Bringing Differential Private SGD to Practice: On the Independence of Gaussian Noise and the Number of Training Rounds

Marten Van Dijk [*1] **Nhuong V. Nguyen** [*2 3] **Toan N. Nguyen** [2 3] **Phuong Ha Nguyen** [4] **Lam M. Nguyen** [5]

Abstract

In DP-SGD each round communicates a local SGD update which leaks some new information about the underlying local data set to the outside world. In order to provide privacy, Gaussian noise with standard deviation σ is added to local SGD updates after performing a clipping operation. We show that for attaining (ϵ, δ)-differential privacy σ can be chosen equal to $\sqrt{2(\epsilon + \ln(1/\delta))/\epsilon}$ for $\epsilon = \Omega(T/N^2)$, where T is the total number of rounds and N is equal to the size of the local data set. In many existing machine learning problems, N is always large and $T = O(N)$. Hence, σ becomes "independent" of any $T = O(N)$ choice with $\epsilon = \Omega(1/N)$. This means that our σ *only depends on* N *rather than* T. As shown in our paper, this differential privacy characterization allows one to *a-priori* select parameters of DP-SGD based on a fixed privacy budget (in terms of ϵ and δ) in such a way to optimize the anticipated utility (test accuracy) the most. This ability of planning ahead together with σ's independence of T (which allows local gradient computations to be split among as many rounds as needed, even for large T as usually happens in practice) leads to a *proactive DP-SGD algorithm* that allows a client to balance its privacy budget with the accuracy of the learned global model based on local test data. We notice that the current state-of-the art differential privacy accountant method based on f-DP has a closed form for computing the privacy loss for DP-SGD. However, due to its interpretation complexity, it cannot be used in a simple way to plan ahead. Instead, accountant methods are only used for keeping track of how privacy budget has been spent (after the fact).

1. Introduction

Privacy leakage is a big problem in the big-data era. Solving a learning task based on big data intrinsically means that only through a collaborative effort sufficient data is available for training a global model with sufficient clean accuracy (utility). Federated learning is a framework where a learning task is solved by a loose federation of participating devices/clients which are coordinated by a central server (McMahan et al., 2017). Clients, who use own local data to participate in a learning task by training a global model, want to have privacy guarantees for their local proprietary data. For this reason DP-SGD (Abadi et al., 2016) was introduced as it adapts distributed Stochastic Gradient Descent (SGD) with Differential Privacy (DP). Here, differential privacy comes from adding Gaussian noise to local (client-computed) mini-batch SGD updates (called Local DP) and this wrings with being able to learn an accurate global model; more Gaussian noise leads to better differential privacy guarantees, but hurts accuracy. Since the most important mission in machine learning is achieving a good accuracy, the added Gausian noise cannot be too large and is constrained. There are two main problems:

• Each client wants to be able to *plan ahead* meaning that one can a-priori optimize hyper parameters given a target accuracy (utility) and target differential privacy budget.

• Each local SGD update with Gaussian DP noise communicated to the central server leaks privacy about a client's local data set. If this leakage, for example, composes linearly over communication rounds, then only a small number of communication rounds is possible in order to satisfy a target differential privacy budget. However, smaller number of communication rounds implies slower convergence to a reduced accuracy and one's target accuracy may not be attained. In order to achieve a good accuracy the number of

[*]Equal contribution [1]Centrum Wiskunde & Informatica, Amsterdam, Netherlands [2]Department of Computer Science and Engineering, University of Connecticut, Storrs, CT, USA [3]**AUTHORERR: Missing \icmlaffiliation.** [4]eBay Inc., San Jose, CA 95125 [5]IBM Research, Thomas J. Watson Research Center, Yorktown Heights, NY10598, USA. Correspondence to: Marten Van Dijk <marten.van.dijk@cwi.nl>, Nhuong V. Nguyen <nhuong.nguyen@uconn.edu>, Toan N. Nguyen <toan.nguyen@uconn.edu>, Phuong Ha Nguyen <phuongha.ntu@gmail.com>, Lam M. Nguyen <lamnguyen.mltd@gmail.com>.

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communication rounds is generally large.

In this paper we introduce theory that allows ahead planning and shows that the number of rounds can actually be quite large; we address the two main open problems as described above at the same time.

The current state-of-the-art differential privacy accountant method by (Zhu et al., 2021) based on the Gaussian differential privacy framework of (Dong et al., 2021) allows a client to keep track of how much "differential privacy" has been spent up to the current round. The client knows when to stop computing and stop sending local SGD updates to the central server as soon as its "privacy budget" has been depleted. The accountant method gives a precise characterization of the spent privacy budget.

However, the accountant method cannot be used to a-priori select parameters and anticipate whether sufficient accuracy can be achieved within the privacy budget available for the client. We note that the accountant method and the precise Gaussian differential privacy theory does give an expression of the differential privacy guarantee as a function/expression of parameters representing the local data set size N, representing the mini-batch or sample size s during local SGD iterations, the total number of rounds T, and the added Gaussian noise per communicated SGD update represented by a standard deviation σ. Problematic is that this expression is in a sense unwieldy in that it does not admit simple closed form. A simple closed form (or a sufficiently tight simple closed form approximation) is needed for optimizing (with respect to accuracy) the parameters σ, s, and T, given N and given a target differential privacy budget. Our aim is to *a-priori select parameters* in order to achieve the best accuracy of the final global model given the allocated privacy budget.

• We non-trivially improve the analysis of the moment accountant method in (Abadi et al., 2016) and show for the first time that (ϵ, δ)-differential privacy can be achieved for

$$\sigma = \sqrt{2(\epsilon + \ln(1/\delta))/\epsilon}$$

in practical parameter settings with (a) a reasonable DP guarantee by choosing $\delta \leq 1/N$ and ϵ smaller than 0.5, where (b) the total number K of gradient computations over all local rounds performed on the local data set is at least a constant ($\approx e/2$) times $\sqrt{\epsilon \ln(1/\delta)}$ epochs (of size N), and (c) T is at least another constant (≈ 2) times $(K/N)^2/\epsilon$. Condition (c) turns out to be the limiting constraint, but only shows a lower bound on T suggesting that T can be freely chosen as large as K. Analysis of this constraint shows that, for $\epsilon = \Omega(1/N)$, σ is "independent" of any $T = O(N)$ (see abstract).

• Simulations show a significantly smaller ϵ compared to current state of the art. For example, our theory applies to $\epsilon = 0.15$ for the non-convex problem of the simple neural network LeNet (LeCun et al., 1998) with cross entropy loss function for image classification of MNIST (LeCun & Cortes, 2010) at a test accuracy of 93%, compared to 98% without differential privacy. Notice that (Abadi et al., 2016) reports for a 60-dimensional PCA projection layer with a single 1,000-unit ReLU hidden layer for MNIST 90% test accuracy for $\epsilon = 0.5$ and 95% test accuracy for $\epsilon = 2$, both for $\delta = 10^{-5} = 0.6/N$.

• We discuss and detail in Supplemental Material a proactive DP-SGD framework where the client adapts the added Gaussian noise according to measured test accuracy in order to achieve the target accuracy while at the same time being in control not to cross its differential privacy budget. This provides a blueprint for the engineer who, without a-priori knowledge, wants to use our adaptive hyperparameter selection for DP-SGD. Here we use a varying sample (mini-batch) sequence in DP-SGD (this is covered by our theory). Also, the framework (and theory) applies to the asynchronous SGD framework.

Outline: In order to set up the proper background in Section 2, we first define (ϵ, δ)-differential privacy in Section 2.1, next discuss DP-SGD as introduced by (Abadi et al., 2016) in Section 2.2, and finally explain how f-DP and Gaussian DP relates to the results in this paper in Section 2.3. In Section 3 we explain our main theory with in Section 3.1 its usage in realizing planning ahead with proactive DP-SGD and in Section 3.2 the interpretation of the lower bound on T, where we discuss the independence of σ and T (that is, we explain there is no practical prohibitive limit to the total number of local training rounds in DP-SGD and that this can be chosen independent of the required differential privacy with corresponding Gaussian noise according to our formula). Experiments are in Section 4. The asynchronous SGD framework, detailed differential privacy proofs and analysis, and additional experiments with extra details are in Supplemental Material.

2. Differential Private SGD (DP-SGD)

2.1. Differential Privacy

Differential privacy (Dwork et al., 2006b; Dwork, 2011; Dwork et al., 2014; 2006a) defines privacy guarantees for algorithms on databases, in our case a client's sequence of mini-batch gradient computations on his/her training data set. The guarantee quantifies into what extent the output of a client (the collection of updates communicated to the server) can be used to differentiate among two adjacent training data sets d and d' (i.e., where one set has one extra element compared to the other set).

Definition 2.1. A randomized mechanism $\mathcal{M} : D \to R$ is (ϵ, δ)-DP (Differentially Private) if for any adjacent d and

d' in D and for any subset $S \subseteq R$ of outputs,

$$Pr[\mathcal{M}(d) \in S] \leq e^{\epsilon} Pr[\mathcal{M}(d') \in S] + \delta,$$

where the probabilities are taken over the coin flips of mechanism $\mathcal{M}$.

When using differential privacy in machine learning we typically use $\delta = 1/N$ (or $1/(10N)$) inversely proportional with the data set size N.

In order to prevent data leakage from inference attacks in machine learning (Lyu et al., 2020) such as the deep leakage from gradients attack (Zhu et al., 2019; Zhao et al., 2020; Geiping et al., 2020) or the membership inference attack (Shokri et al., 2017; Nasr et al., 2019; Song et al., 2019) a range of privacy-preserving methods have been proposed. Privacy-preserving solutions for federated learning are Local Differential Privacy (LDP) solutions (Abadi et al., 2016; Bhowmick et al., 2019; Naseri et al., 2021; Truex et al., 2019; Hao et al., 2020; Duchi et al., 2014) and Central Differential Privacy (CDP) solutions (Naseri et al., 2021; Geyer et al., 2018; McMahan et al., 2018; Papernot et al., 2018; Yu et al., 2019). In LDP, the noise for achieving differential privacy is computed locally at each client and is added to the updates before sending to the server – *in this paper we also consider LDP*. In CDP, a *trusted* server aggregates received client updates into a global model; in order to achieve differential privacy the server adds noise to the global model before communicating it to the clients.

2.2. DP-SGD

We analyse the Gaussian based differential privacy method, called DP-SGD, of (Abadi et al., 2016), depicted in Algorithm 1 *in a distributed setting* (see Appendix A). Rather than using the gradient $\nabla f(\hat{w}, \xi)$ itself, DP-SGD uses its clipped version $[\nabla f(\hat{w}, \xi)]_C$ where $[x]_C = x/\max\{1, \|x\|/C\}$. Clipping is needed because in general we cannot assume a bound C on the gradients (for example, the bounded gradient assumption is in conflict with strong convexity (Nguyen et al., 2018)), yet the added gradients need to be bounded by some constant C in order for the DP analysis of (Abadi et al., 2016) to go through.

DP-SGD uses a mini-batch approach where before the start of the i-th local round a random min-batch of sample size s_i is selected out of a local data set d of size $|d| = N$. Here, we slighty generalize DP-SGD's original formulation which uses a constant $s_i = s$ sample size sequence, while our analysis will hold for a larger class of sample size sequences. The inner loop maintains the sum U of gradient updates where each of the gradients correspond to the same local model $\hat{w}$ until it is replaced by a newer global model at the start of the outer loop. At the end of each local round the sum of updates U is obfuscated with Gaussian noise

Algorithm 1 DP-SGD: Local Model Updates with Differential Privacy

procedure LOCALSGDWITHDP(d)
  for $i \in \{0, \ldots, T-1\}$ **do**
    Receive the current global model $\hat{w}$ from Server.
    Uniformly sample a random set $\{\xi_h\}_{h=1}^{s_i} \subseteq d$
    $h = 0, U = 0$
    while $h < s_i$ **do**
      $g = [\nabla f(\hat{w}, \xi_h)]_C$
      $U = U + g$
      h++
    end while
    $n \leftarrow \mathcal{N}(0, C^2\sigma^2\mathbf{I})$
    $U = U + n$
    Send (i, U) to the Server.
  end for
end procedure

$\mathcal{N}(0, C^2\sigma^2)$ added to each vector entry, and the result is transmitted to the server. The server adds U times the round step size $\bar{\eta}_i$ to its global model $\hat{w}$. As soon as all clients have submitted their updates, the resulting new global model $\hat{w}$ is broadcast to all clients, who in turn replace their local models with the newly received global model.

2.3. Tight Analysis using Gaussian DP

Appendix C summarizes the recent work by (Dong et al., 2021) that introduces the f-DP framework based on hypothesis testing. f-DP has (ϵ, δ)-DP as a special case in that a mechanism is (ϵ, δ)-DP if and only if it is $f_{\epsilon,\delta}$-DP with $f_{\epsilon,\delta}(\alpha) = \min\{0, 1 - \delta e^{\epsilon}\alpha, (1 - \delta - \alpha)e^{-\epsilon}\}$. They prove that DP-SGD is $C_{s/N}(G_{(\sigma/2)^{-1}})^{\otimes T}$-DP where $C_{s/N}$ is an operator representing the effect of subsampling, $G_{(\sigma/2)^{-1}}$ is a Gaussian function characterizing the differential privacy (called Gaussian DP) due to adding Gaussian noise, and operator $\otimes T$ describes composition over T rounds. $C_{s/N}(G_{(\sigma/2)^{-1}})^{\otimes T}$-DP can be translated in a tight (ϵ, δ)-DP formulation.

This leads in (Zhu et al., 2021) to a differential privacy accountant (using a complex characteristic function based on taking the Fourier transform) for a client to understand when to stop helping the server to learn a global model. The goal of this paper is to a-priori understand how to, given a privacy budget and given a utility that we wish to achieve, optimize parameters σ, s and T for best utility and minimal privacy leakage given a data set of size N. As a method for keeping track (account for) spent privacy budget, differential privacy accountant does not give us this insight.

Towards understanding how to a-priori set parameters for best utility and minimal privacy leakage, the tight f-DP formulation for DP-SGD can be translated into sharp pri-

vacy guarantees. However, cited from (Dong et al., 2021), "the disadvantage is that the expressions it yields are more unwieldy: they are computer evaluable, so usable in implementations, but do not admit simple closed form." And it is a simple closed form or simple approximation of a closed form that is needed for an a-priori direct interpretation of how the different parameters σ, s and T impact the privacy budget.

The question remains of how to a-priori select concrete parameters σ, s, and T given concrete parameters for N, a given privacy budget and utility that we wish to achieve. In this paper we decided to generalize the proof method of (Abadi et al., 2016) rather than working with the complex integrals that provide the exact characterization of f-DP for the DP-SGD algorithm. This approach, as we will demonstrate, allows us to derive a concrete (non-asymptotic) simple form which turns out to show into large extent the independence of the privacy budget on T, which is not immediately understood from (Abadi et al., 2016), see Theorem B.1 in Supplemental Material, and the f-DP framework discussed above. The advantage of our result is that it is easy to interpret; clients do not need to fully rely on an accountant method to keep track of spent privacy budget, they can use our theory to a-priori set parameters such that privacy budget will be well-spent. (In addition, our theory holds for variable sized sample sequences, which is needed in our proactive DP-SGD, see Appendix E for details.)

3. Main Thm: Independence between σ and T

On the dependence of σ on T: As discussed in (Abadi et al., 2016), in order to control privacy leakage over an increasing number of training rounds, we need to increase the magnitude, or standard deviation σ, of the added noise per round. Increasing σ makes each round leakage smaller and as a result the aggregated leakage becomes less. For a fixed privacy leakage budget this allows us to add more entries, in other words, we may increase T. The maximum possible T increases as σ increases.

This dependence of the amount of Gaussian noise σ on the number of training rounds T may impose an impractical upper bound on T: First, σ cannot be too large. The global model suffers from aggregated noise from each local update. In particular, the global model includes σ noise from the final training round. If σ is too large, then this leads to a low accuracy global model. Second, if σ cannot be too large, then the number of training rounds T cannot be too large because otherwise privacy leakage aggregates too much over all training rounds. But this implies that the global model may receive too few local SGD updates; local SGD updates from different clients are communicated through the global model to one another using too few interactions/rounds (frequent exchange is especially important when local data sets are heterogeneous, hence, different clients produce distinctive updates that are specific to their local data sets). This leads again to a low accuracy global model. This intuition/reasoning indicates that Differential Private SGD (DP-SGD) can only be practical for subtle parameter settings and data sets that allow 'fast enough' training/convergence to a good accuracy global model. In general, DP-SGD is expected to be much less competitive compared to using other existing privacy techniques (such as secure multiparty computation even though this is less efficient and/or trusted execution environments which impose believe in a larger trusted computing base).

Main contribution: *We show that DP-SGD is actually competitive by non-trivially improving the analysis of the moment accountant method in (Abadi et al., 2016).* We state our main theorem below. It shows that (ϵ, δ)-differential privacy can be achieved for $\sigma = \sqrt{2(\epsilon + \ln(1/\delta))/\epsilon}$ in practical parameter settings with (a) a reasonable DP guarantee by choosing $\delta \leq 1/N$ and ϵ smaller than 0.5, where (b) the total number K of gradient computations over all local rounds performed on the local data set is at least a small constant times $\sqrt{\epsilon \ln(1/\delta)}$ epochs (of size N), and (c) T is at least another small constant times $(K/N)^2/\epsilon$.

Theorem 3.1. *Let σ and (ϵ, δ) satisfy the relation*

$$\sigma = \sqrt{2(\epsilon + \ln(1/\delta))/\epsilon} \text{ with } \delta \leq 1/N \text{ and } \epsilon < 0.5. \quad (1)$$

For sample size sequence $\{s_i\}_{i=0}^{T-1}$ the total number of local SGD iterations is equal to $K = \sum_{i=0}^{T-1} s_i$. We define $k = K/N$ as the total number of local SGD iterations measured in epochs (of size N). Related to the sample size sequence we define the mean $\bar{s}$ and maximum s_{max} and their quotient θ as

$$\bar{s} = \frac{1}{T}\sum_{i=0}^{T-1} s_i = \frac{K}{T}, \quad s_{max} = \max\{s_i\} \text{ and } \quad \theta = \frac{s_{max}}{\bar{s}}.$$

Let γ be the smallest solution satisfying

$$\gamma \geq \frac{2^4 \cdot \bar{\alpha}}{1-\bar{\alpha}} \left(\frac{\sigma}{(1-\sqrt{\bar{\alpha}})^2} + \frac{1}{\sigma(1-\bar{\alpha}) - 2e\sqrt{\bar{\alpha}}} \frac{e^3}{\sigma} \right) e^{3/\sigma^2} + \frac{2}{1-\bar{\alpha}} \quad \text{with } \bar{\alpha} = \frac{\epsilon}{\gamma k}.$$

Parameter $\gamma = 2 + O(\bar{\alpha})$, which is close to 2 for small $\bar{\alpha}$. We assume data sets of size $N \geq 10000$ and sample size sequences with $\theta \leq 6.85$. If

$$k \geq \sqrt{2\epsilon \ln(1/\delta)} \cdot e/(\gamma\theta) \text{ and} \quad (2)$$

$$T \geq \frac{\gamma\theta^2}{\epsilon} \cdot k^2, \quad (3)$$

then Algorithm 1 is (ϵ, δ)-differentially private.

All our theory, including the above theorem, holds in the asynchronous SGD framework of Appendix A. In Appendix A we provide a more general *asynchronous mini-batch SGD algorithm* (which follows Hogwild!'s philosophy (Recht et al., 2011; De Sa et al., 2015; Zhang et al., 2016; Nguyen et al., 2018; Leblond et al., 2018; Nguyen et al., 2021)) *with DP*. The asynchronous setting allows clients to adapt their sample sizes to their processing speed and communication latency.

We notice that polynomial increasing sample size sequences $s_i \sim qNi^p$ have $\bar{s} \approx [qNT^{p+1}/(p+1)]/T$ and $s_{max} = qNT^p$, hence, $\theta = 1 + p$. This shows that our theory covers e.g. linear increasing sample size sequences as discussed in (Nguyen et al., 2021), where is explained how this implies reduced round communication – another metric which one may trade-off against accuracy and total local number K of gradient computations.

The proof of the above theorem is in Appendix B and follows a sequence of steps: We discuss the analysis of (Abadi et al., 2016) and explain where we will improve. This leads to an improved analysis with a first generally applicable Theorem B.2. As a consequence we derive a simplified characterization in the form of Theorem B.4. Finally, we introduce more coarse bounds in order to extract the more readable Theorem 3.1. We notice that the simulations in Section 4 are based on parameters that satisfy constraints (28, 29, 30, 58) of Theorem B.4 as this leads to slightly better results.

One order of magnitude smaller ϵ: In order to attain an accuracy comparable to the non-DP setting where no noise is added, the papers cited in Section 2.1 generally require large ϵ (such that σ can be small enough) – which gives a weak privacy posture (a weak bound on the privacy loss). For example, when considering LDP (see Section 2.1), 10% deduction in accuracy yields only $\epsilon = 50$ in (Bhowmick et al., 2019) and $\epsilon = 10.7$ in (Naseri et al., 2021), while (Truex et al., 2019; Hao et al., 2020) show solutions for $\epsilon = 0.5$.

The theory presented in this paper allows relatively small Gaussian noise for small ϵ: We only need to satisfy the main equation (58). For example, in Section 4 simulations for the LIBSVM data set show $(\epsilon = 0.05, \delta = 1/N)$-DP is possible while achieving good accuracy with $\sigma \approx 20$. Such small ϵ is a significant improvement over existing literature and gives us significant more trust in that DP offers appropriate privacy.

Interpretation of Theorem 3.1: In Section 3.1 we discuss how we use Theorem 3.1 and implement *planning ahead*, that is, a-priori selection of hyper parameters given a target accuracy and target differential privacy budget. We discuss how this leads to proactive DP-SGD. In Section 3.2 we interpret and discuss constraint (3) in detail; here, we explain that σ depends on N rather than T. The other requirements need much less discussion: A practical usable differential privacy budget has $\delta \leq 1/N$ and $\epsilon \leq 0.5$. And lower bound (2) on K is generally satisfied in practice as usually K is 50 or 100s of epochs.

3.1. Planning Ahead

Expression (1) allows us to plan ahead. To see this, we first discuss the concept of a utility graph where a 'best-case' accuracy is depicted as a function of noise σ and clipping constant C in DP-SGD (see Section 4). In DP-SGD the last round of local updates is aggregated into an update of the global model, after which the global model is finalized. This means that the Gaussian noise added to a client local update of its last round is directly added as a perturbation to the final global model. We have a best-case scenario if we neglect the added noise of all previous rounds. That is, the 'best-case' accuracy for DP-SGD is the accuracy of a global model which is trained using SGD without DP after which Gaussian noise is added. The utility graph depicts this 'best-case' accuracy.

To generate the graph, we fix a diminishing learning rate η_t (step size) from round to round and we fix the total number K of local gradient computations that will be performed. Based on local training data and a-priori knowledge, a local client can run SGD locally without any added DP mechanism. This learns a local model w^* and we compare how much accuracy is sacrificed by adding Gaussian noise $n \sim \mathcal{N}(0, C^2\sigma^2\mathbf{I})$; that is, we compute and depict the ratio $F(w^* + n)/F(w^*)$. This teaches us the range of σ and C combinations that may lead to sufficient accuracy (say at most a 10% drop).

We notice that the server may start DP-SGD based on an architecture and model for another related problem that is already known. This model is used as a starting point to train the new model for the new problem (transfer learning). The utility graph of the model for the other related problem can be computed or is already known and a range of acceptable σ and C combinations can be given to the clients by the server at the start of their DP-SGD. Here, we assume that the utility graph of the other related problem transfers to the new problem.

We note that the Gaussian noise scales with the product $C\sigma$ and this suggests that lowering C allows larger σ, which leads to a better differential privacy guarantee according to (1). However, we cannot lower the clipping constant C arbitrarily: Clipping the gradient implies rescaling its norm to a lower value and intuitively this can be seen as having a similar effect when a (factor C) lower learning rate (step size) η_t is used during SGD updates. In our experiments we implement a diminishing step size sequence, however, in

order to obtain practical convergence we need an initial step size which is 'large enough'. Hence, we cannot start local SGD updates using a 'very small' clipping constant C. This observation does suggest that keeping the step size more constant (and not diminishing) over rounds together with a diminishing clipping constant may allow good convergence leading to sufficient accuracy/utility. This means that σ can increase from round to round, which implies that the privacy budget is being depleted at a lower decreasing rate. In this paper, however, we base our analysis on Theorem B.2 which we have only been able to prove for constant σ from round to round. For this reason, the theory in this paper needs to start with a clipping constant C large enough in order to initiate convergence and we do not decrease C from round to round. This means that both C and σ are constant throughout DP-SGD and the range of σ and C combinations learned from the utility graph is limited.

If there exists a σ and C combination for which the utility graph expects good enough accuracy of the final global model and for which formula (1) indicates an acceptable differential privacy guarantee, then we can choose parameters s and T freely in that their product $K = sT$, which represents the total number of local gradient computations over all rounds, is large enough to establish convergence (given the learning rate / step size sequence η_t and used clipping constant C). Experiments show that K being in the order of 50-100s epochs of gradient computations is sufficient. We optimize s and T for best accuracy of the final model. If all local data sets are iid, that is, come from the same source distribution, then the larger s and the smaller T seems best. So, for best accuracy in the iid case, we suggest to meet condition (3) with equality.

Proactive DP-SGD: The above describes how our main formula (1) can be used to implement planning ahead, that is, given a local data set size N with DP budget $(\epsilon_{target}, \delta_{target})$ and a prediction/test accuracy target acc_{target} (of the final global model) we can a-priori derive/set hyper-parameters σ, s, and T. We generalize this towards proactive DP-SGD in Appendix E where the client uses a private test data set to evaluate the accuracy of received global models. If the test accuracy does not converge towards an acceptable value $\geq acc_{target}$ at a noticeable rate from round to round, then the client adapts σ to a slightly smaller value (if allowed by the DP budget) and recomputes hyper parameters. Our theory is general enough to accommodate varying (in this case increasing) sample size sequences (we can use Theorem B.4 of which Theorem 3.1 is a consequence as it has no concrete restriction on θ). Our proactive DP-SGD is designed to attain the best DP guarantee given acc_{target} within our framework.

The main advantage of proactive DP-SGD is that without a-prior knowledge proactive DP-SGD learns how to fine tune parameters itself in order to reach a target test accuracy while remaining within a target privacy budget. Proactive DP-SGD does not need one to run other (distributed) experiments that potentially leak privacy in order to be able to select 'good' hyper parameters.

3.2. Interpreting the Limiting Constraint on T

Lower bound (3) on T shows that we may increase T without having to increase σ – in this sense σ does not depend on T anymore. This important discovery brings DP-SGD to practice because, in order to make the trained model have high accuracy even for large T as happens in practice, σ does not need to increase with T as indicated by our formula (1). In fact T *can become as large as* K (corresponding to sample size $s = 1$) implying that for $K = T$, lower bound (3) on T translates into ϵ at least a small constant $\gamma\theta^2$ times K/N^2 (leading to the interpretation in the abstract). First, this shows that in theory ϵ can be chosen very small for large enough data sets N and practical K (although as explained above, very small ϵ lead to too large σ implying low accuracy, hence, our requirement on accuracy likely leads to a more strict lower bound on ϵ). Second, requiring ϵ at least a constant $\gamma\theta^2$ times K/N^2 with $K = kN$ (that is, K is equal to k epochs of gradient computations) is equivalent to requiring N at least a constant $\gamma\theta^2$ times k/ϵ. For typical values of k (100s of epochs), this means

$$N = \Omega(1/\epsilon).$$

Our σ therefore depends on N rather than T because the constraints on selecting ϵ and δ depends only on N as explained above. For improved differential privacy (smaller ϵ), we need a larger data set (larger N). This observation comes natural since selecting (subsampling) a random batch of local training data among a larger data set helps in making round computation less dependent on an a-priori selected single local training data sample and being able to differentiate whether such a local training data sample is part of the larger local training data set is what differential privacy is about.

In Section 3.1 we explained that with respect to achieving good utility (sufficient clean accuracy) experiments show that reducing the so-called clipping constant up to some value leads to convergence to a global model that is less sensitive to noise. Reducing the clipping constant can mean a magnitude larger K in order for DP-SGD to converge (because a smaller clipping constant implies that locally computed gradients contribute less to the global model). This implies a larger number k of epochs of local gradient computations, hence, the local data set needs to be larger for differential privacy to hold (as N is required to be at least a constant $\gamma\theta^2$ times k/ϵ, see above).

We stress that in our theory T cannot grow arbitrarily large

as it is restricted by $K = kN$. Also k cannot grow arbitrarily large since $N \geq \gamma\theta^2 \cdot k/\epsilon$. This upper bound on k does impose a constraint after which (ϵ, δ)-DP cannot be guaranteed – so, K and, hence, T cannot increase indefinitely without violating the privacy budget. Here we notice that repeated use of the same data set over multiple learning problems (one after another) is allowed as long as the number of epochs gradient computing is $k \leq \frac{\epsilon N}{\gamma\theta^2}$. Hence, the larger N the more collaborative learning tasks the client can participate in. For typical values $\epsilon = 0.2$, $\gamma\theta^2 \approx 2$, and a data set of size $N = 10000$ we have $k \leq 1000$, which accommodates about 10-20 learning tasks.

As a final remark, Section C.2 analyses the tightness of Theorem 3.1 in an asymptotic setting: The lower bound on T in Theorem 3.1 can be reduced by at most a factor $4\gamma\theta^2$.

4. Experiments

Our goal is to show that the more general asynchronous differential privacy framework (asynchronous DP-SGD which includes DP-SGD of Algorithm 1) of Appendix A ensures a strong privacy guarantee, i.e, can work with very small ϵ (and $\delta = 1/N$), while having a good convergence rate to good accuracy. We refer to Appendix D for simulation details and complete parameter settings.

Objective function. We summarize experimental results of our asynchronous DP-SGD framework for strongly convex, plain convex and non-convex objective functions with *constant sample size sequences*. As the plain convex objective function we use logistic regression: The weight vector w and bias value b of the logistic function can be learned by minimizing the log-likelihood function J:

$$J = -\sum_{i=1}^{N}[y_i \cdot \log(\bar{\sigma}_i) + (1 - y_i) \cdot \log(1 - \bar{\sigma}_i)],$$

where N is the number of training samples (x_i, y_i) with $y_i \in \{0, 1\}$, and $\bar{\sigma}_i = 1/(1 + e^{-(w^{\mathrm{T}} x_i + b)})$ is the sigmoid function. The goal is to learn a vector/model w^* which represents a pair $\bar{w} = (w, b)$ that minimizes J. Function J changes into a strongly convex problem by adding ridge regularization with a regularization parameter $\lambda > 0$, i.e., we minimize $\hat{J} = J + \frac{\lambda}{2} \|\bar{w}\|^2$ instead of J. For simulating non-convex problems, we choose a simple neural network (LeNet) (LeCun et al., 1998) with cross entropy loss function for image classification.

Asynchronous DP-SGD setting. The experiments are conducted with 5 compute nodes and 1 central server. For simplicity, the compute nodes have iid datasets.

Data sets. All our experiments are conducted on LIBSVM (Chang & Lin, 2011)[1] and MNIST (LeCun & Cortes, 2010) [2] data sets.

4.1. Utility graph

Since we do not have a closed form to describe the relation between the utility of the model (i.e., prediction accuracy) and σ, we propose a heuristic approach to learn the range of σ from which we may select σ for finding the best (ϵ, δ)-DP. The utility graphs – Figures 1a, 2a and 3a – show the fraction of test accuracy between the model $F(w + n)$ over the original model $F(w)$ (without noise), where $n \sim \mathcal{N}(0, C^2\sigma^2\mathbf{I})$ for various values of the clipping constant C and noise standard deviation σ. Intuitively, the closer $F(w + n)/F(w)$ to 1, the better accuracy wrt to $F(w)$. Note that w can be any solution and in the utility graphs, we choose $w = w^*$ with w^* being near to an optimal solution.

The smaller C, the larger σ can be, hence, ϵ can be smaller which gives stronger privacy. However, the smaller C, the more iterations (larger K) are needed for convergence. And if selected too small, no fast enough convergence is possible.

In next experiments we use clipping constant $C = 0.1$, which gives a drop of at most 10% in test accuracy for $\sigma \leq 20$ for both strongly convex and plain convex objective functions. To keep the test accuracy loss $\leq 10\%$ in the non-convex case, we choose $C = 0.025$ and $\sigma \leq 12$.

4.2. Asynchronous DP-SGD with different constant sample size

Figure 1b and Figure 2b illustrate the test accuracy of our asynchronous DP-SGD with various constant sample sizes for the strongly convex and plain convex cases. Here, we use privacy budget $\epsilon = 0.04945$ and noise $\sigma = 19.2$. It is clear that with $s = 1$, the algorithm shows a bad test accuracy though this constant sample size has the maximum communication rounds. When we use a bigger constant sample size s, for example, $s = 26$, our algorithm can achieve the desired performance, when compared to other constant sample sizes.[3] The experiment is extended to the non-convex case as shown in Figure 3b, where we can see a similar pattern. Experimental results for other data sets are in Appendix D in Supplemental Material. This confirms that our DP-SGD framework can converge to a decent accuracy while achieving a very small privacy budget ϵ.

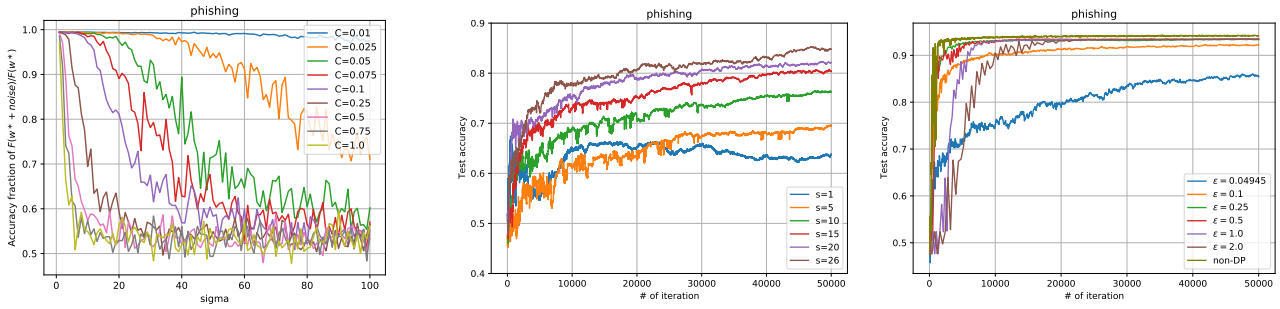
4.3. Asynchronous DP-SGD with different levels of privacy budget

Figure 1c and Figure 2c show that our DP-SGD framework converges to better accuracy if ϵ is slightly larger. E.g., in the

[1] https://www.csie.ntu.edu.tw/ cjlin/libsvmtools/datasets/binary.html

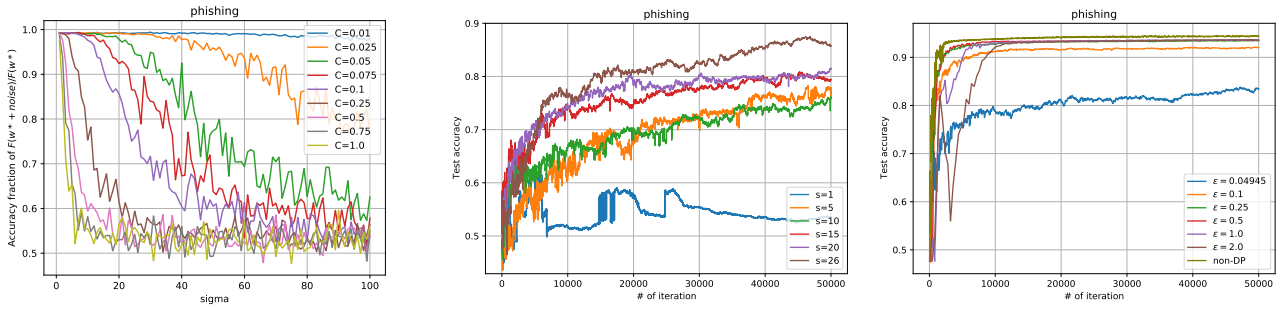
[2] http://yann.lecun.com/exdb/mnist/

[3] $s = 26$ meets the lower bound on T; a larger s would violate this lower bound because of the relatively small data set size.



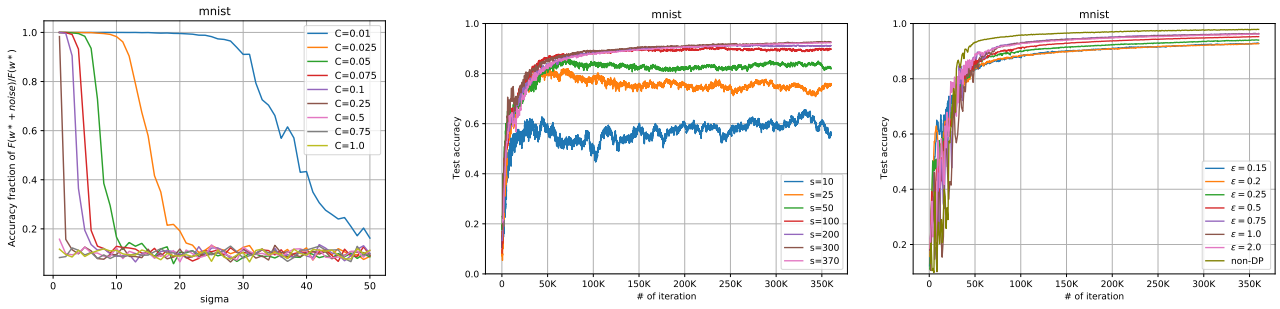
(a) (b) (c)

Figure 1. Strongly convex. (a) Utility graph, (b) Different s, (c) Different ϵ



(a) (b) (c)

Figure 2. Plain convex. (a) Utility graph, (b) Different s, (c) Different ϵ



(a) (b) (c)

Figure 3. Non-convex. (a) Utility graph, (b) Different s, (c) Different ϵ

strongly convex case, privacy budget $\epsilon = 0.04945$ achieves test accuracy 86% compared to 93% without differential privacy (hence, no added noise); $\epsilon = 0.1$, still significantly smaller than what is reported in literature, achieves test accuracy 91%. Figure 3c shows the test accuracy of our asynchronous DP-SGD for different privacy budgets ϵ in the non-convex case. For $\epsilon = 0.15$, our framework can achieve the test accuracy about 93%, compared to 98% without differential privacy. These figures again confirm the effectiveness of our DP-SGD framework, which can obtain a strong differential privacy guarantee.

5. Conclusion

We proved a new differential privacy guarantee for DP-SGD which is into large extent insensitive to the total number of communication rounds, attains significantly smaller ϵ than what has been reported in literature (implying a sensible privacy guarantee), and does this for reasonable DP noise such that test accuracy does not suffer much. We showed how the local data set size N plays an important role; it bounds $\epsilon = \Omega(1/N)$ for $O(1)$ epochs local gradient evaluations. Our theory is general in that it applies to asynchronous DP-SGD (like Hogwild!) and it analyses arbitrary sequences of sample sizes. We show how our theory allows

planning ahead, i.e., being able to a-priori optimize hyper parameters given a target accuracy and target differential privacy budget. In our proactive DP-SGD framework it is possible for local clients to adaptively trade-off privacy budget with test accuracy. Clients can also control the privacy leakage of their local data set over multiple learning tasks. The above indicates that DP-SGD is competitive with other solutions such as distributed DP which employs a secure function (such as a secure shuffler) implemented by secure Multi Party Computation (which is communication intense) and/or a Trusted Execution Environment (which requires trust in a larger Trusted Computing Base).

References

Abadi, M., Chu, A., Goodfellow, I., McMahan, H. B., Mironov, I., Talwar, K., and Zhang, L. Deep learning with differential privacy. In *Proceedings of the 2016 ACM SIGSAC Conference on Computer and Communications Security*, pp. 308–318. ACM, 2016.

Bhowmick, A., Duchi, J., Freudiger, J., Kapoor, G., and Rogers, R. Protection against reconstruction and its applications in private federated learning, 2019.

Chang, C.-C. and Lin, C.-J. LIBSVM: A library for support vector machines. *ACM Transactions on Intelligent Systems and Technology*, 2:27:1–27:27, 2011.

De Sa, C. M., Zhang, C., Olukotun, K., and Ré, C. Taming the wild: A unified analysis of hogwild-style algorithms. In *NIPS*, pp. 2674–2682, 2015.

Dong, J., Roth, A., and Su, W. Gaussian differential privacy. *Journal of the Royal Statistical Society*, 2021.

Duchi, J. C., Jordan, M. I., and Wainwright, M. J. Local privacy, data processing inequalities, and statistical minimax rates, 2014.

Dwork, C. A firm foundation for private data analysis. *Communications of the ACM*, 54(1):86–95, 2011.

Dwork, C., Kenthapadi, K., McSherry, F., Mironov, I., and Naor, M. Our data, ourselves: Privacy via distributed noise generation. In *Annual International Conference on the Theory and Applications of Cryptographic Techniques*, pp. 486–503. Springer, 2006a.

Dwork, C., McSherry, F., Nissim, K., and Smith, A. Calibrating noise to sensitivity in private data analysis. In *Theory of cryptography conference*, pp. 265–284. Springer, 2006b.

Dwork, C., Roth, A., et al. The algorithmic foundations of differential privacy. *Foundations and Trends® in Theoretical Computer Science*, 9(3–4):211–407, 2014.

Geiping, J., Bauermeister, H., Dröge, H., and Moeller, M. Inverting gradients – how easy is it to break privacy in federated learning? In *NIPS*, 2020.

Geyer, R. C., Klein, T., and Nabi, M. Differentially private federated learning: A client level perspective, 2018.

Hao, M., Li, H., Luo, X., Xu, G., Yang, H., and Liu, S. Efficient and privacy-enhanced federated learning for industrial artificial intelligence. *IEEE Transactions on Industrial Informatics*, 16(10):6532–6542, 2020. doi: 10.1109/TII.2019.2945367.

Krizhevsky, A., Sutskever, I., and Hinton, G. E. Imagenet classification with deep convolutional neural networks. *Advances in neural information processing systems*, 25: 1097–1105, 2012.

Leblond, R., Pedregosa, F., and Lacoste-Julien, S. Improved asynchronous parallel optimization analysis for stochastic incremental methods. *JMLR*, 19(1):3140–3207, 2018.

LeCun, Y. and Cortes, C. MNIST handwritten digit database. 2010. URL http://yann.lecun.com/exdb/mnist/.

LeCun, Y., Bottou, L., Bengio, Y., and Haffner, P. Gradient-based learning applied to document recognition. *Proceedings of the IEEE*, 86(11):2278–2324, 1998.

Lyu, L., Yu, H., and Yang, Q. Threats to federated learning: A survey, 2020.

McMahan, B., Ramage, D., Talwar, K., and Zhang, L. Learning differentially private recurrent language models. In *International Conference on Learning Representations (ICLR)*, 2018. URL https://openreview.net/pdf?id=BJ0hF1Z0b.

McMahan, H. B., Moore, E., Ramage, D., Hampson, S., and y Arcas, B. A. Communication-efficient learning of deep networks from decentralized data, 2017.

Naseri, M., Hayes, J., and Cristofaro, E. D. Toward robustness and privacy in federated learning: Experimenting with local and central differential privacy, 2021.

Nasr, M., Shokri, R., and Houmansadr, A. Comprehensive privacy analysis of deep learning: Passive and active white-box inference attacks against centralized and federated learning. In *2019 IEEE Symposium on Security and Privacy (SP)*, pp. 739–753, 2019. doi: 10.1109/SP.2019.00065.

Nguyen, L., Nguyen, P. H., Dijk, M., Richtárik, P., Scheinberg, K., and Takác, M. Sgd and hogwild! convergence without the bounded gradients assumption. In *International Conference on Machine Learning*, pp. 3750–3758. PMLR, 2018.

Nguyen, N., Nguyen, T., Nguyen, P. H., Tran-Dinh, Q., Nguyen, L., and Dijk, M. Hogwild! over distributed local data sets with linearly increasing mini-batch sizes. In *International Conference on Artificial Intelligence and Statistics*, pp. 1207–1215. PMLR, 2021.

Papernot, N., Song, S., Mironov, I., Raghunathan, A., Talwar, K., and Úlfar Erlingsson. Scalable private learning with pate. In *International conference on learning representations*, 2018.

Recht, B., Re, C., Wright, S., and Niu, F. Hogwild: A lock-free approach to parallelizing stochastic gradient descent. In *Advances in neural information processing systems*, pp. 693–701, 2011.

Roux, N., Schmidt, M., and Bach, F. A stochastic gradient method with an exponential convergence rate for finite training sets. In Pereira, F., Burges, C. J. C., Bottou, L., and Weinberger, K. Q. (eds.), *Advances in Neural Information Processing Systems*, volume 25. Curran Associates, Inc., 2012.

Shokri, R., Stronati, M., Song, C., and Shmatikov, V. Membership inference attacks against machine learning models. In *2017 IEEE Symposium on Security and Privacy (SP)*, pp. 3–18. IEEE, 2017.

Song, L., Shokri, R., and Mittal, P. Membership inference attacks against adversarially robust deep learning models. In *2019 IEEE Security and Privacy Workshops (SPW)*, pp. 50–56, 2019. doi: 10.1109/SPW.2019.00021.

Truex, S., Baracaldo, N., Anwar, A., Steinke, T., Ludwig, H., Zhang, R., and Zhou, Y. A hybrid approach to privacy-preserving federated learning. In *Proceedings of the 12th ACM Workshop on Artificial Intelligence and Security*, pp. 1–11, 2019.

Wasserman, L. and Zhou, S. A statistical framework for differential privacy. *Journal of the American Statistical Association*, 105(489):375–389, 2010.

Yu, L., Liu, L., Pu, C., Gursoy, M. E., and Truex, S. Differentially private model publishing for deep learning. *2019 IEEE Symposium on Security and Privacy (SP)*, May 2019. doi: 10.1109/sp.2019.00019. URL http://dx.doi.org/10.1109/SP.2019.00019.

Zhang, H., Hsieh, C.-J., and Akella, V. Hogwild++: A new mechanism for decentralized asynchronous stochastic gradient descent. In *2016 IEEE 16th International Conference on Data Mining (ICDM)*, pp. 629–638. IEEE, 2016.

Zhao, B., Mopuri, K. R., and Bilen, H. idlg: Improved deep leakage from gradients, 2020.

Zhu, L., Liu, Z., and Han, S. In *Advances in Neural Information Processing Systems*, volume 32, 2019.

Zhu, Y., Dong, J., and Wang, Y.-X. Optimal accounting of differential privacy via characteristic function. *arXiv preprint arXiv:2106.08567*, 2021.

Bringing Differential Private SGD to Practice: On the Independence of Gaussian Noise and the Number of Training Rounds Appendix

A. Asynchronous Mini-Batch DP-SGD

The optimization problem for training many Machine Learning (ML) models using a training set $\{\xi_i\}_{i=1}^m$ of m samples can be formulated as a finite-sum minimization problem as follows

$$\min_{w\in\mathbb{R}^d}\left\{F(w)=\frac{1}{m}\sum_{i=1}^{m}f(w;\xi_i)\right\}. \tag{4}$$

The objective is to minimize a loss function with respect to model parameters w. This problem is known as empirical risk minimization and it covers a wide range of convex and non-convex problems from the ML domain, including, but not limited to, logistic regression, multi-kernel learning, conditional random fields and neural networks.

We want to solve (4) in a distributed setting where many clients have their own local data sets and the finite-sum minimization problem is over the collection of all local data sets. A widely accepted approach is to repeatedly use the Stochastic Gradient Descent (SGD) recursion

$$w_{t+1}=w_t-\eta_t\nabla f(w_t;\xi), \tag{5}$$

where w_t represents the model after the t-th iteration; w_t is used in computing the gradient of $f(w_t;\xi)$, where ξ is a data sample randomly selected from the data set $\{\xi_i\}_{i=1}^m$ which comprises the union of all local data sets.

This approach allows each client to perform local SGD recursions for the ξ that belong to the client's local data set. The updates as a result of the SGD recursion (5) are sent to a centralized server who aggregates all received updates and maintains a global model. The server regularly broadcasts its most recent global model so that clients can use it in their local SGD computations. This allows each client to use what has been learned from the local data sets at the other clients. This leads to good accuracy of the final global model.

Each client is doing SGD recursions for a batch of local data. These recursions together represent a local round and at the end of the local round the sum of local model updates, i.e., the addition of computed gradients, is transmitted to the server. The server in turn adds the received sum of local updates to its global model – and once the server receives new sums from all clients, the global model is broadcast to each of the clients. When considering privacy, we are concerned about how much information these sums of local updates reveal about the used local data sets. Each client wants to keep its local data set as private as possible with respect to the outside world which observes round communication (the outside world includes all other clients as well).

Rather than reducing the amount of round communication such that less sensitive information is leaked, differential privacy (Dwork et al., 2006b; Dwork, 2011; Dwork et al., 2014; 2006a) offers a solution in which each client-to-server communication is obfuscated by noise. If the magnitude of the added noise is not too much, then a good accuracy of the global model can still be achieved albeit at the price of more overall SGD iterations needed for achieving good accuracy. On the other hand, only if the magnitude of the added noise is large enough, then good differential privacy guarantees can be given.

When using Gaussian based differential privacy, each local round produces a sum of locally computed gradients based on the local data set, which is transmitted to the central server after adding Gaussian noise (Abadi et al., 2016). Because of this, each local training round communicates information about the corresponding local data set. Privacy leakage aggregates over multiple training rounds, and the total amount of privacy leakage from the local data set increases as the total number T of local training rounds increases.

Algorithms[4] 2, 3, and 4 explain in pseudo code our asynchronous LDP approach. It is based on the Hogwild! (Recht et al., 2011) recursion

$$w_{t+1} = w_t - \eta_t \nabla f(\hat{w}_t; \xi_t), \tag{6}$$

where $\hat{w}_t$ represents the vector used in computing the gradient $\nabla f(\hat{w}_t; \xi_t)$ and whose vector entries have been read (one by one) from an aggregate of a mix of previous updates that led to w_j, $j \leq t$. In a single-thread setting where updates are done in a fully consistent way, i.e. $\hat{w}_t = w_t$, yields SGD with diminishing step sizes $\{\eta_t\}$.

Recursion (6) models asynchronous SGD. The amount of asynchronous behavior that can be tolerated is given by some function $\tau(t)$, see (Nguyen et al., 2018) where this is analysed for strongly convex objective functions: We say that the sequence $\{w_t\}$ is consistent with delay function τ if, for all t, vector $\hat{w}_t$ includes the aggregate of the updates up to and including those made during the $(t - \tau(t))$-th iteration, i.e.,

$$\hat{w}_t = w_0 - \sum_{j \in \mathcal{U}} \eta_j \nabla f(\hat{w}_j; \xi_j)$$

for some $\mathcal{U}$ with $\{0, 1, \ldots, t - \tau(t) - 1\} \subseteq \mathcal{U}$.

In Algorithm 4 the local SGD iterations all compute gradients based on the same local model $\hat{w}$, which gets substituted by a newer global model $\hat{v}_k$ as soon as it is received by the interrupt service routine ISRRECEIVE. As explained in ISRRECEIVE $\hat{v}_k$ includes all the updates from all the clients up to and including their local rounds $\leq k$. This shows that locally the delay τ can be estimated based on the current local round i together with k. Depending on how much delay can be tolerated SETUP defines $\Upsilon(k, i)$ to indicate whether the combination (k, i) is permissible (i.e., the corresponding delay aka asynchronous behavior can be tolerated). It has been shown that for strongly convex objective functions (without DP enhancement) the convergence rate remains optimal even if the delay $\tau(t)$ is as large as $\approx \sqrt{t/\ln t}$ (Nguyen et al., 2018). Similar behavior has been reported for plain convex and non-convex objective functions in (Nguyen et al., 2021).

In Algorithm 4 we assume that messages/packets never drop; they will be resent but can arrive out of order. This guarantees that we get out of the "while $\Upsilon(k, i)$ is false loop" because at some moment the server receives all the updates in order to broadcast a new global model $\hat{v}_{k+1}$ and once received by ISRRECEIVE this will increment k and make $\Upsilon(k, i)$ true which allows LOCALSGDWITHDP to exit the wait loop. As soon as the wait loop is exited we know that all local gradient computations occur when $\Upsilon(k, i)$ is true which reflect that these gradient computations correspond to delays that are permissible (in that we still expect convergence of the global model to good accuracy).

Algorithm 2 Client – Local model with Differential Privacy

1: **procedure** SETUP(n):

Initialize sample size sequence $\{s_i\}_{i=0}^{T}$, (diminishing) round step sizes $\{\bar{\eta}_i\}_{i=0}^{T}$, and a default global model $\hat{v}_0$ to start with.

Define a permissible delay function $\Upsilon(k, i) \in \{\textbf{True}, \textbf{False}\}$ which takes the current local round number i and the round number k of the last received global model into account to find out whether local SGD should wait till a more recent global model is received. $\Upsilon(\cdot, \cdot)$ can also make use of knowledge of the sample size sequences used by each of the clients.

2: **end procedure**

In this paper we analyse the Gaussian based differential privacy method of (Abadi et al., 2016). We use their clipping method; rather than using the gradient $\nabla f(\hat{w}, \xi)$ itself, we use its clipped version $[\nabla f(\hat{w}, \xi)]_C$ where $[x]_C = x/\max\{1, \|x\|/C\}$. Also, we use the same mini-batch approach where before the start of the i-th local round a random min-batch of sample size s_i is selected. During the inner loop the sum of gradient updates is maintained where each of the gradients correspond to the same local model $\hat{w}$ until it is replaced by a newer global model. In supplementary material B we show that this is needed for proving DP guarantees and that generalizing the algorithm by locally implementing the Hogwild! recursion itself (which updates the local model each iteration) does not work together with the DP analysis. So, our approach only uses the Hogwild! concept at a global round by round interaction level.

At the end of each local round the sum of updates U is obfuscated with Gaussian noise; Gaussian noise $\mathcal{N}(0, C^2\sigma_i^2)$ is added to each vector entry. In this general description σ_i is round dependent, but our DP analysis in Supplementary Material

[4] Our pseudocode uses the format from (Nguyen et al., 2021).

Algorithm 3 Client – Local model with Differential Privacy

procedure ISRRECEIVE($\hat{v}_k$):

This Interrupt Service Routine is called whenever a new broadcast global model $\hat{v}_k$ is received from the server. Once received, *the client's local model $\hat{w}$ is replaced with $\hat{v}_k$* (if no more recent global model $\hat{v}_{>k}$ was received out of order before receiving this $\hat{v}_k$)

The server broadcasts global model $\hat{v}_k$ for global round number k once the updates corresponding to local round numbers $\leq k-1$ from *all* clients have been received and have been aggregated into the global model. The server aggregates updates from clients into the current global model as soon as they come in. This means that $\hat{v}_k$ includes all the updates from all the clients up to and including their local round numbers $\leq k-1$ and potentially includes updates corresponding to later round numbers from subsets of clients. The server broadcasts the global round number k together with $\hat{v}_k$.

end procedure

Algorithm 4 Client – Local model with Differential Privacy

procedure LOCALSGDWITHDP(d)
  $i = 0$, $\hat{w} = \hat{v}_0$
  while True do
    while $\Upsilon(k, i) =$ **False do** nothing **end** ▷ k is the global round at the server.
    Uniformly sample a random set $\{\xi_h\}_{h=1}^{s_i} \subseteq d$
    $h = 0$, $U = 0$
    while $h < s_i$ **do**
      $g = [\nabla f(\hat{w}, \xi_h)]_C$
      $U = U + g$
      h++
    end while
    $n \leftarrow \mathcal{N}(0, C^2\sigma_i^2\mathbf{I})$
    $U = U + n$
    $\hat{w} = \hat{w} + \bar{\eta}_i \cdot U$
    Send (i, U) to the Server.
    i++
  end while
end procedure

B must from some point onward assume a constant $\sigma = \sigma_i$ over all rounds. The noised U times the round step size $\bar{\eta}_i$ is added to the local model after which a new local round starts again.

The noised U is also transmitted to the server who adds U times the round step size $\bar{\eta}_i$ to its global model $\hat{v}$. As soon as all clients have submitted their updates up to and including their local rounds $\leq k-1$, the global model $\hat{v}$, denoted as $\hat{v}_k$, is broadcast to all clients, who in turn replace their local models with the newly received global model. Notice that $\hat{v}_k$ may include updates from a subset of clients that correspond to local rounds $\geq k$. We refer to (Nguyen et al., 2021) for pseudo code of these computations at the server.

The presented algorithm adapts to asynchronous behavior in the following two ways: We explained above that the broadcast global models $\hat{v}_k$ themselves include a mix of received updates that correspond to local rounds $\geq k$ – this is due to asynchronous behavior. Second, the sample size sequence $\{s_i\}$ does not necessarily need to be fixed a-priori during SETUP (the round step size sequence $\{\bar{\eta}_i\}$ does need to be fixed a-priori). In fact, the client can adapt its sample sizes s_i on the fly to match its speed of computation and communication latency. This allows the client to adapt its local mini-batch SGD to its asynchronous behavior due to the scheduling of its own resources. Our DP analysis holds for a wide range of varying sample size sequences.

We notice that adapting sample size sequences on a per client basis still fits the same overall objective function as long as all local data sets are iid: This is because iid implies that the execution of the presented algorithm can be cast in a single Hogwild! recursion where the ξ_h are uniformly chosen from a common data source distribution $\mathcal{D}$. This corresponds to the

stochastic optimization problem

$$\min_{w\in\mathbb{R}^d} \{F(w) = \mathbb{E}_{\xi\sim\mathcal{D}}[f(w;\xi)]\},$$

which defines objective function F (independent of the locally used sample size sequences). Local data sets being iid in the sense that they are all, for example, drawn from car, train, boat, etc images benefit from DP in that car details (such as an identifying number plate), boat details, etc. need to remain private.

B. Differential privacy proofs

This appendix provides the proof of Theorem 3.1. It follows a sequence of steps: In Section B.1 we discuss the analysis of (Abadi et al., 2016) and explain where we will improve. This leads in Section B.2 to an improved analysis yielding a first generally applicable Theorem B.2; DP definitions/tools with a key lemma (generalized from (Abadi et al., 2016)) are discussed in Section B.2.1 and the proof of Theorem B.2 is in Section B.2.2. As a consequence we derive in Section B.3 a simplified characterization in the form of Theorem B.4. Finally, we introduce more coarse bounds in order to extract the more readable Theorem 3.1 in Section B.4.

B.1. DP-SGD Analysis by Abadi et al.

(Abadi et al., 2016) proves the following theorem (rephrased using our notation substituting $q = s/N$):

Theorem B.1. *There exist constants c_1 and c_2 so that given a sample size sequence $s_i = s$ and number of steps T, for any $\epsilon < c_1 T(s/N)^2$, Algorithm 1 is (ϵ,δ)-differentially private for any $\delta > 0$ if we choose*

$$\sigma \geq c_2 \frac{(s/N)\cdot\sqrt{T\ln(1/\delta)}}{\epsilon}.$$

The theorem suggests a necessary dependence between σ and T where σ scales with $\sqrt{T}$ for fixed s and N. If this is indeed necessary, then in order to achieve good accuracy for a given σ we need to increase the number of rounds T, but this requires us to increase σ, which increases T, etc. This reasoning would imply a subtle setting of σ and T for a given privacy budget (ϵ,δ) that leads to 'best' accuracy. Here, σ must be small enough so that it does not causes too much noise in the final global model. But this implies that the 'best' accuracy is restricted by only being able to use a limited number of training rounds T (since T cannot be too large for small enough $\sigma = O(\sqrt{T})$).

The interpretation of Theorem B.1, however, is more subtle: The condition on ϵ in Theorem B.1 is equivalent to

$$1/\sqrt{c_1} < z \text{ where } z = (s/N)\cdot\sqrt{T/\epsilon}.$$

Substituting this into the bound for σ yields

$$\sigma \geq (c_2\cdot z)\cdot\sqrt{\frac{\ln(1/\delta)}{\epsilon}}. \tag{7}$$

This formulation only depends on T through the definition of z. Notice that z may be as small as $1/\sqrt{c_1}$. In fact, it is unclear how z depends on T since T is equal to the total number K of gradient computations over all local rounds performed on the local data set divided by the mini-batch size s, i.e., $T = K/s$, hence, $z = (K/N)\cdot\sqrt{1/(T\epsilon)}$. This shows that for fixed K and N, we can increase T as long as $1/\sqrt{c_1} < z$, or equivalently,

$$T < c_1(K/N)^2/\epsilon \tag{8}$$

(notice that the original constraint on ϵ in Theorem B.1 directly translates into this upper bound on T by using $T = K/s$). Since σ cannot be chosen too large (otherwise the final global model has too much noise), ϵ, see (7), cannot be very small. Therefore, this more precise analysis still puts an upper bound on T which is in general much less than K for practically sized large data sets (K equals the maximum possible number of rounds for mini-batch size $s = 1$).

Rather than applying Theorem B.1, we can directly use the moment accountant method of its proof to analyse specific parameter settings. It turns out that T can be much larger than upper bound (8). In this paper we formalize this insight (by showing that 'constants' c_1 and c_2 can be chosen as functions of T and other parameters) and show a lower bound on σ which does not depend on T at all – in fact z in (7) can be characterized as a constant independent of any parameters. This will show that σ can remain small up to a lower bound that only depends on the privacy budget. We can freely increase T (up to K if needed) in order to improve accuracy.

B.2. A General Improved DP-SGD Analysis

We generalize Theorem B.1 (Abadi et al., 2016):

Theorem B.2. *We assume that $\sigma = \sigma_i$ with $\sigma \geq 216/215$ for all rounds i. Let*

$$r = r_0 \cdot 2^3 \cdot \left(\frac{1}{1-u_0} + \frac{1}{1-u_1} \frac{e^3}{\sigma^3} \right) e^{3/\sigma^2} \text{ with } u_0 = \frac{2\sqrt{r_0 \sigma}}{\sigma - r_0} \text{ and } u_1 = \frac{2e\sqrt{r_0 \sigma}}{(\sigma - r_0)\sigma},$$

where r_0 is such that it satisfies

$$r_0 \leq 1/e, \; u_0 < 1, \text{ and } u_1 < 1.$$

Let the sample size sequence satisfy $s_i/N \leq r_0/\sigma$. For $j = 1, 2, 3$ we define $\hat{S}_j$ (resembling an average over the sum of j-th powers of s_i/N) with related constants ρ and $\hat{\rho}$:

$$\hat{S}_j = \frac{1}{T} \sum_{i=0}^{T-1} \frac{s_i^j}{N(N-s_i)^{j-1}}, \quad \frac{\hat{S}_1 \hat{S}_3}{\hat{S}_2^2} \leq \rho \text{ and } \frac{\hat{S}_1^2}{\hat{S}_2} \leq \hat{\rho}.$$

Let $\epsilon = c_1 T \hat{S}_1^2$. Then, Algorithm 4 is (ϵ, δ)-differentially private if

$$\sigma \geq \frac{2}{\sqrt{c_0}} \frac{\sqrt{\hat{S}_2 T(\epsilon + \ln(1/\delta))}}{\epsilon} \text{ where } c_0 = c(c_1) \text{ with } c(x) = \min \left\{ \frac{\sqrt{2r\rho x + 1} - 1}{r\rho x}, \frac{2}{\hat{\rho} x} \right\}.$$

This generalizes Theorem B.1 where all $s_i = s$ are constant. First, Theorem B.2 covers a much broader class of sample size sequences that satisfy bounds on their 'moments' $\hat{S}_j$ (this is more clear as a consequence of Theorem B.2). Second, our detailed analysis provides a tighter bound in that it makes the relation between "constants" c_0 and c_1 explicit, contrary to (Abadi et al., 2016). Exactly due to this relation $c_0 = c(c_1)$ we are able to prove in Supplemental Material B.3 Theorem B.4 as a consequence of Theorem B.2 by considering the case $c(c_1) = 2/(\hat{\rho} c_1)$.

In order to prove Theorem B.2, we first set up the differential privacy framework of (Abadi et al., 2016) in Appendix B.2.1. Here we enhance a core lemma by proving a concrete bound rather than an asymptotic bound on the so-called λ-th moment which plays a crucial role in the differential privacy analysis. The concrete bound makes explicit the higher order error term in (Abadi et al., 2016).

In Appendix B.2.2 we generalize Theorem B.1 of (Abadi et al., 2016) by proving Theorem B.4 using the core lemma of Appendix B.2.1.

B.2.1. Definitions and Main Lemma

We base our proofs on the framework and theory presented in (Abadi et al., 2016). In order to be on the same page we repeat and cite word for word their definitions:

For neighboring databases d and d', a mechanism $\mathcal{M}$, auxiliary input $\texttt{aux}$, and an outcome o, define the privacy loss at o as

$$c(o; \mathcal{M}, \texttt{aux}, d, d') = \ln \frac{Pr[\mathcal{M}(\texttt{aux}, d) = o]}{Pr[\mathcal{M}(\texttt{aux}, d') = o]}.$$

For a given mechanism $\mathcal{M}$, we define the λ-th moment $\alpha_{\mathcal{M}}(\lambda; \texttt{aux}, d, d')$ as the log of the moment generating function evaluated at the value λ:

$$\alpha_{\mathcal{M}}(\lambda; \texttt{aux}, d, d') = \ln \mathbf{E}_{o \sim \mathcal{M}(\texttt{aux}, d)}[\exp(\lambda \cdot c(o; \mathcal{M}, \texttt{aux}, d, d'))].$$

We define

$$\alpha_{\mathcal{M}}(\lambda) = \max_{\texttt{aux}, d, d'} \alpha_{\mathcal{M}}(\lambda; \texttt{aux}, d, d')$$

where the maximum is taken over all possible $\texttt{aux}$ and all the neighboring databases d and d'.

We first take Lemma 3 from (Abadi et al., 2016) and make explicit their order term $O(q^3\lambda^3/\sigma^3)$ with $q = s_{i,c}$ and $\sigma = \sigma_i$ in our notation. The lemma considers as mechanism $\mathcal{M}$ the i-th round of gradient updates and we abbreviate $\alpha_{\mathcal{M}}(\lambda)$ by $\alpha_i(\lambda)$. The auxiliary input of the mechanism at round i includes all the output of the mechanisms of previous rounds (as in (Abadi et al., 2016)).

For the local mini-batch SGD the mechanism $\mathcal{M}$ of the i-th round is given by

$$\mathcal{M}(\texttt{aux}, d) = \sum_{h=0}^{s_i-1} [\nabla f(\hat{w}, \xi_h)]_C + \mathcal{N}(0, C^2\sigma_i^2\mathbf{I}),$$

where $\hat{w}$ is the local model at the start of round i which is replaced by a new global model $\hat{v}$ as soon as a new $\hat{v}$ is received from the server (see ISRReceive), and where ξ_h are drawn from training data d, and $[.]_C$ denotes clipping (that is $[x]_C = x/\max\{1, \|x\|_2/C\}$). In order for $\mathcal{M}$ to be able to compute its output, it needs to know the global models received in round i and it needs to know the starting local model $\hat{w}$. To make sure $\mathcal{M}$ has all this information, $\texttt{aux}$ represents the collection of all outputs generated by the mechanisms of previous rounds $< i$ together with the global models received in round i itself.

In the next subsection we will use the framework of (Abadi et al., 2016) and apply its composition theory to derive bounds on the privacy budget (ϵ, δ) for the whole computation consisting of T rounds that reveal the outputs of the mechanisms for these T rounds as described above.

We remind the reader that s_i/N is the probability of selecting a sample from a sample set (batch) of size s_i out of a training data set d' of size $N = |d'|$; σ_i corresponds to the $\mathcal{N}(0, C^2\sigma_i^2\mathbf{I})$ noise added to the mini-batch gradient computation in round i (see the mechanism described above).

Lemma B.3. *Assume a constant $r_0 < 1$ and deviation $\sigma_i \geq 216/215$ such that $s_i/N \leq r_0/\sigma_i$. Suppose that λ is a positive integer with*

$$\lambda \leq \sigma_i^2 \ln \frac{N}{s_i\sigma_i}$$

and define

$$U_0(\lambda) = \frac{2\sqrt{\lambda r_0/\sigma_i}}{\sigma_i - r_0} \text{ and } U_1(\lambda) = \frac{2e\sqrt{\lambda r_0/\sigma_i}}{(\sigma_i - r_0)\sigma_i}.$$

Suppose $U_0(\lambda) \leq u_0 < 1$ and $U_1(\lambda) \leq u_1 < 1$ for some constants u_0 and u_1. Define

$$r \quad = \quad r_0 \cdot 2^3 \left(\frac{1}{1-u_0} + \frac{1}{1-u_1}\frac{e^3}{\sigma_i^3}\right) \exp(3/\sigma_i^2).$$

Then,

$$\alpha_i(\lambda) \leq \frac{s_i^2\lambda(\lambda+1)}{N(N-s_i)\sigma_i^2} + \frac{r}{r_0} \cdot \frac{s_i^3\lambda^2(\lambda+1)}{N(N-s_i)^2\sigma_i^3}.$$

Proof. The start of the proof of Lemma 3 in (Abadi et al., 2016) implicitly uses the proof of Theorem A.1 in (Dwork et al., 2014), which up to formula (A.2) shows how the 1-dimensional case translates into a privacy loss that corresponds to the 1-dimensional problem defined by μ_0 and μ_1 in the proof of Lemma 3 in (Abadi et al., 2016), and which shows at the end of the proof of Theorem A.1 (p. 268 (Dwork et al., 2014)) how the multi-dimensional problem transforms into the 1-dimensional problem. In the notation of Theorem A.1, $f(D) + \mathcal{N}(0, \sigma^2\mathbf{I})$ represents the general (random) mechanism $\mathcal{M}(D)$, which for Lemma 3 in (Abadi et al., 2016)'s notation should be interpreted as the batch computation

$$\mathcal{M}(d) = \sum_{h\in J} f(d_h) + \mathcal{N}(0, \sigma^2\mathbf{I})$$

for a random sample/batch $\{d_h\}_{h\in J}$. Here, $f(d_h)$ (by abuse of notation – in this context f does not represent the objective function) represent clipped gradient computations $\nabla f(\hat{w}; d_h)$ where $\hat{w}$ is the last received global model with which round i starts (Lemma 3 in (Abadi et al., 2016) uses clipping constant $C = 1$, hence $\mathcal{N}(0, C^2\sigma^2\mathbf{I}) = \mathcal{N}(0, \sigma^2\mathbf{I})$).

Let us detail the argument of the proof of Lemma 3 in (Abadi et al., 2016) in order to understand what flexibility is possible: We consider two data sets $d = \{d_1, \ldots, d_{N-1}\}$ and $d' = d + \{d_N\}$, where $d_N \notin d$ represents a new data base element so that d and d' differ in exactly one element. The size of d' is equal to N. We define vector x as the sum

$$x = \sum_{J \setminus \{N\}} f(d_i).$$

Let

$$z = f(d_N).$$

If we consider data set d, then sample set $J \subseteq \{1, \cdots, N-1\}$ and mechanism $\mathcal{M}(d)$ returns

$$\mathcal{M}(d) = \sum_{h \in J} f(d_h) + \mathcal{N}(0, \sigma^2 \mathbf{I}) = \sum_{h \in J \setminus \{N\}} f(d_h) + \mathcal{N}(0, \sigma^2 \mathbf{I}) = x + \mathcal{N}(0, \sigma^2 \mathbf{I}).$$

If we consider data set d', then $J \subseteq \{1, \cdots, N\}$ contains d_N with probability $q = |J|/N$ ($|J| = s_i$ is the sample size used in round i). In this case mechanism $\mathcal{M}(d')$ returns

$$\mathcal{M}(d') = \sum_{h \in J} f(d_h) + \mathcal{N}(0, \sigma^2 \mathbf{I}) = f(d_N) + \sum_{h \in J \setminus \{N\}} f(d_h) + \mathcal{N}(0, \sigma^2 \mathbf{I}) = z + x + \mathcal{N}(0, \sigma^2 \mathbf{I})$$

with probability q. It returns

$$\mathcal{M}(d') = \sum_{h \in J} f(d_h) + \mathcal{N}(0, \sigma^2 \mathbf{I}) = \sum_{h \in J \setminus \{N\}} f(d_h) + \mathcal{N}(0, \sigma^2 \mathbf{I}) = x + \mathcal{N}(0, \sigma^2 \mathbf{I})$$

with probability $1 - q$. Combining both cases shows that $\mathcal{M}(d')$ represents a mixture of two Gaussian distributions (shifted over a vector x):

$$\mathcal{M}(d') = x + (1 - q) \cdot \mathcal{N}(0, \sigma^2 \mathbf{I}) + q \cdot \mathcal{N}(z, \sigma^2 \mathbf{I}).$$

This high dimensional problem is transformed into a single dimensional problem at the end of the proof of Theorem A.1 (p. 268 (Dwork et al., 2014)) by considering the one dimensional line from point x into the direction of z, i.e., the line through points x and $x + z$; the one dimensional line maps x to the origin 0 and $x + z$ to $\|z\|_2$. $\mathcal{M}(d)$ as wells as $\mathcal{M}(d')$ projected on this line are distributed as

$$\mathcal{M}(d) \sim \mu_0 \text{ and } \mathcal{M}(d') \sim (1 - q)\mu_0 + q\mu_1,$$

where

$$\mu_0 \sim \mathcal{N}(0, \sigma^2) \text{ and } \mu_1 \sim \mathcal{N}(\|z\|_2, \sigma^2).$$

In (Abadi et al., 2016) as well as in this paper the gradients are clipped (their Lemma 3 uses clipping constant $C = 1$) and this implies

$$\|z\|_2 = \|f(d_N)\|_2 \leq C = 1.$$

Their analysis continues by assuming the worst-case in differential privacy, that is,

$$\mu_1 \sim \mathcal{N}(1, \sigma^2).$$

Notice that the above argument analyses a local mini-batch SGD computation. Rather than using a local mini-batch SGD computation, can we use clipped SGD iterations which continuously update the local model:

$$\hat{w}_{h+1} = \hat{w}_h - \eta_h \nabla [f(\hat{w}_h, \xi_h)]_C.$$

This should lead to faster convergence to good accuracy compared to a local minibatch computation. However, the above arguments cannot proceed[5] because (in the notation used above where the d_h, $h \in J$, are the ξ_h, $h \in \{0, \ldots, s_i - 1 = |J| - 1\}$) selecting sample d_N in iteration h does not only influence the update computed in iteration h but also influences all iterations after h till the end of the round (because $f(d_N)$ updates the local model in iteration h which is used in the

[5]Unless we assume a general upper bound on the norm of the Hessian of the objective function which should be large enough to cover a wide class of objective functions and small enough in order to be able to derive practical differential privacy guarantees.

iterations that come after). Hence, the dependency on d_N is directly felt by $f(d_N)$ in iteration h and indirectly felt in the $f(d_j)$ that are computed after iteration h. This means that we cannot represent distribution $\mathcal{M}(d')$ as a clean mix of Gaussian distributions with a mean z, whose norm is bounded by the clipping constant.

The freedom which we do have is replacing the local model by a newly received global model. This is because the updates $f(d_h)$, $h \in J$, computed locally in round i have not yet been transmitted to the server and, hence, have not been aggregated into the global model that was received. In a way the mechanism $\mathcal{M}(d)$ is composed of two (or multiple if more newer and newer global models are received during the round) sums

$$\mathcal{M}(d) = \sum_{h \in J_0} f_0(d_h) + \sum_{h \in J_1} f_1(d_h) + \mathcal{N}(0, \sigma^2 \mathbf{I}),$$

where $J = J_0 \cup J_1$ and J_0 represent local gradient computations, shown by $f_0(.)$, based on the initial local model $\hat{w}$ and J_1 represent the local gradient computations, shown by $f_1(.)$, based on the newly received global model $\hat{v}$ which replaces $\hat{w}$. As one can verify, the above arguments are still valid for this slight adaptation. As in Lemma 3 in (Abadi et al., 2016) we can now translate our privacy loss to the 1-dimensional problem defined by $\mu_0 \sim \mathcal{N}(0, C^2\sigma^2)$ and $\mu_1 \sim \mathcal{N}(C, C^2\sigma^2)$ for $\|\nabla f(.,.)\|_2 \leq C$ as in the proof of Lemma 3 (which after normalization with respect to C gives the formulation of Lemma 3 in (Abadi et al., 2016) for $C = 1$).

The remainder of the proof of Lemma 3 analyses μ_0 and the mix $\mu = (1-q)\mu_0 + q\mu_1$ leading to bounds for the expectations (3) and (4) in (Abadi et al., 2016) which only depend on μ_0 and μ_1. Here, q is the probability of having a special data sample ξ (written as d_N in the arguments above) in the batch. In our algorithm $q = s_i/N$. So, we may adopt the statement of Lemma 3 and conclude for the i-th batch computation

$$\alpha_i(\lambda) \leq \frac{s_i^2 \lambda(\lambda+1)}{N(N-s_i)\sigma_i^2} + O\left(\frac{s_i^3 \lambda^3}{N^3 \sigma_i^3}\right).$$

In order to find an exact expression for the higher order term we look into the details of Lemma 3 of (Abadi et al., 2016). It computes an upper bound for the binomial tail

$$\sum_{t=3}^{\lambda+1} \binom{\lambda+1}{t} \mathbb{E}_{z\sim\nu_1}[((\nu_0(z) - \nu_1(z))/\nu_1(z))^t], \tag{9}$$

where

$$\begin{aligned} & \mathbb{E}_{z\sim\nu_1}[((\nu_0(z) - \nu_1(z))/\nu_1(z))^t] \\ \leq\ & \frac{(2q)^t (t-1)!!}{2(1-q)^{t-1}\sigma^t} + \frac{q^t}{(1-q)^t \sigma^{2t}} + \frac{(2q)^t \exp((t^2-t)/(2\sigma^2))(\sigma^t (t-1)!! + t^t)}{2(1-q)^{t-1}\sigma^{2t}} \\ =\ & \frac{(2q)^t (t-1)!!(1+\exp((t^2-t)/(2\sigma^2)))}{2(1-q)^{t-1}\sigma^t} + \frac{q^t(1+(1-q)2^t \exp((t^2-t)/(2\sigma^2))t^t)}{2(1-q)^t\sigma^{2t}} \end{aligned} \tag{10}$$

Since $t \geq 3$, we have the coarse upper bounds

$$1 \leq \frac{\exp((t^2-t)/(2\sigma^2))}{\exp((3^2-3)/(2\sigma^2))} \text{ and } 1 \leq \frac{(1-q)2^t \exp((t^2-t)/(2\sigma^2))t^t}{(1-q)2^3 \exp((3^2-3)/(2\sigma^2))3^3}.$$

By defining c as 1 plus the maximum of these two bounds,

$$c = 1 + \frac{\max\{1, 1/((1-q)\cdot 216)\}}{\exp(3/\sigma^2)},$$

we have (10) at most

$$\leq \frac{(2q)^t (t-1)!! c \exp((t^2-t)/(2\sigma^2))}{2(1-q)^{t-1}\sigma^t} + \frac{q^t c(1-q)2^t \exp((t^2-t)/(2\sigma^2))t^t}{2(1-q)^t\sigma^{2t}}. \tag{11}$$

Generally (for practical parameter settings as we will find out), $q \leq 1 - 1/216$ which makes $c \leq 2$. In the remainder of this proof, we use $c = 2$ and assume $q \leq 215/216$. In fact, assume in the statement of the lemma that $\sigma = \sigma_i \geq 216/215$ which together with $q = s_i/N \leq r_0/\sigma_i$ and $r_0 < 1$ implies $q \leq 215/216$.

After multiplying (11) with the upper bound for

$$\binom{\lambda+1}{t} \leq \frac{\lambda+1}{\lambda}\frac{\lambda^t}{t!}$$

and noticing that $(t-1)!!/t! \leq 1$ and $t^t/t! \leq e^t$ we get the addition of the following two terms

$$\frac{\lambda+1}{\lambda}\frac{\lambda^t(2q)^t\exp((t^2-t)/(2\sigma^2))}{(1-q)^{t-1}\sigma^t} + \frac{\lambda+1}{\lambda}\frac{\lambda^t q^t(1-q)2^t\exp((t^2-t)/(2\sigma^2))e^t}{(1-q)^t\sigma^{2t}}.$$

This is equal to

$$(1-q)\frac{\lambda+1}{\lambda}\left(\frac{\lambda 2q\exp((t-1)/(2\sigma^2))}{(1-q)\sigma}\right)^t + (1-q)\frac{\lambda+1}{\lambda}\left(\frac{\lambda q 2\exp(1+(t-1)/(2\sigma^2))}{(1-q)\sigma^2}\right)^t. \tag{12}$$

We notice that by using $t \leq \lambda+1$, $\lambda/\sigma^2 \leq \ln(1/(q\sigma))$ (assumption), and $q = s_{i,c}/N_c \leq r_0/\sigma$ we obtain

$$\frac{\lambda 2q\exp((t-1)/(2\sigma^2))}{(1-q)\sigma} \leq \frac{\lambda 2q\exp(\lambda/(2\sigma^2))}{(1-q)\sigma} \leq \frac{2\sqrt{\lambda q}}{(1-q)\sigma} = \frac{2\sqrt{\lambda r_0/\sigma}}{\sigma - r_0} = U_0(\lambda)$$

and

$$\frac{\lambda q 2\exp(1+(t-1)/(2\sigma^2))}{(1-q)\sigma^2} \leq \frac{\lambda q 2e\exp(\lambda/(2\sigma^2))}{(1-q)\sigma^2} \leq \frac{2e\sqrt{\lambda q}}{(1-q)\sigma^2} = \frac{2e\sqrt{\lambda r_0/\sigma}}{(\sigma-r_0)\sigma} = U_1(\lambda).$$

Together with our assumption on $U_0(\lambda)$ and $U_1(\lambda)$, this means that the binomial tail (9) is upper bounded by the two terms in (12) after substituting $t = 3$, with the two terms multiplied by

$$\sum_{j=0}^{\infty} U_0(\lambda)^j = \frac{1}{1-U_0(\lambda)} \leq \frac{1}{1-u_0} \text{ and } \sum_{j=0}^{\infty} U_1(\lambda)^j = \frac{1}{1-U_1(\lambda)} \leq \frac{1}{1-u_1}$$

respectively. For (9) this yields the upper bound

$$\begin{aligned} &\frac{1}{1-u_0}(1-q)\frac{\lambda+1}{\lambda}\left(\frac{\lambda 2q\exp(1/\sigma^2)}{(1-q)\sigma}\right)^3 + \frac{1}{1-u_1}(1-q)\frac{\lambda+1}{\lambda}\left(\frac{\lambda q 2\exp(1+1/\sigma^2)}{(1-q)\sigma^2}\right)^3 \\ \leq\ &\left(\frac{1}{1-u_0}2^3\exp(3/\sigma^2) + \frac{1}{1-u_1}\frac{2^3\exp(3+3/\sigma^2)}{\sigma^3}\right)\cdot\frac{\lambda^2(\lambda+1)q^3}{(1-q)^2\sigma^3}. \end{aligned}$$

By the definition of r, we obtain the bound

$$\leq \frac{r}{r_0}\cdot\frac{\lambda^2(1+\lambda)q^3}{(1-q)^2\sigma^3},$$

which finalizes the proof.

B.2.2. Proof of Theorem B.2

The proof Theorem B.2 follows the line of thinking in the proof of Theorem 1 in (Abadi et al., 2016). Our theorem applies to varying sample/batch sizes and for this reason introduces moments $\hat{S}_j$. Our theorem explicitly defines the constant used in the lower bound of σ – this is important for proving our second (main) theorem in the next subsection.

Theorem B.2 assumes $\sigma = \sigma_i$ for all rounds i with $\sigma \geq 216/215$; constant $r_0 \leq 1/e$ such that $s_i/N \leq r_0/\sigma$; constant

$$r \quad = \quad r_0 \cdot 2^3 \left(\frac{1}{1-u_0} + \frac{1}{1-u_1} \frac{e^3}{\sigma^3} \right) \exp(3/\sigma^2), \tag{13}$$

where

$$u_0 = \frac{2\sqrt{r_0\sigma}}{\sigma - r_0} \text{ and } u_1 = \frac{2e\sqrt{r_0\sigma}}{(\sigma - r_0)\sigma}$$

are both assumed < 1.

For $j = 1, 2, 3$ we define[6]

$$\hat{S}_j = \frac{1}{T} \sum_{i=0}^{T-1} \frac{s_i^j}{N(N-s_i)^{j-1}} \text{ with } \frac{\hat{S}_1 \hat{S}_3}{\hat{S}_2^2} \leq \rho, \ \frac{\hat{S}_1^2}{\hat{S}_2} \leq \hat{\rho}.$$

Based on these constants we define

$$c(x) = \min \left\{ \frac{\sqrt{2r\rho x + 1} - 1}{r\rho x}, \frac{2}{\hat{\rho} x} \right\}.$$

Let $\epsilon = c_1 T \hat{S}_1^2$. We want to prove Algorithm 4 is (ϵ, δ)-differentially private if

$$\sigma \geq \frac{2}{\sqrt{c_0}} \frac{\sqrt{\hat{S}_2 T(\epsilon + \ln(1/\delta))}}{\epsilon} \text{ where } c_0 = c(c_1).$$

Proof. For $j = 1, 2, 3$, we define

$$S_j = \sum_{i=0}^{T-1} \frac{s_i^j}{N(N-s_i)^{j-1}\sigma_i^j} \text{ and } S_j' = \frac{1}{T} \sum_{i=0}^{T-1} \frac{s_i^j \sigma_i^j}{N(N-s_i)^{j-1}}.$$

(Notice that $S_1' \leq r_0$.) Translating Lemma B.3 in this notation yields (we will verify the requirement/assumptions of Lemma B.3 on the fly below)

$$\sum_{i=0}^{T-1} \alpha_i(\lambda) \leq S_2 \lambda(\lambda+1) + \frac{r}{r_0} S_3 \lambda^2 (\lambda+1).$$

The composition Theorem 2 in (Abadi et al., 2016) shows that our algorithm for client c is (ϵ, δ)-differentially private for

$$\delta \quad \geq \quad \min_{\lambda} \exp \left(\sum_{i=0}^{T-1} \alpha_i(\lambda) - \lambda\epsilon \right),$$

where T indicates the total number of batch computations and the minimum is over positive integers λ. Similar to their proof we choose λ such that

$$S_2 \lambda(\lambda+1) + \frac{r}{r_0} S_3 \lambda^2 (\lambda+1) - \lambda\epsilon \leq -\lambda\epsilon/2. \tag{14}$$

This implies that we can choose δ as small as $\exp(-\lambda\epsilon/2)$, i.e., if

$$\delta \geq \exp(-\lambda\epsilon/2), \tag{15}$$

then we have (ϵ, δ)-differential privacy. After dividing by the positive integer λ, inequality (14) is equivalent to the inequality

$$S_2(\lambda+1) + \frac{r}{r_0} S_3 \lambda(1+\lambda) \leq \epsilon/2,$$

[6] s_i^j denotes the j-th power $(s_i)^j$.

which is equivalent to

$$(\lambda+1)\left(1+\frac{r}{r_0}\frac{S_3}{S_2}\lambda\right)\leq\frac{\epsilon}{2S_2}.$$

This is in turn implied by

$$\lambda+1\leq c_0\frac{\epsilon}{2S_2} \tag{16}$$

together with

$$c_0\frac{\epsilon}{2S_2}\left(1+\frac{r}{r_0}\frac{S_3}{S_2}c_0\frac{\epsilon}{2S_2}\right)\leq\frac{\epsilon}{2S_2},$$

or equivalently,

$$c_0\left(1+\frac{r}{2r_0}\cdot c_0\cdot\frac{S_3}{S_2^2}\epsilon\right)\leq 1. \tag{17}$$

We use

$$\epsilon=c_1\cdot T\hat{S}_1^2=c_1\cdot S_1S_1' \tag{18}$$

(for constant $\sigma_i=\sigma$). This translates our requirements (16) and (17) into

$$\lambda+1\leq\frac{c_0c_1}{2}\frac{S_1S_1'}{S_2}\text{ and} \tag{19}$$

$$c_0\left(1+\frac{r}{2r_0}\cdot c_0c_1\frac{S_1S_3}{S_2^2}S_1'\right)\leq 1. \tag{20}$$

Since we assume

$$\frac{S_1S_3}{S_2^2}=\frac{\hat{S}_1\hat{S}_3}{\hat{S}_2^2}\leq\rho$$

and since we know that $S_1'\leq r_0$, requirement (20) is implied by

$$c_0(1+\frac{r\rho}{2}\cdot c_0c_1)\leq 1,$$

or equivalently

$$c_1\leq\frac{1-c_0}{\frac{r\rho}{2}c_0^2}. \tag{21}$$

Also notice that for constant $\sigma_i=\sigma$ we have $S_1'=S_1\sigma^2/T$. Together with

$$\frac{S_1^2}{S_2}=\frac{\hat{S}_1^2}{\hat{S}_2}T\leq\hat{\rho}T$$

we obtain from (19)

$$\lambda+1\leq\frac{c_0c_1}{2}\frac{S_1S_1'}{S_2}\leq\frac{c_0c_1}{2}\hat{\rho}\sigma^2. \tag{22}$$

Generally, if

$$c_1\leq\frac{2}{\hat{\rho}c_0}, \tag{23}$$

then (22) implies $\lambda\leq\sigma^2$: Hence, (a) for our choice of u_0 and u_1 in this theorem, $U_0(\lambda)\leq u_0$ and $U_1(\lambda)\leq u_1$ as defined in Lemma B.3, and (b) the condition $\lambda\leq\sigma_i^2\ln\frac{N_c}{s_{i,c}\sigma_i}$ is satisfied (by assumption, $\frac{N_c}{s_{i,c}\sigma_i}\geq 1/r_0\geq e$). This implies that Lemma B.3 is indeed applicable.

For the above reasons we strengthen the requirement on ϵ (conditions (21) and (23) with (18)) to

$$\epsilon\leq\min\left\{\frac{1-c_0}{\frac{r\rho}{2}c_0^2},\frac{2}{\hat{\rho}c_0}\right\}\cdot S_1S_1'$$

For constant $\sigma_i = \sigma$, we have

$$S_1 S_1' = T\hat{S}_1^2,$$

hence, we need

$$\epsilon \leq \min\left\{\frac{1-c_0}{\frac{r\rho}{2}c_0^2}, \frac{2}{\hat{\rho}c_0}\right\} \cdot T\hat{S}_1^2 \tag{24}$$

Summarizing (24), (16), and (15) for some positive integer λ proves (ϵ, δ)-differential privacy.

Condition (15) (i.e., $\exp(-\lambda\epsilon/2) \leq \delta$) is equivalent to

$$\ln(1/\delta) \leq \frac{\lambda\epsilon}{2} \tag{25}$$

If

$$\lambda = \lfloor c_0 \frac{\epsilon}{2S_2} \rfloor - 1 \tag{26}$$

is positive, then it satisfies (16) and we may use this λ in (25). This yields the condition

$$\ln(1/\delta) \leq \left(\lfloor c_0 \frac{\epsilon}{2S_2} \rfloor - 1\right)\frac{\epsilon}{2},$$

which is implied by

$$\ln(1/\delta) \leq \left(c_0 \frac{\epsilon}{2S_2} - 2\right)\frac{\epsilon}{2} = \frac{c_0}{4S_2}\epsilon^2 - \epsilon.$$

For constant $\sigma_i = \sigma$ we have $S_2 = \hat{S}_2 T/\sigma^2$ and the latter inequality is equivalent to

$$\sigma \geq \frac{2}{\sqrt{c_0}} \frac{\sqrt{\hat{S}_2}\sqrt{T(\epsilon + \ln(1/\delta))}}{\epsilon}. \tag{27}$$

Summarizing, if (24), (27), and the lambda value (26) is positive, then this shows (ϵ, δ)-differential privacy.

The condition (26) being positive follows from

$$\frac{4S_2}{c_0} \leq \epsilon.$$

Substituting $S_2 = \hat{S}_2 T/\sigma^2$ yields the equivalent condition

$$\frac{4T\hat{S}_2}{\sigma^2 c_0} \leq \epsilon$$

or

$$\sigma \geq \frac{2}{\sqrt{c_0}}\sqrt{\hat{S}_2}\frac{\sqrt{T\epsilon}}{\epsilon},$$

which is implied by (27). Summarizing, if (24) and (27), then this shows (ϵ, δ)-differential privacy. Notice that (27) corresponds to Theorem 1 in (Abadi et al., 2016) where all s_i are constant implying $\sqrt{\hat{S}_2} = q/\sqrt{1-q}$ in their notation.

We are interested in a slightly different formulation: Given

$$c_1 = \min\left\{\frac{1-c_0}{\frac{r\rho}{2}c_0^2}, \frac{2}{\hat{\rho}c_0}\right\}$$

what is the maximum possible c_0 (which minimizes σ implying more fast convergence to an accurate solution). We need to satisfy $c_0 \leq 2/(\hat{\rho}c_1)$ and

$$\frac{r\rho}{2}c_1 c_0^2 + c_0 - 1 \leq 0,$$

that is,

$$(c_0 + 1/(r\rho c_1))^2 \leq 1/\left(\frac{r\rho}{2}c_1\right) + 1/(r\rho c_1)^2,$$

or

$$c_0 \leq \sqrt{1/\left(\frac{r\rho}{2}c_1\right) + 1/\left(r\rho c_1\right)^2} - 1/(r\rho c_1) = \frac{\sqrt{2r\rho c_1 + 1} - 1}{r\rho c_1}.$$

We have

$$c_0 = \min\left\{\frac{\sqrt{2r\rho c_1 + 1} - 1}{r\rho c_1}, 2/(\hat{\rho} c_1)\right\} = c(c_1).$$

This finishes the proof.

B.3. A Simplified Characterization

So far, we have generalized Theorem B.1 in Appendix B in a non-trivial way by analysing increasing sample size sequences, by making explicit the higher order error term in (Abadi et al., 2016), and by providing a precise functional relationship among the constants c_1 and c_2 in Theorem B.1. The resulting Theorem B.2 is still hard to interpret. The next theorem is a consequence of Theorem B.2 and brings us the interpretation we look for.

Theorem B.4. *For sample size sequence $\{s_i\}_{i=0}^{T-1}$ the total number of local SGD iterations is equal to $K = \sum_{i=0}^{T-1} s_i$. We define the mean $\bar{s}$ and maximum s_{max} and their quotient θ as*

$$\bar{s} = \frac{1}{T}\sum_{i=0}^{T-1} s_i = \frac{K}{T}, \quad s_{max} = \max\{s_0, \ldots, s_{T-1}\}, \quad \textit{and} \quad \theta = \frac{s_{max}}{\bar{s}}.$$

We define

$$h(x) = \left(\sqrt{1 + (e/x)^2} - e/x\right)^2, \quad g(x) = \min\left\{\frac{1}{ex}, h(x)\right\},$$

and denote by γ the smallest solution satisfying

$$\gamma \geq \frac{2}{1-\bar{\alpha}} + \frac{2^4 \cdot \bar{\alpha}}{1-\bar{\alpha}}\left(\frac{\sigma}{(1-\sqrt{\bar{\alpha}})^2} + \frac{1}{\sigma(1-\bar{\alpha}) - 2e\sqrt{\bar{\alpha}}}\frac{e^3}{\sigma}\right)e^{3/\sigma^2} \textit{ with } \bar{\alpha} = \frac{\epsilon N}{\gamma K}.$$

If the following requirements are satisfied:

$$\bar{s} \leq \frac{g\left(\sqrt{2(\epsilon + \ln(1/\delta))/\epsilon}\right)}{\theta} \cdot N, \tag{28}$$

$$\epsilon \leq \gamma h(\sigma) \cdot \frac{K}{N}, \tag{29}$$

$$\epsilon \geq \gamma\theta^2 \cdot \frac{K}{N} \cdot \frac{\bar{s}}{N}, \textit{ and} \tag{30}$$

$$\sigma \geq \sqrt{2(\epsilon + \ln(1/\delta))/\epsilon}, \tag{31}$$

then Algorithm 4 is (ϵ, δ)-differentially private.

Its proof follows from analysing the requirements stated in Theorem B.2. We will focus on the case where $c(x) = \frac{2}{\hat{\rho}x}$, which turns out to lead to practical parameter settings as discussed in the main body of the paper.

Requirement on r – (34): In Theorem B.2 we use

$$r = r_0 \cdot 2^3 \cdot \left(\frac{1}{1-u_0} + \frac{1}{1-u_1}\frac{e^3}{\sigma^3}\right)e^{3/\sigma^2}$$

with

$$u_0 = \frac{2\sqrt{r_0\sigma}}{\sigma - r_0} \text{ and } u_1 = \frac{2e\sqrt{r_0\sigma}}{(\sigma - r_0)\sigma},$$

where r_0 is such that it satisfies

$$r_0 \leq 1/e, \; u_0 < 1, \text{ and } u_1 < 1. \tag{32}$$

In our application of Theorem B.2 we substitute $r_0 = \alpha\sigma$. This translates the requirements of (32) into

$$\alpha \leq \frac{1}{e\sigma},\ \alpha < 1,\ \text{and}\ \sigma > \frac{2e\sqrt{\alpha}}{1-\alpha}. \tag{33}$$

As we will see in our derivation, we will require another lower bound (38) on σ. We will use (38) together with

$$\alpha \leq \frac{1}{e\sqrt{2(\epsilon + \ln(1/\delta))/\epsilon}}, \alpha < 1,\ \text{and}\ \sqrt{2(\epsilon + \ln(1/\delta))/\epsilon} > \frac{2e\sqrt{\alpha}}{1-\alpha}$$

to imply the needed requirement (33). These new bounds on α are in turn equivalent to

$$\begin{aligned} &\alpha \leq g(\epsilon, \delta) \text{ where} \\ &g(\epsilon, \delta) = \min\left\{ \frac{\sqrt{\epsilon}}{e\sqrt{2(\epsilon + \ln(1/\delta))}}, \left(\sqrt{1 + \frac{e^2\epsilon}{2(\epsilon + \ln(1/\delta))}} - \frac{e\sqrt{\epsilon}}{\sqrt{2(\epsilon + \ln(1/\delta))/\epsilon}}\right)^2 \right\} \end{aligned} \tag{34}$$

(notice that this implies $\alpha < 1$).

Substituting $r_0 = \alpha\sigma$ in the formula for r yields the expression

$$r \quad = \quad 2^3 \cdot \left(\frac{\sigma}{(1-\sqrt{\alpha})^2} + \frac{1}{\sigma(1-\alpha) - 2e\sqrt{\alpha}} \frac{e^3}{\sigma} \right) \cdot e^{3/\sigma^2} (1-\alpha)\alpha. \tag{35}$$

Requirement on s_i/N – (36): In Theorem B.2 we also require $s_i/N \leq r_0/\sigma$ which translates into

$$s_i/N \leq \alpha. \tag{36}$$

Requirement on σ – (38) and (39): In Theorem B.2 we restrict ourselves to the case where function $c(x)$ attains the minimum $c(x) = 2/(\hat{\rho}x)$. This happens when

$$\frac{\sqrt{2r\rho x + 1} - 1}{r\rho x} \geq \frac{2}{\hat{\rho}x}.$$

This is equivalent to

$$x \geq 2r\frac{\rho}{\hat{\rho}^2} + \frac{2}{\hat{\rho}}. \tag{37}$$

Notice that in the lower bound for σ in Theorem B.2 we use $c_0 = c(x)$ for $x = c_1$, where c_1 is implicitly defined by

$$\epsilon = c_1 T \hat{S}_1^2$$

or equivalently

$$c_1 = \frac{\epsilon}{T\hat{S}_1^2}.$$

To minimize ϵ, we want to minimize $c_1 = x$. That is, we want $c_1 = x$ to match the lower bound (37). This lower bound is smallest if we choose the smallest possible ρ (due to the linear dependency of the lower bound on ρ). Given the constraint on ρ this means we choose

$$\rho = \frac{\hat{S}_1\hat{S}_3}{\hat{S}_2^2}.$$

For $c_1 = x$ satisfying (37) we have

$$c_0 = c(c_1) = \frac{2}{\hat{\rho}x}.$$

Substituting this in the lower bound for σ attains

$$\sigma \geq \frac{2}{\sqrt{c(c_1)}} \frac{\sqrt{\hat{S}_2 T(\epsilon + \ln(1/\delta))}}{\epsilon} = \sqrt{\frac{\hat{\rho}\hat{S}_2}{\hat{S}_1^2}} \sqrt{2(\epsilon + \ln(1/\delta))/\epsilon}.$$

In order to yield the best test accuracy we want to choose the smallest possible σ. Hence, we want to minimize the lower bound for σ and therefore choose the smallest $\hat{\rho}$ given its constraints, i.e.,

$$\hat{\rho} = \frac{\hat{S}_1^2}{\hat{S}_2}.$$

This gives

$$\sigma \geq \sqrt{2(\epsilon + \ln(1/\delta))/\epsilon}. \tag{38}$$

Notice that this lower bound implies $\sigma \geq 216/215$ and for this reason we do not state this as an extra requirement.

Our expressions for ρ, $\hat{\rho}$, and c_1 with $x = c_1$ shows that lower bound (37) holds if and only if

$$\epsilon \geq \left(2r \frac{\hat{S}_3}{\hat{S}_1} + 2\hat{S}_2 \right) T. \tag{39}$$

Requirement implying (39): The definition of moments $\hat{S}_j$ imply

$$\hat{S}_1 = \frac{K}{TN}$$

and, since $s_i/N \leq \alpha < 1$,

$$\hat{S}_j \leq \alpha^j/(1-\alpha)^{j-1}.$$

Lower bound (39) on ϵ is therefore implied by

$$\epsilon \geq 2r \frac{\alpha^3}{(1-\alpha)^2} \frac{T^2 N}{K} + 2\frac{\alpha^2}{1-\alpha} T. \tag{40}$$

We substitute

$$T = \beta \frac{K}{N} \tag{41}$$

in (40) which yields the requirement

$$\epsilon \frac{N}{K} \geq \frac{2r}{\alpha(1-\alpha)^2} (\alpha^2\beta)^2 + \frac{2}{1-\alpha} (\alpha^2\beta). \tag{42}$$

This inequality is implied by the combination of the following two inequalities:

$$\alpha^2\beta \leq \frac{\epsilon N}{\gamma K} \tag{43}$$

and

$$1 \geq \frac{2r}{\alpha(1-\alpha)^2} \frac{\epsilon N}{K} \frac{1}{\gamma^2} + \frac{2}{1-\alpha} \frac{1}{\gamma}. \tag{44}$$

Inequality (44) is equivalent to

$$\gamma \geq \frac{2r}{\alpha(1-\alpha)^2} \frac{\epsilon N}{\gamma K} + \frac{2}{1-\alpha}. \tag{45}$$

This implies

$$\gamma \geq \frac{2}{1-\alpha} \geq 2.$$

Also notice that

$$\frac{1}{\beta} = \frac{K}{TN} = \hat{S}_1 \leq \alpha$$

from which we obtain

$$1 \leq \alpha\beta.$$

Let us define

$$\bar{\alpha} = \frac{\epsilon N}{\gamma K}. \tag{46}$$

Inequalities $\gamma \geq 2$ and $1 \leq \alpha\beta$ together with (43) and the definition of $\bar{\alpha}$ imply

$$\alpha \leq \alpha^2\beta \leq \frac{\epsilon N}{\gamma K} = \bar{\alpha} \leq \frac{\epsilon N}{2K}. \tag{47}$$

We will require

$$\bar{\alpha} < 1 \tag{48}$$

and also $\sigma(1-\bar{\alpha}) - 2e\sqrt{\bar{\alpha}} > 0$ i.e,

$$\sigma > \frac{2e\sqrt{\bar{\alpha}}}{1-\bar{\alpha}}. \tag{49}$$

Bounds (48) and (49) are equivalent to

$$\bar{\alpha} \leq h(\sigma) \text{ where } h(\sigma) = \left(\sqrt{1+(e/\sigma)^2} - e/\sigma\right)^2. \tag{50}$$

With condition (50) in place we may derive the upper bound

$$\begin{aligned} & \frac{2r}{\alpha(1-\alpha)^2} \\ = \; & \frac{2^4}{1-\alpha}\left(\frac{\sigma}{(1-\sqrt{\alpha})^2} + \frac{1}{\sigma(1-\alpha) - 2e\sqrt{\alpha}}\frac{e^3}{\sigma}\right)e^{3/\sigma^2} \\ \leq \; & \frac{2^4}{1-\bar{\alpha}}\left(\frac{\sigma}{(1-\sqrt{\bar{\alpha}})^2} + \frac{1}{\sigma(1-\bar{\alpha}) - 2e\sqrt{\bar{\alpha}}}\frac{e^3}{\sigma}\right)e^{3/\sigma^2} \end{aligned}$$

because all denominators are decreasing functions in α and remain positive for $\alpha \leq \bar{\alpha}$. Similarly,

$$\frac{2}{1-\alpha} \leq \frac{2}{1-\bar{\alpha}}.$$

These two upper bounds combined with (46) show that (45) is implied by choosing

$$\gamma = \gamma(\sigma, \epsilon N/K),$$

where $\gamma(\sigma, \epsilon N/K)$ is defined as the smallest solution of γ satisfying

$$\begin{aligned} \gamma \quad \geq \; & \frac{2}{1-\bar{\alpha}} + \\ & \frac{2^4\cdot\bar{\alpha}}{1-\bar{\alpha}}\left(\frac{\sigma}{(1-\sqrt{\bar{\alpha}})^2} + \frac{1}{\sigma(1-\bar{\alpha}) - 2e\sqrt{\bar{\alpha}}}\frac{e^3}{\sigma}\right)e^{3/\sigma^2}, \end{aligned} \tag{51}$$

where $\bar{\alpha} = (\epsilon N/K)/\gamma$. The smallest solution γ will meet (51) with equality. For this reason the minimal solution γ will be at most the right hand side of (51) where γ is replaced by its lower bound 2; this is allowed because this increases $\bar{\alpha}$ to the upper bound in (47) and we know that the right hand side of (51) increases in $\bar{\alpha}$ up to the upper bound in (47) if the upper bound satisfies

$$\frac{\epsilon N}{2K} \leq h(\sigma).$$

This makes requirement (50) slightly stronger – but in practice this stronger requirement is already satisfied because K is several epochs of N iterations making $\frac{\epsilon N}{2K} \ll 1$ while $\sigma \gg 1$ for small ϵ implying that $h(\sigma)$ is close to 1.

Notice that $\gamma = 2 + O(\bar{\alpha})$, hence, for small $\bar{\alpha}$ we have $\gamma \approx 2$. A more precise asymptotic analysis reveals

$$\gamma = 2 + (2 + 2^4 \cdot \left(\sigma + \frac{e^3}{\sigma^2}\right)e^{3/\sigma^2})\bar{\alpha} + O(\bar{\alpha}^{3/2}).$$

Relatively large $\bar{\alpha}$ closer to 1 will yield $\gamma \gg 2$.

Summarizing

$$\{(41), (43), (46), (50), (51)\} \Rightarrow (39).$$

Combining all requirements – resulting in (53), (54), and (38), or equivalently (56), (57), and (38): The combination of requirements (41) and (43) is equivalent to

$$\alpha \leq \sqrt{\frac{\epsilon}{\gamma T}} \tag{52}$$

(notice that T and β are not involved in any of the other requirements including those discussed earlier in this discussion, hence, we can discard (41) and substitute this in (43)). The combination of (46), (50), and (51) is equivalent to

$$\frac{\epsilon N}{\gamma K} \leq h(\sigma) \text{ with } \gamma = \gamma\left(\sigma, \frac{\epsilon N}{K}\right) \tag{53}$$

(for the definition of $h(.)$ see (50) and for $\gamma(.,.)$ see (51)).

We may now combine (52), (34), and (36) into a single requirement

$$s_i/N \leq \min\left\{g(\epsilon, \delta), \sqrt{\frac{\epsilon}{\gamma T}}\right\} \tag{54}$$

(for the definition of $g(.,.)$ see (34)). This shows that (53), (54), and (38) (we remind the reader that the last condition is the lower bound on $\sigma \geq \sqrt{2(\epsilon + \ln(1/\delta))/\epsilon)}$ implies (ϵ, δ)-DP by Theorem B.2.

Let us rewrite these conditions. We introduce the mean $\bar{s}$ of all s_i defined by

$$\bar{s} = \frac{1}{T}\sum_{i=0}^{T-1} s_i = \frac{K}{T}$$

and we introduce the maximum s_{max} of all s_i defined by

$$s_{max} = \max\{s_0, \ldots, s_{T-1}\}.$$

We define θ as the fraction

$$\theta = \frac{s_{max}}{\bar{s}}. \tag{55}$$

This notation allows us to rewrite

$$s_i/N \leq \sqrt{\frac{\epsilon}{\gamma T}}$$

from (54) as

$$\gamma \frac{K}{N}\frac{\bar{s}}{N}\theta^2 \leq \epsilon.$$

From this we obtain that the requirements (53) and (54) are equivalent to

$$\gamma\left(\sigma, \frac{\epsilon N}{K}\right) \cdot \frac{K}{N}\frac{\bar{s}}{N}\theta^2 \leq \epsilon \leq \gamma\left(\sigma, \frac{\epsilon N}{K}\right) \cdot h(\sigma)\frac{K}{N} \tag{56}$$

and

$$\theta\bar{s} \leq g(\epsilon, \delta)N. \tag{57}$$

This alternative description shows that (56), (57), and (38) with definitions for $h(.)$, $\gamma(.,.)$, $g(.,.)$, and θ in (50), (51), (34), and (55) implies (ϵ, δ)-DP. This proves Theorem B.4 (after a slight rewrite of the definitions of functions $h(.)$ and $g(.,.)$).

B.4. Proof of the Main Theorem

Theorem B.4 can already be used to a-priori set hyperparameters given DP and accuracy targets. Still, as discussed below, by making slight approximations (leading to slightly stronger constraints) we obtain the easy to interpret Theorem 3.1 discussed in Section 3.

We set σ as large as possible with respect to the accuracy we wish to have. Given this σ we want to max out on our privacy budget. That is, we satisfy (31) with equality,

$$\sigma = \sqrt{\frac{2(\epsilon + \ln(1/\delta))}{\epsilon}}. \tag{58}$$

We discuss (58) with constraints (28), (29), and (30) below:

Replacing (28) and (29): In practice, we need a sufficiently strong DP guarantee, hence, $\delta \leq 1/N$ and ϵ is small enough, typically ≤ 0.5. This means that we will stretch σ to at least $\sqrt{2 + 4 \ln N}$. A local data set of size $N = 10000$ requires $\sigma \geq 6.23$; a local data set of size $N = 100000$ requires $\sigma \geq 6.93$. For such $\sigma \geq 6$ we have $h(\sigma) \geq h(6) = 0.42$ (since $h(\sigma)$ is increasing in σ). (For reference, $h(10) = 0.58$, and for $\sigma \gg 1$ we have $h(\sigma) \approx 1$.) From (58) we infer that $g(\sqrt{2(\epsilon + \ln(1/\delta))/\epsilon}) = g(\sigma) = \min\{1/(e\sigma), h(\sigma)\}$. One can verify that $h(\sigma) - 1/(e\sigma)$ is positive and increasing for $\sigma \geq 2.5$, hence, $g(\sigma) = 1/(e\sigma)$ for $\sigma \geq 6$. This reduces requirement (28) to $\bar{s} \leq N/(e\sigma\theta)$ and requirement (29) to $\epsilon \leq 0.42 \cdot \gamma K/N$. Notice that (30) in combination with $\epsilon \leq \frac{\gamma\theta}{e\sigma} K/N$ implies condition $\bar{s} \leq N/(e\sigma\theta)$. This implies that (28) and (29) are satisfied for $\epsilon \leq \min\{0.42 \cdot \gamma, \gamma\theta/(e\sigma)\} \cdot \frac{K}{N}$ or, equivalently, $K \geq \epsilon \cdot \max\{2.4/\gamma, e\sigma/(\gamma\theta)\}$ epochs of size N. If $\theta \leq 6.85$, then $\sigma \geq 6 \geq 0.88 \cdot \theta = 2.4 \cdot \theta/e$, hence, $\max\{2.4/\gamma, e\sigma/(\gamma\theta)\} = e\sigma/(\gamma\theta)$ and this reduces the condition on K to

$$K \geq \epsilon\sigma \cdot \frac{e}{\gamma\theta} = \sqrt{2\epsilon(\epsilon + \ln(1/\delta))} \cdot e/(\gamma\theta) \text{ epochs of size } N,$$

where the equality follows from (58). In practical settings, K consists of multiple (think 50 or 100s of) epochs (of size N) computation and this is generally true. We conclude that (28) and (29) are automatically satisfied by (30) for general practical settings with $\delta \leq 1/N$, ϵ typically smaller than 0.5, $N \geq 10000$, $\theta \leq 6.85$, and $K \geq \sqrt{2\epsilon(\epsilon + \ln(1/\delta))} \cdot e/(\gamma\theta)$ epochs.

Remaining constraint (30): By using (58), (30) can be equivalently recast as an upper bound on σ,

$$\sigma \leq \sqrt{\frac{2(\epsilon + \ln(1/\delta))}{\gamma\theta^2 \cdot (K/N) \cdot (\bar{s}/N)}}.$$

Here, γ is a function of σ because γ depends on ϵ in $\bar{\alpha}$ which is a function of σ through (58). However, the definition of γ shows that for small ϵ, γ is close to 2 and this gives $\sqrt{\ln(1/\delta)/(\theta^2 \cdot (K/N) \cdot (\bar{s}/N))}$ as a good approximation of the upper bound. Substituting $\bar{s} = K/T$ yields

$$\sigma \leq \frac{N\sqrt{T}}{K} \sqrt{\frac{2(\epsilon + \ln(1/\delta))}{\gamma\theta^2}}. \tag{59}$$

For $\gamma \approx 2$ and $\theta = 1$ (constant sample size), this upper bound compares to taking $c_2 z \approx \sqrt{2}$ in (7); we go beyond the analysis presented in (Abadi et al., 2016) in a non-trivial way.

If $N\sqrt{T}/K$ is large enough, larger than the relatively small $\sigma\sqrt{\theta^2(\gamma/2)/(\epsilon + \ln(1/\delta))}$, then upper bound (59) is satisfied. That is, for given K and N, we need T to be large enough, or equivalently the mean sample/mini-batch size $\bar{s} = K/T$ small enough. Squaring both sides of (59) and moving terms yields the equivalent lower bound

$$T \geq \frac{\gamma}{2} \frac{\sigma^2\theta^2}{\epsilon + \ln(1/\delta)} \cdot (K/N)^2,$$

which after substituting (58) gives

$$T \geq \frac{\gamma\theta^2}{\epsilon} \cdot (K/N)^2.$$

In other words T is at least a factor $\gamma\theta^2/\epsilon$ larger than the square of the overall amount of local SGD computations measured in epochs (of size N). Notice that we have a lower bound on T rather than an upper bound as in (8) from the theorem presented in (Abadi et al., 2016).

C. Tight Analysis using Gaussian DP

(Dong et al., 2021) explain an elegant alternative DP formulation based on hypothesis testing. From the attacker's perspective, it is natural to formulate the problem of distinguishing two neighboring data sets d and d' based on the output of a DP mechanism $\mathcal{M}$ as a hypothesis testing problem:

$$H_0: \text{ the underlying data set is } d \quad \text{versus} \quad H_1: \text{ the underlying data set is } d'.$$

We define the Type I and Type II errors by

$$\alpha_\phi = \mathbf{E}_{o\sim\mathcal{M}(d)}[\phi(o)] \text{ and } \beta_\phi = 1 - \mathbf{E}_{o\sim\mathcal{M}(d')}[\phi(o)],$$

where ϕ in $[0,1]$ denotes the rejection rule which takes the output of the DP mechanism as input. We flip a coin and reject the null hypothesis with probability ϕ. The optimal trade-off between Type I and Type II errors is given by the trade-off function

$$T(\mathcal{M}(d), \mathcal{M}(d'))(\alpha) = \inf_\phi\{\beta_\phi \ : \ \alpha_\phi \leq \alpha\},$$

for $\alpha \in [0,1]$, where the infimum is taken over all measurable rejection rules ϕ. If the two hypothesis are fully indistinguishable, then this leads to the trade-off function $1-\alpha$. We say a function $f \in [0,1] \to [0,1]$ is a trade-off function if and only if it is convex, continuous, non-increasing, and $f(x) \leq 1-x$ for $x \in [0,1]$. We define a mechanism $\mathcal{M}$ to be f-DP if

$$T(\mathcal{M}(d), \mathcal{M}(d')) \geq f$$

for all neighboring d and d'. Proposition 2.5 in (Dong et al., 2021) is an adaptation of a result in (Wasserman & Zhou, 2010) and states that a mechanism is (ϵ,δ)-DP if and only if the mechanism is $f_{\epsilon,\delta}$-DP, where

$$f_{\epsilon,\delta}(\alpha) = \min\{0, 1-\delta-e^\epsilon\alpha, (1-\delta-\alpha)e^{-\epsilon}\}.$$

We see that f-DP has the (ϵ,δ)-DP formulation as a special case. It turns out that the DP-SGD algorithm can be tightly analysed by using f-DP.

Gaussian DP: In order to proceed (Dong et al., 2021) first defines Gaussian DP as another special case of f-DP as follows: We define the trade-off function

$$G_\mu(\alpha) = T(\mathcal{N}(0,1), \mathcal{N}(\mu,1))(\alpha) = \Phi(\Phi^{-1}(1-\alpha) - \mu),$$

where Φ is the standard normal cumulative distribution of $\mathcal{N}(0,1)$. We define a mechanism to be μ-Gaussian DP if it is G_μ-DP. Corollary 2.13 in (Dong et al., 2021) shows that a mechanism is μ-Gaussian DP if and only if it is $(\epsilon, \delta(\epsilon))$-DP for all $\epsilon \geq 0$, where

$$\delta(\epsilon) = \Phi(-\frac{\epsilon}{\mu} + \frac{\mu}{2}) - e^\epsilon\Phi(-\frac{\epsilon}{\mu} - \frac{\mu}{2}). \tag{60}$$

Subsampling: Besides implementing Gaussian noise, DP-SGD also uses sub-sampling: For a data set d of N samples, $\texttt{Sample}_s(d)$ selects a subset of size s from d uniformly at random. We define convex combinations

$$f_p(\alpha) = pf(\alpha) + (1-p)(1-\alpha)$$

with corresponding p-sampling operator

$$C_p(f) = \min\{f_p, f_p^{-1}\}^{**},$$

where the conjugate g^* of a function g is defined as

$$g^*(y) = \sup_x\{yx - g(x)\}.$$

Theorem 4.2 in (Dong et al., 2021) shows that if a mechanism $\mathcal{M}$ on data sets of size N is f-DP, then the subsampled mechanism $\mathcal{M} \circ \texttt{Sample}_s$ is $C_{s/N}(f)$-DP.

Composition: The tensor product $f \otimes g$ for trade-off functions $f = T(P, Q)$ and $g = T(P', Q')$ is well-defined by

$$f \otimes g = T(P \times P', Q \times Q').$$

Let $y_i \leftarrow \mathcal{M}_i(\texttt{aux}, d)$ with $\texttt{aux} = (y_1, \ldots, y_{i-1})$. Theorem 3.2 in (Dong et al., 2021) shows that if $\mathcal{M}_i(\texttt{aux}, .)$ is f-DP for all $\texttt{aux}$, then the composed mechanism $\mathcal{M}$, which applies $\mathcal{M}_i$ in sequential order from $i = 1$ to $i = T$, is $f^{\otimes T}$-DP.

Tight Analysis DP-SGD: We are now able to formulate the differential privacy guarantee of DP-SGD since it is a composition of subsampled Gaussian DP mechanisms. Theorem 5.1 in (Dong et al., 2021) states that DP-SGD in Algorithm 1 is[7]

$$C_{s/N}(G_{(\sigma/2)^{-1}})^{\otimes T}\text{-DP}.$$

Since each of the theorems and results from (Dong et al., 2021) enumerated above are exact, we have a tight analysis.

Our Goal: We want to understand the behavior of the DP guarantee in terms of s, N, T, and σ. Our goal is to have an easy interpretation of the DP guarantee so that we can select "good" parameters s, N, T, and σ a-priori; good in terms of achieving at least our target accuracy without depleting our privacy budget. If we know how the differential privacy budget is being depleted over DP-SGD iterations, then we can optimize parameter settings in order to attain best performance, that is, best accuracy of the final global model (the most important target when we work with machine learning modelling). According to our best knowledge, all the current-state-of-the art privacy accountants do not allow us to achieve this goal because they are only privacy loss accountants and do not offer ahead-planning. It is not sufficient to only rely on a differential privacy accountant (see e.g., (Zhu et al., 2021) as follow-up work of (Dong et al., 2021)) for a client to understand when to stop helping the server to learn a global model.

When talking about accuracy, we mean how much loss in prediction/test accuracy is sacrificed by fixing a σ (and clipping constant C). Our theory maps σ directly to an (ϵ, δ)-DP guarantee independent of the number of rounds T. This allows use to characterize the trade-off between accuracy and privacy budget. All the current-state-of-the art privacy loss frameworks do not offer this property.

We notice that (Dong et al., 2021) makes an effort to interpret the $C_{s/N}(G_{(\sigma/2)^{-1}})^{\otimes T}$-DP guarantee. Their Corollary 5.6 provides a precise expression based on integrals, themselves again depending on $p = s/N$ and $\mu = (\sigma/2)^{-1}$ in our notation. This still does not lead to the intuition we seek as we cannot extract how to select parameters σ, s and T given a data set of size N, given a privacy budget, and given a utility that we wish to achieve. We further explain this point in next paragraphs.

In what follows, we seek a relationship between σ, s, T, ϵ, δ, and N for Gaussian DP based on Corollary 5.4 in (Dong et al., 2021). Corollary 5.4 in (Dong et al., 2021) provides an **asymptotic analysis** which is a step forward to the kind of easy to understand interpretation we seek for: It states that if both $T \to \infty$ and $N \to \infty$ such that $s\sqrt{T}/N \to c$ for some constant $c > 0$ (and where s is a function of N that may tend to ∞ as well), then the DP-SGD algorithm (in this paper with the factor 2 differing σ) is μ-Gaussian DP for

$$\mu = (c/2) \cdot \tau^{-1} \text{ with } \tau^{-1} = \sqrt{8} \cdot \sqrt{e^{(\sigma/2)^{-2}} \Phi(3(\sigma/2)^{-1}/2) + 3\Phi(-(\sigma/2)^{-1}/2) - 2}.$$

In Section C.1 we show that $\tau^{-1} = \sigma^{-1} + O(\sigma^{-2})$ and we show that for $\mu \leq \epsilon \leq 1$, μ-Gaussian DP translates to the DP-SGD algorithm being (ϵ, δ)-DP for $\delta \ll \epsilon \ll 1$ if

$$\tau \quad \approx \quad \frac{(c/2)\sqrt{2(\ln(1/\delta) + \ln(\epsilon) - O(\ln\ln(1/\delta)))}}{\epsilon} \text{ with } s\sqrt{T}/N \to c.$$

We see a similar $s\sqrt{T}/N$ dependency in Theorem B.1 by (Abadi et al., 2016). The difference is that Theorem B.1 holds in a **non-asymptotic** setting. That is, T and N do not need to tend to ∞ whereas the expression above does require taking such a limit. Of course, one can analyse the convergence rate of achieving the limit μ given T and N tending to infinity. When doing such an analysis one may find expressions of Gaussian DP guarantees as a function of T and N that hold for all concrete values of T and N. This may lead to results that strengthen our Theorem B.1 (we leave this as an open problem). It

[7] Their DP-SGD algorithm uses noise $\mathcal{N}(0, C^2(2\sigma)^2\mathbf{I})$ compared to $\mathcal{N}(0, C^2\sigma^2\mathbf{I})$ in our version of the DP-SGD algorithm.

is clear that the above asymptotic result is still insufficient for our purpose: How do we a-priori select concrete parameters σ, s, and T given concrete parameters for N, a given privacy budget and utility that we wish to achieve?

In this paper we decided to generalize the proof method of Theorem B.1 rather than working with the complex integrals that provide the exact characterization of f-DP for the DP-SGD algorithm as stated above. This approach allows us to obtain the non-asymptotic result of Theorem 3.1 which shows into large extent the independence of T, which is not immediately understood from Theorem B.1 and the corollary discussed above. As future work we propose to use the f-DP framework to figure out how tight the presented analysis is in this paper – since the discussed corollary above and Theorem B.1 are up to constants equivalent (in the asymptotic setting) and since our generalization makes constants in Theorem B.1 explicit as function of parameters s, T, and N, we expect our analysis to be tight up to a small constant factor. Section C.2 shows a first result on the tightness of our Theorem 3.1. The advantage of our result is that it is easy to interpret and we do not need to fully rely on an accountant method to keep track of spent privacy budget while participating in learning a global model based on local data.

C.1. Translation to (ϵ, δ)-DP

We first observe that by using $e^x = 1 + x + O(x^2)$ and $\Phi(x) = \frac{1}{2} + \frac{e^{-x^2/2}}{\sqrt{2\pi}}(x + O(x^3)) = \frac{1}{2} + \frac{x}{\sqrt{2\pi}} + O(x^3)$, a first order approximation of τ^{-1} is equal to $\sigma^{-1} + O(\sigma^{-2})$ (hence, $\tau^{-1} \approx \sigma^{-1}$ for large σ).

For $x \geq 0$, we have the approximation

$$\Phi(-x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \left(\frac{1}{x} - \frac{1}{x^3} + O(\frac{1}{x^5}) \right).$$

Let $y \leq x$. Together with $-(x-y)^2/2 = 2xy - (x+y)^2/2$ we derive

$$\begin{aligned}
\Phi(-x+y) - e^{2xy}\Phi(-x-y) &= \frac{e^{-(x-y)^2/2}}{\sqrt{2\pi}}(\frac{1}{x-y} - \frac{1}{(x-y)^3} + O(\frac{1}{(x-y)^5}) \\
&\quad - \frac{e^{-(x-y)^2/2}}{\sqrt{2\pi}}(\frac{1}{x+y} - \frac{1}{(x+y)^3} + O(\frac{1}{(x+y)^5})) \\
&= \frac{e^{-(x-y)^2/2}}{\sqrt{2\pi}}(\frac{2y}{x^2-y^2} - \frac{6yx^2+2y^3}{(x^2-y^2)^3} + O(\frac{1}{x^5})) \\
&= \frac{e^{-x^2(1-y/x)^2/2}}{\sqrt{2\pi}}(\frac{2y}{x^2(1-(y/x)^2)} + O(\frac{y}{x^4} + \frac{1}{x^5}))
\end{aligned}$$

If we assume

$$\mu \leq \epsilon \leq 1,$$

then $(\mu/2)/(\epsilon/\mu) = \mu^2/(2\epsilon) \leq \epsilon/2 \leq 1$, $\epsilon/\mu \geq 1$, and $\epsilon \leq 1$. We can use the above formulas and approximate (60) as follows:

$$\begin{aligned}
\delta &= \Phi(-\frac{\epsilon}{\mu} + \frac{\mu}{2}) - e^{\epsilon}\Phi(-\frac{\epsilon}{\mu} - \frac{\mu}{2}) \\
&= \frac{e^{-(\epsilon/\mu)^2(1-\mu^2/(2\epsilon))^2/2}}{\sqrt{2\pi}} \left(\frac{\mu^3}{\epsilon^2(1-(\mu^2/(2\epsilon))^2)} + O(\frac{\mu^5}{\epsilon^5}) \right).
\end{aligned}$$

This gives

$$1/\mu = \frac{\sqrt{2(\ln(1/\delta) + \ln(\frac{\mu^3}{\sqrt{2\pi}(\epsilon^2-\mu^4/4)} + O((\frac{\mu}{\epsilon})^5)))}}{\epsilon - \mu^2/2},$$

hence,

$$\tau = \frac{(c/2)\sqrt{2(\ln(1/\delta) + \ln(\frac{\mu^3}{\sqrt{2\pi}(\epsilon^2-\mu^4/4)} + O((\frac{\mu}{\epsilon})^5)))}}{\epsilon - \mu^2/2} \text{ with } s\sqrt{T}/N \to c \text{ and } \tau^{-1} = \sigma^{-1} + O(\sigma^{-2}).$$

For small ϵ, we can approximate τ as

$$\tau \approx \frac{(c/2)\sqrt{2(\ln(1/\delta) + \ln(\frac{\epsilon}{\sqrt{2\pi}}(\frac{\mu}{\epsilon})^3 + O((\frac{\mu}{\epsilon})^5)))}}{\epsilon}.$$

Since $(c/2)\tau^{-1} = \mu \leq \epsilon$, we may write $\mu = \epsilon/b$ for some $b \geq 1$. Notice that $(c/2)/\epsilon = \tau/b$. This leads to the approximation

$$b \approx \sqrt{2(\ln(1/\delta) + \ln(\frac{\epsilon}{\sqrt{2\pi}}\frac{1}{b^3} + O(\frac{1}{b^5})))}.$$

By default $\delta = 1/N$ and, see Section 3.2, $N = \Omega(1/\epsilon)$, that is, $\epsilon = \Omega(1/N)$ (also notice that good accuracy can only be achieved for σ small enough, that is, ϵ is generally orders of magnitude larger than $1/N$). So, we may assume $\epsilon \gg \delta$. Then substituting $\sqrt{2\ln(1/\delta)}$ for b at the right hand side yields

$$b \approx \sqrt{2(\ln(1/\delta) + \ln(\epsilon) - O(\ln\ln(1/\delta)))}.$$

Substituting back in the expression for τ proves for $\delta \ll \epsilon \ll 1$,

$$\tau \approx \frac{(c/2)\sqrt{2(\ln(1/\delta) + \ln(\epsilon) - O(\ln\ln(1/\delta)))}}{\epsilon} \text{ with } s\sqrt{T}/N \to c \text{ and } \tau^{-1} = \sigma^{-1} + O(\sigma^{-2}).$$

C.2. On the Tightness of the Lower Bound on T

We consider the special case where T meets its lower bound (3): $T = \frac{\gamma\theta^2}{\epsilon} \cdot k^2$. Notice that our experiments show that this is a good setting for best accuracy; we also target this in our adaptive DP-SGD. We wonder into what extent we can relax the constraint on T in Theorem 3.1. That is, how much can the lower bound on T be reduced?

To answer this question, let

$$T = \frac{a}{\epsilon} \cdot k^2$$

and let's define

$$c := \sqrt{\frac{\epsilon}{a}} = \frac{k}{\sqrt{T}} = \frac{K}{\sqrt{T}N} = \frac{s\sqrt{T}}{N},$$

where we consider a constant step size s (hence, $\theta = 1$). Substituting this in the final formula for τ of Section C.1 yields for $\epsilon \gg \delta$

$$\tau \approx \sqrt{\frac{\ln(1/\delta) + \ln(\epsilon) - O(\ln\ln(1/\delta))}{2a\epsilon}}. \tag{61}$$

Here, $\tau \approx \sigma$ for $\sigma \gg 1$. Since Section C.1 provides a tight analysis, we know that for $\epsilon \gg \delta$ and $\sigma \gg 1$ with $s\sqrt{T}/N \to c$ for $N \to \infty$ and $T \to \infty$, we cannot achieve a σ smaller than the right hand side of (61). If Theorem 3.1 holds true for $T \geq \frac{a}{\epsilon} \cdot k^2$, then $\sigma = \sqrt{2(\epsilon + \ln(1/\delta))/\epsilon}$ can be achieved. According to the previous argument this must be at least the right hand side of (61) and we conclude that $2 \geq 1/(2a)$. This shows that $a \geq 1/4$ and at best Theorem 3.1 holds for $T \geq \frac{1}{4\epsilon} \cdot k^2$, hence, at most a factor $4\gamma\theta^2$ smaller than the current lower bound on T for which Theorem 3.1 is proven. We conclude that the lower bound on T is tight up to at most a factor $4\gamma\theta^2$.

This raises the question, can we prove a lower σ for (ϵ, δ) than the one of Theorem 3.1 if we require a more restrictive lower bound on T, that is, $T \geq \frac{a}{\epsilon} \cdot k^2$ where $a \geq \gamma\theta^2$ (a larger lower bound)? That is, can we prove a lower σ for larger $a \geq 1$; the current theorem proves the current $\sigma = \sqrt{2(\epsilon + \ln(1/\delta))/\epsilon}$ for $a = 1$. It may very well be the case that we can prove a σ characterization less than $\sqrt{2(\epsilon + \ln(1/\delta))/\epsilon}$ if $a > 1$. We do not know into what extent our formula for σ can be improved (made smaller) for larger $T \geq \frac{a}{\epsilon} \cdot k^2$. We hypothesize that such an improvement will transform the current simple formula into a complex one in order to account for the dependency on a (which may restrict our capability of being able to plan ahead). We expect that Theorem 3.1 provides a good first order approximation of the dependency on a. We leave the above question as an open problem.

D. Experiments

We provide experiments to support our theoretical findings, i.e., convergence of our proposed asynchronous distributed learning framework with differential privacy (DP) to a sufficiently accurate solution. We cover strongly convex, plain convex and non-convex objective functions over iid local data sets.

We introduce our experimental set up in Section D.1. Section D.2 provides utility graphs for different data sets and objective functions. A utility graph helps choosing the maximum possible noise σ, in relation to the value of the clipping constant C, for which decent accuracy can be achieved. Section D.3 provides detailed experiments for our asynchronous differential privacy SGD framework (asynchronous DP-SGD) with different types of objective functions (i.e., strongly convex, plain convex and non-convex objective functions), different types of constant sample size sequences and different levels of privacy guarantees (i.e., different privacy budgets ϵ).

All our experiments are conducted on LIBSVM (Chang & Lin, 2011)[8] , MNIST (LeCun & Cortes, 2010) [9], and CIFAR10[10] data sets.

D.1. Experiment settings

Simulation environment. For simulating the asynchronous DP-SGD framework, we use multiple threads where each thread represents one compute node joining the training process. The experiments are conducted on Linux-64bit OS, with 16 cpu processors, and 32Gb RAM.

Objective functions. Equation (62) defines the plain convex logistic regression problem. The weight vector w and the bias value b of the logistic function can be learned by minimizing the log-likelihood function J:

$$J = -\sum_{i=1}^{N}[y_i \cdot \log(\bar{\sigma}_i) + (1 - y_i) \cdot \log(1 - \bar{\sigma}_i)], \text{ (plain convex)} \tag{62}$$

where N is the number of training samples (x_i, y_i) with $y_i \in \{0, 1\}$ and $\bar{\sigma}_i$ is defined by

$$\bar{\sigma}_i = \frac{1}{1 + e^{-(w^{\mathrm{T}} x_i + b)}},$$

which is the sigmoid function with parameters w and b. Our goal is to learn a vector w^* which represents a pair $\bar{w} = (w, b)$ that minimizes J.

Function J can be changed into a strongly convex problem $\hat{J}$ by adding a regularization parameter $\lambda > 0$:

$$\hat{J} = -\sum_{i=1}^{N}[y_i \cdot \log(\sigma_i) + (1 - y_i) \cdot \log(1 - \sigma_i)] + \frac{\lambda}{2} \|w\|^2, \text{ (strongly convex).}$$

where $\bar{w} = (w, b)$ is vector w concatenated with bias value b. In practice, the regularization parameter λ is set to $1/N$ (Roux et al., 2012).

For simulating non-convex problems, we choose a simple neural network (Letnet) (LeCun et al., 1998) for MNIST data set and AlexNet (Krizhevsky et al., 2012) for CIFAR10 data set with cross entropy loss function for image classification.

The loss functions for the strong, plain, and non-convex problems represent the objective function $F(.)$.

Parameter selection. The parameters used for our distributed algorithm with Gaussian based differential privacy for strongly convex, plain convex and non-convex objective functions are described in Table 1. The clipping constant C is set to 0.1 for strongly convex and plain convex problems and 0.025 for non-convex problem (this turns out to provide good utility).

For the plain convex case, we can use diminishing step size schemes $\frac{\eta_0}{1+\beta \cdot t}$ or $\frac{\eta_0}{1+\beta \cdot \sqrt{t}}$. In this paper, we focus our experiments for the plain convex case on $\frac{\eta_0}{1+\beta \cdot \sqrt{t}}$. Here, η_0 is the initial step size and we perform a systematic grid search on parameter $\beta = 0.001$ for strongly convex case and $\beta = 0.01$ for both plain convex and non-convex cases. Moreover, most of the

[8]https://www.csie.ntu.edu.tw/ cjlin/libsvmtools/datasets/binary.html
[9]http://yann.lecun.com/exdb/mnist/
[10]https://www.cs.toronto.edu/ kriz/cifar.html

Table 1. Common parameters of asynchronous DP-SGD framework with differential privacy

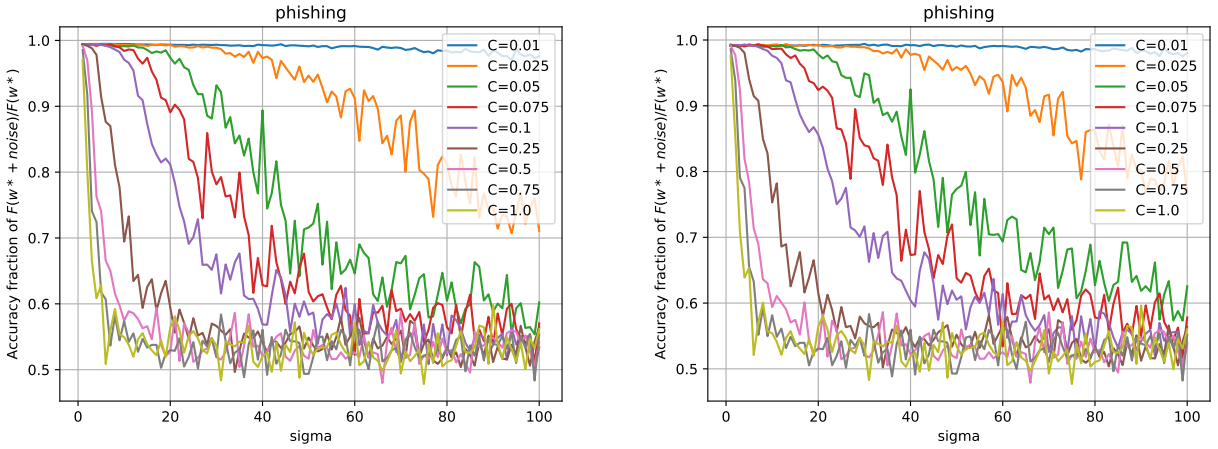
	# of clients n	Diminishing step size $\bar{\eta}_t$	Regular λ	Clipping constant C
Strongly convex	5	$\frac{\eta_0}{1+\beta t}$ ‡	$\frac{1}{N}$	0.1
Plain convex	5	$\frac{\eta_0}{1+\beta t}$ or $\frac{\eta_0}{1+\beta\sqrt{t}}$	N/A	0.1
Non-convex	5	$\frac{\eta_0}{1+\beta\sqrt{t}}$	N/A	0.025

‡ The i-th round step size $\bar{\eta}_i$ is computed by substituting $t = \sum_{j=0}^{i-1} s_j$ into the diminishing step size formula.

experiments are conducted with 5 compute nodes and 1 central server. When we talk about accuracy (from Figure 7 and onward), we mean test accuracy defined as the fraction of samples from a test data set that get accurately labeled by the classifier (as a result of training on a training data set by minimizing a corresponding objective function).

D.2. Utility graph

The purpose of a utility graph is to help us choose, given the value of the clipping constant C, the maximum possible noise σ for which decent accuracy can be achieved. A utility graph depicts the test accuracy of model $F(w^* + n)$ over $F(w^*)$, denotes as accuracy fraction, where w^* is a near optimal global model and $n \sim \mathcal{N}(0, C^2\sigma^2\mathbf{I})$ is Gaussian noise. This shows which maximum σ can be chosen with respect to allowed loss in expected test accuracy, clipping constant C and standard deviation σ.



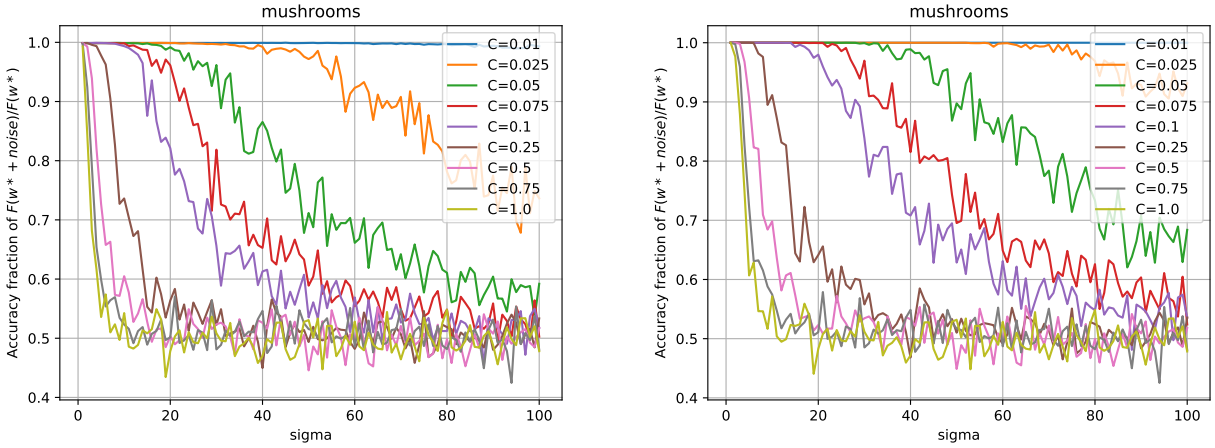
(a) Strong convex.

(b) Plain convex.

Figure 4. Utility graph with various gradient norm C and noise level σ

As can be seen from Figure 4 and Figure 5, for clipping constant $C = 0.1$, we can choose the maximum σ somewhere in the range $\sigma \in [18, 22]$ if we want to guarantee there is at most about 10% accuracy loss compared to the (near)-optimal solution without noise. Another option is $C = 0.075$, where we can tolerate $\sigma \in [18, 30]$ yielding the same accuracy loss guarantee. When the gradient bound C gets smaller, our DP-SGD can tolerate bigger noise, i.e, bigger values of σ. However, we need to increase the number K of iterations during the training process when C is smaller in order to converge and gain a specific test accuracy – this is the trade-off. For simplicity, we intentionally choose $C = 0.1$, $\sigma \leq 20$ and expected test accuracy loss about 10% for our experiments with strongly convex and plain convex objective functions.

The utility graph is extended to the non-convex objective function in Figure 6. To keep the test accuracy loss less or equal to 10% (of the final test accuracy of the original model w^*), we choose $C = 0.025$ and noise level $\sigma \leq 12$ for MNIST data set (as shown in Figure 6a) and $C = 0.025$ and noise level $\sigma \leq 6.572$ for CIFAR10 data set (as shown in Figure 6b). For simplicity, we use this parameter setting for our experiments with the non-convex problem.



(a) Strong convex.

(b) Plain convex.

Figure 5. Utility graph with various gradient norm C and noise level σ

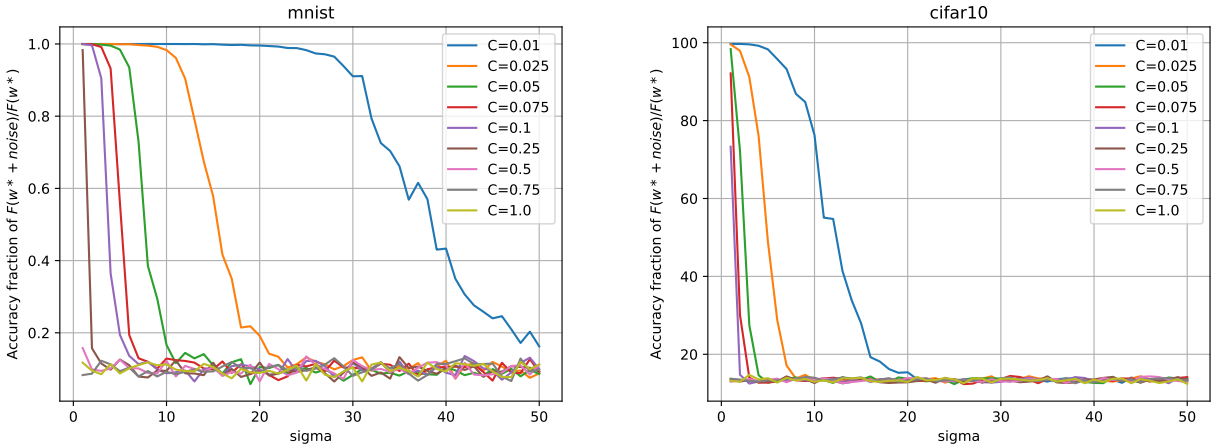


Figure 6. Utility graph with various gradient norm C and noise level σ for MNIST and CIFAR10 data sets.

D.3. Asynchronous distributed learning with differential privacy

We consider the asynchronous DP-SGD framework with strongly convex, plain convex and non-convex objective functions for different settings, i.e., different levels of privacy budget ϵ and different constant sample size sequences.

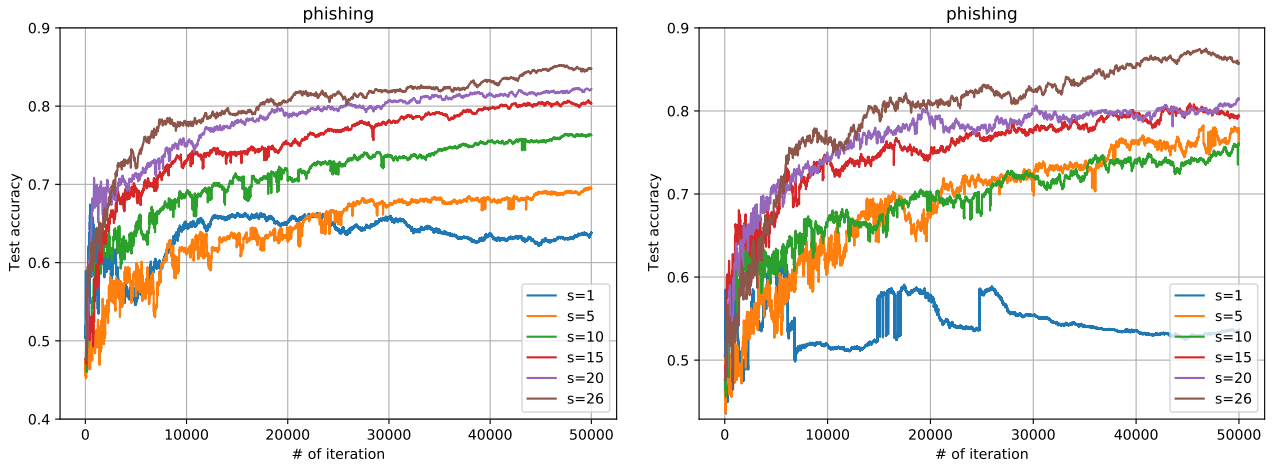
D.3.1. Asynchronous DP-SGD with different constant sample size sequences

The purpose of this experiment is to investigate which is the best constant sample size sequence $s_i = s$. This experiment allows us to choose a decent sample size sequence that will be used in our subsequent experiments. To make the analysis simple, we consider our asynchronous DP-SGD framework with $\Upsilon(k, i)$ defined as false if and only if $k < i - 1$, i.e., compute nodes are allowed to run fast and/or have small communication latency such that broadcast global models are at most 1 local round in time behind (so different clients can be asynchronous with respect to one another for 1 local round). We also use iid data sets. The detailed parameters are in Table 2.

The results from Figure 7 to Figure 8 confirm that our asynchronous DP-SGD framework can converge under a very small

Table 2. Basic parameter setting for strongly and plain convex problems

Parameter	Value	Note
$\bar{\eta}_0$	0.1	initial stepsize
N_c	$10,000$	# of data points
K	$50,000$	# of iterations
ϵ	0.04945	
σ	19.29962	
δ	0.0001	
C	0.1	clipping constant
s	$\{1, 5, 10, 15, 20, 26\}$	constant sample size sequence
dataset	LIBSVM	iid dataset
n	5	# of nodes
Υ	$k \geq i - 1$	$1-$asynchronous round



(a) Strong convex.

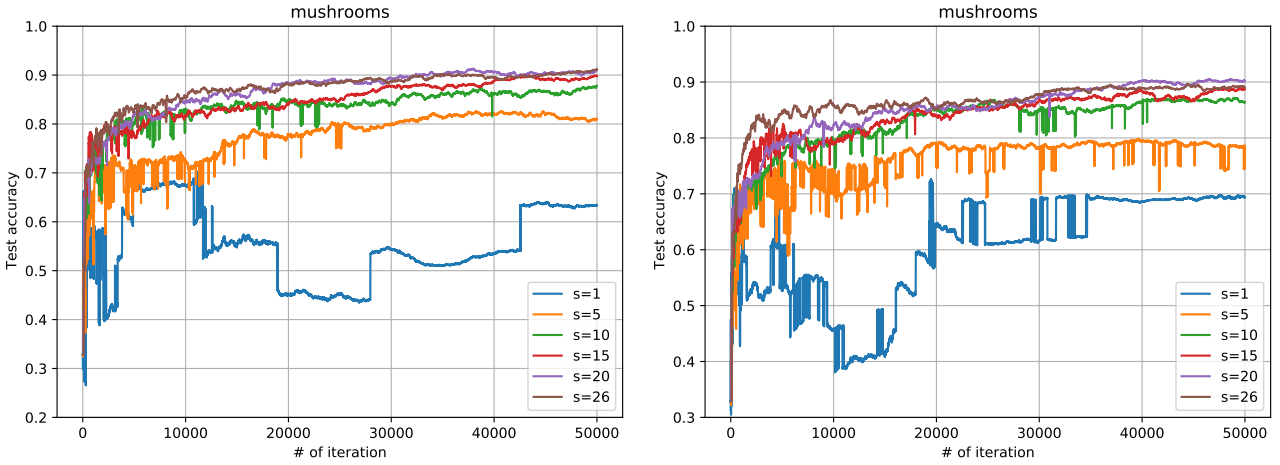
(b) Plain convex.

Figure 7. Effect of different constant sample size sequences

privacy budget. When the constant sample size $s = 1$, it is clear that the DP-SGD algorithm does not achieve good accuracy compared to other constant sample sizes even though this setting has the maximum number of communication rounds. When we choose constant sample size $s = 26$ (this meets the upper bound for constant sample sizes for our small $N = 10,000$ and small $\epsilon \approx 0.05$, see Theorem B.4), our DP-SGD framework converges to a decent test accuracy, i.e, the test accuracy loss is expected less than or equal to 10% when compared to the original mini-batch SGD without noise. In conclusion, this experiment demonstrates that our asynchronous DP-SGD with diminishing step size scheme and constant sample size sequence works well under DP setting, i.e, our asynchronous DP-SGD framework can gain differential privacy guarantees while maintaining an acceptable accuracy.

We also conduct the experiment for the non-convex objective function with MNIST and CIFAR10 data sets.

The detailed parameter settings can be found in Table 3 and Table 4. Here, we again consider our asynchronous setting where each compute node is allowed to run fast and/or has small communication latency such that broadcast global models are at most 1 local round in time behind. As can be seen from Figure 9 (with MNIST data set), our proposed asynchronous DP-SGD still converges under small privacy budget. Moreover, when we use the constant sample size $s = 370$ (this meets the upper bound for constant sample sizes for our small $N = 60,000$ and small $\epsilon \approx 0.15$, see Theorem B.4), we can significantly reduce the communication cost compared to other constant sample sizes while keeping the test accuracy loss within 10%. The constant sample size $s = 10$ (as well as $s \leq 10$) shows a worse performance while this setting requires more communication rounds, compared to other constant sample sizes. We can observe the same pattern for CIFAR10 data



(a) Strong convex.

(b) Plain convex.

Figure 8. Effect of different constant sample size sequences

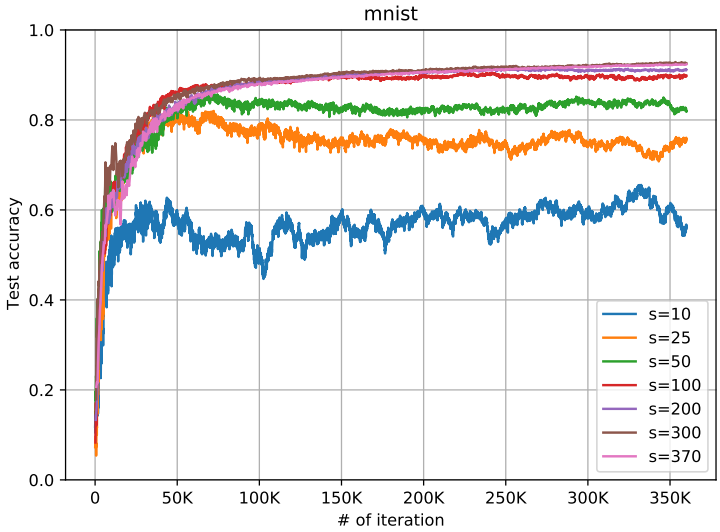
Table 3. Basic parameter setting for non-convex problem with MNIST data set

Parameter	Value	Note
$\bar{\eta}_0$	0.1	initial stepsize
N_c	$60,000$	# of data points
K	$360,000$	# of iterations
ϵ	0.15007	
σ	12.10881	
δ	$1.667 \cdot 10^{-5}$	
C	0.025	clipping constant
s	$\{10, 25, 50, 100, 200, 300, 370\}$	constant sample size sequence
dataset	MNIST	iid dataset
n	5	# of nodes
Υ	$k \geq i - 1$	$1-$asynchronous round

Table 4. Basic parameter setting for non-convex problem for CIFAR10 data set

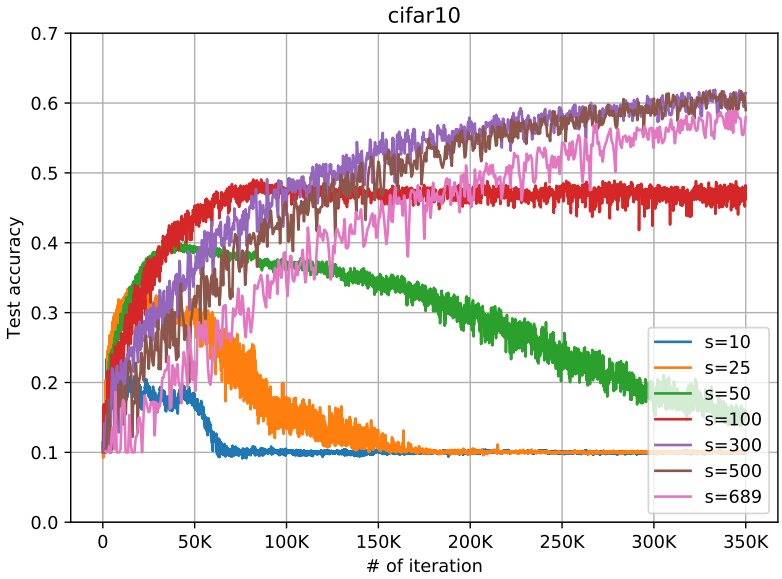
Parameter	Value	Note
$\bar{\eta}_0$	0.1	initial stepsize
N_c	$50,000$	# of data points
K	$350,000$	# of iterations
ϵ	0.50102	
σ	6.572	
δ	$2 \cdot 10^{-5}$	
C	0.025	clipping constant
s	$\{10, 25, 50, 100, 300, 500, 689\}$	constant sample size sequence
dataset	CIFAR10	iid dataset
n	5	# of nodes
Υ	$k \geq i - 1$	$1-$asynchronous round

set as shown in Figure 10, where we can choose the constant sample size $s \leq 689$ with $N = 50,000$ data points and $\epsilon \approx 0.5$. While the constant sample size s satisfying $300 \leq s \leq 689$, the test accuracy gets the highest level while the constant



Non-convex.

Figure 9. Effect of different constant sample size sequences



Non-convex.

Figure 10. Effect of different constant sample size sequences.

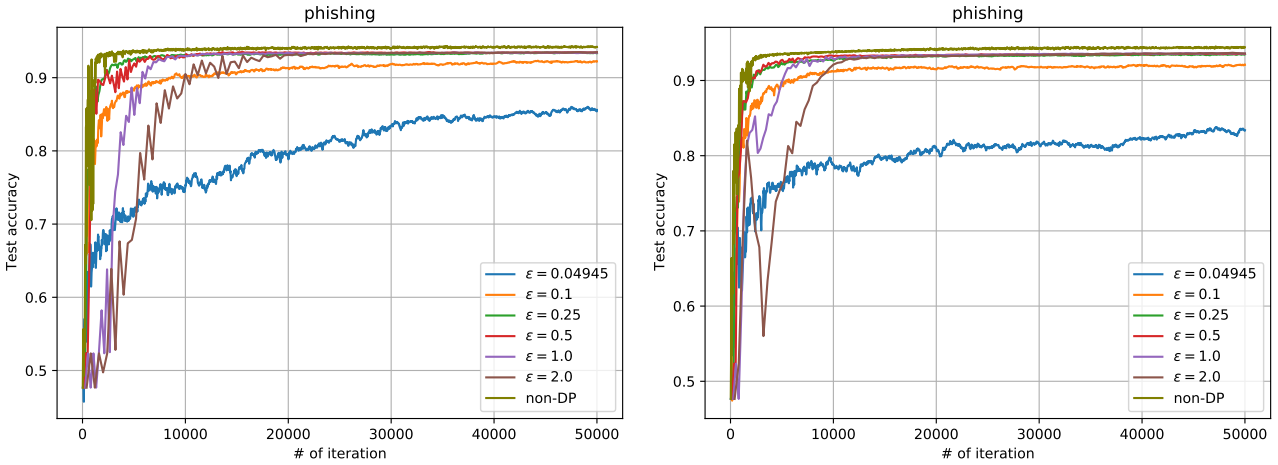
sample size $s \leq 50$ deteriorates the performance of accuracy significantly. This figure again confirms the effectiveness of our asynchronous DP-SGD framework towards a strong privacy guarantee for all types of objective function.

D.3.2. Asynchronous DP-SGD with different levels of privacy budget

We conduct the following experiments to compare the effect of our DP-SGD framework for different levels of privacy budget ϵ including the non-DP setting (i.e., no privacy at all, hence, no noise). The purpose of this experiment is to show that the test accuracy degradation is at most 10% even if we use very small ϵ. The detailed constant sample sequence s and noise level σ based on Theorem B.4 are illustrated in Table 5. Other parameter settings, such as initial stepsize η_0, are kept the same as in Table 2.

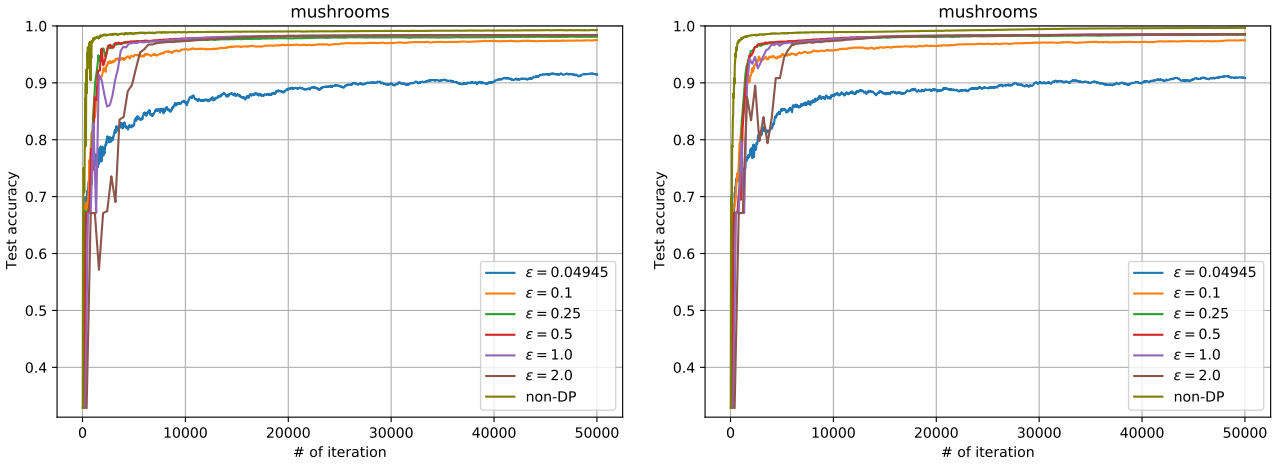
Table 5. Different privacy budget settings for strongly and plain convex problems

Privacy budget (ϵ, δ)	σ	Sample size s
$(0.04945, 0.0001)$	19.29962	26
$(0.1, 0.0001)$	13.06742	55
$(0.25, 0.0001)$	8.59143	103
$(0.5, 0.0001)$	6.05868	168
$(1.0, 0.0001)$	4.27273	265
$(2.0, 0.0001)$	3.03241	400



(a) Strong convex. (b) Plain convex.

Figure 11. Effect of different levels of privacy budgets ϵ and non-DP settings



(a) Strong convex. (b) Plain convex.

Figure 12. Effect of different levels of privacy budgets ϵ and non-DP settings

As can be seen from Figures 11 and Figure 12, the test accuracy degradation is about 10% for $\epsilon = 0.04945$ compared to the other graphed privacy settings and non-DP setting. Privacy budget $\epsilon = 0.1$, still significant smaller than what is reported in literature, comes very close to the maximum attainable test accuracy of the non-DP setting.

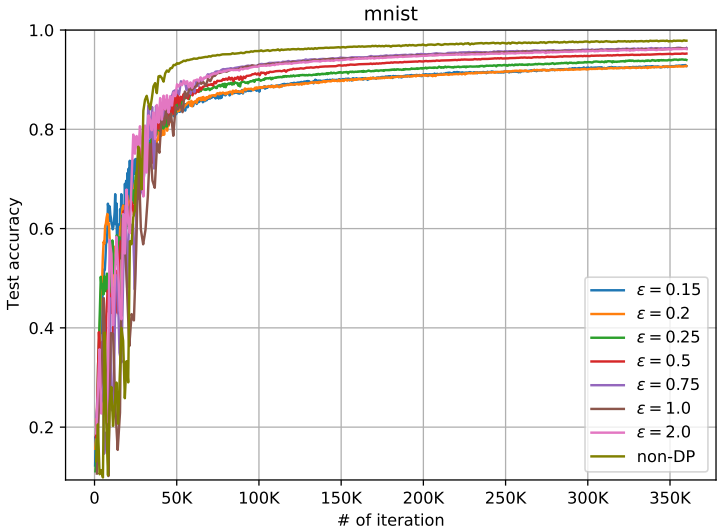
We ran the same experiment for the non-convex objective function. The detailed setting of different privacy budgets is shown in Table 6. Note that we also set the asynchronous behavior to be 1 asynchronous round, and the total of iterations on each compute node is $K = 360,000$. Other parameter settings for the non-convex case, such as initial stepsize η_0, are kept the same as in Table 3. As can be seen from Figure 13 with MNIST data set, the test accuracy loss with $\epsilon \approx 0.15$ is less than 10% (the expected test accuracy degradation from utility graph at Figure 6). Another pattern can be found in Figure 14. By selecting $\epsilon \approx 0.5$ for CIFAR10 data set, the test accuracy reduces less than 10%, compared to the non-DP setting. Note that we use AlexNet for CIFAR10, which shows ≈ 0.74 maximum test accuracy in practice[11].

Table 6. Different privacy budget settings for non-convex problem for MNIST data set

Privacy budget (ϵ, δ)	σ	Sample size s
$(0.15007, 1.667 \cdot 10^{-5})$	12.10881	370
$(0.2, 1.667 \cdot 10^{-5})$	10.48452	460
$(0.25, 1.667 \cdot 10^{-5})$	9.37379	543
$(0.5, 1.667 \cdot 10^{-5})$	6.63120	889
$(0.75, 1.667 \cdot 10^{-5})$	5.41887	1168
$(1.0, 1.667 \cdot 10^{-5})$	4.69244	1409
$(2.0, 1.667 \cdot 10^{-5})$	3.31648	2159

Table 7. Different privacy budget settings for non-convex problem for CIFAR10 data set

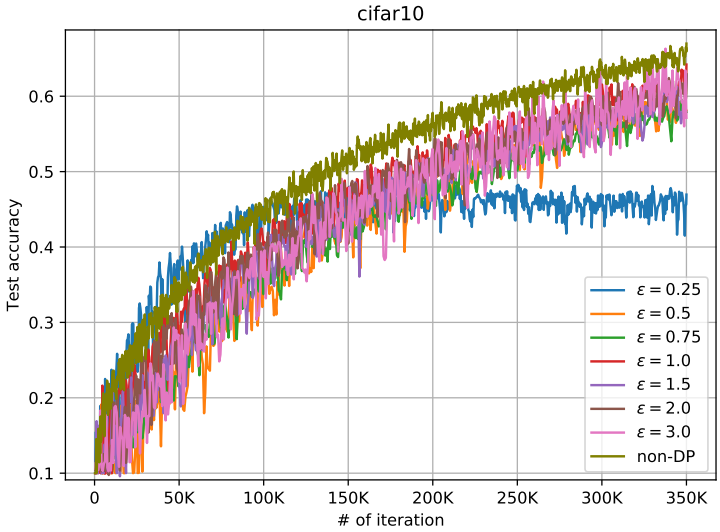
Privacy budget (ϵ, δ)	σ	Sample size s
$(0.25, 2.0 \cdot 10^{-5})$	9.29838	417
$(0.5, 2.0 \cdot 10^{-5})$	6.57192	689
$(0.75, 2.0 \cdot 10^{-5})$	5.36937	909
$(1.0, 2.0 \cdot 10^{-5})$	4.65014	1099
$(1.5, 2.0 \cdot 10^{-5})$	4.16111	1267
$(2.0, 2.0 \cdot 10^{-5})$	3.28831	1690
$(3.0, 2.0 \cdot 10^{-5})$	2.68273	1994



Non-convex.

Figure 13. Effect of different levels of privacy budgets ϵ and non-DP settings

[11] https://github.com/icpm/pytorch-cifar10



Non-convex.

Figure 14. Effect of different levels of privacy budgets ϵ and non-DP settings

These figures again confirm the effective performance of our DP-SGD framework, which not only conserves strong privacy, but also keeps a decent convergence rate to good accuracy, even for a very small privacy budget.

E. Towards Using Proactive DP-SGD in Practice

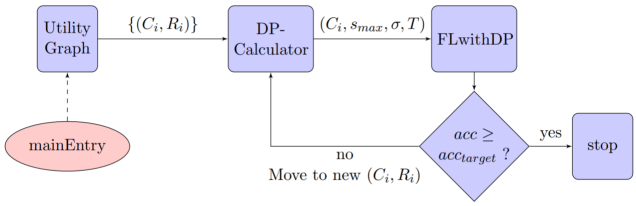


Figure 15. The flow chart of our proactive (asynchronous) DP-SGD framework.

In order to apply our theory in practice, we introduce a flow chart of our asynchronous DP-SGD learning from a client's perspective in Figure 15. Each client wants to participate in the collective learning of a global model that achieves a sufficient target accuracy acc_{target} with respect to their test data set. That is, when a client tests the final global model against its own private test set, then the client is satisfied if an accuracy acc_{target} is achieved. Such accuracy can only be achieved for certain combinations of noise σ and clipping constant C. In particular, the final round introduces Gaussian noise with deviation σ and this leads to inherent inaccuracy of the final global model. Experiments in Section 4 have shown that reducing the clipping constant C allows a larger σ for attaining a good target accuracy acc_{target} (if a sufficient larger number K of gradient computations have been executed). Here, we note that C cannot be reduced indefinitely without hurting accuracy; this is because a reduced clipping constant into some extent plays the role of a reduced step size (or learning rate) and we know that convergence to an accurate global model must start with a large enough step size. Therefore, before starting any learning, we need to understand how utility relates to σ and the clipping constant C. This is reflected in the functionality "Utility Graph" in the flow chart of Figure 15. Based on learning a local model (and based on a-priori information from learning models on similar data sets), "Utility Graph" produces a set of pairs $\{(C_i, R_i)\}$ together with a total number K of gradient computations: For various clipping constants C_i, a range R_i of possible σ is output. That is, if $\sigma \in R_i$ for clipping constant $C = C_i$, then there is a good indication that this will lead to target accuracy acc_{target} of the final model after the client has contributed K local gradient computations.

A second functionality in the flow chart is the "DP Calculator". This calculator takes the pair (C_i, R_i) that allows the maximum possible σ in R_i. This σ defines the best possible (ϵ, δ)-DP curve through our formula $\sqrt{2(\epsilon + \ln(1/\delta))/\epsilon}$. That is, it allows the "smallest" possible (ϵ, δ) pairs. The "DP Calculator" checks whether the maximum σ allows the target differential privacy budget of the client defined by $(\epsilon_{target}, \delta_{target})$. If not, then the client cannot participate in the collective learning. If the target differential privacy budget does fit, then, given σ, $\epsilon = \epsilon_{target}$, $\delta = \delta_{target}$, K, and the client's data set size N, the "DP Calculator" computes the maximum possible sample size s_{max} according to the conditions in Theorem B.4. This in turn results in a number of rounds $T = K/s_{max}$. We propose to choose the maximum possible sample size as this leads to the best accuracy/utility in our experiments. This is because s_{max} leads to the smallest number of rounds T, hence, the smallest number of times noise is added and aggregated into the global model at the server. (Also, as a secondary objective, a smaller number of rounds means less round communication.)

As soon as the "DP Calculator" has calculated all parameters, the client executes T rounds of DP-SGD with sample (mini-batch) size $s = s_{max}$. This is represented by the "FLwithDP" functionality in the flow chart of Figure 15. Once the T rounds are done, the client estimates the accuracy acc of the last received global model based on the client's test data set. Next, the client checks whether it is at least the target accuracy acc_{target}. If so, then the client stops participating. That is, with each new global model received by the server, the clients test whether the accuracy is satisfactory; if not, then the client will want to participate again. If the test accuracy acc is not sufficient, then the "DP Calculator" will work with a new (C_i, R_i). The next subsection details the computations by the "DP Calculator" and explains the feedback loop in the flow chart of Figure 15.

We remark that the client can use a complimentary differential privacy accountant to keep track of the exact privacy budget that has been spent.

E.1. DP Calculator

We propose each local client to take control over its own privacy budget while making sure the locally measured test accuracy of the final global model is acceptable. The main idea is to start with an initial $\sigma = \sigma_0$ with appropriate clipping constant $C = C_0$ and an estimated number $K = K_0$ of local gradient computations needed for convergence to "sufficient" test accuracy (utility). For local data set size N, we want to compute proper parameter settings including the batch size $s = s_0$ for each round and the total number of rounds $T = T_0$ (with $K = sT$). Once the T rounds are finished, local test data is used to compute the test accuracy of the final global model. If the accuracy is not satisfactory, then σ must be reduced to a lower σ_1 (and we may re-tune to a larger clipping constant C_1). This leads to an additional (estimated) number of local K_1 gradient computations that need to be executed. The lower σ corresponds to worse differential privacy since a lower σ is directly related to a higher ϵ for given $\delta = 1/N$. The local client is in control of what ϵ is acceptable – and if needed, the local client simply stops participating helping the central server learn a global model.

In order to apply our theory, we pretend as if the initial T rounds used the lower $\sigma = \sigma_1$ – this means that our analysis provides an advantage to the adversary as we assume less noise is used compared to what was initially actually used. Hence, the resulting DP guarantee for σ_1 will hold for all $K = K_0 + K_1$ local gradient computations. We use the new $\sigma = \sigma_1$ and K together with badge size s_0 for the first T_0 rounds to compute a new parameter setting for the next rounds; this includes the number T_1 of additional rounds (making $T = T_0 + T_1$) and the new batch size s_1. The new batch size implies a new average $\bar{s} = (s_0 T_0 + s_1 T_1)/(T_0 + T_1)$ as well as a new variation $\theta_1 > 1$ of the sequence of batch sizes.

Once all $T = T_0 + T_1$ rounds are finished (or equivalently all $K = K_0 + K_1$ local gradient computations are finished), the local client again computes the test accuracy of the last global model. If not acceptable σ is reduced again and we repeat the above process. If the test accuracy is acceptable, then the local client stops participating, that is, the local client stops gradient computations but continues to receive global models from the central server. As soon as the local client measures a new unacceptable local test accuracy, the client will continue the above process and starts a new series of rounds based on σ.

Stopping participation and later continuing if needed best fits learning problems over large data: Here, each local client samples its own local data set according to the 'client's behavior'. The local client wants to prevent as much leakage of its privately selected local data set as possible. Notice that each local data set is too small for a local client to learn a global model on its own – this is why local clients need to unite in a joint effort to learn a global model (by using distributed SGD). Assuming all samples are iid (all local data sets are themselves sampled from a global distribution), the final global model is not affected by having more or less contribution from local clients (as a result of different stopping and continuation patterns). Notice that if local data sets would be heterogeneous, then the final global model corresponds to a mix of all heterogeneous data sets and here it matters how much each local client participates (as this influences the mix).

The above procedure describes a proactive method for adjusting σ to lower values if the locally measured test accuracy is not satisfactory. Of course, the local client sets an a-priori upper bound ϵ_{target} on the ϵ (with $\delta = \delta_{target} = 1/N$), its privacy budget. This privacy budget cannot be exceeded, even if the local test accuracy becomes unsatisfactory.

We notice that our theory is general in that it can be used to analyse varying sequences of batch sizes, which is needed for our proactive method. We now describe in detail how to calculate parameter settings according to our theorems:

Suppose the local client has already computed for $T_0 + T_1 + \ldots + T_{j-1}$ rounds with badge sizes $s_0, s_1, \ldots, s_{j-1}$, hence, $K_0 = s_0T_0, K_1 = s_1T_1, \ldots, K_{j-1} = s_{j-1}T_{j-1}$. The local client sets/fixes the total K_j of gradient computations it wants to compute over the next T_j rounds. We want to compute a new s_j and T_j. Notice that $s_j = K_j/T_j$ and $\bar{s} = \sum_{i=0}^{j} s_iT_i/T$, where $T = \sum_{i=0}^{j} T_i$. We want to find a suitable s_j.

We start our calculation with $s_j = 1$ and we rerun our calculation for bigger batch sizes until we reach a maximum. Given a choice $s_j = s$, we execute the following steps (we base the calculator on the slightly more complex but more accurate Theorem B.4 of Appendix B):

1. Set $\delta = 1/N$, compute $T_j = K_j/s_j$ given the input values K_j and s_j, compute the corresponding $\bar{s}$ (see above) together with corresponding $\theta = \max\{s_i\}/\bar{s}$. Compute $K = \sum_{i=0}^{j} K_i$.
2. Set $\gamma = 2$ (because $\gamma = 2 + O(\bar{\alpha})$) as the initial value.
3. According to Theorem B.4, we compute ϵ, σ, and $\bar{\alpha}$ as follows:
   - Based on inequality (30), set ϵ as small as possible, that is, $\epsilon = \gamma\theta^2\bar{s}\frac{K}{N^2}$.
   - We distinguish two cases:
     $j = 0$**:** In case we want to determine s_0, we compute $\sigma_0 = \sigma$ where σ meets (31) with equality, that is, $\sigma = \sqrt{2(\epsilon + \log 1/\delta)/\epsilon}$.
     $j > 0$**:** During previous computations we already selected a σ_{j-1}. As described above, we only perform these calculations if the corresponding test-accuracy is not satisfactory. For this reason we want a lower $\sigma_j < \sigma_{j-1}$. The local client chooses a smaller σ_j with possibly a larger clipping constant C_j for which better accuracy within K_j local gradient computing steps is expected. We compute ϵ as a solution of $\sigma_j = \sqrt{2(\epsilon + \log 1/\delta)/\epsilon}$ and set ϵ to the maximum of this solution and the minimal possible $\epsilon = \gamma\theta^2\bar{s}\frac{K}{N^2}$ computed above.
   - Compute $\bar{\alpha} = \frac{\epsilon N}{\gamma K}$.
4. Recompute the new $\gamma_{new} = \frac{2}{1-\bar{\alpha}} + \frac{2^4\cdot\bar{\alpha}}{1-\bar{\alpha}}\left(\frac{\sigma}{(1-\sqrt{\bar{\alpha}})^2} + \frac{1}{\sigma(1-\bar{\alpha})-2e\sqrt{\bar{\alpha}}}\frac{e^3}{\sigma}\right)e^{3/\sigma^2}$.
5. Repeat steps 3 and 4 with γ replaced by γ_{new} until $\gamma_{new} - \gamma \leq 0.0001\gamma$, that is, γ has converged sufficiently.
6. The resulting set of parameters $(\epsilon, \delta, \sigma, \gamma, \theta, K, N)$ can only be used if inequalities (28) and (29) are satisfied and $\epsilon \leq \epsilon_{target}$.
   - If these conditions are satisfied, then we save the parameters (s, ϵ, σ) and rerun the above calculation for bigger sample size s. Otherwise, we output (s, ϵ, σ) of the previous run (as this corresponds to the maximum s and thus minimal number of communication rounds) and terminate: We set $\sigma_j = \sigma$, $s_j = s$, and $T_j = K_j/s_j$. Our theory proves that we satisfy $(\epsilon, \delta = 1/N)$-differential privacy.
   - It may be that even the minimal batch size $s_j = 1$ does not result in valid parameters (s, ϵ, σ). This means that the local client cannot participate any more otherwise its required differential privacy guarantee cannot be met.