EE3110 - Probability Foundations for Electrical Engineers
Tutorial - Week 4

Please submit solutions to the 2 starred questions in moodle for assignment submission by **Sept 22, 11:59 PM**.

---

1. Let X be a random variable with probability density
$$f(x) = \begin{cases} c(1-x^2) & -1 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$
   (a) What is the value of c?
   (b) What is the cumulative distribution function of X?

2. Let X and Y be identical and independent exponential random variables with parameter λ. Define $U = X + Y$, $V = X - Y$. Find the joint pdf of U, V and the marginal pdfs of U and of V.

3. Let X and Y be independent geometric random variables with $X \sim \text{Geo}\,(1/2)$ and $Y \sim \text{Geo}\,(1/3)$. Evaluate the following:
   (a) $\mathbb{P}(X > Y)$
   (b) $\mathbb{P}(X = Y)$

4. Let N be a geometric random variable with parameter p and X_i, $i = 1, 2, \ldots$, be *i.i.d* Poisson random variables with mean λ. Find the p.m.f of $Y = \sum_{i=1}^{N} X_i$, i.e. Y is a random variable obtained by taking a sum of geometric number of independent Poisson random variables.

5.* Let X be a Poisson random variable with mean $\lambda > 0$. Given $X = x$, the random variable Y is Binomial with parameters (x, p). Define $Z = X - Y$. (a) What are the (marginal) distributions of Y and Z? (b) Are Y and Z independent?

6. Let X_1 and X_2 be independent Poisson random variables with parameters $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively. Define $X = X_1 + X_2$. Find the p.m.f. of X.

7. Let X be a discrete random variable with distribution $\{\overset{1/4}{-1}, \overset{1/8}{0}, \overset{1/8}{1}, \overset{1/2}{2}\}$.
   (a) Let the random variable Y be defined by $Y = X^2$. Calculate the probability mass function of Y.
   (b) Find a function f such that $f(X)$ is uniformly distributed.

8. Let X be a random variable that takes values from 0 to 9 with equal probability 1/10.
   (a) Find the p.m.f of the random variable $Y = X \bmod 3$. [7 *mod* 4 = 3 *(remainder)*]
   (b) Find the p.m.f of the random variable $Y = 5 \bmod (X + 1)$.

9. We measure the resistance R of each resistor in a production line and accept only the resistors whose resistance is between 96 and 104 ohms. Find the percentage of the accepted units, if
   (a) R is uniform between 95 and 105 ohms.
   (b) R is normal with $\mu = 100$ and $\sigma = 2$ ohms.

10.* Let X and Y be independent exponential random variables with parameters $\lambda = 1$ and $\mu = 3$. Then, if $Z = min(X, Y)$ is a random variable
   (a) the value of $\mathbb{P}[X = Z]$ is?
   (b) is it true that the random variable $Z = min(X, Y)$ is independent of the event $X < Y$?

11. Suppose that the joint distribution of X and Y is uniform over a set A in the xy-plane. For which of the following sets A are X and Y independent?
   (a) A circle with a radius of 1 and with its center at the origin
   (b) A square with vertices at the four points (0, 0), (1, 1), (0, 2), and (-1, 1)
   (c) A rectangle with vertices at the four points (0, 0), (0, 3), (1, 3), and (1, 0)

12. Let X and Y have the joint pmf

$$p_{XY}(x, y) = \frac{C}{(x + y - 1)(x + y)(x + y + 1)}, \quad x, y \in \{1, 2, 3, ..\}.$$

   Determine the following:
   (a) The value of C
   (b) The (marginal) pmfs of X and Y
   (c) The pmf of $Z = X + Y$
   (d) $\mathbb{P}(X = Y)$ (express the answer in summation form)

13. Let X be Geometric(p). Let $X_i = I(X > i)$ for $i \in \{1, 2, \ldots\}$, i.e. $X_i = 1$ if $X > i$, and 0 otherwise. Find the joint PMF of (X_i, X_j) for $i, j \in \{1, 2, \ldots\}$. Are X_i and X_j independent?

14. Consider a triangle with height h. Let X be the distance from a point randomly chosen within the triangle to the base of the triangle. What is the CDF and the PDF of X?

15. Consider two continuous random variables Y and Z, and a random variable X that is equal to Y with probability p and to Z with probability $1 - p$. (a) Show that the PDF of X is given by

$$f_X(x) = p f_Y(x) + (1 - p) f_Z(x)$$

   (b) Calculate the CDF of the two-sided exponential random variable that has PDF given by

$$f_X(x) = \begin{cases} p\lambda e^{\lambda x}, & \text{if } x < 0 \\ (1 - p)\lambda e^{-\lambda x}, & \text{if } x \geq 0 \end{cases}$$

   where $\lambda > 0$ and $0 < p < 1$.

16. A defective coin minting machine produces coins whose probability of heads is a random variable P with PDF

$$f_P(p) = \begin{cases} pe^p, & p \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

A coin produced by this machine is selected and tossed repeatedly, with successive tosses assumed independent. (a) Find the probability that a coin toss results in heads. (b) Given that a coin toss resulted in heads, find the conditional PDF of P. (c) Given that the first coin toss resulted in heads, find the conditional probability of heads on the next toss.

17. Athletes compete one at a time at the high jump. Let X_j be how high the j^{th} jumper jumped, with $X_1, X_2, ...$ i.i.d. with a continuous distribution. We say that the j^{th} jumper set a record if X_j is greater than all of $X_{j-1}, ..., X_1$. Is the event "the 110th jumper sets a record" independent of the event "the 111th jumper sets a record"? Justify your answer by finding the relevant probabilities in the definition of independence and with an intuitive explanation.

18. Let $U \sim \text{Unif}(0,1)$. Using U, construct a r.v. X whose PDF is $\lambda e^{-\lambda x}$ for $x > 0$ (and 0 otherwise), where $\lambda > 0$ is a constant, then X is said to have a Exponential distribution; this distribution is of great importance in engineering, chemistry, survival analysis, and elsewhere.