Strang Method to solve RTE

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September 23, 2025

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1 Introduction

We present a numerical method for solving the radiative transfer equation (RTE) in scattering media. The approach leverages operator splitting to separate the RTE into two physically distinct components: streaming along characteristics and local collision interactions. We employ the Strang splitting scheme for time integration, which provides second-order accuracy while maintaining computational efficiency through exact solutions of the split operators.

2 General idea of splitting

Consider the abstract evolution equation

$$\frac{\partial u}{\partial t} = (A + B)u, \tag{1}$$

where A and B are two operators that in general do not commute.

2.1 Problem with the exponential of a sum

The formal solution of (1) is

$$u(t) = e^{t(A+B)}u(0).$$

However, computing the exponential of a sum is nontrivial if A and B do not commute. The Baker–Campbell–Hausdorff (BCH) formula shows

$$e^{tA}e^{tB} = e^{t(A+B)+\frac{t^2}{2}[A,B]+\mathcal{O}(t^3)}.$$

Noncommutativity produces extra commutator terms and reduces accuracy.

2.2 Strang splitting

To achieve second-order accuracy one uses the symmetric Strang splitting [1]:

$$e^{t(A+B)} = e^{\frac{t}{2}A}\, e^{tB}\, e^{\frac{t}{2}A} + \mathcal{O}(t^3). \tag{2}$$

This formula employs only simple exponentials e^{tA} and e^{tB}, yet guarantees a global error of order $\mathcal{O}(t^2)$.

So, the solution can be found in a series of small steps Δt in time:

$$u(t + \Delta t) = e^{\frac{\Delta t}{2}A}\, e^{\Delta tB}\, e^{\frac{\Delta t}{2}A}u(t).$$

3 Radiative transfer equation

We consider the specific intensity

$$I(\boldsymbol{r}, \hat{\boldsymbol{s}}, t), \qquad \boldsymbol{r} = (x, y, z), \quad \hat{\boldsymbol{s}} \in S^2.$$

3.1 Formulation

The radiative transfer equation reads

$$\frac{1}{c}\frac{\partial I}{\partial t} + \hat{\boldsymbol{s}} \cdot \nabla I = -\mu_t I + \mu_s \int_{4\pi} p(\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{s}}') I(\boldsymbol{r}, \hat{\boldsymbol{s}}', t)\, d\Omega' + \eta(\boldsymbol{r}, \hat{\boldsymbol{s}}, t), \tag{3}$$

with $\mu_t = \mu_a + \mu_s$.

The phase function is Henyey–Greenstein:

$$p(\cos\theta) = \frac{1}{4\pi} \frac{1 - g^2}{(1 + g^2 - 2g\cos\theta)^{3/2}}, \qquad g \in (-1, 1). \tag{4}$$

3.2 Operators

Define

$$\begin{aligned} \mathcal{L} I &= -\hat{\boldsymbol{s}} \cdot \boldsymbol{\nabla} I, \qquad \text{(streaming)} \\ \mathcal{C} I &= -\mu_t I + \mu_s \int_{4\pi} p(\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{s}}') I(\hat{\boldsymbol{s}}')\, d\Omega' + \eta. \qquad \text{(collision)} \end{aligned}$$

4 Splitting idea for RTE

We apply Strang splitting (2) to the decomposition

$$\frac{1}{c}\frac{\partial I}{\partial t} = \mathcal{L} I + \mathcal{C} I,$$

with $\mathcal{L}$ and $\mathcal{C}$ from Section 3.2.

4.1 Action of the collision operator

Expand intensity at a fixed spatial cell in spherical harmonics:

$$I(\boldsymbol{r}, \hat{\boldsymbol{s}}, t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}(\boldsymbol{r}, t)\, Y_{\ell m}(\hat{\boldsymbol{s}}). \tag{5}$$

Project the source in the same way:

$$\eta(\boldsymbol{r}, \hat{\boldsymbol{s}}, t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} b_{\ell m}(\boldsymbol{r}, t)\, Y_{\ell m}(\hat{\boldsymbol{s}}). \tag{6}$$

For the Henyey–Greenstein phase function, the scattering operator acts diagonally in this basis:

$$\int_{4\pi} p(\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{s}}')\, Y_{\ell m}(\hat{\boldsymbol{s}}')\, d\Omega' = \chi_\ell\, Y_{\ell m}(\hat{\boldsymbol{s}}), \qquad \chi_\ell = g^\ell.$$

Hence the coefficients evolve as

$$\frac{da_{\ell m}}{dt} = c\Big[-\mu_t a_{\ell m} + \mu_s \chi_\ell a_{\ell m} + b_{\ell m} \Big]. \tag{7}$$

Each (ℓ, m) mode is independent, so the half collision step $\exp(\frac{\Delta t}{2}\mathcal{C})$ is simply a set of scalar ODE solves, given in closed form in Section 5.

Thus the exponential operator acts modewise:

$$\left(e^{\frac{c\Delta t}{2}\mathcal{C}} I\right)(\hat{\boldsymbol{s}}) = \sum_{\ell,m} \left[e^{\lambda_\ell \Delta t/2}\, a_{\ell m}(\boldsymbol{r}, t) + \frac{e^{\lambda_\ell \Delta t/2} - 1}{\lambda_\ell}\, c\, b_{\ell m}(\boldsymbol{r}, t) \right] Y_{\ell m}(\hat{\boldsymbol{s}}).$$

4.2 Action of the streaming operator

The streaming operator translates intensity along rays:

$$\frac{\partial}{\partial t} I(\boldsymbol{r}, \hat{\boldsymbol{s}}, t) = -c\, \hat{\boldsymbol{s}} \cdot \nabla I(\boldsymbol{r}, \hat{\boldsymbol{s}}, t). \tag{8}$$

This has the exact solution

$$I(\boldsymbol{r}, \hat{\boldsymbol{s}}, t + \Delta t) = I(\boldsymbol{r} - c\Delta t\, \hat{\boldsymbol{s}},\ \hat{\boldsymbol{s}}, t). \tag{9}$$

Thus $\exp(\Delta t \mathcal{L})$ is a pure shift operator in space, leaving angular coefficients unchanged.

4.3 Combined Strang step

To integrate (3) we apply Strang splitting (2):

$$I(\boldsymbol{r}, \hat{\boldsymbol{s}}, t_{n+1}) = e^{\frac{c\Delta t}{2}\mathcal{C}}\, e^{c\Delta t\,\mathcal{L}}\, e^{\frac{c\Delta t}{2}\mathcal{C}}\, I(\boldsymbol{r}, \hat{\boldsymbol{s}}, t_n).$$

Collision steps act diagonally on spherical harmonic modes $a_{\ell m}$ via (7), while the streaming step shifts each angular node according to (9). This clear separation of actions is what makes the method efficient.

4.4 Explicit form of the two substeps

The Strang update consists of two distinct operator actions:

Collision half step. At fixed spatial position $\boldsymbol{r}$, expand intensity into spherical harmonics with coefficients $a_{\ell m}(t)$. From (7) each mode satisfies

$$\frac{da_{\ell m}}{dt} = \lambda_\ell a_{\ell m} + c\, b_{\ell m}, \qquad \lambda_\ell = c(-\mu_t + \mu_s g^\ell).$$

This linear ODE has the exact solution over time τ:

$$a_{\ell m}(t+\tau) = e^{\lambda_\ell \tau} a_{\ell m}(t) + \frac{e^{\lambda_\ell \tau} - 1}{\lambda_\ell}\, c\, b_{\ell m},$$

(with the limit τ if $\lambda_\ell = 0$). Thus

$$\left(e^{\frac{c\Delta t}{2}\mathcal{C}} I\right)(\hat{\boldsymbol{s}}) = \sum_{\ell,m} \left[e^{\lambda_\ell \Delta t/2} a_{\ell m}(t) + \frac{e^{\lambda_\ell \Delta t/2} - 1}{\lambda_\ell}\, c\, b_{\ell m} \right] Y_{\ell m}(\hat{\boldsymbol{s}}).$$

Streaming full step. The operator $\mathcal{L}$ generates pure translations along rays. Its exact action over Δt is

$$\left(e^{c\Delta t \mathcal{L}} I\right)(\boldsymbol{r}, \hat{\boldsymbol{s}}) = I(\boldsymbol{r} - c\Delta t\, \hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}, t).$$

That is, each angular component remains unchanged, while the spatial distribution is shifted backwards along direction $\hat{\boldsymbol{s}}$ by distance $c\Delta t$.

In summary, the Strang step alternates between an *angular update* (diagonal in spherical harmonics) and a *spatial shift* (exact translation along characteristics).

5 Numerical scheme

5.1 Discretization

Let for given time t

$$I[q,k,j,i] \approx I(x_i, y_j, z_k, s_q, t),$$

where s_q is a direction labelled by q.

5.2 Collision step (HG via spherical harmonics)

Let $\{(\hat{\boldsymbol{s}}_q, w_q)\}_{q=1}^{N_\Omega}$ be the angular quadrature. At each spatial cell (i,j,k) we project the nodal intensities onto spherical harmonics up to $L_{\max}$:

$$a_{\ell m}[k,j,i] \;=\; \sum_{q=1}^{N_\Omega} w_q \; I[q,k,j,i] \; Y^*_{\ell m}(\hat{\boldsymbol{s}}_q), \qquad 0 \le \ell \le L_{\max}, \; |m| \le \ell. \tag{10}$$

For Henyey–Greenstein, the scattering operator is diagonal in (ℓ, m) with eigenvalues $\chi_\ell = g^\ell$. Define

$$\lambda_\ell \;=\; c\big(-\mu_t + \mu_s \chi_\ell\big), \qquad b_{\ell m}[k,j,i] \;=\; c \sum_{q=1}^{N_\Omega} w_q \; \eta[q,k,j,i] \; Y^*_{\ell m}(\hat{\boldsymbol{s}}_q). \tag{11}$$

A half collision step over $\Delta t/2$ is updated *exactly* modewise by

$$a^\star_{\ell m} \;=\; e^{\lambda_\ell \Delta t/2}\, a_{\ell m} \;+\; \phi(\lambda_\ell, \tfrac{\Delta t}{2})\, b_{\ell m}, \qquad \phi(\lambda,\tau) = \begin{cases} \dfrac{e^{\lambda\tau} - 1}{\lambda}, & \lambda \neq 0, \\ \tau, & \lambda = 0. \end{cases} \tag{12}$$

Transform back to nodal angles:

$$I^\star[q,k,j,i] \;=\; \sum_{\ell=0}^{L_{\max}} \sum_{m=-\ell}^{\ell} a^\star_{\ell m}[k,j,i] \; Y_{\ell m}(\hat{\boldsymbol{s}}_q). \tag{13}$$

5.3 Streaming step (semi-Lagrangian)

The streaming update $I \mapsto e^{c\Delta t\,\mathcal{L}} I$ is a shift along characteristics. For each (q,i,j,k) let the foot point be

$$\boldsymbol{r}^{\text{foot}}_{i,j,k,q} \;=\; (x_i, y_j, z_k) - c\Delta t\, \hat{\boldsymbol{s}}_q.$$

Then

$$I^{\star\star}[q,k,j,i] \;=\; \mathrm{Interp}\big(I^{\star}[q,\cdot,\cdot,\cdot],\; \boldsymbol{r}^{\mathrm{foot}}_{i,j,k,q}\big)\,, \tag{14}$$

where Interp denotes spatial interpolation on the (x, y, z) grid (linear/monotone cubic/WENO as desired). Inflow boundary conditions are applied whenever $\boldsymbol{r}^{\mathrm{foot}}$ exits the domain.

5.4 Full algorithm

Given $I^n[q, k, j, i] = I(\boldsymbol{r}_{i,j,k}, \hat{\boldsymbol{s}}_q, t_n)$:

1. **Half collision:** apply (10)–(13) with $\Delta t/2$ to obtain $I^{\star}$.
2. **Full streaming:** apply (14) with Δt to obtain $I^{\star\star}$.
3. **Half collision:** repeat (10)–(13) on $I^{\star\star}$ (using η at t_{n+1} or a centered value) to obtain I^{n+1}.

6 A pseudocode

Algorithm 1 Strang-split RTE step (HG scattering): from $I^n[q,k,j,i]$ to $I^{n+1}[q,k,j,i]$

Require: Arrays $\mu_t[k,j,i]$, $\mu_s[k,j,i]$, source $\eta[q,k,j,i]$; directions $\{\hat{\boldsymbol{s}}_q, w_q\}_{q=1}^{N_\Omega}$; Δt, c, $L_{\max}$

Precompute $\chi_\ell = g^\ell$ for $\ell = 0, \ldots, L_{\max}$

procedure HalfCollision(I)
  for all (k,j,i) **do** ▷ project to SH
    for $\ell = 0$ to $L_{\max}$ **do**
      for $m = -\ell$ to ℓ **do**
        $a_{\ell m} \leftarrow \sum_{q=1}^{N_\Omega} w_q\, I[q,k,j,i]\, Y_{\ell m}^*(\hat{\boldsymbol{s}}_q)$
        $b_{\ell m} \leftarrow \sum_{q=1}^{N_\Omega} w_q\, \eta[q,k,j,i]\, Y_{\ell m}^*(\hat{\boldsymbol{s}}_q)$
      end for
    end for
    for $\ell = 0$ to $L_{\max}$ **do**
      for $m = -\ell$ to ℓ **do** ▷ analytic mode update
        $\lambda_\ell \leftarrow c\big(-\mu_t[k,j,i] + \mu_s[k,j,i]\,\chi_\ell\big)$
        $\tau \leftarrow \Delta t/2$
        $\phi \leftarrow \begin{cases} (e^{\lambda_\ell \tau} - 1)/\lambda_\ell, & \lambda_\ell \neq 0 \\ \tau, & \lambda_\ell = 0 \end{cases}$
        $a_{\ell m}^\star \leftarrow e^{\lambda_\ell \tau}\, a_{\ell m} + c\,\phi\, b_{\ell m}$
      end for
    end for
    for $q = 1$ to N_Ω **do** ▷ reconstruct to nodal angles
      $I^\star[q,k,j,i] \leftarrow \sum_{\ell=0}^{L_{\max}} \sum_{m=-\ell}^{\ell} a_{\ell m}^\star\, Y_{\ell m}(\hat{\boldsymbol{s}}_q)$
    end for
  end for
  return $I^\star$
end procedure

procedure Stream($I^\star$)
  for $q = 1$ to N_Ω **do**
    for all (k,j,i) **do** ▷ semi-Lagrangian backtrace
      $\boldsymbol{r}_{\text{foot}} \leftarrow (x_i, y_j, z_k) - c\Delta t\, \hat{\boldsymbol{s}}_q$
      $I^{\star\star}[q,k,j,i] \leftarrow \text{Interp}(I^\star[q,\cdot,\cdot,\cdot],\, \boldsymbol{r}_{\text{foot}})$
      if $\boldsymbol{r}_{\text{foot}}$ outside domain **then** apply inflow BC
    end for
  end for
  return $I^{\star\star}$
end procedure

$I^{(1)} \leftarrow$ HalfCollision(I^n) ▷ collision half-step
$I^{(2)} \leftarrow$ Stream($I^{(1)}$) ▷ full streaming
$I^{n+1} \leftarrow$ HalfCollision($I^{(2)}$) ▷ collision half-step

7 A pseudocode in numpy

```
# Precompute spherical harmonics matrix Y[l,m,q]
# shape: (Lmax+1, 2*Lmax+1, N_omega)

# Half collision step
A = einsum('q,ijkl,q->lmijk', w, I, conj(Y))   # project to a_lm
lambda_l = c * (-mu_t + mu_s * g**l)
phi = where(lambda_l != 0,
            (exp(lambda_l*dt/2)-1)/lambda_l,
            dt/2)

A_new = exp(lambda_l*dt/2)*A + c*phi*B         # analytic update
I_star = einsum('lmijk,lmq->qijk', A_new, Y)   # reconstruct

# Streaming step
r_foot = grid_coords - c*dt*directions[q]
I_starstar = interpolate(I_star, r_foot)

# Final half collision step
A = project(I_starstar)
A_new = exp(lambda_l*dt/2)*A + c*phi*B
I_new = reconstruct(A_new)
```

References

[1] G. Strang, *On the construction and comparison of difference schemes*, SIAM Journal on Numerical Analysis, 5(3):506–517, 1968. See also https://en.wikipedia.org/wiki/Strang_splitting.