# EE3110 - Probability Foundations for Electrical Engineers
# Tutorial solutions - Week 2

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1. Let A be the event that the batch will be accepted. Then, $A = A_1 \cap A_2 \cap A_3 \cap A_4$ where $A_i$ is the event that the $i^{th}$ item is not defective. Hence, we have

$$\begin{aligned}\mathbb{P}(A) &= \mathbb{P}(A_1)\mathbb{P}(A_2|A_1)\mathbb{P}(A_3|A_2 \cap A_1)\mathbb{P}(A_4|A_1 \cap A_2 \cap A_3) \\ &= \frac{95}{100} \times \frac{94}{99} \times \frac{93}{98} \times \frac{92}{97} = 0.812.\end{aligned}$$

2. Assignment.

3. Let $A$ be the event that the first toss is a head and $B$ be the event that second toss is a head. Shipra claims that $\mathbb{P}(A \cap B|A) \geq \mathbb{P}(A \cap B|A \cup B)$. Let us analyze the claim:

$$\mathbb{P}(A \cap B|A) = \frac{\mathbb{P}(A \cap B \cap A)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}. \tag{1}$$

$$\mathbb{P}(A \cap B|A \cup B) = \frac{\mathbb{P}((A \cap B) \cap (A \cup B))}{\mathbb{P}(A \cup B)} = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A \cup B)}. \tag{2}$$

Since, $\mathbb{P}(A \cup B) \geq \mathbb{P}(A)$, we have (1) greater than (2) and hence Shipra is correct irrespective of the coin being fair or not.
A generalization of Shipra's reasoning is that if $A'$, $B'$, and $C'$ are events such that $B' \subset C'$ and $A' \cap B' = A' \cap C'$ then the event $A'$ is at least as likely if we know that $B'$ has occurred; than if we know that $C'$ has occurred.

4. (a) Aim: To show B suggests A if and only if A suggests B.

Forward implication: If A suggests B, then B suggests A.
We have:

$$P\left(B|A\right) = \frac{P\left(A|B\right)P(B)}{P(A)} > P(B)$$

Since the probabilities are assumed to be non-zero, the above equation can be re-written as:

$$P\left(A|B\right) > P(A)$$

which is exactly the definition of B suggesting A. Hence, the forward implication is proved. A similar proof can be used for the backward implication.

(b) Aim: To show B suggests A if and only if $B^c$ does not suggest A.

Forward implication: If $B^c$ does not suggest A, then B suggests A.
We have:

$$P\left(A|B^c\right) = \frac{P\left(B^c|A\right)P(A)}{P(B^c)} < P(A)$$

From this equation, we have:

$$P\left(B^c|A\right) = 1 - P\left(B|A\right) < P(B^c) = 1 - P(B)$$

$$P\left(B|A\right) > P(B)$$

Hence, we've shown A suggests B. Using (1), we can tell that B suggests A. Hence, we've proved the forward implication. The same proof can be started from the end and reversed to prove the backward implication.

(c) Given:
P(Treasure is in first place) $= \beta$, P(Treasure is in second place) $= 1 - \beta$.
P(Finding treasure given treasure is present) $= p$

Event A: Not finding the treasure in first place
Event B: Treasure is in the second place

Aim: To show that A suggests B. Event A can occur in two cases:

1. Treasure is not present at the first place (or)
2. Treasure is present but it was not found

Clearly, we can tell that event B is a proper subset of event A (Event that treasure is in the second place means it is not in the first place which is part of the event that treasure is not found in the first place).
It is a proper subset since event A can happen even if event B is not true (First place contains the treasure but it was not found)

Thus, we have $B \subset A$. Hence, $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1 > P(A)$. We've shown that B suggests A. Using (1), we can show A too suggests B. Hence proved.

5. Let $A$ be the event that Alice does not find her paper in drawer $i$. Since the paper is in drawer $i$ with probability $p_i$, and her search is successful with probability $d_i$, the multiplication rule yields $\mathbf{P}(A^c) = p_i d_i$, so that $\mathbf{P}(A) = 1 - p_i d_i$. Let $B$ be the event that the paper is in drawer $j$. If $j \neq i$, then $A \cap B = B, \mathbf{P}(A \cap B) = \mathbf{P}(B)$, and we have

$$\mathbf{P}(B \mid A) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A)} = \frac{\mathbf{P}(B)}{\mathbf{P}(A)} = \frac{p_j}{1 - p_i d_i}$$

Similarly, if $i = j$, we have

$$\mathbf{P}(B \mid A) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A)} = \frac{\mathbf{P}(B)\mathbf{P}(A \mid B)}{\mathbf{P}(A)} = \frac{p_i (1 - d_i)}{1 - p_i d_i}$$

6. Let $A$ be the event that the first $n-1$ tosses produce an even number of heads, and let $E$ be the event that the $n$th toss is a head. We can obtain an even number of heads in $n$ tosses in two distinct ways: 1) there is an even number of heads in the first $n-1$ tosses, and the $n$th toss results in tails: this is the event $A \cap E^c$; 2) there is an odd number of heads in the first $n-1$ tosses, and the $n$th toss results in heads: this is the event $A^c \cap E$. Using also the independence of $A$ and $E$,

$$\begin{aligned} q_n &= \mathbf{P}\left((A \cap E^c) \cup (A^c \cap E)\right) \\ &= \mathbf{P}(A \cap E^c) + \mathbf{P}(A^c \cap E) \\ &= \mathbf{P}(A)\mathbf{P}(E^c) + \mathbf{P}(A^c)\mathbf{P}(E) \\ &= (1-p)q_{n-1} + p(1 - q_{n-1}) \end{aligned}$$

We now use induction. For $n = 0$, we have $q_0 = 1$, which agrees with the given formula for $q_n$. Assume, that the formula holds with $n$ replaced by $n-1$, i.e.,

$$q_{n-1} = \frac{1 + (1-2p)^{n-1}}{2}$$

Using this equation, we have

$$\begin{aligned} q_n &= p(1 - q_{n-1}) + (1-p)q_{n-1} \\ &= p + (1-2p)q_{n-1} \\ &= p + (1-2p)\frac{1 + (1-2p)^{n-1}}{2} \\ &= \frac{1 + (1-2p)^n}{2} \end{aligned}$$

so the given formula holds for all $n$.

7. (a) Let $A$ be the event that doubles are rolled. $\#(\cdot)$ denote the number of elements of any event.

$$\#(A) = 6, \text{ so } \quad \mathbb{P}(A) = 1/6$$

(b) Let $B$ be the event that the roll results in a sum of 4 or less. We need to calculate $\mathbb{P}(A|B)$

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

As we have $\#(B) = 6$ and $\#(A \cap B) = 2$ so

$$\begin{aligned}\mathbb{P}(A|B) &= \frac{2/36}{6/36} \\ &= 1/3\end{aligned}$$

(c) Let C be the event that at least one die roll is a 6.

$$\#(C) = 11, \text{ so } \quad \mathbb{P}(C) = 11/36$$

(d) Let $D$ be the event that the two dice land on different numbers.
$\#(D) = 30$ and $\#(C \cap D) = 10$ so

$$\begin{aligned}\mathbb{P}(C|D) &= \frac{10/36}{30/36} \\ &= 1/3\end{aligned}$$

8.

$$\begin{aligned}P(ill|test+) &= \frac{P(test+|ill)(P(ill)}{P(test+|ill)(P(ill)+P(test+|healthy)(P(healthy)} \\ &= \frac{\frac{99}{100}.10^{-5}}{\frac{99}{100}.10^{-5}+\frac{1}{100}.(1-10^{-5})} \\ &\approx \frac{1}{1101}.\end{aligned}$$

The chance of being ill is rather small. Indeed it is more likely that the test was incorrect.

9. Let $A$ denote the event that the family has at least one boy, and $B$ denote the event that it has at least one girl. Then,

$\Pr(B) = 1 - (1/2)^n$,
$\Pr(A \cap B) = 1 - \Pr(\text{All Boys}) - \Pr(\text{All Girls}) = 1 - (1/2)^n - (1/2)^n = 1 - (1/2)^{n-1}$

Hence,

$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)} = \frac{1-(1/2)^{n-1}}{1-(1/2)^n}$$

10. Let $X$ and $Y$ denote the number of tosses required on the first experiment and second experiment, respectively. Then $X = n$ if and only if the first $n-1$ tosses of the first experiment are tails and the $n$th toss is a head, which has probability $(1/2)^n$. Furthermore, $Y > n$ if and only if the first $n$ tosses of the second experiment are all tails, which also has probability $(1/2)^n$. Hence,

$$\Pr(Y > X) = \textstyle\sum_{n=1}^{\infty} Pr(Y > n \mid X = n)Pr(X = n)$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^n}.\frac{1}{2^n} = \sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{1}{3}$$

11. Using total probability theorem, we know that,

$$A = (A \cap C) \cup (A \cap C^c)$$

Note that by the definition of $P_B$ and the fact that it is indeed another probability map (i.e., satisfies the axioms of probability), we have,

$$\begin{aligned} P(A|B) = P_B(A) &= P_B((A \cap C) \cup (A \cap C^c)) \\ &= P_B(A \cap C) + P_B(A \cap C^c) \\ &= P_B(C)P_B(A|C) + P_B(C^c)P_B(A|C^c) \end{aligned}$$

By the fact that

$$P_B(A|C) = P(A|B \cap C)$$

we have,

$$P(A|B) = P(C|B)P(A|B \cap C) + P(C^c|B)P(A|B \cap C^c)$$

Comparing with the expression given in the question,

$$\alpha = P(C|B) = \tfrac{P(C)}{P(B)} \text{ (as C is a subset of B).}$$

12. rth urn contains $r-1$ red and $N-r$ blue balls. Hence the total number of balls $= N-1$.

(a)

$$\begin{aligned} P(2^{nd}\, ball\, is\, blue|r^{th}\, urn\, is\, chosen) &= P(2^{nd}\, is\, blue|r^{th}\, urn, 1^{st}\, is\, red).P(1^{st}\, is\, red|r^{th}\, urn) \\ &+ P(2^{nd}\, is\, blue|r^{th}\, urn, 1^{st}\, is\, blue).P(1^{st}\, is\, blue|r^{th}\, urn) \\ &= \frac{N-r}{N-2}.\frac{r-1}{N-1} + \frac{N-r-1}{N-2}.\frac{N-r}{N-1} = \frac{N-r}{N-1} \end{aligned}$$

$$\begin{aligned} P(second\, ball\, is\, blue) &= \sum_{r=1}^{N} P(second\, ball\, is\, blue|r^{th}\, urn\, is\, selected).P(r^{th}\, urn\, is\, selected) \\ &= \sum_{r=1}^{N} \frac{N-r}{N-1}.\frac{1}{N} = \frac{1}{2} \end{aligned}$$

(b)

$$P(second\, ball\, is\, blue|first\, ball\, is\, blue) = \frac{P(both\, the\, balls\, are\, blue)}{P(first\, ball\, is\, blue)}$$

$$
\begin{aligned}
P(first\, ball\, is\, blue) &= \sum_{r=1}^{N} P(first\, ball\, is\, blue|r^{th}\, urn\, is\, selected).P(r^{th}\, urn\, is\, selected) \\
&= \sum_{r=1}^{N} \frac{N-r}{N-1}.\frac{1}{N} = \frac{1}{2}
\end{aligned}
$$

$$
P(both\, the\, balls\, blue|r^{th}\, urn\, is\, selected) = \begin{cases} \frac{(N-r)(N-r-1)}{(N-1)(N-2)}, & \text{r=1,...,N-2} \\ 0, & \text{r=N} \end{cases}
$$

$$
\begin{aligned}
P(both\, the\, balls\, blue) &= \sum_{r=1}^{N-2} \frac{(N-r)(N-r-1)}{(N-1)(N-2)}.\frac{1}{N} \\
&= \frac{1}{(N)(N-1)(N-2)} \sum_{r=1}^{N-2} (N-r)(N-r-1) \\
&= \frac{1}{3}
\end{aligned}
$$

$$
\begin{aligned}
P(second\, ball\, is\, blue|first\, ball\, is\, blue) &= \frac{P(both\, the\, balls\, are\, blue)}{P(first\, ball\, is\, blue)} \\
&= \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}
\end{aligned}
$$

13. (a)

$$
\begin{aligned}
P(k^{th} \text{ symbol received correctly}) &= P(0 \text{ transmitted and } 0 \text{ received}) \\
&\quad + P(1 \text{ transmitted and } 1 \text{ received}) \\
&= P(0 \text{ transmitted})P((0 \text{ received})|(0 \text{ transmitted})) \\
&\quad + P(1 \text{ transmitted})P((1 \text{ received})|(1 \text{ transmitted})) \\
&= (1-p)(1-\epsilon_0) + (p)(1-\epsilon_1)
\end{aligned}
$$

(b)

$$
\begin{aligned}
P(\text{string 1101 received correctly}) &= P(1101 \text{ received}|1101 \text{ transmitted}) \\
&= (1-\epsilon_1)(1-\epsilon_1)(1-\epsilon_0)(1-\epsilon_1)
\end{aligned}
$$

(c) **Majority Decision Decoder:** For the transmission of 'one symbol', transmitter will send the same symbol 3 times. So the receiver will receive 3 symbols, out of 3 symbols, 1 is received 2 times, receiver will decode that 'one symbol' as 1, otherwise 0.

$$
\begin{aligned}
P(k^{th} \text{ symbol decoded correctly}) &= P(0 \text{ transmitted and } 0 \text{ decoded}) \\
&\quad + P(1 \text{ transmitted and } 1 \text{ decoded}) \\
&= P(0 \text{ transmitted})P(0 \text{ decoded}|0 \text{ transmitted}) \\
&\quad + P(1 \text{ transmitted})P(1 \text{ decoded}|1 \text{ transmitted}) \\
&= P(0 \text{ transmitted})P(000 \text{ or } 001 \text{ or } 010 \text{ or } 100|0 \text{ transmitted}) \\
&+ P(1 \text{ transmitted})P(111 \text{ or } 110 \text{ or } 101 \text{ or } 011|1\text{transmitted}) \\
&= (1-p)((1-\epsilon_0)^3 + 3(\epsilon_1)(1-\epsilon_0)^2)) \\
&\quad + (p)((1-\epsilon_1)^3 + 3(\epsilon_0)(1-\epsilon_1)^2))
\end{aligned}
$$

14. Let the events be defined as:
A = {An Aircraft is present}, A' = {Aircraft is not present}
B = {The Radar generates an Alarm}, B' = {The radar don't generates an alarm}

The conditional probabilities are:
$P(B|A) = 0.99$ : The radar generates an alarm given aircraft is present.
$P(B'|A) = 0.11$ : The radar gives no alarm when the aircraft is present.

$P(B|A') = 0.10$ : False alarm by the Radar.
$P(B'|A') = 0.90$ : No alarm when aircraft is not present.

    1. $P(A) = 0.165$, $P(A') = 0.835$
    P(not present, false alarm)= $P(A' \cap B) = P(A')P(B|A') = 0.835 * 0.10 = 0.0835$
    P(present, No detection)= $P(A \cap B') = P(A)P(B'|A) = 0.165 * 0.01 = 0.00165$

    2. $P(A) = 0.35$, $P(A') = 0.65$
    P(not present, false alarm)= $P(A' \cap B) = P(A')P(B|A') = 0.65 * 0.10 = 0.065$
    P(present, No detection)= $P(A \cap B') = P(A)P(B'|A) = 0.35 * 0.01 = 0.0035$

    3. The efficicency is defined as:
    P(Aircraft present given Radar generates alarm)= $P(A|B) = P(A) * P(B|A)/P(B)$

15. Let $D$ is the event that a person is HIV positive, and $T$ is the event that the person tests positive.

$$\mathbb{P}(D \mid T) = \frac{\mathbb{P}(D \cap T)}{\mathbb{P}(T)} = \frac{(0.99)(0.003)}{(0.99)(0.003) + (0.01)(0.997)} \approx 23\%$$

A short reason why this surprising result holds is that the error in the test is much greater than the percentage of people with HIV. This leads to a lot of false positives than "expected".

16. The game can be described as having probability 1/2 of winning 1 dollar and a probability 1/2 of losing 1 dollar. A player begins with a given number of dollars, and intends to play the game repeatedly until the player either goes broke or increases his holdings to $N$ dollars.

For any given amount $n$ of current holdings, the conditional probability of reaching $N$ dollars before going broke is independent of how we acquired the $n$ dollars, so there is a unique probability $\mathbb{P}(N \mid n)$ of reaching $N$ on the condition that we currently hold $n$ dollars. Of course, for any finite N we see that $\mathbb{P}(N \mid 0) = 0$ and $\mathbb{P}(N \mid N) = 1$. The problem is to determine the values of $\mathbb{P}(N \mid n)$ for $n$ between 0 and $N$.

We are considering this setting for $N = 200$, and we would like to find $\mathbb{P}(200 \mid 50)$. Denote $y(n) := \mathbb{P}(200 \mid n)$, which is the probability you get to 200 without first getting ruined if you start with $n$ dollars. We saw that $y(0) = 0$ and $y(200) = 1$. Suppose the player has $n$ dollars at the moment, the next round will leave the player with either $n + 1$ or $n - 1$ dollars, both with probability 1/2. Thus the current probability of winning is the same as a weighted average of the probabilities of winning in player's two possible next states. So we can safely say that,

$$y(n) = \frac{1}{2}y(n+1) + \frac{1}{2}y(n-1).$$

Multiplying by 2 , and subtracting $y(n) + y(n-1)$ from each side, we have

$$y(n+1) - y(n) = y(n) - y(n-1)$$

This says that slopes of the graph of $y(n)$ on the adjacent intervals are constant (remember that $x$ must be an integer). In other words, the graph of $y(n)$ must be a line. Since $y(0) = 0$ and $y(200) = 1$, we have $y(n) = n/200$, and therefore $y(50) = 1/4$. Another way to see what the function $y(n)$ is to use the telescoping sum as follows

$$\begin{aligned} &y(n) = y(n) - y(0) = (y(n) - y(n-1)) + \ldots + (y(1) - y(0)) \\ &= n(y(1) - y(0)) = ny(1) \end{aligned}$$

since the all these differences are the same, and $y(0) = 0$. To find $y(1)$ we can use the fact that $y(200) = 1$, so $y(1) = 1/200$, and therefore $y(n) = n/200$ and $y(50) = 1/4$.

17. Assignment.