二维非齐次热传导方程的向后 Euler ADI 格式

作业:

$$\begin{cases} \dfrac{\partial u}{\partial t}-(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2})=-\dfrac{3}{2}e^{\frac{1}{2}(x+y)-t} & ,\quad 0<x,y<1,0<t\leq 1 \\ u(x,y,0)=e^{\frac{1}{2}x-t} & ,\quad 0<x,y<1 \\ u(0,y,t)=e^{\frac{1}{2}y-t},u(1,y,t)=e^{\frac{1}{2}(1+y)-t}, & ,\quad 0\leq y\leq 1,0\leq t\leq 1 \\ u(x,0,t)=e^{\frac{1}{2}x-t},u(x,1,t)=e^{\frac{1}{2}(1+x)-t}, & ,\quad 0<x<1,0\leq t\leq 1 \end{cases}$$

该问题的精确解为 $u(x,y,t)=e^{\frac{1}{2}(x+y)-t}$.
定义误差为

$$E_\infty(h,\tau)=\max_{\substack{1\leq i,j\leq m-1\\ 1\leq k\leq n}}|u_{i,j}^k-u(x_i,y_j,t_k)|$$

用向后 Euler ADI 格式求下述问题的数值解并对数值解、精度和误差阶进行相应的数值分析。

解：
将 x 和 y m 等分，将 t n 等分, 记 $h=\dfrac{1}{m},\tau=\dfrac{1}{n}$
$x_i=ih,0\leq i\leq m$ $y_j=jh,0\leq j\leq m$ $t_k=k\tau,0\leq k\leq n$
差分格式为

$$\begin{cases} (I-\tau\delta_x^2)u_{i,j}^{k+\frac{1}{2}}=u_{i,j}^k+\tau f_{i,j}^{k+1} & (1a)\\ (I-\tau\delta_y^2)u_{i,j}^{k+1}=u_{i,j}^{k+\frac{1}{2}} & (1b)\end{cases}$$

其中 $Iu_{i,j}^k=u_{i,j}^k$,$u_{i,j}^{k+\frac{1}{2}}$ 为中间层。$\delta_x^2 u_{i,j}=\dfrac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}$. $\delta_y^2 u_{i,j}=\dfrac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{h^2}$.
(1a) 可以写成矩阵形式

$$Au_{i,j}^{k+\frac{1}{2}}=B_1^{k+1}$$

.

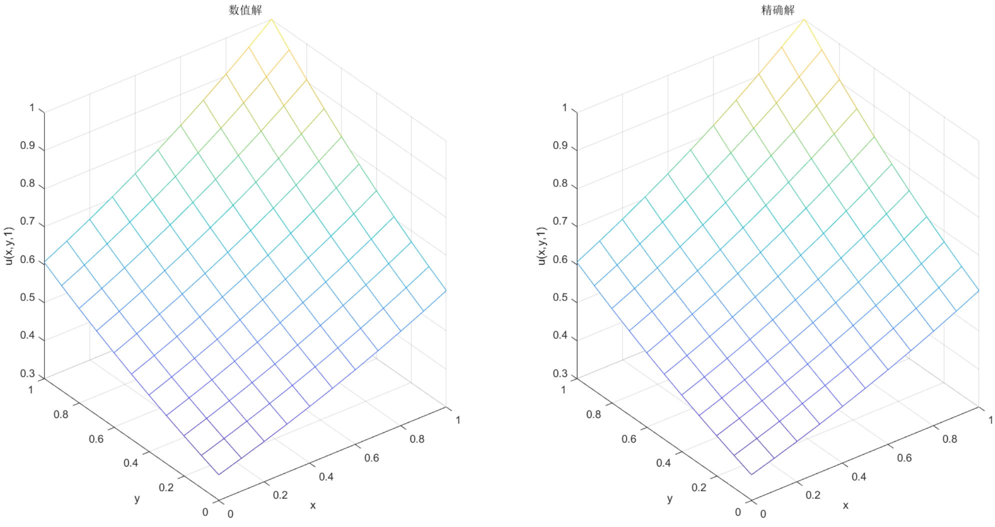


图 **1** $\tau=\dfrac{1}{100},h=\dfrac{1}{10}$ 时 **t=1** 处的数值解和精确解

$$A=\begin{bmatrix}1+2\gamma & -\gamma & & \\ -\gamma & 1+2\gamma & -\gamma & \\ & \ddots & \ddots & \ddots \\ & & -\gamma & 1+2\gamma\end{bmatrix}$$

$$B_1^{k+1}=[u_{1,j}^k,u_{2,j}^k,\cdots,u_{m-1,j}^k]^T+\tau[f_{1,j}^{k+1},f_{2,j}^{k+1},\cdots,f_{m-1,j}^{k+1}]^T+\gamma[u_{0,j}^{k+\frac{1}{2}},0,\cdots,0,u_{m,j}^{k+\frac{1}{2}}]^T.$$

(1b) 可写成矩阵形式

$$Au_{i,j}^{k+\frac{1}{2}}=B_2^{k+1}$$

.

$$B_2^{k+1}=[u_{i,1}^{k+\frac{1}{2}}+\gamma u_{i,0}^{k+1},u_{i,2}^{k+\frac{1}{2}},\cdots,u_{i,m-2}^{k+\frac{1}{2}},u_{i,m-1}^{k+\frac{1}{2}}+\gamma u_{i,m}^{k+1}]^T$$

解题程序运行于 Matlab 2018a

$\tau=\dfrac{1}{10},h=\dfrac{1}{10}$ 时 t=1 处的数值解和精确解见图1，非常接近。

部分节点处的数值解、精确解和误差见表1.

t=1 时，取不同步长时的误差见图2, 步长越小，误差越小。

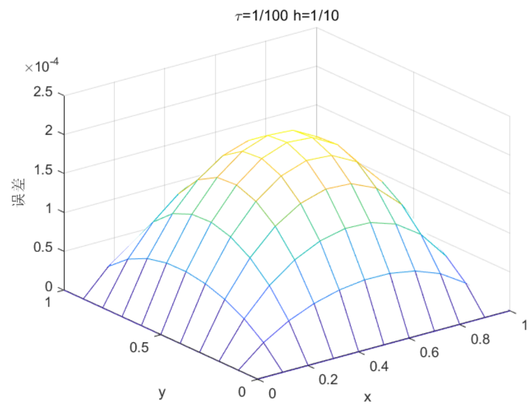
取不同步长时的最大误差和最大误差的比见表2,h 变为原来的 2 倍，τ 变为原来的 4 倍，最大误差变为原来的 4 倍。

表 **1**　部分节点处的数值解、精确解和误差

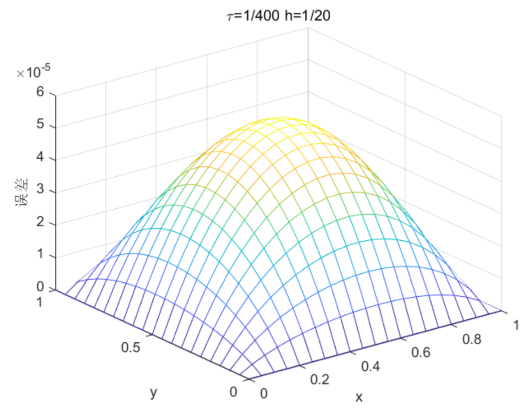
t,x,y	数值解	精确解	误差
0.1,0.5,0.5	1.492229	1.491825	4.0480E-04
0.2,0.5,0.5	1.350296	1.349859	4.3716E-04
0.3,0.5,0.5	1.221809	1.221403	4.0652E-04
0.4,0.5,0.5	1.105540	1.105171	3.6952E-04
0.5,0.5,0.5	1.000335	1.000000	3.3462E-04
0.6,0.5,0.5	0.905140	0.904837	3.0282E-04
0.7,0.5,0.5	0.819005	0.818731	2.7401E-04
0.8,0.5,0.5	0.741066	0.740818	2.4793E-04
0.9,0.5,0.5	0.670544	0.670320	2.2434E-04
1.0,0.5,0.5	0.606734	0.606531	2.0299E-04

表 **2**　不同步长时的最大误差和最大误差的比

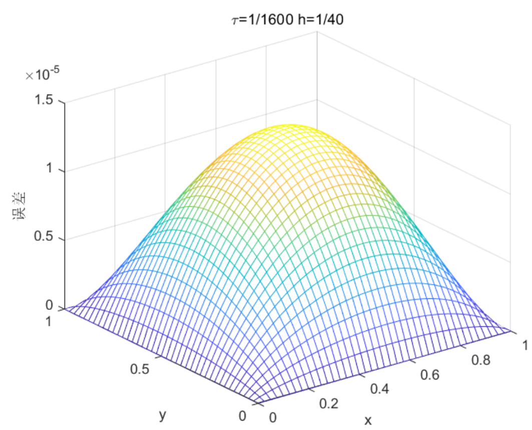
h,τ	$E_\infty(h,\tau)$	$E_\infty(2h,2\tau)/E_\infty(h,\tau)$
1/10,1/100	4.40E-04	*
1/20,1/400	1.16E-04	3.7873
1/40,1/1600	2.94E-05	3.9525
1/80,1/6400	7.38E-06	3.9842



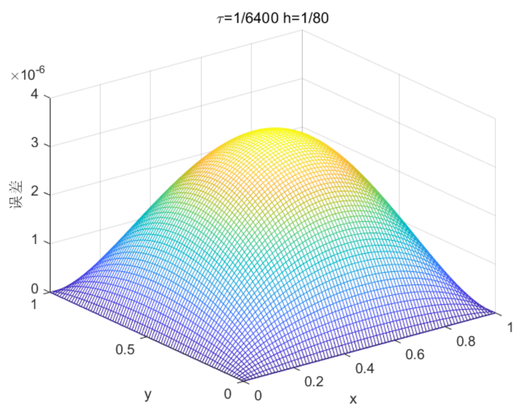
(a) $h = 1/10, \tau = 1/100$ 时的误差



(b) $h = 1/20, \tau = 1/400$ 时的误差



(c) $h = 1/40, \tau = 1/1600$ 时的误差



(d) $h = 1/180, \tau = 1/6400$ 时的误差

图 **2**　**t=1** 时的误差图