二维非齐次热传导方程的向后 Crank-Nicolson ADI 格式

作业:

$$\begin{cases} \dfrac{\partial u}{\partial t} - (\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}) = -\dfrac{3}{2}e^{\frac{1}{2}(x+y)-t} & , \quad 0 < x, y < 1, 0 < t \leq 1 \\ u(x,y,0) = e^{\frac{1}{2}x-t} & , \quad 0 < x, y < 1 \\ u(0,y,t) = e^{\frac{1}{2}y-t}, u(1,y,t) = e^{\frac{1}{2}(1+y)-t}, & , \quad 0 \leq y \leq 1, 0 \leq t \leq 1 \\ u(x,0,t) = e^{\frac{1}{2}x-t}, u(x,1,t) = e^{\frac{1}{2}(1+x)-t}, & , \quad 0 < x < 1, 0 \leq t \leq 1 \end{cases}$$

该问题的精确解为 $u(x,y,t) = e^{\frac{1}{2}(x+y)-t}$.

定义误差为

$$E_\infty(h,\tau) = \max_{\substack{1\leq i,j\leq m-1 \\ 1\leq k\leq n}} |u_{i,j}^k - u(x_i, y_j, t_k)|$$

用向后 Crank-Nicolson ADI 格式求下述问题的数值解并对数值解、精度和误差阶进行相应的数值分析。

解：

将 x 和 y m 等分，将 t n 等分, 记 $h = \dfrac{1}{m}, \tau = \dfrac{1}{n}$

$x_i = ih, 0 \leq i \leq m$ $y_j = jh, 0 \leq j \leq m$ $t_k = k\tau, 0 \leq k \leq n$

令 $\gamma = \dfrac{\tau}{h^2}$

P-R 差分格式为

$$\begin{cases} (I - \dfrac{\tau}{2}\delta_x^2)\overline{u_{i,j}} = (I + \dfrac{\tau}{2}\delta_y^2)u_{i,j}^k + \dfrac{\tau}{2}f_{i,j}^{k+\frac{1}{2}}, 1 \leq i \leq m-1 & \text{(1a)} \\ (I - \dfrac{\tau}{2}\delta_y^2)u_{i,j}^{k+1} = (I + \dfrac{\tau}{2}\delta_x^2)\overline{u_{i,j}} + \dfrac{\tau}{2}f_{i,j}^{k+\frac{1}{2}}, 1 \leq j \leq m-1 & \text{(1b)} \end{cases}$$

其中 $Iu_{i,j}^k = u_{i,j}^k$,$\overline{u_{i,j}}$ 为中间层。$\delta_x^2 u_{i,j} = \dfrac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$.

$\delta_y^2 u_{i,j} = \dfrac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2}$.

中间层的初值为

$$\begin{cases} \overline{u_{0,j}} = \dfrac{1}{2}(u_{0,j}^k + u_{0,j}^{k+1}) - \dfrac{\tau}{4}(\delta_y^2 u_{0,j}^{k+1} - \delta_y^2 u_{0,j}^k) & \text{(2a)} \\ \overline{u_{m,j}} = \dfrac{1}{2}(u_{m,j}^k + u_{m,j}^{k+1}) - \dfrac{\tau}{4}(\delta_y^2 u_{m,j}^{k+1} - \delta_y^2 u_{m,j}^k) & \text{(2b)} \end{cases}$$

(1a) 可以写成矩阵形式

$$A\overline{u_{i,j}} = B_1^{k+1}$$

.

$$A = \begin{bmatrix} 1+\gamma & -\frac{\gamma}{2} & & \\ -\frac{\gamma}{2} & 1+\gamma & -\frac{\gamma}{2} & \\ & \ddots & \ddots & \ddots \\ & & -\frac{\gamma}{2} & 1+\gamma \end{bmatrix}$$

$$B_1^{k+1} = \frac{\gamma}{2} \cdot \begin{bmatrix} u_{1,j-1}^{k-1} \\ u_{2,j-1}^{k-1} \\ \vdots \\ u_{m-1,j-1}^{k-1} \end{bmatrix} + (1-\gamma) \cdot \begin{bmatrix} u_{1,j}^{k-1} \\ u_{2,j}^{k-1} \\ \vdots \\ u_{m-1,j}^{k-1} \end{bmatrix} + \frac{\gamma}{2} \cdot \begin{bmatrix} u_{1,j+1}^{k-1} \\ u_{2,j+1}^{k-1} \\ \vdots \\ u_{m-1,j+1}^{k-1} \end{bmatrix} + \frac{\tau}{2} \cdot \begin{bmatrix} f_{1,j}^{k+\frac{1}{2}} \\ f_{2,j}^{k+\frac{1}{2}} \\ \vdots \\ f_{m-1,j}^{k+\frac{1}{2}} \end{bmatrix}$$

(1b) 可写成矩阵形式

$$Au_{i,j}^{k+1} = B_2^{k+1}$$

.

$$B_2^{k+1} = \frac{\gamma}{2} \cdot \begin{bmatrix} \overline{u_{i-1,1}} \\ \overline{u_{i-1,2}} \\ \vdots \\ \overline{u_{i-1,m-1}} \end{bmatrix} + (1-\gamma) \cdot \begin{bmatrix} \overline{u_{i,1}} \\ \overline{u_{i,2}} \\ \vdots \\ \overline{u_{i,m-1}} \end{bmatrix} + \frac{\gamma}{2} \cdot \begin{bmatrix} \overline{u_{i+1,1}} \\ \overline{u_{i+1,2}} \\ \vdots \\ \overline{u_{i+1,m-1}} \end{bmatrix} + \frac{\tau}{2} \cdot \begin{bmatrix} f_{i,1}^{k+\frac{1}{2}} \\ f_{i,2}^{k+\frac{1}{2}} \\ \vdots \\ f_{i,m-1}^{k+\frac{1}{2}} \end{bmatrix}$$

两个方程的系数矩阵均为三对角矩阵。

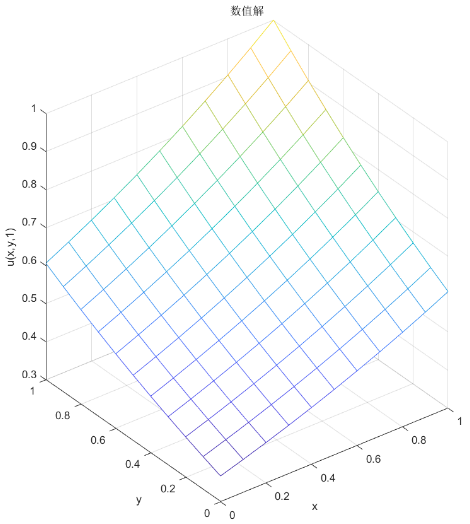
解题程序运行于 Matlab 2018a

$\tau = \dfrac{1}{10}, h = \dfrac{1}{10}$ 时 t=1 处的数值解和精确解见图1，非常接近。

部分节点处的数值解、精确解和误差见表1.

t=1 时，取不同步长时的误差见图2, 步长越小，误差越小。

取不同步长时的最大误差和最大误差的比见表2,h 变为原来的 2 倍，τ 变为原来的 2 倍，最大误差变为原来的 4 倍, 符合 $O(\tau^2 + h^2)$ 的截断误差。



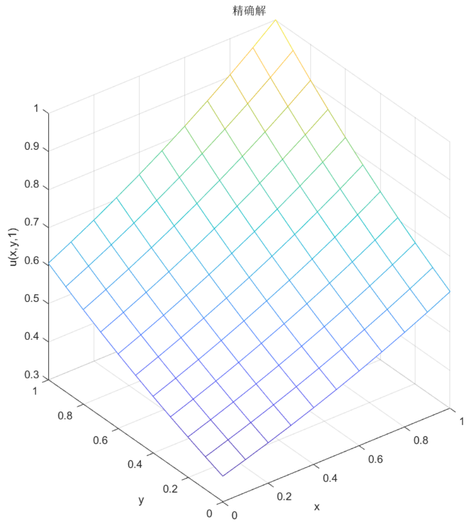
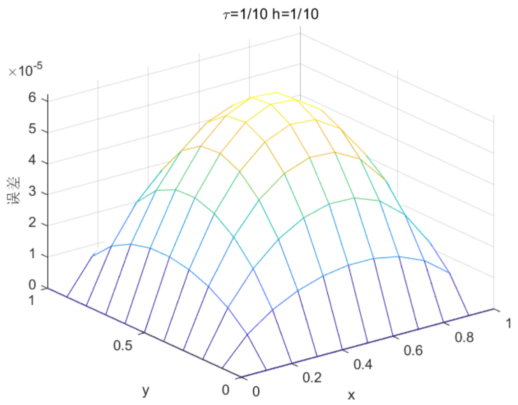


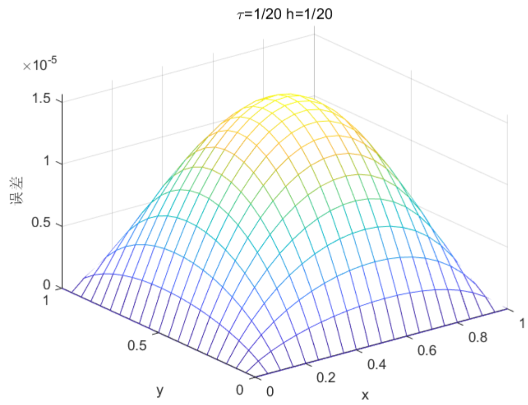
图 **1**　$\tau=\dfrac{1}{10},h=\dfrac{1}{10}$ 时 **t=1** 处的数值解和精确解

表 **1**　部分节点处的数值解、精确解和误差

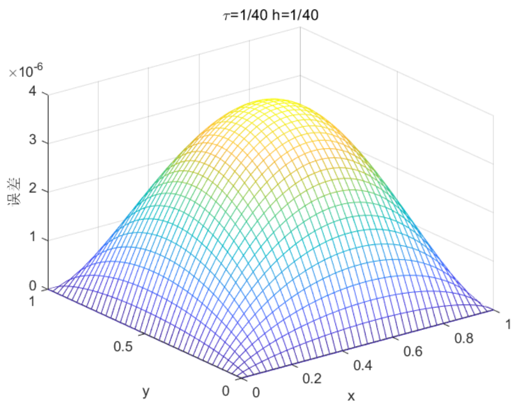
t,x,y	数值解	精确解	误差
0.1,0.5,0.5	1.491952	1.491825	1.2683E-04
0.2,0.5,0.5	1.349996	1.349859	1.3688E-04
0.3,0.5,0.5	1.221528	1.221403	1.2516E-04
0.4,0.5,0.5	1.105285	1.105171	1.1366E-04
0.5,0.5,0.5	1.000103	1.000000	1.0285E-04
0.6,0.5,0.5	0.904930	0.904837	9.3073E-05
0.7,0.5,0.5	0.818815	0.818731	8.4216E-05
0.8,0.5,0.5	0.740894	0.740818	7.6202E-05
0.9,0.5,0.5	0.670389	0.670320	6.8951E-05
1.0,0.5,0.5	0.606593	0.606531	6.2389E-05



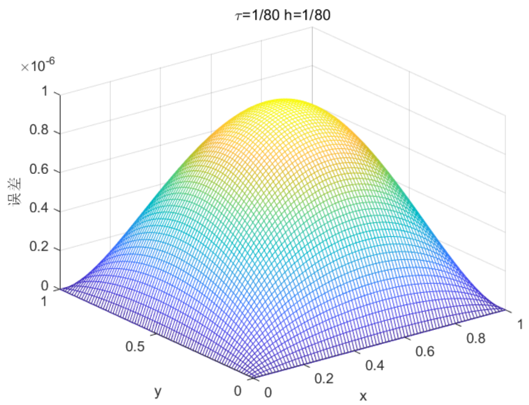
(a) $h = 1/10, \tau = 1/10$ 时的误差



(b) $h = 1/20, \tau = 1/20$ 时的误差



(c) $h = 1/40, \tau = 1/40$ 时的误差



(d) $h = 1/180, \tau = 1/80$ 时的误差

图 **2**　**t=1** 时的误差图

表 **2**　不同步长时的最大误差和最大误差的比

h, τ	$E_\infty(h,\tau)$	$E_\infty(2h,2\tau)/E_\infty(h,\tau)$
1/10,1/10	1.37E-04	*
1/20,1/20	3.44E-05	3.9795
1/40,1/40	8.57E-06	4.0113
1/80,1/80	2.15E-06	3.9937