EE3110 - Probability Foundations for Electrical Engineers
Tutorial solutions - Week 1

---

1. (a) By using a Venn diagram, it can be seen that for any sets S and T, we have

$$S = (S \cap T) \cup (S \cap T^C) \tag{1}$$

Alternatively, argue that an x must belong to either T or to T^C, so x belongs to S if and only if it belongs to $S \cap T$ or to $S \cap T^C$. Apply 1 with $S = A^C$ and $T = B$ to obtain the first relation

$$A^C = (A^C \cap B) \cup (A^C \cap B^C) \tag{2}$$

Interchange the roles of A and B to obtain the second relation

$$B^C = (A \cap B^C) \cup (A^C \cap B^C) \tag{3}$$

(b) By De Morgan's law, we have

$$(A \cap B)^C = A^C \cup B^C$$

and by using 2 and 3, we obtain

$$\begin{aligned}(A \cap B)^C &= ((A^C \cap B) \cup (A^C \cap B^C)) \cup ((A \cap B^C) \cup (A^C \cap B^C)) \\ &= (A^C \cap B) \cup (A^C \cap B^C) \cup ((A \cap B^C)\end{aligned} \tag{4}$$

(c)

$$\begin{aligned} A &= \{1, 3, 5\} \\ B &= \{1, 2, 3\} \\ \implies A \cap B &= \{1, 3\} \\ \therefore (A \cap B)^C &= \{2, 4, 5, 6\} \end{aligned}$$

and

$$\begin{aligned} A^C \cap B &= \{2\} \\ A^C \cap B^C &= \{4, 6\} \\ A \cap B^C &= \{5\} \end{aligned}$$

Thus, 4 holds.

2. Let G and C be the events that the chosen student is a genius and a chocolate lover, respectively. We have $P(G)$=0.6, $P(C)$=0.7 and $P(G \cap C)$=0.4. The probability that a randomly chosen student is neither a genius nor a chocolate lover is given by,

$$\begin{aligned} P(G^C \cap C^C) &= 1 - P(G \cup C) \\ &= 1 - (P(G) + P(C) - P(G \cap C)) \\ &= 1 - (0.6 + 07 - 0.4) \\ &= 0.1 \\ \implies P(G^C \cap C^C) &= 0.1 \end{aligned}$$

3. (a) Since $\Omega = \cup_{i=1}^{n} S_i$ and $\boldsymbol{P}(A \cap \Omega) = \boldsymbol{P}(A)$, we get

$$\boldsymbol{P}(A) = \boldsymbol{P}(A \cap \Omega) = \boldsymbol{P}(A \cap (\cup_{i=1}^{n} S_i)) = \cup_{i=1}^{n} \boldsymbol{P}(A \cap S_i)$$

As events $S_1, ..., S_n$ are disjoint, the sets $(A \cap S_i)$ are disjoint. Hence, union can be written as the sum of disjoint sets, which gives the following result,

$$\boldsymbol{P}(A) = \cup_{i=1}^{n} \boldsymbol{P}(A \cap S_i) = \sum_{i=1}^{n} \boldsymbol{P}(A \cap S_i)$$

(b) The events $B \cap C$, $B^c \cap C$, $B \cap C^c$, $B^c \cap C^c$ are disjoint and form a partition of Ω, so using the result from part (a), we get,

$$\boldsymbol{P}(A) = \boldsymbol{P}(A \cap B \cap C) + \boldsymbol{P}(A \cap B^c \cap C) + \boldsymbol{P}(A \cap B \cap C^c) + \boldsymbol{P}(A \cap B^c \cap C^c) \tag{5}$$

Event $A \cap B$ can be written as the union of two disjoint events as follows,

$$(A \cap B) = (A \cap B \cap C) + (A \cap B \cap C^c)$$

So,

$$\boldsymbol{P}(A \cap B) = \boldsymbol{P}(A \cap B \cap C) + \boldsymbol{P}(A \cap B \cap C^c) \tag{6}$$

Similarly,

$$\boldsymbol{P}(A \cap C) = \boldsymbol{P}(A \cap B \cap C) + \boldsymbol{P}(A \cap B^c \cap C) \tag{7}$$

Combining (1),(2),(3) we get,

$$\boldsymbol{P}(A) = \boldsymbol{P}(A \cap B) + \boldsymbol{P}(A \cap C) + \boldsymbol{P}(A \cap B^c \cap C^c) - \boldsymbol{P}(A \cap B \cap C)$$

4. (a)

$$\boldsymbol{P}(A \cap B) = \boldsymbol{P}(A) + \boldsymbol{P}(B) - \boldsymbol{P}(A \cup B) \geq \boldsymbol{P}(A) + \boldsymbol{P}(B) - 1$$
$$(A \cap B) \subseteq A, (A \cap B) \subseteq B \Rightarrow \boldsymbol{P}(A \cap B) \leq min[\boldsymbol{P}(A), \boldsymbol{P}(B)]$$

(b)

$$\boldsymbol{P}(A \cap B) \geq \boldsymbol{P}(A) + \boldsymbol{P}(B) - 1 \geq 1 - \delta + 1 - \delta - 1 = 1 - 2\delta$$

5. The sample space is $S = \{1, 2, 3, 4, 5, 6\}$.
$P(\{1\}) = P(\{3\}) = P(\{5\}) = x\ (say)$
$P(\{2\}) = P(\{4\}) = P(\{6\}) = y\ (say)$.
As per question

$$y = 2x \tag{8}$$
$$3x + 3y = 1 \tag{9}$$

Solving the equations we get $x = 1/9$ and $y = 2/9$ which defines the probability model.
Let A be the event that the outcome is less than 4, $A = \{1, 2, 3\}$.
$P(A) = 2x + y = 4/9$.

6. Upper bound for $P(A \cap B)$: Occurs when the smaller event is contained in the larger event. $(i.e) when\ B \subset A$ (In this question)

$$\begin{aligned} P(A \cap B) &\leq min\{P(A), P(B)\} \\ &= P(B) \\ &= \frac{1}{3} \end{aligned}$$

Lower bound for $P(A \cap B)$: Occurs when both the events are as disjoint as possible(as less intersection as possible). (In this question) A and B cannot be completely disjoint, as $P(A) + P(B) > 1$. So, to make the intersection portion small, let's take $A \cup B = \Omega$

$$\begin{aligned} P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ &\geq P(A) + P(B) - P(\Omega) && (\because P(A \cup B) \leq P(\Omega)) \\ &= \frac{3}{4} + \frac{1}{3} - 1 \\ &= \frac{1}{12} \end{aligned}$$

$\implies \frac{1}{12} \leq P(A \cap B) \leq \frac{1}{3}$

$$\begin{aligned} &\implies \frac{1}{12} \leq P(A) + P(B) - P(A \cup B) \leq \frac{1}{3} \\ &\implies \frac{1}{12} \leq \frac{3}{4} + \frac{1}{3} - P(A \cup B) \leq \frac{1}{3} \\ &\implies \frac{3}{4} \leq P(A \cup B) \leq 1 \end{aligned}$$

7. Let $P = A \cup (\cap_{i \in I} B_i)$ and $Q = \cap_{i \in I}(A \cup B_i)$
   In order to prove $P = Q$, we have to show $P \subseteq Q$ and $Q \subseteq P$
   Let x be an element in P, then there are two possibilities:

   1. Either x belongs to A in which case x belongs to all sets $A \cup B_n$ implying x to be in Q.
   2. or x belongs to all sets of B_n, in which case it belongs to all sets $A \cup B_n$, making again x belonging to Q.

   Hence,

   $$P \subseteq Q \tag{10}$$

   Now, if x belongs to Q, then x belongs to $A \cup B_n$.
   If $x \in A$, then $x \in P$ or x must be in every set $B_n \implies x \in P$
   Hence,

   $$Q \subseteq P \tag{11}$$

   From 10 and 11, $P = Q$

8. As we know, the probability of any event is ≤ 1

   $$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \leq 1 \tag{12}$$
   $$\Rightarrow \quad \mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1 \tag{13}$$

   $$\mathbb{P}(\cap_{i \in \{1 \dots n\}} A_i) = 1 - \mathbb{P}((\cap_{i \in \{1 \dots n\}} A_i)^c) \tag{14}$$
   $$= 1 - \mathbb{P}(\cup_{i \in \{1 \dots n\}} A_i^c) \tag{15}$$

   Now, using the union bound, we have

   $$\mathbb{P}(\cap_{i \in \{1 \dots n\}} A_i) \geq 1 - \sum_{i=1}^{n} \mathbb{P}(A_i^c) \tag{16}$$
   $$= 1 - \sum_{i=1}^{n} 1 - \mathbb{P}(A_i) \tag{17}$$
   $$= \mathbb{P}(A_1) + \mathbb{P}(A_2) + \dots + \mathbb{P}(A_n) - (n-1) \tag{18}$$

9. The presence of the blue balls is irrelevant in this problem since whenever a blue ball is drawn, it is ignored. Therefore, looking at only the red and white balls, there are $\binom{r+w}{r}$ possible positions that the red balls could occupy in the ordering as they are drawn. Therefore, the probability that they will be in the first r positions is $1/\binom{r+w}{r}$.

10. It is impossible to place exactly $n-1$ letters in the correct envelopes, because if $n-1$ letters are placed correctly, then the n^{th} letter must also be placed correctly.

11. First, we need to calculate the probability that no two people share the same birthday. Then, we will determine when this probability drops below 50%.

    So, for k people, the probability $P(\text{no shared birthday})$ is given by:

$$P(\text{no shared birthday}) = \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \cdots \times \frac{365-k+1}{365}$$

    This can be simplified to:

$$P(\text{no shared birthday}) = \frac{365!}{(365-k)! \cdot 365^k} = \prod_{i=0}^{k-1} \left(1 - \frac{i}{365}\right)$$

    Using the Approximation $\ln(1-x) \approx -x$ for small x, we have the product as:

$$\ln\left(P(\text{no shared birthday})\right) = \sum_{i=0}^{k-1} \ln\left(1 - \frac{i}{365}\right) \approx \sum_{i=0}^{k-1} -\frac{i}{365}$$

    Thus, exponentiating both sides to get back to the probability, we have:

$$P(\text{no shared birthday}) \approx \exp\left(-\frac{k(k-1)}{2 \cdot 365}\right)$$

    We want this to be less than 0.5; solving the quadratic equation gives us the least integer satisfying this is $k = 23$. It is non-trivial that with just 23 people in a room, there is over a 50% chance that at least two of them share the same birthday!

12. P_4 will reach the semifinal if and only if none of P_1, P_2, P_3 is in the same quarter of the final draw as P_4. The probability of this event E is,

$$P(E) = \frac{\binom{12}{3}}{\binom{15}{3}} = 44/91$$

13. There are 7^5 possible outcomes in the sample space. If the five passengers are to get off on different floors, the first passenger can get off on any one of the seven floors, and the second passenger can then get off on any one of the other six floors, etc. Thus, the probability is

$$\frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{7^5} = \frac{360}{2401}.$$

14. We derive a recursion for the probability p_i that a white ball is chosen from the i th jar. We have, using the total probability theorem,

$$p_{i+1} = \frac{m+1}{m+n+1} p_i + \frac{m}{m+n+1}(1-p_i) = \frac{1}{m+n+1} p_i + \frac{m}{m+n+1},$$

starting with the initial condition $p_1 = m/(m+n)$. Thus, we have

$$p_2 = \frac{1}{m+n+1} \cdot \frac{m}{m+n} + \frac{m}{m+n+1} = \frac{m}{m+n}.$$

More generally, this calculation shows that if $p_{i-1} = m/(m+n)$, then $p_i = m/(m+n)$. Thus, we obtain $p_i = m/(m+n)$ for all i.

15. If we put $A = a$ and $B = b$ for some $a \neq b$. Now, if it were to be that $f(a) = f(b)$, then we see that $f(A) \cap f(B) = \{f(a)\} \neq \phi$, which is a contradiction to the hypothesis. Therefore, our choice of a, b being arbitrary, f must be injective.

16. Define $f : \{0,1\} \to \{0,1\}$ by $f(0) = f(1) = 0$. Setting $A = \{0\}$ and $B = \{1\}$, we have that $f(A \cap B) = \phi$. On the other hand, $f(A) \cap f(B) = \{0\}$.

17.

$$\begin{gathered}
P(A' \cap B \cap C) = P(B \cap C) - P(A \cap B \cap C) \\
P(B \cap C) = P(B \mid C) \times P(C) \\
P(B \cap C) = 1/3 \times 1/2 = 1/6 \\
P(A \cap B \cap C) = P(A \mid B \cap C) \times P(B \cap C) \\
P(A \cap B \cap C) = 1/4 \times 1/6 = 1/24 \\
P(A' \cap B \cap C) = P(B \cap C) - P(A \cap B \cap C \\
P(A' \cap B \cap C) = 1/6 - 1/24 = 1/8
\end{gathered}$$

18. Probability of sample space is equal to 1. Hence

$$\begin{gathered}
p^2 + p - 1 = 0 \\
p = 0.6180
\end{gathered}$$