1. **Demuestra que las componentes de velocidad y aceleración en un sistema coordenado esférico son las siguientes:**

$$v_r = \dot{r}$$

$$v_\theta = r\dot{\theta}$$

$$v_\varphi = r\sin\theta\dot{\varphi}$$

$$a_r = \ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\varphi}^2$$

$$a_\theta = r\ddot{\theta} - 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\dot{\varphi}^2$$

$$a_\varphi = r\sin\theta\ddot{\varphi} + 2\dot{r}\sin\theta\dot{\varphi} + 2r\cos\theta\dot{\theta}\dot{\varphi}$$

Considera que:

$$\mathbf{r}(t) = \hat{\mathbf{r}}(t)r(t) = [\hat{\boldsymbol{\imath}}\sin\theta(t)\cos\varphi(t) + \hat{\boldsymbol{\jmath}}\sin\theta(t)\sin\varphi(t) + \hat{\mathbf{k}}\cos\theta(t)]r(t) = \hat{\mathbf{e}}_r r(t)$$

SOLUCIÓN:

La velocidad se describe como:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(r(t)\hat{\mathbf{e}}_r)$$

y por la regla de la cadena

$$\mathbf{v} = \dot{r}\hat{\mathbf{e}}_r + r\dot{\hat{\mathbf{e}}}_r$$

entonces como la derivada total del vector unitario de r es

$$\dot{\hat{\mathbf{e}}}_r = \frac{\partial\hat{\mathbf{e}}_r}{\partial r}\dot{r} + \frac{\partial\hat{\mathbf{e}}_r}{\partial\theta}\dot{\theta} + \frac{\partial\hat{\mathbf{e}}_r}{\partial\varphi}\dot{\varphi}$$

recordando que

$$\hat{\mathbf{e}}_r = \sin\theta\cos\varphi\hat{\boldsymbol{\imath}} + \sin\theta\sin\varphi\hat{\boldsymbol{\jmath}} + \cos\theta\hat{\mathbf{k}}$$

$$\hat{\mathbf{e}}_\theta = \cos\theta\cos\varphi\hat{\boldsymbol{\imath}} + \cos\theta\sin\varphi\hat{\boldsymbol{\jmath}} - \sin\theta\hat{\mathbf{k}}$$

$$\hat{\mathbf{e}}_\varphi = -\sin\varphi\hat{\boldsymbol{\imath}} + \cos\varphi\hat{\boldsymbol{\jmath}}$$

tenemos

$$\dot{\hat{\mathbf{e}}}_r = (0)\dot{r} + (\cos\theta\cos\varphi\hat{\boldsymbol{\imath}} + \cos\theta\sin\varphi\hat{\boldsymbol{\jmath}} - \sin\theta\hat{\mathbf{k}})\dot{\theta} + (-\sin\theta\sin\varphi\hat{\boldsymbol{\imath}} + \sin\theta\cos\varphi\hat{\boldsymbol{\jmath}})\dot{\varphi}$$

que reordenando términos y usando lo anteriormente recordado

$$= (\cos\theta\cos\varphi\hat{\boldsymbol{\imath}} + \cos\theta\sin\varphi\hat{\boldsymbol{\jmath}} - \sin\theta\hat{\mathbf{k}})\dot{\theta} + \sin\theta(-\sin\varphi\hat{\boldsymbol{\imath}} + \cos\varphi\hat{\boldsymbol{\jmath}})\dot{\varphi}$$

$$= \dot{\theta}\hat{\mathbf{e}}_\theta + \sin\theta\dot{\varphi}\hat{\mathbf{e}}_\varphi$$

y así

$$\mathbf{v} = \dot{r}\hat{\mathbf{e}}_r + r[\dot{\theta}\hat{\mathbf{e}}_\theta + \sin\theta\dot{\varphi}\hat{\mathbf{e}}_\varphi] = \dot{r}\hat{\mathbf{e}}_r + r\dot{\theta}\hat{\mathbf{e}}_\theta + r\sin\theta\dot{\varphi}\hat{\mathbf{e}}_\varphi$$

$$\therefore \qquad v_r = \dot{r} \qquad v_\theta = r\dot{\theta} \qquad v_\varphi = r\sin\theta\dot{\varphi} \qquad \blacksquare$$

ahora para la aceleración, se tiene que

$$\mathbf{a} = \dot{\mathbf{v}} = \dot{(\dot{r}\hat{\mathbf{e}}_\mathbf{r})} + \dot{(r\dot{\theta}\hat{\mathbf{e}}_\theta)} + \dot{(r\sin\theta\dot{\varphi}\hat{\mathbf{e}}_\varphi)}$$

y de nuevo por la regla de la cadena

$$= \ddot{r}\hat{\mathbf{e}}_r + \dot{r}\dot{\hat{\mathbf{e}}}_r + \dot{r}\dot{\theta}\hat{\mathbf{e}}_\theta + r\ddot{\theta}\hat{\mathbf{e}}_\theta + r\dot{\theta}\dot{\hat{e}}_\theta + \dot{r}\sin\theta\dot{\varphi}\hat{\mathbf{e}}_\varphi + r\cos\theta\dot{\theta}\dot{\varphi}\hat{\mathbf{e}}_\varphi + r\sin\theta\ddot{\varphi}\hat{\mathbf{e}}_\varphi + r\sin\theta\dot{\varphi}\dot{\hat{e}}_\varphi$$

por lo que necesitamos $\dot{\hat{e}}_\theta$ y $\dot{\hat{e}}_\phi$

$$\dot{\hat{e}}_\theta = \frac{\partial\hat{\mathbf{e}}_\theta}{\partial r}\dot{r} + \frac{\partial\hat{\mathbf{e}}_\theta}{\partial\theta}\dot{\theta} + \frac{\partial\hat{\mathbf{e}}_\theta}{\partial\varphi}\dot{\varphi}$$

$$= (0)\dot{r} + (-\sin\theta\cos\varphi\hat{\boldsymbol{\imath}} + -\sin\theta\sin\varphi\hat{\boldsymbol{\jmath}} - \cos\theta\hat{\mathbf{k}})\dot{\theta} + (-\cos\theta\sin\varphi\hat{\boldsymbol{\imath}} + \cos\theta\cos\varphi\hat{\boldsymbol{\jmath}})\dot{\varphi}$$

que reordenando

$$= -(\sin\theta\cos\varphi\hat{\boldsymbol{\imath}} + \sin\theta\sin\varphi\hat{\boldsymbol{\jmath}} + \cos\theta\hat{\mathbf{k}})\dot{\theta} + \cos\theta(-\sin\varphi\hat{\boldsymbol{\imath}} + \cos\varphi\hat{\boldsymbol{\jmath}})\dot{\varphi}$$

$$= -\dot{\theta}\hat{\mathbf{e}}_r + \cos\theta\dot{\varphi}\hat{\mathbf{e}}_\varphi$$

$$\dot{\hat{e}}_\phi = \frac{\partial\hat{\mathbf{e}}_\varphi}{\partial r}\dot{r} + \frac{\partial\hat{\mathbf{e}}_\varphi}{\partial\theta}\dot{\theta} + \frac{\partial\hat{\mathbf{e}}_\varphi}{\partial\varphi}\dot{\varphi}$$

$$= (0)\dot{r} + (0)\dot{\theta} + (-\cos\varphi\hat{\boldsymbol{\imath}} - \sin\varphi\hat{\boldsymbol{\jmath}})\dot{\varphi}$$

y recordando que

$$\hat{\boldsymbol{\imath}} = \sin\theta\cos\varphi\hat{\boldsymbol{e}}_r + \cos\theta\cos\hat{\mathbf{e}}_\theta - \sin\varphi\hat{\boldsymbol{e}}_\varphi$$

$$\hat{\boldsymbol{\jmath}} = \sin\theta\sin\varphi\hat{\mathbf{e}}_r + \cos\theta\sin\hat{\mathbf{e}}_\theta + \cos\varphi\hat{\mathbf{e}}_\varphi$$

$$\hat{\boldsymbol{k}} = \cos\theta\hat{\mathbf{e}}_r - \sin\theta\hat{\mathbf{e}}_\theta$$

entonces

$$\dot{\hat{e}}_\phi = (-\cos\varphi(\sin\theta\cos\varphi\hat{\mathbf{e}}_r+\cos\theta\cos\hat{\mathbf{e}}_\theta-\sin\varphi\hat{\mathbf{e}}_\varphi)-\sin\varphi(\sin\theta\sin\varphi\hat{\mathbf{e}}_r+\cos\theta\sin\hat{\mathbf{e}}_\theta+\cos\varphi\hat{\mathbf{e}}_\varphi))\dot{\varphi}$$

$$= (-\sin\theta\cos^2\varphi\hat{\mathbf{e}}_r+\cos\theta\cos^2\varphi\hat{\mathbf{e}}_\theta-\cancel{\sin\varphi\cos\varphi\hat{\mathbf{e}}_\varphi}-\sin\theta\sin^2\varphi\hat{\mathbf{e}}_r+\cos\theta\sin^2\hat{\mathbf{e}}_\theta+\cancel{\sin\varphi\cos\varphi\hat{\mathbf{e}}_\varphi})\dot{\varphi}$$

$$= (-\sin\theta\cancel{(\cos^2\varphi+\sin^2\varphi)}\hat{\mathbf{e}}_r + \cos\theta\cancel{(\cos^2\varphi+\sin^2\varphi)}\hat{\mathbf{e}}_\theta)\dot{\varphi}$$

$$= (-\sin\theta\hat{\mathbf{e}}_r + \cos\theta\hat{\mathbf{e}}_\theta)\dot{\varphi}$$

por fin sustituyendo

$$= \ddot{r}\hat{\mathbf{e}}_r+\dot{r}[\dot{\theta}\hat{\mathbf{e}}_\theta+\sin\theta\dot{\varphi}\hat{\mathbf{e}}_\varphi]+\dot{r}\dot{\theta}\hat{\mathbf{e}}_\theta+r\ddot{\theta}\hat{\mathbf{e}}_\theta+r\dot{\theta}[-\dot{\theta}\hat{\mathbf{e}}_r+\cos\theta\dot{\varphi}\hat{\mathbf{e}}_\varphi]+\dot{r}\sin\theta\dot{\varphi}\hat{\mathbf{e}}_\varphi+r\cos\theta\dot{\theta}\dot{\varphi}\hat{\mathbf{e}}_\varphi+r\sin\theta\ddot{\varphi}\hat{\mathbf{e}}_\varphi$$

$$+r\sin\theta\dot{\varphi}[(-\sin\theta\hat{\mathbf{e}}_r + \cos\theta\hat{\mathbf{e}}_\theta)\dot{\varphi}]$$

$$= \ddot{r}\hat{\mathbf{e}}_r+\dot{r}\dot{\theta}\hat{\mathbf{e}}_\theta+\dot{r}\sin\theta\dot{\varphi}\hat{\mathbf{e}}_\varphi+\dot{r}\dot{\theta}\hat{\mathbf{e}}_\theta+r\ddot{\theta}\hat{\mathbf{e}}_\theta-r\dot{\theta}\dot{\theta}\hat{\mathbf{e}}_r+r\dot{\theta}\cos\theta\dot{\varphi}\hat{\mathbf{e}}_\varphi+\dot{r}\sin\theta\dot{\varphi}\hat{\mathbf{e}}_\varphi+r\cos\theta\dot{\theta}\dot{\varphi}\hat{\mathbf{e}}_\varphi+r\sin\theta\ddot{\varphi}\hat{\mathbf{e}}_\varphi$$

$$-r\sin\theta\dot{\varphi}\sin\theta\hat{\mathbf{e}}_r\dot{\varphi} + r\sin\theta\dot{\varphi}\cos\theta\dot{\varphi}\hat{\mathbf{e}}_\theta$$

$$[\ddot{r}-r\dot{\theta}^2-r\sin\theta\dot{\varphi}^2\sin\theta]\hat{\mathbf{e}}_r+[2\dot{r}\dot{\theta}+r\ddot{\theta}+r\sin\theta\dot{\varphi}^2\cos\theta]\hat{\mathbf{e}}_\theta+[2\dot{r}\sin\theta\dot{\varphi}+2r\dot{\theta}\cos\theta\dot{\varphi}+r\sin\theta\ddot{\varphi}]\hat{\mathbf{e}}_\varphi$$

$$\therefore \quad a_r = \ddot{r}-r\dot{\theta}^2-r\sin^2\theta\dot{\varphi}^2 \quad a_\theta = r\ddot{\theta}-2\dot{r}\dot{\theta}-r\sin\theta\cos\theta\dot{\varphi}^2 \quad a_\varphi = r\sin\theta\ddot{\varphi}+2\dot{r}\sin\theta\dot{\varphi}+2r\cos\theta\dot{\theta}\dot{\varphi}$$

■

2. **Evalúa las siguientes expresiones en un sistema de coordenadas cilíndrico:**

$$\nabla \times \ln r\hat{\mathbf{e}}_z \qquad \nabla \ln r \qquad \nabla \cdot (r\hat{\mathbf{e}}_r + z\hat{\mathbf{e}}_z)$$

SOLUCIÓN:

Primero debemos recordar que

$$\mathbf{r} = \hat{\boldsymbol{\imath}} r \cos\varphi + \hat{\boldsymbol{\jmath}} r \operatorname{sen}\varphi + \hat{\mathbf{k}} z$$

y usando lo siguiente

$$\frac{\partial \mathbf{r}}{\partial r} = \hat{\boldsymbol{\imath}} \cos\varphi + \hat{\boldsymbol{\jmath}} \operatorname{sen}\varphi$$

$$\frac{\partial \mathbf{r}}{\partial \varphi} = -\hat{\boldsymbol{\imath}} r \operatorname{sen}\varphi + \hat{\boldsymbol{\jmath}} r \cos\varphi$$

$$\frac{\partial \mathbf{r}}{\partial z} = \hat{\mathbf{k}}$$

tenemos que

$$h_r = \|\frac{\partial \mathbf{r}}{\partial \rho}\| = \sqrt{\cos^2\psi + \operatorname{sen}^2\psi} = 1$$

$$h_\varphi = \|\frac{\partial \mathbf{r}}{\partial \varphi}\| = \sqrt{r^2 \operatorname{sen}^2\varphi + r^2\cos^2\varphi} = r\sqrt{\operatorname{sen}^2\varphi + \cos^2\varphi} = r$$

$$h_z = \|\frac{\partial \mathbf{r}}{\partial z}\| = \sqrt{1} = 1$$

Lo que usaremos en todo este ejercicio, ahora, por definición

$$\nabla \times \ln r\hat{\mathbf{e}}_z = \frac{1}{h}\begin{vmatrix} h_r\hat{e}_r & h_\varphi\hat{e}_\varphi & h_z\hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial u_z} \\ (0)h_r & (0)h_\varphi & \ln r h_z \end{vmatrix}$$

$$= \frac{h_r\hat{\mathbf{e}}_r}{h}\frac{\partial}{\partial \varphi}(\ln r h_z) - \frac{h_\varphi\hat{\mathbf{e}}_\varphi}{h}\frac{\partial}{\partial r}(\ln r h_z)$$

$$= \frac{\hat{\mathbf{e}}_r}{h_\varphi h_z}\frac{\partial}{\partial \varphi}(\ln r h_z) - \frac{\hat{\mathbf{e}}_\varphi}{h_r h_z}\frac{\partial}{\partial r}(\ln r h_z)$$

sustituyendo

$$= \frac{\hat{\mathbf{e}}_r}{r}\cancel{\frac{\partial}{\partial \varphi}(\ln r)} - \frac{\hat{\mathbf{e}}_\varphi}{1}\frac{\partial}{\partial r}(\ln r)$$

$$= -\frac{\hat{\mathbf{e}}_\varphi}{r}$$

de nuevo por definición

$$\nabla \ln r = \sum \frac{\hat{\mathbf{e}}_i}{h_i}\frac{\partial}{\partial u_i}(\ln r) = \frac{\hat{\mathbf{e}}_r}{h_r}\frac{\partial}{\partial r}(\ln r) + \frac{\hat{\mathbf{e}}_\varphi}{h_\varphi}\cancel{\frac{\partial}{\partial \varphi}(\ln r)} + \frac{\hat{\mathbf{e}}_z}{h_z}\cancel{\frac{\partial}{\partial z}(\ln r)}$$

$$= \hat{\mathbf{e}}_r\frac{\partial}{\partial r}(\ln r) = \frac{\hat{\mathbf{e}}_r}{r}$$

y ya por último

$$\nabla \cdot (r\hat{\mathbf{e}}_r + z\hat{\mathbf{e}}_z) = \frac{1}{h}\frac{\partial}{\partial r}\left(\frac{(r\hat{\mathbf{e}}_r + z\hat{\mathbf{e}}_z)h}{h_r}\right) + \frac{1}{h}\frac{\partial}{\partial \varphi}\left(\frac{(r\hat{\mathbf{e}}_r + z\hat{\mathbf{e}}_z)h}{h_\varphi}\right) + \frac{1}{h}\frac{\partial}{\partial z}\left(\frac{(r\hat{\mathbf{e}}_r + z\hat{\mathbf{e}}_z)h}{h_z}\right)$$

sustituyendo

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{(r\hat{\mathbf{e}}_r + z\hat{\mathbf{e}}_z)r}{1}\right) + \frac{1}{r}\cancel{\frac{\partial}{\partial \varphi}\left(\frac{(r\hat{\mathbf{e}}_r + z\hat{\mathbf{e}}_z)\cancel{r}}{\cancel{r}}\right)} + \frac{1}{\cancel{r}}\frac{\partial}{\partial z}\left(\frac{(r\hat{\mathbf{e}}_r + z\hat{\mathbf{e}}_z)\cancel{r}}{1}\right)$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left((r^2\hat{\mathbf{e}}_r + zr\hat{\mathbf{e}}_z)\right) + \frac{\partial}{\partial z}\left((r\hat{\mathbf{e}}_r + z\hat{\mathbf{e}}_z)\right) = \frac{1}{r}(2r\hat{\mathbf{e}}_r + z\hat{\mathbf{e}}_z) + \hat{\mathbf{e}}_z$$

$$2\hat{\mathbf{e}}_r + (\frac{z}{r} + 1)\hat{\mathbf{e}}_z$$

3. **Para una esfera de radio r, calcula el volumen en un sistema de coordenadas oblatas. Considera que $a = 1$.**

SOLUCIÓN:

Primero recordemos las reglas de transformación de este sistema

$$x = a\cosh\xi\cos\eta\cos\phi$$
$$y = a\cosh\xi\cos\eta\sin\phi$$
$$z = a\sinh\xi\sin\eta$$

y de una vez calculemos los factores de escala

$$h_\xi = \|\frac{\partial \mathbf{r}}{\partial \xi}\| = \sqrt{a^2\cos^2\eta\cos^2\phi\sinh^2\xi + a^2\cos^2\eta\sin^2\phi\sinh^2\xi + a^2\sin^2\eta\cosh^2\xi}$$

$$= a\sqrt{\cos^2\eta\sinh^2\xi\cancel{(\cos^2\phi + \sin^2\phi)} + \sin^2\eta\cosh^2\xi}$$

$$= a\sqrt{\cos^2\eta\sinh^2\xi + \sin^2\eta(\sinh^2\xi + 1)}$$

$$= a\sqrt{\sinh^2\xi\cancel{(\sin^2\eta + \cos^2\eta)} + \sin^2\eta} = a\sqrt{\sinh^2\xi + \sin^2\eta}$$

$$h_\eta = \|\frac{\partial \mathbf{r}}{\partial \eta}\| = \sqrt{a^2\cosh^2\xi\sin^2\eta\cos^2\phi + a^2\cosh^2\xi\sin^2\eta\sin^2\phi + a^2\sinh^2\xi\cos^2\eta}$$

$$= a\sqrt{\cosh^2\xi\sin^2\eta\cancel{(\cos^2\phi + \sin^2\phi)} + \sinh^2\xi\cos^2\eta}$$

$$= a\sqrt{\sin^2\eta(\sinh^2\xi + 1) + \sinh^2\xi\cos^2\eta}$$

$$= a\sqrt{\sinh^2\xi\cancel{(\sin^2\eta + \cos^2\eta)} + \sin^2\eta} = h_\xi$$

$$h_\phi = \|\frac{\partial \mathbf{r}}{\partial \phi}\| = \sqrt{a^2\cosh^2\xi\cos^2\eta\,\mathrm{sen}^2\phi + a^2\cosh^2\xi\cos^2\eta\cos^2\phi}$$

$$= a\sqrt{\cosh^2\xi\cos^2\eta\cancel{(\mathrm{sen}^2\phi + \cos^2\phi)}} = a\cosh\xi\cos\eta$$

o resumiendo

$$h_\xi = a\sqrt{\sinh^2\xi + \sin^2\eta} = h_\eta \qquad\qquad h_\phi = a\cosh\xi\cos\eta$$

ahora para obtener el volumen

$$\iiint F(x,y,z)dV = \iiint F(\xi,\eta,\phi)dV'$$

donde $dV = h_x h_y h_z dxdydz = dxdydz$, $dV = h_\xi h_\eta h_\phi d\xi d\eta d\phi$ (diferencial de volumen construidas en las notas) y $F = 1$ para integrar sobre la superficie y así obtener su volumen

para obtener los límites de integración, sustituyamos las reglas de transformación en la esfera de radio r con $a = 1$

$$x^2 + y^2 + z^2 = r^2$$

$$\cosh^2\xi\cos^2\eta\cos^2\phi + \cosh^2\xi\cos^2\eta\sin^2\phi + \sinh^2\xi\sin^2\eta = r^2$$

$$\cosh^2\xi\cos^2\eta\cancel{(\sin^2\phi + \cos^2\phi)} + \sinh^2\xi\sin^2\eta = r^2$$

$$(\sinh^2\xi + 1)\cos^2\eta + \sinh^2\eta\sin^2\eta = r^2$$

$$\sinh^2\xi\cos^2\eta + \cos^2\eta + \sinh^2\eta\sin^2\eta = r^2$$

$$\sinh^2\xi\cancel{(\cos^2\eta + \sin^2\eta)} + \cos^2\eta = r^2$$

$$\sinh^2\xi + \cos^2\eta = r^2$$

y despejando ξ

$$\xi = \sinh^{-1}\sqrt{r^2 - \cos^2\eta}$$

y entonces usando esto en los limites de integración, tenemos

$$\int_{\sinh^{-1}(-r)}^{\sinh^{-1}(r)}\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}(\sinh^2\xi + \sin^2\eta)\cosh\xi\cos\eta d\phi d\eta d\xi$$

$$\int_{\sinh^{-1}(-r)}^{\sinh^{-1}(r)}\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\sinh^2\xi\cosh\xi\cos\eta d\phi d\eta d\xi d\xi + \int_{\sinh^{-1}(-r)}^{\sinh^{-1}(r)}\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\sin^2\eta\cosh\xi\cos\eta d\phi d\eta d\xi$$

$$2\pi\left[\int_{\sinh^{-1}(-r)}^{\sinh^{-1}(r)}\sinh^2\xi\cosh\xi d\xi\int_{-\pi/2}^{\pi/2}\cos\eta d\eta + \int_{\sinh^{-1}(-r)}^{\sinh^{-1}(r)}\cosh\xi d\xi\int_{-\pi/2}^{\pi/2}\sin^2\eta\cos\eta d\eta\right]$$

$$2\pi\left[2\int_{\sinh^{-1}(-r)}^{\sinh^{-1}(r)}\sinh^2\xi\cosh\xi d\xi + \frac{2}{3}\int_{\sinh^{-1}(-r)}^{\sinh^{-1}(r)}\cosh\xi d\xi\right]$$

$$4\pi\left[\int_{\sinh^{-1}(-r)}^{\sinh^{-1}(r)}\sinh^2\xi\cosh\xi d\xi + \frac{1}{3}\int_{\sinh^{-1}(-r)}^{\sinh^{-1}(r)}\cosh\xi d\xi\right]$$

$$4\pi\left[\frac{\sinh^3(\sinh^{-1}r) - \sinh^3(\sinh^{-1}r)}{3} + \cancel{\frac{\sinh(\sinh^{-1}r) - \sinh(\sinh^{-1}r)}{3}}\right] = \frac{4\pi}{3}r^3$$

4. **La inductancia magnética $\hat{\mathbf{B}}$ es el rotacional del potencial magnético$\hat{\mathbf{A}}$. Supongamos que en un sistema coordenado bipolar, $\hat{\mathbf{A}} = -c\eta\hat{\mathbf{e}}_z$. Calcula $\hat{\mathbf{B}}$. Este problema describe el caso de dos alambres que conducen el mismo valor de corriente en direcciones paralelas y opuestas al eje z.**

   SOLUCIÓN:

   la transformación cartesiana del sistema coordenado cilíndrico bipolar es

$$x = \frac{a\sinh\eta}{\cosh\eta - \cos\xi}$$

$$y = \frac{a \sin \xi}{\cosh \eta - \cos \xi}$$

$$z = z$$

$$h_\xi = \|\frac{\partial \mathbf{r}}{\partial \xi}\| = \sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2}$$

y por regla de la cadena

$$= \sqrt{\frac{a^2 \sinh^2 \eta \sin^2 \xi}{(\cosh \eta - \cos \xi)^4} + \frac{(a \cos \xi (\cosh \eta - \cos \xi) - a \sin^2 \xi)^2}{(\cosh \eta - \cos \xi)^4}}$$

$$= \frac{a}{(\cosh \eta - \cos \xi)^2} \sqrt{\sinh^2 \eta \sin^2 \xi + (\cosh \eta \cos \xi - \cancel{(\cos^2 \xi + \sin^2 \xi)})^2}$$

$$= \frac{a}{(\cosh \eta - \cos \xi)^2} \sqrt{\sinh^2 \eta \sin^2 \xi + (\cosh \eta \cos \xi - 1)^2}$$

$$= \frac{a}{(\cosh \eta - \cos \xi)^2} \sqrt{\sinh^2 \eta \sin^2 \xi + \cosh^2 \eta \cos^2 \xi - 2 \cosh \eta \cos \xi + 1}$$

$$= \frac{a}{(\cosh \eta - \cos \xi)^2} \sqrt{(\cosh^2 \eta - 1) \sin^2 \xi + \cosh^2 \eta \cos^2 \xi - 2 \cosh \eta \cos \xi + 1}$$

$$= \frac{a}{(\cosh \eta - \cos \xi)^2} \sqrt{\cosh^2 \eta \cancel{(\sin^2 \xi + \cos^2 \xi)} - \sin^2 \xi - 2 \cosh \eta \cos \xi + 1}$$

$$= \frac{a}{(\cosh \eta - \cos \xi)^2} \sqrt{\cosh^2 \eta - 2 \cosh \eta \cos \xi + (- \sin^2 \xi + 1)}$$

$$= \frac{a}{(\cosh \eta - \cos \xi)^2} \sqrt{\cosh^2 \eta - 2 \cosh \eta \cos \xi + \cos^2 \xi}$$

$$= \frac{a}{(\cosh \eta - \cos \xi)^{\cancel{2}}} \cancel{\sqrt{(\cosh \eta - \cos \xi)^2}} = \frac{a}{\cosh \eta - \cos \xi}$$

$$h_\eta = \|\frac{\partial \mathbf{r}}{\partial \eta}\| = \sqrt{\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 + \left(\frac{\partial z}{\partial \eta}\right)^2}$$

$$= \sqrt{\frac{(a \cosh \eta (\cosh \eta - \cos \xi) - a \sinh \eta \cosh \eta)^2}{(\cosh \eta - \cos \xi)^4} + \frac{a^2 \sin^2 \xi \sinh^2 \eta}{(\cosh \eta - \cos \xi)^4}}$$

$$\frac{a}{(\cosh \eta - \cos \xi)^2} \sqrt{(\cosh \eta (\cosh \eta - \cos \xi) - \sinh \eta \cosh \eta)^2 + \sin^2 \xi \sinh^2 \eta}$$

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no supe como llegar a esto :(

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$$\frac{a}{\cosh\eta - \cos\xi} = h_\xi$$

$$h_z = \|\frac{\partial \mathbf{r}}{\partial z}\| = \sqrt{\left(\frac{\partial x}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial z}{\partial z}\right)^2} = \sqrt{1} = 1$$

$$\nabla \times \mathbf{A} = \frac{1}{h}\begin{vmatrix} h_\xi \hat{e}_\xi & h_\eta \hat{e}_\eta & h_z \hat{e}_z \\ \frac{\partial}{\partial \xi} & \frac{\partial}{\partial \eta} & \frac{\partial}{\partial z} \\ (0)h_r & (0)h_\varphi & -c\eta h_z \end{vmatrix}$$

$$= \frac{(\cosh\eta - \cos\xi)^2}{a^2}[h_\xi \frac{\partial}{\partial \eta}(-c\eta h_z)\hat{\mathbf{e}}_\xi - h_\eta \cancel{\frac{\partial}{\partial \xi}(-c\eta h_z)}\hat{\mathbf{e}}_\eta]$$

$$= -\frac{(\cosh\eta - \cos\xi)^{\cancel{2}}}{a^{\cancel{2}}}\cancel{\frac{a}{\cosh\eta - \cos\xi}} c\hat{\mathbf{e}}_\xi = \frac{c(\cos\xi - \cosh\eta)}{a}\hat{\mathbf{e}}_\xi$$

5. **A partir de la definición de la función Beta $B(m,n)$, demuestra que:**

$$B(m,n)B(m+n,k) = B(n,k)B(n+k,m)$$

SOLUCIÓN:

Usando la indentidad $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, multiplicando por $\frac{\Gamma(n+k)}{\Gamma(n+k)}$ y reordenando

$$B(m,n)B(m+n,k) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\frac{\Gamma(m+n)\Gamma(k)}{\Gamma(m+n+k)} = \frac{\Gamma(n+k)}{\Gamma(n+k)}\frac{\Gamma(m)\Gamma(n)}{\cancel{\Gamma(m+n)}}\frac{\cancel{\Gamma(m+n)}\Gamma(k)}{\Gamma(m+n+k)}$$

$$= \frac{\Gamma(n)\Gamma(k)}{\Gamma(n+k)}\frac{\Gamma(n+k)\Gamma(m)}{\Gamma((n+k)+m)} = B(n,k)B(n+k,m) \qquad \blacksquare$$