EE3110 - Probability Foundations for Electrical Engineers
Tutorial - Week 3

Please submit solutions to the 2 starred questions in moodle for assignment submission by **Sept 9, 11:59 PM**.

---

1. Let X be a random variable with law $f(x) = C\frac{2^x}{x!}$ for $x \in \mathbb{N}$. Then, the probability that X is even is approximately ______.

2. Show that for any continuous random variable
$$P[X = x] = 0$$
for any point x in its range.

3. Let $F :\to [0, 1]$ be defined as follows. For some $-\infty < a < \infty$,
$$\begin{aligned} F(x) &= 0 \;\text{ for } x \leq a \\ &= 0.5(x - a) \text{ for } a < x \leq a + 2 \\ &= 1 \;\text{ for } x > a + 2. \end{aligned}$$
Is it possible to have a discrete random variable with $F(\cdot)$ as its CDF? Justify your answer.

4. Suppose a small aircraft arrives at a certain airport according to a Poisson process with rate λ =8 per hour.
    1. What is the probability that **exactly** 6 small aircrafts arrive during a 1-hour period?
    2. What is the probability that **atmost** 5 small aircrafts arrive during a 2-hour period?
    3. What is the probability that **atleast** 6 small aircrafts arrive during a 2-hour period?

5. Let $\lambda > 0, k \geq 1, n \geq \lambda$ and $p = \frac{\lambda}{n}$. Let $A_k = \{k \text{ successes in } n \text{ Bernoulli}(p) \text{ trials}\}$. Show that:
$$\lim_{n\to\infty} P(A_k) = \frac{e^{-\lambda}\ \lambda^k}{k!}.$$

6. Suppose a die is rolled twice. What are the possible values that the following random variables can take?
    1. The maximum value to appear in the two rolls
    2. The minimum value to appear in the two rolls
    3. The sum of two rolls
    4. The value of the first roll minus the value of the second roll

7. Suppose a coin having probability 0.7 of coming up heads is tossed three times. Let X denote the number of heads that appear in the three tosses. Determine the probability mass function of X.

8.* **Happy Birthday again!** You go to a party with 500 guests. What is the probability that exactly one other guest has the same birthday as you? Calculate this exactly and also approximately by using the Poisson PMF. (For simplicity, exclude birthdays on February 29.)

9. **Form of the Poisson PMF** Let X be a Poisson random variable with parameter λ. Show that the PMF $p_X(k)$ increases monotonically with k up to the point where k reaches the largest integer not exceeding λ, and after that point decreases monotonically with k.

10. *(Buffon's Needle)* If a short needle of length l is dropped on a paper that is ruled with equally spaced lines of distance $d \geq l$, find the probability that the needle comes to lie in a position where it crosses one of the lines.

11. A family has 5 natural children and has adopted 2 girls. Each natural child has equal probability of being a girl or a boy, independent of the other children. Find the PMF of the number of girls out of the 7 children.

12. **The matchbox problem - inspired by Banach's smoking habits.** A smoker mathematician carries one matchbox in his right pocket and one in his left pocket. Each time he wants to light a cigarette, he selects a matchbox from either pocket with probability $p = 1/2$, independent of earlier selections. The two matchboxes have initially n matches each. What is the PMF of the number of remaining matches at the moment when the mathematician reaches for a match and discovers that the corresponding matchbox is empty? How can we generalize to the case where the probabilities of a left and a right pocket selection are p and $1 - p$, respectively?

13. Recursive computation of the binomial PMF. Let X be a binomial random variable with parameters n and p. Show that its PMF can be computed by starting with $p_X(0) = (1-p)^n$. and then using the recursive formula

$$p_X(k+1) = \frac{p}{1-p} \cdot \frac{n-k}{k+1} \cdot p_X(k), \quad k = 0, 1 \ldots, n-1$$

14. A computer transmits 3 digital messages of 12 million bits of information each. Each bit has a probability of one one-billionth that it will be incorrectly received, independent of all other bits. What is the probability that at least 2 of 3 messages will be received error free?

15.* **Memorylessness** Let X be a discrete random variable following the Geometric PMF with parameter p. Show the following:

$$\mathbb{P}(X = n + k | X > n) = \mathbb{P}(X = k). \tag{1}$$

This means that given you have already waited for n trials for a success the probability that you will wait for k more trials is same as the probability of waiting for k trials from the first trial, i.e., the random variable doesn't remember the past.

16. Consider a probability space with sample space $\Omega = \{(i, j) : i, j \in 0, 1, 2, \ldots\}$ and probability function $\mathbb{P}(\{(i, j)\}) = C\frac{\lambda^i \mu^j}{i!j!}$.

(a) Find C.

(b) Define $X : \Omega \to \mathbb{R}$ as $X(\{(i, j)\}) = i + j$. Find the range of X and the p.m.f of X.

17. Let X be a random day of the week, coded so that Monday is 1, Tuesday is 2, etc. (so X takes values 1, 2,..., 7, with equal probabilities). Let Y be the next day after X (again represented as an integer between 1 and 7). Do X and Y have the same distribution? What is $P(X < Y)$?

18. Find an example of two discrete random variables X and Y (on the same sample space) such that X and Y have the same distribution (i.e., same PMF and same CDF), but the event X = Y never occurs.