Дз # 3

Упражнение 0.1.

Найдите d и d^2 от следующих функций

1) $u = f(x+y, z) = f(v, w)$

$$du = \frac{du}{dx} \cdot dx + \frac{du}{dy} \cdot dy + \frac{du}{dz} \cdot dz =$$

$$= \left(\frac{du}{dv} \cdot 1 + \frac{du}{dw} \cdot 0\right) dx + \left(\frac{du}{dv} \cdot 1 + \frac{du}{dw} \cdot 0\right) \cdot dy +$$

$$+ \left(\frac{du}{dv} \cdot 0 + \frac{du}{dw} \cdot 1\right) \cdot dz = \frac{du}{dv} \cdot dx + \frac{du}{dv} \cdot dy +$$

$$+ \frac{du}{dw} \cdot dz = \frac{du}{dv} \cdot (dx+dy) + \frac{du}{dw} \cdot dz$$

$$d^2u = d\left(\frac{du}{dv} (dx+dy) + \frac{du}{du} \cdot dz\right) =$$

$$= (dx+dy)\, d\left(\frac{du}{dv}\right) + dz \cdot d\left(\frac{du}{dw}\right) = (dx+dy)$$

$$\left(\frac{d^2u}{dv^2} \cdot (dx+dy) + \frac{d^2u}{dw\,dv} \cdot dz\right) +$$

$+dz \cdot \left(\frac{d^2u}{dvdw} \cdot (dx+dy) + \frac{d^2u}{dw^2} dz \right) =$

$= \frac{d^2u}{dv^2} \cdot (dx+dy)^2 + \left[\frac{d^2u}{dwdv} + \frac{d^2u}{dvdw} \right] \cdot (dx+dy)dz +$

$+ \frac{d^2u}{dw^2} dz^2 = \frac{d^2u}{dv^2} \cdot (dx+dy)^2 + 2 \frac{d^2u}{dvdw} \cdot (dx+dy)dz +$

$+ \frac{d^2u}{dw^2} \cdot (dz)^2$

4) $u = f(xyz) = f(v)$

$du = \frac{du}{dx} dx + \frac{du}{dy} dy + \frac{du}{dz} dz =$

$= \frac{du}{dv} yz dx + \frac{du}{dv} \cdot xz dy + \frac{du}{dv} \cdot xy dz =$

$= \frac{du}{dv} \cdot (yz dx + xz dy + xy dz)$

$d^2u = \frac{d^2u}{dv^2} \cdot (yz dx + xz dy + xy dz)^2 +$

$+ \frac{du}{dv} \cdot (z dx dy + y dx dz + z dx dy + x dy dz + y dx dz + x dy dz)$

$= \frac{d^2u}{dv^2} \cdot (yz dx + xz dy + xy dz)^2 + 2 \frac{du}{dv} \cdot (x dy dz + y dx dz +$
$+ z dx dy)$

3) $u = f(x^2+y^2,\ x^2-y^2,\ 2xy) = f(v, w, h)$

$$du = \left(\frac{du}{dv}\cdot 2x + \frac{du}{dw}\cdot 2x + \frac{du}{dh}\cdot 2y\right)dx +$$

$$+\left(\frac{du}{dv}\cdot 2y - \frac{du}{dw}\cdot 2y + \frac{du}{dh}\cdot 2x\right)\cdot dy =$$

$$= \frac{du}{dv}\cdot(2x\,dx + 2y\,dy) + \frac{du}{dw}\cdot(2x\,dx - 2y\,dy) +$$

$$+\frac{du}{dh}\cdot(2y\,dx + 2x\,dy) = 2\frac{du}{dv}\cdot(x\,dx + y\,dy) + 2\cdot\frac{du}{dw}\cdot$$

$$\cdot(x\,dx - y\,dy) + 2\frac{du}{dh}\cdot(y\,dx + x\,dy)$$

$$d^2u = 2\left(2\frac{d^2u}{dv^2}\cdot(x\,dx+y\,dy) + 2\frac{d^2u}{dw\,dv}\cdot(x\,dx-y\,dy)\right.$$

$$\left.+2\frac{d^2u}{dh\,dv}\cdot(y\,dx+x\,dy)\right)\cdot(x\,dx+y\,dy) +$$

$$+2\left(2\cdot\frac{d^2u}{dv\,dw}\cdot(x\,dx+y\,dy) + 2\frac{d^2u}{dw^2}(x\,dx-y\,dy)\right.$$

$$\left.+2\cdot\frac{d^2u}{dh\,dw}\cdot(y\,dx+x\,dy)\right)\cdot(x\,dx-y\,dy) +$$

$$2\left(2\frac{d^2u}{dv\,dh}\cdot(x\,dx+y\,dy) + 2\cdot\frac{d^2u}{dw\,dh}\cdot(x\,dx-y\,dy)+\right.$$

$$+2\frac{d^2u}{dh^2}\cdot(y\,dx+x\,dy)\Big)\cdot(y\,dx+x\,dy)+$$

$$+2\cdot\frac{du}{dv}\cdot((dx)^2+(dy)^2)+2\cdot\frac{du}{dw}\cdot((dx)^2-(dy)^2)+$$

$$+2\frac{du}{dh}\cdot(dx\,dy+dx\,dy)=4\cdot\frac{d^2u}{dV^2}\cdot(x\,dx+y\,dy)^2$$

$$+4\cdot\frac{d^2u}{dw^2}\cdot(x\,dx-y\,dy)^2+4\cdot\frac{d^2u}{dh^2}\cdot(y\,dx+x\,dy)^2+$$

$$+8\cdot\frac{d^2u}{dv\,dw}\cdot(x^2(dx)^2-y^2(dy)^2)+8\cdot\frac{d^2u}{dv\,dh}\cdot(x\,dx+y\,dy)\cdot$$

$$\cdot(y\,dx+x\,dy)+8\cdot\frac{d^2u}{dw\,dh}\cdot(x\,dx-y\,dy)(y\,dx+x\,dy)+$$

$$+2\cdot\frac{du}{dV}((dx)^2+(dy)^2)+2\cdot\frac{du}{dw}\cdot((dx)^2-dy)^2)+$$

$$+4\cdot\frac{du}{dh}\cdot dx\,dy.$$

Упражнение 0.2

Найдите d^n от следующих функций

$u = f(ax + by + cz) = f(v)$

$du = \frac{du}{dv} \cdot (a\,dx + b\,dy + c\,dz)$

$d^2 u = \frac{d^2u}{dv^2} \cdot (a\,dx + b\,dy + c\,dz)^2 + \frac{du}{dv} \cdot 0$

$d^n u = \frac{d^n u}{dv^n} \cdot (a\,dx + b\,dy + c\,dz)^n$

докажем по индукции:

$n = 1 \quad du = \frac{du}{dv} \cdot (a\,dx + b\,dy + c\,dz)$

$\exists\, d^n u = \frac{d^n u}{dv^n} \cdot (a\,dx + b\,dy + c\,dz)^n$

$d^{n+1} u = \frac{d^{n+1} u}{dv^{n+1}} \cdot (a\,dx + b\,dy + c\,dz)^{n+1} + \frac{d^n u}{dv^n} \cdot 0$

$= \frac{d^{n+1} u}{dv^{n+1}} \cdot (a\,dx + b\,dy + c\,dz)^{n+1}$

$u = f(ax, by, cz) = f(v, w, h)$

$$du = \frac{du}{dv} \cdot a\,dx + \frac{du}{dw} \cdot b\,dy + \frac{du}{dh} \cdot c\,dz$$

$$d^2u = \frac{d^2u}{dv^2} \cdot a^2 (dx)^2 + \frac{d^2u}{dw\,dv} \cdot ab\,dx\,dy + \frac{d^2u}{dh\,dv} \cdot ac\,dx\,dz$$

$$+ \frac{d^2u}{dv\,dw} \cdot ab\,dx\,dy + \frac{d^2u}{dw^2} \cdot b^2 (dy)^2 + \frac{d^2u}{dh\,dw} \cdot bc\,dy\,dz +$$

$$+ \frac{d^2u}{dh\,dv} \cdot ac\,dx\,dz + \frac{d^2u}{dh\,dw} \cdot bc\,dy\,dz + \frac{d^2u}{dh^2} \cdot c^2 (dz)^2 =$$

$$= \frac{d^2u}{dv^2} \cdot a^2 (dx)^2 + \frac{d^2u}{dw^2} \cdot b^2 (dy)^2 + \frac{d^2u}{dh^2} \cdot c^2 (dz)^2 +$$

$$+ \frac{d^2u}{dv\,dw} \cdot 2ab\,dx\,dy + \frac{d^2u}{dv\,dh} \cdot 2ac\,dx\,dz + \frac{d^2u}{dw\,dh} \cdot$$

$$\cdot 2bc\,dy\,dz$$

$$d^n u = \sum_{\substack{i,j,k \geq 0 \\ i+j+k=n}} \frac{d^n u}{dv^i\,dw^j\,dh^k} \cdot \frac{n!}{i!\,j!\,k!} \cdot a^i b^j c^k \cdot$$

$$\cdot (dx)^i (dy)^j (dz)^k$$

Докажем по индукции

$n = 1$

$$du = \frac{du}{dv} \cdot a\,dx + \frac{du}{dw} \cdot b\,dy + \frac{du}{dh} \cdot c\,dz$$

$$\exists\ d^n u = \sum_{\substack{i,j,k \geq 0 \\ i+j+k=n}} \frac{d^n u}{dv^i\, dw^j\, dh^k} \cdot \frac{n!}{i!\, j!\, k!} \cdot a^i b^j c^k$$

$$\cdot (dx)^i\ (dy)^j\ (dz)^k$$

$$d^{n+1} u = \sum_{\substack{i,j,k \geq 0 \\ i+j+k=n}} \frac{n!}{i!\, j!\, k!} \cdot a^i b^j c^k \cdot (dx)^i \cdot$$

$$\cdot (dy)^j\ (dz)^k \cdot \Big(\frac{d^{n+1} u}{dv^{i+1}\, dw^j\, dh^k} \cdot a\,dx + \frac{d^{n+1} u}{dv^i\, dw^{j+1}\, dh^k} \cdot$$

$$\cdot b\,dy + \frac{d^{n+1} u}{dv^i\, dw^j\, dh^{k+1}} \cdot c\,dz \Big) = \sum_{\substack{i',j',k' > 0 \\ i'+j'+k'=n+1}} \frac{d^{n+1} u}{dv^{i'}\, dw^{j'}\, dh^{k'}}$$

$$\cdot C(i', j', k') \cdot a^{i'} b^{j'} c^{k'} \cdot (dx)^{i'}\ (dy)^{j'}\ (dz)^{k'}$$

а) $i', j', k' > 0$

$$C(i', j', k') = \frac{n!}{(i'-1)!\,j'!\,k'!} + \frac{n!}{i'!\,(j'-1)!\,k'!} +$$

$$+ \frac{n!}{i'!\,j'!\,(k'-1)!} = \frac{n!\,(i'+j'+k')}{(i'!\,j'!\,k'!)} = \frac{(n+1)!}{i'!\,j'!\,k'!}$$

б) $i' = 0 \quad j', k' > 0$

$$C(0, j', k') = \frac{n!}{(j'-1)!\,k'!} + \frac{n!}{j'!\,(k'-1)!} = \frac{n!\,(j'+k')}{j'!\,k'!}$$

$$= \frac{(n+1)!}{\underset{1=0!}{\underset{\parallel}{i'!}}\,j'!\,k'!}$$

$i' = 0, \ j' = 0, \quad k' > 0$

$$C(0, 0, k') = \frac{n!}{(k'-1)!} = \frac{n!\,k'}{k'!} = \frac{(n+1)!}{\underset{0!}{\underset{\parallel}{\underset{1}{\underset{\parallel}{i'!}}}}\,\underset{0!}{\underset{\parallel}{\underset{1}{\underset{\parallel}{j'!}}}}\,k'!}$$

Доказано