Equations for system

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We follow [1] very very closely.

Domain

Our domain Ω is $d = 2$ or 3-dimensional, and partitioned into Ω_f and Ω_p, with $\Gamma_{fp} = \Omega_f \cap \Omega_p$ being the $(d-1)$-dimensional interface. The boundary $\partial\Omega$ is partitioned into $\Gamma_f = \partial\Omega \cap \partial\Omega_f$ and $\Gamma_p = \partial\Omega \cap \partial\Omega_p$. We assume each region is connected, reasonably smooth and all that.

Unknowns

The unknowns of the system and the corresponding test functions are:

- $\mathbf{u}_f$, $\mathbf{v}_f$: free flow fluid velocity. Defined on Ω_f.
- $\mathbf{u}_p$, $\mathbf{v}_p$: porous flow fluid velocity. Defined on Ω_p.
- p_f, w_f: free flow fluid pressure. Defined on Ω_f.
- p_p, w_p: porous flow fluid pressure. Defined on Ω_p.
- $\boldsymbol{\eta}_p$, $\boldsymbol{\xi}_p$: displacement. Defined on Ω_p.
- λ_Γ, μ_Γ: normal stress balance Lagrange multiplier. Defined on Γ_{fp}. In [1], denoted λ, μ_h.

Parameters

μ_f fluid viscosity (denoted μ in the [1])

λ_p, μ_p Lamé parameters. Denoted μ in [1].

α Biot-Willis constant

K Permeability tensor. Symmetric, bounded, positive definite. I take it to be scalar.

α_{BJS} Friction coefficient

s_0 Storage coefficient

$B = \frac{\mu_f \alpha_{BJS}}{\sqrt{K}}$ The coefficient in the BJS condition. Could be zero.

Notation

- $\mathbf{n}_f$, $\mathbf{n}_p$ are the outward unit normal vectors to $\partial\Omega_f, \partial\Omega_p$.
- $\boldsymbol{\tau}_{f,j}, j = 1, \ldots, d-1$ is an orthogonal system of unit tangent vectors at Γ_{fp}.
- $\mathbf{D}(\mathbf{v}) := \frac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^T)$
- $\boldsymbol{\sigma}_f(\mathbf{u}_f, p_f) := -p_f\mathbf{I} + 2\mu_f\mathbf{D}(\mathbf{u}_f)$
- $\boldsymbol{\sigma}_p(\boldsymbol{\eta}_p, p_p) := \lambda_p(\nabla\cdot\boldsymbol{\eta}_p)\mathbf{I} + 2\mu_p\mathbf{D}(\boldsymbol{\eta}_p) - \alpha p_p\mathbf{I}$

Next, here are a bunch of bilinear forms used in the problem:

- $a_f(\mathbf{u}_f, \mathbf{v}_f) = (2\mu\mathbf{D}(\mathbf{u}_f), \mathbf{D}(\mathbf{v}_f))_{\Omega_f}$
- $a_p^d(\mathbf{u}_p, \mathbf{v}_p) = \left(\mu K^{-1}\mathbf{u}_p, \mathbf{v}_p\right)_{\Omega_p}$
- $a_p^e(\boldsymbol{\eta}_p, \boldsymbol{\xi}_p) = \left(\mu_p\mathbf{D}(\boldsymbol{\eta}_p), \mathbf{D}(\boldsymbol{\xi}_p)\right)_{\Omega_p} + \left(\lambda_p\nabla\cdot\boldsymbol{\eta}_p, \nabla\cdot\boldsymbol{\xi}_p\right)_{\Omega_p}$
- $b_f(\mathbf{v}_f, w_f) = -\left(\nabla\cdot\mathbf{v}_f, w_f\right)_{\Omega_f}$
- $b_p(\mathbf{v}_p, w_p) = -\left(\nabla\cdot\mathbf{v}_p, w_p\right)_{\Omega_p}$
- $a_{BJS}(\mathbf{u}_f, \boldsymbol{\eta}_p; \mathbf{v}_f, \boldsymbol{\xi}_p) = \frac{\mu\alpha_{BJS}}{\sqrt{K}}\sum_{j=1}^{d-1}\left((\mathbf{u}_f - \boldsymbol{\eta}_p)\cdot\boldsymbol{\tau}_j, (\mathbf{v}_f - \boldsymbol{\eta}_p)\cdot\boldsymbol{\tau}_j\right)_{\Gamma_{fp}}$
- $b_{\Gamma_{fp}}(\mathbf{v}_f, \mathbf{v}_p, \boldsymbol{\xi}_p; \mu_\Gamma) = b_{\Gamma_{fp}}^{\mathbf{v}_f}(\mathbf{v}_f, \mu_\Gamma) + b_{\Gamma_{fp}}^{\mathbf{v}_p}(\mathbf{v}_p, \mu_\Gamma) + b_{\Gamma_{fp}}^{\boldsymbol{\xi}_p}(\boldsymbol{\xi}_p, \mu_\Gamma)$
- $b_{\Gamma_{fp}}^{\mathbf{v}_f}(\mathbf{v}_f, \mu_\Gamma) = (\mathbf{v}_f\cdot\mathbf{n}_f, \mu_\Gamma)_{\Gamma_{fp}}$
- $b_{\Gamma_{fp}}^{\mathbf{v}_p}(\mathbf{v}_p, \mu_\Gamma) = (\mathbf{v}_p\cdot\mathbf{n}_p, \mu_\Gamma)_{\Gamma_{fp}}$
- $b_{\Gamma_{fp}}^{\boldsymbol{\xi}_p}(\boldsymbol{\xi}_p, \mu_\Gamma) = \left(\boldsymbol{\xi}_p\cdot\mathbf{n}_p, \mu_\Gamma\right)_{\Gamma_{fp}}$

Strong formulation

These are all ignoring body force and source terms. So it's okay to put stuff on the right hand sides if you want to.

Stokes (applies in Ω_f):

$$-\nabla \cdot \boldsymbol{\sigma}_f(\mathbf{u}_f, p_f) = 0 \tag{1a}$$

$$\nabla \cdot \mathbf{u}_f = 0 \tag{1b}$$

Darcy (applies in Ω_p):

$$\mathbf{u}_p = -\frac{K}{\mu}\nabla p_p \tag{2}$$

Biot (applies in Ω_p):

$$-\nabla \cdot \boldsymbol{\sigma}_p(\boldsymbol{\eta}_p, p_p) = 0 \tag{3a}$$

$$\partial_t \left(s_0 p_p + \alpha \nabla \cdot \boldsymbol{\eta}_p\right) + \nabla \cdot \mathbf{u}_p = 0 \tag{3b}$$

Interface conditions

Conservation of mass:

$$\mathbf{u}_f \cdot \mathbf{n}_f + \left(\partial_t \boldsymbol{\eta}_p + \mathbf{u}_p\right) \cdot \mathbf{n}_p = 0 \tag{4}$$

Balance of stress :

$$-(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \mathbf{n}_f = p_p + C_p = \lambda \tag{5}$$

$$\boldsymbol{\sigma}_f \mathbf{n}_f + \boldsymbol{\sigma}_p \mathbf{n}_p = 0 \tag{6}$$

BJS condition:

$$-(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \boldsymbol{\tau}_j = B\left(\mathbf{u}_f - \partial_t \boldsymbol{\eta}_p\right) \cdot \boldsymbol{\tau}_j \tag{7}$$

[1] does not have the constant C_p in (5), but their interface is 'actually flat' and not a vessel wall with muscles extering force of their own, so that's reasonable.

Variational formulation

Having used a backward Euler discretization of the time derivative, [1] obtain the following variational formulation

$$
\begin{aligned}
a_f(\mathbf{u}_f, \mathbf{v}_f) + & b_f(\mathbf{v}_f, p_f) + a_p^e(\boldsymbol{\eta}_p, \boldsymbol{\xi}_p) \\
& + \alpha b_p(\boldsymbol{\xi}_p, p_p) + a_p^d(\mathbf{u}_p, \mathbf{v}_p) + b_p(\mathbf{v}_p, p_p) \\
& + b_{\Gamma_{fp}}\left(\mathbf{v}_f, \mathbf{v}_p, \boldsymbol{\xi}_p; \lambda\right) + a_{BJS}\left(\mathbf{u}_f, \frac{\boldsymbol{\eta}_p}{\Delta t}; \mathbf{v}_f, \boldsymbol{\xi}_p\right) \\
= a_{BJS}\left(0, \frac{\boldsymbol{\eta}_p^{n-1}}{\Delta t}; \mathbf{v}_f, \boldsymbol{\xi}_p\right) & + (\mathbf{v}_p, C_p \mathbf{n}_p)_{\Gamma_{fp}} + (\boldsymbol{\sigma}_f \mathbf{n}_f, \mathbf{v}_f)_{\Gamma_f} \\
& + (\boldsymbol{\sigma}_p \mathbf{n}_p, \boldsymbol{\xi}_p)_{\Gamma_p} + (p_p \mathbf{n}_p, \mathbf{v}_p)_{\Gamma_p}
\end{aligned} \tag{8a}
$$

$$
\begin{aligned}
& \left(s_0 \frac{p_p}{\Delta t}, w_p\right)_{\Omega_p} - \alpha b_p\left(\frac{\boldsymbol{\eta}_p}{\Delta t}, w_p\right) - b_p(\mathbf{u}_p, w_p) - b_f(\mathbf{u}_f, w_f) \\
= & \left(s_0 \frac{p_p^{n-1}}{\Delta t}, w_p\right)_{\Omega_p} - \alpha b_p\left(\frac{\boldsymbol{\eta}_p^{n-1}}{\Delta t}, w_p\right)
\end{aligned} \tag{8b}
$$

$$
b_{\Gamma_{fp}}\left(\mathbf{u}_f, \mathbf{u}_p, \frac{\boldsymbol{\eta}_p}{\Delta t}; \mu_\Gamma\right) = b_{\Gamma_{fp}}^{\boldsymbol{\xi}_p}\left(\frac{\boldsymbol{\eta}_p^{n-1}}{\Delta t}; \mu_\Gamma\right) \tag{8c}
$$

Here the unknowns with no superscript mean the unknowns at time n (e.g. $\mathbf{u}_f = \mathbf{u}_f^n$).

Derivation of weak form

Briefly, (8a) is (1a) multiplied by $\mathbf{v}_f$ integrated over Ω_f; (3a) multiplied by $\boldsymbol{\xi}_p$ integrated over Ω_p; and (2) multiplied $\mathbf{v}_p$ integrated over Ω_p. The interface conditions are also all used.

To be more detailed, multiply (1a) by $\mathbf{v}_f$ and integrate over Ω_f. By standard vector calculus,

$$
\int_{\Omega_f} -\mathbf{v}_f \cdot \nabla \cdot \boldsymbol{\sigma}_f = \int_{\Omega_f} \boldsymbol{\sigma}_f : \nabla \mathbf{v}_f - \int_{\partial\Omega_f} \boldsymbol{\sigma}_f \mathbf{n}_f \cdot \mathbf{v}_f .
$$

Now, as the $\partial\Omega_f$ consists of Γ_f and Γ_{fp}, the boundary term splits into an integral over Γ_f which we can move to the RHS by applying boundary conditions [1], and an interface term $-I_{\mathbf{v}_f} = -\int_{\Gamma_{fp}} \boldsymbol{\sigma}_f \mathbf{n}_f \cdot \mathbf{v}_f$.

[1] Specifically, Dirichlet conditions on $\mathbf{u}_f$ or Neumann conditions on $\boldsymbol{\sigma}_f$.

Before proceeding, we expand[2]

$$\begin{aligned}\int_{\Omega_f} \boldsymbol{\sigma}_f \colon \nabla \mathbf{v}_f &= \int_{\Omega_f} (-p_f \mathbf{I} + 2\mu_f \mathbf{D}(\mathbf{u}_f)) \colon \nabla \mathbf{v}_f \\ &= \int_{\Omega_f} -p_f \nabla \cdot \mathbf{v}_f + \int_{\Omega_f} 2\mu_f \mathbf{D}(\mathbf{u}_f) \colon \nabla \mathbf{v}_f \\ &= b_f(\mathbf{v}_f, p_f) + a_f(\mathbf{u}_f, \mathbf{v}_f)\end{aligned}$$

So the contribution here is

$$a_f(\mathbf{u}_f, \mathbf{v}_f) + b_f(\mathbf{v}_f, p_f) - I_{\mathbf{v}_f} - (\boldsymbol{\sigma}_f \mathbf{n}_f, \mathbf{v}_f)_{\Gamma_f}.$$

Next, multiply (3a) by $\boldsymbol{\xi}_p$ and integrate over Ω_p. By exactly the same argument,

$$\int_{\Omega_p} -\boldsymbol{\xi}_p \cdot \nabla \cdot \boldsymbol{\sigma}_p = \int_{\Omega_p} \boldsymbol{\sigma}_p \colon \nabla \boldsymbol{\xi}_p - \int_{\partial\Omega_p} \boldsymbol{\sigma}_p \mathbf{n}_p \cdot \boldsymbol{\xi}_p.$$

Again, the boundary term splits into an integral over Γ_p where we need boundary conditions [3] and an interface term $-I_{\boldsymbol{\xi}_p} = -\int_{\Gamma_{fp}} \boldsymbol{\sigma}_p \mathbf{n}_p \cdot \boldsymbol{\xi}_p$. Expanding,

$$\begin{aligned}\int_{\Omega_p} \boldsymbol{\sigma}_p \colon \nabla \boldsymbol{\xi}_p &= \int_{\Omega_p} \left(\lambda_p (\nabla \cdot \boldsymbol{\eta}_p)(\nabla \cdot \boldsymbol{\xi}_p) + 2\mu_p \mathbf{D}(\boldsymbol{\eta}_p) \colon \nabla \boldsymbol{\xi}_p\right) - \alpha p_p \nabla \cdot \boldsymbol{\xi}_p \\ &= a_p^e(\boldsymbol{\eta}_p, \boldsymbol{\xi}_p) + \alpha b_p(\boldsymbol{\xi}_p, p_p)\end{aligned}$$

So the contribution is

$$a_p^e(\boldsymbol{\eta}_p, \boldsymbol{\xi}_p) + \alpha b_p(\boldsymbol{\xi}_p, p_p) - I_{\boldsymbol{\xi}_p} - (\boldsymbol{\sigma}_p \mathbf{n}_p, \boldsymbol{\xi}_p)_{\Gamma_p}.$$

Next, multiply (2) by $\frac{\mu}{K}\mathbf{v}_p$ and integrate over Γ_p. Integration by parts yields

$$\int_{\Omega_p} \frac{\mu}{K} \mathbf{v}_p \cdot \mathbf{u}_f - \int_{\Omega_p} p_p \nabla \cdot \mathbf{v}_p = \int_{\partial\Omega_p} p_p \mathbf{v}_p \cdot \mathbf{n}_p$$

The boundary term splits into an integral over Γ_p where we need boundary conditions [4] and an interface term $I_{\mathbf{v}_p} = \int_{\Gamma_{fp}} p_p \mathbf{v}_p \cdot \mathbf{n}_p$, so the contribution is

$$a_p^d(\mathbf{u}_p, \mathbf{v}_p) + b_p(\mathbf{v}_p, p_p) + I_{\mathbf{v}_p} - (p_p \mathbf{n}_p, \mathbf{v}_p)_{\Gamma_p}.$$

[2] The equality $\mathbf{D}(\mathbf{u}_f) \colon \nabla \mathbf{v}_f = \mathbf{D}(\mathbf{u}_f) \colon \mathbf{D}(\mathbf{v}_f)$ looks false because $\nabla \mathbf{v}_f \neq \mathbf{D}(\mathbf{v}_f)$, but $\mathbf{D}(\mathbf{u}_f)$ is symmetric, so $\mathbf{D}(\mathbf{u}_f) \colon \nabla \mathbf{v}_f = \mathbf{D}(\mathbf{u}_f) \colon \nabla \mathbf{v}_f^T = \mathbf{D}(\mathbf{u}_f) \colon \mathbf{D}(\mathbf{v}_f)$.

[3] Specifically, Dirichlet conditions on $\boldsymbol{\eta}_p$ or Neumann conditions on $\boldsymbol{\sigma}_p$.

[4] Specifically, Dirichlet conditions on $\mathbf{u}_p$ or Neumann conditions on p_p.

Next, once we add all these equations together, we will have to handle the sum of the interface terms

$$-I_{\mathbf{v}_f} - I_{\boldsymbol{\xi}_p} + I_{\mathbf{v}_p} = -\int\limits_{\Gamma_{fp}} \boldsymbol{\sigma}_f \mathbf{n}_f \cdot \mathbf{v}_f - \int\limits_{\Gamma_{fp}} \boldsymbol{\sigma}_p \mathbf{n}_p \cdot \boldsymbol{\xi}_p + \int\limits_{\Gamma_{fp}} p_p \mathbf{v}_p \cdot \mathbf{n}_p$$

We start with $\int\limits_{\Gamma_{fp}} p_p \mathbf{v}_p \cdot \mathbf{n}_p$. By (5), $p_p = \lambda_\Gamma - C_p$ on Γ_{fp}, so

$$I_{\mathbf{v}_p} = \int\limits_{\Gamma_{fp}} p_p \mathbf{v}_p \cdot \mathbf{n}_p = \int\limits_{\Gamma_{fp}} \lambda_\Gamma \mathbf{v}_p \cdot \mathbf{n}_p - \int\limits_{\Gamma_{fp}} C_p \mathbf{v}_p \cdot \mathbf{n}_p$$

The second term can then go live on the right hand side. Next, we treat the other two interface terms. By (6), $\boldsymbol{\sigma}_f \mathbf{n}_f = -\boldsymbol{\sigma}_p \mathbf{n}_p$, so we have that

$$I_{\mathbf{v}_f} + I_{\boldsymbol{\xi}_p} = \int\limits_{\Gamma_{fp}} (\mathbf{v}_f - \boldsymbol{\xi}_p) \cdot \boldsymbol{\sigma}_f \mathbf{n}_f$$

Next, note that the BJS condition (7) gives us information on the tangential component of $\boldsymbol{\sigma}_f \mathbf{n}_f$, while (5) gives us information on the normal component. As $\mathbf{n}_f, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2$ form an orthonormal system, we have that[5]

$$\begin{aligned}
&\mathbf{v}_f - \boldsymbol{\xi}_p = \mathbf{n}_f((\mathbf{v}_f - \boldsymbol{\xi}_p) \cdot \mathbf{n}_f) + \sum_j \boldsymbol{\tau}_j((\mathbf{v}_f - \boldsymbol{\xi}_p) \cdot \boldsymbol{\tau}_j) \\
\implies \quad &(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot (\mathbf{v}_f - \boldsymbol{\xi}_p) = ((\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \mathbf{n}_f)\left((\mathbf{v}_f - \boldsymbol{\xi}_p) \cdot \mathbf{n}_f\right) \\
&\qquad + \sum_j ((\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \boldsymbol{\tau}_j)((\mathbf{v}_f - \boldsymbol{\xi}_p) \cdot \boldsymbol{\tau}_j)
\end{aligned}$$

By (5), $(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \mathbf{n}_f = -\lambda$, and by (7), $(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \boldsymbol{\tau}_j = -B\left(\mathbf{u}_f - \partial_t \boldsymbol{\eta}_p\right) \cdot \boldsymbol{\tau}_j$, so

$$\begin{aligned}
(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot (\mathbf{v}_f - \boldsymbol{\xi}_p) = &-\lambda\left((\mathbf{v}_f - \boldsymbol{\xi}_p) \cdot \mathbf{n}_f\right) \\
&- \sum_j \left(B\left(\mathbf{u}_f - \partial_t \boldsymbol{\eta}_p\right) \cdot \boldsymbol{\tau}_j\right)\left((\mathbf{v}_f - \boldsymbol{\xi}_p) \cdot \boldsymbol{\tau}_j\right)
\end{aligned}$$

We can then use that $\mathbf{n}_f = -\mathbf{n}_p$ to write $\lambda(\mathbf{v}_f - \boldsymbol{\xi}_p) \cdot \mathbf{n}_f = \lambda(\mathbf{v}_f \cdot \mathbf{n}_f + \boldsymbol{\xi}_p \cdot \mathbf{n}_p)$. Putting it all together,

[5] This is just use of the fact when a vector $\mathbf{v}$ is written in a basis $\mathbf{e}_i$, the coefficient of $\mathbf{e}_i$

$$
\begin{aligned}
-(I_{\mathbf{v}_f} + I_{\boldsymbol{\xi}_p}) + I_{\mathbf{v}_p} &= \int\limits_{\Gamma_{fp}} \lambda \left(\mathbf{v}_f \cdot \mathbf{n}_f + (\boldsymbol{\xi}_p + \mathbf{v}_p) \cdot \mathbf{n}_p\right) \\
&+ \int\limits_{\Gamma_{fp}} \left(B \left(\mathbf{u}_f - \partial_t \boldsymbol{\eta}_p\right) \cdot \boldsymbol{\tau}_j\right) \left((\mathbf{v}_f - \boldsymbol{\xi}_p) \cdot \boldsymbol{\tau}_j\right) \\
&- \int\limits_{\Gamma_{fp}} C_p \mathbf{v}_p \cdot \mathbf{n}_p \\
= b_{\Gamma_{fp}} \left(\mathbf{v}_f, \mathbf{v}_p, \boldsymbol{\xi}_p; \lambda\right) &+ a_{BJS} \left(\mathbf{u}_f, \partial_t \boldsymbol{\eta}_p; \mathbf{v}_f, \boldsymbol{\xi}_p\right) - \left(\mathbf{v}_p, C_p \mathbf{n}_p\right)_{\Gamma_{fp}}
\end{aligned}
$$

We have now derived all the necessary identities, and summing them all yields the following:

$$
\begin{aligned}
a_f(\mathbf{u}_f, \mathbf{v}_f) &+ b_f(\mathbf{v}_f, p_f) + a_p^e(\boldsymbol{\eta}_p, \boldsymbol{\xi}_p) \\
&+ \alpha b_p(\boldsymbol{\xi}_p, p_p) + a_p^d(\mathbf{u}_p, \mathbf{v}_p) + b_p(\mathbf{v}_p, p_p) \\
&+ b_{\Gamma_{fp}} \left(\mathbf{v}_f, \mathbf{v}_p, \boldsymbol{\xi}_p; \lambda\right) + a_{BJS} \left(\mathbf{u}_f, \partial_t \boldsymbol{\eta}_p; \mathbf{v}_f, \boldsymbol{\xi}_p\right) \\
= (\mathbf{v}_p, C_p \mathbf{n}_p)_{\Gamma_{fp}} &+ (\boldsymbol{\sigma}_f \mathbf{n}_f, \mathbf{v}_f)_{\Gamma_f} + (\boldsymbol{\sigma}_p \mathbf{n}_p, \boldsymbol{\xi}_p)_{\Gamma_p} + (p_p \mathbf{n}_p, \mathbf{v}_p)_{\Gamma_p}
\end{aligned}
$$

If $\partial_t \boldsymbol{\eta}_p$ is now discretized by a backward Euler difference, this is exactly (8a).

The next two are not as bad. (8b) is just (1b) multiplied by w_f integrated over Ω_f plus (3b) multiplied by w_p integrated over Darcy.

Doing the above (no integration of parts needed) yields

$$
s_0 \left(\partial_t p_p, w_p\right)_{\Omega_p} - \alpha b_p(\partial_t \boldsymbol{\eta}_p, w_p) - b_p(\mathbf{u}_p, w_p) - b_f(\mathbf{u}_f, w_f)
$$

Once the ∂_t's are discretized by a backward Euler difference, this is exactly (8b).

Finally, (8c) is obtained by taking (4), multiplying by μ_Γ and integrating over Γ_{fp}. This yields

$$
b_{\Gamma_{fp}}(\mathbf{u}_f, \mathbf{u}_p, \partial_t \boldsymbol{\eta}_p; \mu_\Gamma) = 0
$$

which, when $\partial_t \boldsymbol{\eta}_p$ is discretized using a Backward Euler difference, yields (8c).

Boundary conditions

As the derivation of the weak form shows, I need :

- Dirichlet conditions on $\mathbf{u}_f$ or Neumann conditions on $\boldsymbol{\sigma}_f$ on Γ_f
- Dirichlet conditions on $\boldsymbol{\eta}_p$ or Neumann conditions on $\boldsymbol{\sigma}_p$ on Γ_p

- Dirichlet conditions on $\mathbf{u}_p$ or Neumann conditions on p_p on Γ_p.

In their numerical experiment (section 7.2), [1] use the domain shown in figure The Darcy boundary Γ_p is partitioned into the left part Γ_p^{L} and the

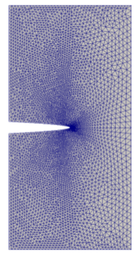

Figure 1: Darcy domain from [1]. Stokes domain is the removed 'finger'.

remainder $\Gamma_p^{\neg\mathrm{L}}$ in the obvious way. Physically, I think Γ_p^{L} is above ground and the other part is below ground or something.

As boundary conditions, they use:

- $\mathbf{u}_f = 10\mathbf{n}_f$ on Γ_f
- $\mathbf{u}_p \cdot \mathbf{n}_p = 0$ on Γ_p^{left}
- $p_p = 1000$ on $\Gamma_p^{\neg\mathrm{left}}$ (maybe this should be on Γ_p^{left} instead?)
- $\boldsymbol{\eta}_p \cdot \mathbf{n}_p = 0$ on $\Gamma_p^{\neg\mathrm{left}}$
- $(\boldsymbol{\sigma}_p \mathbf{n}_p) \cdot \boldsymbol{\tau}_p = 0$ on $\Gamma_p^{\neg\mathrm{left}}$

Thoughts

- Kent offered very gentle scepticism about using a 3-field formulation, and suggested not having $\mathbf{u}_p$ as an unknown, using $\mathbf{u}_p = \nabla p_p$ to remove it. I thought [1] had some opinion on this, but on closer reading I can't find it, so maybe that's from another article. I should read up on this.
- Why is there a 2 in front of μ in equation (1a)? If μ is just divided by 2 that's fine, but then I need to divide my choice by μ accordingly.

References

[1] Ambartsumyan, Ilona, et al. , *"A Lagrange multiplier method for a Stokes-Biot fluid-poroelastic structure interaction model."* , arXiv preprint arXiv:1710.06750 (2017).