EE3110 - Probability Foundations for Electrical Engineers
Tutorial - Week 1

Please submit solutions to the 2 starred questions in moodle for assignment submission by **Aug 11, 11:59 PM**.

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1. Let A and B be two sets.
   (a) Show that
   $$A^c = (A^c \cap B) \cup (A^c \cap B^c)$$
   $$B^c = (A \cap B^c) \cup (A^c \cap B^c)$$
   (b) Show that
   $$(A \cap B)^c = (A^c \cap B) \cup (A \cap B^c) \cup (A^c \cap B^c)$$
   (c) Consider rolling a fair six-sided die. Let A be the set of outcomes where the roll is an odd number. Let B be the set of outcomes where the roll is less than 4. Calculate the sets on both sides of the equality in part (b), and verify that the equality holds.

2. Out of the students in a class, 60 % are geniuses, 70% love chocolate, and 40% fall into both categories. Determine the probability that a randomly selected student is neither a genius nor a chocolate lover.

3. A partition of the sample space Ω is a collection of disjoint events $S_1, S_2, ..., S_n$ such that $\Omega = \cup_{i=1}^{n} S_i$.
   (a) Show that for any event A,
   $$\boldsymbol{P}(A) = \sum_{i=1}^{n} \boldsymbol{P}(A \cap S_i)$$
   (b) Use part(a) to show that for any events A, B, C, we get
   $$\boldsymbol{P}(A) = \boldsymbol{P}(A \cap B) + \boldsymbol{P}(A \cap C) + \boldsymbol{P}(A \cap B^c \cap C^c) - \boldsymbol{P}(A \cap B \cap C)$$

4. Let A and B be two events such that $\boldsymbol{P}(A) > \boldsymbol{P}(B) > 0$ and $\boldsymbol{P}(A) + \boldsymbol{P}(B) \geq 1$.
   (a) Determine upper and lower bounds on $\boldsymbol{P}(A \cap B)$ in terms of $\boldsymbol{P}(A), \boldsymbol{P}(B)$.
   (b) When $\boldsymbol{P}(A) \geq 1 - \delta$ and $\boldsymbol{P}(B) \geq 1 - \delta$, express the lower bound on $\boldsymbol{P}(A \cap B)$ in terms of δ.

5. A six-sided die is loaded in a way that each even face is twice as likely as each odd face. All even faces are equally likely, as are all odd faces. Construct a probabilistic model for a single roll of this die and find the probability that the outcome is less than 4.

6.* Let A and B be events with probabilities $\mathbb{P}(A) = \frac{3}{4}$ and $\mathbb{P}(B) = \frac{1}{3}$. Show that $\frac{1}{12} \leq \mathbb{P}(A \cap B) \leq \frac{1}{3}$. Find the corresponding bounds for $\mathbb{P}(A \cup B)$.

7. Prove the identity

$$A \cup (\cap_{i\in I} B_i) = \cap_{i\in I}(A \cup B_i)$$

8. (a) Prove that for any two events A and B, we have

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1.$$

   (b) Generalize to the case of n events $A_1, A_2, \cdots, A_n$ by showing that

$$\mathbb{P}\left(A_1 \cap A_2 \cap \cdots A_n\right) \geq \mathbb{P}(A_1) + \mathbb{P}(A_2) + \cdots + \mathbb{P}(A_n) - (n-1).$$

9.* Suppose that a box contains r red balls, w white balls, and b blue balls. Suppose also that balls are drawn from the box one at a time, at random, without replacement. What is the probability that all r red balls will be obtained before any white balls are obtained?

10. If n letters are placed at random in n envelopes, what is the probability that exactly $n-1$ letters will be placed in the correct envelopes?

11. What is the smallest number k such that in a group of k people, there is a greater than 50% chance that at least two people share the same birthday? (Assume there are exactly 365 days in a year and each day is equally likely to be someone's birthday. Use the approximation $ln(1-x) \approx -x$ for small x)

12. Sixteen players, $P_1, P_2, \ldots, P_{16}$, compete in a knockout tournament. It is known that whenever players P_i and P_j face each other, P_i will win if $i < j$. Assuming the players are paired randomly in each round, what is the probability that player P_4 reaches the semi-finals?

13. An elevator in a building starts with five passengers and stops at seven floors. If every passenger is equally likely to get off at each floor and all the passengers leave independently of each other, what is the probability that no two passengers will get off at the same floor?

14. Each of k jars contains m white and n black balls. A ball is randomly chosen from jar 1 and transferred to jar 2, then a ball is randomly chosen from jar 2 and transferred to jar 3, etc. Finally, a ball is randomly chosen from jar k. Show that the probability that the last ball is white is the same as the probability that the first ball is white, i.e., it is $m/(m+n)$.

15. For $X, Y \subset \mathbb{R}$, let $f : X \to Y$ be a function such that, for every pair of disjoint subsets $A, B \subset X$, $f(A) \cap f(B) = \phi$. Is f an injective function? Justify.

16. Let $f : X \to Y$ be an arbitrary function and $X, Y \subset \mathbb{R}$. Then, for any $A, B \subset X$, is it true that $f(A) \cap f(B) \subset f(A \cap B)$? Justify.

17. The events A, B, and C satisfy: $P(A|B \cap C) = 1/4$, $P(B|C) = 1/3$, and P(C) = 1/2. Calculate $P(A^c \cap B \cap C)$.

18. An experiment has only two outcomes. The first has probability p to occur, the second probability p^2. What is p?