Division of Computing Science
Department of Information Technology
Uppsala University

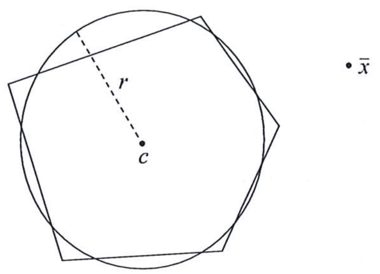
Written examination in 1TD184 Optimization

- Date: 2023-01-10, 08.00-13.00
- Allowed tools: Pocket calculator, one A4 paper with notes (computer typed, font size minimum 10 pt).
- Maximum number of points: 36 (18 to pass).
- All assumptions and answers *must* be motivated for full points.

(1) Describe the following concepts: 1) Gauss-Newton method, 2) Reduced gradient and reduced Hessian. (4p)

(2) We have a polytope defined by linear inequalities $Ax \leq b$, and a sphere with center c and radius r. See the figure for an illustration for two dimensions. We are also given a point $\bar{x}$ that is not in the polytope or the sphere. We would like to find the point in the intersection of the polytope and sphere with minimum distance to $\bar{x}$.

Note: For a) and b) below, explain clearly the variables, constraints, and objective function. Moreover, the mathematical models may not assume that we know the extreme points of the polytope, and they may not contain an infinite number of variables or constraints.



a) Formulate the optimization problem mathematically. (1p)

b) If the polytope is completely inside the sphere, then we can ignore the polytope and the task becomes much easier. Formulate an optimization problem, such that solving it tells whether or not the entire polytope is inside the sphere. (2p)

c) Examine if your models are convex. (2p)

(3) True or false (answers *must* be motivated):

a) For a one-dimensional function, the steepest descent and Newton directions will always be the same. (1p)

b) The step size found by Armijo line search with backtracking will be smaller than the optimal step size. (1p)

c) For a minimization problem, the search direction generated in sequential quadratic programming (SQP) is a descent direction. (1p)

d) For a linear programming problem in standard form with n variables and m constraints, a basic feasible solution will never have more than m positive-valued variables. (1p)

(4) Consider function $f(x_1, x_2) = ax_1^2 + x_2^2 - x_1x_2$, where $a > 1$.

a) Find the stationary point and explain why this point is a minimum. (2p)

b) For $x_1 = -1$ and $x_2 = -1$, what is the search direction that points at the minimum? Is this the steepest descent direction? (2p)

c) If we would apply compass search at $x_1 = -1$ and $x_2 = -1$, which of the directions in this method are descent? (2p)

d) Derive the Newton direction[1] for any point $(x_1, x_2)^T$. Does the direction always point at the minimum? No matter the answer, explain the underlying reason. (2p)

e) In Quasi-Newton methods, there is a so called secant condition involving two points. Why is this condition imposed? Next, for point $(0, 0)^T$ and any other point $(x_1, x_2)^T$, formulate the corresponding secant condition for the above function. (2p)

(5) We have the following constrained optimization problem.

$$
\begin{aligned}
\min \quad & \frac{1}{2}(x_1 - 3)^2 + (x_2 + 2)^2 \\
& x_1 - x_2 \geq 0 \\
& -x_1 - x_2 \geq -4 \\
& x_2 \geq 0
\end{aligned}
$$

a) Consider point $(0, 0)^T$. Show that the point cannot be optimal. (2p)

b) There are two active constraints at $(0, 0)^T$. Consider an active-set method that keeps one of the constraints active. Which constraint is kept active? What would be the search direction? What is the optimal (and feasible) step size for this direction? (2p)

c) For the new point obtained by taking the step size, show it does satisfy the first-order necessary condition, and also show it is indeed a minimum. (2p)

d) Suppose we only consider the constraint kept active in b), and ignore the other constraints. Use the logarithmic barrier function to turn the problem into an unconstrained one. Does the optimum of the unconstrained problem approach the optimum in c) when the barrier parameter approaches zero? (2p)

(6) We have the following linear programming (LP) problem (in standard form) and its dual LP, where all the coefficients are strictly positive.

(Primal) LP	Dual
$\min \sum_{i=1}^{n} c_i x_i$	$\max\ by$
$\sum_{i=1}^{n} a_i x_i = b$	$a_i y \leq c_i, i = 1, 2, \ldots, n$
$x_i \geq 0, i = 1, 2, \ldots, n$	

a) For the primal LP, how many extreme points (vertices) exist? (1p)

b) For the primal LP, suppose we pick a variable x_k as the basic variable, under what condition is this an optimal basis? If it is not optimal, how many iterations would it take for the simplex method to find the optimum? (2p)

c) The dual LP has one single variable but multiple constraints. Derive the optimum solution of the dual. Can we use the dual LP optimum (instead of the simplex method) to find the optimum to the primal LP? (2p)

Good Luck!

[1] For a 2×2 matrix $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, its inverse matrix is given by $\frac{1}{\alpha\delta - \beta\gamma} \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix}$.