**EE2025 Engineering Eelectromagnetics**
**Tutorial 3 (RKS 5.0 to 5.7)**

## Part A

1. The electric field of a plane wave in a non-magnetic, charge-free medium is given by

$$\vec{E} = (a\hat{x} - 2\hat{y} + 5\hat{z})e^{j(\omega t - 2x - 10y + 6z)} \text{ [V/m]}$$

(a) Write down the wave vector $\vec{k}$ of the plane wave.

(b) Calculate the unknown constant $a$.

(c) The corresponding magnetic field $\vec{H}$.

(d) From the $\vec{H}$ field, prove that the angle between $\vec{E}$ and $\vec{H}$ fields is equal to $90^o$.

Consider $f = 1$ GHz and $\epsilon_r = 2.5$.

**Solution:** (a) From the given electric field expression we can write the dot product of wave vector $\vec{k}$ and the position vector $\vec{r}$ as

$$\vec{k}.\vec{r} = 2x + 10y - 6z \Rightarrow \vec{k} = 2\hat{x} + 10\hat{y} - 6\hat{z} \text{ [1/m]}$$

(b) For the unknown constant, we know that $\vec{k}.\vec{E} = 0$

$$(2\hat{x} + 10\hat{y} - 6\hat{z}).(a\hat{x} - 2\hat{y} + 5\hat{z}) = 0$$

$$2a - 20 - 30 = 0 \Rightarrow a = 25$$

(c) The magnetic field in terms of wave vector $\vec{k}$ and electric field can be given as

$$\vec{H} = \frac{1}{\omega\mu}(\vec{k} \times \vec{E})$$

$$\omega\mu = (2\pi 10^9) \times (4\pi 10^{-7}) = 800\pi^2 \,;\ \vec{H} = \frac{1}{800\pi^2}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 2 & 10 & -6 \\ 25 & -2 & 5 \end{vmatrix} e^{j(\omega t - 2x - 10y + 6z)}$$

$$\vec{H} = \frac{1}{800\pi^2}(38\hat{x} - 160\hat{y} - 254\hat{z})e^{j(\omega t - 2x - 10y + 6z)} \text{ [A/m]}$$

(d) The angle between the vectors $\vec{E}$ and $\vec{H}$ field is written as

$$\cos\theta = \frac{\vec{E}.\vec{H}}{|\vec{E}||\vec{H}|}$$

$$\vec{E}.\vec{H} = (25 \times 38) + (160 \times 2) - (254 \times 5) = 0$$

Therefore, the angle $\theta = \pi/2$

2. A light beam is incident from air to a medium with a dielectric constant of 4 and relative permeability of 100. If the angle of incidence is 60°, find the angle of reflection and angle of refraction.

**Solution:**

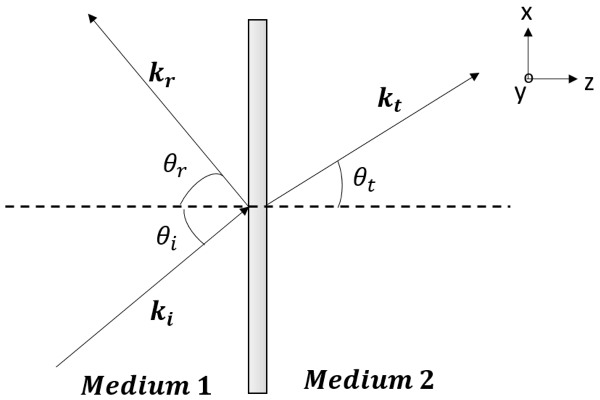
The angle of reflection $\theta_r$ = angle of incidence $\theta_i = 60°$.

From the Snell's law,

$$\begin{aligned} \sqrt{\mu_1\epsilon_1}\sin\theta_i &= \sqrt{\mu_2\epsilon_2}\sin\theta_t \\ \Rightarrow \sqrt{\mu_0\epsilon_0}\sin 60° &= \sqrt{\mu_0(100)\epsilon_0(4)}\sin\theta_t \\ \sin\theta_t &= \frac{\sin 60°}{20} \\ &= 0.0433 \\ \Rightarrow \quad \text{Angle of refraction} \quad \theta_t &= 2.48° \end{aligned}$$

3. A light beam is incident from air to a medium with a dielectric constant $\epsilon$ and relative permeability $\mu$ at an angle to the interface. Prove that the incident ray, reflected ray and transmitted ray lie on the same plane of incidence.

**Solution:**




We can apply the boundary condition for the continuity of fields at the interface. The boundary condition at the interface for the electric and magnetic field follow the generic format given below,

$$()e^{i(\mathbf{k_i}\cdot\mathbf{r}-\omega t)} + ()e^{i(\mathbf{k_r}\cdot\mathbf{r}-\omega t)} = ()e^{i(\mathbf{k_t}\cdot\mathbf{r}-\omega t)}$$

The boundary condition should be satisfied at all points of time and at all points in space. This means it should be satisifed at z = 0 also. This can be satisfied at z = 0 only if,

$$\mathbf{k_i}\cdot\mathbf{r} = \mathbf{k_r}\cdot\mathbf{r} = \mathbf{k_t}\cdot\mathbf{r}$$

This means the incident ray, the reflected ray and the transmitted ray all lie on the same plane.

4. A uniform plane wave is incident from the air onto glass at an angle from the normal of 30°. Determine the fraction of the incident power reflected and transmitted for

   (a) parallel polarization and

   (b) perpendicular polarization.

   Glass has a refractive index $n_2 = 1.45$.

**Solution:**

First, we apply Snell's law to find the transmission angle using $n = 1$ for air to get,

$$\theta_2 = sin^{-1}\left(\frac{sin\,30^\circ}{1.45}\right) = 20.2^\circ$$

Now, for parallel polarization, use the formula for the reflection coefficient:

$$\Gamma_{||} = \frac{\eta_2 cos\,20.2^\circ - \eta_1 cos\,30^\circ}{\eta_2 cos\,20.2^\circ + \eta_1 cos\,30^\circ} = -0.144$$

Therefore the fraction of incident power which is reflected is:

$$|\Gamma_{||}|^2 = 0.021$$

The fraction of the power that is transmitted is

$$1 - |\Gamma_{||}|^2 = 0.979$$

similarly for s-polarization, we have

$$\Gamma_{\perp} = \frac{\eta_2 sec\,20.2^\circ - \eta_1 sec\,30^\circ}{\eta_2 sec\,20.2^\circ + \eta_1 sec\,30^\circ} = -0.222$$

The reflected power fraction is thus

$$|\Gamma_{\perp}|^2 = 0.049$$

Thus the fraction of the power that is transmitted is

$$1 - |\Gamma_{\perp}|^2 = 0.951$$

5. The electric field of a uniform plane in the air is given by

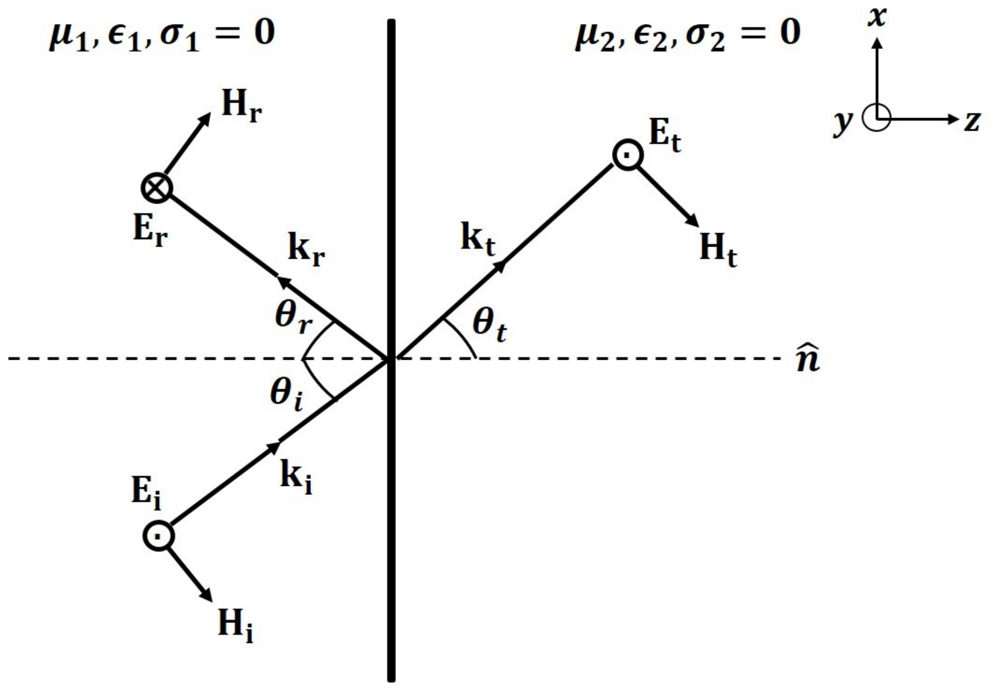
$$\vec{E_i} = 100\cos(2\pi f t - 3x - 4z)\hat{y} \text{ [V/m]},$$

   is incident on a dielectric slab ($z \geq 0$) with $\mu_r = 1, \epsilon_r = 3$ and $\sigma = 0$, at an angle $\theta_i$.

   (i) Sketch the dielectric interface with corresponding orientations of electric field and magnetic fields for incident $(\vec{E_i}, \vec{H_i})$, reflected $(\vec{E_r}, \vec{H_r})$ and transmitted wave $(\vec{E_t}, \vec{H_t})$ and find the polarization of the wave.

   (ii) Calculate the frequency $f$, angle of incidence $(\theta_i)$, reflection $(\theta_r)$ and transmission $(\theta_t)$.

**Solution:** (i)




Here the electric field is oriented along $\hat{y}$ direction, which is perpendicular to the plane of incidence ($xz-$plane). Therefore the wave is perpendicularly polarized (sometimes also called as TE polarization)

(ii) From the electric field expression the wave vector of the incident wave can be expressed as,

$$\vec{k_i} = 3\hat{x} + 4\hat{z} \Rightarrow \beta_1 = |\vec{k_i}| = \omega\sqrt{\mu\epsilon}$$

$$\beta_1 = |\vec{k_i}| = \omega\sqrt{\mu\epsilon} = \omega = \frac{\beta_1}{\sqrt{\mu_0\epsilon_0}} = \beta_1 c$$

$$\omega = \sqrt{3^2 + 4^2} \times 3 \times 10^8 = 15 \times 10^8 \,[\text{rad/s}] \Rightarrow f \approx 238.7\,\text{MHz}$$

From the figure, the wavevector can be written as

$$\vec{k_i} = k_{ix}\hat{x} + k_{iz}\hat{z} = \beta_1 \sin\theta_i \hat{x} + \beta_1 \cos\theta_i \hat{z}$$

Therefore,

$$\beta_1 \sin\theta_i = 3\,;\ \beta_1 \cos\theta_i = 4 \Rightarrow \tan\theta_i = 3/4 \Rightarrow \theta_i = 0.643\,[\text{rad}] = 36.84^o$$

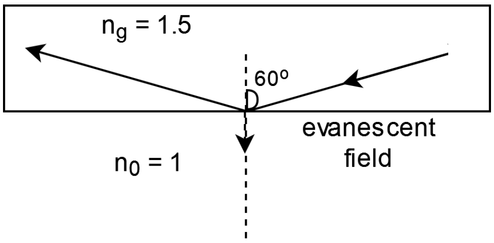
From the law of reflection, $\theta_r = \theta_i = 36.84^o$

From the law of refraction (Snell's law) for a pure dielectric media

$$\sqrt{\epsilon_{r1}}\sin\theta_i = \sqrt{\epsilon_{r2}}\sin\theta_t \Rightarrow \theta_t = \sin^{-1}\left(\sqrt{\frac{\epsilon_{r1}}{\epsilon_{r2}}}\sin\theta_i\right)$$

$$\theta_t = \sin^{-1}\left(\sqrt{\frac{1}{3}}\sin\theta_i\right) = 0.353\,[\text{rad}] = 20.22^o$$

6. A beam of light from an argon laser ($\lambda = 500\,\text{nm}$) traveling in a glass block ($n_g = 1.5$) is totally internally reflected at the flat air–glass interface. If the beam strikes the interface at $60°$ to the normal, how deep will the light penetrate the air before its amplitude drops to about 36.8% of its value at the interface?




**Solution:**

The evanescent wave decreases exponentially from the interface between media. The expression for the exponentially decaying term (in z-direction) for the evanescent wave is

$$e^{-z\frac{2\pi}{\lambda}\sqrt{n_g^2 sin^2\theta_i - n_0^2}}$$

Here, $\lambda$ is the wavelength of the light, and $\theta_i$ is the angle of incidence of the light at the interface.

The length at which the evanescent wave has decreased to an amplitude of 36.8% (1/e) of its value at the interface is

$$e^{-z\frac{2\pi}{\lambda}\sqrt{n_g^2 sin^2\theta_i - n_0^2}} = e^{-1}$$

$$z\frac{2\pi}{\lambda}\sqrt{n_g^2 sin^2\theta_i - n_0^2} = 1$$

$$z = \frac{\lambda}{2\pi}(n_g^2 sin^2\theta_i - n_0^2)^{-1/2}$$

$$z = \frac{500}{2\pi}(1.5^2 sin^2 60^o - 1^2)^{-1/2} = 96\ nm$$

7. A left-hand circularly polarized wave impinges at an interface between two different media with an angle of $45°$. Determine the state of polarisation of the reflected and transmitted wave:

    1. If the wave travels from air to a perfect conductor.
    2. If the wave travels from air to a dielectric medium (non-magnetic) of $\epsilon_r = 2.5$.

**Solution:**

Any arbitrary polarised incident wave can be expressed as

$$\vec{E_i} = \vec{E_{i||}} + \vec{E_{i\perp}}e^{j\phi}$$

Similarly,

$$\vec{E_r} = \vec{E_{r||}} + \vec{E_{r\perp}} = \Gamma_{||}\vec{E_{i||}} + \Gamma_{\perp}\vec{E_{i\perp}}e^{j\phi}$$

$$\vec{E_t} = \vec{E_{t||}} + \vec{E_{t\perp}} = \tau_{||}\vec{E_{i||}} + \tau_{\perp}\vec{E_{i\perp}}e^{j\phi}$$

Given the wave is LCP $\implies |E_{i||}| = |E_{i\perp}|$ and $\phi = \pi/2$

1. If the wave travels from air to a perfect conductor:
   Perfect conductor $\implies \sigma \to \inf \implies \eta_2 = 0$ and hence $\Gamma_{||} = 1$ and $\Gamma_{\perp} = -1$. Thus,

   $$\vec{E_r} = \vec{E_{i||}} - \vec{E_{i\perp}}e^{j\pi/2} = \vec{E_{i||}} + \vec{E_{i\perp}}e^{j\pi}e^{j\pi/2} = \vec{E_{i||}} + \vec{E_{i\perp}}e^{j\pi}e^{j\pi/2} = \vec{E_{i||}} + \vec{E_{i\perp}}e^{-j\pi/2}$$

   $\therefore$ Reflected wave is **right circularly polarised** since $|E_{i||}| = |E_{i\perp}|$

   In this case, there is no transmitted wave his case as the transmission medium is a perfect conductor.

2. If the wave travels from air to a medium of $\epsilon_r = 2.5$ :
   In this case, $\eta_1 = \eta_0$ and $\eta_2 = \frac{\eta_0}{\sqrt{\epsilon_r}}$
   $\theta_i = 45°$ and thus $\theta_t = \sin^{-1}(\frac{n_1}{n_2}\sin\theta_i) = \sin^{-1}(\frac{1}{\sqrt{2.5}}\sin\theta_i) = 26.56°$

   $$\Gamma_{||} = \frac{\eta_1\cos\theta_i - \eta_2\cos\theta_t}{\eta_1\cos\theta_i + \eta_2\cos\theta_t} = \frac{\cos\theta_i - \frac{1}{\sqrt{\epsilon_r}}\cos\theta_t}{\cos\theta_i + \frac{1}{\sqrt{\epsilon_r}}\cos\theta_t} = 0.11$$

   $$\Gamma_{\perp} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} = \frac{\frac{1}{\sqrt{\epsilon_r}}\cos\theta_i - \cos\theta_t}{\frac{1}{\sqrt{\epsilon_r}}\cos\theta_i + \cos\theta_t} = -0.33$$

   Thus $\vec{E_r} = 0.11\vec{E_{i||}} - 0.33\vec{E_{i\perp}}e^{j\pi/2} = 0.11\vec{E_{i||}} + 0.33\vec{E_{i\perp}}e^{-j\pi/2}$

   $\therefore$ Reflected wave is **right elliptically polarised**

   Similarly,

   $$\tau_{||} = \frac{2\eta_2\cos\theta_i}{\eta_1\cos\theta_i + \eta_2\cos\theta_t} = \frac{2\cos\theta_i}{\sqrt{\epsilon_r}\cos\theta_i + \cos\theta_t} = 0.70$$

   $$\tau_{\perp} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} = \frac{2\cos\theta_i}{\cos\theta_i + \sqrt{\epsilon_r}\cos\theta_t} = 0.67$$

   Thus $\vec{E_t} = 0.70\vec{E_{i||}} + 0.67\vec{E_{i\perp}}e^{j\pi/2}$

   $\therefore$ Transmitted wave is **left elliptically polarised**

8. A right circularly polarised plane wave in air is incident at Brewster's angle on a semi-infinite slab of plexiglass ($\epsilon_r = 3.45$). Determine the fraction of incident power that is reflected and transmitted at the interface.

**Solution:** Incident at the Brewster's angle and $\epsilon_r = 3.45$
$\theta_i = \theta_B = \tan^{-1}\sqrt{\epsilon_r} = 61.7^o$
Now using Snell's law we have
$n_1 \sin(\theta_B) = n_2 \sin(\theta_t)$; where $n_2 = \sqrt{\epsilon_r}$
From here, $\theta_t = 28.3^o$
When incident at the Brewster's angle then all the parallel polarised are transmitted (i.e. $\Gamma_{||} = 0$)
$\Gamma_\perp = \frac{\eta_2 \cos\theta_B - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_B + \eta_1 \cos\theta_t} \Rightarrow$ From here, $\Gamma_\perp = -0.55$

Now the reflected power transfer is $|\Gamma|^2 = 0.303$
Reflected Power fraction is 0.303 or 30.3%

Since the wave is circularly polarised, the perpendicular polarised component represents one half of the total incident wave power and so the fraction of the total power that is reflected is $0.303/2 = 0.152$ i.e. 15.2%
The fraction of power that is transmitted is the remaining 84.8%

## Part B

9. In question 5, calculate the reflected electric field ($\vec{E_r}$) and transmitted magnetic field ($\vec{H_t}$).

**Solution:** The reflection and the transmission coefficients for perpendicular polarization are

$$\Gamma_\perp = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}\,;\, \tau_\perp = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}.$$

The intrinsic impedance of the media are

$$\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}} = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 377\,\Omega\,;\, \eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}} = \frac{377}{\sqrt{3}} \approx 217.6\,\Omega$$

$$\Gamma_\perp = -0.34\,;\, \tau_\perp = 0.66$$

The reflected electric field can be written as

$$\vec{E_r} = E_{0r}\cos(2\pi f t - \vec{k_r}.\vec{r}) = \Gamma_\perp E_{0i}\cos(2\pi f t - \vec{k_r}.\vec{r})$$

$$\vec{k_r} = \beta_1 \sin\theta_r \hat{x} - \beta_1 \cos\theta_r \hat{z} = 3\hat{x} - 4\hat{z}$$

$$\vec{E_r} = -34\cos(15 \times 10^8 t - 3x + 4z)\hat{y}\,[\text{V/m}]$$

The transmitted electric field can be written as

$$\vec{E_t} = E_{0t}\cos(2\pi f t - \vec{k_t}.\vec{r}) = \tau_\perp E_{0i}\cos(2\pi f t - \vec{k_t}.\vec{r})$$

$$\vec{k_t} = \beta_2 \sin\theta_t \hat{x} + \beta_2 \cos\theta_t \hat{z}\,;\, \beta_2 = \omega\sqrt{\mu_2 \epsilon_2} = 8.66\,[\text{rad/m}]$$

$$\vec{k_{tx}} = \beta_2 \sin\theta_t = 3\,;\, \vec{k_{tz}} = \beta_2 \cos\theta_t = 8.124 \Rightarrow \vec{k_t} = 3\hat{x} + 8.124\hat{z}$$

$$\vec{E}_t = 66\cos\left(15\times 10^8 t - 3x - 8.124z\right)\hat{y}\ [\mathrm{V/m}]$$

For transmitted magnetic field

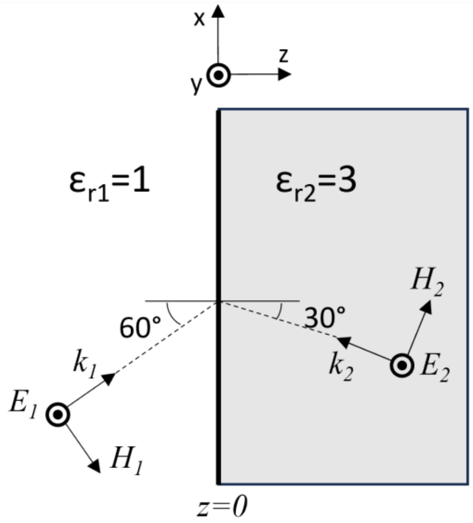
$$\vec{H}_t = \frac{1}{\omega\mu}(\vec{k}_t \times \vec{E}_t) = \frac{1}{600\pi}\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ 3 & 0 & 8.124\\ 0 & E_{ty} & 0\end{vmatrix} = \frac{1}{600\pi}\left(-8.124E_{ty}\hat{x} + 3E_{ty}\hat{z}\right)$$

$$\vec{H}_t = (-0.28\hat{x} + 0.105\hat{z})\cos\left(15\times 10^8 t - 3x - 8.124z\right)\hat{y}\ [\mathrm{A/m}]$$

10. Consider two perpendicularly polarized plane waves, of the same frequency $\omega$, incident from either side of the dielectric ($\sigma = 0$ and $\mu_r = 1$) interface as shown in the figure. The electric field of the two incident waves are out of phase as in the following expressions.

$$\vec{E}_1 = E_0 e^{-j\mathbf{k_1}\cdot\mathbf{r}}\hat{y}$$
$$\vec{E}_2 = -E_0 e^{-j\mathbf{k_2}\cdot\mathbf{r}}\hat{y}$$

Find the net electric field in the region $z > 0$ after interaction with the dielectric interface.




**Solution:** The reflected and refracted angles can be calculated using Snell's law (phase matching along 'x'). For the wave on the left (wave1), the angle of transmission into the second medium is given by:

$$1\sin(60°) = \sqrt{3}\sin(\theta_t) \implies \theta_t = 30°$$

For the wave on the right (wave2), the angle of reflection is the same as the angle of incidence, which is also 30° . Hence, the total field is the sum of the incident field of wave2, and the transmitted and reflected fields of wave1 and wave2 respectively. To estimate the magnitudes of these fields we need to evaluate the reflection and transmission coefficients for both waves using Fresnel's equations. For wave1, the reflection coefficient is

$$\Gamma_1 = \frac{\frac{\eta_2}{\cos(30°)} - \frac{\eta_1}{\cos(60°)}}{\frac{\eta_2}{\cos(30°)} + \frac{\eta_1}{\cos(60°)}} = -0.5$$

and so the transmission coefficient $T_1 = 1 + \Gamma_1 = 0.5$. For wave2, the reflection coefficient $\Gamma_2 = -\Gamma_1 = 0.5 = T_1$. Since the incident fields for wave1 and wave2 are out of phase, the transmitted fields due to wave1 and the reflected fields from wave2 perfectly cancel each other. Therefore, the total field in the region z > 0 is the same as that due to wave2.

$$\vec{E}(z > 0) = -E_0 e^{-j\mathbf{k_2}\cdot\mathbf{r}}\hat{y}$$

$$\vec{H}(z > 0) = -\frac{E_0}{2\eta_2} e^{-j\mathbf{k_2}\cdot\mathbf{r}}(\sqrt{3}\hat{x} + \hat{z})$$

11. You are given three materials: fiberglass ($n = 1.6$), rain erosion paint ($n = 1.6$, thickness=$\lambda_3/16$) and primer ($n = 2.56$, thickness=$\lambda_2/2$), where $n$ is the refractive index of the material.

    Design a cascade of these three materials (propose an arrangement and the thickness of the fiberglass), such that overall transmission coefficient for a normally incident wave is unity. The wave propagates through the air ($n = 1$) into the cascade and then leaves back into the air. Suppose now you use this configuration for a radome which must transmit at least 95% of the incident signal power. Find the value of $n$ for the atmosphere ($n$ of atmosphere varies with height). Assume that the air between the aircraft antenna and the radome walls has $n = 1$ always, i.e air at sea level.

**Solution:**

The following configuration can be used to attain transmission coefficient equal to unity:

| Air ($n = 1$) | Fiber-glass ($n_1 = 1.6$) | Primer ($n_2 = 2.56$) | Erosion Paint ($n_3 = 1.6$) | Air ($n = 1$) |
|---|---|---|---|---|
| | $7\lambda_1/16$ | $\lambda_2/2$ **Fixed** | $\lambda_3/16$ **Fixed** | |

Explanation: The Fiber-glass + Primer + Erosion Paint cascade has effective transmission coefficient equal to 1 and thus the primer in the middle can be ignored simplifying the system to the following:

| Air ($n = 1$) | Fiber-glass ($n_1 = 1.6$) | Erosion Paint ($n_3 = 1.6$) | Air ($n = 1$) |
|---|---|---|---|
| | $7\lambda_1/16$ | $\lambda_3/16$ | |

$$\lambda_1 = \lambda_3 = \lambda$$

$$\textit{total thickness} = \lambda/2$$

This system has air on both sides and slab length is $\lambda/2$, thus has an effective transmission coefficient of unity.

Solve for the transmission coefficient when the air on the right has a variable $n$ to get the following formula for the effective transmission coefficient. Setting length of slab equal to $\lambda/2$:

$$|T| = \frac{t_{12}t_{23}}{1 + r_{21}r_{23}}$$

substituting values in terms of $n$'s ($n_{air} = n_1 = 1$, $n_{glass/paint} = n_2$, $n_{atm} = n$):

$$|T| = \frac{\frac{4n_1n_2}{(n_1+n_2)(n_2+n)}}{\frac{n_1n_2+n_1n+n_2^2+n_2n+n_1n_2+n_2n-n_2^2-n_1n}{(n_1+n_2)(n_2+n)}}$$

$$\Rightarrow |T| = \frac{2}{n+1}$$

The incident power is $\frac{|E_i|^2}{2\eta_1}$ and the transmitted power is $\frac{|T \times E_i|^2}{2\eta_3}$

For transmission of 95% of incident power, $\frac{\frac{|T \times E_i|^2}{2\eta_3}}{\frac{|E_i|^2}{2\eta_1}} = 0.95$

$|T|^2 \times \frac{\eta_1}{\eta_3} = |T|^2 \times n = 0.95$ gives $n = 1.567$

12. Light from a red LED (free space wavelength = 650 nm) is incident normally on an optical sensor embedded in a thick glass slab with relative permeability of 3/8 and dielectric constant of 8.

    a. What color of light in glass would the sensor detect?

    b. What percentage of incident power is reflected at the glass interface?

    c. To eliminate reflections, the glass is coated with a material of dielectric constant of 2, and thickness of 400 nm. Do you think this coating would act as an anti-reflection coating? If yes, justify your answer. If not, suggest an alternative coating (thickness and relative permittivity) to replace this coating.

    d. If the red LED is replaced by a green LED (525 nm) in the above question, quantify the reflection coefficient.

**Solution:**

1. Sensor detects the optical frequency, and the frequency would be the same irrespective of the medium. Hence the color would be RED itself.

2. Reflection coefficient

$$\rho = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

Where:

$$\eta_1 = \eta_0 \quad \text{and} \quad \eta_2 = \eta_0\sqrt{\frac{\mu_r}{\epsilon_r}} = \eta_0\sqrt{\frac{3}{8} \cdot \frac{1}{8}} = \frac{\eta_0\sqrt{3}}{8}$$

Substituting the values, we get:

$$\rho = \frac{\frac{\eta_0\sqrt{3}}{8} - \eta_0}{\frac{\eta_0\sqrt{3}}{8} + \eta_0} = \frac{\sqrt{3} - 8}{\sqrt{3} + 8} \approx -0.644$$

The percentage of power reflected is:

$$\% \text{ of power reflected} = |\rho|^2 \times 100\% \approx 41.48\%$$

3. Let free space be medium 1, the coating be medium 2, and glass be medium 3.
To eliminate reflections:
(a) $\eta_2 = \sqrt{\eta_1\eta_3}$
(b) The thickness of the coating should be an odd multiple of $\frac{\lambda_2}{4}$.

For $\epsilon_r = 2$, $\lambda_2 = \frac{\lambda_0}{\sqrt{\epsilon_r}} = \frac{650\,\text{nm}}{\sqrt{2}} \approx 459.62\,\text{nm}$.

$\frac{\lambda_2}{4} \approx 114.90\,\text{nm}$.

400 nm is not an odd multiple of 114.90 nm, so this coating will not act as an anti-reflection coating.

Design of desired Anti-Reflection Coating (ARC):

We need $\eta_2 = \sqrt{\eta_1\eta_3} = \sqrt{\eta_0\eta_0\sqrt{\frac{3}{8}}} = \eta_0\sqrt{\frac{\sqrt{3}}{8}}$ (1)

Also, $\eta_2 = \frac{\eta_0}{\sqrt{\epsilon_r}}$ (2)

Comparing Eqn. 1 and 2:

$\eta_0\sqrt{\frac{\sqrt{3}}{8}} = \frac{\eta_0}{\sqrt{\epsilon_r}}$

$\Rightarrow \epsilon_r = \frac{8}{\sqrt{3}} \approx 4.619$

$\lambda_2 = \frac{\lambda_0}{\sqrt{\epsilon_r}} = \frac{650\,\text{nm}}{\sqrt{4.619}} \approx 302.44\,\text{nm}$ and $\frac{\lambda_2}{4} \approx 75.61\,\text{nm}$.

Thus, a coating material with a relative permittivity of 4.619 and a thickness of 75.61 nm can be used as an Anti-Reflection Coating.

4. If the red LED is replaced by a green LED (525 nm) in the above question,
the reflection coefficient $\rho$ is: $\rho = \frac{\eta_{\text{in}} - \eta_1}{\eta_{\text{in}} + \eta_1}$,
where

$$\eta_{\text{in}} = \eta_2 \frac{\eta_3 + j\eta_2 \tan(\beta_2 l)}{\eta_2 + j\eta_3 \tan(\beta_2 l)}$$

$$\eta_{\text{in}} = \eta_2 \frac{\eta_3 + j\eta_2 \tan(\frac{2\pi\sqrt{\epsilon_r}}{\lambda_0} \cdot 75.61 \times 10^{-9})}{\eta_2 + j\eta_3 \tan(\frac{2\pi\sqrt{\epsilon_r}}{\lambda_0} \cdot 75.61 \times 10^{-9})} \tag{3}$$

With $\lambda_0$ changed to 525 nm, and given $\eta_1 = \eta_0$, $\eta_2 = \frac{\eta_0}{\sqrt{4.619}}$, and $\eta_3 = \frac{\eta_0\sqrt{3}}{8}$,
we can calculate $\eta_{\text{in}}$ using Eqn. (3):

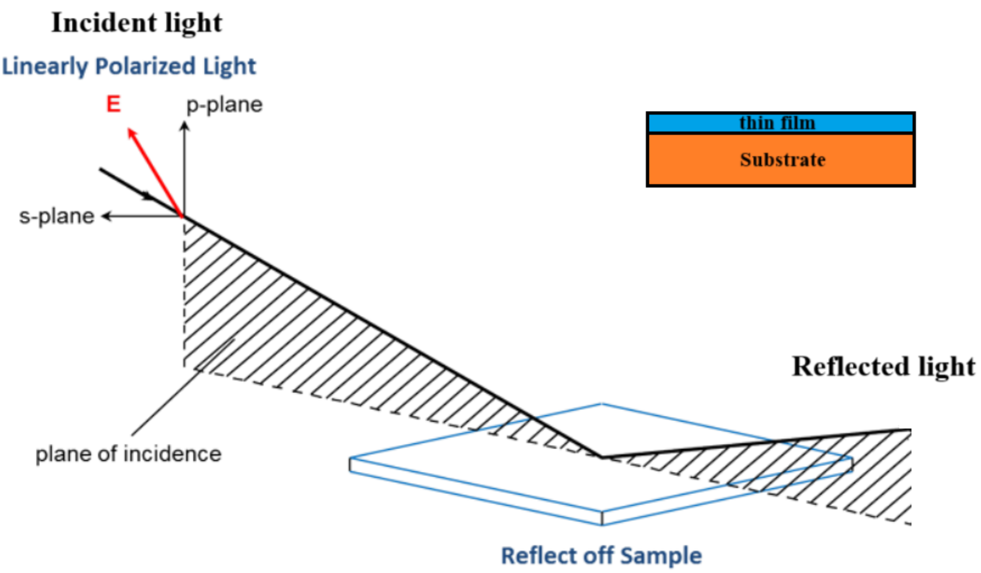
$$\eta_{\text{in}} = \eta_0 \frac{\left(\frac{1}{\sqrt{4.619}}\right)\left(\sqrt{\frac{3}{8}} + j\left(\frac{1}{\sqrt{4.619}}\right)\tan\left(\frac{2\pi\sqrt{4.619}}{525} \times 75.61\right)\right)}{\left(\frac{1}{\sqrt{4.619}}\right) + j\sqrt{\frac{3}{8}}\tan\left(\frac{2\pi\sqrt{4.619}}{525} \times 75.61\right)}$$

$$\eta_{\text{in}} = \eta_0 \frac{0.1 - j0.552}{0.465 - j0.552} \approx (0.674 - 0.387j)\eta_0$$

Therefore,

$$\rho = \frac{(0.674 - 0.387j) - 1}{(0.674 - 0.387j) + 1} = -0.134 - 0.262j = 0.295\angle - 117.09^\circ$$

13. At the CNNP lab in IIT Madras, a spectroscopic ellipsometer is used to characterize thin films by measuring the polarization state of light after reflection from the sample. In this tool, a randomly polarized light from the light source is converted into linearly polarized light at 45°, with respect to the plane of incidence, before being directed onto the sample (see figure). The reflected light's polarisation state is measured using a rotating analyzer and a detector.




(a) Determine the typical polarization state of the reflected light if the angle of incidence ranges between 55° and 75°. Explain how the reflection affects the polarization state of the light.

(b) The primary data in an ellipsometer are the amplitude ratio ($\psi$) and phase difference ($\Delta$) between parallel and perpendicular components of the reflected light. How can these raw data be used to determine the thickness and refractive index of the film?

**Solution:**

(a) The linearly polarised light at 45° has equal contributions for parallel and perpendicular polarisation. At non-normal incidence, the Fresnel reflection coefficients for the parallel and perpendicular polarisation of light are different.

Typically, the reflected light will be elliptically polarised due to the phase difference between the parallel and perpendicular components of the reflected light. The amplitude of the parallel and perpendicular components and the phase shift between them will determine the exact nature of ellipticity.

(b) In ellipsometer, the rotating analyser modulates the intensity of the reflected light ($I(\theta)$) at different angles. $I(\theta)$ is sinusoidal, as it follows Mallus law. $\psi$ and $\Delta$ is extracted from this modulated intensity.

When light hits a thin film, it reflects partially off the top surface (air/film interface) and partially off the bottom surface (film/substrate interface). The overall reflected light is due to the superposition of multiple light waves.

As light travels through the film, it experience a phase shift which is related to the thickness and the refractive index of the film. This phase shift is different for parallel and perpendicular polarisation as the reflection coefficient for each of them is different at interfaces. Also, this phase difference ($\Delta$) increases with thicker films due to increased optical path lengths.

$\psi$ and $\Delta$ are directly related to the Fresnel reflection coefficients as, (see - https://www.jawoollam.com/resources/ellipsometry-tutorial/ellipsometry-measurements)

$$\frac{\Gamma_{||}}{\Gamma_{\perp}} = \tan(\psi)e^{i\Delta}$$

With known angle of incidence, the complex refractive index can be estimated. However, films may have some surface roughness and unwanted oxide layer which can cause inaccurate estimation.

Typically, multiple incidence angle and broadband spectrum light is used for illuminating the sample. Then a model is employed to fit the measured $\psi$ and $\Delta$ values, allowing for a precise estimation of the refractive index and thickness of the film.