EE2025: Engineering Electromagnetics

Tutorial 1: Transmission Lines

July-Nov 2024

- All transmission lines are lossless unless mentioned otherwise.
- If $x << y$, a good engineering approximation is $10x \leq y$.
- Electrical signals in coaxial cables travel at 2×10^8 m/s.

Motivating and Modeling Transmission Lines

1. Oniond is a high school student who can use only the lumped circuit model to solve circuits. He wants to install a CCTV camera right outside his circular house. The video feed is then transmitted through a coaxial cable at radio frequencies (assume the wavelength of transmission is 0.3 m). He processes this signal in the center of the house. How big can the house be if he doesn't have to learn any new concepts? Can Oniond live in such a house?

Solution:

If the transit time for the signal to reach the center of the house from the CCTV camera is sufficiently small compared to the time period of the signal itself, we can use the lumped model approximation. The wavelength of transmission is 0.3 m which means that the frequency of transmission is 1 GHz. So the time period of the signal is 1 ns. The transit time is given by $t = \frac{l}{v} = \frac{l}{2 \times 10^8}$. Using the engineering approximation,

$$\frac{l}{2 \times 10^8} \times 10 \leq 1 \times 10^{-9} \implies l \leq 2 \text{ cm}$$

The diameter of the circular house can be at most 4 cm. It looks like Oniond needs to learn about distributed elements after all.

2. Observe the following models for an element (of length Δz) of a transmission line.

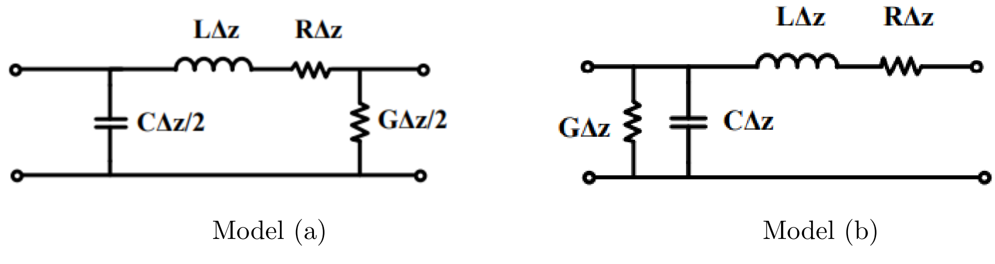


Figure 2: Models for a Transmission Line

Derive the governing equation and the corresponding propogation constant for voltage and current variation across the transmission line in model (a) and model (b). How do the propogation constants differ between model (a) and (b)? Hence find their ratio assuming that all the primary constants remain the same.

Solution: Model (b) is simpler to work with. We'll refer to the $G\Delta z||C\Delta z$ combination as the *shunt* branch and the $R\Delta z + L\Delta z$ series combination as the *main* branch. Let the voltage across the shunt branch be V and the voltage across the main branch be ΔV (this is the incremental voltage in the transmission line for an element). Similarly let the current flowing out of the main branch be $I + \Delta I$. Finally let the current flowing in the shunt branch be ΔI. Applying Ohm's Law on the shunt branch,

$$\Delta I = (G\Delta z + j\omega C\Delta z)V$$

Applying Ohm's Law on the main branch,

$$\Delta V = (R\Delta z + j\omega L\Delta z)(I + \Delta I)$$

In the limiting case,

$$\begin{aligned}\frac{\partial I}{\partial z} &= (G + j\omega C)V \\ \frac{\partial V}{\partial z} &= (R + j\omega L)I \\ \implies \frac{\partial^2 V}{\partial z^2} &= (R + j\omega L)(G + j\omega C)V = \gamma_b^2 V\end{aligned}$$

where γ_b is the propagation constant of the model (b). So $\gamma_b = \sqrt{(R + j\omega L)(G + j\omega C)}$. For model (a) we will assume that the current flowing through the main branch is I_m. Therefore the current flowing through the capactive element is $I_m - I$ and the conductance is $I + \Delta I - I_m$. Applying Ohm's Law on the first shunt:

$$V = \frac{2}{j\omega C\Delta z}(I_m - I)$$

Applying Ohm's Law on the second shunt:

$$I + \Delta I - I_m = (V + \Delta V)\frac{G\Delta z}{2}$$

$$\implies I_m - I = \Delta I - \frac{VG\Delta z}{2} \text{ (Neglecting higher order terms)}$$

$$\implies V = \frac{2}{j\omega C\Delta z}\left(\Delta I - \frac{VG}{2}\Delta z\right)$$

In the limiting case, this becomes

$$V = \frac{2}{j\omega C}\left(\frac{\partial I}{\partial z} - \frac{VG}{2}\right)$$

$$\implies \frac{\partial I}{\partial z} = \frac{(G + j\omega C)}{2}V$$

Applying Ohm's Law on the main branch we get

$$\Delta V = (R\Delta z + j\omega L\Delta z)I_m \approx (R + j\omega L)\Delta z \times I$$

This is because I_m differs from I by a negligible amount. In the limiting case

$$\frac{\partial V}{\partial z} = (R + j\omega L)I$$

$$\implies \frac{\partial^2 V}{\partial z^2} = \frac{(R + j\omega L)(G + j\omega C)}{2}V = \gamma_a^2 V$$

This means that $\gamma_a = \sqrt{\frac{(R + j\omega L)(G + j\omega C)}{2}}$ or $\frac{\gamma_a}{\gamma_b} = \frac{1}{\sqrt{2}}$

Parameters of Transmission Lines

3. A 170 mm long lossless coaxial transmission line is characterized by the primary constants $L = 245$ nH/m and $C = 200$ pF/m. The transmission line is connected to a satellite dish which offers a purely resistive load $Z_L = 100\ \Omega$. The electrical signals are transmitted at 1 GHz.

   (a) Determine the characteristic impedance of the line. Is it purely resistive?

   (b) Determine the impedance looking into the input terminals of the transmission line. Think about why this impedance isn't purely resistive even though both Z_0 and Z_L are.

   (c) Determine the Reflection coefficient and the VSWR.

   (d) I am an engineer who can modify the primary constants of my transmission line. What should I do to optimize performance? Calculate optimal L and C values.

Solution:

(a) Since the line is lossless, $R = 0$, $G = 0$ and $Z_0 = \sqrt{\frac{L}{C}} = 35\ \Omega$. Yes, it is purely resistive (lossless line).

(b)

$$\beta = \omega\sqrt{LC} = 43.98 rad/m$$

Input impedance

$$Z_i = Z_0 \frac{Z_L \cos(\beta l) + jZ_0 \sin(\beta l)}{Z_0 \cos(\beta l) + jZ_L \sin(\beta l)} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)} = 13.9 - j12.1\Omega$$

The impedances Z_0 and Z_L are not equal despite being real. This introduces a phase shift at the boundary which is captured by the quadrature component of the look-in impedance.

(c) Reflection coefficient

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = 0.48$$

VSWR

$$\rho = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} = 2.84$$

(d) The lower my ρ, the better my transmission line. Ideally I want to match my load to the impedance of the line. This reduces the reflection at the interface to 0 and I ensure that all the power I input is delivered to the load. If $L = 200$ nH/m and $C = 20$ pF/m, then my transmission line is optimal.

4. A transmission line operating at 500 MHz has $Z_0 = 80\ \Omega$, $\alpha = 0.04$ Np/m, $\beta = 1.5$ rad/m. Find the line parameters R, L, G and C.

Solution: We know that the characteristic impedance of the line

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

and the propogation constant is

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$\implies Z_0\gamma = 3.2 + 120j = R + j\omega L \implies R = 3.2\Omega/m \text{ and } \omega L = 120$$

$$\implies \frac{\gamma}{Z_0} = 5 \times 10^{-4} + 0.01875j = G + j\omega C \implies G = 0.5 \text{ m℧}/m \text{ and } \omega C = 0.01875$$

Given $\omega = 2\pi \times 500 \times 10^6 \implies L = 38$ nH/m and $C = 5.96$ pF/m.

5. The transmission line shown in the below figure is a two-wire line lead-in from an antenna to a television receiver, where the input impedance offered to the voltage source is 300 Ω

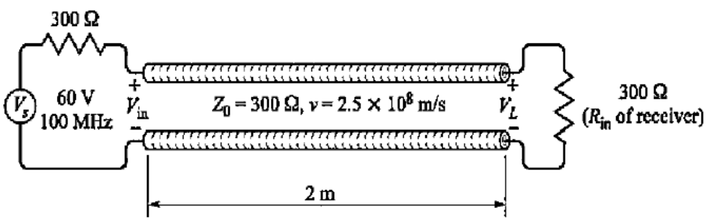


Figure 3: Transmission Line

(a) Find the phase difference between the voltage at the source (V_{in}) and the voltage at the load (V_L).

(b) Find the reflection coefficient and voltage standing wave ratio on the line.

(c) Suppose a second receiver, also having an impedance of 300 Ω, is connected across the line in parallel with the first receiver. Find the new reflection coefficient and the voltage standing wave ratio on the line.

(d) Calculate the impedance seen by the AC voltage source before and after the introduction of the second reciever.

Solution:

(a) Phase difference $= \beta l = \frac{2\pi f}{v} l = 5\ rad$

(b) $\Gamma_L = 0, \rho = 1$ (characteristic impedance of the line = load impedance)

(c) After connecting the second receiver, load impedance Z_L= 150 Ω (parallel combination of two receivers). Then

$$\Gamma_L = -\frac{1}{3}, \rho = 2$$

(d) Before the introduction of the second reciever, the impedance seen at V_{in} is 300 Ω. Therefore the AC source sees 600 Ω. Impedance after the introduction of the second receiver seen at V_{in} is

$$Z_L^{`} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)} = 465 - j205\Omega$$

Therefore the AC source sees $765 - j205\Omega$. We're simply adding the resistances in series here because once we find the input impedance at V_{in}, we can use the lumped circuit model.

6. A transmission line of characteristic impedance 600 Ω is terminated by a reactance of $j150$ Ω. Find the input impedance of a section 25 cm long at a frequency of 300 MHz. Use the velocity of the wave as 3×10^8 m/s. Is this impedance capacitive or inductive? Can I change the characteristic impedance of my trasmission line to change that?

Solution: Given $Z_o = 600\ \Omega$, $Z_L = j150\ \Omega$
$\lambda = \frac{300}{300} m = 1m$
$\beta l = \frac{2\pi}{\lambda} l = \frac{\pi}{2}$
The input impedance of a purely reactive circuit is given by

$$Z_{in} = Z_0 \left[\frac{Z_L cosh(\pi/2) + Z_0 sinh(\pi/2)}{Z_0 cosh(\pi/2) + Z_L sinh(\pi/2)} \right] = \frac{Z_0^2}{Z_L} = -j2400\Omega$$

The impedance is purely capacitive. No matter how I change the characteristic impedance Z_0, I cannot change the capacitive nature of input impedance. This shows that at special lengths of the transmission lines I can transform my load impedance independent of the transmission line parameters. This forms the basis of impedance transformers.

Voltages and Currents in Transmission Lines

7. A 300 Ω transmission line is 0.8 m long and terminated with a short circuit. The line operates in the air with a wavelength of 0.3 m and is lossless.

(a) If the input voltage amplitude is 10 V, what is the maximum value of voltage amplitude on the line?

(b) What is the current amplitude in the short circuit?

Solution:

(a) The net voltage on any point on the line is the sum of forward and backward wave voltage.

$$V(l) = V^+ e^{j\beta l} + V^- e^{-j\beta l}$$

Since the line is short-circuited, $\Gamma_L = -1$ and $V^- = -V^+$.

$$V(l) = j2V^+ \sin(\beta l)$$

Maximum value of voltage amplitude $= 2|V^+|$
At the input end ($l = 0.8\ m$)

$$V_l = j2V^+ \sin\left(\frac{2\pi}{0.3} \times 0.8\right) = -j1.73V^+$$

Given that the input voltage amplitude is 10 V, thus$|V^+| = 5.78\ V$ and so the maximum value of voltage is 11.56 V

(b) Current at the load end

$$I(0) = \frac{V^+}{Z_0} - \frac{V^-}{Z_0} = \frac{2V^+}{Z_0}$$

Current amplitude in the short circuit (load end) is $= \frac{2|V^+|}{Z_0} = 38.5\ mA$

8. A 50 Ω transmission line is terminated to a load with an unknown impedance. The voltage standing wave ratio on the line is $\rho = 2.4$ and a voltage maximum occurs at a distance $\lambda/8$ from the load.

   (a) Determine the load impedance.

   (b) How far is the first minimum from the load (write the answer as a multiple of λ).

Solution:

(a) $|\Gamma_L| = \frac{\rho-1}{\rho+1} = 0.41$

$$V(l) = V^+ e^{j\beta l}\left[1 + |\Gamma_L| e^{j(\Phi_L - 2\beta l)}\right]$$

For voltage maximum $\Phi_L - 2\beta l = 2m\pi$ and for the first maximum, $m = 0$ and $\Phi_L = 2\beta l = \frac{\pi}{2}\ rad$

$$\Gamma_L = |\Gamma_L| e^{j\Phi_L} = j0.41$$

$$Z_L = Z_0 \frac{1 + \Gamma_L}{1 - \Gamma_L} = 35.5 - j35$$

(b) The second voltage maxima occurs at a distance of $\lambda/2$ from the first maxima. The first minimum occurs in between at a distance $\lambda/4$ from the first maximum. Position of first minimum $= \lambda/8 + \lambda/4 = 0.375\ \lambda$

9. On a lossless 50 Ω transmission line, measurements at the load gave VSWR = 5. The results of three voltage measurements made on the line are as follows:

   (a) The first maximum from the load is found at a distance of 0.25 m from the load.

   (b) The next maximum is found at 0.75 m from the load.

   (c) $V_L = 100\ V$.

   Calculate Z_L, V^+, V^-,I^+,I^-

Solution:

$$|\Gamma_L| = \frac{\rho - 1}{\rho + 1} = 2/3$$

The distance between the first and second maximum $= \lambda/2 = 0.5\ m$, $\lambda = 1\ m$

$$V(l) = V^{+}e^{j\beta l}\left[1 + |\Gamma_L|e^{j(\Phi_L - 2\beta l)}\right]$$

For voltage maximum $\Phi_L - 2\beta l = 2m\pi$ and for the first maximum, $m = 0$ and $\Phi_L = 2\beta l = \pi\ rad$

$$\Gamma_L = |\Gamma_L|e^{j\Phi_L} = -2/3$$

$$Z_L = Z_0\frac{1+\Gamma_L}{1-\Gamma_L} = 10\Omega$$

$$V_L = V(0) = V^{+}[1+\Gamma_L] = 100V$$

$V^{+} = 300\ V$, $V^{-} = \Gamma_L V^{+} = -200\ V$, $I^{+} = \frac{V^{+}}{Z_0} = 6\ A$, $I^{-} = \frac{V^{-}}{Z_0} = -4\ A$

10. The Γ_L on a 500 m long transmission line has a phase angle of 210^0. If the operating wavelength is 150m, what would be the number of voltage maxima formed on the line?

Solution: Maxima to maxima gap $=\frac{\lambda}{2} = 75$ m
we know (From the solution of Question 5)

$$\phi - 2\beta x_{max} = 2m\pi$$

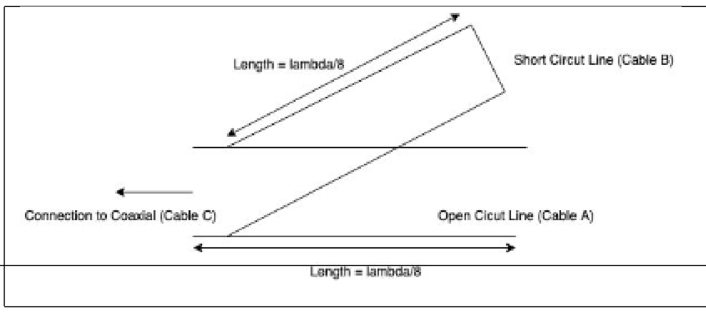
for m=0, we get $x_{max} = 43.75$ m
$\therefore$ first maxima will be formed at a distance of 43.75m

As periodicity of maximas is $\frac{\lambda}{2} = 75$ m

$\therefore$ successive maxima will be formed at a distance of 75 m from each other
$\implies$ the Total number of voltage maxima formed on a 500 m long line would be 43.75+n75<500
on solving n=6, thus a total of 7 voltage maximas (including 1 at 43.75 m) will be formed on the given transmission line

11. You have two pieces of coaxial cable both of length $\lambda/8$ (call them cables A and B) where λ is the operating wavelength. One of them is short-circuited and the other is left open. Both of them are connected in parallel at the end of yet another coaxial line (call it cable C). What will be the VSWR seen on the cable C? Show the configuration using a diagram.

Solution:



The diagram is shown in above figure. Since cable A is open-circuited and cable B is short-circuited, the input impedance is $Z_A = -jZ_0 \cot\beta l$, $Z_B = jZ_0 \tan\beta l$. For the given length ($l = \lambda/8$), $Z_A = -Z_B$. Thus the load impedance $Z_L = \infty$, $\Gamma_L = 1$ and VSWR $= \infty$.

Lossy Transmission Lines

12. A 50 Ω low-loss transmission line has a loss of 1.5 dB/m. The velocity of the voltage wave is 2×10^8 m/sec. A parallel resonant circuit at 2 GHz is to be designed with the section of the line.

   (a) Find the input impedance of the line of the above design.

   (b) Find the quality factor and 3 dB bandwidth.

Solution:

(a) The wavelength of the line is

$$\lambda = \frac{v}{f} = \frac{2 \times 10^8}{2 \times 10^9} = 0.1 \text{ m}$$

The phase constant is

$$\beta = \frac{2\pi}{\lambda} = 20\pi \text{ rad/m}$$

$$\alpha = 1.5 \text{ dB/m} \Rightarrow 1.5/8.68 = 0.173 \text{ neper/m}$$

In the parallel resonance; the length of the section for

(i) for short circuit,

$$l_{sc} = \frac{\lambda}{4}, \frac{3\lambda}{4}, ... = 0.025,\ 0.075,\ ... \text{ m}$$

The input impedance is given by;

$$Z_{in} = \frac{Z_0}{\alpha l_{sc}} = 11.57 \text{ k}\Omega \quad (\text{for } l_{sc} = \ 0.025 \text{ m})$$

(ii) for open circuit;

$$l_{oc} = \frac{\lambda}{2}, \frac{2\lambda}{2}, ... = 0.05,\ 0.1,\ ... \text{ m}$$

The input impedance is given by;

$$Z_{in} = \frac{Z_0}{\alpha l_{oc}} = 5.79 \text{ k}\Omega \quad (\text{for } l_{oc} = \ 0.05 \text{ m})$$

(b) The quality factor is given by

$$Q = \frac{\beta}{2\alpha} = \frac{20\pi}{2 \times 0.173} = 181.8$$

The 3-dB bandwidth is given by;

$$BW = \frac{f_0}{Q} \approx 11 \text{ MHz}$$

13. An RG-59U coaxial cable has a (power) loss of 10 dB per 100 m of length. A 10 V - 3 A (both peak-to-peak values) signal is generated using a function generator and connected to one end of 50 meters of the above cable. On the other side, the cable is impedance matched to a set top box unit. Find the power delivered to the load. If the cable was not impedance matched can you solve the problem?

Solution:

The voltage at source end, $V_s = 10$ V and the current at source end, I_s =3 A. The load end of the coaxial cable is impedance matched. Loss coefficient of the coaxial cable = 10 dB per 100 m. Total loss in 50 m long coaxial cable = $10\left(\frac{50}{100}\right) = 5$ dB.

Voltage at load end: $-20\log\left(\frac{V_l}{V_s}\right) = 5 \Rightarrow V_l = 10 \times 10^{-\frac{5}{20}} = 5.62$ V.

Current at load end: $-20\log\left(\frac{I_l}{I_s}\right) = 5 \Rightarrow I_l = 3 \times 10^{-\frac{5}{20}} = 1.69$ A.

Power delivered to the load $=\frac{1}{2}$ $5.62 \times 1.69 = 4.75$ W. If it was not impedance matched some power is reflected back at the interface of the transmission line end and the load. We need to know the propogation constant and the load impedance to calculate the power delivered.

Impedance Matching

14. Two lossless two-wire lines of length d and d_1 are added together as shown in Fig. 4 for impedance matching at $\lambda = 100$ cm. If the characteristic impedance of both the lines are $Z_0 = 200\Omega$, find the possible values of d and d_1 to provide a matched load. (Note that the un-shaded and shaded conductor are both parts of the same transmission line, for example they can be the inner and outer conductor of a coaxial cable.)

Solution:

The equivalent circuit is as in Fig 5 Z_1 = impedance of line shorted = jZ_0tan(βd_1)
Z_2 = input impedance of line terminated in $100\Omega = Z_0(\frac{Z_L+jZ_0tan(\beta d)}{Z_0+jZ_Ltan(\beta d)})$

Thus Z_0 = jZ_0tan(βd_1) + $Z_0(\frac{Z_L+jZ_0tan(\beta d)}{Z_0+jZ_Ltan(\beta d)})$

Equating real and imaginary parts of both sides:

$1 = \frac{Z_LZ_0(1+tan^2(\beta d))}{Z_0^2+Z_L^2tan^2(\beta d)}$

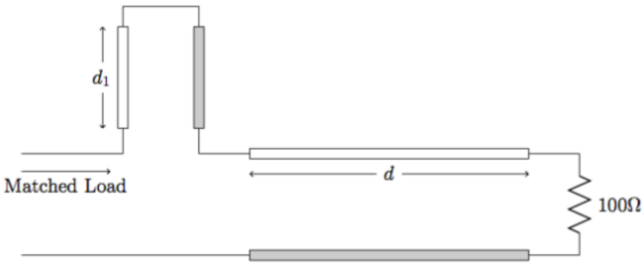


Figure 4

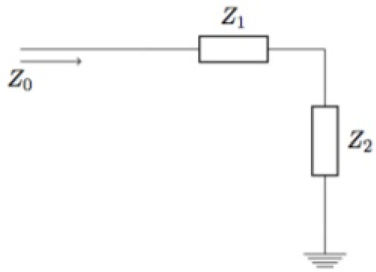


Figure 5

$\tan(\beta d_1) = \frac{(Z_L^2 - Z_0^2)tan(\beta d)}{Z_0^2 + Z_L^2 tan^2(\beta d)}$

$\tan(\beta d) = \pm\sqrt{\frac{Z_0}{Z_L}} = \pm\sqrt{2}$

d = 15.2 cm

$\tan(\beta d_1) = \mp(\sqrt{\frac{Z_0}{Z_L}} - \sqrt{\frac{Z_L}{Z_0}}) = \mp(\sqrt{2} - \sqrt{\frac{1}{2}})$

d_1 = 9.8 cm

15. Given the system in (Fig. 6) is operating with $\lambda = 100cm$ and $Z_0 = 300\Omega$. If $d_1 = 10cm$, $d = 25cm$, and the system is matched to 300 Ω, find Z_L?

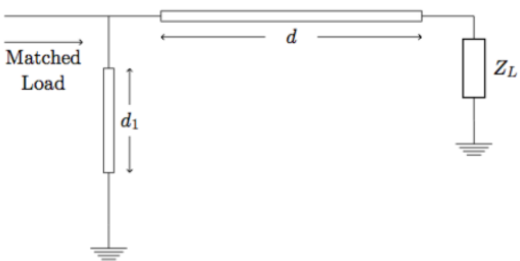
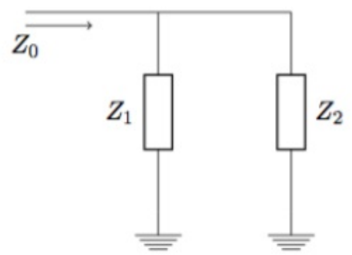


Figure 6

Solution: The equivalent circuit is



Z_1 = impedance of the stub = impedance of a 10cm line shorted = $jZ_0\tan(\beta d_1)$ = j217.963
$Z_0 = Z_1||Z_2 = \frac{Z_1 Z_2}{Z_1+Z_2}$ and hence $Z_2 = \frac{Z_1 Z_0}{Z_1 - Z_0}$ = 103.64-j142.658Ω
But Z_2 (input impedance of line terminated in Z_L) $= Z_0(\frac{Z_L + jZ_0 tan(\beta d)}{Z_0 + jZ_L tan(\beta d)})$
Thus, $Z_L = Z_0(\frac{Z_2 - jZ_0 tan(\beta d)}{Z_0 - jZ_2 tan(\beta d)})$ = 300 + j412.91Ω

Power Transfer

16. According to the maximum power transfer theorem, maximum time averaged power is transferred from a source with internal impedance Z_g to a load, Z_L when $Z_g = Z_L^*$. A 50 MHz generator with an internal impedance (Z_g) of 50 Ω is connected to an impedance 50−j25 Ω. How would you ensure maximum power transfer in this case using a lossless transmission line of characteristic impedance 100 Ω, and what should be the minimum length of the transmission line element ? Assume $v = 2 \times 10^8$m/s as wave velocity.

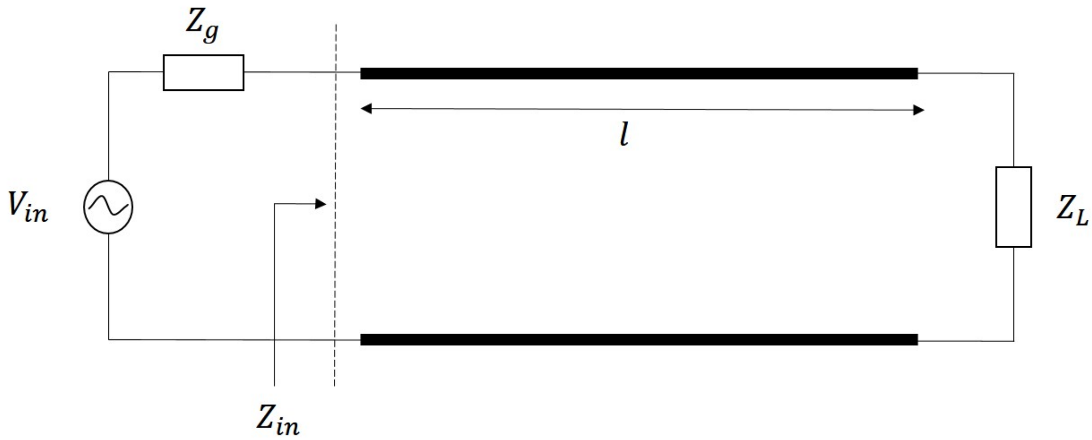


Figure 7: Impedance matching using a transmission line of length l

Solution:

Impedance matching can be achieved by inserting a transmission line of length l such that the impedance seen at the input end, Z_{in} is equal to Z_g as shown in Figure. 7

i.e.

$$Z_{in} = Z_o \frac{Z_L + jZ_o tan(\beta l)}{Z_o + jZ_L tan(\beta l)} = 50\,\Omega$$

Given, Z_o = 100 Ω, Z_L = 50−j25 Ω

$$\frac{Z_{in}}{Z_o} = \frac{Z_L + jZ_o tan(\beta l)}{Z_o + jZ_L tan(\beta l)} = \frac{50}{100} = \frac{1}{2}$$

$$2Z_L + 2jZ_o tan(\beta l) = Z_o + jZ_L tan(\beta l)$$

$$100 - j50 + 200j tan(\beta l) = 100 + j(50 - j25) tan(\beta l)$$

$$-j50 + j150 tan(\beta l) = 25 tan(\beta l)$$

Equating real parts, we get

$tan(\beta l) = 0$, Or $\beta l = n\pi \Rightarrow l = \lambda\ /2$; since $\beta = 2\pi/\lambda$

Equating imaginary parts, we get,

$tan(\beta l) = 1/3 \Rightarrow l = 0.051\lambda$

Here, both the solutions should be compatible. i.e., the length of the transmission line should be such that both conditions are satisfied. Since it is not satisfying, we can not implement impedance matching in this method.

17. In the above question, is there some other method that can be employed for ensuring maximum power transfer?

Solution:

Another method is to connect a short circuited stub in between the source and load as shown in Figure. 8. Here,

$$Z_{stub} = Z_{sc} = jZ_o tan(\beta l)$$

This Z_{stub} is in series with Z_L. Now the input impedance as seen by the source is the sum of the stub and load impedance. i.e.

$Z_{in} = Z_{stub} + Z_L$

For impedance matching, $Z_{in} = Z_{stub} + Z_L = 50\ \Omega$

$jZ_o tan(\beta l)$ + 50 - j25 = 50

$jZ_o tan(\beta l) = j25 \Rightarrow 100 tan(\beta l) = 25$

$\beta l = \tan^{-1}(1/4) \Rightarrow \beta l = 0.245$

$l = 0.04\lambda$

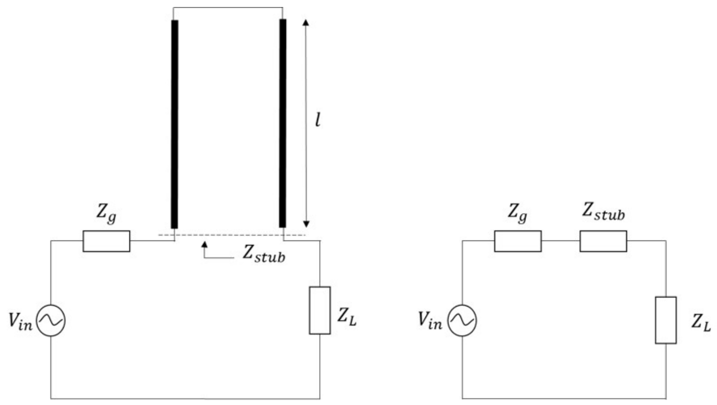


Figure 8: Impedance matching using a short circuited stub of length l and its equivalent circuit

18. Calculate the average power dissipated by each resistor in the circuit shown in Fig. 11.

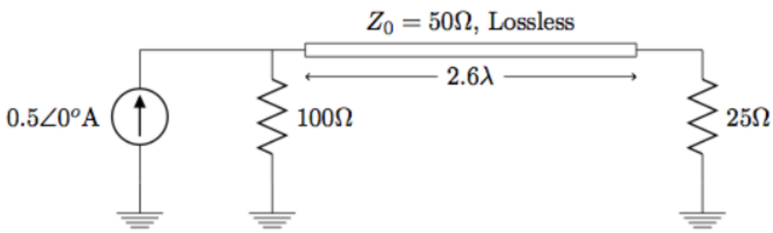
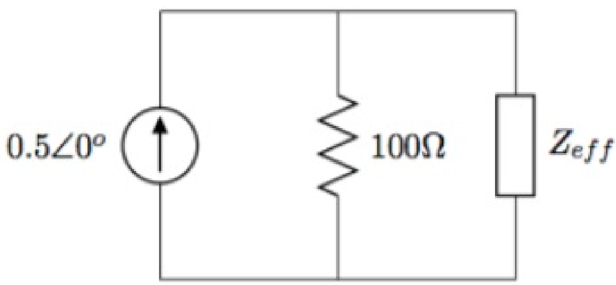


Figure 9

Solution: The circuit is equivalent to



Z_{eff} is basically the input impedance of the transmission line terminated in 25Ω, which can be written as
$= 50(\frac{25+j50tan(360\times 2.6)}{50+j25tan(360\times 2.6)}) = 33.74+j24.06\Omega$
The current through the 100Ω resistor $= I_1 = 0.5(\frac{33.74+j24.06}{100+33.74+j24.06}) = 0.137+j0.065 A$
Thus average power dissipated in $100\Omega = P_{100} = \frac{1}{2}|I_1|^2$ x 100 =1.15 W
The voltage across the current source = voltage across $100\Omega = I_1$ x 100 = 13.7+j6.5 V
Average Power dissipated in source = $P_s = \frac{1}{2}\ Re[V_s I_s^*]$ = 9.27x0.5 =3.42 W
Thus power dissipated in $Z_{eff} = P_s$ - P_{100} = 2.27 W

19. For a 50-Ω lossless transmission line terminated in a load impedance Z_L= (100+ j50) Ω,

determine the fraction of the average incident power reflected by the load. What is the magnitude of the average reflected power if $|V_0^+| = 1$ V (peak)?

Solution:

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{(100 + j50) - 50}{(100 + j50) + 50} = \frac{50 + j50}{150 + j50} = (0.4067 + j0.2446).$$

Fraction of reflected power,

$$|\Gamma|^2 = (0.45)^2 = 20\%.$$

$$P_a^r v = |\Gamma|^2 \frac{|V_0^+|^2}{2Z_0} = \frac{0.21}{250} = 2mW$$

Quarter Wavelength Transformer

20. The 5G testbed of IIT Madras is developing hardware for the next generation of Indian cellular systems that operates at a center frequency of 6GHz. The antennas they develop have an input impedance of 120 ohms, and the source has an internal impedance of 50 ohms. To test out the system, the source and antennas are kept in the lab, and a quarter wave transformer is used to match the source and load. What should be the optimal impedance of the quarter wave line and what is its minimum length? Assume velocity $v = 2.5 \times 10^8$ m/s wherever necessary.

Solution: Given, $f = 6\text{GHz}, Z_g = 50\Omega, Z_L = 120\Omega$. A quarter wave transformer is used to match the source and load. This means the minimum length of the TL is $l = \lambda/4$ and therefore $\tan(\beta l) = \infty$. Let the characteristic impedance of the quarter wave line be Z_0. For matching the source and the load, we need the transformed load impedance $Z(\lambda/4) = Z_g^* = 50\Omega$.

$$Z(\lambda/4) = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)} = \frac{Z_0^2}{Z_L} = Z_g$$

Therefore, the characteristic impedance of the quarter wave line is
$Z_0 = \sqrt{Z_g Z_L} = \sqrt{120 \times 50} \approx 77.45\Omega$
Let us now find the operating wavelength.
$\lambda = v/f = \frac{2.5\times10^8}{6\times10^9} = \frac{1}{24}$ m. Therefore, length of the line is $l = \lambda/4 = \frac{1}{96}$ m $= \frac{25}{24}$ cm $\approx$ 1.04 cm.