EE2025: Engineering Electromagnetics

Tutorial 2: Maxwell's equation, Boundary conditions and plane wave propagation

July-Nov 2024

Maxwell's Equations and Boundary Conditions

1. Show that charge density ρ_v satisfies,

$$\frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\epsilon}\rho_v = 0$$

Is this true for all media?

Solution:

From Gauss's law, we have,

$$\nabla \cdot \mathbf{D} = \rho_v,$$

where $\mathbf{D} = \epsilon \mathbf{E}$

$$\implies \nabla \cdot \mathbf{E} = \frac{\rho_v}{\epsilon}$$

From continuity equation and Ohm's law, we can write,

$$\nabla \cdot \mathbf{J} = \nabla \cdot \sigma \mathbf{E} = \sigma \nabla \cdot \mathbf{E} = -\frac{\partial \rho_v}{\partial t}$$

$$\implies \sigma \frac{\rho_v}{\epsilon} = -\frac{\partial \rho_v}{\partial t}$$

Or we can write,

$$\frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\epsilon}\rho_v = 0$$

The above differential equation holds true for a linear, homogeneous, isotropic medium. This is because $\mathbf{J} = \sigma \rho$ (Ohm's Law) would be valid only in such a medium.

2. A slab of perfect dielectric material ($\epsilon_r = 2$) is placed in a microwave oven. The oven produces an electric field (as well as a magnetic field). Assume that the electric field intensity is uniform in space and sinusoidal in time and is in the direction perpendicular to the surface of the slab. The microwave oven operates at a frequency of 2.45 GHz and produces an electric field intensity with amplitude 500 V/m inside the dielectric, calculate the displacement current density in the dielectric.

Solution: The electric field inside the slab can be given by

$$\mathbf{E} = \hat{n} E_0 \sin(\omega t)\,[\mathrm{V/m}] \Rightarrow \mathbf{D} = \hat{n}\epsilon_0 \epsilon_r E_0 \sin(\omega t)\,\left[\mathrm{C/m^2}\right]$$

where $\hat{n}$ is the direction perpendicular to the slab surface and E_0 is the field amplitude.

The displacement current density is given by

$$\mathbf{J_d} = \frac{\partial \mathbf{D}}{\partial t} = \hat{n}\epsilon_0\epsilon_r \omega E_0 \cos(\omega t)\,\left[\mathrm{A/m^2}\right]$$

Given $\epsilon_r = 2$, $E_0 = 500$ V/m, $\omega = 2\pi \times 2.45 \times 10^9$ rad/sec,

$$\therefore \mathbf{J_d} = 136.3 \cos(4.9 \times 10^9 \pi t)\,\left[\mathrm{A/m^2}\right]$$

3. a) Show that the ratio of the amplitudes of the conduction current density and the displacement current density is $\sigma/\omega\epsilon$ for the applied field $\mathbf{E} = \mathbf{E}_m cos\omega t$. Assume $\mu = \mu_0$.

   b) Assuming that sea water has $\mu = \mu_0$, $\epsilon = 81\epsilon_0$, $\sigma = 20$ S/m, determine the frequency at which the conduction current density is ten times the displacement current density in magnitude.

   c) A parallel-plate capacitor with plate area of 5 cm^2 and plate separation of 3 mm has a voltage 50 $\sin(10^3 t)$ V applied to its plates. Calculate the displacement current assuming $\epsilon = 2\epsilon_o$.

Solution:

a) The displacement vector

$$\mathbf{D} = \epsilon\mathbf{E} = \epsilon\mathbf{E}_m cos\omega t$$

and the displacement current density is

$$\mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t} = -\omega\epsilon\mathbf{E}_m sin\omega t = \omega\epsilon\mathbf{E}_m cos(\omega t + \pi/2)$$

The conduction current density is

$$\mathbf{J}_c = \sigma\mathbf{E} = \sigma\mathbf{E}_m cos\omega t$$

Using these two results we find

$$\frac{|\mathbf{J}_c|}{|\mathbf{J}_d|} = \frac{\sigma\mathbf{E}_m}{\omega\epsilon\mathbf{E}_m} = \frac{\sigma}{\omega\epsilon}.$$

b)

$$\frac{J_c}{J_d} = \frac{\sigma}{\omega\epsilon} = 10 \tag{1}$$

$$\implies \omega = 2\pi f = \frac{\sigma}{10\epsilon} = \frac{20}{10 \times 81\epsilon_0}$$

Solving, we get $f = 0.44$ GHz

c)

$$D = \epsilon E = \epsilon\frac{V}{d}$$

$$J_d = \frac{\partial D}{\partial t} = \frac{\epsilon}{d}\frac{dV}{dt}$$

Hence,

$$I_d = J_d \cdot A = \frac{\epsilon A}{d}\frac{dV}{dt} = C\frac{dV}{dt}$$

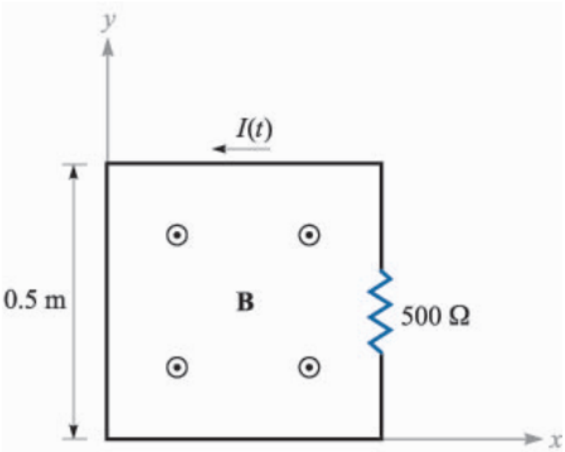
So, value of I_d is given by

$$I_d = \frac{\epsilon A}{d}\frac{dV}{dt} = 2 \cdot \frac{10^{-9}}{36\pi} \times \frac{5 \times 10^{-4}}{3 \times 10^{-3}} \times 10^3 \cdot 50 \cos 10^3 t$$

$$= 147.4 \cos 10^3 t \,[\text{nA}]$$

4. A perfectly conducting filament containing a small 500Ω resistor is formed into a square,

as illustrated by Fig. 2. Find $I(t)$ if (a) $\mathbf{B}$ = 0.3 cos(120πt - 30°)$\mathbf{a}_z$ T; (b) $\mathbf{B}$ = 0.4 cos[$\pi(ct - y)$]$\mathbf{a}_z$ μT, where $c = 3 \times 10^8$ m/s.



Solution:

Area of the square loop = 0.5 × 0.5 = 0.25 m^2

$\texttt{emf} = -\frac{d\phi_B}{dt}$, R = 500Ω and $d\mathbf{S} = dxdy\hat{a_z}$

(a) $\phi_B = \int_S \mathbf{B} \cdot d\mathbf{S} = \int_S 0.3cos(120\pi t - 30°)\mathbf{a}_z \cdot d\mathbf{S} = 0.3cos(120\pi t - 30°) \cdot (0.25)$

$\implies \phi_B$ = 0.075 cos (120πt - 30°) [Wb]

$\texttt{emf} = -\frac{d\phi_B}{dt} = 28.27\sin(120\pi t - 30^o)$ [V]

$\therefore I(t) = \frac{emf}{R} = 0.057\sin(120\pi t - 30°)$ [A]

(b) $\phi_B = \int_S \mathbf{B} \cdot \mathbf{dS} = \int_S 0.4\cos(\pi(ct - y)) \cdot 10^{-6}\mathbf{a}_z \cdot \mathbf{dS}$

$= \int_0^{0.5}\int_0^{0.5} 0.4\cos(\pi ct - \pi y) \cdot 10^{-6} dxdy$

$\implies \phi_B = 0.5 \times 0.4 \times 10^{-6}[\sin(\pi ct - \pi y)]_0^{0.5} \times \frac{-1}{\pi}$

$= \frac{-0.2\times10^{-6}}{\pi}[sin(\pi ct - 0.5\pi) - sin(\pi ct)]$ [Wb]

$\texttt{emf} = -\frac{d\phi_B}{dt} = -0.2c \times 10^{-6}[\cos(\pi ct - 0.5\pi) - \cos(\pi ct)]$ [V]

$\therefore I(t) = \frac{emf}{R} = 0.12[\cos(\pi ct - 0.5\pi) - \cos\pi ct]$ [A]

5. What values of A and β are required if the two fields given below satisfy Maxwell's equations in a linear, isotropic, homogeneous medium with $\epsilon_r = \mu_r = 4$ and $\sigma = 0$?

$$\mathbf{E} = 120\pi cos(10^6\pi t - \beta x)\hat{a_y}V/m$$

$$\mathbf{H} = A\pi cos(10^6\pi t - \beta x)\hat{a_z}A/m$$

Assume there are no current or charge densities in space.

Solution: Intrinsic impedance η can be obtained as:

$\eta = \eta_0\sqrt{\frac{\mu_r}{\epsilon_r}} = 120\pi\ \Omega$

$\beta = \frac{\omega\mu}{\eta} = \frac{10^6\times\pi\times4\pi\times10^{-7}\times4}{120\pi} = 0.0419$ rad/m
$\frac{|\mathbf{E}|}{|\mathbf{H}|} = \eta \implies \frac{120\pi}{A\pi} = 120\pi$

$A = \frac{1}{\pi} = 0.318$

6. Consider a point on the surface of a perfect conductor. The electric field intensity at that point is $\mathbf{E} = (500\hat{x} - 300\hat{y} + 600\hat{z})\cos 10^7 t$ and medium surrounding the conductor is characterized by $\mu_r = 5$ and $\epsilon_r = 10$ and $\sigma = 0$.
(a) Find a unit vector normal to the conductor at that point of the conductor surface.
(b) Find the instantaneous surface charge density at the point.

Solution: (a) Since we have a perfect conductor, Electric field at any point will be normal and directed out of the surface. So it will be given by outward normal electric field vector at that point.

$$\implies \hat{n} = \frac{\mathbf{E}(t=0)}{|E(t=0)|}$$

$$\implies \hat{n} = \frac{5\hat{x} - 3\hat{y} + 6\hat{z}}{\sqrt{(5^2+9+36)}} = 0.60\hat{x} - 0.36\hat{y} + 0.72\hat{z}$$

(b) Since it is a metallic conductor, no field will exist inside the metal. Hence using boundary conditions, surface charge density is given by

$$(\mathbf{D_1}.\hat{n} - \mathbf{D_2}.\hat{n}) = \rho_s$$

Here D_2 will be zero as it is zero within the metallic conductor $\therefore$ the instantaneous surface charge density will be

$$(\mathbf{D_1}.\hat{n}) = \rho_s$$

$\implies \rho_s = 10\epsilon_0(500\hat{x} - 300\hat{y} + 600\hat{z})(\cos 10^7 t).(0.60\hat{x} - 0.36\hat{y} + 0.72\hat{z}) = 74\cos 10^7 t$ [nC/m^2]

7. Find out $\mathbf{B}$ if the electric field is given as $\mathbf{E}(x,y,z,t) = 0.2\sin(10\pi y)\cos(6\pi 10^9 t - \beta z)\hat{\mathbf{x}}$ in vacuum.

Solution:
Using the trigonometric identity, we get

$$\mathbf{E} = 0.1(\sin(6\pi 10^9 t + 10\pi y - \beta z) + \sin(-6\pi 10^9 t + 10\pi y + \beta z))\hat{\mathbf{x}}$$

$\beta = \frac{\omega}{c} = 10\pi\sqrt{3}\text{radm}^{-1}$
Using Maxwell's equations,
$\mathbf{B} = \frac{0.1}{3\times10^8}((\sin(6\pi 10^9 t + 10\pi y - 10\pi\sqrt{3}z)(\frac{\sqrt{3}}{2}\hat{\mathbf{j}} + \frac{1}{2}\hat{\mathbf{k}})$
$-(\sin(-6\pi 10^9 t + 10\pi y + 10\pi\sqrt{3}z)(-\frac{\sqrt{3}}{2}\hat{\mathbf{j}} + \frac{1}{2}\hat{\mathbf{k}}))\text{T}$

8. **Snell's law:** A light beam is incident from air to a medium with a dielectric constant 4 and relative permeability 100 . If the angle of incidence is 60°. Find the angle of reflection and angle of refraction.

Solution: The angle of reflection θ_r = angle of incidence $\theta_i = 60°$.

From the Snell's law,

$$\begin{aligned}
\sqrt{\mu_1\epsilon_1}\sin\theta_i &= \sqrt{\mu_2\epsilon_2}\sin\theta_t \\
\Rightarrow \sqrt{\mu_0\epsilon_0}\sin 60° &= \sqrt{\mu_0(100)\epsilon_0(4)}\sin\theta_t \\
\sin\theta_t &= \frac{\sin 60°}{20} \\
&= 0.0433 \\
\Rightarrow \quad \text{Angle of refraction} \quad \theta_t &= 2.48°
\end{aligned}$$

9. a) Show that the ratio of the amplitudes of the conduction current density and the displacement current density is $\sigma/\omega\epsilon$ for the applied field $\mathbf{E} = \mathbf{E}_m cos\omega t$. Assume $\mu = \mu_0$.
   b) Assuming that sea water has $\mu = \mu_0$, $\epsilon = 81\epsilon_0$, $\sigma = 20$ S/m, determine the frequency at which the conduction current density is ten times the displacement current density in magnitude.

Solution:

a) The displacement vector

$$\mathbf{D} = \epsilon\mathbf{E} = \epsilon\mathbf{E}_m cos\omega t$$

and the displacement current density is

$$\mathbf{J}_d = \frac{\partial\mathbf{D}}{\partial t} = -\omega\epsilon\mathbf{E}_m sin\omega t = \omega\epsilon\mathbf{E}_m cos(\omega t + \pi/2)$$

The conduction current density is

$$\mathbf{J}_c = \sigma\mathbf{E} = \sigma\mathbf{E}_m cos\omega t$$

Using these two results we find

$$\frac{|\mathbf{J}_c|}{|\mathbf{J}_d|} = \frac{\sigma\mathbf{E}_m}{\omega\epsilon\mathbf{E}_m} = \frac{\sigma}{\omega\epsilon}.$$

b)

$$\frac{J_c}{J_d} = \frac{\sigma}{\omega\epsilon} = 10 \tag{2}$$

$$\Longrightarrow \omega = 2\pi f = \frac{\sigma}{10\epsilon} = \frac{20}{10\times 81\epsilon_0}$$

Solving, we get $f = 0.44$ GHz

Electromagnetic waves in conductors

10. a) Show that the skin depth in a poor conductor ($\sigma << \omega\epsilon$) is $\frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}$ (independent of frequency). Find the skin depth (in meters) for pure water (use static value of ($\sigma = \frac{1}{2.5\times10^5}$), ($\mu = \mu_0$) and ($\epsilon_r = 80.1$)).

    b) Show that the skin depth in a good conductor ($\sigma >> \omega\epsilon$) is $\frac{\lambda}{2\pi}$ (where λ is the wavelength in the conductor). Find the skin depth (in nanometers) for a typical metal (where $\sigma = 10^7(\Omega m)^{-1}$, for visible range frequency $\omega = 10^{15}$ and assume $\epsilon = \epsilon_0$ and $\mu = \mu_0$).

Solution:

a) Expression of propagation constant

$$\gamma = \alpha + i\beta$$

$$\alpha = \omega\sqrt{\frac{\epsilon\mu}{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}-1}$$

Use the binomial expansion for the square root in the above equation since we have ($\sigma << \omega\epsilon$).

$$\alpha = \omega\sqrt{\frac{\epsilon\mu}{2}}\sqrt{1+\frac{1}{2}\left(\frac{\sigma}{\omega\epsilon}\right)^2-1} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} = \frac{1}{d}$$

Where, d is the skin depth, skin depth is independent on frequency for poor conductor.

Skin depth for pure water for the given value is 1.19×10^4m.

b) We expand the α since we have given ($\sigma >> \omega\epsilon$) for good conductor.

$$\alpha = \omega\sqrt{\frac{\epsilon\mu}{2}}\sqrt{\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}-1}$$

$$\alpha = \omega\sqrt{\frac{\epsilon\mu}{2}}\sqrt{\left(\frac{\sigma}{\omega\epsilon}\right)} = \sqrt{\frac{\omega\mu\sigma}{2}}$$

For good conductor, $\alpha = \beta$, Hence $\lambda = \frac{2\pi}{\alpha}$

$$\alpha = \frac{2\pi}{\lambda} = \frac{1}{d}$$

Skin depth for the given metal is 13 nm.

[Priyanka]

Poynting theorem, Wave Velocity, and Average Power

11. If the electric and magnetic field in a medium are given by $\mathbf{E} = 3\sin(t-5z)\,\hat{x}$ and $\mathbf{H} = 4\cos(t-5z)\,\hat{y}$, then calculate (at $z = 0$) the
a) the instantaneous power density,

b) instantaneous power transmitted through a surface with an area of $5\,m^2$ at $z = 0$ and the normal pointing in $\hat{z}$ direction, and
c) total energy carried by the wave through the given surface from $t = 0\,s$ to $t = 5\,s$.

Solution: a) The **instantaneous** power density at $z = 0$ is given by

$$\begin{aligned}\mathbf{S} &= \mathbf{E} \times \mathbf{H} \\ &= 3\sin(t) \times 4\cos(t)\,\hat{z} = 12\sin(t)\cos(t)\,\hat{z}\end{aligned}$$

Note: In this question, you have been asked to compute the instantaneous power density. Therefore, you should not use the phasor formula.

b) Power through an area of $5\,m^2$ is given by

$$P = \int \mathbf{S} \cdot \hat{n}\, ds = 60\sin(t)\cos(t)$$

c) Total energy carried by the wave is given by

$$E = \int_0^5 P(t)\,dt = \int_0^5 60\sin(t)\cos(t)\,dt = 15(1 - \cos(10))$$

12. A laptop manufacturer wants to shield her laptop such that the energy from its high frequency clock doesn't radiate to the outside world. Assuming a clock rate of 2.45 GHz, what should be the thickness of the metal cladding on the laptop cover and weight of metal required, given standard laptop dimension of 15" x 11" if she uses

    1. silver ($\sigma = 6.2 \times 10^7$ S/m, density=10.49 g/cm^3)
    2. gold ($\sigma = 4.1 \times 10^7$ S/m, density=19.32 g/cm^3)
    3. copper ($\sigma = 5.8 \times 10^7$ S/m, density=8.96 g/cm^3)

    [Note : Assume the required thickness as the depth where the amplitude of E field attenuates by 99%.]

Solution: Given $f = 2.45GHz$
To shield the laptop, thickness of the metal cladding should be such that E field has undergone 99% attenuation.

$$\begin{aligned}E_0 e^{-z/\delta} &= (1 - 0.99) \times E_0 \\ \implies z &= ln(100) \times \delta\end{aligned}$$

Hence, given an for 99% attenuation, the field has to travel a distance of $ln(100)$ times the skin depth.

*Skin depth for any metal with conductivity σ at frequency f, $\delta = \frac{1}{\sqrt{f\pi\sigma\mu}}$

(a) silver ($\sigma = 6.2 \times 10^7 S/m$)
$\delta = \frac{1}{\sqrt{\pi \times 2.45 \times 10^9 \times 6.2 \times 10^7 \times 4\pi \times 10^{-7}}} = 1.2913\ \mu m$
Thickness of silver $= ln(100) \times 1.2913 = 5.9468\ \mu m$

(b) gold ($\sigma = 4.1 \times 10^7 S/m$)
$\delta = \frac{1}{\sqrt{\pi \times 2.45 \times 10^9 \times 4.1 \times 10^7 \times 4\pi \times 10^{-7}}} = 1.588\ \mu m$
Thickness of gold $= ln(100) \times 1.2913 = 5.3129\ \mu m$

(c) copper ($\sigma = 5.8 \times 10^7 S/m$)
$\delta = \frac{1}{\sqrt{\pi \times 2.45 \times 10^9 \times 5.8 \times 10^7 \times 4\pi \times 10^{-7}}} = 1.335\ \mu m$
Thickness of copper $= ln(100) \times 1.2913 = 6.1485\ \mu m$

Reference for density values: https://www.coolmagnetman.com/magconda.htm

Conclusion: Find the weights yourselves!!. Choose the cheapest for base models, and lightest for deluxe models!

13. In a lossless and nonmagnetic medium, the E field of EM wave emitted by a cell phone is given as

$$\mathbf{E}(t) = 4\sin(2\pi \times 10^7 t - 0.8x)\hat{z}\ \ mV/m$$

Find

1. Relative permittivity (ϵ_r) and intrinsic impedance (η) of the medium.
2. The time averaged Poynting vector associated with the wave
3. Total power received through a rectangular area of a window of 1 m and 1.5 m, in the $x = 0$ plane
4. Total power received through the same window if it is placed in the plane $2x + y = 5$

Solution:

1. Converting the electric field into phasor form

$$\mathbf{E}(t) = Im(\mathbf{E}e^{j\omega t}) \Rightarrow \mathbf{E} = 4e^{-j0.8x}\hat{z}\ \ mV/m$$

For a lossless, nonmagnetic medium $\beta = \omega\sqrt{\mu_0\epsilon_0\epsilon_r} = 0.8\ \ rad/m$

$$\sqrt{\epsilon_r} = \frac{\beta}{\omega\sqrt{\mu_0\epsilon_0}} = 3.8, \epsilon_r = 14.6$$

$$\eta = \sqrt{\frac{\mu_0}{\epsilon_0\epsilon_r}} = 98.6\ \ \Omega$$

2. From Maxwell's equation, $\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$

$$\begin{aligned}\mathbf{H} &= \frac{\nabla \times \mathbf{E}}{-j\omega\mu} \\ &= \frac{1}{-j\omega\mu}\left[-\frac{\partial E_z}{\partial x}\hat{z}\right] \\ &= -\frac{4}{\eta}e^{-j0.8x}\hat{y}\ \ mA/m\end{aligned}$$

The time averaged Poynting vector

$$\mathbf{P_{av}} = \frac{1}{2}Re(\mathbf{E} \times \mathbf{H}^*) = 81\hat{x}\ \ nW/m^2$$

3. Total power received through the rectangular window in $x = 0$ plane

$$P = \int_s \mathbf{P_{av}}.d\mathbf{a}$$

Since the power flow is normal to the rectangular area,

$$P = 81 \times 10^{-9} \times 1 \times 1.5 = 120 \ \ nW$$

4. On the plane $2x + y = 5$, $\hat{n} = \frac{2\hat{x}+\hat{y}}{\sqrt{5}}$

$$\begin{aligned} P &= \int_s \mathbf{P_{av}}.d\mathbf{a} \\ &= 81 \times 10^{-9} \times 1 \times 1.5 \times \frac{2}{\sqrt{5}} \\ &= 107.33 \text{ nW} \end{aligned}$$

14. Show that the instantaneous Poynting Vector of a circularly polarised plane wave propagating in a lossless medium is a constant that does not vary with time and space. Compare the results with that in the case of a linearly polarised wave propagating in a lossless medium.

Solution: For a circularly polarised plane propagating in a lossless medium, the instantaneous Electric field

$$\begin{aligned} \mathbf{E}(t) &= Re(E_0 e^{j(\omega t - \beta z)}\hat{x} + E_0 e^{j(\omega t - \beta z - \pi/2)}\hat{y}) \\ &= \hat{x}E_0 \cos(\omega t - \beta z) + \hat{y}E_0 \sin(\omega t - \beta z) \ \ \text{V/m} \end{aligned}$$

Then instantaneous magnetic field is given by,

$$\mathbf{H}(t) = \hat{y}\frac{E_0}{\eta}\cos(\omega t - \beta z) - \hat{x}\frac{E_0}{\eta}\sin(\omega t - \beta z) \text{ A/m}$$

Then the instantaneous Poynting vector

$$\mathbf{P} = \mathbf{E}(t) \times \mathbf{H}(t) = \hat{z}\frac{E_0^2}{\eta}$$

which is a constant that does not vary with space and time.

Consider a linearly polarised wave propagating in a loss less medium.

$$\begin{aligned} \mathbf{E}(t) &= Re(E_0 e^{j(\omega t - \beta z)}\hat{x} + 2E_0 e^{j(\omega t - \beta z)}\hat{y}) \\ &= \hat{x}E_0 \cos(\omega t - \beta z) + \hat{y}2E_0 \cos(\omega t - \beta z) \ \ \text{V/m} \end{aligned}$$

The instantaneous magnetic field

$$\mathbf{H}(t) = \hat{y}\frac{2E_0}{\eta}\cos(\omega t - \beta z) - \hat{x}\frac{E_0}{\eta}\cos(\omega t - \beta z) \ \ A/m$$

Then the instantaneous Poynting vector

$$\mathbf{P} = \mathbf{E}(t) \times \mathbf{H}(t) = \hat{z}\frac{4E_0^2}{\eta}\cos^2(\omega t - \beta z)$$

which varies with space and time.

15. In a dielectric medium, a wave has electric and magnetic fields given as,

$$\mathbf{E} = (j\hat{x} + 2\hat{y} - j\hat{z}) \exp\left[-j\pi(x+z)\right] V/m$$

$$\mathbf{H} = \frac{1}{60\pi}(-\hat{x} + j\hat{y} + \hat{z}) \exp\left[-j\pi(x+z)\right] A/m$$

Show that $\mathbf{E}, \mathbf{H}$ and wavevector ($\mathbf{k}$) forms an orthogonal triad. Find

(a) phase constant of the wave
(b) velocity of the wave
(c) frequency of the wave

Solution: Consider the equation
$\mathbf{E} = E_0 e^{-j\beta z}\hat{x}$
The wave has phase constant of β and the wave travels in $\hat{z}$ direction and the electric field is along x direction. In the given question the wave is travelling in $\frac{1}{\sqrt{2}}(\hat{x}+\hat{z})$ direction and hence we can express the $\mathbf{E}$ as
$\mathbf{E} = \bar{E}e^{-j(\beta_x x + \beta_y y)}$
wavevector($\mathbf{k}$) is given as $\beta\hat{n}$, where $\hat{n}$ is the direction of wave propagation. To check $\mathbf{E}, \mathbf{H}$ and $\mathbf{k}$ to forms an orthogonal triad, we need to verify the following conditions
$\hat{n} \cdot \mathbf{E} = 0$
$\hat{n} \cdot \mathbf{H} = 0$
and Electric field and magnetic field should be orthogonal to each other
From the field expressions, we get
$\hat{n} = \frac{1}{\sqrt{2}}(\hat{x}+\hat{z})$
Solving for the conditions,
$\hat{n} \cdot \mathbf{E} = (\frac{1}{\sqrt{2}})(\hat{x}+\hat{z}) \cdot (j\hat{x} + 2\hat{y} - j\hat{z}) \exp\left[-j\pi(x+z)\right] = (j\pi - j\pi)\exp\left[-j\pi(x+z)\right] = 0$
$\hat{n} \cdot \mathbf{H} = (\frac{1}{\sqrt{2}})(\hat{x}+\hat{z}) \cdot \frac{1}{60\pi}(-\hat{x} + j\hat{y} + \hat{z}) \exp\left[-j\pi(x+z)\right] = (-\pi+\pi)\exp\left[-j\pi(x+z)\right] = 0$
And $\mathbf{E} \cdot \mathbf{H} = 0$. Hence $\mathbf{E}$, $\mathbf{H}$ and $\mathbf{k}$ forms an orthogonal triad.

(a) phase constant of the wave
$\beta = \sqrt{(k_x)^2 + (k_z)^2} = \sqrt{2}\pi$ rad/s

(b) velocity of the wave
$|\eta| = \frac{|\mathbf{E}|}{|\mathbf{H}|} = \sqrt{2} \times 60\pi\Omega = \sqrt{\frac{\mu}{\epsilon}} = \frac{1}{\sqrt{\epsilon_r}} \times 120\pi$
$\sqrt{\epsilon_r} = \sqrt{2}$
$v = \frac{c}{\sqrt{\epsilon_r}} = 2.12 \times 10^8$ m/s

(c) frequency of the wave
$\frac{\omega}{\beta} = v$
$f = \frac{\beta \times v}{2\pi} = 150$ MHz

16. ***Determining polarisation states from field vector*** **:** Describe the nature of polarisation of the following electric fields (linear, circular or elliptical).

(a) $E = E_0\cos(\omega t - \beta z)\hat{x} - E_0\cos(\omega t - \beta z)\hat{y}$
(b) $E = E_0\sin(\omega t - \beta z)\hat{x} + E_0\cos(\omega t - \beta z + \frac{\pi}{4})\hat{y}$
(c) $E = \mathfrak{Re}(E_0 \exp^{j(\omega t - \beta z)}\,\hat{x} - jE_0 \exp^{j(\omega t - \beta z)}\,\hat{y})$
(d) $E = E_1\cos(\omega t - \beta z)\hat{x} - E_2\cos(\omega t - \beta z + \frac{\pi}{2})\hat{y} \qquad (E_1 \neq E_2)$

What is the polarisation state of sunlight? How do sunglasses help to reduce glare?

Solution:

The general equation of the locus of the tip of the field E is

$$\frac{E_x^2}{E_{x0}^2} - \frac{2E_xE_y cos\,\phi}{E_{x0}E_{y0}} + \frac{E_y^2}{E_{y0}^2} = sin^2\phi$$

Here, the x-component of the field is $E_x = E_{x0}\,cos\,\omega t$ and y-component is $E_y = E_{y0}cos(\omega t + \phi)$

(a) In this case the phase difference ϕ between the x- and y-component is 180^o. The tip of the electric field vector E draws a straight line irrespective of the amplitudes of the two field components. Hence **linear polarisation**

(b) E_{x0} and E_{y0} are equal but the phase difference between the field components is not equal to 0, $\pm\frac{\pi}{2}$ or π. Hence **elliptical polarisation**

(c) The field components given in complex exponential form can be written as,

$$E_0 \exp^{j(\omega t - \beta z)}\,\hat{x} - E_0 \exp^{j(\omega t - \beta z)} \exp^{j\frac{\pi}{2}}\,\hat{y}$$

The real part of this term (which is of physical significance) can be written as,

$$E = E_0 \cos(\omega t - \beta z)\hat{x} - E_0 \cos(\omega t - \beta z + \frac{\pi}{2})\hat{y}$$

$E_{x0} = E_{y0}$ and $\phi = -\frac{\pi}{2}$. Hence **circular polarisation**

(d) In this case, $\phi = -\frac{\pi}{2}$ but $E_{x0} \neq E_{y0}$. Hence **elliptical polarisation**

Sunlight as such is randomly polarised. But when the light interacts with the molecules and particles in the atmosphere it gets partially polarised due to scattering, reflection etc.

Sunglasses are mostly used to prevent glare due to the reflection of light from horizontal surfaces. When light reflects from a horizontal surface, most of the reflected light will be horizontally polarised. Sunglasses are made with a special material that blocks this horizontally polarised light. This way, they help to reduce glare.

17. Suppose, we have electromagnetic waves propagating in z-direction and electric field components in xy plane are given as:

$$E_x = E_{0x}cos(\omega t), \text{ and } E_y = E_{0y}cos(\omega t + \phi)$$

The axial ratio (AR) is defined as the ratio of major axis and minor axis. The AR in terms of E_{0x}, E_{0y} and ϕ is expressed as:

$$\mathrm{AR} = \sqrt{0x \quad E\,0y}$$

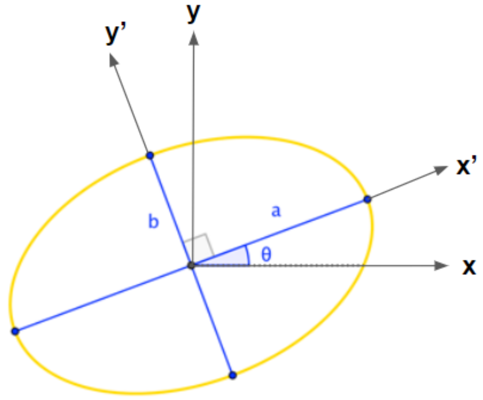


Figure 1: Tilted ellipse

Solution: Expressing E_y in terms of E_x gives:

$$E_y = \frac{E_{0y}E_x}{E_{0x}}\cos\phi - E_{0y}\sqrt{1-\frac{E_x^2}{E_{0x}^2}}\sin\phi \tag{3}$$

After rearranging the terms and simplifying:

$$\frac{E_x^2}{E_{0x}^2\sin^2\phi} - \frac{2E_xE_y\cos\phi}{E_{0x}E_{0y}\sin^2\phi} + \frac{E_y^2}{E_{0y}^2\sin^2\phi} = 1 \tag{4}$$

The general equation for a tilted ellipse as shown in figure 2 is:

$$E_x^2\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right) - E_xE_y\sin(2\theta)\left(\frac{1}{a^2}-\frac{1}{b^2}\right) + E_y^2\left(\frac{\sin^2\theta}{a^2}+\frac{\cos^2\theta}{b^2}\right) = 1 \tag{5}$$

Comparing (4) and (5):

$$\sin(2\theta)\left(\frac{1}{a^2}-\frac{1}{b^2}\right) = \frac{2\cos\phi}{E_{0x}E_{0y}\sin^2\phi} \tag{6}$$

$$\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2} = \frac{1}{E_{0x}^2\sin^2\phi} \tag{7}$$

$$\frac{\sin^2\theta}{a^2}+\frac{\cos^2\theta}{b^2} = \frac{1}{E_{0y}^2\sin^2\phi} \tag{8}$$

Solving for θ, a^2+b^2 and a^2-b^2:

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2E_{0x}E_{0y}\cos\phi}{E_{0y}^2-E_{0x}^2}\right) \tag{9}$$

$$a^2+b^2 = a^2b^2\left(\frac{E_{0x}^2+E_{0y}^2}{E_{0x}^2E_{0y}^2\sin^2\phi}\right) \tag{10}$$

$$a^2-b^2 = \frac{a^2b^2}{cos(2\theta)}\left(\frac{E_{0x}^2-E_{0y}^2}{E_{0x}^2E_{0y}^2\sin^2\phi}\right) \tag{11}$$

$$\mathrm{AR} = \frac{a}{b} \implies \frac{\mathrm{AR}^2 + 1}{\mathrm{AR}^2 - 1} = \frac{a^2 + b^2}{a^2 - b^2} \tag{12}$$

Dividing (10) by (11) and then using $\cos(\tan^{-1}(x)) = \frac{1}{\sqrt{1+x^2}}$, we obtain:

$$\frac{\mathrm{AR}^2 + 1}{\mathrm{AR}^2 - 1} = \frac{E_{0x}^2 + E_{0y}^2}{\sqrt{E_{0x}^4 + E_{0y}^4 + 2E_{0x}^2 E_{0y}^2 \cos(2\phi)}} \tag{13}$$

On simplyfying:

$$\mathrm{AR} = \sqrt{\frac{E_{0x}^2 + E_{0y}^2 + \sqrt{E_{0x}^4 + E_{0y}^4 + 2E_{0x}^2 E_{0y}^2 \cos(2\phi)}}{E_{0x}^2 + E_{0y}^2 - \sqrt{E_{0x}^4 + E_{0y}^4 + 2E_{0x}^2 E_{0y}^2 \cos(2\phi)}}} \tag{14}$$

Ideally,

1. For circularly polarized wave, $\mathrm{AR} = 1$.
2. For elliptically polarized wave, $1 < \mathrm{AR} < \infty$.
3. For linearly polarized wave, $\mathrm{AR} = \infty$.

18. Suppose, for practical purposes, assume that AR can be tolerated upto 3 dB for circularly polarized wave, AR above 40 dB is a linearly polarized wave and anything in between 3 dB and 40 dB is an elliptically polarized wave. Using given assumption, determine the polarization (mention right or left handed in case of circular and elliptical) and AR for the following:

    (a) $E = 10\cos(\omega t - \beta z)\hat{x} + 11\cos(\omega t - \beta z + 80°)\hat{y}$
    (b) $E = 20\sin(\omega t - \beta z)\hat{x} + 1\cos(\omega t - \beta z + 10°)\hat{y}$
    (c) $E = 3\cos(\omega t - \beta z)\hat{x} + 1\cos(\omega t - \beta z + 60°)\hat{y}$
    (d) $E = 10\cos(\omega t - \beta z)\hat{x} + 9\cos(\omega t - \beta z - 75°)\hat{y}$
    (e) $E = 2\cos(\omega t - \beta z + 20°)\hat{x} + 2\cos(\omega t - \beta z + 100°)\hat{y}$

    Note : $\mathrm{AR}\ (\mathrm{dB}) = 20\log_{10}(\mathrm{AR})$

Solution: $\mathrm{AR}\ (\mathrm{dB}) = 10\log_{10}(\mathrm{AR})$

(a) $\mathrm{AR} = 1.92$ dB, left handed circularly polarized.

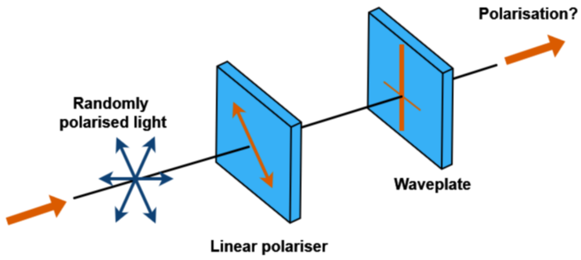
(b) $\mathrm{AR} = 41.24$ dB, linearly polarized.

(c) $\mathrm{AR} = 11.05$ dB, left handed elliptically polarized.

(d) $\mathrm{AR} = 2.47$ dB, right handed circularly polarized.

(e) Phase difference, $\phi = (100° - 20°) = 80°$. Therefore $\mathrm{AR} = 1.52$ dB, left handed circularly polarized.

19. ***Manipulating polarisation states using devices*** **:** A randomly polarized light propagating in the z direction traverses through a linear polariser and a waveplate as shown in the figure.



The polarizer's transmission axis (only the component of light aligned with this axis passes through) is oriented at a 45^o angle to the x-axis. The waveplate is made up of a special material that exhibits a low refractive index (n_1) for the y-polarised light and a high refractive index (n_2) for the x-polarised light. If the thickness of the optical device is

$$d = \frac{\lambda}{4(n_2 - n_1)}$$

What is the polarisation state of the light after the second element? (λ is the wavelength of the light)

Solution: After the polariser, the light is linearly polarised with equal components along the x and y directions. Now this field passes through the waveplate.

The phase acquired by the y-component of light is $\frac{2\pi}{\lambda} * n_2 * d = \frac{2\pi * n_2}{4(n_2 - n1)}$

The phase acquired by the x-component of light is $\frac{2\pi}{\lambda} * n_1 * d = \frac{2\pi * n_1}{4(n_2 - n1)}$

The phase difference between the x- and y-component is $\frac{\pi}{2}$ and the magnitude of the field components are equal. Hence the light after the second element will be **circularly polarised**. The slow axis (axis with high refractive index) adds an additional phase to its component. So y-component lags behind the x-component by $\frac{\pi}{2}$ and the polarisation is RCP.

The waveplate is a **quarter-wave plate**.

20. ***Application in 3D movies*** **:** In the previous question, if the light after the waveplate again passes through another waveplate and then through a linear polariser (both identical as in the previous case) as shown below:

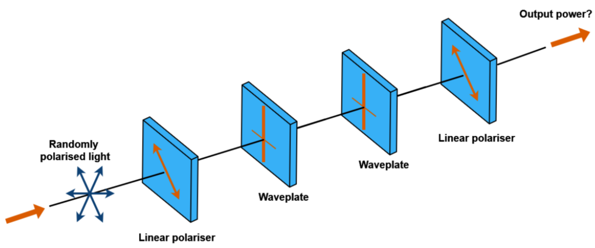


Figure 2: Polarizers and waveplates

How much amount of light will pass through the final polariser? Can you connect the concepts discussed above with polarized 3D technology?

Solution:

The second waveplate again adds an additional $\frac{\pi}{2}$ phase shift to the x-component of the field. Hence the output will be linearly polarised with a -45^o orientation. So no light will pass through the second polariser.

The two quarter-wave plates act as a **half-wave plate** which **rotates the input polarised light by 90^o**.

One way of creating the effect of 3D image is through the use of polarised filters. In 3D movies, two slightly offset perspectives of the same scene are projected onto the screen simultaneously (**stereoscopic 3D**). Circular polarization is often utilized, with one projection polarized in RCP and the other in LCP. Each eye is equipped with glasses that contain corresponding circular polarized filters. These filters ensure that each eye sees only one of the polarized projections. The human brain is able to combine these two images into a 3D image (**stereoscopic vision**).

Linearly polarized projections are used in some 3D systems as an alternative to circular polarization. However, the movement of the head while watching will result in leakage of the image between the eyes, which can lead to a reduced 3D effect.