{span}(v, w, 0) = \operatorname{span}(v, w)$ and thus $\begin{pmatrix}1\\2\\0\\1\\2\end{pmatrix}$ and $\begin{pmatrix}0\\0\\1\\-1\\1\end{pmatrix}$ form a basis of the row space of A.

(iii) It is easy to see that the first and third column of the reduced echelon form of A form a maximal linearly independent subset of the column vectors of the reduced echelon form of A. By a result from the lecture, the same is then true for A. Therefore, $\begin{pmatrix}1\\1\\2\end{pmatrix}$ and $\begin{pmatrix}0\\0\\1\end{pmatrix}$ form a maximal linearly independent subset of the columns of A, thus a basis of the column space of A.

Possible solution 1b: (ii) It is easy to see that the first two rows of A are equal therefore if we denote the rows of A by u, v, w, respectively. Then $\operatorname{span}(u, v, w) = \operatorname{span}(u, w)$. Since u and w are obviously linearly independent as they are not linear multiples of each other, it follows that a basis of the row space of A is given by $\left\{\begin{pmatrix}1\\2\\0\\1\\2\end{pmatrix},\begin{pmatrix}2\\4\\1\\1\\5\end{pmatrix}\right\}$.

(iii) We know from the lecture that the dimension of the row space of A is equal to the dimension of the column space of A. By (ii) this number is 2. Therefore, it suffices to find two linearly independent columns of A, which will then form a basis of the column space of A. A basis of the column space of A is therefore given by

$$\left\{\begin{pmatrix}2\\2\\4\end{pmatrix},\begin{pmatrix}2\\2\\5\end{pmatrix}\right\}.$$

(i) By the lecture we know that the sum of the dimension of the null space of A and the dimension of the row space of A is given by the number of columns of A, i.e. $\dim N(A) + 2 = 5$. Therefore, $\dim N(A) = 3$. It thus suffices to find three linearly independent vectors in the null space of A. It is straightforward to check that $\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\1\\-1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\-3\\-2\\1\end{pmatrix}$ are in the null space of A.

Possible solution 1c: (i) see solutions 1a or 1b.

(ii) The row space of A is equal to the column space of $A^T = \begin{pmatrix}1&1&2\\2&2&4\\0&0&1\\1&1&1\\2&2&5\end{pmatrix}$. Applying Gaussian elimination (elementary row operations) to A^T we obtain

$$\begin{pmatrix}1&1&2\\2&2&4\\0&0&1\\1&1&1\\2&2&5\end{pmatrix} \overset{II-2I,IV-I,V-2I}{\rightsquigarrow} \begin{pmatrix}1&1&2\\0&0&0\\0&0&1\\0&0&-1\\0&0&1\end{pmatrix} \overset{IV+III,V-III}{\rightsquigarrow} \begin{pmatrix}1&1&2\\0&0&0\\0&0&1\\0&0&0\\0&0&0\end{pmatrix} \overset{III\leftrightarrow II}{\rightsquigarrow} \begin{pmatrix}1&1&2\\0&0&1\\0&0&0\\0&0&0\\0&0&0\end{pmatrix}.$$

Here the first and third column of the row echelon form of A^T are linearly independent. Therefore the first and the third column of A^T are linearly independent. Thus, a basis for the row space of A is given by $\left\{\begin{pmatrix}1\\2\\0\\1\\2\end{pmatrix}, \begin{pmatrix}2\\4\\1\\1\\5\end{pmatrix}\right\}$.

(iii) The column space of A does not change under elementary column operations. Therefore, applying elementary column operations to A we obtain

$$\begin{pmatrix}1&2&0&1&2\\1&2&0&1&2\\2&4&1&1&5\end{pmatrix} \overset{II-2I,IV-I,V-2I}{\rightsquigarrow} \begin{pmatrix}1&0&0&0&0\\1&0&0&0&0\\2&0&1&-1&1\end{pmatrix} \overset{IV+III,V-III}{\rightsquigarrow} \begin{pmatrix}1&0&0&0&0\\1&0&0&0&0\\2&0&1&0&0\end{pmatrix}.$$

It follows that a basis for the column space of A is given by $\left\{\begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$.

Exercise 2. Let $V = P_2(\mathbb{R})$ be the space of polynomials of degree at most 2 (You don't have to prove that this is a vector space). Let $U = \left\{ p(x) = a_0 + a_1 x + a_2 x^2 \,\middle|\, \int_0^1 p(x)dx = 0 \right\}$.

(i) Prove that U is a subspace of V.
(ii) Give a basis for U. Justify your answer.

Possible solution 2a: (i) To check that U is a subspace of V we have to check the three conditions:

- $U \neq \emptyset$,
- If $p, q \in U$ then $p + q \in U$,
- If $p \in U$ and $\lambda \in \mathbb{R}$ then $\lambda p \in U$.

To check that $U \neq \emptyset$ note that $0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \in U$ since $\int_0^1 0dx = 0$.
To prove that U is closed under addition assume that $p, q \in U$. Therefore $\int_0^1 p(x)dx = 0$ and $\int_0^1 q(x)dx = 0$. It follows that

$$\int_0^1 (p+q)(x)dx = \int_0^1 p(x) + q(x)dx = \int_0^1 p(x)dx + \int_0^1 q(x)dx = 0.$$

Therefore $p + q \in U$.
To prove that U is closed under scalar multiplication assume that $p \in U$ and $\lambda \in \mathbb{R}$. Therefore $\int_0^1 p(x)dx = 0$. It follows that

$$\int_0^1 (\lambda p)(x)dx = \int_0^1 \lambda p(x)dx = \lambda \int_0^1 p(x)dx = \lambda 0 = 0.$$

Therefore $\lambda p \in U$.

(ii) Computing the integral (see solution 2b (i)) $\int_0^1 p(x)dx = a_0 + \frac{a_1}{2} + \frac{a_2}{3}$ we see that being in U imposes one linear condition on the coefficients of $p(x)$. Therefore $\dim U = \dim P_2(\mathbb{R}) - 1 = 3 - 1 = 2$. Thus, it suffices to find two linearly independent polynomials which are in U. Computation shows that

$$\int_0^1 2x - 1dx = 0 \text{ and } \int_0^1 3x^2 - 1dx = 0$$

Obviously $2x - 1$ and $3x^2 - 1$ are linearly independent as they are not scalar multiples of each other. It follows that $2x - 1$ and $3x^2 - 1$ form a basis of U.

Possible solution 2b: (i) We compute that

$$\int_0^1 p(x)dx = \int_0^1 a_0 + a_1 x + a_2 x^2 dx = \left[a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 \right]_0^1 = a_0 + \frac{a_1}{2} + \frac{a_2}{3}.$$

It follows that

$$U = \left\{ a_0 + a_1 x + a_2 x^2 \mid a_0 + \frac{a_1}{2} + \frac{a_2}{3} = 0 \right\}.$$

For checking that U is a subspace of V we have to check that

- $0 \in U$,
- If $p, q \in U$ then $p + q \in U$,

- If $p \in U$, $\lambda \in \mathbb{R}$, then $\lambda p \in U$.

To check that $0 \in U$ note that $0 = 0 + 0x + 0x^2$ which satisfies $0 + \frac{0}{2} + \frac{0}{3} = 0$.
To check that U is closed for addition let $p(x) = a_0 + a_1x + a_2x^2$ and $q(x) = b_0 + b_1x + b_2x^2$ be elements of U. This means that

$$a_0 + \frac{a_1}{2} + \frac{a_2}{3} = 0$$
$$b_0 + \frac{b_1}{2} + \frac{b_2}{3} = 0$$

By definition $(p + q)(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$. We check that this is in U:

$$(a_0 + b_0) + \frac{a_1 + b_1}{2} + \frac{a_2 + b_2}{3} = \left(a_0 + \frac{a_1}{2} + \frac{a_2}{3}\right) + \left(b_0 + \frac{b_1}{2} + \frac{b_2}{3}\right) = 0 + 0 = 0$$

It follows that $p + q \in U$.
To check that U is closed for scalar multiplication let $p(x) = a_0 + a_1x + a_2x^2$ be in U. This means that $a_0 + \frac{a_1}{2} + \frac{a_2}{3} = 0$. By definition $(\lambda p)(x) = (\lambda a_0) + (\lambda a_1)x + (\lambda a_2)x^2$. We check that this is in U:

$$(\lambda a_0) + \frac{\lambda a_1}{2} + \frac{\lambda a_2}{3} = \lambda\left(a_0 + \frac{a_1}{2} + \frac{a_2}{3}\right) = \lambda \cdot 0 = 0$$

Therefore $\lambda p \in U$.

(ii) We have shown in (i) that

$$U = \left\{a_0 + a_1x + a_2x^2 \mid a_0 + \frac{a_1}{2} + \frac{a_2}{3} = 0\right\}$$

Setting $a_1 = 1, a_2 = 0$ and $a_1 = 0, a_2 = 1$ we obtain the polynomials $p_1(x) = -\frac{1}{2} + x$ and $p_2(x) = -\frac{1}{3} + x^2$. These are linearly independent since $\lambda_1 p_1(x) + \lambda_2 p_2(x) = 0$ for all x implies by comparing coefficients that

$$-\frac{1}{2}\lambda_1 - \frac{1}{3}\lambda_2 = 0$$
$$\lambda_1 = 0$$
$$\lambda_2 = 0.$$

Furthermore they span U as every $p(x)$ in U can be written as

$$\left(-\frac{a_1}{2} - \frac{a_2}{3}\right) + a_1x + a_2x^2 = a_1p_1(x) + a_2p_2(x).$$

Exercise 3. (i) Let V be a vector space. Let $u, v, w \in V$. Prove that if $u + w = v + w$ then $u = v$. In each step indicate which of the vector space axioms you are using.

(ii) Are the following matrices in $M_{2\times 2}(\mathbb{R})$ linearly independent? Justify your answer.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix}$$

Possible solution 3a: (i) Let $u, v, w \in V$ such that $u + w = v + w$. (By (A4)) there exists a vector $-w \in V$ with $w + (-w) = 0_V$. Adding $(-w)$ to both sides of the equation $u + w = v + w$ we obtain

$$(u + w) + (-w) = (v + w) + (-w)$$

By (A1), i.e. associativity of vector addition, we can move the brackets and obtain

$$u + (w + (-w)) = v + (w + (-w))$$

By (A4) we obtain

$$u + 0_V = v + 0_V$$

By (A3), $v' + 0_V = v'$ for all $v' \in V$ we obtain

$$u = v.$$

This proves the claim.

(ii) The three matrices $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, and $C = \begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix}$ are linearly independent if and only if $C \notin \operatorname{span}(A, B)$, $B \notin \operatorname{span}(A, C)$ and $A \notin \operatorname{span}(B, C)$. We see that $C = A + 2B$. Therefore $C \in \operatorname{span}(A, B)$ and thus A, B are linearly dependent.

Possible solution 3b: (i) Let $u, v, w \in V$ such that $u + w = v + w$. Add the vector $(-1)w$ to the equation. Then

$$(u + w) + (-1)w = (v + w) + (-1)w.$$

By (SM2) we obtain that $w = 1w$. Therefore

$$(u + 1w) + (-1)w = (v + 1w) + (-1)w.$$

By (A1), associativity of vector addition, we can move the brackets and obtain

$$u + (1w + (-1)w) = v + (1w + (-1)w).$$

Using (SM4), distributivity, we obtain that

$$u + (1 + (-1))w = v + (1 + (-1))w.$$

And by the rules of addition in $\mathbb{R}$ we obtain

$$u + 0w = v + 0w.$$

Not strictly an axiom, but in the lecture we have shown that from the axioms it follows that $0w = 0_V$. Therefore,

$$u + 0_V = v + 0_V.$$

Using (A3), $v' + 0_V = v'$ for all $v' \in V$ we obtain

$$u = v.$$

This proves the claim.

(ii) The three matrices are linearly independent if and only if the equation

$$\lambda_1 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

has only the trivial solution.
Comparing the entries of the matrices on the left hand side and the right hand side this gives the linear system of equations

$$\begin{aligned} \lambda_1 + \lambda_3 &= 0 \\ \lambda_1 + \lambda_2 + 3\lambda_3 &= 0 \\ \lambda_2 + 2\lambda_3 &= 0 \\ \lambda_1 + \lambda_2 + 3\lambda_3 &= 0 \end{aligned}$$

Using Gaussian elimination (elementary row operations) we obtain

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 0 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix} \overset{II-I, IV-I}{\rightsquigarrow} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \overset{III-II, IV-II}{\rightsquigarrow} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore a parametric solution is given by $\left\{ \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} -s \\ -2s \\ s \end{pmatrix} \,\middle|\, s \in \mathbb{R} \right\}$. In particular the equation has not only the trivial solution (e.g. also $\lambda_1 = -1, \lambda_2 = -2, \lambda_3 = 1$ is a solution).

Exercise 4. (i) Let V, W be vector spaces. Give the definition of what it means for a function $f\colon V \to W$ to be linear.

(ii) Let $v_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$. Consider the basis $B = \{v_1, v_2\}$ of $\operatorname{span}(v_1, v_2)$. Determine the coordinate vector $[v]_B$ of $v = \begin{pmatrix} -2 \\ 5 \\ 1 \end{pmatrix}$ with respect to B.

(iii) Let v, v_1, v_2 be as in (ii). Suppose $g\colon \mathbb{R}^3 \to \mathbb{R}^2$ is a linear function such that $g(v_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $g(v_2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. What is $g(v)$? Justify your answer.

Possible solution 4a: (i) A function $f\colon V \to W$ is called linear if
- $f(v + v') = f(v) + f(v')$ for all $v, v' \in V$, and
- $f(\lambda v) = \lambda f(v)$ for all $v \in V$, $\lambda \in \mathbb{R}$.

(ii) To see what the coordinate vector of v with respect to $\{v_1, v_2\}$ is we need to find λ_1, λ_2 such that

$$\begin{pmatrix} -2 \\ 5 \\ 1 \end{pmatrix} = \lambda_1 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Using Gaussian elimination (elementary row operations) we see that

$$\left(\begin{array}{cc|c} 2 & 1 & -2 \\ 1 & 2 & 5 \\ 1 & 1 & 1 \end{array}\right) \overset{II-\frac{1}{2}I, III-\frac{1}{2}I}{\rightsquigarrow} \left(\begin{array}{cc|c} 2 & 1 & -2 \\ 0 & \frac{3}{2} & 6 \\ 0 & \frac{1}{2} & 2 \end{array}\right) \overset{\frac{2}{3}II, III-\frac{1}{3}\cdot II}{\rightsquigarrow} \left(\begin{array}{cc|c} 2 & 1 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{array}\right)$$

It follows that $\lambda_2 = 4$ and $\lambda_1 = \frac{1}{2}(-2-4) = -3$. Therefore

$$[v]_B = \begin{pmatrix} -3 \\ 4 \end{pmatrix}.$$

(iii) In (ii) we computed that $v = -3v_1 + 4v_2$. Since g is linear it follows that $g(v) = g(-3v_1 + 4v_2) = -3g(v_1) + 4g(v_2) = -3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

Possible solution 4b: (i) On the fifth exercise sheet the alternative definition was given that a function f is linear if
- $f(0_V) = 0_W$, and
- $f(av + (1-a)v') = af(v) + (1-a)f(v')$ for all $a \in \mathbb{R}$ and all $v, v' \in V$.

(ii) The coordinate vector of v with respect to $\{v_1, v_2\}$ is given by the vector $\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}$ in $\mathbb{R}^2$ such that $\lambda_1 v_1 + \lambda_2 v_2 = v$. It is easy to see that $v = -3v_1 + 4v_2$. Therefore $[v]_B = \begin{pmatrix}-3\\ 4\end{pmatrix}$.

(iii) Since g is a linear map $\mathbb{R}^3 \to \mathbb{R}^2$ it follows that g is given by multiplication by a matrix $A \in M_{2\times 3}(\mathbb{R})$. According to the exercise this matrix satisfies $A\begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$ and $A\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$. Let $A = (a_{ij})_{ij}$. Then we obtain the linear system of equations

$$\begin{aligned}2a_{11} + a_{12} + a_{13} &= 1\\ 2a_{21} + a_{22} + a_{23} &= 0\\ a_{11} + 2a_{12} + a_{13} &= 1\\ a_{21} + 2a_{22} + a_{23} &= 1\end{aligned}$$

Applying Gaussian elimination (using elementary row operations and ordering the variables as $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}$ we obtain

$$\left(\begin{array}{cccccc|c} 2 & 1 & 1 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 2 & 1 & 1 & 0\\ 1 & 2 & 1 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1 & 2 & 1 & 1\end{array}\right) \overset{III-\frac{1}{2}I, IV-\frac{1}{2}II}{\rightsquigarrow} \left(\begin{array}{cccccc|c} 2 & 1 & 1 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 2 & 1 & 1 & 0\\ 0 & \frac{3}{2} & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2}\\ 0 & 0 & 0 & 0 & \frac{3}{2} & \frac{1}{2} & 1\end{array}\right).$$

It follows that

$$\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\end{pmatrix} = \begin{pmatrix}\frac{1}{3}(1-a_{13}) & \frac{1}{3}(1-a_{13}) & a_{13}\\ -\frac{1}{3}(1+a_{23}) & \frac{1}{3}(2-a_{23}) & a_{23}\end{pmatrix}.$$

We now compute

$$\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\end{pmatrix}\begin{pmatrix}-2\\5\\1\end{pmatrix} = \begin{pmatrix}\frac{1}{3}(1-a_{13}) & \frac{1}{3}(1-a_{13}) & a_{13}\\ -\frac{1}{3}(1+a_{23}) & \frac{1}{3}(2-a_{23}) & a_{23}\end{pmatrix}\begin{pmatrix}-2\\5\\1\end{pmatrix} = \begin{pmatrix}1\\4\end{pmatrix}$$

Therefore $g(v) = \begin{pmatrix}1\\4\end{pmatrix}$.

Possible solution 4c: (i) A function f is linear if and only if $f(\lambda v + \mu v') = \lambda f(v) + \mu f(v')$ for all $v, v' \in V$ and all $\lambda, \mu \in \mathbb{R}$.

(ii) Suppose that $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \operatorname{span}(v_1, v_2)$. To find out what the coordinate vector of $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is we have to find λ_1, λ_2 such that

$$\lambda_1 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Using Gaussian elimination (elementary row operations) we obtain

$$\left(\begin{array}{cc|c} 2 & 1 & x \\ 1 & 2 & y \\ 1 & 1 & z \end{array}\right) \overset{II-\frac{1}{2}I, III-\frac{1}{2}I}{\rightsquigarrow} \left(\begin{array}{cc|c} 2 & 1 & x \\ 0 & \frac{3}{2} & y-\frac{1}{2}x \\ 0 & \frac{1}{2} & z-\frac{1}{2}x \end{array}\right) \overset{III-\frac{1}{3}II}{\rightsquigarrow} \left(\begin{array}{cc|c} 2 & 1 & x \\ 0 & \frac{3}{2} & y-\frac{1}{2}x \\ z-\frac{1}{3}y-\frac{1}{3}x & & \end{array}\right)$$

Therefore, for $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to be in $\operatorname{span}(v_1, v_2)$ we need that $z - \frac{1}{3}y - \frac{1}{3}x = 0$. For our given vector $\begin{pmatrix} -2 \\ 5 \\ 1 \end{pmatrix}$. This is satisfied. We can continue (assuming that $z - \frac{1}{3}y - \frac{1}{3}x = 0$):

$$\overset{\frac{2}{3}II}{\rightsquigarrow} \left(\begin{array}{cc|c} 2 & 1 & x \\ 0 & 1 & \frac{2}{3}y-\frac{1}{3}x \\ 0 & 0 & 0 \end{array}\right) \overset{I-II}{\rightsquigarrow} \left(\begin{array}{cc|c} 2 & 0 & \frac{4}{3}x-\frac{2}{3}y \\ 0 & 1 & \frac{2}{3}y-\frac{1}{3}x \\ 0 & 0 & 0 \end{array}\right) \overset{\frac{1}{2}I}{\rightsquigarrow} \left(\begin{array}{cc|c} 1 & 0 & \frac{2}{3}x-\frac{1}{3}y \\ 0 & 1 & \frac{2}{3}y-\frac{1}{3}x \\ 0 & 0 & 0 \end{array}\right)$$

Therefore, $\lambda_1 = \frac{2}{3}x - \frac{1}{3}y$ and $\lambda_2 = \frac{2}{3}y - \frac{1}{3}x$. Plugging in $x = -2, y = 5, z = 1$ as given we obtain $\lambda_1 = -3, \lambda_2 = 4$. It follows that $[v]_B = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$.

(iii) For the exercise it suffices to restrict g to a function $g\colon \operatorname{span}(v_1, v_2) \to \mathbb{R}^2$. Immediately by the statement of the exercise we see that $[g]_{E\leftarrow B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. It follows that $[g(v)]_E = [g]_{E\leftarrow B}[v]_B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$. Therefore $g(v) = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.