EE3110 - Probability Foundations for Electrical Engineers
Tutorial solutions - Week 3

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1. We have,

$$\mathbb{P}[\Omega] = 1$$
$$\Rightarrow \sum_{x=1}^{\infty} f(x) = \sum_{x=1}^{\infty} C\frac{2^x}{x!} = 1$$

   Solving for C, we obtain $C = \frac{1}{e^2-1}$
   To find $\mathbb{P}[X\ is\ even] = \mathbb{P}[X = 2k : k \in \mathbb{N}]$
   Therefore,

$$\begin{aligned}\mathbb{P}[X\ is\ even] = \sum_{k=1}^{\infty} C\frac{2^{2k}}{(2k)!} &= C\sum_{i=0}^{\infty}\frac{2^i+(-2)^i}{2(i!)} - 1 \\ &= C\left(\frac{e^2+e^{-2}}{2} - 1\right) \\ &= \frac{1-e^{-2}}{2}\end{aligned}$$

2. Since

$$\bigcap_{n\in\mathbb{N}}\left[x-\frac{1}{n}, x+\frac{1}{n}\right] = \{x\},$$

   by the continuity of probability, we have,

$$P[X = x] = \lim_{\epsilon\to 0} P[x-\epsilon < X \leq x].$$

   Now, since

$$P[x-\epsilon < X \leq x] = F_X(x) - F_X(x-\epsilon),$$

   it follows from the definition of a continuous random variable, that,

$$P[X = x] = \lim_{\epsilon\to 0} F_X(x) - F_X(x-\epsilon) = \lim_{\epsilon\to 0}\int_{x-\epsilon}^{x} f_X(x)dx = 0.$$

3. No! CDF must have at least one point of discontinuity in the range of a discrete random variable.

4.
   1. t=1, μ=λ*t = 8, P(X=6) = $\frac{e^{-8}8^6}{6!}$ = 0.122
   2. t=2, μ=λ*t = 16, P(X$\leq$5) =[P(X=0)+P(X=1)+..+P(X=5)] = 0.001384
   3. t=2, μ=λ*t = 16, P(X$\geq$6) = 1 - [P(X$\leq$5)] = 0.9986

5. Given A_K is k success in n Bernoulli(p) trials, it implies binomial distribution as follows:

$$\begin{aligned}
\mathbb{P}(A_k) &= \binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k} \\
&= \frac{n(n-1)\dots(n-k+1)}{k!}\frac{\lambda^k}{n^k}\left(1-\frac{\lambda}{n}\right)^{n-k} \\
&= \frac{\lambda^k}{k!}\frac{n(n-1)\dots(n-k+1)}{n^k}\left(1-\frac{\lambda}{n}\right)^{n}\left(1-\frac{\lambda}{n}\right)^{-k} \\
&= \frac{\lambda^k}{k!}1(1-\frac{1}{n})\dots(1-\frac{k-1}{n})\left(1-\frac{\lambda}{n}\right)^{n}\left(1-\frac{\lambda}{n}\right)^{-k} \\
&= \frac{\lambda^k}{k!}\prod_{r=1}^{k-1}\left(1-\frac{r}{n}\right)\left(1-\frac{\lambda}{n}\right)^{n}\left(1-\frac{\lambda}{n}\right)^{-k}
\end{aligned}$$

Standard limit results imply that

$$\lim_{n\to\infty}\left(1-\frac{r}{n}\right) = 1 \quad \text{for all } r \geq 1$$

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-k} = 1 \quad \text{for all } \lambda \geq 0, k \geq 1;$$

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n} = e^{-\lambda} \quad \text{for all } \lambda \geq 0$$

As $\mathbb{P}(A_k)$ is a finite product of such expressions, the result is now immediate using the properties of limits.

$$\lim_{n\to\infty}\mathbb{P}(A_k) = \frac{e^{-\lambda}\,\lambda^k}{k!}.$$

6. a) The maximum value to appear in the two rolls: $X \in \{1, 2, 3, 4, 5, 6\}$
   b) The minimum value to appear in the two rolls: $X \in \{1, 2, 3, 4, 5, 6\}$
   c) The sum of two rolls: $X \in \{2, 3, \dots, 12\}$
   d) The value of the first roll minus the value of the second roll: $X \in \{-5, -4, \dots, 4, 5\}$

7. Given that the probability of getting heads is 0.7 and the probability of getting tails is 0.3, let X represent the number of heads that appear in three tosses.

   According to the Binomial distribution, the probability mass function of X is given by:

$$p_X(x) = \binom{n}{x}p^x(1-p)^{n-x}$$

   Given $n = 3$, $X \in \{0, 1, 2, 3\}$, and the probability mass function of X is:

$$p_X(x) = \binom{3}{x}p^x(1-p)^{3-x}$$

8. The number of guests that have the same birthday as you is binomial with p= 1/365 and n= 499. Thus the probability that exactly one other guest has the same birthday is

$$\binom{499}{1} \times \frac{1}{365} \times \left(\frac{364}{365}\right)^{498}$$

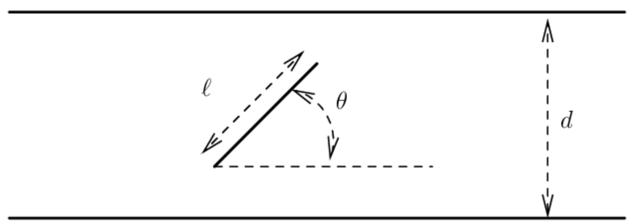
   Let $\lambda = np = 499/365 \approx 1.367$. The Poisson approximation is $e^{-\lambda}\lambda = e^{-1.367} \times 1.367 \approx 0.3483$, which closely agrees with the correct probability based on the binomial.

9. Using the expression for the Poisson PMF, we have, for $k \geq 1$

$$\frac{p_X(k)}{p_X(k-1)} = \frac{\lambda^k \cdot e^{-\lambda}}{k!} \times \frac{(k-1)!}{\lambda^{k-1} \cdot e^{-\lambda}} = \frac{\lambda}{k}$$

Thus if $k \leq \lambda$ the ratio is greater or equal to 1, and it follows that $p_X(k)$ is monotonically increasing. Otherwise, the ratio is less than one, and $p_X(k)$ is monotonically decreasing, as required

10. Let us say the needle drops to lie at an angle of θ away from the horizontal, where θ will be in the range of $0 \leq \theta \leq \pi/2$ (We will ignore the case where the needle comes to lie with a negative slope, since that case is the same as that of positive slope and symmetric).



The needle that lies with angle θ has a height $lsin\theta$, and the probability that such a needle crosses one of the horizontal lines of distance d is $\frac{lsin\theta}{d}$.

Thus, we get the probability by averaging over all the possible angles θ :

$$p = \frac{\int_0^{\frac{\pi}{2}} \frac{lsin\theta}{d} d\theta}{\int_0^{\frac{\pi}{2}} d\theta} = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{lsin\theta}{d} d\theta = \frac{2l}{\pi d} [-cos\theta]_0^{\frac{\pi}{2}}$$

$$p = \frac{2l}{\pi d}$$

11. We will use the PMF for the number of girls among the natural children together with the formula for the PMF of a function of a random variable. Let N be the number of natural children that are girls. Then N has a binomial PMF

$$p_N(k) = \begin{cases} \binom{5}{k} \cdot \left(\frac{1}{2}\right)^5 & , if 0 \leq k \leq 5 \\ 0 & \text{, otherwise.} \end{cases}$$

Let G be the number of girls out of the 7 children, so that $G = N + 2$. By applying the formula for the PMF of a function of a random variable, we have

$$p_G(g) = \sum_{n|n+2=g} p_N(n) = p_N(g-2).$$

Thus

$$p_G(g) = \begin{cases} \binom{5}{g-2} \cdot \left(\frac{1}{2}\right)^5 & , if 2 \leq g \leq 7 \\ 0 & \text{, otherwise.} \end{cases}$$

12. Let X be the number of matches that remain when a matchbox is found empty. For $k = 0, 1, \ldots n$, let L_k (or R_k) be the event that an empty box is first discovered in the left (respectively. right) pocket while the number of matches in the right (respectively, left) pocket is k at that time. The PMF of X is

$$p_X(k) = \mathbf{P}(L_k) + \mathbf{P}(R_k). \quad k = 0, 1, \ldots, n$$

Viewing a left and a right pocket selection as a "success" and a "failure," respectively, $\mathbf{P}(L_k)$ is the probability that there are n successes in the first $2n-k$ trials, and trial $2n-k+1$ is a success. or

$$\mathbf{P}(L_k) = \frac{1}{2}\binom{2n-k}{n}\left(\frac{1}{2}\right)^{2n-k}. \quad k = 0, 1, \ldots, n$$

By symmetry. $\mathbf{P}(L_k) = \mathbf{P}(R_k)$. so

$$p_X(k) = \mathbf{P}(L_k) + \mathbf{P}(R_k) = \binom{2n-k}{n}\left(\frac{1}{2}\right)^{2n-k}, \quad k = 0.1, \ldots n$$

In the more general case, where the probabilities of a left and a right pocket selection are p and $1-p$, using a similar reasoning. we obtain

$$\mathbf{P}(L_k) = p\binom{2n-k}{n}p^n(1-p)^{n-k}, \quad k = 0.1, \ldots n$$

and

$$\mathbf{P}(R_k) = (1-p)\binom{2n-k}{n}p^{n-k}(1-p)^n, \quad k = 0, 1, \ldots, n$$

which yields

$$\begin{aligned} p_X(k) &= \mathbf{P}(L_k) + \mathbf{P}(R_k) \\ &= \binom{2n-k}{n}\left(p^{n+1}(1-p)^{n-k} + p^{n-k}(1-p)^{n+1}\right). \quad k = 0.1, \ldots n \end{aligned}$$

13. For $k = 0, 1, \ldots, n-1$, we have

$$\frac{p_X(k+1)}{p_X(k)} = \frac{\binom{n}{k+1}p^{k+1}(1-p)^{n-k-1}}{\binom{n}{k}p^k(1-p)^{n-k}} = \frac{p}{1-p} \cdot \frac{n-k}{k+1}$$

14. Since $n = 12,000,000$ is very large and $p = \frac{1}{1,000,000,000}$ is very small, so it is appropriate to use a Poisson Approximation where $\lambda = np = 0.012$.

    A message is error-free if there is not a single misread bit, so the probability that a given message will be received without an error is $e^{-0.012}$.

    Now we can think of each message being like a Bernoulli trial with probability $p = e^{-0.012}$, so that the number of messages correctly received is then like a **Binomial**$(3, e^{-0.012})$.

    Therefore the probability of receiving at least 2 error free messages is:

$$\binom{3}{3} * p^3 * (1-p)^0 + \binom{3}{2} * p^2 * (1-p)^1 = 0.9996$$

    There is about a 99.96 percentage chance that at least two of the messages will be correctly received.

15. for X which is geometric with parameter p, its PMF is :

$$\mathbb{P}(X = k) = (1-p)^{k-1}p \quad k = 1, 2, 3.....$$

$$\mathbb{P}(X = n+k | X > n) = \frac{\mathbb{P}(X = n+k, X > n)}{\mathbb{P}(X > n)}$$

The intersection of the events $X = n + k$ and $X > n$, is simply $X = n + k$

$$\mathbb{P}(X = n + k | X > n) = \frac{\mathbb{P}(X = n + k)}{\mathbb{P}(X > n)}$$

$$\mathbb{P}(X > n) = \sum_{i=n+1}^{\infty} \mathbb{P}(X = i) = \sum_{i=n+1}^{\infty} (1-p)^{i-1}p$$

Let $j = i - n$, then

$$\mathbb{P}(X > n) = \sum_{j=1}^{\infty}(1-p)^{j+n-1}p = (1-p)^n \sum_{j=1}^{\infty}(1-p)^{j-1}p = (1-p)^n \cdot 1$$

$$\begin{aligned}
\mathbb{P}(X = n + k | X > n) &= \frac{\mathbb{P}(X = n + k)}{\mathbb{P}(X > n)} \\
&= \frac{(1-p)^{n+k-1}p}{(1-p)^n} \\
\therefore \mathbb{P}(X = n + k | X > n) &= (1-p)^{k-1}p = \mathbb{P}(X = K)
\end{aligned}$$

16. a) We have, $\mathbb{P}[\Omega] = 1$

$$\begin{aligned}
&\implies \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{C\lambda^i \mu^j}{i!j!} = 1 \\
&\implies C\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} \sum_{j=0}^{\infty} \frac{\mu^j}{j!} = 1 \\
&\implies C\left[e^{\lambda}\right]\left[e^{\mu}\right] = 1 \\
&\implies C = e^{-(\lambda+\mu)}
\end{aligned}$$

b) $X : \Omega \to \mathbb{R}$ as $X(\{(i, j)\}) = i + j$ and $i, j \in \{0, 1, 2, 3, 4, ...\}$. So, $i + j \in \{0, 1, 2, 3, 4, ...\}$ this implies $\Omega_x = \{0, 1, 2, 3, 4, ...\}$
Now, p.m.f of X is

$$\begin{aligned}
\mathbb{P}[X = k] &= \mathbb{P}\left[(i, j) : i + j = k\right] \\
&= \mathbb{P}\left[(i, j) : i = l, j = k - l, \forall\ l \in 0, 1, 2, 3, ..., k\right] \\
&= \mathbb{P}\left[\bigcup_{l=0}^{k}\{(l, k - l)\}\right] \\
&= \sum_{l=0}^{k} \mathbb{P}\left[\{(l, k - l)\}\right] \\
&= \sum_{l=0}^{k} \frac{C\lambda^l \mu^{k-l}}{l!(k-l)!} \\
p_X(x) &= \sum_{l=0}^{k} \frac{e^{-(\lambda+\mu)}\lambda^l \mu^{k-l}}{l!(k-l)!}
\end{aligned}$$

17. They have the same distribution since Y is also equally likely to represent any day of the week, but $P(X < Y) = P(X \neq 7) = 6/7$.

18. For a simple example, let $X \leftarrow \mathit{Bernoulli}(1/2)$ (i.e., X can be thought of as a fair coin flip), and let $Y = 1 - X$. Then Y is also Bernoulli(1/2) by symmetry, but $X = Y$ is impossible. A more general example is to let $X \leftarrow \mathit{Binomial}(n, 1/2)$ and $Y = n - X$, where n is any odd number (think of this as interchanging the definitions of "success" and "failure"!).