EE3110 - Probability Foundations for Electrical Engineers
Supplementary Tutorial solutions - Week 4

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1. Solution. (a) The joint PDF of X and Y is

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = \frac{1}{2\pi}e^{-(x^2+y^2)/2}$$

We first find the joint CDF of R and Θ. Fix some $r > 0$ and some $\theta \in [0.2\pi]$. and let A be the set of points (x,y) whose polar coordinates $(\bar{r},\bar{\theta})$ satisfy $0 \leq \bar{r} \leq r$ and $0 \leq \bar{\theta} \leq \theta$; note that the set A is a sector of a circle of radius r. with angle θ. We have

$$F_{R.\Theta}(r,\theta) = \mathbf{P}(R \leq r, \Theta \leq \theta) = \mathbf{P}((X,Y) \in A)$$
$$= \frac{1}{2\pi}\int\int_{(x.y)\in A} e^{-(x^2+y^2)/2}dxdy = \frac{1}{2\pi}\int_0^\theta \int_0^r e^{-\bar{r}^2/2}\bar{r}d\bar{r}d\bar{\theta}$$

where the last equality is obtained by transforming to polar coordinates. We then differentiate, to find that

$$f_{R,\Theta}(r.\theta) = \frac{\partial^2 F_{R,\Theta}}{\partial r \partial \bar{\theta}}(r.\theta) = \frac{r}{2\pi}e^{-r^2/2}. \quad r \geq 0, \theta \in [0, 2\pi]$$

Thus,

$$f_R(r) = \int_0^{2\pi} f_{R,\Theta}(r,\theta)d\theta = re^{-r^2/2}, \quad r \geq 0$$

Furthermore,

$$f_{\Theta|R}(\theta \mid r) = \frac{f_{R,\Theta}(r,\theta)}{f_R(r)} = \frac{1}{2\pi}, \quad \theta \in [0, 2\pi]$$

Since the conditional PDF $f_{\Theta|R}$ of Θ is unaffected by the value of the conditioning variable R. it follows that it is also equal to the unconditional PDF f_Θ. In particular, $f_{R,\Theta}(r,\theta) = f_R(r)f_\Theta(\theta)$, so that R and Θ are independent.

(b) Let $t \geq 0$. We have

$$\mathbf{P}\left(R^2 \geq t\right) = \mathbf{P}(R \geq \sqrt{t}) = \int_{\sqrt{t}}^\infty re^{-r^2/2}dr = \int_{t/2}^\infty e^{-u}du = e^{-t/2}$$

where we have used the change of variables $u = r^2/2$. By differentiating, we obtain

$$f_R(t) = \frac{1}{2}e^{-t/2}, \quad t \geq 0$$

2. First observe that y is an integer

$$
\begin{aligned}
F_Y(y) &= P(\lfloor X \rfloor \leq y) \\
&= \int_0^{y+1} \lambda e^{-\lambda x} dx \\
&= 1 - e^{-\lambda(y+1)} \\
f_Y(y) &= F_Y(y) - F_Y(y-1) = e^{-\lambda y}(1 - e^{-\lambda}) \\
F_R(r) &= P(R \leq r) = \sum_{n \in \mathbb{N}} \int_n^{n+r} \lambda e^{-\lambda x} dx \\
&= (1 - e^{-\lambda r}) \sum_{n \in \mathbb{N}} e^{-\lambda n} = \frac{1 - e^{-\lambda r}}{1 - e^{-\lambda}} \\
f_R(r) &= \frac{\lambda}{1 - e^{-\lambda}} e^{-\lambda x} \\
\lim_{r \to 0} f_R(r) &= e^{-\lambda r}
\end{aligned}
$$

3. Using the independence of the RVs, the CDF of the maximum RV is given as:

$$P(X_{(n)} \leq \tau) = P(X_1 \leq \tau, X_2 \leq \tau, ..., X_n \leq \tau) = (P(X_i \leq \tau))^n$$

$$P(X_i \leq \tau) = \int_0^{\tau} e^{-x} dx = 1 - e^{-\tau}$$

   Hence, we have:

$$P(X_{(n)} \leq \tau) = (1 - e^{-\tau})^n$$

   Putting $\tau = x + log(n)$, we have:

$$P(X_{(n)} - log(n) \leq x) = \left(1 - e^{-(x + log(n))}\right)^n = \left(1 + \frac{-e^{-x}}{n}\right)^n$$

   Clearly, using the limit definition of the exponential, we have:

$$\lim_{n \to \infty} P(X_{(n)} - log(n) \leq x) = \exp(-e^{-x})$$

4. Let X be the waiting time and Y be the number of customers found. For $x < 0$, we have $F_X(x) = 0$. For $x \geq 0$,

$$F_X(x) = P(X \leq x) = \frac{1}{2}(P(X \leq x \mid Y = 0) + \frac{1}{2}P(X \leq x \mid Y = 1))$$

   We have,

$$
\begin{aligned}
P(X \leq x \mid Y = 0) &= 1, \\
P(X \leq x \mid Y = 1) &= 1 - e^{-\lambda x}
\end{aligned}
$$

   Thus,

$$F_X(x) = \frac{1}{2} \cdot (2 - e^{\lambda x}), \quad x \geq 0.$$

5. (a) Let $Y = X_1$. The transformation is $Y = X_1$ and $Z = X_1 - X_2$. The inverse is $x_1 = y$ and $x_2 = y - z$. The Jacobian has absolute value 1. The joint p.d.f. of (Y, Z) is

$$g(y, z) = \exp(-y - (y - z)) = \exp(-2y + z)$$

for $y > 0$ and $z < y$. (b) The marginal p.d.f. of Z is

$$\int_{\max\{0,z\}}^{\infty} \exp(-2y + z)dy = \frac{1}{2}\exp(z)\exp(-2\max\{0, z\}) = \frac{1}{2}\begin{cases}\exp(-z) & \text{if } z \geq 0\\ \exp(z) & \text{if } z < 0\end{cases}$$

The conditional p.d.f. of $Y = X_1$ given $Z = 0$ is the ratio of these two with $z = 0$, namely

$$g_1(x_1 | 0) = 2\exp(-2x_1), \text{ for } x_1 > 0$$

(c) Let $Y = X_1$. The transformation is now $Y = X_1$ and $W = X_1/X_2$. The inverse is $x_1 = y$ and $x_2 = y/w$. The Jacobian is

$$J = \det\begin{pmatrix} 1 & 0 \\ 1/w & -y/w^2 \end{pmatrix} = -\frac{y}{w^2}$$

The joint p.d.f. of (Y, W) is

$$g(y, w) = \exp(-y - y/w)y/w^2 = y\exp(-y(1 + 1/w))/w^2$$

for $y, w > 0$.

6. With the convention that $\sqrt{r^2 - u^2} = 0$ when $r^2 - u^2 < 0$, we have that,

$$F(r, x) = \mathbb{P}[R \leq r, X \leq x] = \frac{2}{\pi}\int_{-r}^{x}\sqrt{r^2 - u^2}\,du \qquad \text{and}$$

$$f(r, x) = \frac{\partial^2 F}{\partial r\,\partial x} = \frac{2r}{\pi\sqrt{r^2 - x^2}}, \quad |x| < r < 1$$

substituting $r = \frac{\sqrt{3}}{\pi}$ and $x = \frac{1}{\pi}$ the result is $2.45 \times \pi^{-1}$.