

Strang Method to solve RTE

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1 Introduction

We present a numerical method for solving the radiative transfer equation (RTE) in scattering media. The approach leverages operator splitting to separate the RTE into two physically distinct components: streaming along characteristics and local collision interactions. We employ the Strang splitting scheme for time integration, which provides second-order accuracy while maintaining computational efficiency through exact solutions of the split operators.

2 General idea of splitting

Consider the abstract evolution equation

$$\frac{\partial u}{\partial t} = (A + B)u, \quad (1)$$

where A and B are two operators that in general do not commute.

2.1 Problem with the exponential of a sum

The formal solution of (1) is

$$u(t) = e^{t(A+B)}u(0).$$

However, computing the exponential of a sum is nontrivial if A and B do not commute. The Baker–Campbell–Hausdorff (BCH) formula shows

$$e^{tA}e^{tB} = e^{t(A+B) + \frac{t^2}{2}[A,B] + \mathcal{O}(t^3)}.$$

Noncommutativity produces extra commutator terms and reduces accuracy.

2.2 Strang splitting

To achieve second-order accuracy one uses the symmetric Strang splitting [1]:

$$e^{t(A+B)} = e^{\frac{t}{2}A} e^{tB} e^{\frac{t}{2}A} + \mathcal{O}(t^3). \quad (2)$$

This formula employs only simple exponentials e^{tA} and e^{tB} , yet guarantees a global error of order $\mathcal{O}(t^2)$.

So, the solution can be found in a series of small steps Δt in time:

$$u(t + \Delta t) = e^{\frac{\Delta t}{2}A} e^{\Delta t B} e^{\frac{\Delta t}{2}A} u(t).$$

3 Radiative transfer equation

We consider the specific intensity

$$I(\mathbf{r}, \hat{\mathbf{s}}, t), \quad \mathbf{r} = (x, y, z), \quad \hat{\mathbf{s}} \in S^2.$$

3.1 Formulation

The radiative transfer equation reads

$$\frac{1}{c} \frac{\partial I}{\partial t} + \hat{\mathbf{s}} \cdot \nabla I = -\mu_t I + \mu_s \int_{4\pi} p(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}') I(\mathbf{r}, \hat{\mathbf{s}}', t) d\Omega' + \eta(\mathbf{r}, \hat{\mathbf{s}}, t), \quad (3)$$

with $\mu_t = \mu_a + \mu_s$.

The phase function is Henyey–Greenstein:

$$p(\cos \theta) = \frac{1}{4\pi} \frac{1 - g^2}{(1 + g^2 - 2g \cos \theta)^{3/2}}, \quad g \in (-1, 1). \quad (4)$$

3.2 Operators

Define

$$\begin{aligned} \mathcal{L}I &= -\hat{\mathbf{s}} \cdot \nabla I, & (\text{streaming}) \\ \mathcal{C}I &= -\mu_t I + \mu_s \int_{4\pi} p(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}') I(\hat{\mathbf{s}}') d\Omega' + \eta. & (\text{collision}) \end{aligned}$$

4 Splitting idea for RTE

We apply Strang splitting (2) to the decomposition

$$\frac{1}{c} \frac{\partial I}{\partial t} = \mathcal{L}I + \mathcal{C}I,$$

with \mathcal{L} and \mathcal{C} from Section 3.2.

4.1 Action of the collision operator

Expand intensity at a fixed spatial cell in spherical harmonics:

$$I(\mathbf{r}, \hat{\mathbf{s}}, t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}(\mathbf{r}, t) Y_{\ell m}(\hat{\mathbf{s}}). \quad (5)$$

Project the source in the same way:

$$\eta(\mathbf{r}, \hat{\mathbf{s}}, t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} b_{\ell m}(\mathbf{r}, t) Y_{\ell m}(\hat{\mathbf{s}}). \quad (6)$$

For the Henyey–Greenstein phase function, the scattering operator acts diagonally in this basis:

$$\int_{4\pi} p(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}') Y_{\ell m}(\hat{\mathbf{s}}') d\Omega' = \chi_{\ell} Y_{\ell m}(\hat{\mathbf{s}}), \quad \chi_{\ell} = g^{\ell}.$$

Hence the coefficients evolve as

$$\frac{da_{\ell m}}{dt} = c \left[-\mu_t a_{\ell m} + \mu_s \chi_{\ell} a_{\ell m} + b_{\ell m} \right]. \quad (7)$$

Each (ℓ, m) mode is independent, so the half collision step $\exp(\frac{\Delta t}{2} \mathcal{C})$ is simply a set of scalar ODE solves, given in closed form in Section 5.

Thus the exponential operator acts modewise:

$$(e^{\frac{c\Delta t}{2} \mathcal{C}} I)(\hat{\mathbf{s}}) = \sum_{\ell, m} \left[e^{\lambda_{\ell} \Delta t/2} a_{\ell m}(\mathbf{r}, t) + \frac{e^{\lambda_{\ell} \Delta t/2} - 1}{\lambda_{\ell}} c b_{\ell m}(\mathbf{r}, t) \right] Y_{\ell m}(\hat{\mathbf{s}}).$$

4.2 Action of the streaming operator

The streaming operator translates intensity along rays:

$$\frac{\partial}{\partial t} I(\mathbf{r}, \hat{\mathbf{s}}, t) = -c \hat{\mathbf{s}} \cdot \nabla I(\mathbf{r}, \hat{\mathbf{s}}, t). \quad (8)$$

This has the exact solution

$$I(\mathbf{r}, \hat{\mathbf{s}}, t + \Delta t) = I(\mathbf{r} - c\Delta t \hat{\mathbf{s}}, \hat{\mathbf{s}}, t). \quad (9)$$

Thus $\exp(\Delta t \mathcal{L})$ is a pure shift operator in space, leaving angular coefficients unchanged.

4.3 Combined Strang step

To integrate (3) we apply Strang splitting (2):

$$I(\mathbf{r}, \hat{\mathbf{s}}, t_{n+1}) = e^{\frac{c\Delta t}{2}\mathcal{C}} e^{c\Delta t \mathcal{L}} e^{\frac{c\Delta t}{2}\mathcal{C}} I(\mathbf{r}, \hat{\mathbf{s}}, t_n).$$

Collision steps act diagonally on spherical harmonic modes $a_{\ell m}$ via (7), while the streaming step shifts each angular node according to (9). This clear separation of actions is what makes the method efficient.

4.4 Explicit form of the two substeps

The Strang update consists of two distinct operator actions:

Collision half step. At fixed spatial position \mathbf{r} , expand intensity into spherical harmonics with coefficients $a_{\ell m}(t)$. From (7) each mode satisfies

$$\frac{da_{\ell m}}{dt} = \lambda_{\ell} a_{\ell m} + c b_{\ell m}, \quad \lambda_{\ell} = c(-\mu_t + \mu_s g^{\ell}).$$

This linear ODE has the exact solution over time τ :

$$a_{\ell m}(t + \tau) = e^{\lambda_{\ell} \tau} a_{\ell m}(t) + \frac{e^{\lambda_{\ell} \tau} - 1}{\lambda_{\ell}} c b_{\ell m},$$

(with the limit τ if $\lambda_{\ell} = 0$). Thus

$$(e^{\frac{c\Delta t}{2}\mathcal{C}} I)(\hat{\mathbf{s}}) = \sum_{\ell, m} \left[e^{\lambda_{\ell} \Delta t/2} a_{\ell m}(t) + \frac{e^{\lambda_{\ell} \Delta t/2} - 1}{\lambda_{\ell}} c b_{\ell m} \right] Y_{\ell m}(\hat{\mathbf{s}}).$$

Streaming full step. The operator \mathcal{L} generates pure translations along rays. Its exact action over Δt is

$$(e^{c\Delta t \mathcal{L}} I)(\mathbf{r}, \hat{\mathbf{s}}) = I(\mathbf{r} - c\Delta t \hat{\mathbf{s}}, \hat{\mathbf{s}}, t).$$

That is, each angular component remains unchanged, while the spatial distribution is shifted backwards along direction $\hat{\mathbf{s}}$ by distance $c\Delta t$.

In summary, the Strang step alternates between an *angular update* (diagonal in spherical harmonics) and a *spatial shift* (exact translation along characteristics).

5 Numerical scheme

5.1 Discretization

Let for given time t

$$I[q, k, j, i] \approx I(x_i, y_j, z_k, s_q, t),$$

where s_q is a direction labelled by q .

5.2 Collision step (HG via spherical harmonics)

Let $\{(\hat{\mathbf{s}}_q, w_q)\}_{q=1}^{N_\Omega}$ be the angular quadrature. At each spatial cell (i, j, k) we project the nodal intensities onto spherical harmonics up to L_{\max} :

$$a_{\ell m}[k, j, i] = \sum_{q=1}^{N_\Omega} w_q I[q, k, j, i] Y_{\ell m}^*(\hat{\mathbf{s}}_q), \quad 0 \leq \ell \leq L_{\max}, \quad |m| \leq \ell. \quad (10)$$

For Henyey–Greenstein, the scattering operator is diagonal in (ℓ, m) with eigenvalues $\chi_\ell = g^\ell$. Define

$$\lambda_\ell = c(-\mu_t + \mu_s \chi_\ell), \quad b_{\ell m}[k, j, i] = c \sum_{q=1}^{N_\Omega} w_q \eta[q, k, j, i] Y_{\ell m}^*(\hat{\mathbf{s}}_q). \quad (11)$$

A half collision step over $\Delta t/2$ is updated *exactly* modewise by

$$a_{\ell m}^* = e^{\lambda_\ell \Delta t/2} a_{\ell m} + \phi(\lambda_\ell, \frac{\Delta t}{2}) b_{\ell m}, \quad \phi(\lambda, \tau) = \begin{cases} \frac{e^{\lambda \tau} - 1}{\lambda}, & \lambda \neq 0, \\ \tau, & \lambda = 0. \end{cases} \quad (12)$$

Transform back to nodal angles:

$$I^*[q, k, j, i] = \sum_{\ell=0}^{L_{\max}} \sum_{m=-\ell}^{\ell} a_{\ell m}^*[k, j, i] Y_{\ell m}(\hat{\mathbf{s}}_q). \quad (13)$$

5.3 Streaming step (semi-Lagrangian)

The streaming update $I \mapsto e^{c\Delta t \mathcal{L}} I$ is a shift along characteristics. For each (q, i, j, k) let the foot point be

$$\mathbf{r}_{i,j,k,q}^{\text{foot}} = (x_i, y_j, z_k) - c\Delta t \hat{\mathbf{s}}_q.$$

Then

$$I^{**}[q, k, j, i] = \text{Interp} \left(I^*[q, \cdot, \cdot, \cdot], \mathbf{r}_{i,j,k,q}^{\text{foot}} \right), \quad (14)$$

where Interp denotes spatial interpolation on the (x, y, z) grid (linear/monotone cubic/WENO as desired). Inflow boundary conditions are applied whenever \mathbf{r}^{foot} exits the domain.

5.4 Full algorithm

Given $I^n[q, k, j, i] = I(\mathbf{r}_{i,j,k}, \hat{\mathbf{s}}_q, t_n)$:

1. **Half collision:** apply (10)–(13) with $\Delta t/2$ to obtain I^* .
2. **Full streaming:** apply (14) with Δt to obtain I^{**} .
3. **Half collision:** repeat (10)–(13) on I^{**} (using η at t_{n+1} or a centered value) to obtain I^{n+1} .

6 A pseudocode

Algorithm 1 Strang-split RTE step (HG scattering): from $I^n[q, k, j, i]$ to $I^{n+1}[q, k, j, i]$

Require: Arrays $\mu_t[k, j, i]$, $\mu_s[k, j, i]$, source $\eta[q, k, j, i]$; directions $\{\hat{\mathbf{s}}_q, w_q\}_{q=1}^{N_\Omega}$; $\Delta t, c, L_{\max}$

```

1: Precompute  $\chi_\ell = g^\ell$  for  $\ell = 0, \dots, L_{\max}$ 

2: procedure HALFCOLLISION( $I$ )
3:   for all  $(k, j, i)$  do ▷ project to SH
4:     for  $\ell = 0$  to  $L_{\max}$  do
5:       for  $m = -\ell$  to  $\ell$  do
6:          $a_{\ell m} \leftarrow \sum_{q=1}^{N_\Omega} w_q I[q, k, j, i] Y_{\ell m}^*(\hat{\mathbf{s}}_q)$ 
7:          $b_{\ell m} \leftarrow \sum_{q=1}^{N_\Omega} w_q \eta[q, k, j, i] Y_{\ell m}^*(\hat{\mathbf{s}}_q)$ 
8:       end for
9:     end for
10:    for  $\ell = 0$  to  $L_{\max}$  do
11:      for  $m = -\ell$  to  $\ell$  do ▷ analytic mode update
12:         $\lambda_\ell \leftarrow c(-\mu_t[k, j, i] + \mu_s[k, j, i] \chi_\ell)$ 
13:         $\tau \leftarrow \Delta t/2$ 
14:         $\phi \leftarrow \begin{cases} (e^{\lambda_\ell \tau} - 1)/\lambda_\ell, & \lambda_\ell \neq 0 \\ \tau, & \lambda_\ell = 0 \end{cases}$ 
15:         $a_{\ell m}^* \leftarrow e^{\lambda_\ell \tau} a_{\ell m} + c \phi b_{\ell m}$ 
16:      end for
17:    end for
18:    for  $q = 1$  to  $N_\Omega$  do ▷ reconstruct to nodal angles
19:       $I^*[q, k, j, i] \leftarrow \sum_{\ell=0}^{L_{\max}} \sum_{m=-\ell}^{\ell} a_{\ell m}^* Y_{\ell m}(\hat{\mathbf{s}}_q)$ 
20:    end for
21:  end for
22:  return  $I^*$ 
23: end procedure

24: procedure STREAM( $I^*$ )
25:   for  $q = 1$  to  $N_\Omega$  do
26:     for all  $(k, j, i)$  do ▷ semi-Lagrangian backtrace
27:        $\mathbf{r}_{\text{foot}} \leftarrow (x_i, y_j, z_k) - c\Delta t \hat{\mathbf{s}}_q$ 
28:        $I^{**}[q, k, j, i] \leftarrow \text{Interp}(I^*[q, \cdot, \cdot, \cdot], \mathbf{r}_{\text{foot}})$ 
29:       if  $\mathbf{r}_{\text{foot}}$  outside domain then apply inflow BC
30:     end for
31:   end for
32:   return  $I^{**}$ 
33: end procedure

```

34: $I^{(1)} \leftarrow \text{HALFCOLLISION}(I^n)$ ▷ collision half-step

35: $I^{(2)} \leftarrow \text{STREAM}(I^{(1)})$ ▷ full streaming

7 A pseudocode in numpy

```
# Precompute spherical harmonics matrix Y[l,m,q]
# shape: (Lmax+1, 2*Lmax+1, N_omega)

# Half collision step
A = einsum('q,ijkl,q->lmijk', w, I, conj(Y))    # project to a_lm
lambda_l = c * (-mu_t + mu_s * g**l)
phi = where(lambda_l != 0,
            (exp(lambda_l*dt/2)-1)/lambda_l,
            dt/2)

A_new = exp(lambda_l*dt/2)*A + c*phi*B           # analytic update
I_star = einsum('lmijk,lmq->qijk', A_new, Y)     # reconstruct

# Streaming step
r_foot = grid_coords - c*dt*directions[q]
I_starstar = interpolate(I_star, r_foot)

# Final half collision step
A = project(I_starstar)
A_new = exp(lambda_l*dt/2)*A + c*phi*B
I_new = reconstruct(A_new)
```

References

- [1] G. Strang, *On the construction and comparison of difference schemes*, SIAM Journal on Numerical Analysis, 5(3):506–517, 1968. See also https://en.wikipedia.org/wiki/Strang_splitting.

A Impulsive angular source: pure scattering test

Before addressing the full transport problem, it is instructive to study the purely local angular dynamics governed by the collision operator \mathcal{C} alone,

neglecting spatial streaming:

$$\frac{1}{c} \frac{\partial I(\mathbf{r}, \hat{\mathbf{s}}, t)}{\partial t} = -\mu_t I(\mathbf{r}, \hat{\mathbf{s}}, t) + \mu_s \int_{4\pi} p(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}') I(\mathbf{r}, \hat{\mathbf{s}}', t) d\Omega' + \eta(\mathbf{r}, \hat{\mathbf{s}}, t). \quad (15)$$

This describes how the angular distribution of radiation at a fixed point evolves due to scattering and absorption.

A.1 Point-like directional impulse source

Consider an instantaneous directional source at time $t = 0$:

$$\eta(\mathbf{r}, \hat{\mathbf{s}}, t) = J \delta^3(\mathbf{r}) \delta(t) \delta(\hat{\mathbf{s}} - \hat{\mathbf{s}}_0), \quad (16)$$

representing a burst of intensity J emitted into direction $\hat{\mathbf{s}}_0$. Initially there is no radiation: $I(\hat{\mathbf{s}}, 0^-) = 0$.

Expanding I in spherical harmonics,

$$I(\mathbf{r}, \hat{\mathbf{s}}, t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}(\mathbf{r}, t) Y_{\ell m}(\hat{\mathbf{s}}),$$

and using the eigenproperty of the Henyey–Greenstein kernel

$$\int_{4\pi} p(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}') Y_{\ell m}(\hat{\mathbf{s}}') d\Omega' = \chi_{\ell} Y_{\ell m}(\hat{\mathbf{s}}), \quad \chi_{\ell} = g^{\ell},$$

we obtain independent equations for each angular mode:

$$\frac{da_{\ell m}}{dt} = \lambda_{\ell} a_{\ell m} + c b_{\ell m}(t), \quad \lambda_{\ell} = c(-\mu_t + \mu_s g^{\ell}), \quad (17)$$

where the source coefficients are

$$b_{\ell m}(\mathbf{r}, t) = J \delta^3(\mathbf{r}) \delta(t) Y_{\ell m}^*(\hat{\mathbf{s}}_0). \quad (18)$$

A.2 Exact solution

Integrating (17) across the impulse and using $a_{\ell m}(0^-) = 0$, we find for $t > 0$

$$a_{\ell m}(\mathbf{r}, t) = cJ \delta^3(\mathbf{r}) Y_{\ell m}^*(\hat{\mathbf{s}}_0) e^{\lambda_{\ell} t}. \quad (19)$$

Hence the full angular distribution is

$$I(\mathbf{r}, \hat{\mathbf{s}}, t) = cJ\delta^3(\mathbf{r}) \sum_{\ell, m} Y_{\ell m}^*(\hat{\mathbf{s}}_0) e^{\lambda_\ell t} Y_{\ell m}(\hat{\mathbf{s}}). \quad (20)$$

Applying the spherical addition theorem,

$$\sum_{m=-\ell}^{\ell} Y_{\ell m}^*(\hat{\mathbf{s}}_0) Y_{\ell m}(\hat{\mathbf{s}}) = \frac{2\ell+1}{4\pi} P_\ell(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}_0),$$

we obtain a compact expression:

$$I(\mathbf{r}, \hat{\mathbf{s}}, t) = \frac{cJ}{4\pi} \delta^3(\mathbf{r}) \sum_{\ell=0}^{\infty} (2\ell+1) e^{\lambda_\ell t} P_\ell(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}_0), \quad \lambda_\ell = c(-\mu_t + \mu_s g^\ell). \quad (21)$$

A.3 Physical interpretation

At the instant of emission, $I(\hat{\mathbf{s}}, 0^+) = cJ\delta^3(\mathbf{r})\delta(\hat{\mathbf{s}} - \hat{\mathbf{s}}_0)$. As time progresses, higher multipoles decay faster than lower ones because

$$-\lambda_\ell = c[\mu_t - \mu_s g^\ell] = c[\mu_a + \mu_s(1 - g^\ell)],$$

and this quantity increases with ℓ . Thus the initially narrow beam spreads over the unit sphere: the angular distribution becomes progressively smoother. The process acts as a *diffusion in direction space*, though the eigenvalues are not quadratic in ℓ as for a Laplacian. In the limit of strongly forward scattering ($g \rightarrow 1$) one can approximate

$$\mu_s(1 - g^\ell) \approx D_{\text{rot}} \ell(\ell+1), \quad D_{\text{rot}} \approx \frac{1}{2}c\mu_s(1 - g),$$

so that (21) approaches a rotational diffusion kernel on S^2 .

Rotational-diffusion limit of the HG operator

In the limit of strongly forward scattering ($g \rightarrow 1$), the Henyey–Greenstein phase function produces only small deflections of photon direction. The scattering operator then approaches a continuous rotational diffusion on the unit sphere.

Starting from the HG eigenvalues

$$\chi_\ell = g^\ell, \quad \lambda_\ell = -c[\mu_a + \mu_s(1 - g^\ell)],$$

we expand g^ℓ for $1 - g \ll 1$:

$$g^\ell = e^{\ell \ln g} \simeq e^{-\ell(1-g)} = 1 - \ell(1 - g) + \frac{1}{2}\ell^2(1 - g)^2 - \dots$$

Hence

$$1 - g^\ell \simeq \ell(1 - g) - \frac{1}{2}\ell^2(1 - g)^2 + \dots \quad (22)$$

For small ℓ , the eigenvalues of the Laplace–Beltrami operator on the sphere are $-\ell(\ell + 1) \approx -(\ell^2 + \ell)$. Matching the first-order behaviour of (22) to this spectrum at $\ell = 1$ gives

$$\mu_s(1 - g^1) = D_{\text{rot}} 1 \cdot 2 \quad \Rightarrow \quad D_{\text{rot}} = \frac{1}{2}\mu_s(1 - g).$$

Therefore, for low multipoles and $1 - g \ll 1$,

$$\mu_s(1 - g^\ell) \approx D_{\text{rot}} \ell(\ell + 1), \quad D_{\text{rot}} \simeq \frac{1}{2}\mu_s(1 - g). \quad (23)$$

Multiplying by the transport speed c , the collision operator reduces to

$$\frac{1}{c} \frac{\partial I}{\partial t} \simeq -\mu_a I + D_{\text{rot}} \nabla_\Omega^2 I,$$

which is the Fokker–Planck (rotational-diffusion) equation on the unit sphere. Here ∇_Ω^2 is the angular Laplacian, with eigenvalues $-\ell(\ell + 1)$. This limit provides a continuous description of angular spreading in the regime of very anisotropic scattering, where successive small deflections accumulate to a diffusive random walk in direction space.

A.4 Energy balance and limiting behaviour

Integrating (21) over the sphere gives the total (zeroth moment) intensity:

$$\Phi(t) = \int_{4\pi} I(\hat{\mathbf{s}}, t) d\Omega = \sqrt{4\pi} a_{00}(t) = cJ e^{\lambda_0 t} = cJ e^{-c\mu_a t}, \quad (24)$$

where $\mu_a = \mu_t - \mu_s$ is the absorption coefficient. Thus total energy decays exponentially with rate $c\mu_a$, and is conserved only when $\mu_a = 0$.

If absorption is neglected ($\mu_a = 0$), the monopole mode $\ell = 0$ remains constant, while all anisotropic components $\ell \geq 1$ decay exponentially:

$$I(\hat{\mathbf{s}}, t) \xrightarrow[t \rightarrow \infty]{} \frac{cJ}{4\pi},$$

corresponding to a uniform angular distribution.

A.5 Directional moments

Choosing the z -axis along $\hat{\mathbf{s}}_0$, only $m = 0$ coefficients are nonzero:

$$a_{\ell 0}(t) = cJ \sqrt{\frac{2\ell + 1}{4\pi}} e^{\lambda_\ell t}.$$

The first angular moment (flux along z) is

$$J_z(t) = \int_{4\pi} (\hat{\mathbf{s}} \cdot \hat{\mathbf{e}}_z) I(\hat{\mathbf{s}}, t) d\Omega = \sqrt{\frac{4\pi}{3}} a_{10}(t) = cJ e^{\lambda_1 t}. \quad (25)$$

Its decay rate $-\lambda_1 = c(\mu_t - \mu_s g)$ defines the characteristic time over which directional anisotropy vanishes.

A.6 Summary of decay spectrum

The eigenmodes of the collision operator are the spherical harmonics $Y_{\ell m}$ with eigenvalues

$$\lambda_\ell = -c[\mu_a + \mu_s(1 - g^\ell)], \quad \ell = 0, 1, 2, \dots \quad (26)$$

forming a discrete spectrum of exponentially damped angular modes. The lowest mode $\ell = 0$ describes total energy, while higher ℓ represent progressively finer angular structures. This explicit solution provides a stringent test for the numerical implementation of the collision step: the numerical coefficients $a_{\ell m}(t)$ should follow (19) exactly, and $\ln |a_{\ell m}(t)|$ should vary linearly with slope λ_ℓ .

Numerical validation of the collision step (no streaming)

The impulsive source (omit for compactness $\delta^3(\mathbf{r})$) $\eta(\hat{\mathbf{s}}, t) = J \delta(t) \delta(\hat{\mathbf{s}} - \hat{\mathbf{s}}_0)$ provides an almost ideal benchmark for the collision substep, because the analytic solution is known in closed form for all $t > 0$, see Eqs. (21), (??). A correct implementation of `HalfCollision` must satisfy the following tests:

1. Full angular field $I(\hat{\mathbf{s}}, t)$.

- Initialize the code at $t = 0^+$ with $I(\hat{\mathbf{s}}, 0^+) = cJ \delta(\hat{\mathbf{s}} - \hat{\mathbf{s}}_0)$ projected onto the discrete angular grid.

- Evolve in time using only the collision operator (no streaming).
- Reconstruct the analytic prediction at the same times:

$$I_{\text{ref}}(\hat{\mathbf{s}}, t) = \frac{cJ}{4\pi} \sum_{\ell=0}^{L_{\text{max}}} (2\ell+1) e^{\lambda_{\ell} t} P_{\ell}(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}_0), \quad \lambda_{\ell} = -c[\mu_a + \mu_s(1-g^{\ell})].$$

- Compare $I(\hat{\mathbf{s}}, t)$ from the solver with I_{ref} pointwise in angle.

Convergence check in L_{max} : at very small t the angular profile is sharply peaked and requires many multipoles for accuracy; as t increases the peak broadens, and the truncated Legendre series with moderate L_{max} already matches the numerical angular profile.

2. **Zeroth moment (total energy in the cell).** Compute

$$\Phi_{\text{num}}(t) = \int_{4\pi} I(\hat{\mathbf{s}}, t) d\Omega$$

from the code at each time step and compare to

$$\Phi_{\text{ref}}(t) = cJ e^{-c\mu_a t}.$$

This is an extremely stringent test: for any fixed (μ_a, μ_s, g) , the two curves must agree to machine precision, because $\ell = 0$ is updated analytically in `HalfCollision`.

3. **Decay rates of individual multipoles.** Project the numerical angular distribution onto spherical harmonics at each time step:

$$a_{\ell m}^{\text{num}}(t) = \int_{4\pi} I(\hat{\mathbf{s}}, t) Y_{\ell m}^*(\hat{\mathbf{s}}) d\Omega.$$

For the impulsive source we know the exact behaviour:

$$a_{\ell m}(t) = cJ Y_{\ell m}^*(\hat{\mathbf{s}}_0) e^{\lambda_{\ell} t}, \quad \lambda_{\ell} = -c[\mu_a + \mu_s(1-g^{\ell})].$$

Therefore $\ln |a_{\ell m}^{\text{num}}(t)|$ must be a straight line in t with slope λ_{ℓ} . Fitting a straight line numerically and comparing its slope to λ_{ℓ} directly validates the exponential update in `HalfCollision` for every (ℓ, m) independently.

4. **Legendre moments $M_L(t)$ along the beam axis.** Choose the z -axis along $\hat{\mathbf{s}}_0$, define

$$M_L^{\text{num}}(t) = \int_{4\pi} P_L(\cos \theta) I(\theta, \varphi, t) d\Omega, \quad \cos \theta = \hat{\mathbf{s}} \cdot \hat{\mathbf{s}}_0.$$

Analytically,

$$M_L(t) = cJ e^{\lambda_L t}, \quad \lambda_L = -c[\mu_a + \mu_s(1 - g^L)].$$

Thus each $M_L^{\text{num}}(t)$ must again decay exponentially with the correct rate *specific to that L* . In particular M_0 tests absorption only, M_1 tests the forward-peaked anisotropy, M_2 tests quadrupole damping, etc.

5. **Long-time limit for conservative scattering.** Set $\mu_a = 0$ (no true absorption). Then the analytic solution predicts isotropization:

$$I(\hat{\mathbf{s}}, t) \xrightarrow[t \rightarrow \infty]{} \frac{cJ}{4\pi}.$$

Numerically, at large t the angular field should become flat over all directions to within roundoff, and all higher moments $M_{L \geq 1}(t)$ should decay to zero.

Passing these tests confirms:

- that the projection $\{I \rightarrow a_{\ell m}\}$ and reconstruction $\{a_{\ell m} \rightarrow I\}$ (Eqs. (10)–(13)) are implemented correctly;
- that the analytic exponential update of each harmonic mode (Eq. (12)) is coded without algebraic mistakes;
- and that the chosen angular quadrature is sufficiently accurate to resolve spherical harmonics up to the intended L_{max} .