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BEND|P|Y: PYTHON FRAMEWORK FOR COMPUTING BENDING OF COMPLEX PLATE-BEAM SYSTEMS

MIKAEL MORTENSEN[1,2] AND MIROSLAV KUCHTA[1] AND KENT-ANDRE MARDAL[1,2]

[1] Department of Mathematics, Division of Mechanics, University of Oslo, 0316 Oslo, Norway

[2] Center for Biomedical Computing, Simula Research Laboratory, P.O. Box 134, No-134 Lysaker, Norway

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Abstract. We present a light-weight Python module for computing small deformations of a single plate supported by an arbitrary number of possibly intersecting stiffeners. We show how the problem fits into the framework of abstract saddle point problems and how this abstraction can be exploited for clean design of the code. Stability properties of the resulting linear systems for two different sets of basis functions, namely, the eigenfunctions of the biharmonic operator and specialized Legendre polynomials are discussed.

1 Introduction

In this paper we discuss Galerkin methods for finding the equilibrium state of a physical system formed by a loaded thin plate and several beams which are constrained to deform together with the plate. Let us denote as Ω the subset of the Euclidean plane occupied by the plate. We shall consider the plate supported by k beams labeled by index r from the index set $R = \{1, 2, \cdots, k\}$. Further, each beam is a curve $\Gamma_r = \{x \in \Omega \,|\, x = F_r(s), s \in [s_0, s_1] = \mathcal{I}\}$ that is described by an invertible mapping $F_r : \mathcal{I} \mapsto \Omega$ with Jacobian J_r. Note that for straight beams, which are the main focus of the present work, the Jacobian is simply the length of the beam divided by the length of the interval. In addition we shall for simplicity require that the mapping F_r satisfies two additional constraints $F_r(s_0), F_r(s_1) \in \partial\Omega$. The endpoints of the beam are thus required to lie on the boundary of the plate $\partial\Omega$. Equivalently, the additional constraints

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ensure that no beam ends inside the plate. The equilibrium state of the physical system formed by the plate supported with the beams is found as the solution of a constrained minimization problem

$$u = \min_{v\in V} \mathcal{E}\left(v\right) \quad \text{ subject to } \quad u_0 \circ F_r = u_r, r \in R, \tag{1}$$

where the energy functional $\mathcal{E} : V \mapsto \mathbb{R}$ is defined as

$$\mathcal{E}\left(u\right) = \frac{E_0}{2}\int_{\Omega} \Delta u_0 \, \Delta u_0 + \sum_{r=1}^{k} \frac{E_r}{2}\int_{\mathcal{I}} \frac{\mathrm{d}^2 u_r}{\mathrm{d}s^2}\frac{\mathrm{d}^2 u_r}{\mathrm{d}s^2} J_r^{-3} - \int_{\Omega} f u_0.$$

Here $V = V_0 \times V_1 \times \cdots \times V_k$ is a function space with $k+1$ components. The first component V_0 contains functions that map the plate domain Ω to real numbers and is therefore the space where the plate's vertical displacement u_0 is found. The remaining components in V contain vertical displacements of individual beams u_r, i.e., the scalar functions whose domain is the interval $\mathcal{I}$. The system is loaded by the load f while $E_0, E_r, r \in R$ are constant parameters that, following the Kirchhoff-Love and Euler-Bernoulli hypothesis (see, e.g., [11]), depend on the material and the geometry. Note that the problem (1) has to be closed by suitable boundary conditions. Possible choices are discussed in Section 3

Introducing k Lagrange multipliers $\lambda_k \in Q_k$, where the functions in Q_k map the interval $\mathcal{I}$ to scalars, the constrained problem (1) can be equivalently written as a search for extrema of the Lagrangian $\mathcal{L} : V \times Q \mapsto \mathbb{R}$,

$$\mathcal{L}\left(u, \lambda\right) = \mathcal{E}\left(u\right) + \sum_{r=1}^{k} \int_{\mathcal{I}} \left(u_0 \circ F_r - u_r\right) \lambda_r J_r. \tag{2}$$

The space $Q = Q_1 \times Q_2 \times \cdots \times Q_k$ has k components with r-th component containing a Lagrange multiplier that enforces the matching deformation constraint on the beam with index r. We note that the unconstrained problem (2) is solvable only for suitable pair of spaces V, Q for which the requirements of the Babuška-Brezzi [2, 6] theory are satisfied.

For plate Ω with complex shape the finite element method is arguably the most suitable method to solve the problem (2). If, on the other hand, the domain is simple (e.g. separable), the Galerkin method with globally supported basis functions that take advantage of the structure of the domain (and the underlying PDE) can be applied. Note that the finite element method is an instance of the Galerkin method with trial/test[1] spaces spanned by the functions with local support. Regardless of the basis functions employed, stable discretization of problem (2) requires that the Babuška-Brezzi conditions be satisfied on the constructed finite dimensional spaces. We remark that in addition to stability considerations the finite element discretization of the problem is also complicated by the presence of the biharmonic operator, which requires techniques for fourth order problems

[1] We consider Bubnov-Galerkin method where the trial and test spaces are equal.

(see, e.g., [4]). The approximations of V must be constructed from C^1-continuous elements, e.g., the Argyris element [1], or non-conforming elements like the Morley element [9]. Alternatively, discontinuous elements with suitable stabilization [5] can be used.

Here we shall consider the plate as a simply connected rectangular domain and as such focus on Galerkin methods with test functions having global support. Further, the two selected basis are designed for fourth order problems and thus only the Babuška-Brezzi theory needs to be considered to derive a stable discretization of problem (2). The methods are discussed within a framework for abstract saddle point problems reviewed in Section 2. The framework serves to identify the few common elements (matrices) that are required to easily introduce Galerkin discretizations of (2) based on any set of basis functions. Properties of the two sets of basis functions considered in this paper are then compared in Section 3. Finally in Section 4 the inf-sup condition for the two proposed discretizations is discussed.

2 Abstract framework for saddle point problems

The necessary conditions for the extreme point of the Lagrangian $\mathcal{L}$ defined in (2) form a saddle point system of $2k+1$ equations to be satisfied by $2k+1$ unknowns $(u, \lambda) \in V \times Q$. These equations read

$$
\begin{aligned}
E_0 \int_\Omega \Delta u_0 \, \Delta v_0 - \sum_{r=1}^{k} \int_{\mathcal{I}} v_0 \lambda_r J_r &= \int_\Omega f v_0 \quad \forall v_0 \in V_0, \\
E_r \int_{\mathcal{I}} \frac{\mathrm{d}^2 u_r}{\mathrm{d}s^2} \frac{\mathrm{d}^2 v_r}{\mathrm{d}s^2} J_r^{-3} + \int_{\mathcal{I}} v_r \lambda_r J_r &= 0 \quad \forall v_r \in V_r, r \in R, \\
\int_{\mathcal{I}} (u_r - u_0)\, \mu_r J_r &= 0 \quad \forall \mu_r \in Q_r, r \in R.
\end{aligned} \tag{3}
$$

System (3) fits into the abstract framework for saddle point problems (see eq. Quarteroni [10]). Within the framework, existence and uniqueness of the solution of (3) can be discussed. Here we shall assume that the problem is indeed well-posed and instead use the abstractions of the framework to identify the building blocks for efficient implementation of the Galerkin method for the system. To simplify the notation we let $(\cdot, \cdot)$ denote the L^2-inner product over the plate. Moreover for $r \in R$ the weighted L^2-inner product over the interval $\mathcal{I}$ with the weight J_r is denoted as $(\cdot, \cdot)_r$.

The abstract saddle point problem is defined in terms of bilinear forms $a : V \times V \mapsto \mathbb{R}$, $b : Q \times V \mapsto \mathbb{R}$ and a linear form $L : V \mapsto \mathbb{R}$. To identify these forms in the plate-beam system (3), let $a_0 : V_0 \times V_0 \mapsto \mathbb{R}$ with $a\,(,0)\, u_0 v_0 = E_0\,(\Delta u_0, \Delta v_0)$ and $a_r : V_r \times V_r \mapsto \mathbb{R}$ such that $a_r\,(u_r, v_r) = E_r \left(J_r^{-2} \frac{\mathrm{d}^2 u_r}{\mathrm{d}s^2}, J_r^{-2} \frac{\mathrm{d}^2 v_r}{\mathrm{d}s^2} \right)_r$ for $r \in R$. Moreover we define r bilinear forms $b_r : Q_r \times V \mapsto \mathbb{R}$ by $b_r\,(\lambda_r, u) = (u_0 \circ F_r - u_r, \lambda_r)_r$. It then follows that the bilinear forms a and b are simply

$$a(u,v) = \sum_{r=0}^{k} a_r(u_r, v_r) \quad \text{and} \quad b(\lambda, u) = \sum_{r=1}^{k} b_r(\lambda_r, u).$$

Finally with the linear form $L(u) = (f, u_0)$ the problem (3) is rewritten as an abstract saddle point problem: Find $(u, \lambda) \in V \times Q$ such that

$$a(u,v) + b(\lambda, v) + b(\mu, u) = L(v) \quad \forall (v, \mu) \in V \times Q. \tag{4}$$

To apply the Galerkin method to abstract formulation (4) we construct finite dimensional subspaces $V_{\mathbf{n}} \subset V$ and $Q_{\mathbf{m}} \subset Q$ that approximate V and Q respectively. As such the spaces $V_{\mathbf{n}}, Q_{\mathbf{m}}$ have respectively $k+1$ and k components with their individual dimensions $\dim(V^r_{\mathbf{n}}) = \mathbf{n}_r$, $\dim(Q^r_{\mathbf{m}}) = \mathbf{m}_r$ encoded in multi-indices[2] $\mathbf{n} \in \mathbb{R}^{k+1}, \mathbf{m} \in \mathbb{R}^k$. From this definition it follows that $n = \dim(V_{\mathbf{n}})$ and $m = \dim(Q_{\mathbf{m}})$ are the sizes of the respected multi-indices. Considering the saddle point problem (4) on the constructed subspaces yields a symmetric indefinite linear system

$$\begin{bmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{B}^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix}, \tag{5}$$

for the unknown expansion coefficients $\mathbf{U} \in \mathbb{R}^n, \mathbf{P} \in \mathbb{R}^m$ of the approximate solution $(u_{\mathbf{n}}, \lambda_{\mathbf{m}}) \in V_{\mathbf{n}} \times Q_{\mathbf{m}}$ of (3). Vector $\mathbf{b} \in \mathbb{R}^n$ is obtained by discretizing the linear form L on the space $V_{\mathbf{n}}$.

Matrices $\mathbb{A} \in \mathbb{R}^{n \times n}$, $\mathbb{B} \in \mathbb{R}^{n \times m}$ in (5) inherit the structure of the bilinear forms a, b. Specifically, the structure of a translates into a block diagonal matrix $\mathbb{A}$ consisting of $k+1$ blocks

$$\mathbb{A} = \begin{bmatrix} \mathbb{A}^0 & & & \\ & \mathbb{A}^1 & & \\ & & \ddots & \\ & & & \mathbb{A}^k \end{bmatrix},$$

where the submatrices $\mathbb{A}^r$ are defined as $\mathbb{A}^r_{i,j} = a_r\left(\phi^r_i, \phi^r_j\right)$ for $\phi^r_i, i = 1, 2, \cdots \mathbf{n}_r$ the basis functions of the component $V^r_{\mathbf{n}}$. Moreover, due to structure of b the matrix $\mathbb{B}$ consists of k columns with block structure

$$\mathbb{B} = \begin{bmatrix} \mathbb{C}^1 & \mathbb{C}^2 & \cdots & \cdots & \mathbb{C}^k \\ \mathbb{M}^1 & 0 & \cdots & \cdots & 0 \\ 0 & \mathbb{M}^2 & 0 & \cdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \mathbb{M}^k \end{bmatrix}.$$

[2] A multi-index ι of length n is an n-tuple of positive numbers. The size of multi-index is defined as the sum of its components $\iota_i, i = 1, 2, \cdots, n$.

Each column features two sub-matrices $\mathbb{C}^r \in \mathbb{R}^{n\times\mathbf{m}_r}$ and $\mathbb{M}^r \in \mathbb{R}^{\mathbf{n}_r\times\mathbf{m}_r}$, which enforce the constraint of equal deformation of the plate and r-th beam. The matrix $\mathbb{C}^r$ with entries $\mathbb{C}^r_{i,j} = \left(\phi^0_i \circ F_r, \psi^r_j\right)_r$ enforces the constraint on the plate, while matrix $\mathbb{M}^r$ defined by $\mathbb{M}^r_{i,j} = -\left(\phi^r_i, \psi^r_j\right)_r$ then enforces the constraint on the beam. Note that the $\psi^r_j, j = 1, 2, \cdots, \mathbf{m}_r$ denote the basis functions of the Lagrange multiplier space $Q^r_{\mathbf{m}}$.

It is clear from the structure of the matrices of the linear system (5) that its assembly requires three types of procedures: (i) For every subspace $V^r_{\mathbf{n}}$ a matrix resulting from discretization of the biharmonic operator is needed. Note that for all the components but $V^0_{\mathbf{n}}$ the biharmonic operator is one-dimensional. (ii) For every pair of spaces $V^r_{\mathbf{n}}, Q^r_{\mathbf{m}}, r \in R$ a negative mass matrix $\mathbb{M}^r$ between the two spaces must be computed. (iii) Finally, for every space $Q^r_{\mathbf{m}}$ there must be a procedure for computing matrix $\mathbb{C}^r$, which can be interpreted as a mass matrix between the restriction of $V^0_{\mathbf{n}}$ to beam with label r and the multiplier space $Q^r_{\mathbf{m}}$. The number of procedures is thus significantly reduced if all the spaces involved in the discrete formulation are spanned by the same functions (e.g. all matrices $\mathbb{A}^r, r \in R$ can then be computed by one routine which takes $\mathbf{m}_r$, the size of the required matrix, as an argument). Moreover we shall set $\mathbf{m}_r = \mathbf{n}_r, r \in R$ and thus $V^r_{\mathbf{n}} = Q^r_{\mathbf{m}}$ and all the matrices $\mathbb{M}^r$ are square. A further step to speed up the assembly is a choice of basis functions for which the matrices $\mathbb{M}^r$ and $\mathbb{A}^0, \mathbb{A}^r$ are sparse and their nonzero values can be tabulated. With such basis functions computing the matrices $\mathbb{C}^r$ remains the only bottleneck of the assembly process.

In Bend|P|y[3] the integration of matrices $\mathbb{C}^r$ is built on top of `NumPy`[15] and `SymPy`[14] modules. Specifically, using `SymPy`'s `lambdify` function, each integrand is turned from a symbolic representation to a lambda function which is vectorized for fast evaluation of `NumPy` arrays that contain points of the Gauss-Legendre quadrature. The remaining matrices of the linear system (5) are sparse and efficiently stored in the compressed sparse row containers provided by `scipy.sparse` module. The two sets of basis functions which are currently supported by Bend|P|y are discussed in the next section.

3 Basis functions of the Galerkin method

In order to find a unique solution of the system (3) suitable boundary conditions need to be prescribed on the unknowns (u, λ). Here we shall consider the problem equipped with clamped boundary conditions $u_0 = 0, \partial_n u_0 = 0$ on $\partial\Omega$ together with $u_r = 0, \frac{\mathrm{d}u_r}{\mathrm{d}s} = 0$ on $\partial\mathcal{I}$ and simply supported boundary conditions $u_0 = 0, \Delta u_0 = 0$ on $\partial\Omega$ together with $u_r = 0, \frac{\mathrm{d}^2u_r}{\mathrm{d}s^2} = 0$ on $\partial\mathcal{I}$. With either set of boundary conditions the boundary datum enforced on the multipliers λ_r is the same as that of the beam unknowns u_r. Both boundary conditions are supported in Bend|P|y by constructing test spaces $V_{\mathbf{n}}, Q_{\mathbf{m}}$ such that the respected test functions have the required boundary values.

The clamped boundary conditions are enforced onto the solution by constructing the test spaces of the Galerkin method from functions S_i presented in Shen [13]. Functions

[3] The source code can be obtained from https://github.com/MiroK/lega/tree/bendpy

S_i defined over the interval $\mathcal{I} = [-1, 1]$ are linear combinations of Legendre polynomials where the coefficients of the combinations are chosen such that $S_i = 0, \frac{\mathrm{d}S_i}{\mathrm{d}s} = 0$ in the endpoints of the domain. To span an n-dimensional space $V^r_{\mathbf{n}}, r \in R$ defined over the interval $\mathcal{I}$, the first n functions S_i are taken as the basis. For n^2-dimensional space $V^0_{\mathbf{n}}$ defined over rectangular plate $\Omega = \mathcal{I} \times \mathcal{I}$ we use as the basis functions products of the first n functions in each Cartesian direction. That is, the basis functions have the form $S_i(x) S_j(y), 1 \leq i, j \leq n$. Further the choice of Shen basis functions yields one dimensional bending matrix $\mathbb{A}^r = \mathbb{I}, r \in R$ while the mass matrix of the constructed space is sparse and pentadiagonal. We refer the reader to the original paper [13] for the tabulated non-zero entries of the matrix. Finally the matrix of the two-dimensional biharmonic operator $\mathbb{A}^0 \in \mathbb{R}^{n^2 \times n^2}$ is obtained as a Kronecker product of one-dimensional mass matrix $\mathbb{M} \in \mathbb{R}^n$ and the stiffness matrix $\mathbb{C} \in \mathbb{R}^n$, i.e., $\mathbb{A}^0 = \mathbb{M} \otimes \mathbb{I} + 2\mathbb{C} \otimes \mathbb{C} + \mathbb{I} \otimes \mathbb{M}$. We note that the condition number of matrix $\mathbb{A}^0$ grows exponentially with n.

To enforce simply supported boundary conditions onto the approximate solutions of (3) we consider test spaces constructed from functions $E_k = \left(\frac{2}{\pi}\right)^{1/2} \sin kx$, which are eigenfunctions of the eigenvalue problem for the one-dimensional biharmonic operator

$$\frac{\mathrm{d}^4 u}{\mathrm{d}s^4} = \lambda u \text{ in } \mathcal{I},$$
$$u = \frac{\mathrm{d}^2 u}{\mathrm{d}s^2} = 0 \text{ on } \partial\mathcal{I},$$

with the interval $\mathcal{I} = [0, \pi]$. The corresponding eigenvalues are $\lambda_k = k^4$. In complete analogy to the spaces spanned by functions due to Shen, the basis of the n-dimensional space $V^r_{\mathbf{n}}, r \in R$ over $\mathcal{I}$ is formed by the first n eigenfunctions E_k. The basis of the space $V^0_{\mathbf{n}}$ defined over $\Omega = \mathcal{I} \times \mathcal{I}$ are then formed as tensor products. An attractive property of the eigenfunctions E_k is the fact that all the matrices $\mathbb{A}^0, \mathbb{A}^r, \mathbb{M}^r$ are diagonal. Indeed we have $\mathbb{M}^r = \mathbb{I}$, while $\mathbb{A}^r \in \mathbb{R}^{n \times n}, r \in R$ takes the form $\mathbb{A}^r = \operatorname{diag}(\lambda_k, k = 1, 2, \cdots, n)$. Finally the two-dimensional bending matrix takes the form $\mathbb{A}^0 = \operatorname{diag}(\Lambda_k, k = 1, 2, \cdots, n^2)$ with $\Lambda_{n(i-1)+j} = i^4 + 2i^2 j^2 + j^4, 1 \leq i, j \leq n$. It is clear that the condition number of the matrix grows as n^4.

We shall now compare the convergence properties of the two basis on a simple one-dimensional biharmonic problem

$$\frac{\mathrm{d}^4 u}{\mathrm{d}s^4} = f \text{ in } \omega = (-1, 1) \qquad \text{with } f(x) = \begin{cases} g(x) & x \leq 0 \\ h(x) & x > 0 \end{cases}, \tag{6}$$

which can be viewed as a building block of the constrained plate beam system (3). We remark that the Shen basis solves the problem (6) with clamped boundary conditions, while for the basis of eigenfunctions simply supported boundary conditions are used. The problem is considered with four different manufactured right-hand sides f_i. Each function f_i is determined by two functions, g_i, h_i. This property is symbolically denoted as $f_i = (g_i, h_i)$.

Table 1: The error and convergence rate of the Galerkin method with Shen basis for problem (6) with clamped boundary conditions. For each right-hand side f_i the L^2-norm of the error and the convergence rate (bracketed value) are shown for different polynomial degrees n. Fixed order of convergence for the first three right-hand sides and exponential convergence for f_3 confirm that the convergence rate is determined by the regularity of the solution. Note that for f_3 the error is reduced to machine precision for $n = 18$. Afterwards the convergence rate of the method suffers from finite arithmetics.

n	f_0	f_1	f_2	f_3
4	3.0367E-05(2.34)	1.1478E-05(4.11)	7.5138E-07(8.28)	8.7249E-05(2.65)
6	9.4639E-06(2.88)	2.0211E-06(4.28)	1.1883E-07(4.55)	1.1292E-05(5.04)
8	3.7833E-06(3.19)	5.5737E-07(4.48)	2.7564E-08(5.08)	7.2035E-07(9.57)
10	1.7737E-06(3.39)	1.9868E-07(4.62)	1.0879E-08(4.17)	2.8851E-08(14.42)
12	9.2954E-07(3.54)	8.3836E-08(4.73)	3.5788E-09(6.10)	4.2572E-10(23.12)
14	5.2899E-07(3.66)	3.9893E-08(4.82)	1.1724E-09(7.24)	8.7160E-12(25.23)
16	3.2080E-07(3.75)	2.0775E-08(4.89)	6.8783E-10(3.99)	4.7527E-14(39.03)
18	2.0463E-07(3.82)	1.1004E-08(5.40)	4.3562E-10(3.88)	8.3310E-15(14.78)
20	1.3602E-07(3.88)	5.4562E-09(6.66)	2.5669E-10(5.02)	1.7601E-14(-7.10)
22	9.3564E-08(3.93)	3.2351E-09(5.48)	1.5121E-10(5.55)	2.0605E-13(-25.81)
24	6.3166E-08(4.52)	2.3836E-09(3.51)	7.8326E-11(7.56)	2.8589E-13(-3.76)
26	4.8081E-08(3.41)	1.9361E-09(2.60)	4.5928E-11(6.67)	1.0247E-12(-15.95)
28	3.2319E-08(5.36)	1.2840E-09(5.54)	3.0818E-11(5.38)	1.9021E-12(-8.35)
30	2.0739E-08(6.43)	9.6124E-10(4.20)	1.7277E-11(8.39)	2.7543E-11(-38.74)

With this convention, the right-hand sides considered are $f_0 = (1, 2)$, $f_1 = (1, x + 1)$, $f_2 = (-x^2/2 + x/4 + 3/4, -x^2 + x/4 + 3/4)$ and $f_3 = (\exp x \sin 5x, \exp x \sin 5x)$. Functions f_i belong respectively to spaces $L^2(\omega), H^1(\omega), H^2(\omega)$ and $C^\infty(\omega)$. The smoothness of the solution u of (6) is then four "orders" higher than the given right-hand side. As such, in all but the last case, fixed convergence rate is to be expected from the Galerkin method with Shen basis functions. In the last case the method should converge exponentially. These expectations are confirmed by the results shown in Table 1. On the other hand, a simple error estimate $\|e\|_0 \leq \frac{\|f\|_0}{n^4}$ for the Galerkin method with eigenfunctions E_k shows that the order of convergence of the method should be at least four if the error is to be measured in the L^2-norm. This estimate is confirmed by the results listed in Table 2.

The results presented thus far have validated the Galerkin method with basis of Shen functions and eigenfunctions as viable methods to solve biharmonic equations. Interpreting these as subproblems in the system (3) we claim that the methods are suitable also to tackle the constrained plate-beam problem. At the time of writing we are unable to provide a support for this claim in the form of a convergence study. However as can be seen in Figure 1 the solutions obtained by the numerical methods are physical. The effect of stiffeners is clearly visible in the displacement of the plate. Moreover the constraints remain well respected.

Table 2: The error and convergence rate of the Galerkin method with eigenfunctions for problem (6) with simply supported boundary conditions. Notation from Table 1 is reused. Irrespective of the regularity of the right-hand sides f_i the method converges with fixed order. In agreement with the theoretical analysis, the order is greater then four. Note that for the right-hand side f_2 the observed convergence rate is six. This is due to the fact that the corresponding solution has second derivatives equal to zero on the boundary (cf. functions E_k). Also note that the error for $n = 128$ is of the size of the machine precision and in finite arithmetics cannot be reduced further. Hence for $n = 256$ the rate is no longer six.

n	f_0	f_1	f_2	f_3
8	6.1251E-06(4.11)	5.7545E-06(4.11)	2.8982E-08(5.84)	6.2305E-06(4.49)
16	2.9912E-07(4.36)	2.8123E-07(4.35)	4.1493E-10(6.13)	3.0082E-07(4.37)
32	1.3929E-08(4.42)	1.4545E-08(4.27)	4.8407E-12(6.42)	1.4161E-08(4.41)
64	6.2978E-10(4.47)	5.9356E-10(4.61)	5.6482E-14(6.42)	6.5283E-10(4.44)
128	2.5395E-11(4.63)	2.4049E-11(4.63)	8.5939E-16(6.04)	2.7228E-11(4.58)
256	1.1403E-12(4.48)	1.0861E-12(4.47)	7.0351E-16(0.29)	1.2692E-12(4.42)

4 Condition numbers of plate-beam systems and preconditioning

In this section we briefly discuss a phenomena which is bound to occur in any discretization of a plate-beam system, namely, the condition number of the linear system grows with its size (see e.g. [3]). This growth is due to the saddle point nature of the continuous problem (3) but it can be greatly accelerated by the choice of the discretization. To illustrate this issue let $\mathcal{A}_S$ be the symmetric indefinite matrix in system (5) obtained from discretization of problem (3) by the Shen polynomials. Analogically, let $\mathcal{A}_E$ be the matrix obtained by discretizing the problem with eigenfunctions(odd trigonometric polynomials). Figure 2 shows the spectral condition number κ of both matrices $\mathcal{A}_S, \mathcal{A}_E$ as a function of polynomial degree n. The condition number of matrix $\mathcal{A}_S$ reaches the value of $\kappa = 10^{15}$ very rapidly ($n = 15$). For the same degree the condition number of the eigenfunction matrix $\mathcal{A}_E$ is about ten orders of magnitude smaller. We note that $n = 15$ is a modest degree and to achieve sufficient accuracy of the solution, spaces with larger dimensions might be needed. However with larger spaces the linear systems quickly become stiff and the obtained solution might suffer from round-off errors. This problem can be avoided by solving a preconditioned problem. Here we shall hint on a part of the suitable preconditioner. Note that for simplicity we set $E_r = 1$ for all r.

Investigating the spectra of matrices $\mathcal{A}_S, \mathcal{A}_E$ in Figure 2 it is clear that the growth of the condition number is mainly due to the fact that the for increasing size of the linear system/polynomial degree n its smallest eigenvalue approaches zero. Using results of [12] that build on the work of [7], the smallest eigenvalue defines the discrete Brezzi inf-sup constant β, which determines stability of the discretization. In [12] stability of the finite element discretization of the Stokes(saddle point) system is investigated using the eigenvalue problem for the Schur complement

$$\mathbb{B}^{\mathrm{T}}\mathbb{A}^{-1}\mathbb{B}\mathbf{P} = \beta\mathbb{N}\mathbf{P}, \tag{7}$$

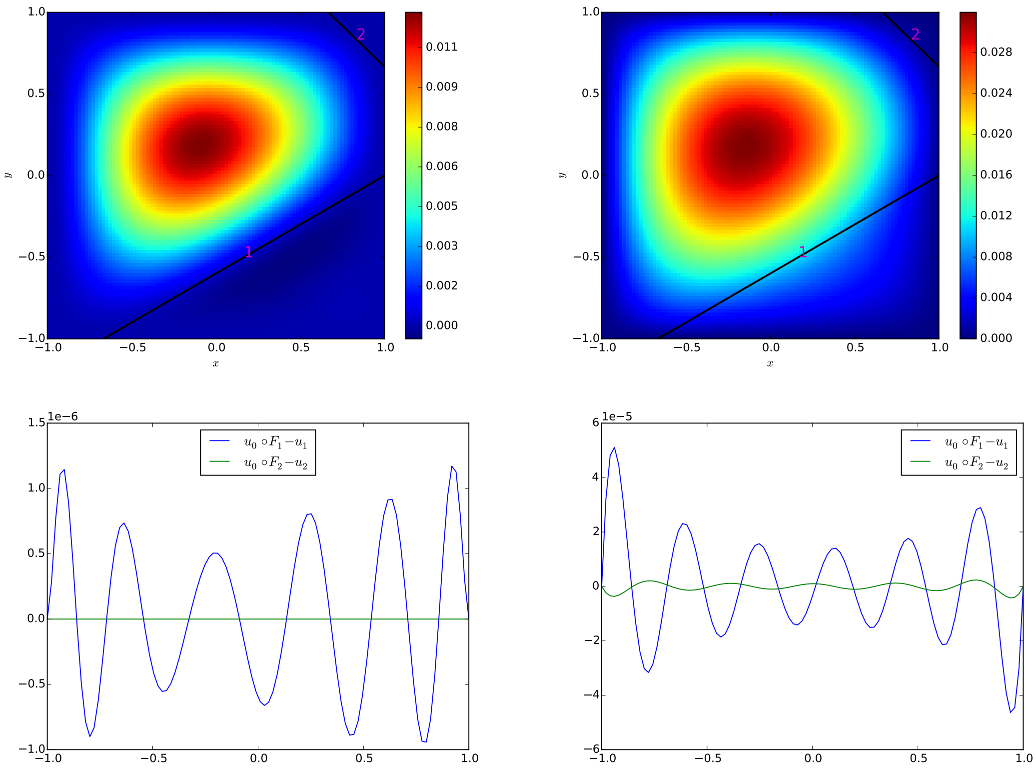

Figure 1: Solution of the plate-beam system (3) obtained by the Galerkin method with Shen functions (left) and eigen functions (right) for $n = 15$. In the top row the vertical displacement u_0 of the plate is plotted. The position of beams is indicated by black lines. Further the index of the beam is shown in magenta. In the bottom row the difference between the plate's displacement on the beams and the beams' displacement is shown. The error of constraints is less than 10^{-4}.

where $\mathbb{N}$ is a symmetric positive definite matrix which represents discretization of the norm on the continuous space Q. The matrix is *a priori* known and the discretization is found stable if there exists a universal lower bound on the smallest of eigenvalues β. That is, stable discretization requires that the smallest (in magnitude) eigenvalues β are bounded from below by a positive constant independent of the size of the linear system/polynomial degree. Here we shall reverse the process: the matrix $\mathbb{N}$ is to be found from the requirement that the eigenvalue with the smallest magnitude remains bounded. Such a matrix can be found by the following algorithm. Let $\mathbb{M}, \mathbb{A}$ the mass matrix and the matrix of the one-dimensional biharmonic operator on $Q_{\mathbf{m}}$. Then $\mathbb{N} = (\mathbb{M}\mathbb{V})\,\Lambda^{-1}\,(\mathbb{M}\mathbb{V})^{\mathrm{T}}$, where $\mathbb{V}$, Λ solve the generalized eigenvalue problem $\mathbb{A}\mathbb{V} = \mathbb{M}\mathbb{V}\Lambda$. Note that the negative power of the diagonal eigenvalue matrix Λ suggests that the continuous spaces for the Lagrange multipliers will be Sobolev spaces with negative index. Also note that with the basis of eigenfunctions we have $\mathbb{M} = \mathbb{V} = \mathbb{I}$ and $\Lambda = \mathbb{A}$. Thus there is no need to solve the generalized eigenvalue problem.

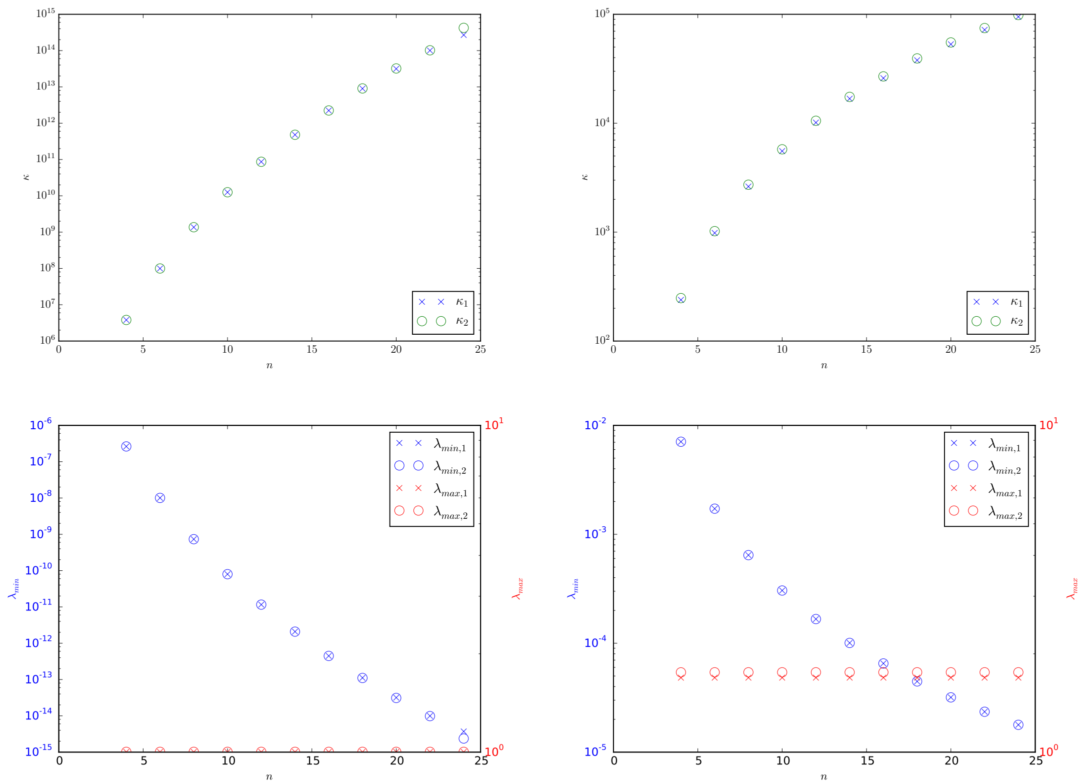

Figure 2: Condition number κ and spectra of matrices $\mathcal{A}_S$ (left column) and $\mathcal{A}_E$ (right column) stemming from discretization of the plate-beam system (3) by Shen polynomials and eigenfunctions with polynomial degrees up to and including n. The top row shows spectral condition number of matrices. Two beam arrangements are considered. Subscripts one corresponds to a single vertical beam $[0,-1]-[0,1]$. Arrangement where additional horizontal beam $[-1,0]-[1,0]$ is added is denoted by subscript two. For both discretizations the condition numbers increase with n. The rate is significantly larger for the Shen basis. Presence of the second beam has little to no effect on the rate and the value of *kappa*. In the bottom row the dependence of the smallest and largest eigenvalues $\lambda_{min}, \lambda_{max}$ on the degree n is shown. The largest eigenvalues are stable, while the smallest ones approach zero and are thus the cause of the growth of the condition number. Consistent with the faster growth of the condition number for matrix $\mathcal{A}_S$ its smallest eigenvalue decreases more rapidly than the smallest eigenvalue of matrix $\mathcal{A}_E$. Note that all plots use logarithmic vertical axis.

The smallest eigenvalue of the problem (7) with constructed matrix $\mathbb{N}$ is shown in Figure 3. The construction has succeeded in stabilizing the smallest eigenvalues as there clearly exists now a positive lower bound independent of the discretization parameter n.

Based on our findings and the theory of operator preconditioners reviewed in [8] or the note of Murphy et al. [16] a possible preconditioner for the plate-beam (5) system is a block diagonal matrix

$$\mathcal{P} = \begin{bmatrix} \mathbb{A}^{-1} & 0 \\ 0 & \mathbb{N}^{-1} \end{bmatrix}. \tag{8}$$

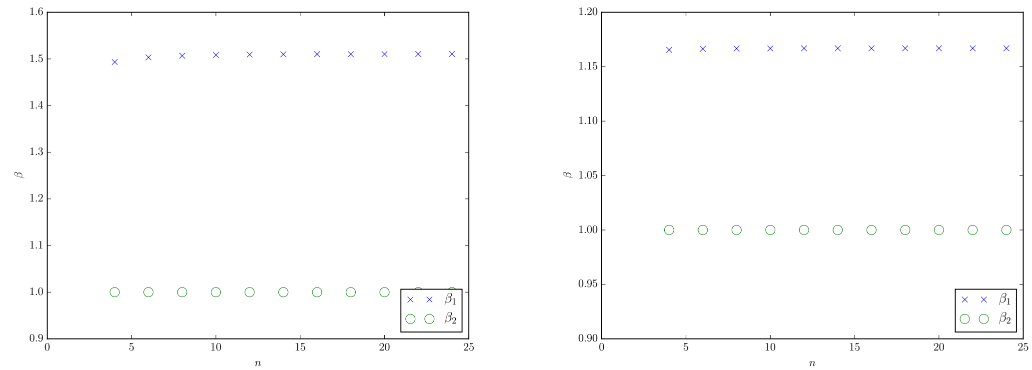

Figure 3: Smallest eigenvalues of the problem (7) with constructed matrix $\mathbb{N}$. On the left the system was discretized with function due to Shen. Result of the system discretized with eigenfunctions are shown on the right. The smallest eigenvalues β_1, β_2 for the system with one and two beams remain constant for all polynomial degrees n.

Here $\mathbb{A}, \mathbb{N}$ are generic matrices from the system (3) and the suggested algorithm that approximates the Schur complement (7). We shall denote $\mathcal{P}_S, \mathcal{P}_E$ the preconditioners obtained from (8) by substituting respectively the matrices obtained from discretizations by Shen polynomials and eigenfunctions. Note that the matrix $\mathcal{P}_S$ is very cheap to compute. In fact it is a diagonal matrix with tabulated values.

The effect of the proposed preconditioner is shown in Figure 4, which shows the spectral condition number and the smallest and largest eigenvalues of the preconditioned systems $\mathcal{P}_S\mathcal{A}_S$ and $\mathcal{P}_E\mathcal{A}_E$. The preconditioners have stabilized the smallest eigenvalues but at the same time linear growth has been introduced to the largest eigenvalues. The growth translates into a linear increase of the condition number of the systems. This is certainly an improvement over the case without the preconditioner (cf. Figure 2). However an ideal preconditioner would yield stable/constant condition numbers. Moreover the preconditioner has to be robust with respect to the material/geometry properties $E_0, E_r, r \in R$ as well as the arrangement of the beams. This aspect of the constructed matrix (8) has not been investigated here.

5 Conclusions

In this paper we have discussed generic mathematical abstractions that lead to efficient implementation of Galerkin methods for the constrained optimization problem describing the coupled deformation of a thin plate supported by an arbitrary arrangement of support beams. Two specific sets of basis functions based on linear combinations of Legendre polynomials and eigenfunctions of the one-dimensional biharmonic operator have been described. Both methods have been implemented in a python package Bend|P|y which was used to explore efficient preconditioners for the resulting linear system. Some elements of the efficient preconditioner have been identified. A complete robust preconditioner is

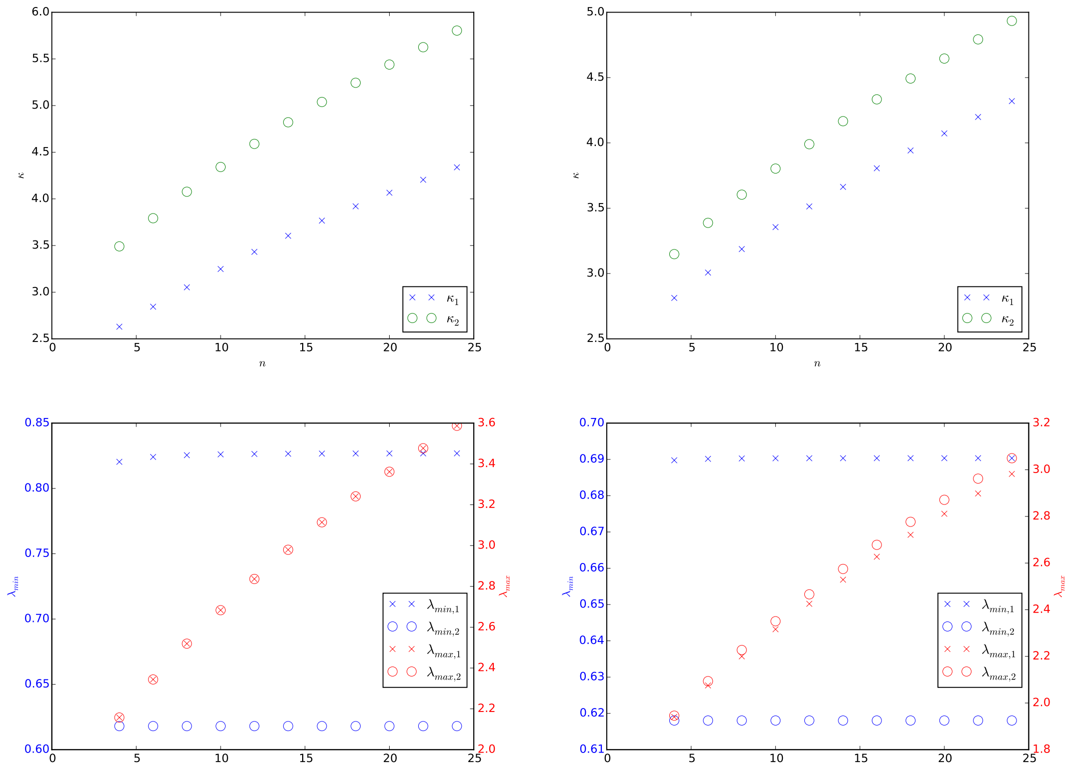

Figure 4: Condition number κ and spectra of matrices $\mathcal{P}_S\mathcal{A}_S$ (left column) and $\mathcal{P}_E\mathcal{A}_E$ (right column) stemming from discretization of the plate-beam system (3) and specialization of (8) by Shen polynomials and eigenfunctions with polynomial degrees up to and including n. Notation from Figure 2 is used. (Top row) For both discretizations the condition numbers grow linearly with n. (Bottom row) The growth is due the growth of the largest eigenvalues of the matrices. Presence of the second beam seems to have negligible effect on the condition number. Note that in all the plots the vertical scale is linear.

currently being pursued and is the topic of future work.

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