EE2025: Engineering Electromagnetics

Tutorial 2: Maxwell's equations, plane wave propagation, and Poynting theorem

July-Nov 2024

Maxwell's Equations and Boundary Conditions

1. Show that charge density ρ_v satisfies,

$$\frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\epsilon}\rho_v = 0$$

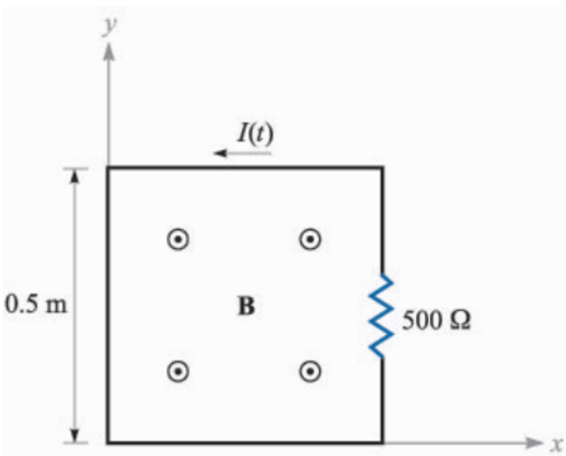
   Is this true for all media?

2. What values of A and β are required if the two fields given below satisfy Maxwell's equations in a linear, isotropic, homogeneous medium with $\epsilon_r = \mu_r = 4$ and $\sigma = 0$?

$$\mathbf{E} = 120\pi cos(10^6\pi t - \beta x)\hat{a_y} V/m$$
$$\mathbf{H} = A\pi cos(10^6\pi t - \beta x)\hat{a_z} A/m$$

   Assume there are no current or charge densities in space.

3. A perfectly conducting filament containing a small 500Ω resistor is formed into a square, as illustrated by the figure below. Find $I(t)$ if (a) $\mathbf{B} = 0.3\cos(120\pi t$ - $30°)\mathbf{a}_z$ T; (b) $\mathbf{B} = 0.4\cos[\pi(ct - y)]\mathbf{a}_z$ μT, where $c = 3 \times 10^8$ m/s.



4. Find out $\mathbf{B}$ if the electric field is given as $\mathbf{E}(x, y, z, t) = 0.2\sin(10\pi y)\cos(6\pi 10^9 t - \beta z)\hat{\mathbf{x}}$ in vacuum.

Polarization

5. Describe the nature of polarisation of the following electric fields (linear, circular or elliptical).

   (a) $E = E_0\cos(\omega t - \beta z)\hat{x} - E_0\cos(\omega t - \beta z)\hat{y}$

   (b) $E = E_0\sin(\omega t - \beta z)\hat{x} + E_0\cos(\omega t - \beta z + \frac{\pi}{4})\hat{y}$

   (c) $E = \mathfrak{Re}(E_0 \exp^{j(\omega t - \beta z)}\hat{x} - jE_0\exp^{j(\omega t - \beta z)}\hat{y})$

   (d) $E = E_1\cos(\omega t - \beta z)\hat{x} - E_2\cos(\omega t - \beta z + \frac{\pi}{2})\hat{y}$ $\quad (E_1 \neq E_2)$

   What is the polarisation state of sunlight? How do sunglasses help to reduce glare?

Wave propagation in conducting and dielectric medium

6. a) Show that the skin depth in a poor conductor ($\sigma << \omega\epsilon$) is $\frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}$ (independent of frequency). Find the skin depth (in meters) for pure water (use static value of ($\sigma = \frac{1}{2.5\times10^5}$), ($\mu = \mu_0$) and ($\epsilon_r = 80.1$)).

   b) Show that the skin depth in a good conductor ($\sigma >> \omega\epsilon$) is $\frac{\lambda}{2\pi}$ (where λ is the wavelength in the conductor). Find the skin depth (in nanometers) for a typical metal (where $\sigma = 10^7(\Omega m)^{-1}$, for visible range frequency $\omega = 10^{15}$ and assume $\epsilon = \epsilon_0$ and $\mu = \mu_0$).

7. A laptop manufacturer wants to shield her laptop such that the energy from its high frequency clock doesn't radiate to the outside world. Assuming a clock rate of 2.45 GHz, what should be the thickness of the metal cladding on the laptop cover and weight of metal required, given standard laptop dimension of 15" x 11" if she uses

   1. silver ($\sigma = 6.2 \times 10^7$ S/m, density=10.49 g/cm^3)
   2. gold ($\sigma = 4.1 \times 10^7$ S/m, density=19.32 g/cm^3)
   3. copper ($\sigma = 5.8 \times 10^7$ S/m, density=8.96 g/cm^3)

   [Note : Assume the required thickness as the depth where the amplitude of E field attenuates by 99%.]

8. In a dielectric medium, a wave has electric and magnetic fields given as,

$$\mathbf{E} = (j\hat{x} + 2\hat{y} - j\hat{z})\exp\left[-j\pi(x+z)\right]V/m$$

$$\mathbf{H} = \frac{1}{60\pi}(-\hat{x} + j\hat{y} + \hat{z})\exp\left[-j\pi(x+z)\right]A/m$$

   Show that $\mathbf{E}, \mathbf{H}$ and wavevector ($\mathbf{k}$) forms an orthogonal triad. Find

   (a) phase constant of the wave
   (b) velocity of the wave
   (c) frequency of the wave

Poynting Theorem

9. If the electric and magnetic field in a medium are given by $\mathbf{E} = 3\sin(t - 5z)\,\hat{x}$ and $\mathbf{H} = 4\cos(t-5z)\,\hat{y}$, then calculate (at $z = 0$) the
   a) the instantaneous power density,
   b) instantaneous power transmitted through a surface with an area of $5\,m^2$ at $z = 0$ and the normal pointing in $\hat{z}$ direction, and
   c) total energy carried by the wave through the given surface from $t = 0\,s$ to $t = 5\,s$.

10. Show that the instantaneous Poynting Vector of a circularly polarised plane wave propagating in a lossless medium is a constant that does not vary with time and space. Compare the results with that in the case of a linearly polarised wave propagating in a lossless medium.

Optional questions

11. A slab of perfect dielectric material ($\epsilon_r = 2$) is placed in a microwave oven. The oven produces an electric field (as well as a magnetic field). Assume that the electric field intensity is uniform in space and sinusoidal in time and is in the direction perpendicular to the surface of the slab. The microwave oven operates at a frequency of 2.45 GHz and produces an electric field intensity with amplitude 500 V/m inside the dielectric, calculate the displacement current density in the dielectric.

12. a) Show that the ratio of the amplitudes of the conduction current density and the displacement current density is $\sigma/\omega\epsilon$ for the applied field $\mathbf{E} = \mathbf{E}_m cos\omega t$. Assume $\mu = \mu_0$.

    b) Assuming that sea water has $\mu = \mu_0$, $\epsilon = 81\epsilon_0$, $\sigma = 20$ S/m, determine the frequency at which the conduction current density is ten times the displacement current density in magnitude.

    c) A parallel-plate capacitor with plate area of 5 cm^2 and plate separation of 3 mm has a voltage 50 $\sin(10^3 t)$ V applied to its plates. Calculate the displacement current assuming $\epsilon = 2\epsilon_o$.

13. Consider a point on the surface of a perfect conductor. The electric field intensity at that point is $\mathbf{E} = (500\hat{x} - 300\hat{y} + 600\hat{z})\cos 10^7 t$ and medium surrounding the conductor is characterized by $\mu_r = 5$ and $\epsilon_r = 10$ and $\sigma = 0$.
    (a) Find a unit vector normal to the conductor at that point of the conductor surface.
    (b) Find the instantaneous surface charge density at the point.

14. a) Show that the ratio of the amplitudes of the conduction current density and the displacement current density is $\sigma/\omega\epsilon$ for the applied field $\mathbf{E} = \mathbf{E}_m cos\omega t$. Assume $\mu = \mu_0$.

    b) Assuming that sea water has $\mu = \mu_0$, $\epsilon = 81\epsilon_0$, $\sigma = 20$ S/m, determine the frequency at which the conduction current density is ten times the displacement current density in magnitude.

15. In a lossless and nonmagnetic medium, the E field of EM wave emitted by a cell phone is given as

    $$\mathbf{E}(t) = 4\sin(2\pi \times 10^7 t - 0.8x)\hat{z} \ \ mV/m$$

    Find

    1. Relative permittivity (ϵ_r) and intrinsic impedance (η) of the medium.
    2. The time averaged Poynting vector associated with the wave
    3. Total power received through a rectangular area of a window of 1 m and 1.5 m, in the $x = 0$ plane
    4. Total power received through the same window if it is placed in the plane $2x + y = 5$

16. **Snell's law:** A light beam is incident from air to a medium with a dielectric constant 4 and relative permeability 100 . If the angle of incidence is 60°. Find the angle of reflection and angle of refraction.

17. Suppose, we have electromagnetic waves propagating in z-direction and electric field components in xy plane are given as:

    $$E_x = E_{0x}cos(\omega t), \text{ and } E_y = E_{0y}cos(\omega t + \phi)$$

The axial ratio (AR) is defined as the ratio of major axis and minor axis. The AR in terms of E_{0x}, E_{0y} and ϕ is expressed as:

$$\mathrm{AR} = \sqrt{\frac{E_{0x}^2 + E_{0y}^2 + \sqrt{E_{0x}^4 + E_{0y}^4 + 2E_{0x}^2 E_{0y}^2 \cos(2\phi)}}{E_{0x}^2 + E_{0y}^2 - \sqrt{E_{0x}^4 + E_{0y}^4 + 2E_{0x}^2 E_{0y}^2 \cos(2\phi)}}}$$

Derive the above expression and write what should be AR ideally for circularly polarized, elliptically polarized and linearly polarized waves.

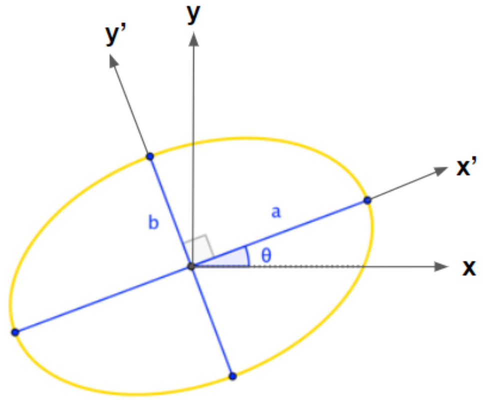


Figure 1: Tilted ellipse

18. Suppose, for practical purposes, assume that AR can be tolerated upto 3 dB for circularly polarized wave, AR above 40 dB is a linearly polarized wave and anything in between 3 dB and 40 dB is an elliptically polarized wave. Using given assumption, determine the polarization (mention right or left handed in case of circular and elliptical) and AR for the following:

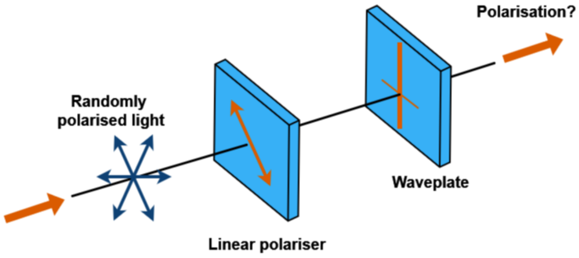
    (a) $E = 10\cos(\omega t - \beta z)\hat{x} + 11\cos(\omega t - \beta z + 80°)\hat{y}$
    (b) $E = 20\sin(\omega t - \beta z)\hat{x} + 1\cos(\omega t - \beta z + 10°)\hat{y}$
    (c) $E = 3\cos(\omega t - \beta z)\hat{x} + 1\cos(\omega t - \beta z + 60°)\hat{y}$
    (d) $E = 10\cos(\omega t - \beta z)\hat{x} + 9\cos(\omega t - \beta z - 75°)\hat{y}$
    (e) $E = 2\cos(\omega t - \beta z + 20°)\hat{x} + 2\cos(\omega t - \beta z + 100°)\hat{y}$

    Note : AR (dB) $= 20\log_{10}(\mathrm{AR})$

19. ***Manipulating polarisation states using devices*** **:** A randomly polarized light propagating in the z direction traverses through a linear polariser and a waveplate as shown in the figure.

    The polarizer's transmission axis (only the component of light aligned with this axis passes through) is oriented at a 45^o angle to the x-axis. The waveplate is made up of a special material that exhibits a low refractive index (n_1) for the y-polarised light and a high refractive index (n_2) for the x-polarised light. If the thickness of the optical device is

$$d = \frac{\lambda}{4(n_2 - n_1)}$$



What is the polarisation state of the light after the second element? (λ is the wavelength of the light)

20. ***Application in 3D movies* :** In the previous question, if the light after the waveplate again passes through another waveplate and then through a linear polariser (both identical as in the previous case) as shown below:

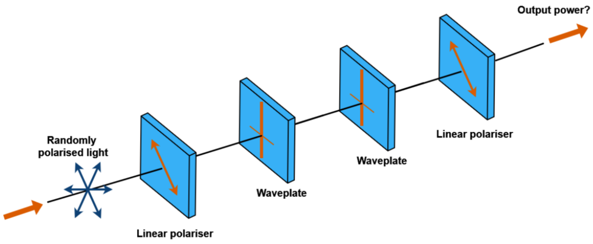


Figure 2: Polarizers and waveplates

How much amount of light will pass through the final polariser? Can you connect the concepts discussed above with polarized 3D technology?