EE2025: Engineering Electromagnetics

Tutorial 4: Antennas

July-Nov 2024

- Use appropriate approximations and work with reasonable assumptions wherever necessary.
- Good familiarity with the Maxwell's equations helps.
- Visualizing the fields in three-dimensional space also helps.

Calculating Field from Potential

1. Recall the expression for $\mathbf{A}(\mathbf{r})$ for a Hertz dipole antenna from class notes. By converting the direction of $\mathbf{A}$, which is $\hat{z}$, into spherical polar coordinates, write out $\mathbf{A}$ fully in spherical polar coordinates. Next, use the definition of $\mathbf{A}$ to derive the magnetic field $\mathbf{H}$ using the expression for curl in spherical coordinates, i.e. $\nabla \times \mathbf{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$.

Solution:

$$\mathbf{A} = A_z \hat{z}$$

$$\mathbf{A} = A_z \cos\theta \hat{r} - A_z \sin\theta \hat{\theta}$$

$$\nabla \times \mathbf{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_z\cos\theta & -A_z\sin\theta & 0 \end{vmatrix}$$

$$\begin{aligned}\nabla \times \mathbf{A} &= \frac{1}{r^2 \sin\theta}(r\sin\theta\hat{\phi}) \left(\frac{\partial}{\partial r}\left(-A_z \sin\theta\right) - \frac{\partial}{\partial \theta}\left(A_z\cos\theta\right) \right) \\ &= \frac{\mu_0 I dl e^{j\omega t}}{4\pi r} \left(\frac{\partial}{\partial r}\left(-e^{-j\beta r}\sin\theta\right) - \frac{\partial}{\partial\theta}\left(\frac{e^{-j\beta\gamma}}{\gamma}\cos\theta\right)\right) \\ &= \frac{\mu_0 I d e^{j\omega t}}{4\pi r}\left(i\beta e^{-j\beta r}\sin\theta + \frac{e^{-j\beta r}}{r}\sin\theta\right) \\ &= \frac{\mu I dl e^{(\omega t - \beta r)}\sin\theta}{4\pi r}\left(j\beta + \frac{1}{r}\right)\end{aligned}$$

$$\frac{\nabla \times \mathbf{A}}{\mu} = \mathbf{H}$$

$$\boxed{\mathbf{H} = \frac{I_0 \cdot dl e^{j(\omega t} - \beta r) \sin\theta}{4\pi r}\left\{ j\beta + \frac{1}{r}\right\}\hat{\phi}}$$

$$\mathbf{E} = \frac{1}{j\omega t}(\nabla \times \mathbf{H})$$

$$\boxed{\mathbf{E} = \frac{I_0 dl e^{j(\omega t - \beta r)}}{4\pi r}\left(\frac{2\cos\theta}{\omega}\left(\frac{\beta}{r^2} - \frac{j}{r^3}\right)\hat{r} + \sin\theta\left(\frac{j\beta^2}{\omega r} + \frac{\beta}{\omega r^2} - \frac{j}{\omega r^3}\right)\hat{\phi}\right)}$$

2. The magnetic vector potential (in phasor form) at point $P(r, \theta, \phi)$ due to a small antenna located at the origin is given by

$$\mathbf{A_s} = \frac{50 e^{-j\beta r}}{r}\hat{x}$$

where $r^2 = x^2 + y^2 + z^2$. Find $E(r, \theta, \phi, t)$ and $H(r, \theta, \phi, t)$ at the far field.

Solution:

Using vector transformation,
$A_{rs} = A_{xs} \sin\theta \cos\phi$,
$A_{\theta s} = A_{xs} \cos\theta \cos\phi$,
$A_{\phi s} = -A_{xs} \sin\phi$

$$\mathbf{A_s} = \frac{50 e^{-j\beta r}}{r} \left(\sin\theta \cos\phi \, \hat{r} + \cos\theta \cos\phi \, \hat{\theta} - \sin\phi \, \hat{\phi} \right)$$

$$\mathbf{H_s} = \frac{\nabla \times \mathbf{A_s}}{\mu}$$

$$\mathbf{H_s} = \Big(\frac{-100 \cos\theta \sin\phi}{\mu r^2 \sin\theta} \, \hat{r} - \frac{50}{\mu r^2} (\sin\theta + j\beta r) \sin\phi \, \hat{\theta} - \frac{50}{\mu r^2} \cos\theta \cos\phi (1 + j\beta r) \, \hat{\phi} \Big) e^{-j\beta r}$$

At the far field, the field is dominated by the radiation field ($\frac{1}{r}$ term). Hence

$$\mathbf{H}_\mathrm{s} = \frac{-j50}{\mu r} \beta e^{-j\beta r} \left(\sin\phi \, \hat{\theta} + \cos\theta \cos\phi \, \hat{\phi} \right)$$

$$\mathbf{E_s} = -\eta \hat{r} \times H_s = \frac{-j50 \beta \eta e^{-j\beta r}}{\mu r} \left(\sin\phi \, \hat{\phi} - \cos\theta \cos\phi \, \hat{\theta} \right)$$

$$\mathbf{H} = \mathrm{Re}\left[H_s e^{j\omega t} \right] = \frac{50}{\mu r} \beta \sin(\omega t - \beta r) \left(\sin\phi \, \hat{\theta} + \cos\theta \cos\phi \, \hat{\phi} \right) \; A/m$$

$$\mathbf{E} = \mathrm{Re}\left[E_s e^{j\omega t} \right] = \frac{-50 \eta \beta}{\mu r} \sin(\omega t - \beta r) \left(- \sin\phi \, \hat{\phi} + \cos\theta \cos\phi \, \hat{\theta} \right) \; V/m$$

Power and Radiation Patterns in a Hertz Dipole

3. A magnetic field strength of $5\,\mu$A/m is required at a point on $\theta = \pi/2$, 2 km from an antenna in air. Neglecting ohmic loss, how much power must the antenna transmit if it is a Hertzian dipole of length $\lambda/25$?

Solution: For a Hertzian dipole,

$$|H_{\phi s}| = \frac{I_\mathrm{o} \beta dl \sin\theta}{4\pi r}, \qquad \text{in the far field}$$

where $dl = \lambda/25$ or $\beta dl = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{25} = \frac{2\pi}{25}$. Hence,

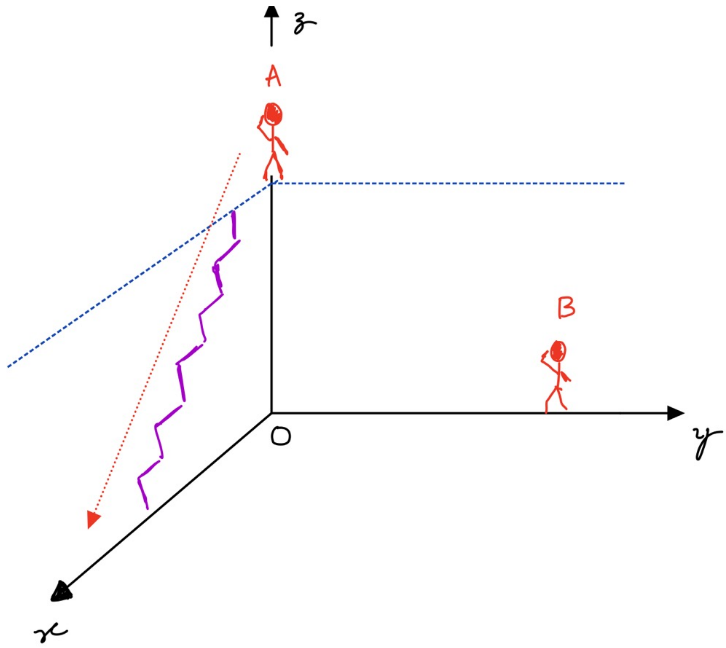
$$5 \times 10^{-6} = \frac{I_\mathrm{o} \cdot \frac{2\pi}{25}(1)}{4\pi\,(2 \times 10^3)} = \frac{I_\mathrm{o}}{10^5}; \; I_\mathrm{o} = 0.5 \text{ A}$$

$$P_\mathrm{rad} = 40\pi^2 \left[\frac{dl}{\lambda} \right]^2 I_\mathrm{o}^2 = \frac{40\pi^2 (0.5)^2}{(25)^2}$$
$$= 158 \text{ mW}$$

4. Consider a Hertzian dipole antenna placed at the origin, two people A and B are communicating over the cellphone and are standing as shown in the figure. How would you orient your dipole

antenna such that the users receive maximum signal strength? Calculate the normalized power of the radiated electric field (far field) in the following planes (try to plot also):

- $\phi = 0°$
- $\theta = 90°$



What happens when one user starts moving towards the z-axis?

Solution: The hertz dipole should be placed symmetrically about the origin and directed along the z-axis. This orientation will provide the maximum E-field along the x-axis and y-axis.

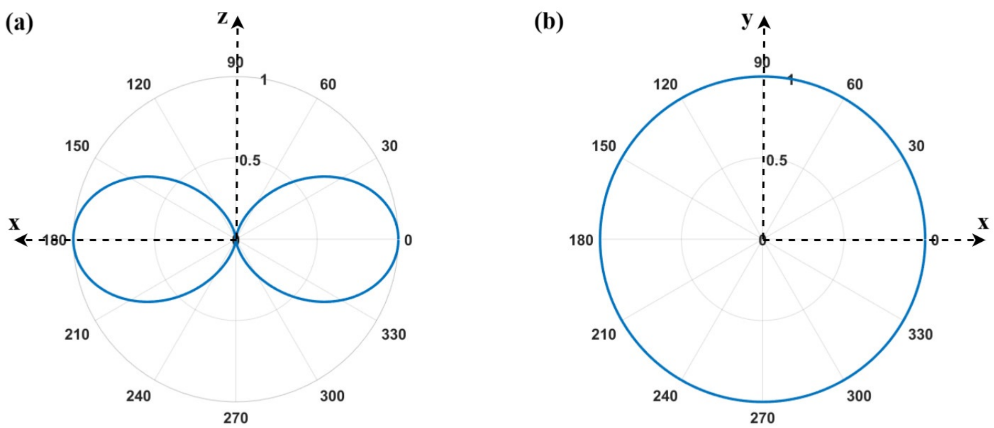
The electric field at the far field for a Hertz dipole oriented along the z-axis is given as,

$$E_\theta = \frac{j\eta\beta I dl}{4\pi r} e^{-j\beta r} \sin\theta$$

Normalized power of the field along given planes

(a) $\phi = 0$ (x-z plane) $\quad |E_\theta|^2 = \sin^2\theta$

(b) $\theta = 90$ (x-y plane) $|E_\theta|^2 = 1$



Assuming both users are standing at the far field of the Hertzian dipole antenna placed at the origin. Initially, both users receive good signal strength. As one of the users starts moving towards the z-axis (in the path as shown in the figure) the signal strength decreases and along the z-axis the signal strength reduces to zero and the call terminates.

5. A Hertzian dipole antenna is 10 mm long and carries a current of 2 A. The dipole is used for cellular telephone communication and radiates at 900 MHz.

   (i) Calculate the total power radiated by the dipole in free space.

   (ii) If the dipole antenna is immersed in water ($\varepsilon_r = 81, \sigma = 0, \mu = \mu_0$), find out the required dipole current so as to maintain the same radiated power as in free space.

   (iii) Calculate the ratio of the radiation resistance in air and in water.

Solution: Wavelength $\lambda = \frac{c}{f} = 0.34$ m.

(i) Radiated power, $P_{\text{rad}} = \frac{I_0^2 \pi \eta}{3} \left(\frac{dl}{\lambda}\right)^2 = 1.366$ W .

(ii) Impedance of water $\eta = 120\pi\sqrt{\frac{1}{\epsilon_r}} = \frac{40\pi}{3}$.

$I_o = \frac{\lambda}{dl}\sqrt{\frac{3P_{rad}}{\pi\eta}} = 6$ A.

(iii) Radiation resistance, $R_{\text{rad}} = \frac{2\pi\eta}{3}\left(\frac{dl}{\lambda}\right)^2$

$\frac{R_{rad\,in\,air}}{R_{rad\,in\,water}} = \frac{\eta_o}{\eta} = 9$

6. **Lorentz Reciprocity Theorem** Consider a volume containing two sets of sources, J_1 and J_2, which each produce fields E_1, H_1 and E_2, H_2, respectively, as shown in Figure 1. Starting from

$$\boldsymbol{\nabla} \cdot (\mathbf{E_1} \times \mathbf{H_2} - \mathbf{E_2} \times \mathbf{H_1}),$$

show that Equation (1) is true for any antenna.

$$\iiint_V \mathbf{J_1} \cdot \mathbf{E_2} dV = \iiint_V \mathbf{J_2} \cdot \mathbf{E_1} dV \tag{1}$$

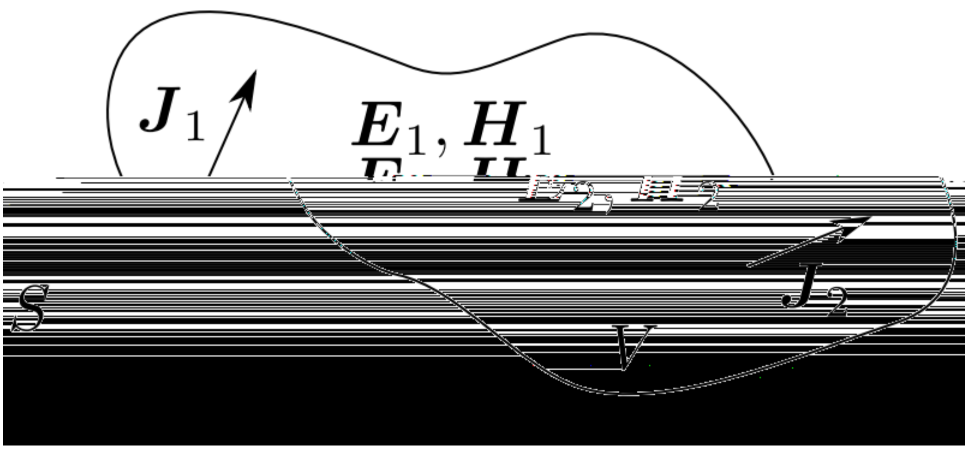


Figure 1: Problem 6

Solution:

Consider the quantity

$$\nabla \cdot (\boldsymbol{E}_1 \times \boldsymbol{H}_2 - \boldsymbol{E}_2 \times \boldsymbol{H}_1), \tag{1}$$

which is expandable using a vector identity as

$$(\nabla \times \boldsymbol{E}_1) \cdot \boldsymbol{H}_2 - (\nabla \times \boldsymbol{H}_2) \cdot \boldsymbol{E}_1 - (\nabla \times \boldsymbol{E}_2) \cdot \boldsymbol{H}_1 + (\nabla \times \boldsymbol{H}_1) \cdot \boldsymbol{E}_2. \tag{2}$$

From Maxwell's curl equations,

$$\begin{aligned}
\nabla \times \boldsymbol{E}_1 &= -j\omega\mu \boldsymbol{H}_1 & (3)\\
\nabla \times \boldsymbol{H}_1 &= j\omega\epsilon \boldsymbol{E}_1 + \boldsymbol{J}_1 & (4)\\
\nabla \times \boldsymbol{E}_2 &= -j\omega\mu \boldsymbol{H}_2 & (5)\\
\nabla \times \boldsymbol{H}_2 &= j\omega\epsilon \boldsymbol{E}_2 + \boldsymbol{J}_2. & (6)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\nabla \cdot (\boldsymbol{E}_1 \times \boldsymbol{H}_2 - \boldsymbol{E}_2 \times \boldsymbol{H}_1) &= -j\omega\mu \boldsymbol{H}_1 \cdot \boldsymbol{H}_2 - j\omega\epsilon \boldsymbol{E}_2 \cdot \boldsymbol{E}_1 - \boldsymbol{J}_2 \cdot \boldsymbol{E}_1 & (7)\\
&\quad + j\omega\mu \boldsymbol{H}_2 \cdot \boldsymbol{H}_1 + j\omega\epsilon \boldsymbol{E}_1 \cdot \boldsymbol{E}_2 + \boldsymbol{J}_1 \cdot \boldsymbol{E}_2 & \\
&= \boldsymbol{J}_1 \cdot \boldsymbol{E}_2 - \boldsymbol{J}_2 \cdot \boldsymbol{E}_1. & (8)
\end{aligned}$$

Since we took the divergence of a quantity in (1), let us now integrate the divergence over the volume of interest:

$$\iiint_V \nabla \cdot (\boldsymbol{E}_1 \times \boldsymbol{H}_2 - \boldsymbol{E}_2 \times \boldsymbol{H}_1) dv' = \iiint_V (\boldsymbol{J}_1 \cdot \boldsymbol{E}_2 - \boldsymbol{J}_2 \cdot \boldsymbol{E}_1) dv' \tag{9}$$

Applying the Divergence Theorem to the left hand side:

$$\oiint_S (\boldsymbol{E}_1 \times \boldsymbol{H}_2 - \boldsymbol{E}_2 \times \boldsymbol{H}_1) \cdot d\boldsymbol{s}' = \iiint_V (\boldsymbol{J}_1 \cdot \boldsymbol{E}_2 - \boldsymbol{J}_2 \cdot \boldsymbol{E}_1) dv'. \tag{10}$$

A more useful form of this theorem, applicable to antennas, is found by noticing that for electric and magnetic fields observed a large distance from a source (e.g., a sphere of infinite radius surrounding an antenna),

- $\boldsymbol{E} \times \boldsymbol{H}$ points in the radial direction normal to the sphere, $\hat{\boldsymbol{n}}$.
- $\boldsymbol{E}$ and $\boldsymbol{H}$ are related through $\boldsymbol{H} = (\hat{\boldsymbol{n}} \times \boldsymbol{E})/\eta$.

Using the latter relation, the integrand on the left hand side of (10) can be re-written as

$$\begin{aligned}
(\boldsymbol{E}_1 \times \boldsymbol{H}_2 - \boldsymbol{E}_2 \times \boldsymbol{H}_1) \cdot \hat{\boldsymbol{n}} dS &= (\hat{\boldsymbol{n}} \times \boldsymbol{E}_1) \cdot \boldsymbol{H}_2 - (\hat{\boldsymbol{n}} \times \boldsymbol{E}_2) \cdot \boldsymbol{H}_1 & (11)\\
&= \eta \boldsymbol{H}_1 \cdot \boldsymbol{H}_2 - \eta \boldsymbol{H}_2 \cdot \boldsymbol{H}_1 & (12)\\
&= 0 & (13)
\end{aligned}$$

Hence,

$$\iiint_V \boldsymbol{J}_1 \cdot \boldsymbol{E}_2 dv' = \iiint_V \boldsymbol{J}_2 \cdot \boldsymbol{E}_1 dv' \tag{14}$$

This is the form of the Reciprocity Theorem that is used in the analysis of receiving antennas.

Directionality in Antennas

7. A $1m$ long dipole is excited by a $5MHz$ current with an amplitude of $5A$. At a distance of $2km$, what is the power density radiated by the antenna along $\theta = 90°$ (broadside direction)?

Solution:

$$S = \frac{\eta_0 k^2 I^2 l^2}{32\pi^2 R^2}\sin^2\theta \tag{2}$$

We know that $\eta_0 = 120\pi$, $\lambda = 60m$, $l = 1m$, $R = 2000m$, $I = 5A$, $k = 2\pi/\lambda$. Substituting these values, we get $S = 8.18 \times 10^{-8} W/m^2$

8. The power radiated by a lossless antenna is 10 watts. The directional characteristics of the antenna is represented by the radiation intensity of $U(\theta,\phi) = B_0\cos^3\theta$ (W/unit solid angle) $0 \le \theta \le \pi/2$ $0 \le \phi \le 2\pi$ the directivity of the antenna (dimensionless and in dB).

Solution: a) Average power is given by
$S_{avg} = \int_0^{2\pi}\int_0^{\pi/2} U(\theta,\phi) sin\theta d\theta d\phi = B_0\int_0^{2\pi}\int_0^{\pi/2} cos^3\theta sin\theta d\theta$
$10 = 2\pi B_0 \int_0^{\pi/2} cos^3\theta sin\theta d\theta d\phi$
$B_0 = 6.36$
$U(\theta,\phi) = 6.36cos^3\theta$
$W = \frac{U}{r^2} = \frac{6.36cos^3\theta}{r^2}$
$W_{max} = \frac{U_{max}}{r^2} = \frac{6.36}{r^2} = 6.36 \times 10^{-6} W/m^2$
Angle at which we have maximum power density is $\theta = 0$

b) $D_0 = \frac{4\pi U_{max}}{S_{avg}} = 8 = 9\text{dB}$

9. Calculate the directivity, total power radiated and radiation resistance of an half-wave dipole (length $= \lambda/2$) antenna with far fields given below:

$$E_\theta \simeq \frac{j\eta I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right]$$

$$H_\phi \simeq \frac{jI_0 e^{-jkr}}{2\pi r}\left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right]$$

Hint: $\int_0^\pi \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}d\theta = 1.21883$

Solution: Time-average power density:

$$\begin{aligned}
\overrightarrow{S_{av}} &= \frac{1}{2}Re(\overrightarrow{\mathbf{E}} \times \overrightarrow{\mathbf{H}}^*) \\
&= \frac{1}{2}Re\left[\frac{j\eta I_0 e^{-jkr}}{2\pi r}\left(\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right)\hat{a_\theta} \times \frac{-jI_0 e^{jkr}}{2\pi r}\left(\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right)\hat{a_\phi}\right] \\
&= \frac{\eta}{2}\left[\frac{I_0}{2\pi r}\left(\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right)\right]^2\hat{a_r}
\end{aligned}$$

Total power radiated:

$$\begin{aligned}
W &= \iint_S \overrightarrow{S_{av}}.\overrightarrow{ds} = \int_0^{2\pi}\int_0^{\pi} \overrightarrow{S_{av}}.(r^2 \sin\theta d\theta d\phi \hat{a_r}) \\
&= \int_0^{2\pi}\int_0^{\pi} \frac{\eta I_0^2}{8\pi^2}\left(\frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right) d\theta d\phi \\
&= \left[\int_0^{2\pi} d\phi\right]\left[\int_0^{\pi} \frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin\theta} d\theta\right]\frac{\eta I_0^2}{8\pi^2} \\
&= \left[2\pi\right]\left[1.21883\right]\frac{\eta I_0^2}{8\pi^2} = \frac{1.21883\eta I_0^2}{4\pi}
\end{aligned}$$

Radiation intensity:

$$U(\theta,\phi) = r^2|\overrightarrow{S_{av}}| = \frac{\eta}{2}\left[\frac{I_0}{2\pi}\left(\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right)\right]^2$$

$$U_{max} = \frac{\eta}{2}\left[\frac{I_0}{2\pi}\right]^2$$

Directivity:

$$D(\theta,\phi) = 4\pi\frac{U(\theta,\phi)}{W} = 1.641\left(\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right)^2$$

(Maximum Directivity = 1.641.)
Radiation Resistance(R_r):

$$\begin{aligned}
W &= \frac{I_0^2 R_r}{2} = \frac{1.21883\eta I_0^2}{4\pi} \\
\implies R_r &= \frac{1.21883\eta}{2\pi} = \frac{1.21883\times 120\pi}{2\pi} = 73.13\Omega
\end{aligned}$$

Half-wave dipoles and mono-poles are the most commonly used antennas. Monopole antennas have only one pole with length $\lambda/4$ and the other pole is replaced with ground plate. Monopole antenna has a radiation resistance of $73.12/2\Omega = 36.56\Omega$ (half of half-wave dipoles as they radiate only to one hemisphere).

10. An antenna has been designed as a half wavelength dipole for use with a TV transmitter at 600 MHz (UHF channel 35) and the transmitter supplies 50 kW to the antenna. The antenna is 6 mm thick and made of aluminium with conductivity $\sigma = 3\times 10^7$ S/m. Consider the radiation resistance of dipole (R_{rad}) is 73.08 Ω and the current is uniform. Calculate:
(a) The radiated power at 600 MHz.
(b) The efficiency of the antenna. ($eff = \frac{P_{rad}}{P_{in}} = \frac{R_{rad}}{R_{rad}+R_{in}}$)

Solution: The length of the antenna is

$$\lambda = \frac{c}{f} = \frac{3\times 10^8}{600\times 10^6} = 0.5 \implies L = 0.25$$

To find the radiated power, we need to find the internal resistance and radiation resistance of the antenna. The radiation resistance (R_{rad}) of the half wave dipole is 73.08 Ω. Internal resistance can be found by using the equation for resistance of a hollow cylinder

$$R_d = \frac{L}{\sigma S}$$

Where S is,

$$S = 2\pi r\delta$$

where δ is the skin depth. In this problem, diameter is given as 6 mm (r=3 mm).

$$\delta = \frac{1}{\sqrt{\pi f \mu_0 \sigma}} = 3.751 \times 10^{-6} m$$

$$R_d = \frac{L}{2\pi r\delta\sigma} = 0.1179\,\Omega$$

$$P_{in} = \frac{I^2(R_{rad} + R_d)}{2} \Longrightarrow I^2 = 1366.16 \Longrightarrow I = 36.96\,A$$

Radiated power

$$P_{rad} = \frac{I^2 R_{rad}}{2} = 49.919\,KW$$

(b) The efficiency of the antenna

$$eff = \frac{P_{rad}}{P_{in}} = 99.84\,\%$$

This is a very high efficiency beacause of its low internal resistance.

11. Four isotropic sources are placed along the z-axis as shown in Fig.2. Assuming that excitations of elements (current fed to the elements) 1 and 2 are +1 and the excitations of elements 3 and 4 are -1 (180^0 out of phase with 1 and 2). Find
(a) The array factor in simplified form.
(b) All the nulls when $d = \frac{\lambda}{2}$.

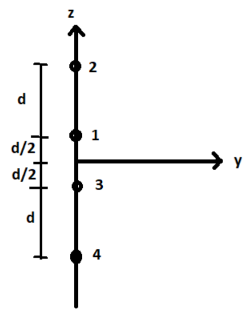


Figure 2: isotropic sources

Solution: (a)

$$E = K\left[\frac{e^{-jkr_2}}{r_2} + \frac{e^{-jkr_1}}{r_1} - \frac{e^{-jkr_3}}{r_3} - \frac{e^{-jkr_4}}{r_4}\right]$$

where,

$$r_1 = r - \frac{d}{2}cos(\theta)$$
$$r_2 = r - \frac{3d}{2}cos(\theta)$$

$$r_3 = r + \frac{d}{2}cos(\theta)$$

$$r_4 = r + \frac{3d}{2}cos(\theta)$$

K is a constant.

$$E = \frac{Ke^{-jkr}}{r}\left[e^{\frac{j3kd}{2}cos(\theta)} + e^{\frac{jkd}{2}cos(\theta)} - e^{-\frac{jkd}{2}cos(\theta)} - e^{-\frac{j3kd}{2}cos(\theta)}\right]$$

For amplitude variations, $r_1 \approx r_2 \approx r_3 \approx r_4 \approx r$

$$AF = 2j[sin(\frac{3kd}{2}cos\theta) + sin(\frac{kd}{2}cos\theta)]$$

Let $x = kdcos\theta$, $y = \frac{kd}{2}cos\theta$

$$AF = 4j[sin(kdcos\theta)cos(\frac{kd}{2}cos\theta)]$$

(b)

$$AF(d = \frac{\lambda}{2}) = 4j[sin(\pi cos\theta)cos(\frac{\pi}{2}cos\theta)]$$

The nulls will be placed at θ such that $AF(\theta) = 0$.

$$\theta_n = 0, 90, 180$$