**EE2025 Quiz I, 29th August 2024, 25points**
Time: 8:00 am to 8:50 am

Instructions:

- Write your name and roll number on your question paper, answer sheet and hand-written (A4) formula sheet. All will have to be submitted.
- Write legibly and clearly.
- While the working out for each problem MUST be shown in the answer sheet, marks will only be given for the final answer -with appropriate units - written in the question paper.
- **Please mark the final numerical answer in a box below each question in the question paper.** For example, if the answer is 10 V, ensure it is written like this: $\boxed{\text{10 V}}$
- **No marks will be awarded if units are not given.**

1. (5 points) If the normalized input impedance on a transmission line is $\bar{z}_{in} = 2 + j1$, determine the following:
   (a) The reflection coefficient at that point on the line.
   (1 mark)
   (b) The magnitude of the reflection coefficient.
   (1 mark)
   (c) The VSWR of the line
   (1 mark)
   (d) Assume that the line is 20 cm long and the wavelength on the line is 8 cm. What is the propagation constant?
   (1 mark)
   (e) What is the normalised load impedance $\bar{z}_L$?
   (1 mark)

**Solution:** a)

$$\Gamma(l) = \frac{\bar{z}_{in} - 1}{\bar{z}_{in} + 1} = \frac{1 + j1}{3 + j1} = 0.4 + 0.2j$$

b) Assuming the line is lossless,

$$|\Gamma(l)| \approx 0.447$$

c)

$$\rho = \frac{1 + |\Gamma|}{1 - |\Gamma|} = 2.617$$

d) $\beta = \pi/4 = 0.785$ rad/m
Note: since the question asked for the propagation constant, we will also accept answers given in terms of 1/m. The units for the phase constant are rad/m.

e) Since the length 20 cm corresponds to a phase of $5\pi$, the normalised load impedance is $\bar{z}_L = 2 + j1$

2. (4 points) FR-4 is a common material that is used for making printed circuit boards (PCBs). An inductance of 0.025 $\mu H$ is to be realised at 6 GHz using an open circuited section of a transmission line on the FR-4 PCB. The wavelength on the line is 2.7 cm. Assume $z_0$ is 150 Ω and that the line is lossless.

(a) What is the reactance that needs to be realised?

(2 marks)

(b) What is the shortest length of the line that can achieve this inductance if the characteristic impedance is 150Ω?

(2 marks)

**Solution:** a) The reactance is $X = \omega L$
$= 2\pi \times 6 \times 10^9 \times 0.025 \times 10^{-6} = 942.5\ \Omega$

b) $\beta = \frac{2\pi}{\lambda} = 232.71$ rad /m

$$l_{oc} = \frac{1}{\beta}\cot^{-1}\left(\frac{-X}{z_0}\right)$$

$$l_{oc} = \frac{1}{\beta}\cot^{-1}\left(\frac{-942.5}{150}\right)$$
$$= 1.282\ cm$$

3. (12 points) You are given a lossless transmission line with a characteristic impedance of $z_0 = 50\Omega$, and it is terminated with a purely resistive load of $z_L = 150\ \Omega$. To achieve impedance matching with the 50 Ω, you decide to add a short-circuited stub in parallel to the load. If you can only add the stub at a distance $\lambda/3$ from the load, answer the following questions, taking note of the fact that lengths may have to be given in terms of the wavelength $\lambda$.

(a) What is the reflection coefficient between the line and the load?

(1 mark)

(b) What is the admittance at the point where the stub is going to be added? Remember, the input admittance is seen at the source side looking towards the load.

(1 mark)

(c) For impedance matching, what should the total admittance at the point where the stub is added?

(2 marks)

(d) If the characteristic impedance of the stub is $z_{stub} = 50\ \Omega$, what is the shortest length of the stub, $l_{stub}$ to achieve impedance matching? Make sure the value you mention is practical.

(1 mark)

(e) If you needed a slighlty longer line, what is the next length that could be used?

(2 marks)

(f) If we wish to deliver 10 mW on average to the load, what is the amplitude of the injected voltage wave ($V^+$) before the stub was added?

(1 mark)

(g) What is the amount of power that is reflected by the load in this case (without stub)?

(1 mark)

(h) If we wish to deliver 10 mW on average to the load, what is the amplitude of the injected voltage wave ($V^+$) after the stub is added, assuming perfect matching was achieved?

(1 mark)

(i) What is the amount of power reflected by the load in this case (with stub)?

(2 marks)

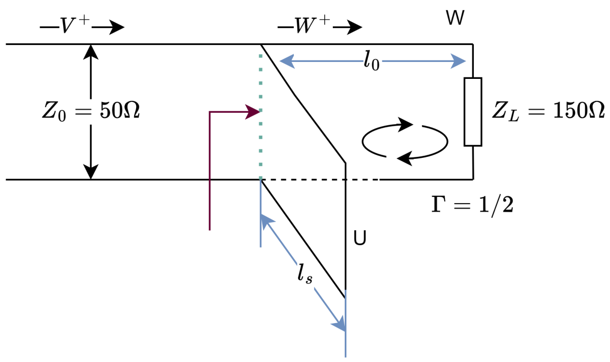


**Figure 1:** Transmission line with single stub

**Solution:** a)

$$\Gamma = \frac{z_2 - z_1}{z_2 + z_1} = \frac{150 - 50}{150 + 50} = 0.5$$

b) $\tan \beta l = -1.7321$
Input impedance at distance l from load is

$$z(l) = z_0 \frac{z_L + jz_0 * tan(\beta l)}{z_0 + jz_L * tan(\beta l)} = 21.4286 + j24.7436\ \Omega$$

Admittance is $Y_l = 0.0200 - j0.0231\ \Omega^{-1}$

c) For impedance matching, the total admittance at the point where the stub is added should equal one divided by the characteristic impedance $\frac{1}{z_0} = 1/50 = \Omega$. That is

$$Y(l) + Y_{stub} = \frac{1}{50} = 0.02\ \Omega^{-1}$$

d) $z_{stub} \times \tan(\beta l_{stub}) = 1/Im(Y(l)) = \frac{-1}{0.0231}$

$l_{stub} = 0.3864\lambda$ (marks given whether unit 'm' is given or not for parts d and e)

e) Add $\lambda/2$, so final length is $0.8864\lambda$ m

f)

$$P_L = \frac{|V^+|^2(1 - |\Gamma|^2)}{2z_0}$$

$$|V^+| = 1.1547V$$

g) Reflected power

$$P_{refl} = \frac{|V^+|^2|\Gamma|^2}{2z_0} = 3.33\ mW$$

h) Refer to figure 1 for the notation used in this explanation. The line between the stub and the load is referred to as W and the stub itself as U. The stub is kept a distance $l_0$ from the load. Assuming perfect matching, the reflection coefficient at the location of the stub is $|\Gamma_{l_0}| =$

0. Since the line is lossless, if we have $P_{at\, l_0} = 10$ mW at the location of the stub, all of that power will be transferred to the load.

$$P_{at\, l_0} = \frac{|V^+|^2(1 - |\Gamma_{l_0}|^2)}{2z_0}$$

Substituting numbers from the problem, we can calculate $|V^+|$

$$10mW = \frac{|V^+|^2(1-0)}{2(50)}.$$

Therefore, the amplitude of the injected wave is

$$|V^+| = 1\ V$$

Note: $|V^+|$ is the magnitude of the wave on the section of the line between the source and the location of the stub.

i) To calculate the reflected power at the load, we use the following equation:

$$P_L = \frac{|W^+|^2(1 - |\Gamma_L|^2)}{2z_0}$$

Important: $|W^+|$ is the magnitude of the wave on the section of the line between the stub and load, labeled W in the figure. The reflection at the load has not changed (as the difference between the impedances at this location is the same as before). So $|\Gamma_L| = 0.5$.

Substituting relevant numbers into this equation and keeping mind that $|\Gamma_L| = 0.5$

$$10mW = \frac{|W^+|^2(1 - |0.5|^2)}{2(50)}$$

$$|W^+| = 1.154\ V$$

And the reflected power at the load is

$$P_{refl} = \frac{|W^+|^2|\Gamma_L|^2}{2z_0}$$

which is also 3.33 mW.

If the question had asked, how much power returns to the source, the answer would have been 0. The presence of the stub is ensuring that the input impedance at the location of the stub matches that of the transmission line, so there is no reflected power (heading to the source from the location of the stub). There is a standing wave in the region between the stub and load, and also in the stub line itself. This is shown in Figure 1. Refer to the discussion at the end of the document for a better understanding of what is happening.

4. (4 points) A 30 m length of a transmission line of characteristic impedance 150 Ω is connected to a 10 m length of line of characteristic impedance 75 Ω that is terminated in a 75 Ω load. The line is to be used to with a signal of frequency 20 MHz. Assume that the velocity on the line is $2 \times 10^8$ $m/s$, and that all the transmission lines are lossless.

(a) Find the VSWR on the 150 Ω line.

(1 mark)

(b) In order to match the 30 m and 10 m lines, it is decided to use a quarter-wave transformer (QWT). What should the characteristic impedance $z_0$ of the QWT be?

(2 marks)

(c) What is the length of the QWT?

(1 mark)

**Solution:** a) Calculate the reflection coefficient $\Gamma$ at the junction between the 150 $\Omega$ and 75 $\Omega$ transmission lines:

$$\Gamma = \frac{z_2 - z_1}{z_2 + z_1} = \frac{75 - 150}{75 + 150} = -1/3$$

$$\rho = \frac{1 + |\Gamma|}{1 - |\Gamma|} = 2$$

b)

$$z_0 = \sqrt{z_1 z_2} = \sqrt{150 \times 75} \approx 106.066\Omega$$

c)

$$\lambda = \frac{v}{f} = \frac{2 \times 10^8}{20 \times 10^6} = 10m$$

Length of line = $\frac{\lambda}{4}$ = 2.5 m

**Discussion regarding solution to questions 3(h) and (i)**

Situation: Lossless TL with $Z_0$ and given a real load $Z_L$ at the end (i.e. $\Gamma_L \in \mathbb{R}$).

Scenario 1: No stub

Phasors are: $V(l) = V^+ e^{j\beta l}(1 + \Gamma_L e^{-j2\beta l})$ where $\Gamma_L = V^-/V^+$, and $I(l) = \frac{V^+}{Z_0} e^{j\beta l}(1 - \Gamma_L e^{-j2\beta l})$. The power delivered to the load is

$$P_1 = \frac{1}{2} Re(V(0)I(0)^*) = \frac{|V^+|^2}{2Z_0}(1 - \Gamma_L^2). \tag{1}$$

Scenario 2: With stub

Once we add a stub, there will be no reflected power going back to the source. However, there will be standing waves on both the load line and the stub line. Let's analyze this in further detail.

Imagine standing at some point on the load line (i.e. between the load and the stub junction on the main TL). There will be a standing wave and the amplitude of the forward component will in general be different from the original $V^+$. Let's call this forward component as $W^+$ and the resulting voltage phasor on this part of the line is:

$$V_w(l) = W^+ e^{j\beta l}(1 + \Gamma_L e^{-j2\beta l}). \tag{2}$$

By a similar logic, the voltage phasor on the stub line will have the following form:

$$V_u(l) = U^+ e^{j\beta l_s}(1 + \Gamma_s e^{-j2\beta l}) \tag{3}$$

where $\Gamma_s$ is the reflection coefficient for a short circuited stub ($\Gamma_s = -1$).

From these we can get the current phasors and then compute complex power as per the usual formula: $V\,I^* = \frac{|V^+|^2}{2Z_0}[(1 - |\Gamma_L|^2) + j2Im(\Gamma_L e^{-j2\beta l})]$ (the general expression derived in class).

The complex power at $l = l_0^-$ must be the same at $l = l_0^+$ (where can it go?). At $l = l_0^+$ we have a matched load condition so that there is no reflected power:

$$P^+ = \frac{|V^+|^2}{2Z_0}. \tag{4}$$

At $l = l_0^-$ power is getting split two ways (load and stub sections), so they must be added to get:

$$P^- = \frac{|W^+|^2}{2Z_0}[(1 - |\Gamma_L|^2) + j2\,Im(\Gamma_L e^{-j2\beta l_0})] + \frac{|U^+|^2}{2Z_0}[j2\,Im((-1)e^{-j2\beta l_s})]. \tag{5}$$

Since $P^- = P^+$, we get the following by matching real and imag parts:

$$\frac{|V^+|^2}{2Z_0} = \frac{|W^+|^2}{2Z_0}(1 - |\Gamma_L|^2), \quad \text{and} \quad \frac{|W^+|^2}{2Z_0} Im(\Gamma_L e^{-j2\beta l_0}) = \frac{|U^+|^2}{2Z_0} Im(e^{-j2\beta l_s}). \tag{6}$$

Simplifying, this gives us (with $\Gamma_L \in \mathbb{R}$):

$$|W^+| = \frac{|V^+|}{\sqrt{1 - |\Gamma_L|^2}}, \quad \text{and } |U^+| = |W^+|\Gamma_L \frac{\sin(\beta l_0)}{\sin(\beta l_s)}. \tag{7}$$

So, the entire power from the source is delivered to the load (via an increased voltage, $|W^+| > |V^+|$), even though it still reflects power. That reflected power is redistributed as reactive power in the load and stub line.

An alternate way of solving this problem is to equate the values of phasor voltages on either side of the junction. This is because we have a parallel combination at the junction, and so the voltages must be the same on each side. This gives us:

$$V^+ e^{j\beta l_0} = W^+ e^{j\beta l_0}(1 + \Gamma_L e^{-j2\beta l_0}) = U^+ e^{j\beta l_s}(1 - e^{-j2\beta l_s}). \tag{8}$$

This gives us $W^+, U^+$ directly in terms of $V+$. Naturally, both approaches give the same answer. The advantage here is that you also get the phase of $W^+$, whereas earlier you only got $|W^+|$.