

Problem 1

Part A: Skip!

Part B: skip!

Problem 2

a) The open loop transfer function is given by

$$G_o(s) = F(s)G(s) = \frac{k}{s+k} \cdot \frac{1}{s+3} = \frac{k}{(s+k)(s+3)}$$

The system is stable if all poles are in the LHP => roots of the denominator is in the RHP. By using $k=1$ the system will be stable as:

$$G_o(s) = \frac{1}{(s+1)(s+3)} \Rightarrow Y(s) = G_o(s)R(s) = \frac{1}{(s+1)(s+3)} \cdot R(s)$$

Answer: I choose to design with $k=1$
(But any $k \geq 1$ will keep the system stable)

b) By using a P feedback controller, $F(s) = k_p$ we get the closed loop transfer function

$$G_{cl}(s) = \frac{F(s)G(s)}{1+F(s)G(s)} = \frac{k_p \cdot \frac{1}{s+3}}{1+k_p \cdot \frac{1}{s+3}} = \frac{k_p}{s+3+k_p}$$

The system is stable if the roots of the denominator is in the RHP. By using $k_p = 1$ we get:

$$G_{cl}(s) = \frac{1}{s+4} \Rightarrow Y(s) = G_{cl}(s)R(s) = \frac{1}{s+4} \cdot R(s)$$

Answer: I choose to design with $k_p = 1$.

Problem 2

c) We know from the lectures that

$$e(t) = r(t) - y(t) \overset{\mathcal{L}}{\Longleftrightarrow} E(s) = R(s) - Y(s)$$

Assuming that the reference signal is a step function and using chosen K and K_p from a) & b):

$$r(t) = \begin{cases} r_0 & t \geq 0 \\ 0 & t < 0 \end{cases} \overset{\mathcal{L}}{\Longleftrightarrow} R(s) = \frac{r_0}{s}$$

Then we calculate the stationary control error by using the Final value theorem:

System a):

$$e_f = \lim_{t\to\infty} e(t) = \lim_{s\to 0} s\,E(s) = \lim_{s\to 0} s\big(R(s) - Y(s)\big) = \lim_{s\to 0} s\big(R(s) - G_a(s)R(s)\big)$$

$$= \lim_{s\to 0} s \cdot \frac{r_0}{s} \frac{s^2+4s+2}{s^2+4s+3} = r_0 \cdot \frac{2}{3}$$

System b)

$$e_f = \lim_{t\to\infty} e(t) = \lim_{s\to 0} s\,E(s) = \lim_{s\to 0} s\big(R(s) - Y(s)\big) = \lim_{s\to 0} s\big(R(s) - G_c(s)R(s)\big)$$

$$= \lim_{s\to 0} s \cdot \frac{r_0}{s} \cdot \frac{(s+3)}{(s+4)} = r_0 \cdot \frac{3}{4}$$

Ansver: The stationary control error is $\frac{2r_0}{3}$ for system a) and $\frac{3r_0}{4}$ for system b), if the reference signal is a step with gain r_0.

Problem 3

a) We know from the lectures that if the determinant of the controllability matrix S is non zero, the system is controllable.

$$\det(S) = \det([B \ AB]) = \det\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = 1 \neq 0$$

=> The system is controllable!

b) When designing with a state feedback controller the closed loop transfer function is given by

$$G_{cl}(s) = C(sI - A + BL)^{-1} B l_0 = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} s-3+5 & 2-2 \\ -1 & s-1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot (-2)$$

$$= \frac{(-2)}{(s+2)(s-1)} \cdot \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} s-1 & 0 \\ 1 & s+2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{-2}{(s+2)(s-1)} \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} s-1 \\ 1 \end{bmatrix} = \frac{(-2)}{(s+2)(s-1)} \cdot (s-1+2) =$$

$$= \frac{-2(s+1)}{(s+2)(s-1)}$$

This controller have the poles $P_1 = -2$ $P_2 = 1$. As $P_2 \geq 0$ it lies in the RHP which makes the system unstable! and therefore not apprpriate.

Problem 3

c) A better controller haves the poles in -2 and have a static gain of 1. This yields the characteristic polynom

$$(s+2)^2 = s^2 + 4s + 4 = 0$$

and $G_{cl}(0) = 1$

To get L we use that the poles of the system is given by

$$0 = \det(sI - A + BL) = \det\begin{bmatrix} s-3+l_1 & 2+l_2 \\ -1 & s-1 \end{bmatrix}$$

$$= (s-3+l_1)(s-1) + 2 + l_2 = s^2 + s(l_1 - 4) + (3 - l_1 + 2 + l_2)$$

Comparing this with $s^2+4s+4=0$ gives:

$$\begin{cases} l_1 - 4 = 4 \\ 5 - l_1 + l_2 = 4 \end{cases} \Rightarrow \begin{cases} l_1 = 8 \\ l_2 = 7 \end{cases} \Rightarrow L = \begin{bmatrix} 8 & 7 \end{bmatrix}$$

This gives the closed loop transfer function

$$G_{cl}(s) = C(sI - A + BL)^{-1} B l_0 = \frac{l_0}{(s+2)^2} \cdot \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} s-1 & -9 \\ 1 & s+5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{l_0}{(s+2)^2} \cdot \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} s-1 \\ 1 \end{bmatrix} = \frac{l_0(s+1)}{(s+2)^2}$$

Wanting $1 = G_{cl}(0) = \frac{l_0}{4} \Rightarrow l_0 = 4$.

<u>Answer</u>: A better controller is given by $L = \begin{bmatrix} 8 & 7 \end{bmatrix}$ and $l_0 = 4$.

Problem 4

a) i)

Pole zero-plot ~~number~~

4: Have a zero in the origin which make the step response go towards 0 => Pairs with step response plot II.

2: Have a zero in the RHP which will make the step response start in the "wrong" direction => Pairs with step response plot III

3: Have poles closer to the origin than pole zero plot number 1, is therfore slower => Pairs with step response plot I

1: Have poles further away from origin, than nr 3, which are also a complex conjugate that can create oscilliators, => Pairs with step respons plot IV

Answer: 1-IV, 2-III, 3-I, 4-II

Problem 4

a) ii)

Pole zero plot:

4: zero in origin => $|G(i0)| = 0$ => Pairs with bode plot B

2: Zero in RHP and have 3 poles and therefor a higher order in the denominator => Creates a "bump" in the magnitude plot => Pairs with bode plot D.

1: Is faster than system 3 => wider band width in magnitude plot => Pairs with bode plot C.

3: Slower tan system 1 => narrower band width in magnitude plot => Pairs with bode plot A.

<u>Answer</u>: 1-C, 2-D, 3-A, 4-B

Problem 4

b) By the sine-in sine-out principle the output will be given by

$$y(t) = |G_{cl}(i3)| \sin(3t + \arg G_{cl}(i3))$$

$$G_{cl}(s) = \frac{G_o(s)}{1+G_o(s)} = \frac{F(s)G(s)}{1+F(s)G(s)}$$

where $F(s) = \frac{1}{5}\left(1+\frac{1}{s}\right)$ and $G(s)$ is given by the bode plots.

$$|F(i\omega)| = \frac{1}{5}\left|1+\frac{1}{i\omega}\right| = \frac{1}{5}\left|1-i\frac{1}{\omega}\right| = \frac{1}{5}\sqrt{1+\frac{1}{\omega^2}}$$

$$\arg F(i\omega) = \arg\frac{1}{5} + \arg\left(1-i\frac{1}{\omega}\right) = 0 + \operatorname{atan}\left(-\frac{1}{\omega}\right) = -\operatorname{atan}\left(\frac{1}{\omega}\right)$$

$$\Rightarrow |F(i3)| = \frac{1}{5}\sqrt{1+\frac{1}{9}} \approx 0.21$$

$$\arg F(i3) = -\operatorname{atan}\left(\frac{1}{3}\right) \approx -0.32$$

Given by the bode plot:

$|G(i3)| \approx 5$

$\arg G(i3) \approx 90^\circ = \frac{\pi}{2}$ $\Rightarrow$

$$|G_o(i3)| = |F(i3)||G(i3)| = 0.21 \cdot 5 = 1.05$$

$$\arg G_o(i3) = \arg(F(i3)G(i3)) = \arg F(i3) + \arg G(i3) = -0.32 + \frac{\pi}{2} = 1.25$$

$$\Rightarrow G_o(i3) = |G_o(i3)| e^{i \arg G_o(i3)} = 0.33 + 0.1i$$

$\Rightarrow$

Next page!

$\Rightarrow \quad 1 + G_o(i3) = 1 + 0{,}33 + 0{,}1i = 1{,}33 + 0{,}1i$

$|1 + G_o(i3)| = |1{,}33 + 0{,}1i| \approx 1{,}33$

$\arg(1 + G_o(i3)) \approx 0{,}08$

$\Rightarrow$

$$|G_u(i3)| = \frac{|G_o(i3)|}{|1+G_o(i3)|} = \frac{1{,}05}{1{,}33} \approx 0{,}79$$

$$\arg G_u(i3) = \arg G_o(i3) - \arg(1+G_o(i3))$$
$$= 1{,}25 - 0{,}08 = 1{,}17$$

$$\Rightarrow \quad Y(t) = |G_u(i3)| \sin(3t + \arg(G_u(i3)))$$
$$= 0{,}79 \sin(3t + 1{,}17) =$$

<u>Answer:</u> The output will after a long time be $0{,}79 \sin(3t + 1{,}17)$

Problem 5

a) We can discretize the system by using ZOH. This gives the following state feedback system

$$x(k+1) = F\,x(k) + G\,u(k)$$
$$y(k) = H\,x(k)$$

Where:

$$F = e^{AT} = \mathcal{L}^{-1}\{(sI-A)^{-1}\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2-9}\cdot\begin{bmatrix} s & 2.25 \\ 4 & s \end{bmatrix}\right\}$$

(using transform tables)

$$\overset{\uparrow}{=} \begin{bmatrix} \cosh(3T) & \frac{2.25}{3}\cdot\sinh(3T) \\ \frac{4}{3}\sinh(3T) & \cosh(3T) \end{bmatrix}$$

As A is invertible:

$$G = A^{-1}(F-I)B = \frac{1}{-9}\cdot\begin{bmatrix} 0 & -2.25 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} \cosh(3T)-1 & \frac{2.25}{3}\cdot\sinh(3T) \\ \frac{4}{3}\sinh(3T) & \cosh(3T)-1 \end{bmatrix}\cdot\begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$

$$= \frac{1}{-9}\begin{bmatrix} 0 & -2.25 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} 0.5(\cosh(3T)-1) \\ \frac{4}{6}\sinh(3T) \end{bmatrix}$$

$$= \frac{-1}{9}\begin{bmatrix} -1.5\sinh(3T) \\ -2(\cosh(3T)-1) \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 1.5\sinh(3T) \\ 2(\cosh(3T)-1) \end{bmatrix}$$

$\Longrightarrow$ Next page

 Problem 5 a) AG-0031-MCZ

$$H = C = \begin{bmatrix} 0 & 0{,}5 \end{bmatrix}$$

Answer: The discrete state space model have the following matrices for a sampling time T

$$F = \begin{bmatrix} \cosh(3T) & \frac{2.25}{3}\sinh(3T) \\ \frac{4}{3}\sinh(3T) & \cosh(3T) \end{bmatrix}$$

$$G = \frac{1}{9}\begin{bmatrix} 1.5\sinh(3T) \\ 2(\cosh(3T)-1) \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & 0{,}5 \end{bmatrix}$$

Problem 5

b) When $T \to 0$ we get the following system matrices: (by using the calculator)

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad H = \begin{bmatrix} 0 & 0.5 \end{bmatrix}$$

This gives the following controllability matrix:

$$S = [G \;\; FG] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \det(S) = 0$$

Furthermore, as the poles of the system is a subset of the eigenvalues of F, $\lambda_{1,2} = 1$, the system cannot be stable as a discrete time system is stable when $|P_i| < 1$.

<u>Answer</u>: The system is not stable and not controllable when $T \to 0\,s$.

When $T = \frac{1}{3}\,s$ we get the following matrices:

$$F = \begin{bmatrix} 1,54 & 0,88 \\ 1,57 & 1,59 \end{bmatrix} \quad G = \begin{bmatrix} 0.20 \\ 0.12 \end{bmatrix} \quad H = \begin{bmatrix} 0 & 0,5 \end{bmatrix}$$

This gives the following stability- and controllability matrix:

$$S = [G \;\; FG] = \begin{bmatrix} 0.20 & 0.41 \\ 0.12 & 0.50 \end{bmatrix}$$

$$O = \begin{bmatrix} H \\ HF \end{bmatrix} = \begin{bmatrix} 0 & 0,5 \\ 0.79 & 0.77 \end{bmatrix}$$

$\Rightarrow$ Next page

Problem 5

$\det(S) = 0{,}05 \neq 0$
$\det(O) \approx -0{,}90 \neq 0$

As $T = 1/3\,s$ will make the system both controllable and observable, the system will also be stable.

Answer: The system will be both stable and controllable, and therefor is $T = 1/3\,s$ prefered.