EE3110 - Probability Foundations for Electrical Engineers
Supplementary Tutorial - Week 4

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1. Let X and Y be independent standard normal random variables. The pair (X, Y) can be described in polar coordinates in terms of random variables $R \geq 0$ and $\Theta \in [0, 2\pi]$, so that

   $$X = R\cos\Theta, \quad Y = R\sin\Theta$$

   (a) Show that Θ is uniformly distributed in $[0, 2\pi]$, that R has the PDF

   $$f_R(r) = re^{-r^2/2}, \quad r \geq 0$$

   and that R and Θ are independent. (The random variable R is said to have a Rayleigh distribution.)
   (b) Show that R^2 has an exponential distribution with parameter $1/2$.

   Note: Using the results in this problem, we see that samples of a normal random variable can be generated using samples of independent uniform and exponential random variables.

2. Let X be an exponentially distributed random variable with parameter λ. Let $Y = \lfloor X \rfloor$ be the largest integer $\leq X$, and let $R = X - Y$ be the fractional part of X. Describe the distributions of Y and R, and find the limit of the PDF of R as $\lambda \to 0$.

3. Let $X_r, 1 \leq r \leq n$, be i.i.d. exponential random variables with parameter 1. Show that $Y = \max\{X_r : 1 \leq r \leq n\}$ satisfies

   $$\lim_{n\to\infty} \mathbb{P}(Y - \log(n) \leq x) = \exp(-e^{-x}).$$

4. Calamity Jane goes to the bank to make a deposit, and is equally likely to find 0 or 1 customer ahead of her. The times of service of these customers are independent and exponentially distributed with parameter λ. What are the CDF and PDF of Jane's waiting time?

5. Let X_1, X_2 be two independent random variables each with p.d.f. $f_1(x) = e^{-x}$ for $x > 0$ and $f_1(x) = 0$ for $x \leq 0$. Let $Z = X_1 - X_2$ and $W = X_1/X_2$
   a. Find the joint p.d.f. of X_1 and Z.
   b. Prove that the conditional p.d.f. of X_1 given $Z = 0$ is

   $$g_1(x_1 \mid 0) = \begin{cases} 2e^{-2x_1} & \text{for } x_1 > 0 \\ 0 & \text{otherwise} \end{cases}$$

   c. Find the joint p.d.f. of X_1 and W.

6. Let (X,Y) denote the coordinates of a point, picked at random uniformly in the unit circle in $\mathbb{R}^2$. Define $R^2 = X^2 + Y^2$. If f denotes the joint PDF of R and X, then, $f\left(\frac{\sqrt{3}}{\pi}, \frac{1}{\pi}\right)$ is?