Equations for system

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We follow [1] very very closely.

Domain

Our domain Ω is $d = 2$ or 3-dimensional, and partitioned into Ω_f and Ω_p, with $\Gamma_{fp} = \Omega_f \cap \Omega_p$ being the $(d-1)$-dimensional interface. The boundary $\partial\Omega$ is partitioned into $\Gamma_f = \partial\Omega \cap \partial\Omega_f$ and $\Gamma_p = \partial\Omega \cap \partial\Omega_p$. We assume each region is connected, reasonably smooth and all that.

Unknowns

The unknowns of the system and the corresponding test functions are:

- $\mathbf{u}_f$, $\mathbf{v}_f$: free flow fluid velocity. Defined on Ω_f.
- $\mathbf{u}_p$, $\mathbf{v}_p$: porous flow fluid velocity. Defined on Ω_p.
- p_f, w_f: free flow fluid pressure. Defined on Ω_f.
- p_p, w_p: porous flow fluid pressure. Defined on Ω_p.
- $\boldsymbol{\eta}_p$, $\boldsymbol{\xi}_p$: displacement. Defined on Ω_p.
- λ_Γ, μ_Γ: normal stress balance Lagrange multiplier. Defined on Γ_{fp}. In [1], denoted λ, μ_h.

Parameters

μ_f fluid viscosity (denoted μ in the [1])

λ_p, μ_p Lamé parameters. Denoted μ in [1].

α Biot-Willis constant

K Permeability tensor. Symmetric, bounded, positive definite. I take it to be scalar.

α_{BJS} Friction coefficient

s_0 Storage coefficient

$B = \frac{\mu_f \alpha_{BJS}}{\sqrt{K}}$ The coefficient in the BJS condition. Could be zero.

Notation

- $\mathbf{n}_f$, $\mathbf{n}_p$ are the outward unit normal vectors to $\partial\Omega_f, \partial\Omega_p$.
- $\boldsymbol{\tau}_{f,j}, j = 1, \ldots, d-1$ is an orthogonal system of unit tangent vectors at Γ_{fp}.
- $\mathbf{D}(\mathbf{v}) := \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T)$
- $\boldsymbol{\sigma}_f(\mathbf{u}_f, p_f) := -p_f \mathbf{I} + 2\mu_f \mathbf{D}(\mathbf{u}_f)$
- $\boldsymbol{\sigma}_p(\boldsymbol{\eta}_p, p_p) := \lambda_p(\nabla \cdot \boldsymbol{\eta}_p)\mathbf{I} + 2\mu_p \mathbf{D}(\boldsymbol{\eta}_p) - \alpha p_p \mathbf{I}$

Next, here are a bunch of bilinear forms used in the problem:

- $a_f(\mathbf{u}_f, \mathbf{v}_f) = (2\mu \mathbf{D}(\mathbf{u}_f), \mathbf{D}(\mathbf{v}_f))_{\Omega_f}$
- $a_p^d(\mathbf{u}_p, \mathbf{v}_p) = \left(\mu K^{-1} \mathbf{u}_p, \mathbf{v}_p\right)_{\Omega_p}$
- $a_p^e(\boldsymbol{\eta}_p, \boldsymbol{\xi}_p) = \left(\mu_p \mathbf{D}(\boldsymbol{\eta}_p), \mathbf{D}(\boldsymbol{\xi}_p)\right)_{\Omega_p} + \left(\lambda_p \nabla \cdot \boldsymbol{\eta}_p, \nabla \cdot \boldsymbol{\xi}_p\right)_{\Omega_p}$
- $b_f(\mathbf{v}_f, w_f) = -\left(\nabla \cdot \mathbf{v}_f, w_f\right)_{\Omega_f}$
- $b_p(\mathbf{v}_p, w_p) = -\left(\nabla \cdot \mathbf{v}_p, w_p\right)_{\Omega_p}$
- $a_{BJS}(\mathbf{u}_f, \boldsymbol{\eta}_p; \mathbf{v}_f, \boldsymbol{\xi}_p) = \frac{\mu \alpha_{BJS}}{\sqrt{K}} \sum_{j=1}^{d-1} \left((\mathbf{u}_f - \boldsymbol{\eta}_p) \cdot \boldsymbol{\tau}_j, (\mathbf{v}_f - \boldsymbol{\xi}_p) \cdot \boldsymbol{\tau}_j\right)_{\Gamma_{fp}}$
- $b_{\Gamma_{fp}}(\mathbf{v}_f, \mathbf{v}_p, \boldsymbol{\xi}_p; \mu_\Gamma) = b_{\Gamma_{fp}}^{\mathbf{v}_f}(\mathbf{v}_f, \mu_\Gamma) + b_{\Gamma_{fp}}^{\mathbf{v}_p}(\mathbf{v}_p, \mu_\Gamma) + b_{\Gamma_{fp}}^{\boldsymbol{\xi}_p}(\boldsymbol{\xi}_p, \mu_\Gamma)$
- $b_{\Gamma_{fp}}^{\mathbf{v}_f}(\mathbf{v}_f, \mu_\Gamma) = (\mathbf{v}_f \cdot \mathbf{n}_f, \mu_\Gamma)_{\Gamma_{fp}}$
- $b_{\Gamma_{fp}}^{\mathbf{v}_p}(\mathbf{v}_p, \mu_\Gamma) = (\mathbf{v}_p \cdot \mathbf{n}_p, \mu_\Gamma)_{\Gamma_{fp}}$
- $b_{\Gamma_{fp}}^{\boldsymbol{\xi}_p}(\boldsymbol{\xi}_p, \mu_\Gamma) = \left(\boldsymbol{\xi}_p \cdot \mathbf{n}_p, \mu_\Gamma\right)_{\Gamma_{fp}}$

Strong formulation

These are all ignoring body force and source terms. So it's okay to put stuff on the right hand sides if you want to.

Stokes (applies in Ω_f):

$$-\nabla \cdot \boldsymbol{\sigma}_f(\mathbf{u}_f, p_f) = 0 \tag{1a}$$

$$\nabla \cdot \mathbf{u}_f = 0 \tag{1b}$$

Darcy (applies in Ω_p):

$$\mathbf{u}_p = -\frac{K}{\mu} \nabla p_p \tag{2}$$

Biot (applies in Ω_p):

$$-\nabla \cdot \boldsymbol{\sigma}_p(\boldsymbol{\eta}_p, p_p) = 0 \tag{3a}$$

$$\partial_t \left(s_0 p_p + \alpha \nabla \cdot \boldsymbol{\eta}_p \right) + \nabla \cdot \mathbf{u}_p = 0 \tag{3b}$$

Interface conditions

Conservation of mass:

$$\mathbf{u}_f \cdot \mathbf{n}_f + \left(\partial_t \boldsymbol{\eta}_p + \mathbf{u}_p \right) \cdot \mathbf{n}_p = 0 \tag{4}$$

Balance of stress :

$$-(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \mathbf{n}_f = p_p + C_p = \lambda \tag{5}$$

$$\boldsymbol{\sigma}_f \mathbf{n}_f + \boldsymbol{\sigma}_p \mathbf{n}_p = 0\, TODO : add C_p here \tag{6}$$

BJS condition:

$$-(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \boldsymbol{\tau}_j = B \left(\mathbf{u}_f - \partial_t \boldsymbol{\eta}_p \right) \cdot \boldsymbol{\tau}_j \tag{7}$$

[1] does not have the constant C_p in (5), but their interface is 'actually flat' and not a vessel wall with muscles extering force of their own, so that's reasonable.

Variational formulation

Having used a backward Euler discretization of the time derivative, [1] obtain the following variational formulation

$$
\begin{aligned}
a_f(\mathbf{u}_f,\mathbf{v}_f) &+ b_f(\mathbf{v}_f,p_f) + a_p^e(\boldsymbol{\eta}_p,\boldsymbol{\xi}_p) \\
&+ \alpha b_p(\boldsymbol{\xi}_p,p_p) + a_p^d(\mathbf{u}_p,\mathbf{v}_p) + b_p(\mathbf{v}_p,p_p) \\
&+ b_{\Gamma_{fp}}\left(\mathbf{v}_f,\mathbf{v}_p,\boldsymbol{\xi}_p;\lambda\right) + a_{BJS}\left(\mathbf{u}_f,\frac{\boldsymbol{\eta}_p}{\Delta t};\mathbf{v}_f,\boldsymbol{\xi}_p\right) \\
= a_{BJS}\left(0,\frac{\boldsymbol{\eta}_p^{n-1}}{\Delta t};\mathbf{v}_f,\boldsymbol{\xi}_p\right) &+ (\mathbf{v}_p,C_p\mathbf{n}_p)_{\Gamma_{fp}} + (\boldsymbol{\sigma}_f\mathbf{n}_f,\mathbf{v}_f)_{\Gamma_f} \\
&+ (\boldsymbol{\sigma}_p\mathbf{n}_p,\boldsymbol{\xi}_p)_{\Gamma_p} + (p_p\mathbf{n}_p,\mathbf{v}_p)_{\Gamma_p}
\end{aligned}
\tag{8a}
$$

$$
\begin{aligned}
&\left(s_0\frac{p_p}{\Delta t},w_p\right)_{\Omega_p} - \alpha b_p\left(\frac{\boldsymbol{\eta}_p}{\Delta t},w_p\right) - b_p(\mathbf{u}_p,w_p) - b_f(\mathbf{u}_f,w_f) \\
=& \left(s_0\frac{p_p^{n-1}}{\Delta t},w_p\right)_{\Omega_p} - \alpha b_p\left(\frac{\boldsymbol{\eta}_p^{n-1}}{\Delta t},w_p\right)
\end{aligned}
\tag{8b}
$$

$$
b_{\Gamma_{fp}}\left(\mathbf{u}_f,\mathbf{u}_p,\frac{\boldsymbol{\eta}_p}{\Delta t};\mu_\Gamma\right) = b_{\Gamma_{fp}}^{\boldsymbol{\xi}_p}\left(\frac{\boldsymbol{\eta}_p^{n-1}}{\Delta t};\mu_\Gamma\right) \tag{8c}
$$

Here the unknowns with no superscript mean the unknowns at time n (e.g. $\mathbf{u}_f = \mathbf{u}_f^n$).

Derivation of weak form

Briefly, (8a) is (1a) multiplied by $\mathbf{v}_f$ integrated over Ω_f; (3a) multiplied by $\boldsymbol{\xi}_p$ integrated over Ω_p; and (2) multiplied $\mathbf{v}_p$ integrated over Ω_p. The interface conditions are also all used.

To be more detailed, multiply (1a) by $\mathbf{v}_f$ and integrate over Ω_f. By standard vector calculus,

$$
\int\limits_{\Omega_f} -\mathbf{v}_f\cdot\nabla\cdot\boldsymbol{\sigma}_f = \int\limits_{\Omega_f} \boldsymbol{\sigma}_f : \nabla\mathbf{v}_f - \int\limits_{\partial\Omega_f} \boldsymbol{\sigma}_f\mathbf{n}_f\cdot\mathbf{v}_f.
$$

Now, as the $\partial\Omega_f$ consists of Γ_f and Γ_{fp}, the boundary term splits into an integral over Γ_f which we can move to the RHS by applying boundary conditions [1], and an interface term $-I_{\mathbf{v}_f} = -\int\limits_{\Gamma_{fp}} \boldsymbol{\sigma}_f\mathbf{n}_f\cdot\mathbf{v}_f$.

[1] Specifically, Dirichlet conditions on $\mathbf{u}_f$ or Neumann conditions on $\boldsymbol{\sigma}_f$.

Before proceeding, we expand[2]

$$\begin{aligned}\int_{\Omega_f} \boldsymbol{\sigma}_f \colon \nabla \mathbf{v}_f &= \int_{\Omega_f} (-p_f \mathbf{I} + 2\mu_f \mathbf{D}(\mathbf{u}_f)) \colon \nabla \mathbf{v}_f \\ &= \int_{\Omega_f} -p_f \nabla \cdot \mathbf{v}_f + \int_{\Omega_f} 2\mu_f \mathbf{D}(\mathbf{u}_f) \colon \nabla \mathbf{v}_f \\ &= b_f(\mathbf{v}_f, p_f) + a_f(\mathbf{u}_f, \mathbf{v}_f)\end{aligned}$$

So the contribution here is

$$a_f(\mathbf{u}_f, \mathbf{v}_f) + b_f(\mathbf{v}_f, p_f) - I_{\mathbf{v}_f} - (\boldsymbol{\sigma}_f \mathbf{n}_f, \mathbf{v}_f)_{\Gamma_f}.$$

Next, multiply (3a) by $\boldsymbol{\xi}_p$ and integrate over Ω_p. By exactly the same argument,

$$\int_{\Omega_p} -\boldsymbol{\xi}_p \cdot \nabla \cdot \boldsymbol{\sigma}_p = \int_{\Omega_p} \boldsymbol{\sigma}_p \colon \nabla \boldsymbol{\xi}_p - \int_{\partial\Omega_p} \boldsymbol{\sigma}_p \mathbf{n}_p \cdot \boldsymbol{\xi}_p.$$

Again, the boundary term splits into an integral over Γ_p where we need boundary conditions [3] and an interface term $-I_{\boldsymbol{\xi}_p} = -\int_{\Gamma_{fp}} \boldsymbol{\sigma}_p \mathbf{n}_p \cdot \boldsymbol{\xi}_p$. Expanding,

$$\begin{aligned}\int_{\Omega_p} \boldsymbol{\sigma}_p \colon \nabla \boldsymbol{\xi}_p &= \int_{\Omega_p} \left(\lambda_p (\nabla \cdot \boldsymbol{\eta}_p)(\nabla \cdot \boldsymbol{\xi}_p) + 2\mu_p \mathbf{D}(\boldsymbol{\eta}_p) \colon \nabla \boldsymbol{\xi}_p\right) - \alpha p_p \nabla \cdot \boldsymbol{\xi}_p \\ &= a_p^e(\boldsymbol{\eta}_p, \boldsymbol{\xi}_p) + \alpha b_p(\boldsymbol{\xi}_p, p_p)\end{aligned}$$

So the contribution is

$$a_p^e(\boldsymbol{\eta}_p, \boldsymbol{\xi}_p) + \alpha b_p(\boldsymbol{\xi}_p, p_p) - I_{\boldsymbol{\xi}_p} - (\boldsymbol{\sigma}_p \mathbf{n}_p, \boldsymbol{\xi}_p)_{\Gamma_p}.$$

Next, multiply (2) by $\frac{\mu}{K}\mathbf{v}_p$ and integrate over Γ_p. Integration by parts yields

$$\int_{\Omega_p} \frac{\mu}{K} \mathbf{v}_p \cdot \mathbf{u}_f - \int_{\Omega_p} p_p \nabla \cdot \mathbf{v}_p = \int_{\partial\Omega_p} p_p \mathbf{v}_p \cdot \mathbf{n}_p$$

The boundary term splits into an integral over Γ_p where we need boundary conditions [4] and an interface term $I_{\mathbf{v}_p} = \int_{\Gamma_{fp}} p_p \mathbf{v}_p \cdot \mathbf{n}_p$, so the contribution is

$$a_p^d(\mathbf{u}_p, \mathbf{v}_p) + b_p(\mathbf{v}_p, p_p) + I_{\mathbf{v}_p} - (p_p \mathbf{n}_p, \mathbf{v}_p)_{\Gamma_p}.$$

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[2]The equality $\mathbf{D}(\mathbf{u}_f) \colon \nabla \mathbf{v}_f = \mathbf{D}(\mathbf{u}_f) \colon \mathbf{D}(\mathbf{v}_f)$ looks false because $\nabla \mathbf{v}_f \neq \mathbf{D}(\mathbf{v}_f)$, but $\mathbf{D}(\mathbf{u}_f)$ is symmetric, so $\mathbf{D}(\mathbf{u}_f) \colon \nabla \mathbf{v}_f = \mathbf{D}(\mathbf{u}_f) \colon \nabla \mathbf{v}_f^T = \mathbf{D}(\mathbf{u}_f) \colon \mathbf{D}(\mathbf{v}_f)$.

[3]Specifically, Dirichlet conditions on $\boldsymbol{\eta}_p$ or Neumann conditions on $\boldsymbol{\sigma}_p$.

[4]Specifically, Dirichlet conditions on $\mathbf{u}_p$ or Neumann conditions on p_p.

Next, once we add all these equations together, we will have to handle the sum of the interface terms

$$-I_{\mathbf{v}_f} - I_{\boldsymbol{\xi}_p} + I_{\mathbf{v}_p} = -\int\limits_{\Gamma_{fp}} \boldsymbol{\sigma}_f \mathbf{n}_f \cdot \mathbf{v}_f - \int\limits_{\Gamma_{fp}} \boldsymbol{\sigma}_p \mathbf{n}_p \cdot \boldsymbol{\xi}_p + \int\limits_{\Gamma_{fp}} p_p \mathbf{v}_p \cdot \mathbf{n}_p$$

We start with $\int\limits_{\Gamma_{fp}} p_p \mathbf{v}_p \cdot \mathbf{n}_p$. By (5), $p_p = \lambda_\Gamma - C_p$ on Γ_{fp}, so

$$I_{\mathbf{v}_p} = \int\limits_{\Gamma_{fp}} p_p \mathbf{v}_p \cdot \mathbf{n}_p = \int\limits_{\Gamma_{fp}} \lambda_\Gamma \mathbf{v}_p \cdot \mathbf{n}_p - \int\limits_{\Gamma_{fp}} C_p \mathbf{v}_p \cdot \mathbf{n}_p$$

The second term can then go live on the right hand side. Next, we treat the other two interface terms. By (6), $\boldsymbol{\sigma}_f \mathbf{n}_f = -\boldsymbol{\sigma}_p \mathbf{n}_p$, so we have that

$$I_{\mathbf{v}_f} + I_{\boldsymbol{\xi}_p} = \int\limits_{\Gamma_{fp}} (\mathbf{v}_f - \boldsymbol{\xi}_p) \cdot \boldsymbol{\sigma}_f \mathbf{n}_f$$

Next, note that the BJS condition (7) gives us information on the tangential component of $\boldsymbol{\sigma}_f \mathbf{n}_f$, while (5) gives us information on the normal component. As $\mathbf{n}_f, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2$ form an orthonormal system, we have that[5]

$$\begin{aligned} \mathbf{v}_f - \boldsymbol{\xi}_p &= \mathbf{n}_f((\mathbf{v}_f - \boldsymbol{\xi}_p) \cdot \mathbf{n}_f) + \sum_j \boldsymbol{\tau}_j((\mathbf{v}_f - \boldsymbol{\xi}_p) \cdot \boldsymbol{\tau}_j) \\ \implies \quad (\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot (\mathbf{v}_f - \boldsymbol{\xi}_p) &= ((\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \mathbf{n}_f)\,((\mathbf{v}_f - \boldsymbol{\xi}_p) \cdot \mathbf{n}_f) \\ &\quad + \sum_j ((\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \boldsymbol{\tau}_j)((\mathbf{v}_f - \boldsymbol{\xi}_p) \cdot \boldsymbol{\tau}_j) \end{aligned}$$

By (5), $(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \mathbf{n}_f = -\lambda$, and by (7), $(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \boldsymbol{\tau}_j = -B\left(\mathbf{u}_f - \partial_t \boldsymbol{\eta}_p\right) \cdot \boldsymbol{\tau}_j$, so

$$\begin{aligned} (\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot (\mathbf{v}_f - \boldsymbol{\xi}_p) = &- \lambda\left((\mathbf{v}_f - \boldsymbol{\xi}_p) \cdot \mathbf{n}_f\right) \\ &- \sum_j \left(B\left(\mathbf{u}_f - \partial_t \boldsymbol{\eta}_p\right) \cdot \boldsymbol{\tau}_j\right)\left((\mathbf{v}_f - \boldsymbol{\xi}_p) \cdot \boldsymbol{\tau}_j\right) \end{aligned}$$

We can then use that $\mathbf{n}_f = -\mathbf{n}_p$ to write $\lambda(\mathbf{v}_f - \boldsymbol{\xi}_p) \cdot \mathbf{n}_f = \lambda(\mathbf{v}_f \cdot \mathbf{n}_f + \boldsymbol{\xi}_p \cdot \mathbf{n}_p)$. Putting it all together,

[5] This is just use of the fact when a vector $\mathbf{v}$ is written in a basis $\mathbf{e}_i$, the coefficient of $\mathbf{e}_i$

$$
\begin{aligned}
-(I_{\mathbf{v}_f} + I_{\boldsymbol{\xi}_p}) + I_{\mathbf{v}_p} &= \int\limits_{\Gamma_{fp}} \lambda \left(\mathbf{v}_f \cdot \mathbf{n}_f + (\boldsymbol{\xi}_p + \mathbf{v}_p) \cdot \mathbf{n}_p\right) \\
&\quad + \int\limits_{\Gamma_{fp}} \left(B \left(\mathbf{u}_f - \partial_t \boldsymbol{\eta}_p\right) \cdot \boldsymbol{\tau}_j\right) \left((\mathbf{v}_f - \boldsymbol{\xi}_p) \cdot \boldsymbol{\tau}_j\right) \\
&\quad - \int\limits_{\Gamma_{fp}} C_p \mathbf{v}_p \cdot \mathbf{n}_p \\
&= b_{\Gamma_{fp}} \left(\mathbf{v}_f, \mathbf{v}_p, \boldsymbol{\xi}_p; \lambda\right) + a_{BJS} \left(\mathbf{u}_f, \partial_t \boldsymbol{\eta}_p; \mathbf{v}_f, \boldsymbol{\xi}_p\right) - \left(\mathbf{v}_p, C_p \mathbf{n}_p\right)_{\Gamma_{fp}}
\end{aligned}
$$

We have now derived all the necessary identities, and summing them all yields the following:

$$
\begin{aligned}
a_f(\mathbf{u}_f, \mathbf{v}_f) &+ b_f(\mathbf{v}_f, p_f) + a_p^e(\boldsymbol{\eta}_p, \boldsymbol{\xi}_p) \\
&+ \alpha b_p(\boldsymbol{\xi}_p, p_p) + a_p^d(\mathbf{u}_p, \mathbf{v}_p) + b_p(\mathbf{v}_p, p_p) \\
&+ b_{\Gamma_{fp}} \left(\mathbf{v}_f, \mathbf{v}_p, \boldsymbol{\xi}_p; \lambda\right) + a_{BJS} \left(\mathbf{u}_f, \partial_t \boldsymbol{\eta}_p; \mathbf{v}_f, \boldsymbol{\xi}_p\right) \\
= (\mathbf{v}_p, C_p \mathbf{n}_p)_{\Gamma_{fp}} &+ (\boldsymbol{\sigma}_f \mathbf{n}_f, \mathbf{v}_f)_{\Gamma_f} + (\boldsymbol{\sigma}_p \mathbf{n}_p, \boldsymbol{\xi}_p)_{\Gamma_p} + (p_p \mathbf{n}_p, \mathbf{v}_p)_{\Gamma_p}
\end{aligned}
$$

If $\partial_t \boldsymbol{\eta}_p$ is now discretized by a backward Euler difference, this is exactly (8a).

The next two are not as bad. (8b) is just (1b) multiplied by w_f integrated over Ω_f plus (3b) multiplied by w_p integrated over Darcy.

Doing the above (no integration of parts needed) yields

$$
s_0 \left(\partial_t p_p, w_p\right)_{\Omega_p} - \alpha b_p(\partial_t \boldsymbol{\eta}_p, w_p) - b_p(\mathbf{u}_p, w_p) - b_f(\mathbf{u}_f, w_f)
$$

Once the ∂_t's are discretized by a backward Euler difference, this is exactly (8b).

Finally, (8c) is obtained by taking (4), multiplying by μ_Γ and integrating over Γ_{fp}. This yields

$$
b_{\Gamma_{fp}}(\mathbf{u}_f, \mathbf{u}_p, \partial_t \boldsymbol{\eta}_p; \mu_\Gamma) = 0
$$

which, when $\partial_t \boldsymbol{\eta}_p$ is discretized using a Backward Euler difference, yields (8c).

Matrix form

Denote by $[c(\mathbf{u}, \mathbf{v})]$ the matrix/vector of the bilinear/linear form c, meaning the matrix with entries $c(\mathbf{e}_i^U, \mathbf{e}_j^V)$, where the test function varies as you move down the rows. Then swapping each argument between test and trial function transposes the matrix, so for example

$$
[b_f(\mathbf{v}_f, p_f)] = [b_f(\mathbf{u}_f, w_f)]^T .
$$

We write our problem in a matrix form, but before doing so, we take a look at what's going on at the interface, as two of our bilinear forms are mixed. First, note that

$$a_{BJS}(\mathbf{u}_f, \boldsymbol{\eta}_p; \mathbf{v}_f, \boldsymbol{\xi}_p) = B \sum_{j=1}^{d-1} (\mathbf{u}_f \cdot \boldsymbol{\tau}_j, \mathbf{v}_f \cdot \boldsymbol{\tau}_j)_{\Gamma_{fp}} - \left(\boldsymbol{\eta}_p \cdot \boldsymbol{\tau}_j, \mathbf{v}_f \cdot \boldsymbol{\tau}_j\right)_{\Gamma_{fp}}$$
$$- \left(\mathbf{u}_f \cdot \boldsymbol{\tau}_j, \boldsymbol{\xi}_p \cdot \boldsymbol{\tau}_j\right)_{\Gamma_{fp}} + \left(\boldsymbol{\eta}_p \cdot \boldsymbol{\tau}_j, \boldsymbol{\xi}_p \cdot \boldsymbol{\tau}_j\right)_{\Gamma_{fp}}$$

splits into $S_{\mathbf{v}_f,\mathbf{u}_f} - S_{\mathbf{v}_f,\boldsymbol{\eta}_p} - S_{\boldsymbol{\xi}_p,\mathbf{u}_f} + S_{\boldsymbol{\xi}_p,\boldsymbol{\eta}_p}$ where $S_{\mathbf{a},\mathbf{b}} = B \sum_{j=1}^{d-1} (\mathbf{b} \cdot \boldsymbol{\tau}_j, \mathbf{a} \cdot \boldsymbol{\tau}_j)_{\Gamma_{fp}}$ (see the Thoughts-section for how this might be conveniently implemented)

Next, looking at the other interface term, $b_{\Gamma_{fp}}(\mathbf{v}_f, \mathbf{v}_p, \boldsymbol{\xi}_p; \lambda_\Gamma)$, note that it splits into sums of mass matrices of normal components. More precisely,

$$b_{\Gamma_{fp}}(\mathbf{v}_f, \mathbf{v}_p, \boldsymbol{\xi}_p; \lambda_\Gamma) = N^f_{\mathbf{v}_f,\lambda_\Gamma} + N^p_{\mathbf{v}_p,\lambda_\Gamma} + N^p_{\boldsymbol{\xi}_p,\lambda_\Gamma}$$

where $N^*_{\mathbf{a},b} = (\mathbf{a} \cdot \mathbf{n}_*, b)_{\Gamma_{fp}}$. With the exception of the RHS term $(\mathbf{v}_p, C_p \mathbf{n}_p)_{\Gamma_{fp}}$, which is easier because it only lives in one function space, these are the only interface terms we need to assemble.

So on the implementation side, the only matrices we need to assemble on the interface are $(\mathbf{a} \cdot \mathbf{n}_*, b)_{\Gamma_{fp}}$ and $B \sum_{j=1}^{d-1} (\mathbf{b} \cdot \boldsymbol{\tau}_j, \mathbf{a} \cdot \boldsymbol{\tau}_j)_{\Gamma_{fp}}$

So on the implementation side, the only matrices we need to assemble on the interface are $(\mathbf{a} \cdot \mathbf{n}_*, b)_{\Gamma_{fp}}$ and $B \sum_{j=1}^{d-1} (\mathbf{b} \cdot \boldsymbol{\tau}_j, \mathbf{a} \cdot \boldsymbol{\tau}_j)_{\Gamma_{fp}}$

Now, abuse notation to let N_*, S_* instead mean the matrices of the bilinear forms defined above, and define a couple more:

Matrices	Vectors
$A_f := [a_f(\mathbf{u}_f, \mathbf{v}_f)]$	$L^{\boldsymbol{\sigma}_f} := \left[(\boldsymbol{\sigma}_f \mathbf{n}_f, \mathbf{v}_f)_{\Gamma_f}\right]$
$A_p^d := \left[a_p^d(\mathbf{u}_p, \mathbf{v}_p)\right]$	
$A_p^e := \left[a_p^e(\boldsymbol{\eta}_p, \boldsymbol{\xi}_p)\right]$	$L^{\boldsymbol{\sigma}_p} := \left[\left(\boldsymbol{\sigma}_p \mathbf{n}_p, \boldsymbol{\xi}_p\right)_{\Gamma_p}\right]$
$B_f := [b_f(\mathbf{v}_f, p_f)]$	$L^{p_p} := \left[(p_p \mathbf{n}_p, \mathbf{v}_p)_{\Gamma_p}\right]$
$B_p^{\mathbf{v}_p} := [b_p(\mathbf{v}_p, p_p)]$	
$B_p^{\boldsymbol{\xi}_p} := \left[b_p(\boldsymbol{\xi}_p, p_p)\right]$	$L^{\Delta p} := \left[(C_p \mathbf{n}_p, \mathbf{v}_p)_{\Gamma_{fp}}\right]$
$M_* :=$ mass matrix of unknown $*$	$\mathbf{u}_f, p_f, \ldots :=$ vector of dofs

Note that although some of these vectors involve the unknowns, but appear on the RHS, they will be known because of BCs.

Then the matrix of our system lookjs like

$$
\left[\begin{array}{ccc|cc|c}
A_p^d & B_p^{\mathbf{v}_p} & & & & N^p_{\mathbf{v}_p,\lambda_\Gamma} \\
-(B_p^{\mathbf{v}_p})^T & \frac{s_0}{\Delta t} M_{p_p} & -\frac{\alpha}{\Delta t}(B_p^{\boldsymbol{\xi}_p})^T & & & \\
 & \alpha B_p^{\boldsymbol{\xi}_p} & A_p^e + S_{\boldsymbol{\xi}_p,\boldsymbol{\eta}_p} & -S_{\boldsymbol{\xi}_p,\mathbf{u}_f} & & N^p_{\boldsymbol{\xi}_p,\lambda_\Gamma} \\
\hline
 & & -\frac{1}{\Delta t} S_{\mathbf{v}_f,\boldsymbol{\eta}_p} & A_f + S_{\mathbf{v}_f,\mathbf{u}_f} & B_f & N^f_{\mathbf{v}_f,\lambda_\Gamma} \\
 & & & -B_f^T & & \\
\hline
(N^p_{\mathbf{v}_p,\lambda_\Gamma})^T & & \frac{1}{\Delta t}(N^p_{\boldsymbol{\xi}_p,\lambda_\Gamma})^T & (N^f_{\mathbf{v}_f,\lambda_\Gamma})^T & &
\end{array}\right]
\left[\begin{array}{c}
\mathbf{u}_p \\ p_p \\ \boldsymbol{\eta}_p \\ \mathbf{u}_f \\ p_f \\ \lambda_\Gamma
\end{array}\right]
=
\left[\begin{array}{c}
L^{p_p} + L^{\Delta p} \\ b^{p_p} \\ L^{\sigma_p} \\ L^{\sigma_f} \\ \\ \frac{1}{\Delta t}(N^p_{\boldsymbol{\xi}_p,\lambda_\Gamma})^T \boldsymbol{\eta}_p^{n-1}
\end{array}\right]
$$

where

$$b^{p_p} = \frac{s_0}{\Delta t} M_{p_p} p_p^{n-1} - \frac{\alpha}{\Delta t}(B_p^{\boldsymbol{\xi}_p})^T \boldsymbol{\eta}_p^{n-1}$$

By scaling, this seems like it could be made symmetric. Specifically, dividing the equation for $\boldsymbol{\xi}_p$ by Δt, flipping the signs of the equations for w_f and w_p gets you there:

$$
\left[\begin{array}{ccc|cc|c}
A_p^d & B_p^{\mathbf{v}_p} & & & & N^p_{\mathbf{v}_p,\lambda_\Gamma} \\
(B_p^{\mathbf{v}_p})^T & -\frac{s_0}{\Delta t} M_{p_p} & \frac{\alpha}{\Delta t}(B_p^{\boldsymbol{\xi}_p})^T & & & \\
 & \frac{\alpha}{\Delta t} B_p^{\boldsymbol{\xi}_p} & \frac{1}{\Delta t} A_p^e + \frac{1}{\Delta t} S_{\boldsymbol{\xi}_p,\boldsymbol{\eta}_p} & -\frac{1}{\Delta t} S_{\boldsymbol{\xi}_p,\mathbf{u}_f} & & \frac{1}{\Delta t} N^p_{\boldsymbol{\xi}_p,\lambda_\Gamma} \\
\hline
 & & -\frac{1}{\Delta t} S_{\mathbf{v}_f,\boldsymbol{\eta}_p} & A_f + S_{\mathbf{v}_f,\mathbf{u}_f} & B_f & N^f_{\mathbf{v}_f,\lambda_\Gamma} \\
 & & & B_f^T & & \\
\hline
(N^p_{\mathbf{v}_p,\lambda_\Gamma})^T & & \frac{1}{\Delta t}(N^p_{\boldsymbol{\xi}_p,\lambda_\Gamma})^T & (N^f_{\mathbf{v}_f,\lambda_\Gamma})^T & &
\end{array}\right]
\left[\begin{array}{c}
\mathbf{u}_p \\ p_p \\ \boldsymbol{\eta}_p \\ \mathbf{u}_f \\ p_f \\ \lambda_\Gamma
\end{array}\right]
=
\left[\begin{array}{c}
L^{p_p} + L^{\Delta p} \\ -b^{p_p} \\ \frac{1}{\Delta t} L^{\sigma_p} \\ L^{\sigma_f} \\ \\ \frac{1}{\Delta t}(N^p_{\boldsymbol{\xi}_p,\lambda_\Gamma})^T \boldsymbol{\eta}_p^{n-1}
\end{array}\right]
$$

This makes the (w_p, p_p) block negative, though.

Now, abuse notation to let N_*, S_* instead mean the matrices of the bilinear forms defined above, and define a couple more:

Matrices

$$A_f := [a_f(\mathbf{u}_f, \mathbf{v}_f)]$$

$$A_p^d := \left[a_p^d(\mathbf{u}_p, \mathbf{v}_p)\right]$$

$$A_p^e := \left[a_p^e(\boldsymbol{\eta}_p, \boldsymbol{\xi}_p)\right]$$

$$B_f := [b_f(\mathbf{v}_f, p_f)]$$

$$B_p^{\mathbf{v}_p} := [b_p(\mathbf{v}_p, p_p)]$$

$$B_p^{\boldsymbol{\xi}_p} := \left[b_p(\boldsymbol{\xi}_p, p_p)\right]$$

$$M_* := \text{mass matrix of unknown } *$$

Vectors

$$L^{\boldsymbol{\sigma}_f} := \left[(\boldsymbol{\sigma}_f \mathbf{n}_f, \mathbf{v}_f)_{\Gamma_f}\right]$$

$$L^{\boldsymbol{\sigma}_p} := \left[\left(\boldsymbol{\sigma}_p \mathbf{n}_p, \boldsymbol{\xi}_p\right)_{\Gamma_p}\right]$$

$$L^{p_p} := \left[(p_p \mathbf{n}_p, \mathbf{v}_p)_{\Gamma_p}\right]$$

$$L^{\Delta p} := \left[(C_p \mathbf{n}_p, \mathbf{v}_p)_{\Gamma_{fp}}\right]$$

$$\mathbf{u}_f, p_f, \ldots := \text{vector of dofs}$$

Note that although some of these vectors involve the unknowns, but appear on the RHS, they will be known because of BCs.

Then the matrix of our system lookjs like

$$\left[\begin{array}{ccc|cc|c}
A_p^d & B_p^{\mathbf{v}_p} & & & & N^p_{\mathbf{v}_p,\lambda_\Gamma} \\
-(B_p^{\mathbf{v}_p})^T & \frac{s_0}{\Delta t} M_{p_p} & -\frac{\alpha}{\Delta t}(B_p^{\boldsymbol{\xi}_p})^T & & & \\
 & \alpha B_p^{\boldsymbol{\xi}_p} & A_p^e & -S_{\boldsymbol{\xi}_p,\mathbf{u}_f} & & N^p_{\boldsymbol{\xi}_p,\lambda_\Gamma} \\
\hline
 & & -\frac{1}{\Delta t} S_{\mathbf{v}_f,\boldsymbol{\eta}_p} & A_f & B_f & N^f_{\mathbf{v}_f,\lambda_\Gamma} \\
 & & & -B_f^T & & \\
\hline
(N^p_{\mathbf{v}_p,\lambda_\Gamma})^T & & \frac{1}{\Delta t}(N^p_{\boldsymbol{\xi}_p,\lambda_\Gamma})^T & (N^f_{\mathbf{v}_f,\lambda_\Gamma})^T & &
\end{array}\right]
\begin{bmatrix} \mathbf{u}_p \\ p_p \\ \boldsymbol{\eta}_p \\ \mathbf{u}_f \\ p_f \\ \lambda_\Gamma \end{bmatrix}
=
\begin{bmatrix} L^{p_p} + L^{\Delta p} \\ R^{p_p} \\ L^{\sigma_p} \\ L^{\sigma_f} \\ \\ \frac{1}{\Delta t}(N^p_{\boldsymbol{\xi}_p,\lambda_\Gamma})^T \boldsymbol{\eta}_p^{n-1} \end{bmatrix}$$

where

$$R^{p_p} = \frac{s_0}{\Delta t} M_{p_p} p_p^{n-1} - \frac{\alpha}{\Delta t}(B_p^{\boldsymbol{\xi}_p})^T \boldsymbol{\eta}_p^{n-1}$$

By scaling the appropriate unknowns, this seems like it could be made symmetric.

Boundary conditions

As the derivation of the weak form shows, I need :

- Dirichlet conditions on $\mathbf{u}_f$ or Neumann conditions on $\boldsymbol{\sigma}_f$ on Γ_f
- Dirichlet conditions on $\boldsymbol{\eta}_p$ or Neumann conditions on $\boldsymbol{\sigma}_p$ on Γ_p
- Dirichlet conditions on $\mathbf{u}_p$ or Neumann conditions on p_p on Γ_p.

In their numerical experiment (section 7.2), [1] use the domain shown in figure The Darcy boundary Γ_p is partitioned into the left part Γ_p^{L} and the remainder $\Gamma_p^{\neg\mathrm{L}}$ in the obvious way. Physically, I think Γ_p^{L} is above ground and the other part is below ground or something.

As boundary conditions, they use:

- $\mathbf{u}_f = 10\mathbf{n}_f$ on Γ_f
- $\mathbf{u}_p \cdot \mathbf{n}_p = 0$ on Γ_p^{left}

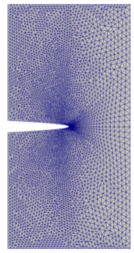

Figure 1: Darcy domain from [1]. Stokes domain is the removed 'finger'.

- $p_p = 1000$ on $\Gamma_p^{\neg\text{left}}$ (maybe this should be on Γ_p^{left} instead?)
- $\boldsymbol{\eta}_p \cdot \mathbf{n}_p = 0$ on $\Gamma_p^{\neg\text{left}}$
- $(\boldsymbol{\sigma}_p \mathbf{n}_p) \cdot \boldsymbol{\tau}_p = 0$ on $\Gamma_p^{\neg\text{left}}$

Thoughts

- Kent offered very gentle scepticism about using a 3-field formulation, and suggested not having $\mathbf{u}_p$ as an unknown, using $\mathbf{u}_p = \nabla p_p$ to remove it. I thought [1] had some opinion on this, but on closer reading I can't find it, so maybe that's from another article. I should read up on this.
- Why is there a 2 in front of μ in equation (1a)? If μ is just divided by 2 that's fine, but then I need to divide my choice by μ accordingly.
- As $\mathbf{n}_f, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2$ form an orthonormal system, $\sum (\mathbf{b} \cdot \boldsymbol{\tau}_j, \mathbf{a} \cdot \boldsymbol{\tau}_j)$ is the normal inner product $(\mathbf{b}, \mathbf{a})$ minus the term $(\mathbf{a} \cdot \mathbf{n}_p) \cdot (\mathbf{b} \cdot \mathbf{n}_p)$ So it can be thought of as the inner product $(\Pi\mathbf{a}, \Pi\mathbf{b})$ where $\Pi\mathbf{a}$ is the projection of the vector $\mathbf{a}$ onto the interface, meaning that the BJS matrix is essentially a "mass matrix with a component missing".

  So if you don't want to mess with tangents, the relation

  $$(\Pi\mathbf{a}, \Pi\mathbf{b}) = (\mathbf{a}, \mathbf{b}) - ((\mathbf{a} \cdot \mathbf{n}_f), (\mathbf{b} \cdot \mathbf{n}_f))$$

  means you're set as long as you can keep track of the interface normal. (In principle it should not even need to have a consistent orientation, but in practice I imagine it might have to.)

References

[1] Ambartsumyan, Ilona, et al. , *"A Lagrange multiplier method for a Stokes-Biot fluid-poroelastic structure interaction model."* , arXiv preprint arXiv:1710.06750 (2017).