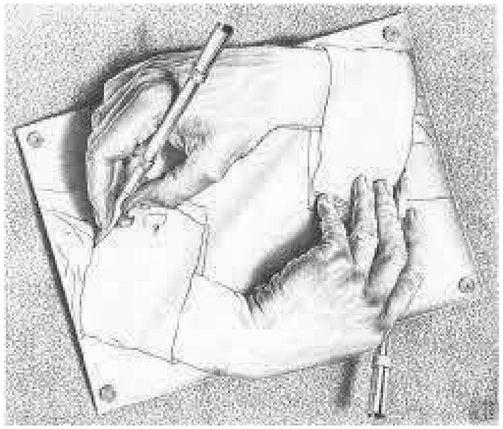
Logic II



Lecture

Review

Propositional modus ponens

Propositional resolution

First order logic

First order modus ponens

First order resolution

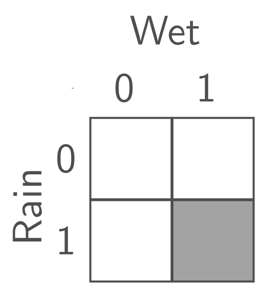
Review: ingredients of a logic

Syntax: defines a set of valid **formulas** (Formulas)

Example: Rain $\wedge$ Wet

Semantics: for each formula f, specify a set of **models** $\mathcal{M}(f)$ (assignments / configurations of the world)

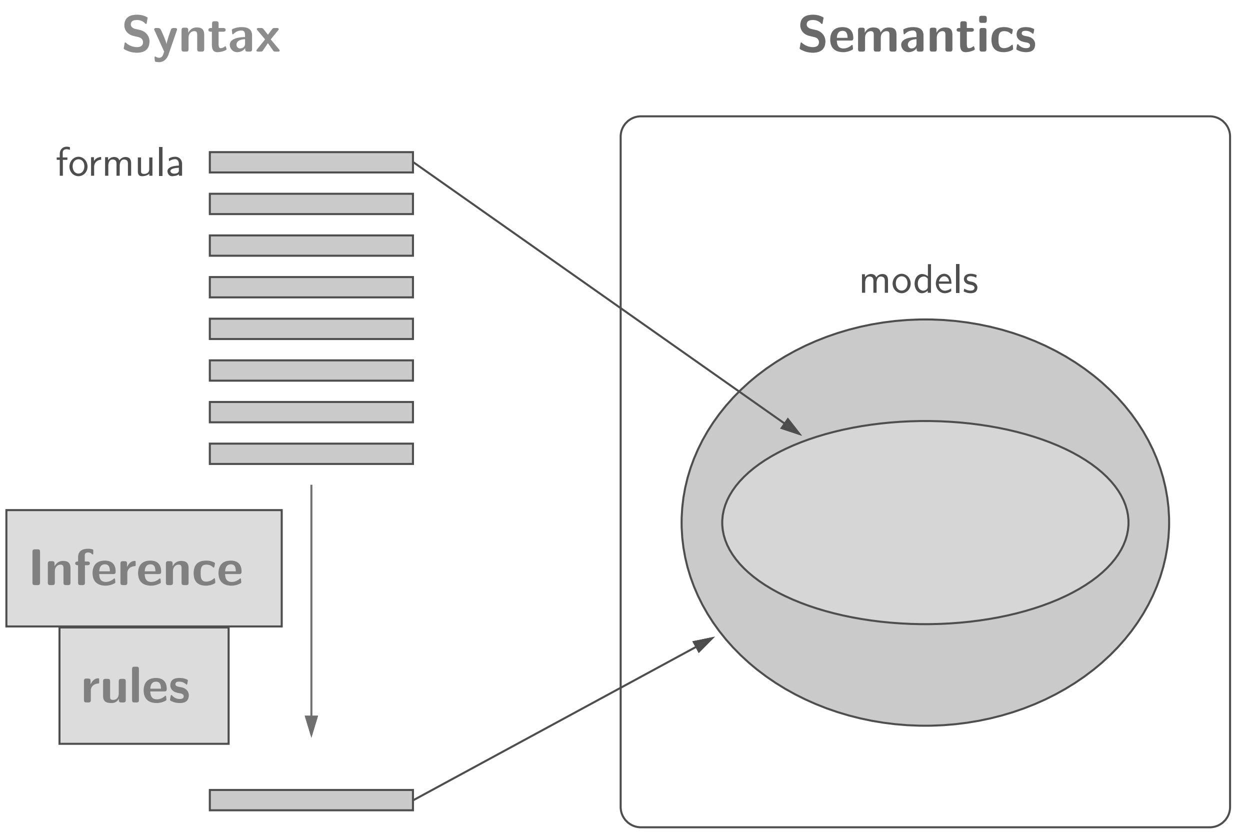
Example:



Inference rules: given KB, what new formulas f can be derived?

Example: from Rain $\wedge$ Wet, derive Rain

Propositional logic



Review: formulas

Propositional logic: any legal combination of symbols

$$(\text{Rain} \wedge \text{Snow}) \rightarrow (\text{Traffic} \vee \text{Peaceful}) \wedge \text{Wet}$$

Review: inference algorithm

Inference algorithm:

$$\text{KB} \xrightarrow{\text{(repeatedly apply inference rules)}} f$$

Definition: modus ponens inference rule

$$\frac{p_1, \quad \cdots \quad , p_k, \quad (p_1 \wedge \cdots \wedge p_k) \to q}{q}$$

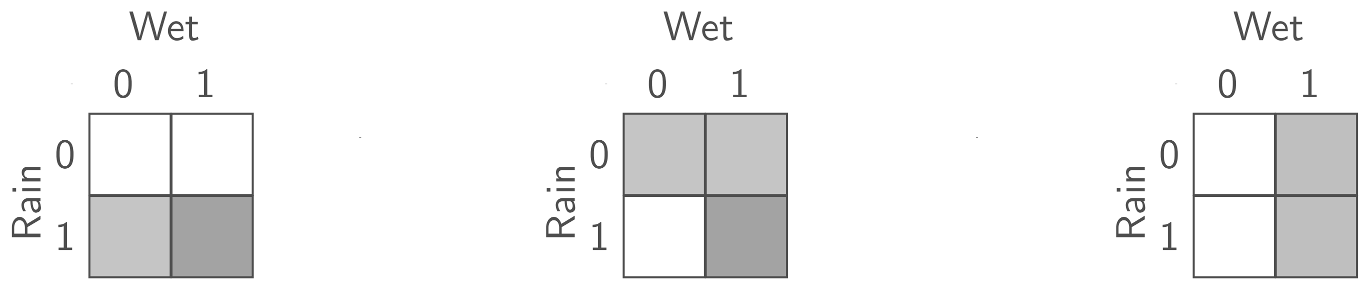
Desiderata: soundness and completeness

entailment (KB $\models f$) derivation (KB $\vdash f$)

Soundness: example

Is $\dfrac{\text{Rain}, \quad \text{Rain} \to \text{Wet}}{\text{Wet}}$ (Modus ponens) sound?

$$\mathcal{M}(\text{Rain}) \qquad \cap \qquad \mathcal{M}(\text{Rain} \to \text{Wet}) \qquad \subseteq? \qquad \mathcal{M}(\text{Wet})$$

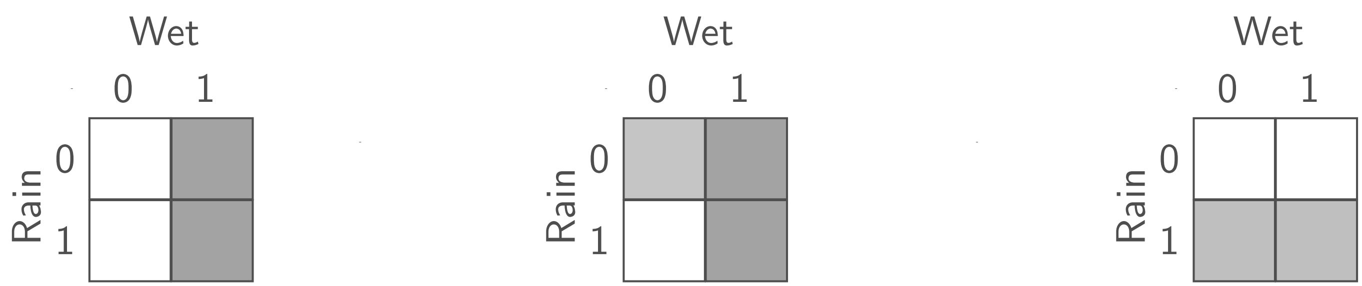


Sound!

Soundness: example

Is $\dfrac{\text{Wet}, \quad \text{Rain} \rightarrow \text{Wet}}{\text{Rain}}$ sound?

$$\mathcal{M}(\text{Wet}) \quad \cap \quad \mathcal{M}(\text{Rain} \rightarrow \text{Wet}) \quad \subseteq ? \quad \mathcal{M}(\text{Rain})$$



Unsound!

Completeness: example

Recall completeness: inference rules derive all entailed formulas (f such that $\text{KB} \models f$)

Example: Modus ponens is incomplete

Setup:

$$\text{KB} = \{\text{Rain}, \text{Rain} \vee \text{Snow} \rightarrow \text{Wet}\}$$

$$f = \text{Wet}$$

$$\text{Rules} = \{\frac{f, \quad f \rightarrow g}{g}\} \text{ (Modus ponens)}$$

Semantically: $\text{KB} \models f$ (f is entailed).

Syntactically: $\text{KB} \not\vdash f$ (can't derive f).

Incomplete!

Fixing completeness

Option 1: Restrict the allowed set of formulas

propositional logic

$\downarrow$

propositional logic with only Horn clauses

Option 2: Use more powerful inference rules

Modus ponens

$\downarrow$

resolution

Lecture

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Definite clauses

Definition: Definite clause

A **definite clause** has the following form:

$$(p_1 \wedge \cdots \wedge p_k) \rightarrow q$$

where $p_1, \dots, p_k, q$ are propositional symbols.

Intuition: if $p_1, \dots, p_k$ hold, then q holds.

Example: $(\text{Rain} \wedge \text{Snow}) \rightarrow \text{Traffic}$

Example: Traffic

Non-example: $\neg\text{Traffic}$

Non-example: $(\text{Rain} \wedge \text{Snow}) \rightarrow (\text{Traffic} \vee \text{Peaceful})$

Horn clauses

Definition: Horn clause

A **Horn clause** is either:

- a definite clause ($p_1 \wedge \cdots \wedge p_k \rightarrow q$)
- a goal clause ($p_1 \wedge \cdots \wedge p_k \rightarrow \text{false}$)

Example (definite): (Rain $\wedge$ Snow) $\rightarrow$ Traffic

Example (goal): Traffic $\wedge$ Accident $\rightarrow$ false

equivalent: $\neg$(Traffic $\wedge$ Accident)

Modus ponens

Inference rule:

Definition: Modus ponens

$$\frac{p_1, \quad \cdots \quad , p_k, \quad (p_1 \wedge \cdots \wedge p_k) \rightarrow q}{q}$$

Example:

Example: Modus ponens

$$\frac{\text{Wet}, \quad \text{Weekday}, \quad \text{Wet} \wedge \text{Weekday} \rightarrow \text{Traffic}}{\text{Traffic}}$$

Completeness of modus ponens

Theorem: Modus ponens on Horn clauses

Modus ponens is **complete** with respect to Horn clauses:

- Suppose KB contains only Horn clauses and p is an entailed propositional symbol.
- Then applying modus ponens will derive p.

Upshot:

KB $\models p$ (entailment) is the same as KB $\vdash p$ (derivation)!

Example: Modus ponens

KB

Rain

Weekday

Rain $\rightarrow$ Wet

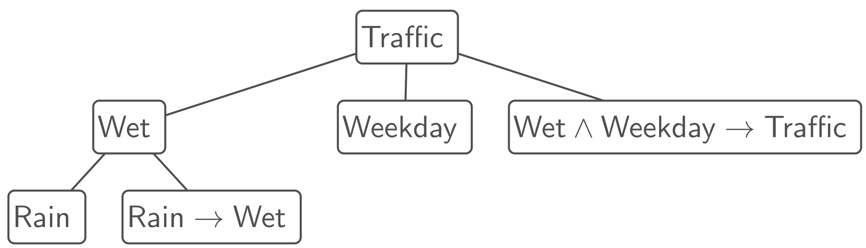
Wet $\wedge$ Weekday $\rightarrow$ Traffic

Traffic $\wedge$ Careless $\rightarrow$ Accident

Definition: Modus ponens

$$\frac{p_1, \quad \cdots \quad, p_k, \quad (p_1 \wedge \cdots \wedge p_k) \rightarrow q}{q}$$

Question: KB $\models$ Traffic $\quad \Leftrightarrow \quad$ KB $\vdash$ Traffic



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Review: tradeoffs

Formulas allowed	Inference rule	Complete?
Propositional logic	modus ponens	no
Propositional logic (only Horn clauses)	modus ponens	yes
Propositional logic	**resolution**	yes

Horn clauses and disjunction

Written with implication	**Written with disjunction**
$A \rightarrow C$	$\neg A \vee C$
$A \wedge B \rightarrow C$	$\neg A \vee \neg B \vee C$

- **Literal**: either p or $\neg p$, where p is a propositional symbol
- **Clause**: disjunction of literals
- **Horn clauses**: at most one positive literal

Modus ponens (rewritten):

$$\frac{A, \quad \neg A \vee C}{C}$$

- Intuition: cancel out A and $\neg A$

Resolution [Robinson, 1965]

General clauses have any number of literals:

$$\neg A \vee B \vee \neg C \vee D \vee \neg E \vee F$$

Example: resolution inference rule

$$\frac{\text{Rain} \vee \text{Snow}, \quad \neg\text{Snow} \vee \text{Traffic}}{\text{Rain} \vee \text{Traffic}}$$

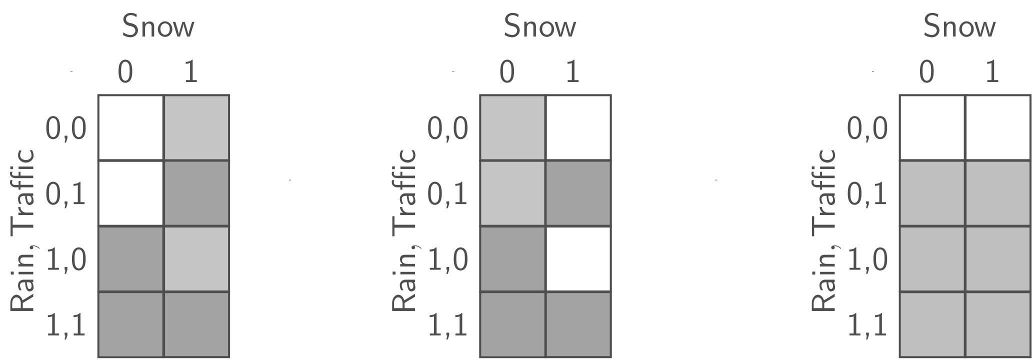
Definition: resolution inference rule

$$\frac{f_1 \vee \cdots \vee f_n \vee p, \quad \neg p \vee g_1 \vee \cdots \vee g_m}{f_1 \vee \cdots \vee f_n \vee g_1 \vee \cdots \vee g_m}$$

Soundness of resolution

$$\frac{\text{Rain} \vee \text{Snow}, \quad \neg\text{Snow} \vee \text{Traffic}}{\text{Rain} \vee \text{Traffic}} \text{ (resolution rule)}$$

$\mathcal{M}(\text{Rain} \vee \text{Snow}) \cap \mathcal{M}(\neg\text{Snow} \vee \text{Traffic}) \subseteq ?\ \mathcal{M}(\text{Rain} \vee \text{Traffic})$



Sound!

Conjunctive normal form

So far: resolution only works on clauses...but that's enough!

Definition: conjunctive normal form (CNF)

A **CNF formula** is a conjunction of clauses.

Example: $(A \vee B \vee \neg C) \wedge (\neg B \vee D)$

Equivalent: knowledge base where each formula is a clause

Proposition: conversion to CNF

Every formula f in propositional logic can be converted into an equivalent CNF formula f':

$$\mathcal{M}(f) = \mathcal{M}(f')$$

Conversion to CNF: example

Initial formula:

$$(\text{Summer} \to \text{Snow}) \to \text{Bizzare}$$

Remove implication ($\to$):

$$\neg(\neg\text{Summer} \vee \text{Snow}) \vee \text{Bizzare}$$

Push negation ($\neg$) inwards (de Morgan):

$$(\neg\neg\text{Summer} \wedge \neg\text{Snow}) \vee \text{Bizzare}$$

Remove double negation:

$$(\text{Summer} \wedge \neg\text{Snow}) \vee \text{Bizzare}$$

Distribute $\vee$ over $\wedge$:

$$(\text{Summer} \vee \text{Bizzare}) \wedge (\neg\text{Snow} \vee \text{Bizzare})$$

Conversion to CNF: general

Conversion rules:

- Eliminate $\leftrightarrow$: $\frac{f \leftrightarrow g}{(f \to g) \wedge (g \to f)}$
- Eliminate $\to$: $\frac{f \to g}{\neg f \vee g}$
- Move $\neg$ inwards: $\frac{\neg (f \wedge g)}{\neg f \vee \neg g}$
- Move $\neg$ inwards: $\frac{\neg (f \vee g)}{\neg f \wedge \neg g}$
- Eliminate double negation: $\frac{\neg \neg f}{f}$
- Distribute $\vee$ over $\wedge$: $\frac{f \vee (g \wedge h)}{(f \vee g) \wedge (f \vee h)}$

Resolution algorithm

Recall: relationship between entailment and contradiction (basically "proof by contradiction")

$$\text{KB} \models f \qquad \longleftrightarrow \qquad \text{KB} \cup \{\neg f\} \text{ is unsatisfiable}$$

Algorithm: resolution-based inference

- Add $\neg f$ into KB.
- Convert all formulas into **CNF**.
- Repeatedly apply **resolution** rule.
- Return entailment iff derive false.

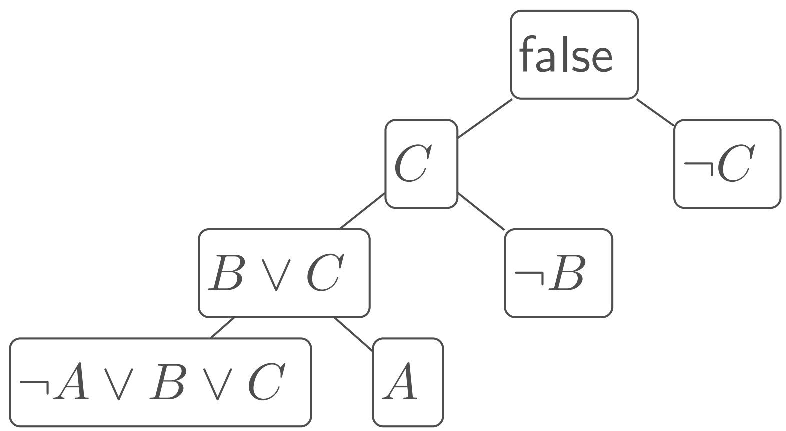
Resolution: example

$$\mathsf{KB}' = \{A \to (B \vee C), A, \neg B, \neg C\}$$

Convert to CNF:

$$\mathsf{KB}' = \{\neg A \vee B \vee C, A, \neg B, \neg C\}$$

Repeatedly apply **resolution** rule:



Conclusion: KB **entails** f

Time complexity

Definition: modus ponens inference rule

$$\frac{p_1, \quad \cdots \quad , p_k, \quad (p_1 \wedge \cdots \wedge p_k) \rightarrow q}{q}$$

- Each rule application adds clause with **one** propositional symbol $\Rightarrow$ linear time

Definition: resolution inference rule

$$\frac{f_1 \vee \cdots \vee f_n \vee p, \quad \neg p \vee g_1 \vee \cdots \vee g_m}{f_1 \vee \cdots \vee f_n \vee g_1 \vee \cdots \vee g_m}$$

- Each rule application adds clause with **many** propositional symbols $\Rightarrow$ exponential time

Summary

Horn clauses	**any clauses**
modus ponens	resolution
linear time	exponential time
less expressive	more expressive

Lecture

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Limitations of propositional logic

Alice and Bob both know arithmetic.

$\text{AliceKnowsArithmetic} \wedge \text{BobKnowsArithmetic}$

All students know arithmetic.

$\text{AliceIsStudent} \rightarrow \text{AliceKnowsArithmetic}$

$\text{BobIsStudent} \rightarrow \text{BobKnowsArithmetic}$

$\dots$

Every even integer greater than 2 is the sum of two primes.

???

Limitations of propositional logic

All students know arithmetic.

AliceIsStudent $\rightarrow$ AliceKnowsArithmetic

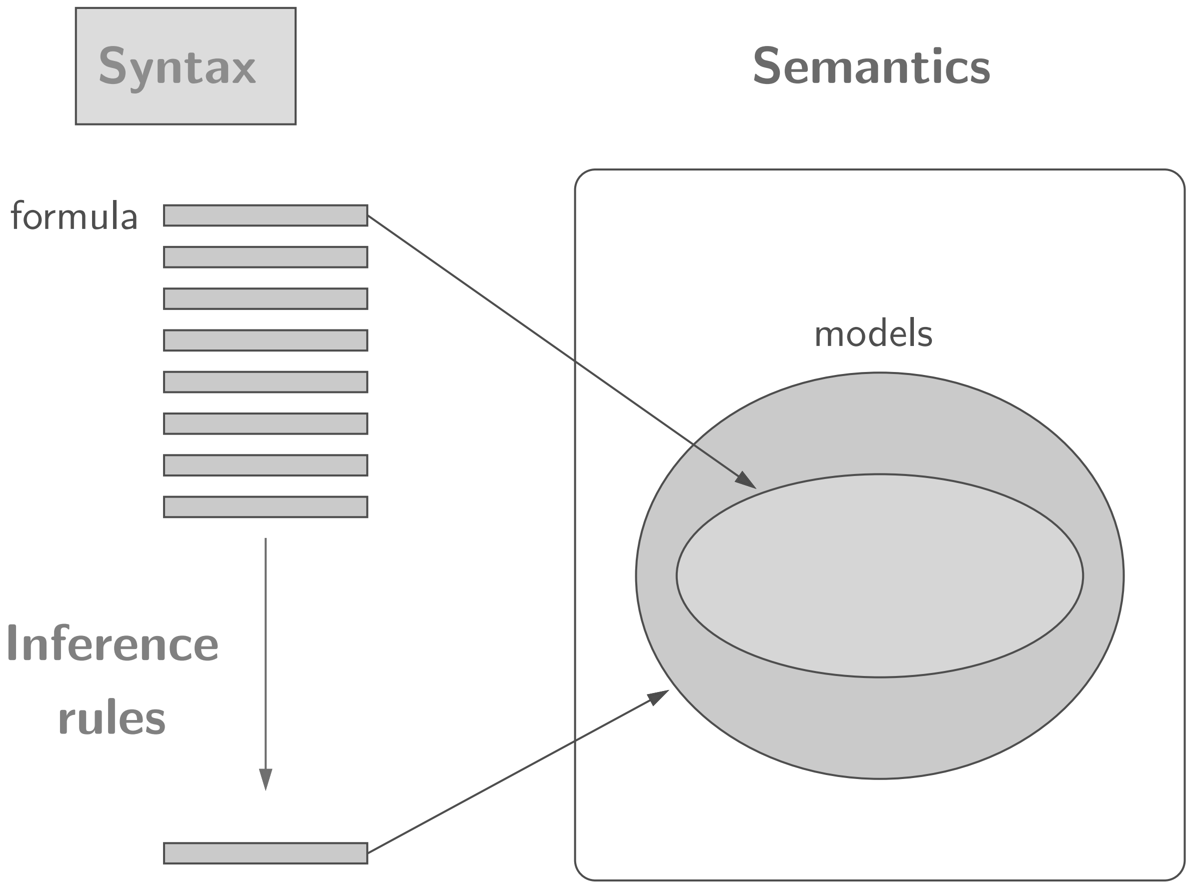
BobIsStudent $\rightarrow$ BobKnowsArithmetic

$\cdots$

Propositional logic is very clunky. What's missing?

- Objects and predicates: propositions (e.g., AliceKnowsArithmetic) have more internal structure (alice, Knows, arithmetic)
- Quantifiers and variables: *all* is a quantifier which applies to each person, don't want to enumerate them all...

First-order logic



First-order logic: examples

Alice and Bob both know arithmetic.

$$\text{Knows}(\text{alice}, \text{arithmetic}) \wedge \text{Knows}(\text{bob}, \text{arithmetic})$$

All students know arithmetic.

$$\forall x \, \text{Student}(x) \rightarrow \text{Knows}(x, \text{arithmetic})$$

Syntax of first-order logic

Terms (refer to objects):

- Constant symbol (e.g., arithmetic)
- Variable (e.g., x)
- Function of terms (e.g., Sum$(3, x)$)

Formulas (refer to truth values):

- Atomic formulas (atoms): predicate applied to terms (e.g., Knows$(x, \text{arithmetic})$)
- Connectives applied to formulas (e.g., Student$(x) \rightarrow$ Knows$(x, \text{arithmetic})$)
- Quantifiers applied to formulas (e.g., $\forall x$ Student$(x) \rightarrow$ Knows$(x, \text{arithmetic})$)

Quantifiers

Universal quantification ($\forall$):

Think conjunction: $\forall x\, P(x)$ is like $P(A) \wedge P(B) \wedge \cdots$

Existential quantification ($\exists$):

Think disjunction: $\exists x\, P(x)$ is like $P(A) \vee P(B) \vee \cdots$

Some properties:

- $\neg \forall x\, P(x)$ equivalent to $\exists x\, \neg P(x)$
- $\forall x\, \exists y\, \text{Knows}(x, y)$ different from $\exists y\, \forall x\, \text{Knows}(x, y)$

Natural language quantifiers

Universal quantification ($\forall$):

Every student knows arithmetic.

$\forall x \, \text{Student}(x) \rightarrow \text{Knows}(x, \text{arithmetic})$

Existential quantification ($\exists$):

Some student knows arithmetic.

$\exists x \, \text{Student}(x) \wedge \text{Knows}(x, \text{arithmetic})$

Note the different connectives!

Some examples of first-order logic

There is some course that every student has taken.

$$\exists y \, \mathsf{Course}(y) \wedge [\forall x \, \mathsf{Student}(x) \rightarrow \mathsf{Takes}(x, y)]$$

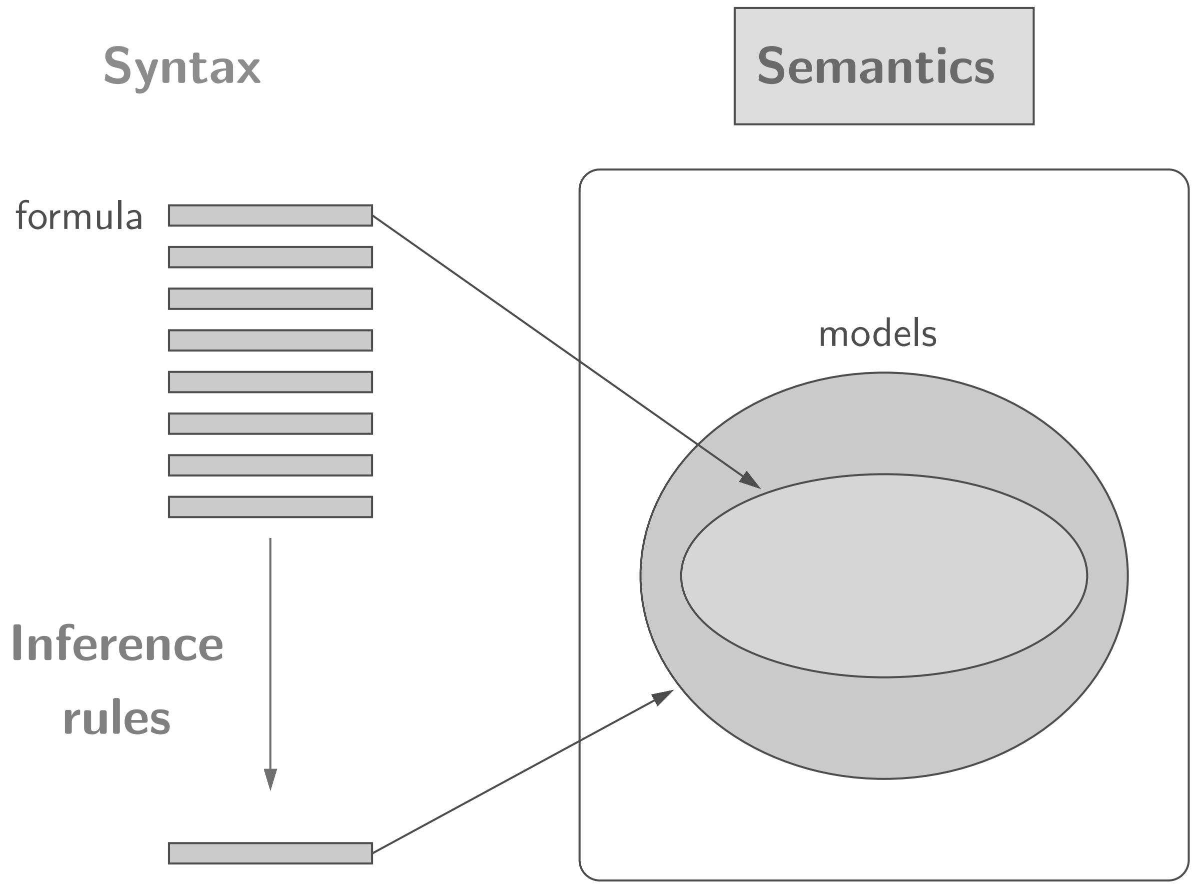
Every even integer greater than 2 is the sum of two primes.

$$\forall x \, \mathsf{EvenInt}(x) \wedge \mathsf{Greater}(x, 2) \rightarrow \exists y \, \exists z \, \mathsf{Equals}(x, \mathsf{Sum}(y, z)) \wedge \mathsf{Prime}(y) \wedge \mathsf{Prime}(z)$$

If a student takes a course and the course covers a concept, then the student knows that concept.

$$\forall x \, \forall y \, \forall z \, (\mathsf{Student}(x) \wedge \mathsf{Takes}(x, y) \wedge \mathsf{Course}(y) \wedge \mathsf{Covers}(y, z)) \rightarrow \mathsf{Knows}(x, z)$$

First-order logic



Models in first-order logic

Recall a model represents a possible situation in the world.

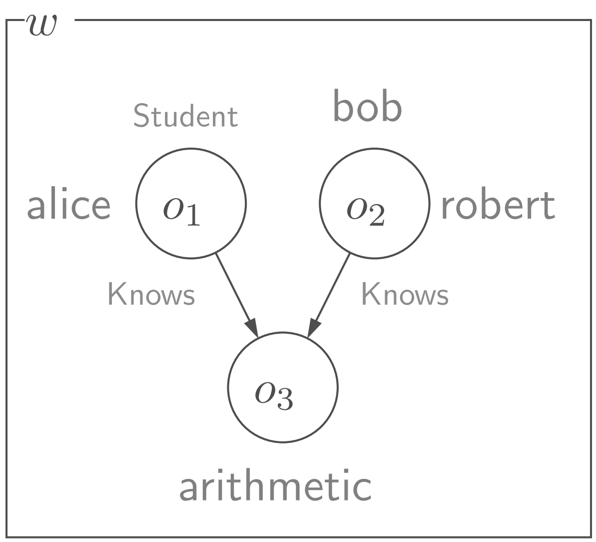
Propositional logic: Model w maps propositional symbols to truth values.

$$w = \{\text{AliceKnowsArithmetic} : 1, \text{BobKnowsArithmetic} : 0\}$$

First-order logic: ?

Graph representation of a model

If only have unary and binary predicates, a model w can be represented as a directed graph:



- Nodes are objects, labeled with constant symbols
- Directed edges are binary predicates, labeled with predicate symbols; unary predicates are additional node labels

Models in first-order logic

Definition: model in first-order logic

A model w in first-order logic maps:

- constant symbols to objects

$w(\text{alice}) = o_1, w(\text{bob}) = o_2, w(\text{arithmetic}) = o_3$

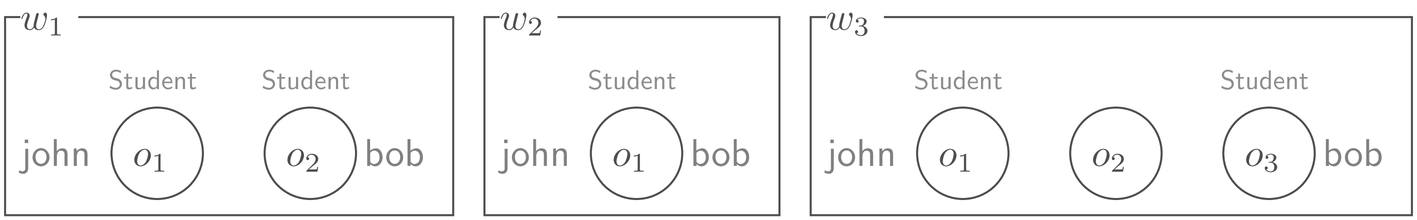
- predicate symbols to tuples of objects

$w(\text{Knows}) = \{(o_1, o_3), (o_2, o_3), \dots\}$

A restriction on models

John and Bob are students.

$$\text{Student}(\text{john}) \wedge \text{Student}(\text{bob})$$



- Unique names assumption: Each object has **at most one** constant symbol. This rules out w_2.
- Domain closure: Each object has **at least one** constant symbol. This rules out w_3.

Point:

$$\text{constant symbol} \longleftrightarrow \text{object}$$

Propositionalization

If one-to-one mapping between constant symbols and objects (**unique names** and **domain closure**),

first-order logic is syntactic sugar for propositional logic:

Knowledge base in first-order logic

$$\text{Student}(\text{alice}) \wedge \text{Student}(\text{bob})$$
$$\forall x\, \text{Student}(x) \rightarrow \text{Person}(x)$$
$$\exists x\, \text{Student}(x) \wedge \text{Creative}(x)$$

Knowledge base in propositional logic

$$\text{Studentalice} \wedge \text{Studentbob}$$
$$(\text{Studentalice} \rightarrow \text{Personalice}) \wedge (\text{Studentbob} \rightarrow \text{Personbob})$$
$$(\text{Studentalice} \wedge \text{Creativealice}) \vee (\text{Studentbob} \wedge \text{Creativebob})$$

Point: use any inference algorithm for propositional logic!

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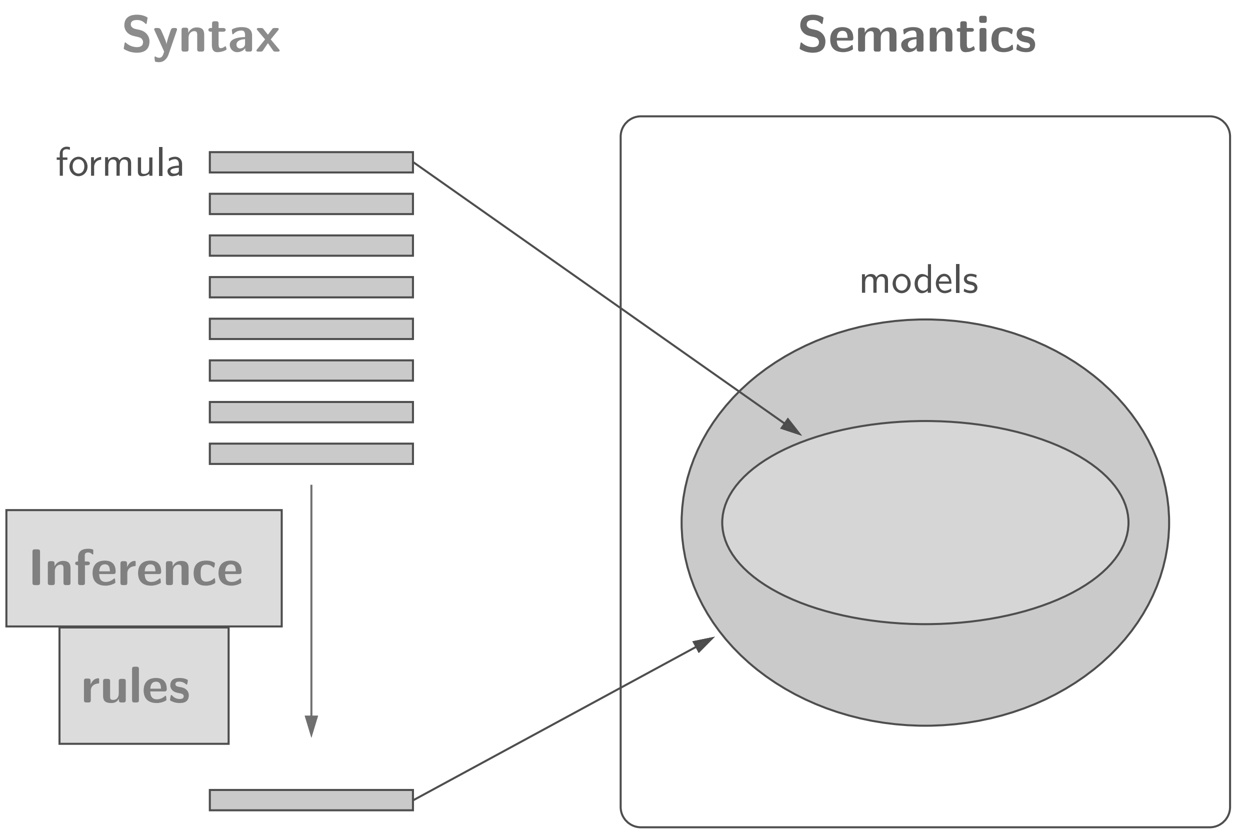

Definite clauses

$$\forall x\, \forall y\, \forall z\, (\mathsf{Takes}(x, y) \wedge \mathsf{Covers}(y, z)) \rightarrow \mathsf{Knows}(x, z)$$

Note: if propositionalize, get one formula for each value to (x, y, z), e.g., $(\mathsf{alice}, \mathsf{cs221}, \mathsf{mdp})$

Definition: definite clause (first-order logic)

A definite clause has the following form:

$$\forall x_1 \cdots \forall x_n\, (a_1 \wedge \cdots \wedge a_k) \rightarrow b$$

for variables $x_1, \dots, x_n$ and atomic formulas $a_1, \dots, a_k, b$ (which contain those variables).

Modus ponens (first attempt)

Definition: modus ponens (first-order logic)

$$\frac{a_1, \dots, a_k \quad \forall x_1 \cdots \forall x_n (a_1 \wedge \cdots \wedge a_k) \rightarrow b}{b}$$

Setup:

Given $P(\text{alice})$ and $\forall x\, P(x) \rightarrow Q(x)$.

Problem:

Can't infer $Q(\text{alice})$ because $P(x)$ and $P(\text{alice})$ don't match!

Solution: substitution and unification

Substitution

Subst$[\{x/\text{alice}\}, P(x)] = P(\text{alice})$

Subst$[\{x/\text{alice}, y/z\}, P(x) \wedge K(x, y)] = P(\text{alice}) \wedge K(\text{alice}, z)$

Definition: Substitution

A substitution θ is a mapping from variables to terms.

Subst$[\theta, f]$ returns the result of performing substitution θ on f.

Unification

Unify[Knows(alice, arithmetic), Knows(x, arithmetic)] = $\{x/$alice$\}$

Unify[Knows(alice, y), Knows(x, z)] = $\{x/$alice$, y/z\}$

Unify[Knows(alice, y), Knows(bob, z)] = fail

Unify[Knows(alice, y), Knows($x, F(x)$)] = $\{x/$alice$, y/F($alice$)\}$

Definition: Unification

Unification takes two formulas f and g and returns a substitution θ which is the most general unifier:

Unify$[f, g] = \theta$ such that Subst$[\theta, f]$ = Subst$[\theta, g]$

or "fail" if no such θ exists.

Modus ponens

Definition: modus ponens (first-order logic)

$$\frac{a'_1, \ldots, a'_k \quad \forall x_1 \cdots \forall x_n (a_1 \wedge \cdots \wedge a_k) \to b}{b'}$$

Get most general unifier θ on premises:

- $\theta = \text{Unify}[a'_1 \wedge \cdots \wedge a'_k, a_1 \wedge \cdots \wedge a_k]$

Apply θ to conclusion:

- $\text{Subst}[\theta, b] = b'$

Modus ponens example

Example: modus ponens in first-order logic

Premises:

$\text{Takes}(\text{alice}, \text{cs221})$

$\text{Covers}(\text{cs221}, \text{mdp})$

$\forall x \, \forall y \, \forall z \, \text{Takes}(x, y) \wedge \text{Covers}(y, z) \rightarrow \text{Knows}(x, z)$

Conclusion:

$\theta = \{x/\text{alice}, y/\text{cs221}, z/\text{mdp}\}$

Derive $\text{Knows}(\text{alice}, \text{mdp})$

Complexity

$$\forall x\, \forall y\, \forall z\, P(x, y, z)$$

- Each application of Modus ponens produces an atomic formula.
- If no function symbols, number of atomic formulas is at most

$$(\text{num-constant-symbols})^{(\text{maximum-predicate-arity})}$$

- If there are function symbols (e.g., F), then infinite...

$$Q(a) \quad Q(F(a)) \quad Q(F(F(a))) \quad Q(F(F(F(a)))) \quad \cdots$$

Complexity

Theorem: completeness

Modus ponens is complete for first-order logic with only Horn clauses.

Theorem: semi-decidability

First-order logic (even restricted to only Horn clauses) is **semi-decidable**.

- If KB $\models f$, forward inference on complete inference rules will prove f in finite time.
- If KB $\not\models f$, no algorithm can show this in finite time.

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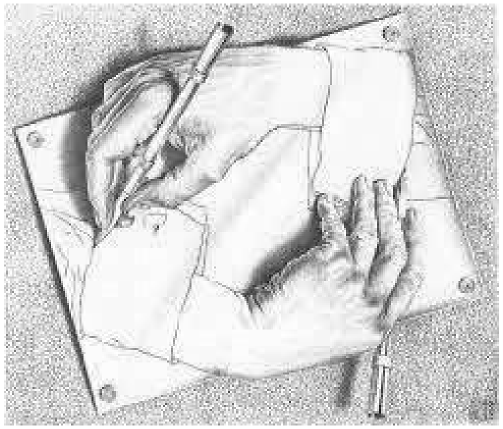
First-order resolution



Resolution

Recall: First-order logic includes non-Horn clauses

$$\forall x \, \mathsf{Student}(x) \rightarrow \exists y \, \mathsf{Knows}(x, y)$$

High-level strategy (same as in propositional logic):

- Convert all formulas to CNF
- Repeatedly apply resolution rule

Conversion to CNF

Input:

$$\forall x\, (\forall y\, \mathsf{Animal}(y) \to \mathsf{Loves}(x, y)) \to \exists y\, \mathsf{Loves}(y, x)$$

Output:

$$(\mathsf{Animal}(Y(x)) \vee \mathsf{Loves}(Z(x), x)) \wedge (\neg \mathsf{Loves}(x, Y(x)) \vee \mathsf{Loves}(Z(x), x))$$

New to first-order logic:

- All variables (e.g., x) have universal quantifiers by default
- Introduce **Skolem functions** (e.g., $Y(x)$) to represent existential quantified variables

Conversion to CNF (part 1)

Anyone who loves all animals is loved by someone.

Input:

$\forall x\, (\forall y\, \text{Animal}(y) \to \text{Loves}(x, y)) \to \exists y\, \text{Loves}(y, x)$

Eliminate implications (old):

$\forall x\, \neg(\forall y\, \neg\text{Animal}(y) \vee \text{Loves}(x, y)) \vee \exists y\, \text{Loves}(y, x)$

Push $\neg$ inwards, eliminate double negation (old):

$\forall x\, (\exists y\, \text{Animal}(y) \wedge \neg\text{Loves}(x, y)) \vee \exists y\, \text{Loves}(y, x)$

Standardize variables (**new**):

$\forall x\, (\exists y\, \text{Animal}(y) \wedge \neg\text{Loves}(x, y)) \vee \exists z\, \text{Loves}(z, x)$

Conversion to CNF (part 2)

$\forall x \, (\exists y \, \text{Animal}(y) \wedge \neg \text{Loves}(x, y)) \vee \exists z \, \text{Loves}(z, x)$

Replace existentially quantified variables with Skolem functions (**new**):

$\forall x \, [\text{Animal}(Y(x)) \wedge \neg \text{Loves}(x, Y(x))] \vee \text{Loves}(Z(x), x)$

Distribute $\vee$ over $\wedge$ (old):

$\forall x \, [\text{Animal}(Y(x)) \vee \text{Loves}(Z(x), x)] \wedge [\neg \text{Loves}(x, Y(x)) \vee \text{Loves}(Z(x), x)]$

Remove universal quantifiers (**new**):

$[\text{Animal}(Y(x)) \vee \text{Loves}(Z(x), x)] \wedge [\neg \text{Loves}(x, Y(x)) \vee \text{Loves}(Z(x), x)]$

Resolution

Definition: resolution rule (first-order logic)

$$\frac{f_1 \vee \cdots \vee f_n \vee p, \quad \neg q \vee g_1 \vee \cdots \vee g_m}{\text{Subst}[\theta, f_1 \vee \cdots \vee f_n \vee g_1 \vee \cdots \vee g_m]}$$

where $\theta = \text{Unify}[p, q]$.

Example: resolution

$$\frac{\text{Animal}(Y(x)) \vee \text{Loves}(Z(x), x), \quad \neg\text{Loves}(u, v) \vee \text{Feeds}(u, v)}{\text{Animal}(Y(x)) \vee \text{Feeds}(Z(x), x)}$$

Substitution: $\theta = \{u/Z(x), v/x\}$.

Summary

Propositional logic		First-order logic
model checking		n/a
	$\Leftarrow$ propositionalization	
modus ponens (Horn clauses)		modus ponens++ (Horn clauses)
resolution (general)		resolution++ (general)

++: unification and substitution

Key idea: variables in first-order logic

Variables yield compact knowledge representations.