Exam in Automatic Control II
Reglerteknik II 5hp (1RT495)

Date: October 21, 2022
Time: 14:00 – 19:00
Venue: Fyrislundsgatan 80, sal 1

Responsible teacher: Hans Rosth
Aiding material: Calculator, mathematical handbooks, textbooks by Glad & Ljung (Reglerteori/Control theory & Reglerteknik). Additional notes in the textbooks are allowed.

Preliminary grades: 23p for grade 3, 33p for grade 4, 43p for grade 5.

Use separate sheets for each problem, i.e. no more than one problem per sheet. Write your exam code on every sheet.

Important: Your solutions should be well motivated unless else is stated in the problem formulation! Vague or lacking motivations may lead to a reduced number of points.
Problem 6 is an alternative to the homework assignments from the autumn semester 2022. (In case you choose to hand in a solution to Problem 6 you will be accounted for the best performance of the homework assignments and Problem 6.)

Good luck!

Laplacetransformtabell

Definition:
För en reellvärd funktion $f(t)$, definierad för $t \geq 0$, ges Laplacetransformen av

$$\mathcal{L}[f(t)] = \int_{0_-}^{\infty} f(t)e^{-st}dt, \quad s \in \mathbb{C}$$

Det är en konvention att använda gemener[1] för tidsfunktioner, och versaler[2] för Laplacetransformer. T.ex. betecknar man Laplacetransformen av $f(t)$ med $F(s)$, d.v.s. $F(s) = \mathcal{L}[f(t)]$. I strikt mening existerar bara Laplacetransformen för $s \in \mathbb{C}$ sådana att integralen i definitionen konvergerar. Om integralen konvergerar för $s = a \in \mathbb{C}$, så konvergerar den för alla s sådana att $\operatorname{Re} s \geq \operatorname{Re} a$. Den *inversa* Laplacetransformen ges då av

$$f(t) = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} e^{st}F(s)ds \quad \text{för} \quad t \geq 0$$

där $\alpha \in \mathbb{R}$ och $\alpha \geq \operatorname{Re} a$.

Operationslexikon

Nr	Laplacetransform	Funktion i tidsplanet	
1	$F(s)$	$f(t)$	
2	$F(s+a)$	$e^{-at}f(t)$	Dämpningssatsen
3	$e^{-as}F(s)$	$\begin{cases} f(t-a) & t-a>0 \\ 0 & t-a<0 \end{cases}$	Förskjutningssatsen
4	$\frac{1}{a}F\left(\frac{s}{a}\right)$	$f(at)$	Töjning
5	$F(as)$	$\frac{1}{a}f\left(\frac{t}{a}\right)$	
6	$\frac{d^nF(s)}{ds^n}$	$(-t)^n f(t)$	Derivering i s-planet
7	$\int_s^{\infty} F(\sigma)d\sigma$	$\frac{f(t)}{t}$	Integration i s-planet
8	$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F_1(\sigma)F_2(s-\sigma)d\sigma$	$f_1(t)f_2(t)$	Faltning i s-planet
9	$F_1(s)F_2(s)$	$\int_{0_-}^{t} f_1(\tau)f_2(t-\tau)d\tau$	Faltning i tidsplanet
10	$sF(s) - f(0_-)$	$\frac{df(t)}{dt}$	Derivering i tidsplanet
11	$s^2F(s) - \left[sf(0_-) + \frac{d}{dt}f(0_-)\right]$	$\frac{d^2f(t)}{dt^2}$	
12	$s^nF(s) - \left[s^{n-1}f(0_-) + \cdots + \frac{d^{n-1}}{dt^{n-1}}f(0_-)\right]$	$\frac{d^nf(t)}{dt^n}$	
13	$\frac{1}{s}F(s)$	$\int_{0_-}^{t} f(\tau)d\tau$	Integration i tidsplanet
14	$\lim\limits_{s\to 0} sF(s)$	$\lim\limits_{t\to\infty} f(t)$	Slutvärdesteoremet
15	$\lim\limits_{s\to\infty} sF(s)$	$\lim\limits_{t\to 0_+} f(t)$	

[1]Små bokstäver.
[2]Stora bokstäver.

Transformlexikon

Nr	Laplacetransform	Funktion i tidsplanet	
1	$F(s)$	$f(t)$	Notation
2	1	$\delta(t)$	Diracpuls
3	$\frac{1}{s}$	1	Konstant eller stegfunktion
4	$\frac{1}{s^2}$	t	Rampfunktion
5	$\frac{1}{s^3}$	$\frac{1}{2}t^2$	Acceleration
6	$\frac{1}{s^{n+1}}$	$\frac{1}{n!}t^n$	
7	$\frac{1}{s+a}$	e^{-at}	Exponentialfunktioner
8	$\frac{1}{(s+a)^2}$	te^{-at}	
9	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$	
10	$\frac{1}{1+as}$	$\frac{1}{a}e^{-\frac{t}{a}}$	
11	$\frac{1}{s(s+a)}$	$\frac{1}{a}(1-e^{-at})$	
12	$\frac{1}{s(1+as)}$	$1-e^{-\frac{t}{a}}$	
13	$\frac{a}{s^2-a^2}$	$\sinh at$	Hyperboliska funktioner
14	$\frac{s}{s^2-a^2}$	$\cosh at$	
15	$\frac{a}{s^2+a^2}$	$\sin at$	Trigonometriska funktioner
16	$\frac{s}{s^2+a^2}$	$\cos at$	
17	$\frac{a}{(s+b)^2+a^2}$	$e^{-bt}\sin at$	Dämpade trig.funktioner
18	$\frac{s+b}{(s+b)^2+a^2}$	$e^{-bt}\cos at$	

Om man vill bestämma den inversa Laplacetransformen för en rationell funktion med nämnarpolynom av gradtal högre än två kan man utnyttja att Laplacetransformen är linjär, genom att använda partialbråksuppdelning. Man kan sedan använda transformlexikonet och inverstransformera varje term för sig.

Exempel:

$$F(s) = \frac{2}{s(s+1)(s+2)} = \frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2} \iff f(t) = \mathcal{L}^{-1}[F(s)] = 1 - 2e^{-t} + e^{-2t}$$

Please use English in your solutions!

Problem 1 The function of a hot air balloon is based on the fact that warm air has lower density than cold air. A simple model of the vertical motion of a hot air balloon, i.e. the change of altitude, is as follows[1]:

i. The rate of the change in altitude, i.e. the vertical speed, denoted with $r(t)$, is modelled as

$$\frac{dr}{dt} = \frac{1}{\tau_1}(w(t) - r(t)) + k\theta(t), \quad k > 0,$$

where $w(t)$ is the vertical wind speed, and $\theta(t)$ is the temperature of the air in the balloon. Here $\tau_1 > 0$ is a time constant.

ii. The temperature is modelled as

$$\frac{d\theta}{dt} = -\frac{1}{\tau_2}\theta(t) + u(t),$$

where $u(t)$ is the amount of heat produced by the burner. Again, $\tau_2 > 0$ is a time constant.

iii. We assume that the vertical wind speed is slowly varying and described by

$$w(t) = \frac{1}{p+\delta}v(t), \qquad \delta > 0,$$

where $v(t)$ is zero mean white noise with intensity $\Phi_v(\omega) = R_v$.

iv. Finally, the altitude $h(t)$ is governed by $\frac{dh}{dt} = r(t)$.

(a) Introduce the state vector $x(t) = \begin{bmatrix}\theta(t) & r(t) & h(t) & w(t)\end{bmatrix}^T$ and give the state space model in the form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Nv(t), \\ z(t) = Mx(t), \end{cases}$$

i.e. determine the matrices and vectors A, B, N and M. We regard

$$z(t) = \begin{bmatrix}z_1(t) & z_2(t)\end{bmatrix}^T = \begin{bmatrix}r(t) & h(t)\end{bmatrix}^T$$

as the output/performance variable. **(5p)**

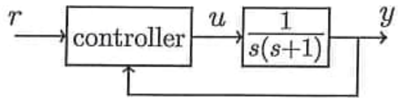
(b) Assume that $u(t) = 0$. Show that $z_1(t) = r(t)$ *is* a stationary stochastic process, but that $z_2(t) = h(t)$ is *not*. **(2p)**

(c) Assume again that $u(t) = 0$. Determine the spectral density, $\Phi_r(\omega)$, of $z_1(t) = r(t)$. **(2p)**

[1]All signals represent deviations from an operating point.

Please use English in your solutions!

Problem 2 The block diagram below shows the implementation af a position servo, based on feedback control of a DC motor. The DC motor is described by the state space model (1).



$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t), \\ y(t) = \begin{bmatrix} -1 & 1 \end{bmatrix} x(t). \end{cases} \tag{1}$$

Initially a continuous-time controller was used, with the control law

$$u(t) = K(r(t) - y(t)), \qquad K \in \mathbb{R}. \tag{2}$$

(a) For which $K \in \mathbb{R}$ is the closed loop system stable when the continuous-time controller (2) is used? **(1p)**

Due to malfunction the original continuous-time controller had to be replaced, and a sampling controller was employed instead. The control law then changed to

$$u(t) = u(kh) \quad \text{for} \quad kh \le t < kh + h, \quad k \in \mathbb{Z},$$
$$\text{and} \quad u(kh) = K(r(kh) - y(kh)). \tag{3}$$

Here h is the sampling period, and $K \in \mathbb{R}$ is a pure gain.

(b) Determine the zero-order-hold sampled, discrete-time version of the state space model (1). **(3p)**

For a certain choice of sampling period the discrete-time representation in (b) will correspond to the following model of the system:

$$Y(z) = G(z)U(z), \qquad G(z) = \frac{(2\alpha - 1)z + 1 - \alpha - \alpha^2}{(z-1)(z-\alpha)}, \qquad \alpha \approx 0.567 \tag{4}$$

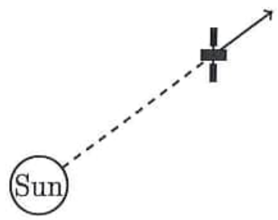
(c) For what values of $K \in \mathbb{R}$ is the closed loop system stable when the sampling controller (3) is used on (4)? **(3p)**

(d) Assume that (3) is used on (4), and that K is chosen so that the closed loop system is stable. What is then the static gain from reference r to output y? **(1p)**

Please use English in your solutions!

Problem 3 A system of radio telescopes is used for observations of an interstellar space probe on its way out of the solar system. The space probe is travelling along a straight line, on a radius starting in the sun, see the figure below. Let $z(t)$ represent the deviation from the calculated trajectory of the space probe. This deviation is modeled with Newton's second law, and is here represented by the state space model (5).

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v_1(t), \\ z(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t), \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + v_2(t), \end{cases} \tag{5}$$



The process noise, v_1, is caused for example by fluctuations in the solar wind, and is considered as a zero mean white noise process with intensity $R_1 = 1$. The measurement noise, v_2, is also considered as zero mean white noise with intensity $R_2 = 1$, and it is uncorrelated with v_1.

Based on the observations the position (in terms of $z(t)$) is estimated with an observer:

$$\dot{\hat{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \left(y(t) - \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{x}(t) \right) \tag{6}$$

(a) Give the state equation for the estimation error $\tilde{x}(t) = x(t) - \hat{x}(t)$, based on (5) and (6). **(2p)**

(b) Compute the covariance matrix $\Pi_{\tilde{x}} = E\tilde{x}\tilde{x}^T$ when the observer gain $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ is used in (6). **(3p)**

(c) What is the smallest value $\Pi_{\tilde{x}}$ can attain, that is, what is the covariance matrix of $\tilde{x}$ for the optimal observer? **(4p)**

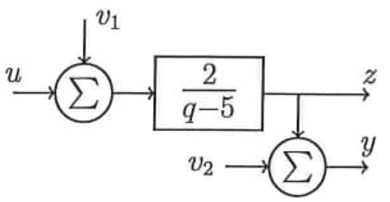
Problem 4 Specify for each of the following statements whether it is true or false. No motivations required — only answers "true"/"false" are considered!

(a) $\Phi(\omega) = \frac{\omega^2}{\omega^4 - 4}$ is the spectrum of a continuous-time stochastic process.
(b) For white noise the covariance function is $r(\tau) = 0$ for $\tau \neq 0$.
(c) A Kalman filter is always stable.
(d) The Nyquist frequency is 50% of the sampling frequency.
(e) MPC controllers are always linear and time invariant.
(f) In MPC the *prediction horizon* > the *control horizon* should hold.

Each correct answer scores +1, each incorrect answer scores −1, and omitted answers score 0 points. (Minimal total score is 0 points.) **(6p)**

Please use English in your solutions!

Problem 5 Consider the following block diagram associated with a discrete-time system.



The system is affected by process and measurement noise denoted by v_1 and v_2, respectively. The noises v_1 and v_2 are uncorrelated zero mean white noises with respective intensities $R_1 = 1$ and $R_2 = 2$.

We seek to control the system by the standard LQG controller

$$u(k) = -L\hat{x}(k|k-1), \tag{7}$$

that minimizes the cost function

$$V = E[z^2 + \rho u^2], \qquad \rho \geq 0.$$

(a) Determine the matrices that describe the state-space representation of the system depicted above in the following standard form

$$\begin{cases} x(k+1) = Fx(k) + Gu(k) + Nv_1(k), \\ z(k) = Mx(k), \\ y(k) = Hx(k) + v_2(k). \end{cases} \tag{8}$$

(2p)

For questions (b)–(d), use your results from (a). In case you have not determined the matrices in (a) you may (for simplicity) consider these to be $F = G = N = M = H = 1$.

(b) Determine the control gain L in (7). **(3p)**

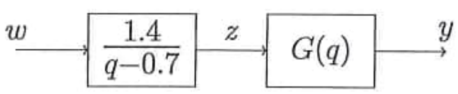
(c) Determine the Kalman filter for obtaining the estimate $\hat{x}(k|k-1)$. **(3p)**

(d) Determine the covariance $\Pi_{\hat{x}}$ associated with $\hat{x}(k|k-1)$ when (7) is used on (8). **(3p)**

Problem 6 *The HW bonus points (from the autumn 2022) are exchangeable for this problem.*

The block diagram below represents a stationary discrete-time stochastic process. The transfer operator $G(q)$ is minimum phase (with $G(1) \geq 0$), and w is zero mean white noise.

$\Phi_w(\omega) = 0.5$



$\Phi_y(\omega) = \frac{98}{1.64 - 1.6\cos\omega}$

(a) Determine the spectral density for z. **(3p)**

(b) Determine the transfer operator $G(q)$. **(4p)**

A short glossary of automatic control

ENGELSKA	SVENSKA
automatic control	reglerteknik
bandwidth	bandbredd
closed loop system	slutet system, återkopplat system
complementary sensitivity function	komplementär känslighetsfunktion
control engineering	reglerteknik
control system	reglersystem
control variable	styrsignal
controllable	styrbar
controller canonical form	styrbar kanonisk form
covariance	kovarians
cross-over frequency	skärfrekvens
dampen	dämpa
damping ratio	relativ dämpning
decoupled	frikopplad
describing function	beskrivande funktion
disturbance	störning, störsignal
eigenvalue	egenvärde
estimate	skattning
estimation error	estimeringsfel, skattningsfel
error	fel, reglerfel
expectation	väntevärde
expected value	väntevärde
feedback	återkoppling
feedforward	framkoppling
frequency	frekvens
gain	förstärkning
gain margin	amplitudmarginal
impulse response	impulssvar
input	insignal
lag filter	fasretarderande länk
lead filter	fasavancerande länk
limit cycle	(stabil) självsvängning
linear	linjär
loop gain	kretsförstärkning
lower limit	nedre gräns
mean	väntevärde, medelvärde
minimum phase	minfas
multivariable	flervariabel
noise	brus
observable	observerbar
observer	observatör
observer canonical form	observerbar kanonisk form
open loop system	öppet system
output	utsignal
overshoot	översläng
pole	pol
phase margin	fasmarginal
reference signal	referenssignal
rise time	stigtid
root locus	rotort
scalar	skalärt
sensitivity function	känslighetsfunktion
setpoint	börvärde, referenssignal
settling time	insvängningstid
singular value	singulärt värde
singular value decompostion	singulärvärdesuppdelning
small gain theorem	lågförstärkningssatsen
spectral density	spektraltäthet
stable	stabil
state	tillstånd
state feedback	tillståndsåterkoppling
state space	tillståndsrum
state space description	tillståndsbeskrivning
state space form	tillståndsform
static gain	statisk förstärkning
step response	stegsvar
time delay	tidsfördröjning
time-invariant	tidsinvariant
transfer function	överföringsfunktion
transient response	transientsvar
unstable	instabil
variance	varians
weighting function	viktfunktion
white noise	vitt brus
zero	nollställe