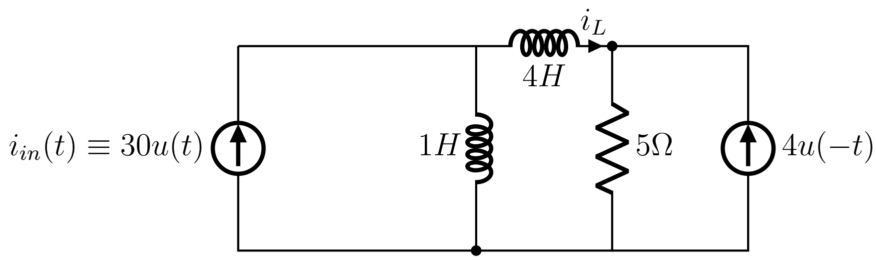
EE2015: Electric Circuits and Networks

Tutorial 7

(September 27, 2024)

1. Evaluate and sketch the current through the 4 H inductor of the circuit given below using time domain techniques. Write the differential equation for the current and find the (a) Zero state and zero input response and (b) natural and forced response.



Solution:

We can ignore the $4u(-t)$ current source for writing the differential equation, since it is accounted for in the initial conditions of the $4H$ inductor, which is equal to $-4A$.

The differential equation is as follows:

$$4\frac{\mathrm{d}i_L}{\mathrm{d}t} + 5i_L = 1\frac{\mathrm{d}(i_{in} - i_L)}{\mathrm{d}t}$$

$$\frac{\mathrm{d}i_L}{\mathrm{d}t} + i_L = \frac{1}{5}\frac{\mathrm{d}i_{in}}{\mathrm{d}t}$$

Using this, we can solve for the ZSR, ZIR, NR, and FR.

(a) **Zero state response:** We assume that the initial conditions are zero, i.e. $i_L(t) = 0A, -\infty \leq t \leq 0^-$. Now, at $t = 0$, the $30A$ source dumps an impulse current into the circuit. Of the $30A$, a current of $\frac{1}{1+4} \cdot 30A = 6A$ is dumped on the $4H$ inductor, thus $i_L(0^+) = 6A$.

At $t = \infty$, all the current is through the $1H$ inductor, so $i_L(\infty) = 0$.

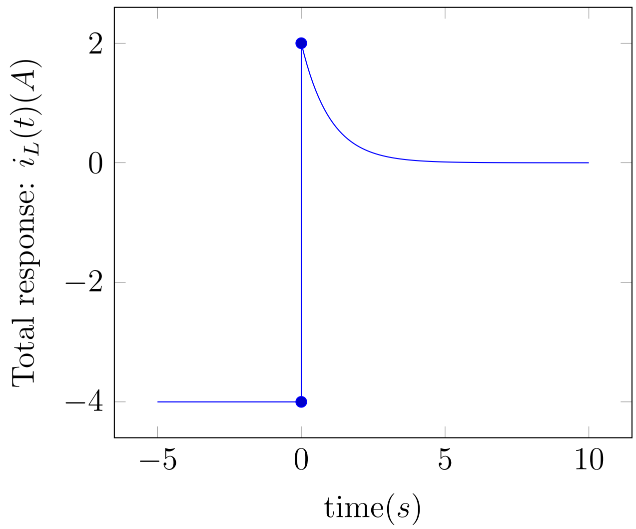
Note that the time constant is equal to $\frac{L_{eq}}{R} = 1s$.

Using this, we conclude that $i_L(t) = 6e^{-\frac{t}{1}}, 0 \leq t \leq \infty$.

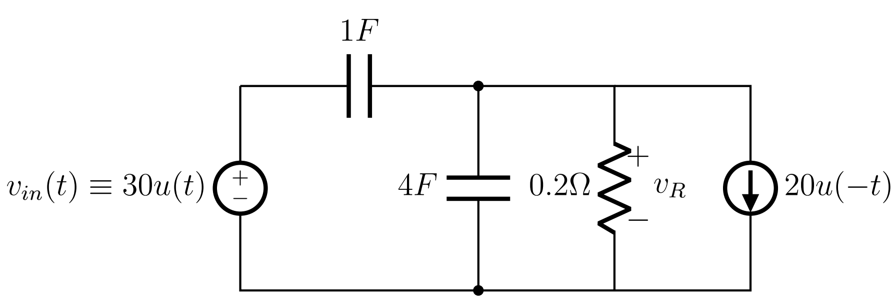
Zero input response: We assume that the input (i.e. $30u(t)$) source is turned off. Now, we have only the initial conditions of $-4A$ current through the inductor, i.e. $i_L(t) = -4A, -\infty \leq t \leq 0^-$. These inductors discharge through the 5Ω resistor, and $i_L(\infty) = 0A$. From this, $i_L(t) = -4e^{-\frac{t}{1}}, 0 \leq t \leq \infty$.

(b) **Forced Response:** This is the i_L's final steady state value, which is just equal to $0A$ (as described in the previous section).

Since the forced response is equal to zero, the natural response would just be equal to the total response, which in turn is equal to ZSR + ZIR. Thus, for the natural response, $i_L(t) = \begin{cases} -4, -\infty \leq t \leq 0^- \\ 2e^{-\frac{t}{1}}, 0^+ \leq t \leq \infty \end{cases} \quad A.$



2. Evaluate and sketch the voltage across the resistor of the circuit given below using time domain techniques. Write the differential equation for the voltage and find the (a) Zero state and zero input response and (b) natural and forced response.



Solution:

Again, the $20u(-t)$ current source can be thought of as initial conditions. This corresponds to v_R having an initial condition equal to $-4V$.

Now, we can ignore the $20u(-t)$ current source for writing the differential equations.

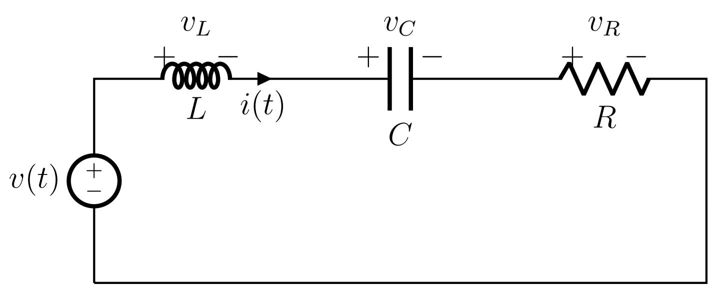
We get: $1\dfrac{\mathrm{d}(v_{in}-v_R)}{\mathrm{d}t} = 4\dfrac{\mathrm{d}v_R}{\mathrm{d}t} + \dfrac{v_R}{0.2}$

$\frac{1}{5}\dfrac{\mathrm{d}v_{in}}{\mathrm{d}t} = \dfrac{\mathrm{d}v_R}{\mathrm{d}t} + v_R$

We note here that this differential equation is a dual of the one obtained in the previous question. Also note that the initial condition of $-4V$ is also the same (we had $-4A$ initial condition in the previous question). Thus, the expressions for voltage $v_R(t)$ in each of the cases is exactly the same as the ones obtained for $i_L(t)$ in the previous question.

$$v_R(t) = \begin{cases} -4, -\infty \le t \le 0^- \\ 2e^{-\frac{t}{1}}, 0^+ \le t \le \infty \end{cases} \quad V$$

3. For the RLC cicuit given below, i) write the differential equation relating these combination of excitation and responses: a) v and i b) v and v_C c) v and v_L. ii) If $i(0^-) = I_0$ and $v_C(0^-) = V_0$, find $i(s)$, $v_C(s)$ and $v_L(s)$ by taking the Laplace transform of the differential equations. iii) In the case of $v_C(t)$, show that the ZSR with $\delta(t)$ as the input is same as the ZIR with initial conditions $v_C(0^-) = 0$ and $i(0^-) = 1/L$



3)

$$v_L = L\frac{di}{dt} \Rightarrow i = \int_0^t \frac{v_L}{L}\,dt + i(0)$$

(Considering only $t=0$ as reference)

$$i = C\frac{dv_c}{dt} \Rightarrow v_c = \int_0^t \frac{i}{C}\,dt + v_c(0)$$

$$v_R = iR$$

$$V = v_L + v_C + v_R$$

i) a) V & i

$$V = L\frac{di}{dt} + \int_0^t \frac{i}{C}\,dt + v_c(0) + iR$$

b) V & v_c

$$V = LC\frac{d^2v_c}{dt^2} + v_c + RC\frac{dv_c}{dt}$$

c) V & v_L

$$V = v_L + \int_0^t \frac{1}{C}\left(\int_0^t \frac{v_L}{L}\,dt + i(0)\right)dt + v_c(0)$$

$$\int_0^t \frac{R}{L}v_L\,dt + i(0)R$$

ii) s-Domain relations (considering zero initial conditions)

i.e $i(0)=0$, $v_c(0)=0$

a) $V(s)$ & $i(s)$

$$\frac{i(s)}{V(s)} = \frac{sC}{s^2LC + sCR + 1}$$

b) $V_c(s)$ & $V(s)$

$$\frac{V_c(s)}{V(s)} = \frac{1}{s^2LC + sCR + 1}$$

c) $V_L(s)$ & $V(s)$

$$\frac{V_L(s)}{V(s)} = \frac{s^2LC}{s^2LC + sCR + 1}$$

iii) ZSR with $v(t) = \delta(t)$

$$V = LC\frac{d^2v_c}{dt^2} + v_c + RC\frac{dv_c}{dt} \quad \rightarrow (1)$$

$V(s) = \mathcal{L}(\delta(t)) = 1$

$$\Rightarrow V_c(s) = \frac{V(s)}{s^2LC + sCR + 1} = \frac{1}{s^2LC + sCR + 1}$$

ZIR with $v_c(0^-) = 0$, $i(0^-) = 1/L$

(1) $\Rightarrow$ $v(t) = 0$ for ZIR

$$0 = LC\left(s^2V_c(s) - s\,v_c(0^-) - v_c'(0^-)\right) + V_c(s) + RC\left(sV_c(s) - v_c(0^-)\right)$$

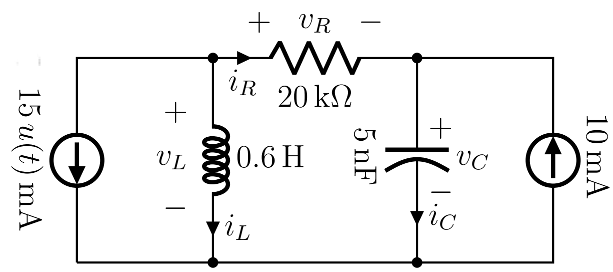
$$\left(\text{as } i = C\frac{dv_c}{dt}\ ;\ i(0^-) = C\,v_c'(0^-) \Rightarrow v_c'(0^-) = \frac{1}{LC}\right)$$

$$V_c(s)\left[s^2LC + sCR + 1\right] = LC\,v_c'(0^-) = 1$$

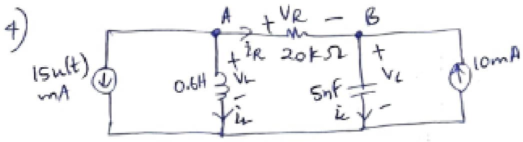
$$\Rightarrow V_c(s) = \frac{1}{s^2LC + sCR + 1}$$

Same response as ZSR with $v(t) = \delta(t)$

4. For the RLC cicuit given on the right, evaluate at $t = 0^+$ the derivative of each current and voltage variable labelled in the figure.



4)



For $t = 0^-$

$v_R = -200V$; $i_R = -10mA$

$v_c = 200V$; $i_c = 0$

$v_L = 0$; $i_L = 10mA$

For $t = 0^+$

$i_L = 10mA$ $(= i_L(0^-))$

$v_c = 200V$ $(= v_c(0^-))$

KCL at node A,

$$i_R + i_L + 15\,u(t)\,mA = 0 \longrightarrow (1)$$

$$i_R(0^+) + 10\,mA + 15\,mA = 0$$

$$\Rightarrow i_R(0^+) = -25\,mA$$

$$\frac{d}{dt}(1) \Rightarrow \left.\frac{di_R}{dt}\right|_{t=0^+} + \left.\frac{di_L}{dt}\right|_{t=0^+} + 15\,\delta(t) \times 10^{-3} = 0$$

$\Bigg($ $v_L = L\dfrac{di_L}{dt}$ & $v_L = v_R + v_C$

gives $v_L(0^+) = i_R(0^+) \times 20 \times 10^3 + 200 = -300V$

$$v_L(0^+) = 0.6 \left.\frac{di_L}{dt}\right|_{t=0^+}$$

$$\Rightarrow \left.\frac{di_L}{dt}\right|_{t=0^+} = -500 \quad \Bigg)$$

$$\boxed{\left.\frac{di_R}{dt}\right|_{t=0^+} = 500 - 15\,\delta(t)\Big|_{t=0^+} \times 10^{-3} \approx 500 \text{ A/s}}$$

KCL at B, $i_R + 10mA = i_c \longrightarrow (2)$ & $i_c(0^+) = C\left.\dfrac{dv_c}{dt}\right|_{t=0^+}$

$i_c(0^+) = -15\,mA$

$$\boxed{\left.\frac{dv_c}{dt}\right|_{t=0^+} = -3 \times 10^6 \text{ V/s}}$$

$$\frac{d}{dt}(2) \Rightarrow \left.\frac{di_R}{dt}\right|_{t=0^+} = \left.\frac{di_c}{dt}\right|_{t=0^+} = 500 \text{ A/s}$$

$$v_L = v_R + v_C$$

$$\frac{d}{dt} v_L \Big|_{t=0^+} = \frac{dv_R}{dt}\Big|_{t=0^+} + \frac{dv_C}{dt}\Big|_{t=0^+}$$

$$= R\,\frac{di_R}{dt}\Big|_{t=0^+} + \frac{dv_C}{dt}\Big|_{t=0^+}$$

$$\frac{dv_L}{dt}\Big|_{t=0^+} = 10\times10^6 - 3\times10^6 = 7\times10^6 \text{ V/s}$$

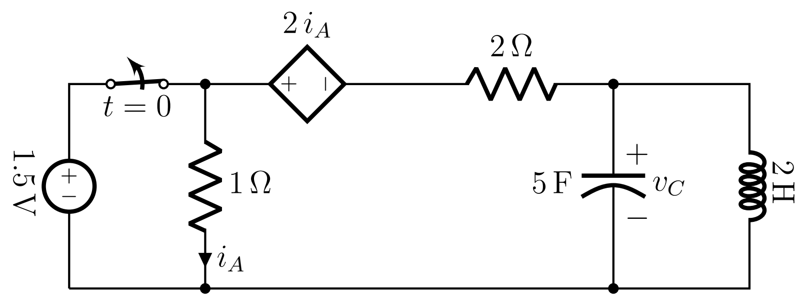
Answers

At $t=0^+$, $\frac{dv_L}{dt} = 7\times10^6$ V/s ; $\frac{di_L}{dt} = -500$ A/s

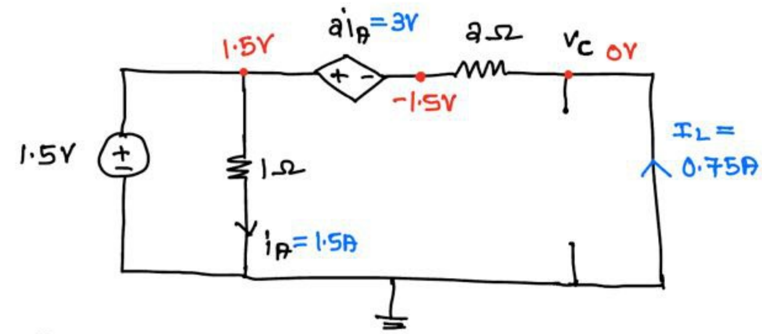
$\frac{dv_R}{dt} = 10\times10^6$ V/s ; $\frac{di_R}{dt} = 500$ A/s

$\frac{dv_C}{dt} = -3\times10^6$ V/s ; $\frac{di_C}{dt} = 500$ A/s

5. For the RLC cicuit given on the right, obtain an expression for the energy stored in the capacitor that is valid for all $t > 0$.

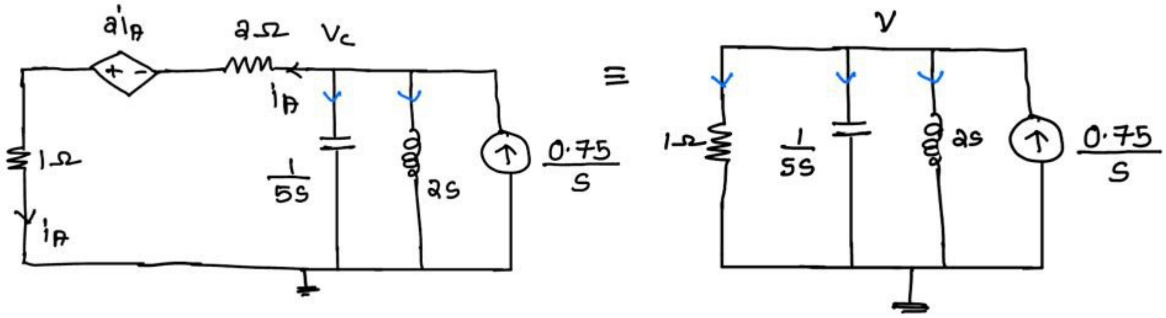


@ $t<0$ (at steady state)



$V_C(0^-) = 0V$
$I_L(0^-) = 0.75A$

@ $t>0$



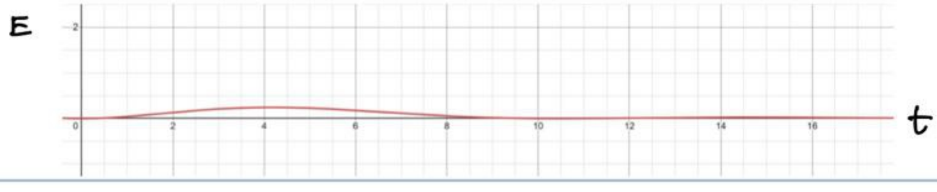
KCL @ V

$$\frac{0.75}{s} = V(s)\left[1 + 5s + \frac{1}{2s}\right]$$

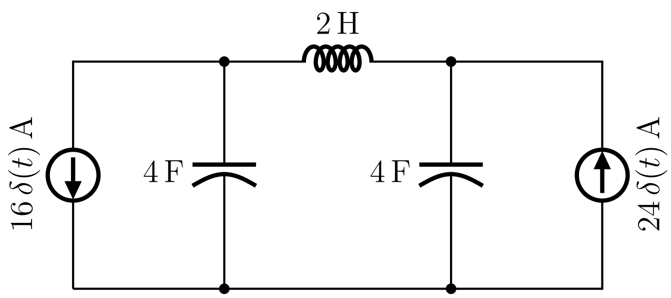
$$V(s) = \frac{0.75}{5s^2 + s + \frac{1}{2}} = \frac{0.15}{s^2 + 0.2s + 0.1} = \frac{0.5 \times 0.3}{(s+0.1)^2 + (0.3)^2}$$

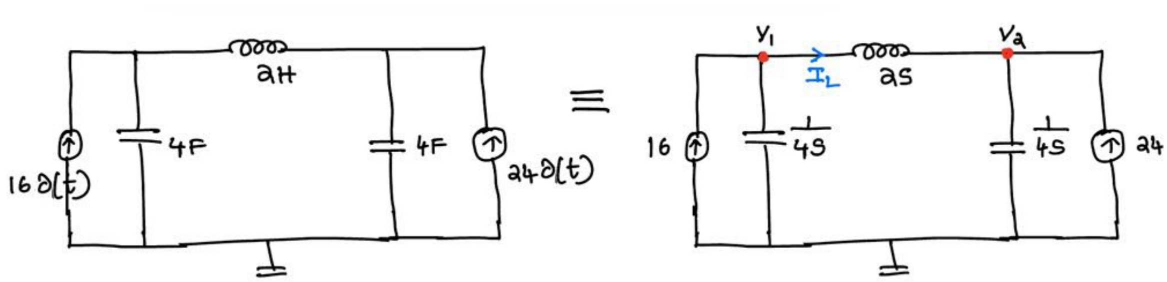
$$v(t) = 0.5\, e^{-0.1t} \sin 0.3t$$

$$\text{Energy}\,(C_{5F}) = \frac{1}{2} \times 5 \times \left(0.5\, e^{-0.1t} \sin 0.3t\right)^2 = 0.625\, e^{-0.2t} \sin^2 0.3t$$



6. Consider the circuit shown on the right. Draw the s-domain representation of the circuit. Find the energy stored in the inductor and each of the capacitors at time $t = \pi$ sec. Assume that the inductor current and capacitor voltages are zero for $t < 0$. Verify your answer by applying the principle of conservation of energy at $t = 0$ and $t = \pi$ sec.





KCL @ V_1 and V_2.

$$-16 + 4sV_1 + \frac{V_1 - V_2}{2s} = 0 \quad — (1)$$

$$-24 + 4s V_2 - \frac{(V_1 - V_2)}{2s} = 0 \quad — (2)$$

Solving (1) and (2),

$$V_1 = \frac{-16s^2+1}{4s^3+s}, \quad V_2 = \frac{24s^2+1}{4s^3+s} \qquad I_L = \frac{-20}{4s^2+1}$$

$$\boxed{\begin{aligned} V_1(t) &= \left(1 - 5\cos\frac{t}{2}\right) u(t) \\ V_2(t) &= \left(1 + 5\cos\frac{t}{2}\right) u(t) \\ I_L(t) &= -10 \sin t/2 \; u(t) \end{aligned}}$$

$V_1(0) = -4$	$V_1(\pi) = 1$
$V_2(0) = 6$	$V_2(\pi) = 1$
$I_2(0) = 0$	$I_2(\pi) = -10$

$$\text{Energy}_{\text{total}}(t) = \frac{1}{2} C V_1^2(t) + \frac{1}{2} C_2 V_2^2(t) + \frac{1}{2} L I_L^2(t)$$

$$\text{Energy}_{\text{total}}(t=0) = \frac{1}{2} \times 4 \times 16 + \frac{1}{2} \times 4 \times 36 + \frac{1}{2} \times 2 \times 0 = 104$$

$$\text{Energy}_{\text{total}}(t=\pi) = \frac{1}{2} \times 4 \times 1 + \frac{1}{2} \times 4 \times 2 + \frac{1}{2} \times 2 \times 100 = 104$$

$$\boxed{\text{Energy}_{\text{total}}(t=0) = \text{Energy}_{\text{total}}(t=\pi), \text{ Hence Verified}}$$