Repeated Gambles with Bundled Options: A Behavioural Model of Probability Misperception

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Abstract

This paper explains the mechanics of toy vending machines in mobile games using prospect theory and naivety. It demonstrates that naive players who perceive independent probabilities in a history-dependent way can be induced to play for longer than optimal, thereby increasing the game provider's profits. Free initial rounds can entice naive players to start a game that will ultimately harm them. Bundling two rounds together increases the likelihood that naive players continue to play due to their probability misperception. Numerical examples further support the feasibility and profitability of this mechanism.

1 Introduction

Gacha games are video games that use a toy vending machine mechanic where players typically use real money to purchase random draws of virtual items. These free-to-play games with in-game purchases currently dominate the mobile games market. As stated in the recent report *Digital Games and Interactive Media: 2020 Year in Review* [1], free-to-play games account for 78% of digital games revenue, with mobile games contributing 75% of the total $98.4 billion revenue generated by free-to-play games. The top two free-to-play titles, both published by Tencent, implement the Gacha mechanic in purchasing character outfits and accessories. In fact, the list of popular mobile games is dominated by those published by Asian companies, with Gacha mechanics being implemented in almost all popular mobile games from Chinese and Japanese developers. The question arises as to how Gacha mechanics attract players, generate profits and become prevalent in mobile games. To answer this question, I use a behavioural model that captures several important features in Gacha games and looks for possible explanations.

These games can be modelled as repeated fair gambles. The fairness comes from the rapid development of online trading platforms, where players can easily trade their game accounts or even in-game items, and evaluate those virtual items accurately through a large number of available quotes. According to early behavioural research, such as the experimental results in Tversky and Kahneman (1992), fair gambles are usually rejected. However, this is not reflective of the reality in Gacha games, as numerous players engage in excessive draws to obtain desired items, resulting in regretful outcomes and a negative net gain. This inconsistency suggests that these players may be irrational (naive) during gameplay, and their behavioural pattern can be linked to the gambler's fallacy or, more generally, the belief in "the law of small numbers" (Tversky and Kahneman, 1971), which is supported by a wealth of evidence in the literature. The gambler's fallacy is the mistaken belief that past outcomes affect future probabilities in a game of chance. This can lead naive players to perceive winning as more likely after a losing streak, encouraging them to continue playing in order to not miss out on a chance to win. This misperception can also be seen as an attempt to bring the sample distribution back to the population distribution, in accordance with the law of small numbers. Thus, I propose several justifiable patterns for the perceived probabilities of naive players during the game without specifying their functional forms.

Gacha games usually provide two features: (1) free draws at the start of the game, and (2) the option to make a certain number of consecutive draws or a single draw on the virtual vending machine interface. This model explains how these features generate profits for the

[1] published by *SuperData*

game provider.

Firstly, to entice naive players to start a game that may otherwise be unattractive, a sample that sufficiently does not resemble the population distribution is necessary. Free initial rounds can effectively generate such a sample, resulting in highly misperceived probabilities for some naive players. By transforming non-participants into participants of the game, and considering the low cost of providing virtual rewards, offering free initial rounds can be a profitable choice for the game provider.

Secondly, as the optimisation problem for this game is highly complex, players may only be able to consider a limited number of future rounds. This narrow-bracketing behaviour further categorises them.

My analysis begins with the simplest scenario where the naive 1-planning player is going to participate in 50-50 fair gambles with 2-draw bundles. Proposition 1 gives the baseline gain-exit behavioural pattern (continue gambling when overall winning but exit the game when losing) when the bundled option is not introduced, which holds true for all reasonable parameter specifications. Proposition 2 then examines the effect of making the 2-draw bundles available in the game, indicating that the naive 1-planning player will be incentivised to continue playing for a longer period of time compared to the baseline result.

Then, I demonstrate that the two propositions for 50-50 fair gambles can each be extended to a typical class of skewed fair gambles, in which the player gains m with probability $\frac{1}{m+1}$ and loses 1 with probability $\frac{1}{m+1}$, at the cost of imposing a restriction on the loss-aversion parameter. However, the generality of these two follow-up propositions is only minimally affected, in that almost all parameter estimations in the literature satisfy the assumption.

The increase in the game provider's profit after introducing the 2-draw bundles is naturally deduced from the aforementioned behaviour patterns of the player. Bundling two rounds together increases the likelihood that a naive 1-planning player continues to play by exacerbating their probability misperception. However, this effect does not apply to other players who plan a longer period of time.

For better understanding, all of these findings are further examined through numerical analysis.

The final step involves examining scenarios in which the naive player plans for more rounds or the game provider offers the option for more bundled rounds. I find hierarchical relationships among players grouped by different bracketing levels. Broader-bracketing naive players are more likely to continue playing, making them more lucrative for the game provider. This is because all possible plans considered by the player who plans for a short term will be encompassed by the player who plans for a longer term.

The paper is organised as follows. In Section 2, I present the theoretical model of Gacha games with naive players and bundled options. The main body of my analysis, including both analytical and numerical findings for naive 1-planning players provided with 2-draw bundles, is presented in Section 3. The general patterns for naive N-planning players are presented in Section 4. I conclude in Section 5.

1.1 Literature review

My model is largely inspired by Barberis (2012). In his model of casino gambling, a player optimises a plan of actions for all possible subsequent gambles using cumulative prospect theory (CPT) value functions (Tversky and Kahneman, 1992) to evaluate outcomes. Depending on the specification of CPT parameters, the player can either start with a loss-exit plan (continue gambling when overall winning but exit the game when losing) and end up playing a gain-exit strategy, or the reverse, in 50-50 fair gambles. Ebert and Strack (2015) pointed out that the CPT probability weighting leads to unrealistic never-stopping results for small and skewed gambles. My model produces a middle-ground result between ambiguous parameter dependency and strict never-stopping behaviour by first proposing properties based on the law of small numbers to replace the CPT probability weighting, and second simplifying the planning process by narrow bracketing.

There is a substantial body of literature on "the law of small numbers" originating from Tversky and Kahneman (1971). This law essentially describes the tendency to overestimate how a probability distribution in a small sample will resemble the population distribution, and has been linked to the gambler's fallacy later. Clotfelter and Cook (1993) and Terrell (1994) provided compelling evidence for the existence of the gambler's fallacy by examining state-run lotteries. Rabin (2002) modelled a believer in the law of small numbers who mistakenly assumes that when drawing from finite signals representing the population distribution, some draws are taken with replacement while others are not. However, there is no definite evidence on the functional form of misperceived probabilities. My model allows for flexibility by assuming only several justifiable properties based on these evidences.

The concept of "narrow barcketing" is firstly introduced in Tversky and Kahneman (1981) to describe the mental procedure of a decision maker who faces multiple decisions and handles them separately. In my model, the planning and optimisation processes can be extremely complex. Therefore, players are assumed to separate future decisions into those that are close and far away, handling the former first by only planning for a limited number of rounds. Empirical research suggests that individuals' decision-making behaviour can vary depending on whether they are making decisions in a combined or separated manner (Simonson, 1990; Gourville, 1998; Gneezy and Potters, 1997). I examine the effect of offering a broader plan to the narrow-bracketing player, and find that it does change the player's behavioural pattern.

The mechanism of Gacha games is not well-studied, as the mobile game industry has only thrived recently. To the best of my knowledge, Gan (2022) is the only existing analytical study based on prospect theory preferences that discusses the optimality of this mechanism. Gan demonstrated that, in the presence of probability weighting, the optimality of selling "loot boxes" with or without the worst-case insurance is determined by whether the player is sophisticated or naive about her own inconsistency. Chen and Fang (2023) modelled the Gacha game differently as a Stackelberg game and demonstrated that the revenue-optimal design is equivalent to the single-item single-bidder Myerson auction.

2 Model

The game conforms to the following structure: there exists a constraint T specifying the maximum number of rounds that can be undertaken, with each round of gamble adhering to identical parameters. At the beginning of each round, the player can opt for the acceptance or rejection of the current round of gamble without any cost. Termination of the game occurs if the player declines participation in the current round or if the maximum allowable rounds have been exhausted. Alternatively, the game proceeds if the player opts to engage in current round, requiring a payment of \$$a$ to participate. There are two potential outcomes for each gamble: either a win U indicating an upward movement in the game tree representation, or a loss D indicating a downward movement. The monetary rewards associated with these outcomes are represented as $B_U > 0$ for a win and $B_D = 0$ for a loss, respectively. Let us denote such a gamble by $h_U, p^U; h_D, p^D$, where p_i is the true probability of outcome $i \in U, D$, yielding a monetary outcome h_i for the player. It is evident that $h_U = B_U - a > 0$ and $h_D = B_D - a < 0$. The gamble is fair from the expectational perspective, demanding that $p^U h_U + p^D h_D = 0$, or equivalently, $p^U B_U + p^D B_D = a$.

A natural and clear representation of such game is a game tree. In the graph, the total monetary outcome is plotted against the number of rounds $t \in \{0, 1, \cdots, T\}$ played. The possible states of the player during the game are represented by tree nodes. The potential evolutionary paths of these states are represented by directed edges connecting tree nodes. Moreover, the node representing a player's state after winning k_U times and losing k_D times can be uniquely determined by the vector (k_U, k_D), since $k_U + k_D = t$ is the number of completed rounds and $k_U h_U + k_D h_D$ is the total monetary gain. Therefore, it is convenient to refer to each node by its unique state vector $\boldsymbol{k}^t = (k_U, k_D)$.

2.1 Probability misperception

A common mistake made by players in repeated gambles is "the gambler's fallacy," which is the non-Bayesian belief that a certain outcome which has just occurred is less likely to happen subsequently. These players are referred to as "naive," and I propose some justifiable patterns of their probability misperception. The explanations of these patterns are based on the more general behavioural concept of "the law of small numbers" (Tversky and Kahneman, 1971): probability distributions in small samples are expected to resemble the population distribution.

Consider a naive player who is losing (winning) overall and believes that small samples resemble the population. Winning (losing) the next round is "expected" because it will bring the sample closer to the population distribution, while losing (winning) the next round is "unexpected" because it further unbalances the sample. This incorrect expectation can lead to biased perceptions of probabilities, and the extent of this misperception depends on how much the current sample deviates from the expectation, i.e., the population distribution. However, bias can be absent when the sample perfectly matches the population distribution or at the beginning of the game when no sample is available, since the naive player does not misinterpret other aspects of the game. Let us denote the naive player's perceived probabilities for next round of gamble on node $\boldsymbol{k}^t$ by $p^U_{\boldsymbol{k}^t}$ and $p^D_{\boldsymbol{k}^t}$, the first property of the naive player's probability misperception is given by:

Property 1 *For any $\boldsymbol{k}^t = (k_U, k_D)$, the naive player's perceived probabilities satisfy*

$$\begin{aligned}
h_U k_U + h_D k_D < 0 &\Rightarrow p^U_{\boldsymbol{k}^t} > p^U, \\
h_U k_U + h_D k_D = 0 &\Rightarrow p^U_{\boldsymbol{k}^t} = p^U, \\
h_U k_U + h_D k_D > 0 &\Rightarrow p^U_{\boldsymbol{k}^t} < p^U.
\end{aligned}$$

Apart from the basic property mentioned above, there may be cases where one sample is perceived to have more or less resemblance to the population distribution than the other by a naive player. As a result, the perceived probabilities on certain node should be adjusted accordingly. The next two properties are derived in this way.

Let us consider the scenario where a naive player on $\boldsymbol{k}^t = (k_U, k_D)$ is experiencing a losing streak. The sample distribution will vary according to the following sequence:

$$(\frac{k_U}{t}, \frac{k_D}{t}), (\frac{k_U}{t+1}, \frac{k_D+1}{t+1}), (\frac{k_U}{t+2}, \frac{k_D+2}{t+2}), \cdots$$

If $k_U \neq 0$, the direction of change along the sequence is monotonic when compared to the population distribution (p^U, p^D). However, the degree of change is attenuating, and the distribution converges to $(0,1)$ for a sufficently long losing streak. The implications of this observation are substantial. Firstly, the losing streak may lead a naive player to believe that they are more likely to win the next round. Secondly, the naive player may perceive the samples as less different from the previous sample along the streak, resulting in smaller further adjustments needed to be made in her mind. Finally, the naive player may think that they are almost sure to win the next round after sufficient losses. For $k_U = 0$, it is possible to retain monotonicity and convergency. This is because a large sample with a distribution of $(0,1)$ is less likely to occur than a small sample with the same distribution. Therefore, the naive player will be more biased and will consider herself almost ensured to win the next round if they have never won for a sufficient number of rounds. These properties of monotonicity, convergency and convexity are summarised as follows:

Property 2 *For any node (k_U, k_D) and any $i \in \{U, D\}$, let us denote the other outcome by $-i$, the naive player's perceived probabilities satisfy*

$$p^i_{(k_i+1,k_{-i})} - p^i_{(k_i,k_{-i})} < 0$$
$$\lim_{k_i \to \infty} p^i_{(k_i,k_{-i})} = 0.$$

If $k_{-i} \neq 0$, they further satisfy

$$p^i_{(k_i+2,k_{-i})} - p^i_{(k_i+1,k_{-i})} > p^i_{(k_i+1,k_{-i})} - p^i_{(k_i,k_{-i})}.$$

Then, consider the scenario where a naive player on node $\boldsymbol{k}^t = (k_U, k_D)$ plays for the next t' rounds, and the aggeregate outcomes in these rounds perfectly match the population distribution, i.e., she wins $t'p^U \in \mathbb{N}^+$ rounds and loses $t'p^D \in \mathbb{N}^+$ rounds. The player receives the same net payoff as on node $\boldsymbol{k}$, but the sample distribution changes from $(\frac{k_U}{t}, \frac{k_D}{t})$ to $(\frac{k_U+t'p^U}{t+t'}, \frac{k_D+t'p^D}{t+t'})$. The new distribution is always closer to the population distribution (p^U, p^D), as long as the player does not have a perfectly matched sample on $\boldsymbol{k}^t$. Furthermore, for sufficiently large values of t', the latter distribution converges to the population distribution. This is because a larger sample size makes the same absolute difference between expected and realized wins or losses less significant. When adding minor differences to a sufficiently large sample that perfectly matches the population distribution, the naive player may feel that the new sample is almost identical to what is expected. These properties are summaries as follows:

Property 3 *For any $i \in \{U, D\}$ and any $\boldsymbol{k}^t = (k_U, k_D)$ such that $k_U h_U + k_D h_D \neq 0$, given any $\boldsymbol{k}^{t'} = (k'_U, k'_D)$ such that $k'_U h_U + k'_D h_D = 0$, the naive player's perceived probabilities* *satisfy*

$$\left|p^i_{\boldsymbol{k}^t} - p^i\right| > \left|p^i_{\boldsymbol{k}^t + \boldsymbol{k}^{t'}} - p^i\right|$$
$$\lim_{t' \to \infty} p^i_{\boldsymbol{k}^t + \boldsymbol{k}^{t'}} = p^i$$

2.2 Planning and optimisation

The decision problem is to determine the choice on each node that may be reached in the future, given that a player always has the freedom to accept or reject the next round. A plan is a specification of choices for all possible nodes. The player then identifies the optimised plan among all possible plans based on an evaluation process of expected outcomes and implements it in the next round. Barberis developed the initial model for casino gambling, but I made two significant modifications.

Firstly, Barberis assumes that a player will always make a plan for subsequent rounds of gambles until the end of the game. However, this assumption is flawed as even small-sized repeated gambles can involve heavy computational load due to the exponential increase in the number of possible plans as the number of rounds increases. A practical solution is to introduce narrow-bracketing behaviour. This means that an N-planning player will only plan for a maximum of N rounds to limit the the complexity of the optimisation problem. The value of N sets an upper bound on the computational complexity, and categorises players based on their planning and optimisation ability. In this paper, I mainly focus on the $N = 1$ case. It is worth noting that Barberis's planning is equivalent to $N \to \infty$.

Secondly, in Barberis's model for casino gambling, the player is assumed to refer to their original wealth level when evaluating a plan. However, in Gacha games, this is different. While bidirectional conversion between real currency and tokens is always feasible in casino gambling, Gacha game players can only top up their virtual account and are unable to convert in-game currency to real money. Therefore, a player with a budget of \$100 can confidently bring \$100 worth of tokens into the casino, but is more likely to only top up for the \$5 draw that they are going to take in Gacha games. When checking their account during the game, a gambler can clearly see their change in wealth relative to their original level, while a Gacha game player is more likely to focus on the gain and loss within their current gamble. Based on this, it can be assumed that the player refers to their current wealth level when making plans.

The planning and optimisation procedures in my model is formally stated below.

An N-planning player with inital wealth w_0 and current state $\boldsymbol{k}^t = (k_U, k_D)$ is faced with the decision of making a plan s^N. This plan is a mapping of each possible node between round t and round $t+N-1$ to two possible actions for the next round: "reject" or "accept". A path of future realisation of outcomes

$$\varphi = (i_{t+1}, i_{t+2}, \cdots, i_{t+\theta}, \cdots, i_{t+\Theta})$$
$$\Theta \leq N, i_{t+\theta} \in \{U, D\}$$

is considered valid for a given plan s^N if the choice on every passed node, starting from $\boldsymbol{k}^t$ and moving according to the sequence of $i_{t+\theta}$ until $\boldsymbol{k}^{t+\Theta-1}$, is to accept next round. If we denote the set of all valid paths for plan s^N by Φ_{s^N}, then the distribution of monetary outcomes from implementing s^N can be written as a random variable

$$X_{s^N} \sim \{H_1, P_1; H_2, P_2; \cdots; H_j, P_j; \cdots\}$$

where H_j represents the possible monetary outcome relative to the current wealth level $w_{\boldsymbol{k}^t} = w_0 + k_U h_U + k_D h_D$, and

$$P_j = \sum_{\substack{\varphi=(i_{t+1}, i_{t+2}, \cdots) \in \Phi_{s^N} \\ h_{i_{t+1}} + h_{i_{t+2}} + \cdots = H_j}} p_{\boldsymbol{k}^t}^{i_{t+1}} p_{\boldsymbol{k}^{t+1}}^{i_{t+2}} \cdots$$

represents the perceived probability of this outcome, which is calculated based on the dynamically perceived probabilities of the player at each node.

To gain a better understanding of these notations for the player's plan, a simple example is given below.

Example 1 *Let us consider a 2-planning player who is currently on node (k_U, k_D). One possible plan for her is to accept the first round and then accept the second round only if she is winning. This plan and its outcome distribution can be expressed using the notations provided above as*

$$s^2 = \{(k_U, k_D), Accept; (k_U + 1, k_D), Accept; (k_U, k_D + 1), Reject\}$$
$$X_{s^2} \sim \{2h_U, p_{(k_U,k_D)}^U p_{(k_U+1,k_D)}^U; h_U + h_D, p_{(k_U,k_D)}^U p_{(k_U+1,k_D)}^D; h_D, p_{(k_U,k_D)}^D\}.$$

Now we can proceed to the optimisation problem for an N-planning player. Let us denote the set of all possible plans s^N by S^N, the player's optimisation problem is given by

$$\max_{s^N \in S^N} V(X_{s^N}; w_{\boldsymbol{k}^t})$$

where $V(\cdot)$ is the value fuction in the following form

$$V(X_{s^N}; w_{\boldsymbol{k}^t}) = \sum_j P_j U(H_j; w_{\boldsymbol{k}^t}),$$

and $U(\cdot)$ is the utility function in the following form

$$\begin{aligned} U(x; w) &= \alpha(w) u(x) \\ \alpha(w) &> 0, \alpha^{'}(w) < 0 \\ u(x) &= \begin{cases} x^r, & x \geq 0 \\ -\lambda(-x)^r, & x < 0 \end{cases} \\ \lambda &> 1, 0 < r < 1. \end{aligned}$$

Here, it is assumed that the level of wealth only affects the utility function as a positive scalar that decreases as wealth increases. This assumption is based on the fact that the relative change in wealth is smaller for a richer person, even if the absolute change is the same. As a result, each player calculates her unadjusted utility using the same method, but adjusts it based on her current wealth level in a way that the player with less wealth experiences a greater change in utility from the same amount of monetary gain or loss. The unadjusted utility function $u(\cdot)$ follows the conventional form in prospect theory (Kahneman and Tversky, 1979). Its two parameters, λ and r, indicate the degree of loss aversion and diminishing sensitivity, respectively.

Finally, the player should follow the optimal plan

$$s^N_{Optimal} = \arg\max_{s^N \in S^N} V(X_{s^N}; w_{\boldsymbol{k}^t}),$$

to make the decision on current node $\boldsymbol{k}^t$. If the decision is to reject, she exits the game. Otherwise, she should a single draw and move to a new state, where she will face a new planning and optimisation problem.

In addition, it is useful to evaluate the acceptability of the gamble by solving the switching probability γ from the following equation

$$\gamma u(h_U) + (1-\gamma) u(h_D) = 0.$$

The unique solution is

$$\gamma = \frac{\lambda}{\lambda + (-\frac{h_U}{h_D})^r} \in (0, 1).$$

A player on node $\boldsymbol{k}^t$ will accept the next round of gamble if and only if her perceived probability of winning is greater than the switching probability, that is $p^U_{\boldsymbol{k}^t} \geq \gamma$, or equivalently $p^D_{\boldsymbol{k}^t} \leq 1 - \gamma$.

2.3 Bundled options

With the single-draw option only, a player can exit the game at the end of any round. In contrast, bundled options function as several consecutive rounds of gambles where the player cannot quit in the middle. In Gacha games, it is common for single draw and consecutive draws to be displayed as two option buttons on the toy vending machine interface. Bundled options can impact the player's planning and optimisation behaviour.

An option of M consecutive gambles is referred to as an M-draw bundle. An N-planning player considers it as an additional possible plan s^M_{Bundle}. The optimisation problem then becomes

$$\max_{s^N \in S^N \cup \{s^M_{Bundle}\}} V(X_{s^N}; w_{\boldsymbol{k}^t}).$$

The only difference from the no-bundle scenario is the set of plans that will be optimised. For a narrow-bracketing player who has $N < M$, the bundled option forces her to consider one specified larger plan. Otherwise, for a broader-bracketing player who has $N \geq M$, she has already planned for longer term and the bundled option does not introduce anything unforeseen. The reason is, the expected outcomes of the bundle are the same as a no-bundle plan $s^N_{M,accept} \in S^N$, where the player always accepts the next round of gamble for the first M rounds and rejects on other nodes. It follows that $X_{s^M_{Bundle}} = X_{s^N_{M,accept}}$ for $N \geq M$. One remaining ambiguity is the case when the bundled option is optimal. I simply assume that the player will always choose the no-bundle equivalent $s^N_{M,accept}$ instead of s^M_{Bundle}. Finally, the optimisation result remains unchanged for the player with $N > M$ after introducing bundled options.

Echoing the previous discussion on computational complexity in Barberis's model, the bundled option in my model only extends the optimisation problem by at most one additional plan. This minor extra effort can always be assumed to be within the player's ability. The impact of this additional plan on the optimisation results is of interest. In the remaining sections, I will focus on how long-term bundles impact the behaviour of players who can only plan for the short term, as well as the behaviour of others.

2.4 Game provider's profit

The cost of providing rewards for game producers is almost negligible since the rewards consist of virtual in-game items. Allowing for the inclusion of other repeated gambles, it is sufficient to require the cost for the provider be strictly lower than the value for the player. Then, for example, my model could also be applied to a promotional event for an industrial producer, where the rewards consist of their own products whose market values exceed their production costs.

Formally, the game provider incurs costs of C_U and C_D to provide winning and losing rewards for providing winning and losing rewards respectively, such that

$$C_D = B_D = 0$$
$$0 \leq C_U < B_U.$$

Use the fair gamble condition $p^U B_U + p^D B_D = a$, the game provider's expected revenue from one round of gamble is positive, that is

$$a - (p^U C_U + p^D C_D) > 0.$$

The total expected revenue is calculated by multiplying the expected revenue per round by the expected number of rounds. Therefore, the game provider has an incentive to encourage players to continue playing in order to increase profits. Additionally, the game provider may offer several free rounds at the start of the game, which will reduce her profit by a fixed amount but will not alter her incentive to encourage players to continue playing.

Note that this model does not apply to casino games where rewards are in real currency; in such cases, the cost of provision always equals the monetary values, and the provider's expected revenue is always zero if games are fair.

3 Analysis on Naive 1-planning Players

The distinguishing feature of a naive 1-planning player, as opposed to other naive N-planning players, is that any plan she devises will be executed in full. This is because she is only responsible for making decisions regarding the current gamble. The player's behaviour is entirely driven by her misperception of probabilities, which provides a clear explanation for the results.

As explained in Section 2.4, it is desirable for the game provider to keep players engaged for as long as possible. However, even a naive player is unbiased at the start of the game and will not accept a fair gamble, because she has prior experience and therefore no source of bias. This raises the question for the game provider: how can naivety be exploited to increase profits? A simple answer is, the game provider can probably "send" the player to the state where she would be willing to accept the next round by offering her several free draws at the beginning to help her build up a sample and thus a source of bias. For simplicity, I assume $p^U_{(0,1)} \geq \gamma$, that is a naive player who loses in the initial free round will accept the next round. It is worth noting that "sending" the player to other state is costly for the game provider in general, so it is feasible only if the costs to offer free rounds could be fully compensated. In Gacha games, it may be reasonable for virtual items to have a zero providing cost. Therefore, to ensure feasibility, I assume zero cost for most of the results. The discussion of providing cost will be presented in the concluding section.

The game provider may wonder if the narrow-bracketing characteristic can be used to increase profits. The 1-planning player gives up the opportunity to improve her payoff by considering larger plans due to the complexity of the optimisation problem. However, she may still assess any specific larger plan presented to them. Then, "forcing" this player to take a more attractive and previously unconsidered plan into consideration may bring her back to the game if she were planning to exit according to the original plan. In order to generate a specific larger plan, the game provider may only be able to bundle several rounds together, as the drawing options are typically not conditional. In this section, I demonstrate that bundling two rounds together increases the likelihood of a naive 1-planning player continuing to play the game, which in turn effectively increases the game provider's overall profit for a typical class of gambles.

3.1 50-50 fair gambles

This is the game setting in Barberis's model of casino gambling. Close connections and key differences can be identified by contrasting the results from my model with his. Without loss of generality, I consider the following repeated gambles:

$$p^U = \frac{1}{2}, h_U = 1,$$
$$p^D = \frac{1}{2}, h_D = -1,$$
$$\gamma = \frac{\lambda}{\lambda + 1}.$$

I start with two lemmas which describe the patterns of acceptance state for next gamble.

Lemma 1 *For a naive* 1*-planning player, if the next round of gamble is accepted on node*

(k_U, k_D), then the node $(k_U, k_D + 1)$ will also be accepted; if the next round of gamble is rejected on node (k_U, k_D), then the node $(k_U, k_D - 1)$ was also rejected.

This lemma is based on the gambler's fallacy, which suggests that a naive player should have a higher expectation of winning in the next round of gamble compared to their current perceived gamble if they were to lose the current one. Therefore, a player should accept the gamble after one more loss if the gamble before losing is already sufficiently appealing, but she should reject the gamble before one more loss if the gamble after losing is still insufficiently appealing.

Lemma 2 *For any given $k_0 \in \mathbb{N}$, there exists a unique $\delta_{k_0} \in \mathbb{N}^+$ such that, a naive 1-planning player on node $(\delta, k_0 + \delta), \delta \in \mathbb{N}$ accepts the next round of gamble for any $\delta < \delta_{k_0}$ while rejects it for any $\delta \geq \delta_{k_0}$, and δ_{k_0} is non-decreasing in k_0.*

This lemma examines the acceptability of nodes in the lower half of the tree where the player has a non-positive net gain. Nodes are categorised based on the player's net gain, determined by k_0. The sequence of nodes on the same iso-net-gain line, i.e., $\delta \in \{0, 1, 2, \cdots\}$ for a given k_0, can always be divided into two parts: all nodes prior to the cut point determined by δ_{k_0} are accepted, and the remaining nodes are rejected. Furthermore, the cut point will not appear in an earlier round when the net loss becomes higher, i.e., non-increasing δ_{k_0}. If these accepted nodes can all be reached during the game, a naive 1-planning player who has lost more would stop later in the game.

The reachability of accepted nodes can actually be proved. Let us first formalise the node reachability using the concept of "valid paths" in Section 2.2 as follows:

1. Reachable: there exists a valid path from $(0, 0)$ to the node, and the node itself is accepted.

2. Stop point: there exists a valid path from $(0, 0)$ to the node, and the node itself is rejected.

3. Unreachable: there is no path from $(0, 0)$ to the node.

Then, the behavioural pattern of the player can be established. For a naive 1-planning player, she must be on either a reachable node or a stop point at any stage of the game. And the player should only exit if she is on a stop point. It is formally stated below:

Proposition 1 *Consider a naive 1-planning player who is to play the game described in this subsection with the single-draw option only.*

1. *In the non-positive-net-gain part of the game tree, every accepted node in Lemma 2 is reachable. Among the remaining nodes, those with a reachable predecessor node are stop points, while the others are unreachable nodes.*

2. *In the positive-net-gain part of the game tree, every node is unreachable except for* $(1, 0)$ *which is a stop point.*

The above proposition demonstrates that the reachable area in the game tree, defined as the smallest area encompassing all reachable nodes and stop points, takes on a stair-step shape (see Figure 1a). This can be viewed as a stronger version of the "gain-exit" outcome documented by Barberis. In addition to finding that the expected length of the game, conditional on exiting with a gain, is less than exiting with a loss, the player never continues to play when she is overall winning. However, she always stays longer if she has lost more. Moreover, a crucial distinction from Barberis's findings is that my results are solely based on probability misperception and one of the properties of prospect theory, namely that losses have a greater impact than gains. Therefore, it will never switch to the "loss-exit" pattern for any reasonable parameter setting in the utility function. In my model, the naive player who has lost more believes that she has been accumulating "good luck" for the future. As a result, she perceives her imaginary lucky streak as long-lasting and is less likely to give up this "opportunity".

Proposition 1 is referred to as the baseline result. The game provider did not consider offering bundles in this game to potentially alter the behavioural patterns of some naive players and increase her profit. The impact of bundled options is revealed in comparison with this baseline result.

The following proposition demonstrates that the naive 1-planning player can be encouraged to continue playing for a longer duration through the use of 2-draw bundles:

Proposition 2 *Let the option of the* 2*-draw bundle available to the naive* 1*-planning player, and refer to the results in Proposition 1 as "previous":*

1. *The player will accept the bundle at some previous stop points in the negative-net-gain part of the game tree. If there is no limit on the maximum number of rounds, there are infinitely many such stop points.*

2. *The player will accept the bundle at every previous reachable node, unless it is not possible to take two more draws on the reachable node.*

The above proposition illustrates the impact of an additional option of the 2-draw bundle on the behaviour of a naive 1-planning player. The reachable area is expanded compared to the no-bundle case (see figure 1b), meaning that the player will be incentivised to continue playing for a longer period of time. It is obvious that the reachable area will never be reduced, because the single-draw option is always available, regardless of the attractiveness of the bundle. The reachable area expands when the accepted bundle's outcome area includes a node that was previously unreachable. The node where the bundled option is accepted must be an adjacent node to a stop point that was previously reachable, or a previous stop point. Intuitively, since the naive 1-planning player underestimates the probability of losing more severely as she is on a losing streak, if she is overall losing now, the worst result is underestimated twice in the bundle but only once in a single draw, while the best result is overestimated twice in the bundle, which misleads her to be more confident about the occurrence of satisfactory results in the bundle than in a single draw. Besides, note that only sufficient conditions are used to prove that the bundle will be accepted on some nodes. The actual reachable area extended by the bundled option may be larger.

3.2 A typical class of gambles

Now, I consider a typical class of repeated gambles as given below:

$$p^U = \frac{1}{m+1}, h_U = m$$
$$p^D = \frac{m}{m+1}, h_D = -1$$
$$\gamma = \frac{\lambda}{\lambda + m^r}$$
$$m \in \mathbb{N}, m \geq 2.$$

It represents the common prize pool in Gacha games where valuable rewards have a low probability of being drawn.

Analogous to Proposition 1, the baseline result for the behaviour pattern of a naive 1-planning player is presented below:

Proposition 3 *Consider a naive* 1*-planning player who is to play the repeated gambles described in this subsection with the single-draw option only.*

1. *In the non-positive-net-gain part of the game tree, every accepted node is reachable. Among the remaining nodes, those with a reachable predecessor node are stop points while the others are unreachable nodes.*

2. *In the positive-net-gain part of the game tree,* $(1, 0)$ *and adjacent nodes to a reachable node are stop points, while the remaining nodes are unreachable.*

The proof follows exactly the same procedure: we first derive patterns of acceptance state, and then demonstrate the reachability of accepted nodes. However, note that there may be stop points other than $(1, 0)$ in the positive-net-gain part of the game tree. This is due to the presence of asymmetric winning and losing payoffs.

Then, analogous to Proposition 2, the naive 1-planning player can be encouraged to continue playing for a longer duration through the use of 2-draw bundles:

Proposition 4 *Let the option of the* 2*-draw bundle available to the naive* 1*-planning player, and refer to the results in Proposition 3 as "previous". Under the assumption that* $\lambda \leq m^{1+r}$ *holdes for the utility function:*

1. *The player will accept the bundle at some previous stop points where the net loss is greater than or equal to* m*. If there is no limit on the maximum number of rounds, there are infinitely many such stop points.*

2. *The player will accept the bundle at every previous reachable node where the net loss is greater than or equal to* m*, unless it is not possible to take two more draws on the reachable node.*

This proposition yields a weaker result than Proposition 2, as it requires additional restrictions. However, given conventional estimations that λ is around 2 and r is not too close to zero, for instance Tversky and Kahneman (1992) estimate $\lambda = 2.25, r = 0.88$, I argue that the results are widely applicable, because the restriction $\lambda \leq m^{1+r}$ applies to a wide range of parameters. For instance, if we reasonably assume $\lambda \leq 3$, then $r > 0.59$ is sufficient for the restriction to hold for any $m \geq 2$; if $m \geq 3$, the restriction holds for any $0 < r < 1$. It is important to note that the requirement for the net loss to be greater than or equal to m can be relaxed to any negative net gain if $\lambda \leq m^r$. This tighter restriction still holds for many reasonable λ and r. For instance, $m \geq 3$ is sufficient for the previously mentioned estimations by Tversky and Kahneman. Again, the proof only uses sufficient conditions, so the actual reachable area extended by the bundled option may be larger.

In summary, the findings and explanations for 50-50 fair gambles are mostly applicable to this typical class of repeated gambles.

3.3 Game providers' profits

The change in the expected profit for the game provider resulting from introducing bundled options is directly derived from the behavioural patterns proposed in previous subsections.

Corollary 1 *In repeated gambles* $\{+m, \frac{1}{m+1}; -1, \frac{m}{m+1}\}, m \in \mathbb{N}^+$*, the introduction of the* 2*-draw bundle in addition to the single-draw option, increases the expected profit of the game provider generated from naive* 1*-planning players.*

The naive 1-planning player's behaviour is affected in a clear way: she will not stop at previous reachable nodes, but may stay longer on previous stop points when the 2-draw bundle is available. Therefore, the expected number of rounds played must increase. Along with the assumption of a positive expected revenue from one draw, the game provider's profit will be higher. These findings indicate that the bundled option is effective in generating higher profits from the most narrowly bracketing naive 1-planning players.

However, from the game provider's perspective, earning more from one group of players does not necessarily increase their overall profit. The provider should consider whether offering a bundled option will have undesirable effects on other players' behaviour. The following result reassures the provider that offering such a bundle is feasible and profitable, as other players who are planning for a longer future have already taken such an option into account.

Corollary 2 *In repeated gambles* $\{+m, \frac{1}{m+1}; -1, \frac{m}{m+1}\}, m \in \mathbb{N}^+$*, the introduction of the* 2*-draw bundle does affect the decisions made by players with a higher order of planning behaviour. In other words, for any* $N \geq 2$*, the* N*-planning player's behaviour is exactly the same in both games with and without.*

The result follows immediately from the player's planning and optimisation process described in Sections 2.2 and 2.3. Therefore, the game provider's expected profit in repeated gambles $\{+m, \frac{1}{m+1}; -1, \frac{m}{m+1}\}, m \in \mathbb{N}^+$ will increase if the option of the 2-draw bundle is available in addition to the single-draw option.

To summarise, the analytical evidence indicates that offering free initial rounds together with bundled options can increase the game provider's profits. The results support the prevalence of these features in the design of toy vending machine mechanics in video games.

3.4 Evidences from numerical analysis

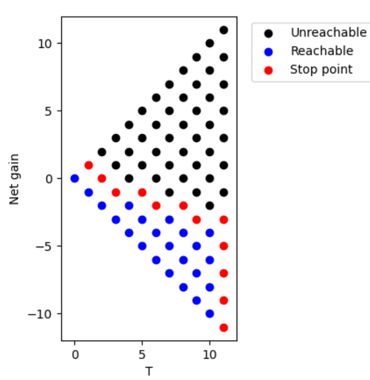
In the previous part of this section, I have explained how two common features in real-world Gacha games, initial free rounds and bundled options, may help the game provider increase its profit. For a better understanding of this mechanism, I conduct a numerical analysis to further support the feasibility and profitability of such a design. The following notations will be used:

Notation	Explanation
NumFree	Number of initial free rounds.
T	Maximum number of rounds that can be played (NumFree excluded).
ExpNB	Expected number of rounds played without bundles (NumFree excluded).
ExpB	Expected number of rounds played with bundles (NumFree excluded).

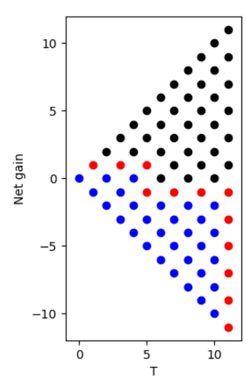
The numerical analysis focuses on the following specification of repeated 50-50 fair gambles:

$$m = 1,$$
$$p^U_{(k_U,k_D)} = \frac{k_D + 1}{k_U + k_D + 2},$$
$$p^D_{(k_U,k_D)} = \frac{k_U + 1}{k_U + k_D + 2},$$
$$NumFree = 1,$$
$$\lambda = 2, r = 0.88.$$

The figures below show the behavioural patterns of a 1-planning player before and after the introduction of 2-draw bundles:



(a) No bundle



(b) 2-draw bundle

Figure 1: Naive 1-planning player's behavioural patterns; $\lambda = 2$, $NumFree = 1$.

The figure below quantifies the effect of 2-draw bundles on the game provider's expected profit from this player, measured by the expected number of rounds played:

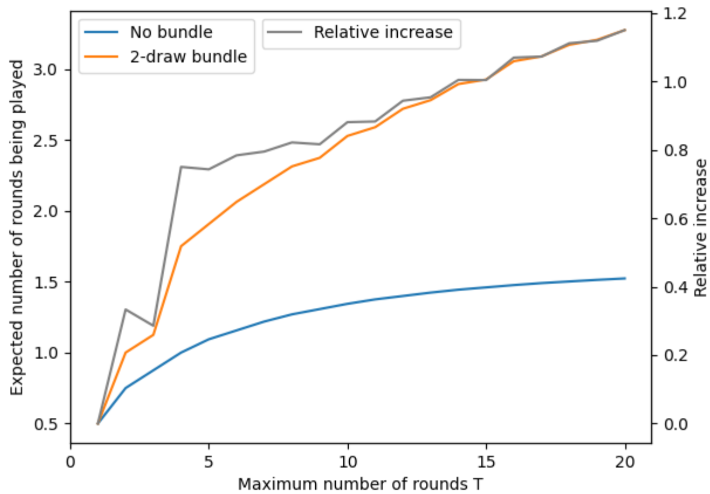


Figure 2: Expected number of rounds being played by a naive 1-planning player with 2-draw bundles; $\lambda = 2$, $NumFree = 1$.

It is evident that the expected number of rounds played by a naive 1-planning player increases logarithmically with or without bundles as the maximum number of rounds increases. The speed of increase is much greater with bundles, resulting in a significant increase in the game provider's profit when 2-draw bundles are offered. A profit increase of over 75% can be achieved if at least 5 rounds are played. Additionally, a relative increase in profit of about 10% can be made each time if 5 more rounds are possible. The relative increase in profit fluctuates but shows an overall upward trend, supporting the behavioural pattern proposed in Proposition 2. The overall upward trend is typically interrupted by a minor downward movement at odd values of T. This happens because, for odd T, the last round must be a single draw if the naive 1-planning player kept taking 2-draw bundles before. Indeed, the naive 1-planning player will keep accepting 2-draw bundles. However, the game restriction prevents the reachable area for the last round from expanding, even if the player would accept the bundle if it were offered. These blocked willing expansions then slightly reduce the relative advantage of the bundled options. However, the overall trend is primarily driven by more expansions that occur an the maximum number of rounds increases.

Based on the analysis of expected profit above, I can discuss the costs of providing rewards. As the cost of the initial free rounds is fixed, and the game provider's expected return from the 1-planning player's later participation in the game increases with the maximum number of rounds, the return will eventually exceed the cost if the game can last longer, under the assumption on lower providing cost than its value. For the numerical setting here, a cost that is less than 50% of the value is enough to make $T = 4$ profitable without bundles.

For $T = 10$, it is profitable even if the cost is 70% as high as the value.

Now, let us examine the impact of the bundle size. The figure below illustrates how the game provider's expected profit from a naive 1-planning player changes with the number of rounds that is bundled in the extra option:

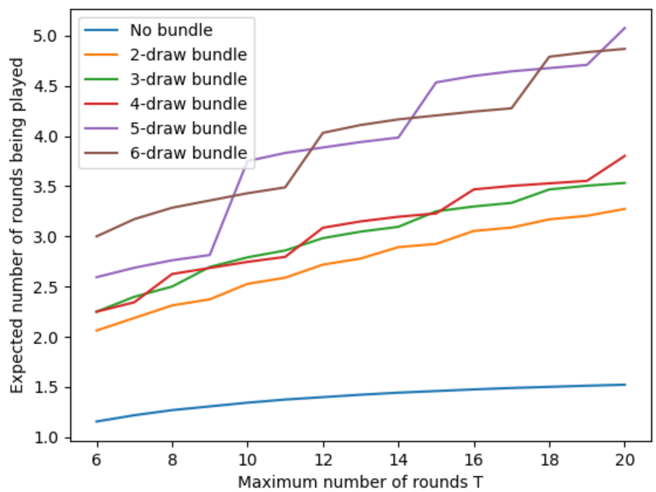


Figure 3: Expected number of rounds being played by a naive 1-planning player with M-draw bundles; $M \in \{2, 3, 4, 5, 6\}$, $\lambda = 2$, $NumFree = 1$.

In general, the number of rounds played by a naive 1-planning player tends to increase when offered larger bundles, such as 2-draw bundles compared to 3-draw or 4-draw bundles, or 3-draw or 4-draw bundles compared to 5-draw or 6-draw bundles. However, the relationship between bundle size and game provider profit is not always clear, as seen in the case of 3-draw versus 4-draw bundles, or 5-draw versus 6-draw bundles. Therefore, increasing the bundle size may not always result in a noticeable improvement in profit. This may explain why real-world Gacha games typically bundle no more than ten rounds together: bundling more rounds together may not significantly increase profits, but it can make the bundle unaffordable for low-wealth players.

4 General Patterns for Naive N-planning Players

The behavioural patterns in the general case of offering M-draw bundles to N-planning players may not be easily described as in previous propositions. However, hierarchical relationships exist among players grouped by different bracketing levels.

The key to the proofs are the following facts derived from the model setting in Section 2.2. Firstly, for any $N \in \mathbb{N}^+$, there exists a unique plan in S^N that results in the naive player choosing to exit the game at current node. This plan is denoted by s^N_{Reject}, because the choice at each node in the plan is to reject. It is evident that $X_{s^N_{Reject}} = (0, 1)$. If a naive player is to follow any plan $s^N \in S^N_{Continue} = S^N \setminus \{s^N_{Reject}\}$, her current decision must be to continue. Secondly, for $N' > N$, any N-round plan s^N can be mapped one-to-one to its unique N'-round equivalent $s^{N'}(s^N)$ by rejecting nodes from round $N+1$ to round N'. In mathematical terms, the following facts hold: (1) $X_{s^{N'}(s^N)} = X_{s^N}$ (2) $s^{N'}(s^N_{Reject}) = s^{N'}_{Reject}$; and (3) $\{s^{N'}(s^N) \mid s^N \in S^N_{Continue}\} \subset S^{N'}_{Continue}$.

I start by further investigating the behavioural patterns outlined in Proposition 1 and 3. Suppose that the game offered effectively induces the naive 1-planning player to participate, i.e., the free initial round is offered and the associated condition is satisfied, then the following relationship holds among naive players with varying levels of narrow bracketing behaviour:

Proposition 5 *In repeated gambles* $\{+m, \frac{1}{m+1}; -1, \frac{m}{m+1}\}, m \in \mathbb{N}^+$*, for any* $N, N' \in \mathbb{N}^+$ *such that* $N' > N$*, the expected number of rounds played by a naive* N'*-planning player is always greater than or equal to that played by a naive* N*-planning player.*

This proposition suggests that a broader-bracketing naive player may behave like a wiseacre. Although she may expect to outperform others by taking a broader view of the future, she is unaware of her naivety. As a result, planning for more rounds only misleads her further.

Then, consider offering an additional option of the general M-draw bundle instead of the 2-draw bundle studied in Proposition 2 and 4. The following relationship holds among naive players with different levels of narrow bracketing behaviour:

Proposition 6 *In repeated gambles* $\{+m, \frac{1}{m+1}; -1, \frac{m}{m+1}\}, m \in \mathbb{N}^+$*, for any* $M, N, N' \in \mathbb{N}^+$ *such that* $N' < N < M$*, if a naive* N*-planning player accepts the* M*-draw bundle at a certain node, then a naive* N'*-planning player also accepts it at the same node.*

Proposition 7 *In repeated gambles* $\{+m, \frac{1}{m+1}; -1, \frac{m}{m+1}\}, m \in \mathbb{N}^+$*, for any* $M, N, N' \in \mathbb{N}^+$ *such that* $N < M$ *and* $N < N'$*, if a naive* N*-planning player rejects the* M*-draw bundle at a certain node, then a naive* N'*-planning player also rejects it at the same node.*

As we already know that planning for more rounds accumulates probability misperception for a naive player, the intuition for such hierarchical structure is quite clear. When comparing it to one's original plan, the more additional misperception brought by the bundle, the more likely it is to be accepted by the naive player. A naive narrow-bracketing player has relatively less misperception than a naive broader-bracketing player before introducing the bundle, so the former is more likely to be affected by the additional bundled option.

Now, let us discuss the implication on the game provider's profit.

According to the characteristics of different players, the Bayesian players are not likely to be misled, so the game provider focuses on exploiting those who believe in the law of small numbers, i.e., the naive players. The probability misperception is triggered by initial free rounds, but later participation is determined by the naive player's personal characteristics. It turns out that those broader-bracketing players are more likely to continue playing, making them a more profitable group.

Note that naivety is a common source of bias. Also, note that narrow-bracketing players only lack the ability to plan for longer periods. Naive players in those less profitable groups are simply not "wise" enough to be wiseacres in those more profitable groups. Then, bundled options are designed to exploit these differences among naive players. If a naive N-planning player finds an M-draw bundle attractive, she can be considered a quasi-M-planning player. This is because her current choice, i.e., the bundle, is a plan with M rounds, although it may not necessarily be optimal for an M-planning player. Therefore, offering bundled options can be viewed as an attempt by the game provider to homogenise the expected profits of naive players, and as a result, the additional improvement in the game provider's profit comes mainly from naive players who merely plan ahead. The corollaries in this section support the idea that these players should be the focus when designing bundled options. It is important to make the bundles attractive to the most narrowly bracketing players, i.e., the naive 1-planning players, otherwise offering bundled options are unlikely to make any profit improvement.

5 Discussion and Conclusion

This paper explains the mechanics of toy vending machines commonly found in mobile games using a behavioural model. In this model, a player with utility based on prospect theory acts according to an optimised plan for a limited number of future rounds in repeated gambles. However, the player may hold a naive belief in the law of small numbers when considering probabilities in future gambles.

I demonstrate that naive players who perceive independent probabilities in a history-dependent way can be induced to play for longer than optimal, resulting in increased profits for the game provider. Two common features in real-world Gacha games — free initial rounds and bundled options, both play important roles in exploiting the naivety of players.

Firstly, offering free initial rounds "entices" naive 1-planning players to start a game that that may ultimately harm them. Those who quit the game later may suffer more. The game provider can exploit the naivety of players by creating a sample through free draws that triggers probability misperception. The costs of providing free draws can be offset by the later game participation of these naive players.

Secondly, bundling two rounds increases the likelihood that naive 1-planning players will continue to play by reinforcing their probability misperception. In addition, this design has the benefit of not affecting the behaviour of other type of players, resulting in an overall increase in profit.

Numerical examples further support these findings, providing quantitative results on the feasibility and profitability of this mechanic. The increasing expected profit in the maximum number of rounds suggests that offering free rounds is likely to be more profitable than its cost. Furthermore, the game provider's profit can be significantly increased by introducing bundled options.

Finally, I demonstrate that some behavioural patterns for general naive M-planning players can be derived from the results for naive 1-planning players. There exists a hierarchical structure of the effects of free initial rounds and bundled options, which is determined by the time range M bracketed by the naive player when planning. Free initial rounds trigger the probability misperception, but naive broader-bracketing players are more likely to continue playing, thereby generating higher profits for the game provider. Bundled options increase the game provider's profits by homogenising naive players and broadening their bracketing range, resulting in higher profits from previously less profitable groups of players.

However, there is still much work to be done in studying mechanisms such as the one used in Gacha games. While playing Gacha games is often equated with gambling, few models have been presented by researchers from this perspective. However, no empirical evidence has been provided to support any of these analytical results. Conducting laboratory experiments or analysing real-world player data in Gacha games will help to examine the theory. Furthermore, this mechanism may be motivated by various behavioural features, such as loss aversion, probability misperception, and narrow bracketing. It will be beneficial to investigate the impact of each of these features in more detail.

References

Barberis, N. (2012). A model of casino gambling. *Management Science*, 58(1):35–51.

Chen, C. and Fang, Z. (2023). Gacha game analysis and design. *Proceedings of the ACM on Measurement and Analysis of Computing Systems*, 7(1):1–45.

Clotfelter, C. T. and Cook, P. J. (1993). The "gambler's fallacy" in lottery play. *Management Science*, 39(12):1521–1525.

Ebert, S. and Strack, P. (2015). Until the bitter end: On prospect theory in a dynamic context. *American Economic Review*, 105(4):1618–33.

Gan, T. (2022). Gacha game: When prospect theory meets optimal pricing. *arXiv preprint arXiv:2208.03602*.

Gneezy, U. and Potters, J. (1997). An experiment on risk taking and evaluation periods. *The quarterly journal of economics*, 112(2):631–645.

Gourville, J. T. (1998). Pennies-a-day: The effect of temporal reframing on transaction evaluation. *Journal of Consumer Research*, 24(4):395–408.

Kahneman, D. and Tversky, A. (1979). Prospect theory: An analysis of decision under risk. *Econometrica*, 47(2):263–292.

Rabin, M. (2002). Inference by believers in the law of small numbers. *The Quarterly Journal of Economics*, 117(3):775–816.

Simonson, I. (1990). The effect of purchase quantity and timing on variety-seeking behavior. *Journal of Marketing research*, 27(2):150–162.

Terrell, D. (1994). A test of the gambler's fallacy: Evidence from pari-mutuel games. *Journal of risk and uncertainty*, 8(3):309–317.

Tversky, A. and Kahneman, D. (1971). Belief in the law of small numbers. *Psychological bulletin*, 76(2):105.

Tversky, A. and Kahneman, D. (1981). The framing of decisions and the psychology of choice. *science*, 211(4481):453–458.

Tversky, A. and Kahneman, D. (1992). Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and uncertainty*, 5(4):297–323.

Appendix

A Omitted proofs

Proof of Lemma 1. By monotonicity in property 2, a losing streak across $(k_U k_D - 1)$ implies that

$$p^D_{(k_U,k_D-1)} > p^D_{(k_U,k_D)} > p^D_{(k_U,k_D+1)}.$$

If (k_U, k_D) is accepted, that is

$$p^D_{(k_U,k_D)} \le \frac{1}{\lambda+1},$$

then we have

$$p^D_{(k_U,k_D+1)} < p^D_{(k_U,k_D)} \le \frac{1}{\lambda+1}$$

which means $(k_U, k_D + 1)$ will also be accepted. Similarly, if (k_U, k_D) is rejected, then we have

$$p^D_{(k_U,k_D-1)} > p^D_{(k_U,k_D)} > \frac{1}{\lambda+1}$$

which means $(k_U, k_D + 1)$ will also be rejected. $\square$

Proof of Lemma 2. For $k_0 = 0$, given that $h_U = -h_D$, by property 1 we know that every node $(\delta, 0+\delta), \delta \in \mathbb{N}^+$ will be rejected because

$$p^D_{(\delta,\delta)} = \frac{1}{2} < \frac{1}{\lambda+1}.$$

But $(0, 0)$ will be accepted by the assumption on initial free round. So, we know that $\delta_0 = 1$.

For $k_0 \in \mathbb{N}^+$, firstly the assumption on perceived probabilities after the initial free round indicates that

$$p^D_{(0,1)} \le \frac{1}{\lambda+1},$$

then by monotonicity in property 2, for any $k_0 \in \mathbb{N}^+$, we have

$$p^D_{(0,k_0)} \le p^D_{(0,1)} \le \frac{1}{\lambda+1} < \frac{1}{2}.$$

Secondly, decomposing $(\delta, k_0 + \delta)$ to $(0, k_0) + (\delta, \delta)$, it is easy to verify $0 \cdot h_U + k_0 h_D \neq 0$ and $\delta h_U + \delta h_D = 0$, then using the result from property 1 that

$$p^D_{(\delta,k_0+\delta)} < p^D, \delta \in \mathbb{N},$$

we can derive from property 3 that $p^D_{(\delta,k_0+\delta)}$ is increasing in δ and

$$\lim_{\delta\to\infty} p^D_{(\delta,k_0+\delta)} = p^D = \frac{1}{2}.$$

Thus, there exists a unique $\delta_{k_0} \in \mathbb{N}$ for given k_0 such that

$$\begin{aligned}p^D_{(0,k_0)} < p^D_{(1,k_0+1)} < \cdots \\ < p^D_{(\delta_{k_0}-1,k_0+\delta_{k_0}-1)} \leq \frac{1}{\lambda+1} < p^D_{(\delta_{k_0},k_0+\delta_{k_0})} \\ < p^D_{(\delta_{k_0}+1,k_0+\delta_{k_0}+1)} < \cdots\end{aligned}$$

which means the naive 1-planning player will accept node $(\delta, k_0+\delta), \delta \in \mathbb{N}$ for any $\delta < \delta_{k_0}$ while reject it for any $\delta \geq \delta_{k_0}$.

Now let $k_0^{'} = k_0 + 1$, by definition of $\delta_{k_0^{'}}$, we know that $(\delta_{k_0^{'}}, k_0^{'} + \delta_{k_0^{'}})$ is rejected. Then by lemma 1, $(\delta_{k_0^{'}}, k_0^{'} + \delta_{k_0^{'}} - 1)$ will also be rejected. Note that this node can be written as $(\delta_{k_0^{'}}, k_0 + \delta_{k_0^{'}})$, suppose $\delta_{k_0^{'}} < \delta_{k_0}$, by definition of δ_{k_0}, this node must be accepted, a contradiction. So, $\delta_{k_0+1} \geq \delta_{k_0}$ always holds, or equivalently, δ_{k_0} is non-decreasing in k_0. $\square$

Proof of Proposition 1. We know that a node (k_U, k_D) has two possible predecessor nodes (k_U-1, k_D) (does not exist if $k_U = 0$) and (k_U, k_D-1) (does not exist if $k_D = 0$). We also know that for $h_U = -h_D$, the player's net gain on node (k_U, k_D) is positive (zero/negative) if and only if k_U is greater than (equal to/less than) k_D.

Suppose the player is on a node (k_U, k_D) with positive net gain, that is $k_U > k_D$. It is obvious that $(1, 0)$ can be reached after initial free round. For any other node, there exist $n_0 \in \mathbb{N}^+$ such that

$$k_U - n_0 = k_D$$

and $n_1 \in \mathbb{N}^+$ such that

$$k_U = k_D + n_1 - 1.$$

Since the player is break-even on both $(k_U - n_0, k_D)$ and $(k_U, k_D + n_1 - 1)$, by property 1, we have

$$p^U_{(k_U-n_0,k_D)} = \frac{1}{2}$$
$$p^D_{(k_U,k_D+n_1-1)} = \frac{1}{2}.$$

Then by monotonicity in property 2, we have

$$p^U_{(k_U-1,k_D)} \le p^U_{(k_U-n_0,k_D)} = \frac{1}{2} < \frac{\lambda}{\lambda+1}$$
$$p^D_{(k_U,k_D-1)} > p^D_{(k_U,k_D+n_1-1)} = \frac{1}{2} > \frac{1}{\lambda+1}.$$

Therefore, the player would have rejected either possibile predecessor node (if it exists) and is never able to reach (k_U, k_D).

The restriction on maximum number of rounds T only forces the player to stop on some otherwise accepted nodes after the last round of gamble. So, we only need to derive results for infinitely many rounds, and the restriction is implemented by changing those reachable nodes representing the player's state at the beginning of round $T+1$ to stop points.

Suppose the player is on a node (k_U, k_D) with non-positive net gain, that is $k_U \le k_D$. This node can be represented in another form $(\delta, k_0+\delta)$ by setting $k_0 = k_D - k_U$ and $\delta = k_U$, where $k_0 \in \mathbb{N}$ and $\delta \in \mathbb{N}$. We first prove that every accpeted node in lemma 2 is actually reachable. For any accepted node $(\delta, k_0+\delta)$, consider the following path

$$(0,0) \to (0,1) \to (0,2) \to \cdots \to (0,k_0)$$
$$(0,k_0) \to (0,k_0+1) \to (1,k_0+1)$$
$$(1,k_0+1) \to (1,k_0+2) \to (2,k_0+2)$$
$$\cdots$$
$$(\delta-1,k_0+\delta-1) \to (\delta,k_0+\delta-1) \to (\delta,k_0+\delta)$$

By the assumption on initial free round and related perceived probabilities, $(0,0)$ and $(0,1)$ are accepted. Other nodes in this path will also be accpeted by lemma 1 and 2. So, this is a valid path, and by definition this accepted node $(\delta, k_0+\delta)$ is reachable. Secondly, for a rejected node with a reachable predecessor node, appending it to the end of the above path for its reachable predecessor still produces a valid path, then by definition it is a reachable stop point. Unreachable nods are therefore those rejected nodes with no reachable predecessor node. □

Proof of Proposition 2. Again we prove the unrestricted case and take the restriction on maximum number of rounds into consideration when necessary.

We first claim that there exist infinitely many previous stop points (k_U, k_D) whose successor node (k_U, k_D+1) is accepted and

$$p^U_{(k_U,k_D)} > \frac{1}{2} + \frac{1}{2}\left(\frac{\lambda}{\lambda+1} - \frac{1}{2}\right).$$

This is because, firstly, by property 1, and also by monotonicity and convergency in property 2, for any $k_U \in \mathbb{N}^+$, there exists an interger $k_D > k_U$ such that

$$p^D_{(k_U,0)} > \cdots > p^D_{(k_U,k_U)} = \frac{1}{2} > \cdots > p^D_{(k_U,k_D)} > \frac{1}{\lambda+1} \geq p^D_{(k_U,k_D+1)},$$

so (k_U, k_D+1) is accepted and (k_U, k_D) is rejected, and (k_U-1, k_D)—which is a predecessor node of (k_U, k_D)—is of the same net gain as (k_U, k_D+1), then by lemma 2, (k_U, k_D) is a stop point whose successor node (k_U, k_D+1) is accepted. Secondly, suppose on this stop point (k_U, k_D) we have

$$p^U_{(k_U,k_D)} \leq \frac{1}{2} + \frac{1}{2}\left(\frac{\lambda}{\lambda+1} - \frac{1}{2}\right),$$

by convexity in property 3, we have

$$p^U_{(k_U,k_D+1)} - p^U_{(k_U,k_D)} < p^U_{(k_U,k_D)} - p^U_{(k_U,k_D-1)},$$

notice that $k_D > k_U$ implies $k_D - 1 \geq k_U$, which means

$$p^U_{(k_U,k_D-1)} \leq \frac{1}{2},$$

then we can compute that

$$p^U_{(k_U,k_D+1)} < 2p^U_{(k_U,k_D)} - p^U_{(k_U,k_D-1)} \leq \frac{\lambda}{\lambda+1}$$

which means (k_U, k_D+1) is rejected, a contradiction and we must have

$$p^U_{(k_U,k_D)} > \frac{1}{2} + \frac{1}{2}\left(\frac{\lambda}{\lambda+1} - \frac{1}{2}\right).$$

Finally, we can find such a (k_U, k_D) for every $k_U \in \mathbb{N}^+$.

Then consider a naive 1-planning player on any stop point (k_U, k_D) with above properties. We know that (k_U, k_D+1) is accepted, that is

$$p^D_{(k_U,k_D+1)} \leq \frac{1}{\lambda+1}.$$

We also know that $k_D - 1 \geq k_U$, so (k_U+1, k_D) is of non-positive net gain, by property 1, we have

$$p^U_{(k_U+1,k_D)} \geq \frac{1}{2}.$$

Thus, note that $u(2) > 0$ and $u(-2) < 0$, this player's value V for a 2-draw bundle on

(k_U, k_D) satisfies

$$\begin{aligned}
\frac{V}{\alpha(w_{(k_U,k_D)})} &= p^U_{(k_U,k_D)} p^U_{(k_U+1,k_D)} u(2) + p^D_{(k_U,k_D)} p^D_{(k_U,k_D+1)} u(-2) \\
&\geq \left[\frac{1}{2} + \frac{1}{2}\left(\frac{\lambda}{\lambda+1} - \frac{1}{2}\right)\right] \cdot \frac{1}{2} \cdot u(2) + \left[\frac{1}{2} - \frac{1}{2}\left(\frac{\lambda}{\lambda+1} - \frac{1}{2}\right)\right] \cdot \frac{1}{\lambda+1} \cdot u(-2) \\
&= \frac{3\lambda+1}{4(\lambda+1)} \cdot \frac{1}{2} \cdot u(2) + \frac{\lambda+3}{4(\lambda+1)} \cdot \frac{1}{\lambda+1} \cdot u(-2).
\end{aligned}$$

We can compute that the coefficient of $u(2)$ is always more than λ times as large as the coefficent of $u(-2)$, that is

$$\frac{\frac{3\lambda+1}{4(\lambda+1)} \cdot \frac{1}{2}}{\frac{\lambda+3}{4(\lambda+1)} \cdot \frac{1}{\lambda+1}} - \lambda = \frac{(\lambda-1)^2}{2(\lambda+3)} > 0$$

always holds for any $\lambda > 1$. By the assumption on the utility function we have $\lambda u(2) + u(-2) = 0$, therefore

$$\frac{V}{\alpha(w_{(k_U,k_D)})} \geq 0$$

or simply $V > 0$, that is a 2 draw bundle will be accepted on (k_U, k_D).

The acceptance of a 2-draw bundle on a previously reachable node (k_U, k_D) is obvious from the fact that

$$p^U_{(k_U,k_D)} \geq \frac{\lambda}{\lambda+1} > \frac{1}{2} + \frac{1}{2}\left(\frac{\lambda}{\lambda+1} - \frac{1}{2}\right).$$

If there is a restriction on the maximum number of rounds that can be played, it is clear that the 2-draw bundle is available as long as it has not come to the last possible round. □

Proof of Proposition 4. Analogous to the proof of Proposition 2, for each $k_U \in \mathbb{N}^+$, we can find a previous stop point (k_U, k_D) in the non-potisitive-net-gain part of the tree whose predecessor node $(k_U, k_D + 1)$ is accepted and

$$p^U_{(k_U,k_D)} > p^U + \frac{1}{2}\left(\frac{\lambda}{\lambda+m^r} - p^U\right)$$

Additionally, we only consider those nodes with sufficiently large net loss (greater than or euqal to m) such that $(k_U + 1, k_D)$ is still in the non-positive-net-gain part of the game tree. This is true for all k_U greater than some positive integer beacuse previouly reachable nodes form a stair-step shape, so there are still infinitely many such stop points in unrestricted case.

Then consider a naive 1-planning player on any of the above (k_U, k_D), her value for a

2-draw bundle is

$$\begin{aligned}\frac{V}{\alpha(w_{(k_U,k_D)})} &= p^U_{(k_U,k_D)}p^U_{(k_U+1,k_D)}u(2m) \\ &+ p^U_{(k_U,k_D)}p^D_{(k_U+1,k_D)}u(m-1) + p^D_{(k_U,k_D)}p^U_{(k_U,k_D+1)}u(m-1) \\ &+ p^D_{(k_U,k_D)}p^D_{(k_U,k_D+1)}u(-2).\end{aligned}$$

By property 1, since (k_U+1,k_D) is of non-positive net gain, we have $p^U_{(k_U+1,k_D)} \geq p^U$ so that

$$p^U_{(k_U,k_D)}p^U_{(k_U+1,k_D)}u(2m) \geq p^U_{(k_U,k_D)}p^U(2m)^r.$$

By monotonicity in property 2, we have

$$p^U_{(k_U,k_D)}p^D_{(k_U+1,k_D)}u(m-1) > p^U_{(k_U,k_D)}p^D_{(k_U,k_D)}(m-1)^r$$

and

$$p^D_{(k_U,k_D)}p^U_{(k_U,k_D+1)}u(m-1) > p^D_{(k_U,k_D)}p^U_{(k_U,k_D)}(m-1)^r.$$

Accepting next round on (k_U,k_D+1) indicates $p^U_{(k_U,k_D+1)}U(m) + p^D_{(k_U,k_D+1)}U(-1) \geq U(0)$, this gives

$$\lambda \leq \frac{p^U_{(k_U,k_D+1)}}{p^D_{(k_U,k_D+1)}}m^r,$$

so we have

$$\begin{aligned}p^D_{(k_U,k_D)}p^D_{(k_U,k_D+1)}u(-2) &= -\lambda p^D_{(k_U,k_D)}p^D_{(k_U,k_D+1)}2^r \\ &\geq -p^D_{(k_U,k_D)}p^U_{(k_U,k_D+1)}(2m)^r \\ &> -p^D_{(k_U,k_D)}p^U_{(k_U,k_D)}(2m)^r\end{aligned}$$

Combining these ineqalities, we have

$$\begin{aligned}\frac{V}{\alpha(w_{(k_U,k_D)})} &> p^U_{(k_U,k_D)} \cdot \left[p^U(2m)^r + 2p^D_{(k_U,k_D)}(m-1)^r - p^D_{(k_U,k_D)}(2m)^r\right] \\ &= p^U_{(k_U,k_D)} \cdot (p^D_{(k_U,k_D)} - p^U)(2m)^r \cdot \left[\frac{2p^D_{(k_U,k_D)}}{p^D_{(k_U,k_D)} - p^U}(\frac{m-1}{2m})^r - 1\right] \\ &= p^U_{(k_U,k_D)} \cdot (p^D_{(k_U,k_D)} - p^U)(2m)^r \cdot \left[\frac{(1-\frac{1}{m})^r}{1-\frac{p^U}{p^D_{(k_U,k_D)}}}2^{1-r} - 1\right]\end{aligned}$$

We know that $0 < r < 1$ and $m \geq 2$, then

$$(1 - \frac{1}{m})^r > 1 - \frac{1}{m}.$$

We also know that (k_U, k_D) is a stop point and

$$p^U_{(k_U,k_D)} > p^U + \frac{1}{2}\left(\frac{\lambda}{\lambda + m^r} - p^U\right),$$

the rejection of this node together with the additional restriction $\lambda \leq m^{1+r}$ indicate that

$$p^D_{(k_U,k_D)} > \frac{m^r}{\lambda + m^r} = \frac{1}{\frac{\lambda}{m^r} + 1} \geq \frac{1}{m+1} = p^U,$$

while the inequality is equivalent to

$$p^D_{(k_U,k_D)} < p^D - \frac{1}{2}\left(\frac{\lambda}{\lambda + m^r} - p^U\right),$$

which implies that

$$\begin{aligned}
1 - \frac{p^U}{p^D_{(k_U,k_D)}} &< 1 - \frac{p^U}{p^D - \frac{1}{2}\left(\frac{\lambda}{\lambda+m^r} - p^U\right)} \\
&= 1 - \frac{1}{m - \frac{m+1}{2}\left(\frac{\lambda}{\lambda+m^r} - \frac{1}{m+1}\right)} \\
&< 1 - \frac{1}{m - \frac{m+1}{2}\left(1 - \frac{1}{m+1}\right)} \\
&< 1 - \frac{1}{m}.
\end{aligned}$$

So, we know that

$$\frac{(1 - \frac{1}{m})^r}{1 - \frac{p^U}{p^D_{(k_U,k_D)}}} > 1,$$

and obviously $2^{1-r} > 1$, then

$$\frac{(1 - \frac{1}{m})^r}{1 - \frac{p^U}{p^D_{(k_U,k_D)}}} 2^{1-r} - 1 > 0.$$

Therefore, we finally conclude that

$$\frac{V}{\alpha(w_{(k_U,k_D)})} > 0$$

or simply $V > 0$, that is a 2-draw bundle will be accepted on (k_U, k_D).

If the net loss on a previously reachable node (k_U, k_D) is greater than or equal to m, the properties for above found stop points are all satisfied from the fact that

$$p^U_{(k_U,k_D)} \geq \frac{\lambda}{\lambda + m^r} > p^U + \frac{1}{2}\left(\frac{\lambda}{\lambda + m^r} - p^U\right),$$

then the acceptance of the 2-draw bundle is obvious. □

Proof of Proposition 5. This can be proved by induction.

At any reachable node for a naive 1-planning player, it is obvious that $s^1_{Optimal} \in S^1_{Continue}$. Then, $s^2(s^1_{Optimal}) \in S^2_{Continue}$, and it must be better than s^2_{Reject} because $V(X_{s^2(s^1_{Optimal})}) = V(X_{s^1_{Optimal}}) > V(X_{s^1_{Reject}}) = V(X_{s^2_{Reject}})$. Therefore, $s^2_{Optimal} \in S^2_{Continue}$, which means that every reachable node for the naive 1-planning player is still reachable for a naive 2-planning player.

However, at any stop point for a naive 1-planning player, although s^2_{Reject} is better than any $s^2 \in \{s^2(s^1) \mid s^1 \in S^1_{Continue}\}$, it is still possible that $s^2_{Optimal} \in S^2_{Continue} \setminus \{s^2(s^1) \mid s^1 \in S^1_{Continue}\}$. Therefore, the stop point for the naive 1-planning player is either a reachable node or a stop point for a naive 2-planning player.

Finally, the reachable area for a naive 2-planning player cannot be smaller than that for a naive 1-planning player, and she is expected to stay in the game for at least the same expected length.

Similarly, the node reachability for naive $(N + 1)$-planning players can be derived from that for naive N-planning players. Reachable nodes will always remain reachable, but stop points may also be reached. Therefore, the expected number of rounds played cannot be reduced. □

Proof of Proposition 6 and 7. It simply comes from the fact that $\{s^{N'}(s^N) \mid s^N \in S^N\} \in S^{N'}$ for any $N' > N$, which means a broader-bracketing player will consider all possible plans that have been considered by a narrow-bracketing player.

If the M-draw bundle is accepted by a naive N-planning player, then $V(X_{s^M_{Bundle}}) > V(X_{s^N(s^{N'})}) = V(X_{s^{N'}})$ for any $s^{N'} \in S^{N'}$ such that $N' < N$, which means the bundle is also optimal for a naive N'-planning player.

If the M-draw bundle is rejected by a naive N-planning player, then $s^N_{Optimal} \in S^N$, which implies $s^{N'}(s^N_{Optimal}) \in S^{N'}$ for any $N' > N$. Then, we have $V(X_{s^{N'}_{Optimal}}) \geq V(X_{s^{N'}(s^N_{Optimal})}) = V(X_{s^N_{Optimal}}) \geq V(X_{s^M_{Bundle}})$, which means the optimal plan for a naive N'-planning player cannot be the bundle. Note that the equal sign could hold because we have assumed that the bundled option is suboptimal to its single-draw equivalent. □