Introduction to Computer Control Systems, 5 credits, 1RT485

Date and Time: 2019-03-19

Place: Bergsbrunnagatan 15, Sal 1.

Teacher on duty: Dave Zachariah.

Allowed aid:

- A basic calculator
- BETA mathematical handbook

Solutions have to be explained in detail and possible to reconstruct.

<u>NB:</u> **Only one problem per sheet.** Write your anonymous exam code on each sheet. Write your name if you do not have an anonymous code.

Best of luck!

Useful results

Laplace transform table

Table 1: Basic Laplace transforms

$f(t)$	$F(s)$	$f(t)$	$F(s)$
unit impulse $\delta(t)$	1	$\sinh(bt)$	$\frac{b}{s^2-b^2}$
unit step $1(t)$	$\frac{1}{s}$	$\cosh(bt)$	$\frac{s}{s^2-b^2}$
t	$\frac{1}{s^2}$	$\frac{1}{2b} t \sin(bt)$	$\frac{s}{(s^2+b^2)^2}$
t^n	$\frac{n!}{s^{n+1}}$	$t\cos(bt)$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
e^{-at}	$\frac{1}{s+a}$	$\frac{\cos(bt)-\cos(at)}{a^2-b^2}; (a^2 \neq b^2)$	$\frac{s}{(s^2+a^2)(s^2+b^2)}$
$\frac{1}{a}(1-e^{-at})$	$\frac{1}{s(s+a)}$	$\frac{\sin(at)+at\cos(at)}{2a}$	$\frac{s^2}{(s^2+a^2)^2}$
$\frac{1}{(n-1)!}t^{n-1}e^{-at}; (n=1,2,3...)$	$\frac{1}{(s+a)^n}$		
$\sin(bt)$	$\frac{b}{s^2+b^2}$		
$\cos(bt)$	$\frac{s}{s^2+b^2}$		
$e^{-at}\sin(bt)$	$\frac{b}{(s+a)^2+b^2}$		
$e^{-at}\cos(bt)$	$\frac{s+a}{(s+a)^2+b^2}$		

Table 2: Properties of Laplace Transforms

$\mathcal{L}[af(t)] = aF(s)$	$\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}$
$\mathcal{L}[f_1(t)+f_2(t)] = F_1(s)+F_2(s)$	$\mathcal{L}\left[t^2 f(t)\right] = \frac{d^2}{ds^2}F(s)$
$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = sF(s)-f(0)$	$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s), \quad n=1,2,3,\dots$
$\mathcal{L}\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s)-sf(0)-f'(0)$	$\mathcal{L}\left[f\left(\frac{t}{a}\right)\right] = aF(as)$
$\mathcal{L}\left[\int f(t)\,dt\right] = \frac{F(s)}{s}+\frac{1}{s}\left[\int f(t)\,dt\right]_{t=0}$	$\mathcal{L}\left[\int_0^t f_1(t-\tau)f(t)d\tau\right] = F_1(s)F_2(s)$
$\mathcal{L}[f(t-a)] = e^{-as}F(s)$	$\mathcal{L}[e^{-at}f(t)] = F(s+a)$

Matrix exponential

$$\boxed{e^{At} \triangleq \mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\}}$$

Open-loop and sensitivity functions

$$G_o(s) = G(s)F_y(s), \qquad S(s) = \frac{1}{1+G_o(s)}, \qquad T(s) = 1-S(s)$$

State-space forms and transfer function relations

- State-space form and transfer function

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad \Rightarrow \quad \boxed{G(s) = C(sI - A)^{-1}B + D}$$

- Associated matrices

$$\mathcal{S} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \qquad \mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

- LTI system with transfer function

$$\boxed{G(s) = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}}$$

i) Observable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ -a_3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ b_3 - a_3 b_0 \\ \vdots \\ b_n - a_n b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

ii) Controllable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_1 - a_1 b_0 & b_2 - a_2 b_0 & \cdots & b_n - a_n b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

- Solution to state-space equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

can be written as

$$\boxed{x(t) = e^{At}x_0 + \int_0^t e^{A\tau} Bu(t - \tau) d\tau}$$

- Observer system

$$\boxed{\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})}$$

Feedback control structures

General linear feedback in Laplace form:

$$\boxed{U(s) = F_r(s)R(s) - F_y(s)Y(s)}$$

Common control structures in this form.

- PID controller:

$$F_y(s) = F_r(s) = F(s) = K_p + \frac{K_i}{s} + K_d s,$$

  where $K_p, K_i, K_d \geq 0$

- Lead-lag controller:

$$F_y(s) = F_r(s) = F(s) = K \left(\frac{\tau_D s + 1}{\beta \tau_D s + 1} \right) \left(\frac{\tau_I s + 1}{\tau_I s + \gamma} \right),$$

  where $K, \tau_D, \tau_I > 0$ and $0 \leq \beta, \gamma < 1$

- State-feedback controller with observer:

$$\begin{aligned} F_r(s) &= \left(1 - L(sI - A + KC + BL)^{-1}B\right)\ell_0 \\ F_y(s) &= L(sI - A + KC + BL)^{-1}K \end{aligned}$$

Discrete-time state-space forms

A continuous time system with zero-order-hold input signal and sample period T can be written in discrete-time as:

$$\boxed{\begin{aligned} x(k+1) &= Fx(k) + Gu(k) \\ y(k) &= Hx(k) \end{aligned}}$$

where

$$\begin{aligned} F &= e^{AT} \\ G &= \int_{\tau=0}^{T} e^{A\tau} d\tau B = \lceil \text{if } A^{-1} \text{ exists} \rceil = A^{-1}(e^{AT} - I)B \\ H &= C \end{aligned}$$

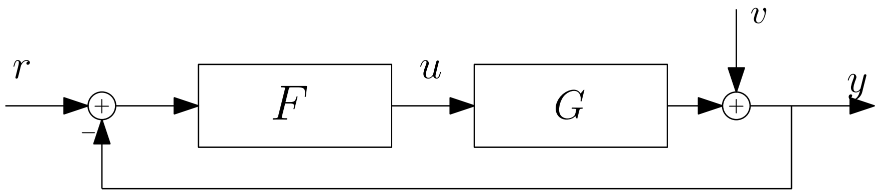
Problem 1: basic questions (6/30)

Answer only *‘true’* or *’false’*. Each correct answer gives 1 point, each wrong answer gives -1 point. Minimum total points for Part A and B is 0 , respectively.

Part A

Note: Write ‘skip’ if your total home assignment score ≥ 8

i) Consider the system illustrated in the figure below. The transfer function from the disturbance v to the output y is called the ‘complementary sensitivity function’.



ii) The following system description is a ‘minimal realization’

$$\dot{x} = \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

iii) The following system

$$G(s) = \frac{s^2 - s - 16}{s^2 + 2s - 15}$$

is not input-output stable.

(3 p)

Part B

Note: Write ‘skip’ if your total home assignment score ≥ 12

i) Consider the controller $U(s) = F(s)(R(s) - Y(s))$, where

$$F(s) = \frac{K_0 s^2 + K_1 s}{s}.$$

The controller corresponds to a PD-controller.

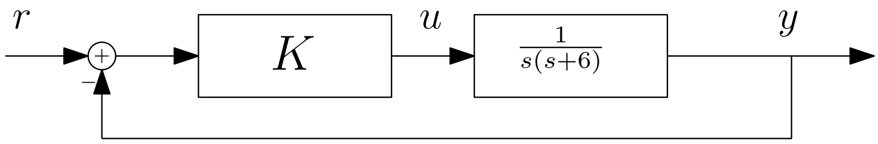
ii) Suppose $G_c(i\omega)$ has a resonance peak at $\omega_0 = 100$ [rad/s], then the bandwidth of the closed-loop system is 100 [rad/s].

iii) The static gain $G_c(0)$ determines the accuracy of the closed-loop system from r to y.

(3 p)

Problem 2 (6/30)

a) We consider controlling a tank system with a P-controller as illustrated in the figure below.



Determine $G_c(s)$ and its poles.

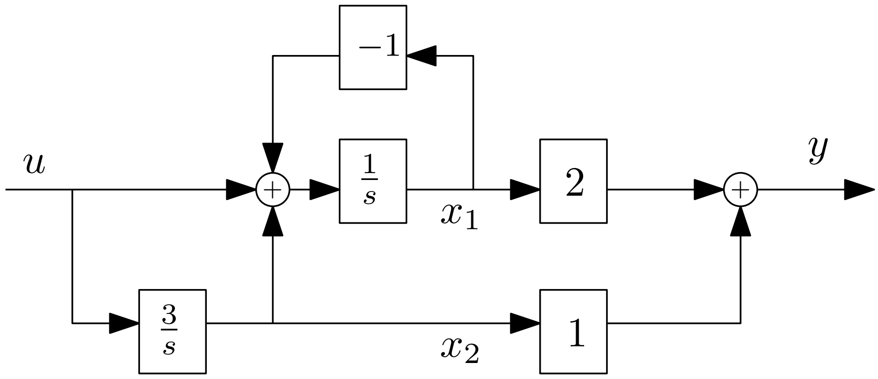
(3 p)

b) For which values of K is the closed-loop system stable?

(3 p)

Problem 3 (6/30)

a) Consider a dynamical system with interacting states, as depicted below. Derive the state-space description of the system.



(3 p)

b) Verify that the system is observable.

(1 p)

b) Determine an observer $\hat{x}$ such that the observer poles are located at -2 and -4, respectively.

(2 p)

Problem 4 (6/30)

A control student investigating a (stable) system $G(s)$ is using the control law $U(s) = F(s)E(s)$. When trying with the proportional controller $F(s) = 1$, the Nyquist curve of the loop gain becomes according to Figure 1 below.

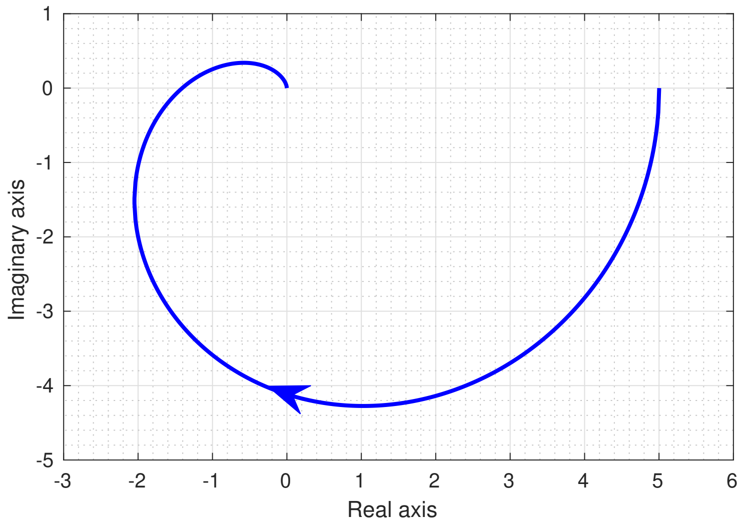


Figure 1: Nyquist curve.

a) Determine the closed-loop static gain $G_c(0)$.

(1 p)

b) In Figure 2, four different pole-zero plots are shown. Poles are indicated by crosses and zeros by rings. One of the plots corresponds to $G_c(s)$. Which one is it? Clearly motivate your answer.

(2 p)

c) The student changes to a PI-controller, $F(s) = K(1 + \frac{\tau}{s})$ where $K, \tau > 0$. Then K and τ are tuned until the step response of the closed loop system oscillates with constant amplitude and frequency $\omega = \tau$. Which value of K was found?

(3 p)

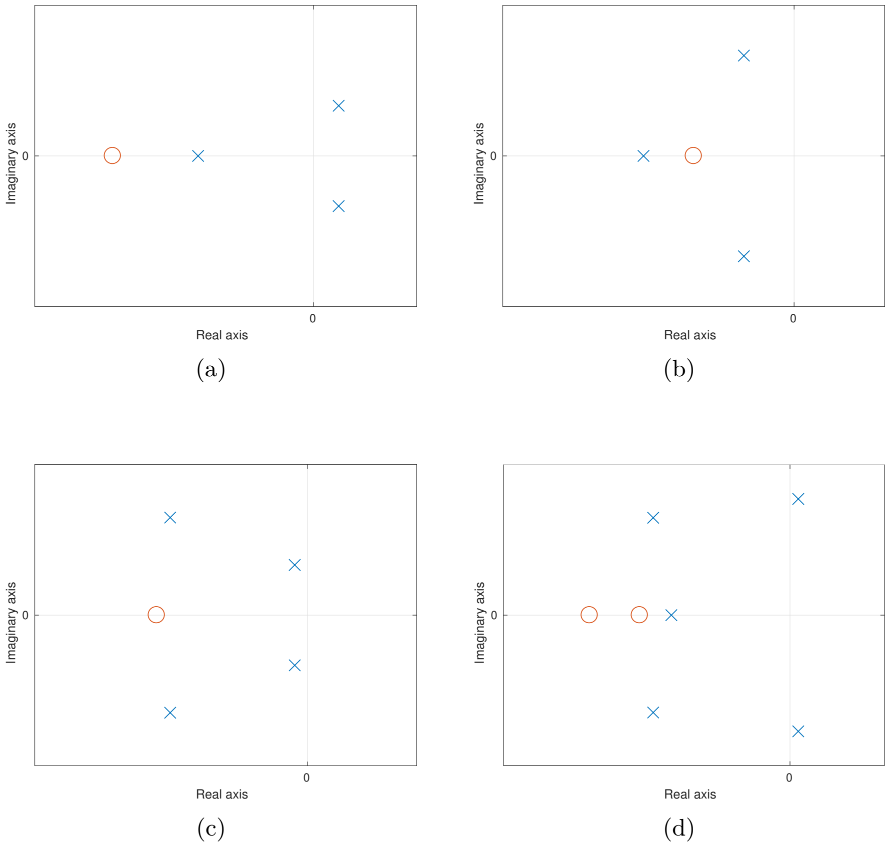


Figure 2: Pole-zero plots.

Proposed solution to problem 4

a) With $F(s) = 1$ we have $G_o(s) = F(s)G(s) = G(s)$. From the Nyquist plot we find $G(0) = 5$, hence

$$G_c(0) = \frac{G(0)}{1 + G(0)} = \frac{5}{6}.$$

b) With $F(s) = 1$, $G_c(s)$ will have the same number of zeros and poles as $G_o(s)$. Since the Nyquist curve goes to the origin from above, we have that $\arg G_o(i\omega) \to -270°$ as $\omega \to \infty$. Hence $G_o(s)$ – and $G_c(s)$ – must have three more poles than zeros, which excludes plot (a) and (b). Furthermore, it is seen that the Nyquist curve encloses -1, which means that the closed-loop system is unstable and must have poles in the right half plane. This excludes plot (c), which leaves plot (d) as the correct answer.

c) Oscillations with constant amplitude means that the system is marginally stable – and since the frequency $\omega = \tau$, we must have $G_o(i\tau) = -1$. Thus, it holds that

$$\begin{aligned} |G_o(i\tau)| &= |G(i\tau)||F(i\tau)| = 1, \\ \arg G_o(i\tau) &= \arg G(i\tau) + \arg F(i\tau) = -180°. \end{aligned}$$

Note that

$$\begin{aligned} \arg F(i\omega) &= \arg K + \arg \left(1 - \frac{\tau}{\omega} i\right) = 0 - \arctan \frac{\tau}{\omega} = -\arctan \frac{\tau}{\omega}, \\ |F(i\omega)| &= |K||(1 - \frac{\tau}{\omega} i)| = K\sqrt{1 + \left(\frac{\tau}{\omega}\right)^2}. \end{aligned}$$

Now, observe that $\arg F(i\tau) = -\arctan 1 = -45°$. Hence, the controller has reduced the argument of $G(i\omega)$ with $45°$ at $\omega = \tau$. In our Nyquist curve, the frequency $\omega = \tau$ must thus correspond to the point where $\arg G(i\omega) = -135°$. Identifying this point we find $|G(i\tau)| = 2\sqrt{2}$. Hence we get

$$1 = |G(i\tau)||F(i\tau)| = 2\sqrt{2}K\sqrt{1 + 1^2} = 4K,$$

and we deduce that $K = 1/4 = 0.25$.

Problem 5 (6/30)

An industrial robot is fixed in a certain spot on the factory floor, but can be turned around on the spot to perform different tasks. A DC-motor, taking a voltage as input, is used to turn the robot around to a desired angle (the output). A DC-motor can be described by the following continuous-time state space model:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

a) Discretize the state-space model (using zero order hold) with sampling time $T > 0$.

(2 p)

b) For what values of T is the discretized system a minimal realization?

(1.5 p)

c) Determine the poles of the discretized system. Is it stable?

(1 p)

d) Assume $T = \log(2)$, where log is the natural logarithm. We can only measure the output, so we need an observer in order to be able to estimate the states of the discrete-time system. Compute the observer gain K for the discrete-time system in a) so that the *continuous time* observer poles are located in -2.

(1.5 p)

Proposed solution to problem 5

a) We use that for ZOH the discretized system is given by

$$F = e^{AT}, \quad G = \int_0^T e^{A\tau} B d\tau, \quad H = C$$

where the matrix exponential can be computed as:

$$e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}]$$

The inverse of $(sI - A)$ for the given system is

$$(sI - A)^{-1} = \left[\begin{array}{cc} s & -1 \\ 0 & s+1 \end{array}\right]^{-1} = \left[\begin{array}{cc} \frac{1}{s} & \frac{1}{s(s+1)} \\ 0 & \frac{1}{s+1} \end{array}\right]$$

Taking the inverse Laplace transform yields

$$F = e^{AT} = \left[\begin{array}{cc} 1 & 1 - e^{-T} \\ 0 & e^{-T} \end{array}\right]$$

Now,we compute G, taking into account that the integral operator can go inside each component of the resulting matrix

$$G = \int_0^T \left[\begin{array}{cc} 1 & 1 - e^{-\tau} \\ 0 & e^{-\tau} \end{array}\right] \left[\begin{array}{c} 0 \\ 1 \end{array}\right] d\tau = \int_0^T \left[\begin{array}{c} 1 - e^{-\tau} \\ e^{-\tau} \end{array}\right] d\tau = \left[\begin{array}{c} T - 1 + e^{-T} \\ 1 - e^{-T} \end{array}\right]$$

b) To determine if the discretized system is a minimal realization we need to check that it is both controllable and observable. This is equivalent to checking if the controllability and observability matrices respectively have non-zero determinants. The controllability matrix is given by

$$S = \left[\begin{array}{cc} G & FG \end{array}\right] = \left[\begin{array}{cc} T - 1 + e^{-T} & T - e^{-T} + e^{-2T} \\ 1 - e^{-T} & e^{-T} - e^{-2T} \end{array}\right].$$

The determinant is

$$\begin{aligned} \det(S) &= (T - 1 + e^{-T})(e^{-T} - e^{-2T}) - (1 - e^{-T})(T - e^{-T} + e^{-2T}) \\ &= 2Te^{-T} - Te^{-2T} - T = -T(e^{-T} - 1)^2 \end{aligned}$$

hence it is non-zero if $T \neq 0$. The observability matrix is given by

$$O = \left[\begin{array}{c} H \\ HF \end{array}\right] = \left[\begin{array}{cc} 1 & 0 \\ 1 & 1 - e^{-T} \end{array}\right].$$

The determinant is

$$\det(O) = 1 - e^{-T}$$

which is also non-zero if $T \neq 0$.

c) The poles of the system are given by

$$\det(\lambda I - F) = \det \begin{pmatrix} \lambda - 1 & e^{-T} - 1 \\ 0 & \lambda - e^{-T} \end{pmatrix} = (\lambda - 1)(\lambda - e^{-T}) = 0.$$

Hence the poles are $\lambda_1 = e^{-T}$ and $\lambda_2 = 1$. The poles of a discretized system are stable if they are located inside the unit circle, that is is $|\lambda| < 1$. The first pole will always be stable since $T > 0$ implies $e^{-T} < 1$. The second pole however is marginally stable. Hence the system is marginally stable.

d) Inserting $T = \log(2)$ in the matrices computed in a) yields

$$F = \begin{bmatrix} 1 & 0.5 \\ 0 & 0.5 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

where K is the gain matrix. The characteristic polynomial for the observer is

$$\begin{aligned} &\det(\lambda I - F + KH) = \det \begin{bmatrix} \lambda - 1 + k_1 & -0.5 \\ k_2 & \lambda - 0.5 \end{bmatrix} \\ &= (\lambda - 1 + k_1)(\lambda - 0.5) + \frac{k_2}{2} \\ &= \lambda^2 + \lambda(-1.5 + k_1) + 0.5 - \frac{k_1}{2} + \frac{k_2}{2} \end{aligned}$$

Since the desired location for the poles in continuous time is $s = -2$, the poles of the corresponding discrete-time system should be located in $p = e^{sT} = e^{-2\log(2)} = 0.25$. Hence the desired characteristic polynomial is

$$(\lambda - 0.25)^2 = \lambda^2 - 0.5\lambda + 0.0625$$

Comparison of the two characteristic polynomials yields

$$\begin{cases} k_1 = 1 \\ k_2 = 0.125 \end{cases}$$