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Let's Make a Deal: The Player's Dilemma

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Implementing conditional probability concepts and understanding their practical implications can be difficult tasks for the student. Textbook problems are often set in artificial situations that neither require nor inspire real-world thinking on the student's part. Recently an old problem appeared in *Parade Magazine* in a new setting that is realistic, widely known, and invariably of great interest. The solution and failed attempts at solution are rich in their lessons in thinking about conditional probability.

KEY WORDS: Conditional probability; Sample space.

In a trio of recent columns, titled "Ask Marilyn," in *Parade Magazine* (vos Savant 1990a,b, 1991) the following question was posed: "Suppose you're on a game show and given a choice of three doors. Behind one is a car; behind the others are goats. You pick door No. 1, and the host, who knows what's behind them, opens No. 3, which has a goat. He then asks if you want to pick No. 2. Should you switch?" (vos Savant 1990b). Marilyn vos Savant, the column author and reportedly holder of the world's highest I.Q., replied in the September article "Yes you should switch. The first door has a 1/3 chance of winning, but the second door has a 2/3 chance." She then went on to give a dubious analogy to explain the choice. In the December article letters from three Ph.D.'s appeared saying that vos Savant's answer was wrong, two of the letters claiming that the correct probability of winning with either remaining door is 1/2. Ms. vos Savant went on to defend her original claim with a false proof and also suggested a false simulation as a method of empirical verification. By the February article a full scale furor had erupted; vos Savant reported "I'm receiving thousands of letters, nearly all insisting I'm wrong . . .. Of the letters from the general public, 92% are against my answer; and of letters from universities, 65% are against my answer." Nevertheless vos Savant does not back down, and for good reason, as, given a certain assumption, her answer is correct. Her methods of proof, however, are not.

This problem, in a variety of other guises, such as the "prisoner's dilemma," may be found in probability textbooks and volumes of mathematical recreations. Mosteller (1965, p. 28) remarked that "Of all the problems people write me about, this one brings in the most letters." Smith (1968, p. 70), who, interestingly, did not give an explicit solution, said "This should be called the Serbelloni problem since it nearly wrecked a conference on theoretical biology at the villa Serbelloni in the summer of 1966; it yields at once to common sense or to Bayes' theorem." Mosteller's solution, like the others we have seen, assumes information that is not given in the problem.

As will be seen, the problem in one form admits an elegant solution that assumes no additional information, as we discovered during a long Monday afternoon of at times animated discussion within our department. Even having found the solution, the discussion continued nearly unabated for the rest of the week as our nonstatistician colleagues in neighboring offices, who could not help but overhear our cries and shouts that first day, came to offer their opinions and analyses. Refusing to believe their arguments wrong, the whirlwind would begin again. Posting the problem on the math club bulletin board increased the clamor as undergraduate students grappled with their arguments. So it was that we repeatedly found what vos Savant's article and the quotes above demonstrate: that this apparently innocuous little problem can be erroneously "solved" in a variety of ways and that the nature of these errors can be quite subtle, at least to those not accustomed to thinking in terms of conditional probability (and sometimes even to those of us who are!). That this problem so entertainingly demonstrates the pitfalls of conditional probability calculations and interpretations alone warrants its wider dissemination for instructional purposes. That we have not previously seen the solution offered in this article and, perhaps, a desire to at least partially put the problem to rest, also motivate us to submit this note.

1. TO SWITCH OR NOT TO SWITCH

We begin by enumerating and discussing the most appealing of the false solutions. To avoid any confusion, here is the situation: The player has chosen door 1, the host has then revealed a goat behind door 3, and the player is now offered the option to switch. Thus is the player having been given additional information, faced with a conditional probability problem. The event of interest is "win by switching"; both "lose by switching" and "win by not switching" are complements of this event. For clarity and equality, we refer to the host as "he" and the player as "she."

Solution F1. If, regardless of the host's action, the player's strategy is to never switch, she will obviously win the car 1/3 of the time. Hence the probability that she wins if she does switch is 2/3.

Solution F2. The sample space is {AGG, GAG, GGA}, each point having probability 1/3, where the triple AGG, for instance, means auto behind door 1, goat behind door 2, and goat behind door 3. The player

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choosing door 1 will win in two of these cases if she switches, hence the probability that she wins by switching is 2/3.

Solution F3. Play the game a few hundred times with the "host" using three cards: two jokers for the goats and an ace for the car. This will verify that the player who switches wins 2/3 of the time.

Solution F4. The original sample space and probabilities are as given in Solution F2. However, since door 3 has been shown to contain a goat, GGA is no longer possible. The remaining two outcomes form the conditional sample space, each having probability (1/3)/(1-(1/3)) = 1/2. Hence the probabilities of winning by switching and by not switching, given that door 3 has a goat, are both 1/2, and the player's choice is a moot one in so far as probability is concerned.

Solution F5. The probability that a player is shown a goat is 1. So conditioning on this event cannot change the probability of 1/3 that door 1 is a winner before a goat is shown; that is, the probability of winning by not switching is 1/3, and by switching is 2/3.

Solution F6. The sample space is {AGG2, AGG3, GAG3, GGA2} where the letter triples have the same meaning as in Solution F2, and the number indicates the door opened by the host. The probabilities for the sample points are, in order, 1/6, 1/6, 1/3, and 1/3. Labeling events W_s = "win by switching" and $D3$ = "goat shown behind door 3," $\Pr(W_s \mid D3) = \Pr(W_s \text{ and } D3)/\Pr(D3) = \Pr(\text{GAG3})/\Pr(\text{AGG3,GAG3}) = (1/3)/(1/6 + 1/3) = 2/3$.

Each of these solutions is, on its face, attractive, and a student or nonprobabilist may be hard pressed to catch some of the flaws. Assigned near the end of a first course in probability (as was done by one of us immediately after the appearance of vos Savant's second article), an instructor is likely to hear them all, and each has its own instructional value.

F1 is immediately appealing, and we found its advocates quite reluctant to capitulate. F1's beauty as a false solution is that it is a true statement! It just does not solve the problem at hand. F1 is a solution to the unconditional problem, which may be stated as follows: "You will be offered the choice of three doors, and after you choose the host will open a different door, revealing a goat. What is the probability that you win if your strategy is to switch?" The distinction between the conditional and unconditional situations here seems to confound many, from whence much of the pedagogic and entertainment value is derived.

Solution F2 is offered by vos Savant in the December article, and may perhaps best be thought of as an attempted solution to the unconditional problem. That it is not a solution to the stated conditional problem is apparent in that the outcome GGA is not in the conditional sample space, since door 3 has been revealed as hiding a goat.

Several people, frustrated by contradictory arguments or failing to believe their arguments wrong, suggested schemes like F3 to settle the issue, which was proposed by vos Savant in the December article (compare, also, the classroom experiment proposed by vos Savant in the February column). It is so appealing because it models F1: This is a correct simulation for the unconditional problem, but not for the conditional problem. The correct simulation for the conditional problem is of course to examine only those trials where door 3 is opened by the host. The modeling of conditional probabilities through repeated experimentation can be a difficult concept for the novice, for whom the careful thinking through of this situation can be of considerable benefit.

In F4 we at last have an attempt to solve the conditional problem. This attempt fails because the original sample space is incorrectly specified. We suspect this may be the solution used by the Ph.D.'s quoted in the December article. This answer is also obtained by the naive statement that, since there are two doors left, one of which contains a goat, each has probability 1/2. By whatever method of determination, 1/2 has in our experience been the most popular "wrong" answer.

Solution F5, like F1, is a true statement that answers a different problem. F5 is incorrect because it does not use the information in the number of the door shown.

Solution F6 (cf. Mosteller 1965) attempts to correct the wrongly specified sample space of F4. One must ask, however, how the probabilities for this sample space are determined. It turns out that this is a correct specification only if one assumes a certain strategy on the part of the host. We will show that the problem can be solved without any assumptions of this type, which is to say the problem can be solved.

Before proceeding, another point not addressed by the question needs to be considered. If the player's original choice does not contain the auto, does the host have the option of revealing the door that does? That is, in the playing of the game, does the host always give the player another chance, or does he sometimes end the game immediately by opening the door with the car? Presumably the host would say something like "Now that you have chosen door 1, the lovely Linda will reveal door 3. Should it be a goat you will be offered the chance to . . .," though this course could be followed without informing the player or audience. Regardless of the answer, we can write down the following sample space and probabilities for the player who chooses door 1 (note that we are assuming here that the host did not have the option of immediately opening the chosen door; see Section 2):

Outcomes:	AGG2	AGG3	GAG2	GAG3	GGA2	GGA3
Probability:	$p_{12}/3$	$p_{13}/3$	$p_{22}/3$	$p_{23}/3$	$p_{32}/3$	$p_{33}/3$,

where p_{ij} = the probability that the host opens door j given that the car is behind door i, so that $p_{i2} + p_{i3} = 1$. The multiplier of 1/3 provides the connection between the conditional p_{ij}'s and the unconditional probabilities of the outcomes and says that the probability that the auto is behind any particular door is 1/3. Thus

$$\Pr(W_s \mid D3) = \Pr(\text{GAG3})/\Pr(\text{AGG3,GAG3})$$

$$= \frac{p_{23}}{p_{13} + p_{23}}.$$

The problem is solved by the assignment of values for the p_{ij}, being properly viewed as a quantification of the host's strategy. For instance, if $p_{12} = p_{22} = p_{32} = p$, say, and $p_{13} = p_{23} = p_{33} = q = 1 - p$ (perhaps a host ignoring the information he has), then $\Pr(W_s \mid D3) = 1/2$ regardless of p and q. So the popular answer of 1/2 can be justified via a plausible scenario, but in a manner that allows more generality to the problem than F4, or any of the other "solutions," has taken. What then are F1–F6 trying to solve?

Though it is not stated in her reader's question, vos Savant makes clear in the September article that the problem she wishes to solve is one in which the host never has the option of showing the car, saying ". . . the host, who knows what's behind the doors and will always avoid the one with the prize . . . ." If these are the rules of the game, which we will call the vos Savant scenario (and which F1–F6 *are* trying to solve), then

$$p_{12} = p, \qquad p_{13} = q = 1 - p,$$
$$p_{22} = p_{33} = 0, \qquad p_{23} = p_{32} = 1.$$

So $\Pr(W_s \mid D3) = 1/(1 + q)$. We then have a quite elegant solution, for $\Pr(W_s \mid D3) \geq 1/2$ for every q. Hence we need not know, or make any assumption about, the host's strategy to state that the answer to the original question is yes. The player should switch, for she can do no worse and may well improve her chances. Incidentally, $\Pr(W_s \mid D3) = 2/3$ iff $p = q = 1/2$, which is solution F6. Though some may argue that this is a natural host strategy, it is clear that this makes an assumption that is not needed to solve the problem.

It is interesting that the answer of 1/2 can occur in the vos Savant scenario. This is the host strategy $q = 1$, which says that the host opens door 3 whenever possible. One would think that such a deterministic strategy could only help the player, but once door 3 is open, just the opposite is true, in the sense that the player is left unable to improve her odds beyond what the opening of the door automatically provides. It is also enlightening to consider, relative to this, the host strategy $q = 0$, as well as the consequences of both of these strategies (or any other q) in the unconditional game, whether the player knows the strategy or not.

The above solution engenders disappointment in some students, who feel that there should be a single answer for $\Pr(W_s \mid D3)$ not depending on unknown probabilities. This provides an excellent opportunity to bring in the Bayesian perspective. For instance, the noninformative prior in the vos Savant scenario makes this probability

$$\int_0^1 \frac{1}{1+q}\, dq = \ln(2) \cong .693.$$

As another example, a player who thinks the host may tend strongly towards $p = q = 1/2$ may take prior $f(q) = 4q$ for $0 < q < 1/2$, $f(q) = 4 - 4q$ for $1/2 \leq q < 1$, which yields posterior $\Pr(W_s \mid D3) = 20 \ln(2) - 12 \ln(3) \cong .680$. Although posterior $\Pr(W_s \mid D3)$ can be made less than 2/3 by priors with large weight on q near 1, it can never be less than 1/2. So Bayesian or non-Bayesian, the decision is the same: The player should switch.

Turning back now to the more general problem and away from the vos Savant scenario, we see that, depending on the host's strategy, it is possible for the player to do better by not switching. This just requires that

$$1 - \frac{p_{23}}{p_{13} + p_{23}} > \frac{1}{2},$$

that is,

$$p_{23} < p_{13}.$$

Unfortunately, since the player will not generally know the host's strategy, she will not know whether the condition holds or not. Thus we cannot answer the question for the player in the general case.

It appears that the only hope for solution in the general case is specification of a prior for (p_{13}, p_{23}). Independent uniform priors on these two probabilities gives posterior $\Pr(W_s \mid D3) = 1/2$ (the popular answer again!).

2. CONCLUSIONS

In general, we cannot answer the question "What is the probability of winning if I switch, given that I have been shown a goat behind door 3?" unless we either know the host's strategy or are Bayesians with a specified prior. Nevertheless, in the vos Savant scenario we can state that it is always better to switch. The fact that $\Pr(W_s \mid D3) \geq 1/2$, regardless of the host's strategy, is the key to the solution.

The unconditional problem is of interest too, for it evaluates the proportion of winners out of all games with the player following a switch strategy. It is instructive to express this as a mixture of the two conditional cases:

$$\Pr(W_s) = \Pr(W_s \mid D3)\Pr(D3) + \Pr(W_s \mid D2)\Pr(D2)$$

$$= \frac{p_{23}}{(p_{13} + p_{23})} \frac{(p_{13} + p_{23})}{3}$$

$$+ \frac{p_{32}}{(p_{12} + p_{32})} \frac{(p_{12} + p_{32})}{3}$$

$$= \frac{p_{23} + p_{32}}{3}.$$

We see that one cannot do better than 2/3 in the unconditional game, and that the vos Savant scenario maximizes the overall efficacy of the switch strategy. This is no surprise, for the vos Savant scenario just says that the host always gives the player another chance. From the host's perspective, keeping informed players from always switching (which would dilute the audience excitement and tension that the offer to switch is supposed to create) requires *not* adopting the vos Savant scenario.

Those who are interested may wish to generalize these results by allowing the host the option of immediately opening the player's chosen door. The considerations for the conditional game do not change, but there are other questions to ask in the unconditional game if the outcome of winning immediately is to be taken into account. One could also consider n doors as does vos Savant in the September article, but we do not see, as she seems to, that this offers any particular insight for the case $n = 3$. Indeed, we found that this approach shows mainly that $n = 3$ is the most interesting case. Other generalizations appear to be of less interest. One possibility is to incorporate prior information on the part of the player as to the location of the car, or, related to this, to allow nonuniform probabilities of assignment of the car to the three doors, but these are unlikely to correspond to a real playing of this particular game show situation.

The intricacies of this simple problem make it an excellent teaching tool, as can be seen from the insights offered by the false solutions F1–F6 and the correct resolution. But be forewarned, should your students know the history of this problem, one will invariably complain, "How do you expect me to solve a problem that stumped scores of Ph.D.'s and confused the world's most intelligent person?"!

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Comment

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Morgan, Chaganty, Dahiya, and Doviak, in their solution to the three door game show problem, conclude with the amusing yet valid question of how to respond to the student who, having been assigned the problem, complains, "How do you expect me to solve a problem that stumped scores of Ph.D.'s and confused the world's most intelligent person?" This question deserves a well considered response and is best answered by separating it into two distinct issues. The first is concerned with clarity of problem definition, and the second is concerned with why sensible and mathematically well-trained people, given that they agree on what the problem is, still get the wrong solution. References to both may be found in historical as well as current literature.

The question of problem definition, undoubtedly a complication for those attempting to solve the three door game show problem, is not without its precedents. For example, in 1899 Joseph Bertrand (Weatherford 1982, p. 56) posed a problem that has come to be known as Bertrand's Paradox, and which may be stated as follows: "for a given circle, what is the probability that a random chord is longer than the side of an inscribed equilateral triangle?" Weatherford noted that "At least three different solutions are possible," and continued:

> 1 If one attends to the end-points of the random chord and computes their possible location, the resulting probability is 1/3.
>
> 2 If one attends to the location of the chord's mid-point along the length of the diameter which bisects it, the probability is 1/2.
>
> 3 Finally, if one asks whether the mid-point of the chord does or does not fall within a concentric circle of appropriate diameter, the probability seems to be 1/4.

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Without a clear understanding of the precise intent of the questioner, there can be no single correct solution to any problem. Thus, with respect to the three door problem, the answer is dependent on the assumptions one makes about the intent of the one who initially posed the question. Marilyn presented what Morgan et al. call the "vos Savant scenario" and proffered the correct answer. Simply put, and quite clear considering her suggestions for simulation procedures in her two later columns, the host is to be viewed as nothing more than an agent of chance who *always* opens a losing door, reveals a goat, and offers the contestant the opportunity to switch to the remaining, unselected door. "Anything else is a different question," she says. That she didn't offer a rigorous, mathematical proof in a popular Sunday supplement does her no discredit. Perhaps some of Marilyn's readers did suspect an alternate game in which the host has an ulterior motive, but none of the published letters even hinted at this possibility. Unfortunately, not being clairvoyant, it is impossible to know what assumptions they may have been operating under. It is reasonable to infer, however, from the bold certainty of their statements, that either they understood precisely the intent of the question and simply got it wrong or that they recognized the further complexities and still got it wrong.