# EE3110 - Probability Foundations for Electrical Engineers
# Tutorial - Week 6

Please submit solutions to the 2 starred questions in moodle for assignment submission by **Nov 8, 11:59 PM**.

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1. **2D transformation of a bivariate Gaussian RV:** Let the joint PDF of two Gaussian random variables $X, Y$ be given as

$$f_{XY}(x,y) = \frac{1}{2\pi(1-\rho^2)} \exp\left[-\left(\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right)\right].$$

   Compute the joint PDF $f_{VW}(v,w)$, where

$$V = \frac{1}{2}(X^2 + Y^2),$$

$$W = \frac{1}{2}(X^2 - Y^2).$$

2. Let $X_1, X_2$ be drawn i.i.d. from the following density

$$f_X(x) = \sqrt{\frac{2}{\pi}} e^{-x^2/2},\ x \geq 0.$$

   Let $W$ be independent of $X_1$ and $X_2$ such that $\mathbb{P}(W = 1) = \mathbb{P}(W = -1) = 1/2$. Let $Y_1 = WX_1$ and $Y_2 = WX_2$.

   1. Find the probability density function of $Y_1$ and sketch it.
   2. Find the joint probability density function of $Y_1$ and $Y_2$. Sketch the region of support.
   3. Find the covariance of $Y_1$ and $Y_2$.
   4. Are $(Y_1, Y_2)$ jointly Gaussian? Are $Y_1$ and $Y_2$ marginally Gaussian? Explain.

3. Let $X$ and $Y$ be i.i.d normal with mean 0 and variance 1. Let $V = 3X + 5Y$ and $W = X + 2Y$. Find the joint pdf of $V$ and $W$.

4. Suppose $X_1, \ldots, X_n \sim$ iid (continuous) Uniform$[0,1]$. Let

$$Y_n = e^{X_1} + \cdots + e^{X_n}.$$

   1. What is the range of values taken by $Y_n$?
   2. Find $E[Y_n]$ and $\mathrm{Var}(Y_n)$.
   3. Approximate $P(|Y_n - E[Y_n]| > \frac{n}{2})$ using Chebyshev's inequality.
   4. Approximate $P(Y_n > 2n)$ using the CLT.

5. Suppose that $X_1, \ldots, X_n$ form a random sample from a normal distribution with unknown mean $\theta$ and variance $\sigma^2$. Assuming that $\theta \neq 0$, determine the asymptotic distribution of $\bar{X}_n^3$.

6. Suppose that $X_1, X_2, \ldots$ are i.i.d. random variables, each of which has m.g.f. $\psi(t)$. Let $Y = X_1 + \cdots + X_N$, where the number of terms $N$ in this sum is a random variable having the Poisson distribution with mean $\lambda$. Assume that $N$ and $X_1, X_2, \ldots$ are independent, and $Y = 0$ if $N = 0$. Determine the m.g.f. of $Y$.

7. Let $\boldsymbol{X}$ be a random variable with multivariate normal distribution $\sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu}$ is the $n$-dimensional mean vector, and $\boldsymbol{\Sigma}$ is the $n$ x $n$ covariance matrix.

   (a) Prove that the random variable $\boldsymbol{AX}$ follows the distribution $\sim N_m(\boldsymbol{A\mu}, \boldsymbol{A\Sigma A^T})$, where $\boldsymbol{A}$ is a $m$ x $n$ matrix.

   (b) For $n = 3$ with $\boldsymbol{\mu^T} = (2, -3, 1)$ and $\boldsymbol{\Sigma} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{pmatrix}$, find the constraint in the 2-dimensional vector $\boldsymbol{a}$ such that $X_2$ and $X_2 - \boldsymbol{a^T} \begin{pmatrix} X_1 \\ X_3 \end{pmatrix}$ are independent.

8. An accountant wants to simplify his bookkeeping by rounding amounts to the nearest integer, for example, rounding 99.53 units and 100.46 units both to 100 units. What is the cumulative effect of this if there are, say, 100 amounts? To study this we model the rounding errors by 100 independent $U(-0.5, 0.5)$ random variables $X_1, X_2, ..., X_{100}$.

   (a) Compute the expectation and the variance of the $X_i$.

   (b) Use Chebyshev's inequality to compute an upper bound for the probability $\mathrm{P}(|X_1 + X_2 + .... + X_{100}| > 10)$ that the cumulative rounding error $X_1, X_2, ..., X_{100}$ exceeds 100 units.

9. Suppose that a two-dimensional random vector $\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ follows a multivariate Gaussian distribution with mean $\boldsymbol{\mu} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and covariance matrix

$$\Sigma = \begin{pmatrix} 4 & 1 \\ 1 & 9 \end{pmatrix}.$$

   1. Find the marginal distribution of $X_1$.
   2. Determine the conditional distribution of $X_2$ given that $X_1 = 0$.

10.* Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed random variables with mean $\mu = 10$ and variance $\sigma^2 = 25$. Suppose $n = 100$. Use the Central Limit Theorem to approximate the probability $P(\bar{X} > 11)$, where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

11. Show that if $(X_1, X_2, X_3)$ is Multivariate Normal, then so is the sub-vector $(X_1, X_2)$.

12. Is it true that if $X = (X_1, ..., X_n)$ and $Y = (Y_1, ..., Y_m)$ are Multivariate Normals with X independent of Y, then the "concatenated" random vector W $= (X_1, ..., X_n, Y_1, ..., Y_m)$ is Multivariate Normal?

13. Consider a random vector $\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$ that follows a trivariate normal (Gaussian) distribution with mean vector $\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$ and covariance matrix

$$\boldsymbol{\Sigma} = \begin{bmatrix} 5 & -1 & 2 \\ -1 & 3 & 1 \\ 2 & 1 & 4 \end{bmatrix}.$$

1. Calculate the conditional mean and variance of $Y_1$ given $Y_2 = 2$.
2. Determine if $Y_1$ and $Y_3$ are uncorrelated.

14. Let $X_1, X_2, \ldots, X_n$ be a sequence of i.i.d. random variables with mean $\mu = 10$ and variance $\sigma^2 = 25$. Define $S_n = \sum_{i=1}^{n} X_i$ and $\overline{X}_n = \frac{S_n}{n}$.

1. According to the Central Limit Theorem, what is the distribution of $\overline{X}_n$ as $n \to \infty$?
2. For $n = 100$, calculate the approximate probability $P(9.5 \leq \overline{X}_n \leq 10.5)$.

15.* The weekly output of a factory is a random variable with mean 50. Then, the probability that a week's output will exceed 75 is at most _____.

16. The total number of students attending class on any given week is a random variable with mean 50. If the variance in a week's attendance is known to be 25, then the probability that the attendance in any week will be between 40 and 60 is at least _____ ?