Equations for system

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We follow [1] very very closely.

Domain

Our domain Ω is $d = 2$ or 3-dimensional, and partitioned into $\Omega_f$and Ω_p, with $\Gamma_{fp} = \Omega_f \cap \Omega_p$ being the $(d-1)$-dimensional interface. The boundary $\partial\Omega$ is partitioned into $\Gamma_f = \partial\Omega \cap \partial\Omega_f$ and $\Gamma_p = \partial\Omega \cap \partial\Omega_p$. We assume each region is connected, reasonably smooth and all that.

Unknowns

The unknowns of the system and the corresponding test functions are:

- $\mathbf{u}_f$, $\mathbf{v}_f$: free flow fluid velocity. Defined on Ω_f.
- $\mathbf{u}_p$, $\mathbf{v}_p$: porous flow fluid velocity. Defined on Ω_p.
- p_f, w_f: free flow fluid pressure. Defined on Ω_f.
- p_p, w_p: porous flow fluid pressure. Defined on Ω_p.
- $\boldsymbol{\eta}_p$, $\boldsymbol{\xi}_p$: displacement. Defined on Ω_p.
- λ_Γ, μ_Γ: normal stress balance Lagrange multiplier. Defined on Γ_{fp}. In [1], denoted λ, μ_h.

Parameters

μ_f fluid viscosity (denoted μ in the [1])

λ_p, μ_p Lamé parameters. Denoted μ in [1].

α Biot-Willis constant

K Permeability tensor. Symmetric, bounded, positive definite. I take it to be scalar.

α_{BJS} Friction coefficient

Notation

- $\mathbf{n}_f$, $\mathbf{n}_p$are the outward unit normal vectors to $\partial\Omega_f, \partial\Omega_p$.
- $\boldsymbol{\tau}_{f,j}, j = 1, \dots, d-1$ is an orthogonal system of unit tangent vectors at Γ_{fp}.
- $\mathbf{D}(\mathbf{v}) := \frac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^T)$
- $\boldsymbol{\sigma}_f(\mathbf{u}_f, p_f) := -p_f\mathbf{I} + 2\mu_f\mathbf{D}(\mathbf{u}_f)$
- $\boldsymbol{\sigma}_p(\boldsymbol{\eta}_p, p_p) := \lambda_p(\nabla\cdot\boldsymbol{\eta}_p) + 2\mu_p\mathbf{D}(\boldsymbol{\eta}_p) - \alpha p_p\mathbf{I}$
- $a_f(\mathbf{u}_f, \mathbf{v}_f) = (2\mu\mathbf{D}(\mathbf{u}_f), \mathbf{D}(\mathbf{v}_f))_{\Omega_f}$
- $a_p^d(\mathbf{u}_p, \mathbf{v}_p) = \left(\mu K^{-1}\mathbf{u}_p, \mathbf{v}_p\right)_{\Omega_p}$
- $a_p^e(\boldsymbol{\eta}_p, \boldsymbol{\xi}_p) = \left(\mu_p\mathbf{D}(\boldsymbol{\eta}_p), \mathbf{D}(\boldsymbol{\xi}_p)\right)_{\Omega_p} + \left(\lambda_p\nabla\cdot\boldsymbol{\eta}_p, \nabla\cdot\boldsymbol{\xi}_p\right)_{\Omega_p}$

Strong formulation

These are all ignoring body force and source terms. So it's okay to put stuff on the right hand sides if you want to.

Stokes (applies in Ω_f):

$$-\nabla\cdot\boldsymbol{\sigma}_f(\mathbf{u}_f, p_f) = 0 \tag{1a}$$
$$\nabla\cdot\mathbf{u}_f = 0 \tag{1b}$$

Darcy (applies in Ω_p):

$$\mathbf{u}_p = -\frac{K}{\mu}\nabla p_p \tag{2}$$

Biot (applies in Ω_p):

$$-\nabla\cdot\boldsymbol{\sigma}_p(\boldsymbol{\eta}_p, p_p) = 0 \tag{3a}$$
$$\frac{\partial}{\partial t}\left(s_0 p_p + \alpha\nabla\cdot\boldsymbol{\eta}_p\right) + \nabla\cdot\mathbf{u}_p = 0 \tag{3b}$$

Interface conditions

Conservation of mass:

$$\mathbf{u}_f\cdot\mathbf{n}_f + \left(\frac{\partial\boldsymbol{\eta}_p}{\partial t} + \mathbf{u}_p\right)\cdot\mathbf{n}_p = 0 \tag{4}$$

Balance of stress (will be enforced using λ_Γ):

$$-(\boldsymbol{\sigma}_f\mathbf{n}_f)\cdot\mathbf{n}_f = p_p, \quad \boldsymbol{\sigma}_f\mathbf{n}_f + \boldsymbol{\sigma}_p\mathbf{n}_p \; = 0, \tag{5}$$

BJS condition:

$$-(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \boldsymbol{\tau}_j = \frac{\mu_f \alpha_{BJS}}{\sqrt{K}} \left(\mathbf{u}_f - \frac{\partial \boldsymbol{\eta}_p}{\partial t} \right) \cdot \boldsymbol{\tau}_j \tag{6}$$

Pressure difference across vessel wall:

$$p_f - p_p = C \tag{7}$$

[1] does not use condition 7.

Boundary conditions

In their numerical experiment (section 7.2), [1] use the domain shown in figure The Darcy boundary $\Gamma_p$is partitioned into the left part Γ_p^{L} and the remainder

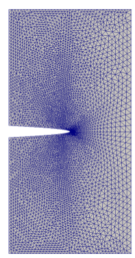

Figure 1: Darcy domain from [1]. Stokes domain is the removed 'finger'.

$\Gamma_p^{\neg\mathrm{L}}$ in the obvious way. Physically, I think Γ_p^{L} is above ground and the other part is below ground or something.

As boundary conditions, they use:

- $\mathbf{u}_f = 10\mathbf{n}_f$ on Γ_f
- $\mathbf{u}_p \cdot \mathbf{n}_p = 0$ on Γ_p^{left}
- $p_p = 1000$ on $\Gamma_p^{\neg\mathrm{left}}$
- $\boldsymbol{\eta}_p \cdot \mathbf{n}_p = 0$ on $\Gamma_p^{\neg\mathrm{left}}$
- $(\boldsymbol{\sigma}_p \mathbf{n}_p) \cdot \boldsymbol{\tau}_p = 0$ on $\Gamma_p^{\neg\mathrm{left}}$

Variational formulation

Having used a backward Euler discretization of the time derivative, [1] obtain the following variational formulation

$$
\begin{aligned}
a_f(\mathbf{u}_f, \mathbf{v}_f) + a_p^d(\mathbf{u}_p, \mathbf{v}_p) + a_p^e(\boldsymbol{\eta}_p, \boldsymbol{\xi}_p) + a_{BJS}\left(\mathbf{u}_f, \frac{\boldsymbol{\eta}_p}{\Delta t}; \mathbf{v}_f, \boldsymbol{\xi}_p\right) & \qquad (8a)\\
+ b_f(\mathbf{v}_f, p_f) + b_p(\mathbf{v}_p, p_p) + \alpha b_p(\boldsymbol{\xi}_p, p_p) &\\
+ b_\Gamma(\mathbf{v}_f, \mathbf{v}_p, \boldsymbol{\xi}_p; \lambda_\Gamma) &\\
= a_{BJS}\left(\mathbf{u}_f, \frac{\boldsymbol{\eta}_p^{n-1}}{\Delta t}; \mathbf{v}_f, \boldsymbol{\xi}_p\right) &\\
\left(s_0 \frac{p_p}{\Delta t}, w_p\right)_{\Omega_p} - \alpha b_p\left(\frac{\boldsymbol{\eta}_p}{\Delta t}, w_p\right) - b_p(\mathbf{u}_p, w_p) - b_f(\mathbf{u}_f, w_f) & \qquad (8b)\\
= \left(s_0 \frac{p_p^{n-1}}{\Delta t}, w_p\right)_{\Omega_p} - \alpha b_p\left(\frac{\boldsymbol{\eta}_p^{n-1}}{\Delta t}, w_p\right)_{\Omega_p} &\\
b_\Gamma\left(\mathbf{u}_f, \mathbf{u}_p; \boldsymbol{\eta}_p \Delta t\right) = b_\Gamma\left(\mathbf{u}_f, \mathbf{u}_p; \boldsymbol{\eta}_p^{n-1}\right) & \qquad (8c)
\end{aligned}
$$

Here the unknowns with no superscript mean the unknowns at time n (e.g. $\mathbf{u}_f = \mathbf{u}_f^n$).

Thoughts

- Kent offered very gentle scepticism about using a 3-field formulation, and suggested not having $\mathbf{u}_p$as an unknown, using $\mathbf{u}_p = \nabla p_p$ to remove it. I thought [1] had some opinion on this, but on closer reading I can't find it, so maybe that's from another article. I should read up on this.
- If I want to use 7, I need to drop an interface condition. The most reasonable thing to drop seem to be the first part of 5, because they both effectively specify p_p. But I don't like the BJS condition because it has too many letters, so I'd like an excuse to drop that instead.

References

[1] Ambartsumyan, Ilona, et al. , *"A Lagrange multiplier method for a Stokes-Biot fluid-poroelastic structure interaction model."* , arXiv preprint arXiv:1710.06750 (2017).