Structural Note on the Variational Section

This model intentionally adopts a quasi-solution $\phi(r)$, rather than an exact solution. The reason is that an exact solution would eliminate the physical significance of the characteristic scale $r = R$, thereby preventing the reproduction of the essential behavior: scale-responsive structure of space.

If an exact solution were enforced, the scale $r = R$ would likely be absorbed into coordinate redefinitions or symmetry-induced degeneracies. In such a case, R would lose its status as a physically anchored scale and instead become a redundant parameter, thereby obscuring any meaningful scale-dependent structure in the resulting dynamics.

Being a quasi-solution is not a weakness but the structural foundation of the model—it is the key to realizing scale-dependent spatial dynamics.

To verify the validity of this quasi-solution, we conducted a variational residual analysis. The results show that the deviation is well-controlled, localized, and physically meaningful. This behavior is also visually illustrated in Fig. 1.

Please do not mistake the use of a quasi-solution for a flaw or a compromise. It is not a shortcoming, but the core of the theory and the very foundation of its structure.

Importantly, the quasi-solution is not adopted for approximation, but for its structural role in deriving the effective cosmological term $\Lambda_{\text{eff}}(r)$.

It does not imply that the entire structure of the universe or the fundamental field equations themselves are quasi-solutions. The aim of this study is to examine whether a quasi-solution term such as $\Lambda_{\text{eff}}(r)$ can function effectively within a universe that is otherwise governed by exact solutions.

For clarity, the term "logarithmic-type" used in this paper does not refer to a strict logarithmic function itself, but rather to a qualitative feature indicating that the functional form exhibits a logarithmic behavior.

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Repulsive Gravity Theory
Based on Scalar Fields and the Effective Λ

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Abstract

In this study, we propose an extension of the Einstein field equations by introducing a repulsive gravity framework based on the scale-responsive geometric repulsion effect of space, derived from a scalar field. This effect is induced by the geometric configuration of the scalar field, from which a scale-dependent effective cosmological term, Λ_{eff}, emerges. The impact of Λ_{eff} depends on spatial scale, potentially redefining the structural properties of gravity, especially at galactic scales and within black hole (BH) interiors. We demonstrate that two key observational and theoretical challenges—the flattening of galactic rotation curves in their outer regions and the avoidance of black hole singularities—can be addressed in a unified and quantitative manner through the common scalar-field-derived term Λ_{eff}.

1 Overview: Galactic Rotation and BH Singularities

Recent astronomical observations have revealed that the rotational velocities of stars in the outer regions of galaxies significantly exceed those predicted by general relativity, even when the observed amount of baryonic matter is taken into account. This discrepancy is known as the *galactic rotation curve problem* and has long been considered difficult to explain without invoking the existence of dark matter [1, 2].

On the other hand, gravitational collapse inside black holes is known, in the classical framework, to lead to a singularity—i.e., a region of infinite curvature—in the limit $r \to 0$. This phenomenon is considered to signal the breakdown or limitation of general relativity itself [3]. The so-called *singularity problem* strongly suggests the necessity of quantum corrections or structural modifications within gravitational theory.

2 Successes and Limitations of the ΛCDM Model

The ΛCDM model, which serves as the standard framework in modern cosmology, has achieved remarkable success in explaining observational data such as the cosmic microwave background (CMB), baryon acoustic oscillations (BAO), and Type Ia supernovae with high precision [4]. However, the cosmological constant Λ is defined as a static and uniform quantity, without incorporating any spatial structure or scale dependence.

Moreover, although ΛCDM relies on invisible components such as dark matter and dark energy, the true nature of these entities remains unknown. They function as auxiliary terms in the theoretical framework, and there has been persistent criticism that they lack a solid basis for physical reality [5].

3 Motivation: Repulsion as the Subject

This study attempts to construct a gravitational theory in which **repulsive response is the subject**, in contrast to general relativity, which treats "gravity as attraction (a manifestation of spacetime curvature)" as its primary perspective. Specifically, we propose a framework that introduces an effective cosmological constant term, $\Lambda_{\text{eff}}(r)$, derived from a scalar field. This enables a unified description of gravitational structures ranging from galactic scales to black hole singularity scales.

It should be noted that this study adopts the stance of preserving the cosmological constant Λ in the Einstein field equations as a true constant. However, its effect manifests in a scale-dependent manner through the spatial configuration of the scalar field $\phi(r)$—exhibiting a logarithmic-type behavior—and is reflected in the observable gravitational structure as $\Lambda_{\text{eff}}(r)$. In this way, the aim is to construct a gravitational theory that retains the philosophical foundation of the ΛCDM model while overcoming its structural limitations.

4 The Scalar Field and the Effective Λ

In this section, we demonstrate that the geometric effective Λ term

$$\Lambda_{\text{eff}}(r) = \frac{4r^2}{(R^2 + r^2)^2}$$

is structurally derived from the variational structure of a spherically symmetric and static scalar field configuration.

Here, R represents the characteristic length scale of each physical regime, while both R and r are treated as dimensionless quantities until calibrated against observations.

We begin with the standard Lorentz-invariant Lagrangian density for a real scalar field:

$$\mathcal{L}_\phi = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - V(\phi),$$

where $V(\phi)$ is the scalar potential.

Here, to ensure the spatial isotropy of the geometric effective term $\Lambda_{\text{eff}}(r)$ and to preserve the characteristic scale R, we restrict the configuration to be spherically symmetric and static, $\phi = \phi(r)$. If anisotropic components were included, the scale R would be absorbed into coordinate redefinitions and would thereby lose its physical meaning.

Thus, the Lagrangian takes the following form:

$$\mathcal{L}_\phi = -\frac{1}{2} \left(\frac{d\phi}{dr} \right)^2 - V(\phi(r)).$$

This formulation does not imply a violation of the Lorentz invariance of the standard Lagrangian structure.

In this context, the isotropic structure of the geometric repulsive tensor—constructed to eliminate spatial anisotropy—is defined as:

$$T_{\mu\nu}^{(\Lambda)} \equiv \Lambda_{\text{eff}}(r) \, g_{\mu\nu}.$$

The scalar field that generates and reproduces this geometric repulsive structure is

$$\phi(r) = \frac{2r}{R^2 + r^2},$$

serving as a *shape function* that characterizes the spatial profile of the repulsive response. Using this expression for $\phi(r)$, the corresponding potential $V(\phi(r))$ expands as follows:

$$V(\phi(r)) = -\frac{1}{4} \cdot \phi^2 \left(4 - R^2 \phi^2 \right) = -\frac{1}{4} \cdot \frac{4r^2}{(R^2 + r^2)^2} \left(4 - \frac{4R^2 r^2}{(R^2 + r^2)^2} \right),$$

$$= \underbrace{-\frac{4r^2}{(R^2 + r^2)^2}}_{\text{Leading term}} + \underbrace{\frac{4R^2 r^4}{(R^2 + r^2)^4}}_{\text{Second-order correction term}}.$$

Here, the second term serves as a correction that slightly modifies the primary structure, but the leading term clearly satisfies the relation:

$$\boxed{V(\phi(r)) \propto -\Lambda_{\text{eff}}(r), \qquad \Lambda_{\text{eff}}(r) \equiv \frac{4r^2}{(R^2 + r^2)^2}.}$$

In particular, in the regime $r \ll R$, the correction term is significantly smaller than the leading term. In the opposite limit $r \gg R$, their asymptotic behaviors are

$$\underbrace{\frac{r^2}{(R^2 + r^2)^2}}_{\text{Leading term}} \sim \frac{1}{r^2}, \qquad \underbrace{\frac{r^4}{(R^2 + r^2)^4}}_{\text{Second-order correction term}} \sim \frac{1}{r^4} \quad,$$

indicating that the contribution from the correction term decays rapidly.

Furthermore, even at the intermediate scale $r = R$, we find:

$$\underbrace{\frac{4R^2}{(2R^2)^2} = \frac{1}{R^2}}_{\text{Leading term}}, \qquad \underbrace{\frac{4R^2 R^4}{(2R^2)^4} = \frac{1}{4R^2}}_{\text{Second-order correction term}} \quad.$$

showing that the correction term remains only one-fourth of the leading term.

Therefore, across all spatial scales, the leading term dominates, and it is demonstrated that $\Lambda_{\text{eff}}(r)$ can be directly and naturally derived from the scalar field potential $V(\phi(r))$.

5 Variational Evaluation of the Scalar Field

In this section, we examine whether the Lagrangian of the scalar field $\phi(r)$—which gives rise to the geometric effective cosmological term $\Lambda_{\text{eff}}(r)$ derived in Section 4—satisfies the Euler–Lagrange extremum condition. In particular, we expect that the extremum condition is locally violated around the characteristic spatial scale $r = R$, and that this violation rapidly diminishes for $r \gg R$. Such behavior is naturally anticipated from the structure of the adopted $\phi(r)$, as it corresponds both to the notion that cosmic acceleration may be closely related to a local violation of the extremum condition and serves as a confirmation that the characteristic scale R is not lost within the structure.

Note that the violation of the extremum condition discussed here does not directly imply a violation of the conservation law $\nabla^\mu T_{\mu\nu} = 0$.

From Section 4, the scalar field and its corresponding potential take the form

$$\phi(r) = \frac{2r}{R^2 + r^2}, \qquad V(\phi(r)) = -\frac{1}{4} \cdot \phi^2 \left(4 - R^2 \phi^2\right).$$

For a spherically symmetric configuration, the Lagrangian density is given by:

$$\mathcal{L}_\phi = -\frac{1}{2} \left(\frac{d\phi}{dr}\right)^2 - V(\phi(r)) = -\frac{1}{2} \left(\frac{d\phi}{dr}\right)^2 + \frac{1}{4} \cdot \phi^2 \left(4 - R^2 \phi^2\right).$$

Thus, the Euler–Lagrange equation derived from this Lagrangian is

$$\frac{d^2\phi}{dr^2} = R^2\phi^3 - 2\phi.$$

Therefore, the deviation from the extremum condition (the variational residual) is defined as follows:

$$\delta(r) \equiv \frac{d^2\phi}{dr^2} - \left(R^2\phi^3 - 2\phi\right).$$

By substituting the adopted scalar field $\phi(r)$ into this expression, the variational residual is explicitly given by:

$$\delta(r) = \left[\frac{4r(r^2 - 3R^2)}{(R^2 + r^2)^3}\right] - \left[\frac{8R^2r^3}{(R^2 + r^2)^3} - \frac{4r}{R^2 + r^2}\right].$$

Based on this expression, the variational residual $\delta(r)$ for the scalar field $\phi(r)$ was numerically evaluated, with the scale fixed at $R = 2$ to clearly highlight the scalar-field structure and its variational behavior. The resulting plot is shown in Fig. 1.
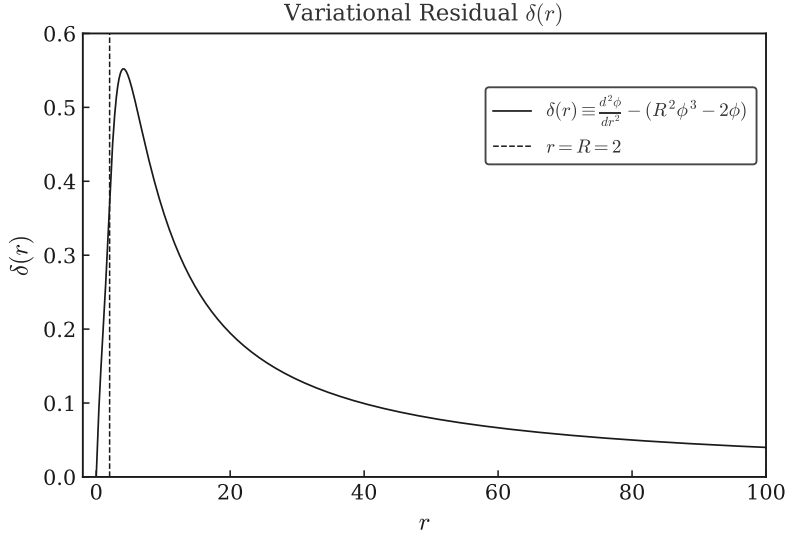

Fig. 1: Variational residual $\delta(r)$ for the scalar field $\phi(r)$.

From Fig. 1, it can be seen that the variational residual $\delta(r)$ begins to locally deviate at the spatial scale $r = R$, reaches a peak shortly thereafter ($r > R$), and then rapidly decays. This result indicates that the scalar field specifically recognizes and responds to $r = R$, and that the violation of the extremum condition is local and non-instantaneous. Such behavior is also consistent with the nature of the sustained cosmic acceleration.

The analysis is performed over the range $r \in [0.0001, 100]$. Within this interval, the maximum value of the variational residual $\delta(r)$ is found to be:

$$\delta_{\mathrm{max}} \approx 0.552 \quad \text{at} \quad r \approx 4.10.$$

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To evaluate the significance of this residual, we define the relative error with respect to the extremum condition as:

$$\epsilon_\phi(r) \equiv \left| \frac{\delta(r)}{\phi(r)} \right|.$$

At the point where the residual peaks ($r \approx 4.10$), the scalar field takes the value $\phi(r) \approx 0.394$, yielding:

$$\epsilon_\phi \approx 1.40.$$

Similarly, at the characteristic scale $r = R = 2$, the relative error is:

$$\epsilon_\phi(R) = \left| \frac{\delta(R)}{\phi(R)} \right| = 0.750 \quad (\delta(R) = 0.375, \ \phi(R) = 0.500).$$

Even around the characteristic scale $r = R$, the relative error remains within a range that can be regarded as a local violation of the extremum condition. The value $\epsilon_\phi(R) = 0.750$ supports the validity of the function as a quasi-solution.

On the other hand, the variational residual $\delta(r)$ reaches its maximum around $r \approx 4.10$, where the relative error rises to approximately $\epsilon_\phi \approx 1.40$. However, since $\phi(r)$ becomes small at this scale, the relative error is numerically overestimated due to the small denominator. Therefore, it is more appropriate to evaluate the validity of the quasi-solution primarily around $r = R$.

Based on these considerations, the scalar field $\phi(r)$ is formulated as a quasi-solution so as to preserve the characteristic scale R—ensured by the spatial isotropy introduced earlier—rather than as an exact solution that would eliminate it.

In this context, the local violation of the variational extremum condition serves as a necessary condition for the excitation of the scalar field $\phi(r)$.

In this paper, we refer to the structure based on the effective cosmological term $\Lambda_{\text{eff}}(r)$, constructed from the quasi-solution $\phi(r)$, as the Λ_{eff} model (or the $f\Lambda$ theory).

6 Consistency with the Observed Λ

The derived scale-dependent effective cosmological term,

$$\Lambda_{\text{eff}}(r) = \frac{4r^2}{(R^2 + r^2)^2},$$

attains its maximum at $r = R$, with the value given by

$$\Lambda_{\text{eff}}^{\max} = \Lambda_{\text{eff}}(r)|_{r=R} = \frac{4R^2}{(R^2 + R^2)^2} = \frac{4R^2}{4R^4} = \frac{1}{R^2}.$$

If this maximum is identified with the observed cosmological constant,

$$\Lambda \approx 10^{-52}\,\mathrm{m}^{-2},$$

then it follows that

$$R = \frac{1}{\sqrt{\Lambda}} \approx 10^{26}\,\mathrm{m},$$

which coincides with the cosmological scale, $\sim 10^{26}\,\mathrm{m}$. Therefore, it is natural to identify this characteristic scale as the effective cosmological scale, $R = \dfrac{1}{\sqrt{\Lambda}} \equiv R_c$.

This result implies that the effective cosmological term Λ_{eff}—derived from the scalar-field structure proposed in this study—is not only naturally consistent with the observed value of the cosmological constant but also provides a physically consistent origin for cosmic repulsion.

7 Extended Einstein Field Equation

The extended Einstein field equation is written as:

$$\Lambda_{\mathrm{eff}}(r)g_{\mu\nu} + G_{\mu\nu} = \kappa\,T_{\mu\nu}^{(\mathrm{matter})}, \qquad \kappa \equiv \frac{8\pi G}{c^4}.$$

In the local regime where $r \ll R_c$ (e.g., at the solar-system scale), the geometrical term $\Lambda_{\mathrm{eff}}(r)g_{\mu\nu}$ becomes negligibly small, and the field equation naturally reduces to the standard Einstein form:

$$G_{\mu\nu} = \kappa\,T_{\mu\nu}^{(\mathrm{matter})}.$$

Therefore, this extended field equation remains consistent with well-established observations precisely explained by general relativity, such as the perihelion precession of Mercury and the gravitational redshift.

Here, the effective term $\Lambda_{\mathrm{eff}}(r) = \dfrac{4r^2}{(R^2 + r^2)^2}$ can be rewritten as:

$$\Lambda_{\mathrm{eff}}(r) = \frac{1}{R^2}\,F\!\left(\frac{r}{R}\right), \qquad F(x) = \frac{4x^2}{(1+x^2)^2}, \quad x \equiv \frac{r}{R}.$$

By applying the Bianchi identity to the extended Einstein field equation, the exchange current J—representing geometry–matter energy transfer (i.e., local non-conservation)—is defined as follows, and in a static, spherically symmetric configuration, only the radial component remains:

$$J_\nu \equiv \nabla^\mu T_{\mu\nu}^{(\mathrm{matter})} = \frac{1}{\kappa}\,\partial_\nu \Lambda_{\mathrm{eff}}, \qquad J_r = \frac{1}{\kappa R^3}\frac{dF}{dx}.$$

Although local conservation is violated, both the radial and total fluxes vanish:

$$\int_0^\infty J_r \, dr = 0, \qquad \Phi(r) = 4\pi r^2 J_r \;\Rightarrow\; \Phi(0) = \Phi(\infty) = 0.$$

Thus, the geometry–matter energy transfer is exactly balanced throughout the volume, ensuring global energy conservation.

Additionally, at cosmological scales $r \sim R_c \sim 10^{26}$ m, where the local non-conservation represented by J_r becomes negligibly small,

$$\left| \frac{d\Lambda_{\text{eff}}}{dr} \right| \sim 10^{-78} \, \text{m}^{-3} \;\Rightarrow\; \Lambda_{\text{eff}} \approx \text{const.},$$

and hence the Friedmann equations are naturally recovered:

$$H^2 \equiv \left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}, \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left(\rho + \frac{3P}{c^2} \right) + \frac{\Lambda c^2}{3}.$$

8 Application to Galactic Rotation Curves

8.1 Gravitational Term: Realistic Linear Approximation

In the central region of a galaxy, the baryonic matter is relatively concentrated, allowing the use of a spherically symmetric and uniform density approximation. Under this assumption, the mass $M(r)$ enclosed within a radius r is given by:

$$\rho = \text{const} \quad \Rightarrow \quad M(r) = \frac{4}{3}\pi r^3 \rho.$$

From the gravitational potential generated by this mass distribution, the rotational velocity is derived as:

$$v_{\text{grav}}^2(r) = \frac{GM(r)}{r} = \frac{G}{r}\left(\frac{4}{3}\pi r^3 \rho \right) = \frac{4}{3}\pi G\rho \, r^2, \quad \Rightarrow \quad v_{\text{grav}} \propto r.$$

The relation $v_{\text{grav}} \propto r$ agrees with the observed linear rise in the inner regions of galaxies.

8.2 Repulsive Term: Rotational Velocity and Flattening

While the gravitational term from the uniform density model dominates in the central region of galaxies, explaining the flattened rotation curves in the outer regions requires a different dynamical contribution.

In this context, a scale-dependent effective Λ term, generated by the spatial configuration of the scalar field $\phi(r)$, yields a geometric repulsive effect that contributes to the

squared rotational velocity as:

$$v_{\text{rep}}^2(r) = v_0^2 \cdot \frac{2r^2}{R^2 + r^2},$$

where:

- v_0 is a characteristic velocity scale (unit: m/s) associated with the repulsive term,

- which becomes dominant in the outer galactic region ($r > R$).

Since this function contributes negligibly near the center and asymptotically approaches a constant value $v_{\text{rep}} \to \sqrt{2}\,v_0$ as $r \to \infty$, it naturally reproduces the observed flattening of galactic rotation curves. Based on observational data, we adopt a representative value of $v_0 \approx 200\,\text{km/s}$.

8.3 Rotational Velocity from Energy Composition

In this model, the gravitational and repulsive contributions to the rotational velocity are regarded as independent energy components. Therefore, the observed rotational velocity is expressed as a quadratic energy sum:

$$v_{\text{total}}^2(r) = v_{\text{grav}}^2(r) + v_{\text{rep}}^2(r) \quad \Rightarrow \quad v_{\text{total}}(r) = \sqrt{v_{\text{grav}}^2(r) + v_{\text{rep}}^2(r)}.$$

This composite structure implies that:

- in the inner region, $v_{\text{grav}} \propto r$ dominates,

- while in the outer region, $v_{\text{rep}} \to \sqrt{2}\,v_0$ flattens the rotation curve.

Overall, this framework demonstrates that the essential features of galactic rotation curves can be reproduced without assuming the existence of dark matter, thereby supporting the validity of the effective Λ term, Λ_{eff}.

8.4 Velocity $v_0 \approx 200\,\text{km/s}$ from the Effective Energy of Λ_{eff}

Based on the result of Section 6 and the vacuum–energy density expression in the ΛCDM model, we obtain the *geometric effective energy density*:

$$\rho_{\Lambda_{\text{eff}}}(R_c) = \rho_\Lambda = \frac{\Lambda c^2}{8\pi G} \approx 10^{-26}\,\text{kg/m}^3.$$

By integrating this geometric effective energy density over a spherically symmetric region of radius $\sim 1000\,\text{kpc} \approx 3.1 \times 10^{22}\,\text{m}$, the corresponding geometric effective mass can be

obtained. Since 10^{22} m $\ll 10^{26}$ m, the effective energy density $\rho_{\Lambda_{\text{eff}}}(R_c)$ can therefore be regarded as nearly constant within this scale range, yielding:

$$M_\Lambda \approx \rho_\Lambda \cdot \frac{4}{3}\pi r^3 \approx 10^{42} \, \text{kg}.$$

If we convert a part of the effective mass energy into kinetic energy, it follows that:

$$E_{\text{rot}} = \frac{1}{2} M_\Lambda v_0^2 \quad \Rightarrow \quad v_0 \approx 200 \, \text{km/s} \approx \sqrt{\frac{2E_{\text{rot}}}{M_\Lambda}}.$$

Therefore, the characteristic velocity $v_0 \approx 200 \, \text{km/s}$ can be naturally estimated within the framework of the Λ_{eff} model, and the corresponding energy $E_{\text{rot}} \approx 2 \times 10^{52} \, \text{J}$ is consistent with the typical observational energy scale $E_{\text{obs}} \sim 10^{52} \, \text{J}$. Considering that the total geometric effective energy is given by $E_\Lambda = M_\Lambda c^2 \approx 10^{59} \, \text{J}$, the magnitude of E_{rot} is sufficiently small in comparison, ensuring energetic consistency.

These results quantitatively support the physical plausibility of the scale-dependent term Λ_{eff}, derived from the scalar field, as an effective cosmological repulsive component.

8.5 Physical Interpretation: Dominant Structures by Region

- **Central region** $(r < R)$:
  The gravitational term dominates, with $v_{\text{grav}} \propto r$.

- **Intermediate region** $(r \approx R)$:
  The repulsive term v_{rep} begins to contribute, marking the transition toward velocity flattening.

- **Outer region** $(r > R)$:
  The repulsive contribution dominates, and the rotation velocity becomes asymptotically flat, reflecting the logarithmic-type structure and scale dependence of Λ_{eff}.

Representative Parameters and Plotting Conditions

- Gravitational constant: $G = 4.302 \times 10^{-6} \, \text{kpc} \cdot (\text{km/s})^2 / M_\odot$

- Mass density (average over the galaxy): $\rho = 10^6 \, M_\odot / \text{kpc}^3$

- Characteristic scale of the galaxy: $R = 5 \, \text{kpc}$

- Characteristic velocity of the galaxy: $v_0 = 200 \, \text{km/s}$

- Radial range for calculation: $r = 0.1 \sim 20 \, \text{kpc}$

Using the above representative values, the numerically plotted rotation velocity profile at the galactic scale is shown in Fig. 2, based on the previously derived composite formula:

$$v_{\text{total}}(r) = \sqrt{v_{\text{grav}}^2(r) + v_{\text{rep}}^2(r)}.$$
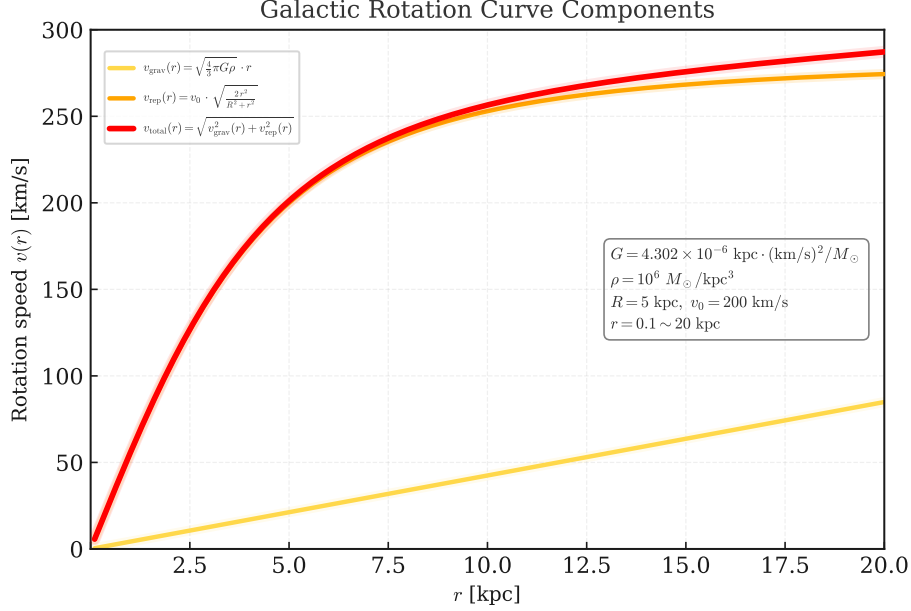

Fig. 2: Numerical simulation of gravitational, repulsive, and total rotational velocity components for a galactic rotation curve.

As clearly demonstrated by the numerical simulation results shown in Fig. 2, it was initially expected that the gravitational term would dominate in the central region $(r < R)$. However, the repulsive component was quantitatively found to exceed the gravitational component at all radii within the galaxies modeled. This indicates that the rotational velocity within galaxies is effectively governed by the scalar-field-induced effective Λ term—characterized by its logarithmic-type structure and defined by the characteristic galactic scale $R \sim \text{kpc} \equiv R_g$.

This result not only explains the observed flattening of rotational velocities in the outer regions of galaxies, but also reveals that the repulsive term significantly contributes even in the central region. It strongly indicates that the scale-dominant behavior of the logarithmic-type repulsive term extends far more broadly than previously anticipated.

Therefore, the galactic rotation curves are systematically reproduced by the repulsive component:

$$v_{\text{rep}}(r) = v_0 \cdot \sqrt{\frac{2r^2}{R^2 + r^2}}.$$

9 BH Singularity Avoidance and the Critical Radius

The repulsive acceleration derived from the scalar field $\phi(r)$ is given by:

$$a_{\text{rep}}(r) = c_0 \cdot \phi(r) = \frac{2c_0 r}{R^2 + r^2}.$$

Here, c_0 is a repulsive potential coefficient (unit: m^2/s^2), representing the characteristic spatial energy scale of the repulsive field. Note that this acceleration formula corresponds uniquely to the rotational velocity function defined in Section 8.2 through the circular motion condition $a = \dfrac{v^2}{r}$:

$$a_{\text{rep}}(r) = \frac{2c_0 r}{R^2 + r^2} \quad \underset{a=\frac{v^2}{r}}{\overset{c_0 = v_0^2}{\Longleftrightarrow}} \quad v_{\text{rep}}^2(r) = \frac{2v_0^2 r^2}{R^2 + r^2} = v_0^2 \cdot \frac{2r^2}{R^2 + r^2}.$$

Inside the black hole, the repulsive acceleration is considered to balance the gravitational acceleration. Under this condition, the balance between the repulsive and gravitational forces is expressed as:

$$a_{\text{rep}}(r) = g(r) \implies \frac{2c_0 r}{R^2 + r^2} = \frac{GM}{r^2}.$$

Here, M denotes the mass of the black hole.

At $r = R = r_{\text{crit}}$, where $a_{\text{rep}}(r)$ attains its maximum $a_{\text{rep}}^{\text{max}}$, the balance condition between the repulsive and gravitational accelerations is expressed as:

$$\frac{c_0}{r_{\text{crit}}} = \frac{GM}{r_{\text{crit}}^2} \implies c_0 = \frac{GM}{r_{\text{crit}}}.$$

Here, the repulsive energy at the critical radius is given by:

$$E_{\text{rep}} = M \cdot c_0 = \frac{GM^2}{r_{\text{crit}}},$$

while the total mass–energy of the black hole is

$$E_{\text{BH}} = Mc^2.$$

Within this setup—where the repulsive response of space is induced by the black hole's own mass–energy—the system can be treated as a spherically symmetric and static closed configuration. By the principle of energy conservation, the repulsive energy accumulated within the black hole cannot exceed its total mass-energy. Accordingly, equating the two energies defines the critical radius at which further collapse becomes energetically

forbidden:
$$E_{\text{rep}} = E_{\text{BH}} \quad \Rightarrow \quad \frac{GM^2}{r_{\text{crit}}} = Mc^2 \quad \Rightarrow \quad r_{\text{crit}} = \frac{GM}{c^2}.$$

This critical radius corresponds exactly to half of the Schwarzschild radius:

$$r_{\text{crit}} = \frac{1}{2}\, r_s, \quad \text{where} \quad r_s = \frac{2GM}{c^2}.$$

Thus, the characteristic scale of a black hole is naturally defined as $R = \frac{1}{2}\, r_s \equiv R_{\text{BH}}$.

Since the condition $M \cdot c_0 = Mc^2$ ($E_{\text{rep}} = E_{\text{BH}}$) is satisfied at the critical radius, it follows that $c_0 = c^2$ at that point. **Consequently, singularities are prohibited by the principle of energy conservation and the speed-of-light limit.** The singularity is therefore avoided not by introducing an external cutoff, but by the internal repulsive structure of space itself.

As a direct geometric consequence, the apparent black hole shadow is characterized by the critical radius $r_{\text{crit}} = \frac{GM}{c^2}$. The photon sphere is located at

$$r_{\text{ph}} = \frac{3}{2}\, r_s = 3\, r_{\text{crit}},$$

and the apparent shadow radius, including gravitational lensing, is

$$b_c = \frac{r_{\text{ph}}}{\sqrt{1 - \dfrac{2\, r_{\text{crit}}}{r_{\text{ph}}}}} = 3\sqrt{3}\, r_{\text{crit}}.$$

Hence, the shadow diameter becomes

$$D = 2\, b_c = 6\sqrt{3}\, r_{\text{crit}},$$

yielding a dimensionless ratio:

$$\frac{D}{r_{\text{crit}}} = 6\sqrt{3} \approx 10.4,$$

which is universally applicable to any black hole without normalization and agrees remarkably well with the Event Horizon Telescope (EHT) measurements, M87* $\approx$ 11.0 and Sgr A* $\approx$ 10.0.

With $a_{\text{rep}}^{\text{max}}$ the repulsive acceleration at r_{crit}, and since $c_0 = c^2$ there, we have:

$$a_{\text{rep}}^{\text{max}} = \frac{c^2}{r_{\text{crit}}}.$$

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Substituting $r_{\text{crit}} = \dfrac{GM}{c^2}$ gives:

$$a_{\text{rep}}^{\max} = \frac{c^2}{\frac{GM}{c^2}} = \frac{c^4}{GM}.$$

Therefore, both the repulsive acceleration and the gravitational acceleration inside a black hole cannot exceed this value.

Here, we take $a_{\text{rep}}^{\max}$ to be equal to the Planck acceleration $a_{\text{P}} \equiv \sqrt{\dfrac{c^7}{\hbar G}}$, and we obtain:

$$\frac{c^4}{GM} = \sqrt{\frac{c^7}{\hbar G}} \implies M = \sqrt{\frac{\hbar c}{G}}.$$

This value corresponds to the Planck mass $m_{\text{P}} \equiv \sqrt{\dfrac{\hbar c}{G}}$, showing that even a black hole compressed to the Planck scale cannot possess a mass exceeding this fundamental limit.

10 Consistency with the Higgs Mechanism

In this section, we demonstrate that the scalar potential of the Λ_{eff} model defined in Section 4,

$$V(\phi(r)) = -\frac{1}{4} \cdot \phi^2\big(4 - R^2\phi^2\big) = -\phi^2 + \frac{1}{4}R^2\phi^4,$$

is directly related to the mass-generating structure of the Higgs mechanism.

To obtain the vacuum expectation value (VEV), we differentiate the potential $V(\phi(r))$ in the Λ_{eff} model:

$$\frac{dV(\phi(r))}{d\phi} = 0 \quad \Rightarrow \quad \phi\big(-2 + R^2\phi^2\big) = 0.$$

Therefore, the solutions are

$$\phi = 0 \quad \text{or} \quad \phi = \pm\frac{\sqrt{2}}{R}.$$

Thus, excluding the trivial solution $\phi = 0$, the vacuum expectation value in the Λ_{eff} model is

$$v = \frac{\sqrt{2}}{R}, \quad \phi = \pm v.$$

From this relation, the characteristic scale R can be expressed in terms of the experimentally determined Higgs vacuum expectation value $v \approx 246\,\text{GeV}$ as

$$R = \frac{\sqrt{2}}{v} = \frac{\sqrt{2}}{246\,\text{GeV}} \approx 5.75 \times 10^{-3}\,\text{GeV}^{-1}.$$

Using the natural unit conversion ($\hbar = c = 1$) with $\mathrm{GeV}^{-1} \approx 1.97 \times 10^{-16}\,\mathrm{m}$, we obtain

$$R \approx 5.75 \times 10^{-3} \times 1.97 \times 10^{-16}\,\mathrm{m} \approx 1.13 \times 10^{-18}\,\mathrm{m}.$$

This value precisely corresponds to the electroweak scale at which the Higgs boson was discovered in experiments at the Large Hadron Collider (LHC), at $\sim 10^{-18}\,\mathrm{m}$. Therefore, it is natural to identify this characteristic scale as the Higgs scale, $R = \dfrac{\sqrt{2}}{v} \equiv R_h$, and the Yukawa coupling can be defined through a geometric relation as

$$y_f = \frac{R_h}{R_f}, \qquad R_f \equiv \frac{1}{m_f},$$

or, taking logarithms, as a distance in log-scale space:

$$\log y_f = \Delta \nu, \qquad \Delta \nu \equiv \log R_h - \log R_f.$$

Here, the potential for the Higgs field in the Standard Model is given by:

$$V_{\mathrm{Higgs}}(\phi) = -\frac{1}{2}\mu^2 \phi^2 + \frac{1}{4}\lambda \phi^4, \qquad v^2 = \frac{\mu^2}{\lambda}, \quad m_h^2 = 2\lambda v^2.$$

By substituting these relations, the potential can be rewritten as:

$$V_{\mathrm{Higgs}}(\phi) = -\frac{1}{4}m_h^2 \phi^2 + \frac{m_h^2}{8v^2}\phi^4.$$

Thus, the Higgs potential can be expressed in a subdivided form:

$$V_{\mathrm{Higgs}}(\phi) = -\left(\frac{m_h}{2}\right)^2 \phi^2 + \frac{1}{4}\left(\frac{\sqrt{2}}{v}\right)^2 \left(\frac{m_h}{2}\right)^2 \phi^4.$$

Here, let us define:

$$\frac{m_h}{2} \equiv \breve{m},$$

and substitute the previously obtained relation $R_h = \dfrac{\sqrt{2}}{v}$:

$$V_{\mathrm{Higgs}}(\phi) = -\breve{m}^2 \phi^2 + \frac{1}{4}R_h^2 \breve{m}^2 \phi^4.$$

Rearranging gives

$$V_{\mathrm{Higgs}}(\phi) = \breve{m}^2 \left(-\phi^2 + \frac{1}{4}R_h^2 \phi^4\right).$$

This suggests the natural choice

$$\breve{m} = 1$$

to bridge mass–dimensionful and dimensionless forms. By adopting $\hbar = c = \check{m} = 1$ and replacing $R_h = \dfrac{\sqrt{2}}{v}$ with a general characteristic scale R, we obtain a universal scalar potential in a mass-dimensionless form:

$$\boxed{V_{\text{Higgs}}(\phi) = -\phi^2 + \frac{1}{4}R^2\phi^4}$$

which can be applied to general scales and is identical to the potential of the Λ_{eff} model $V(\phi(r)) = -\phi^2 + \dfrac{1}{4}R^2\phi^4$.

By assigning $\check{m}$ as the characteristic mass in $V(\phi(r))$ at the Higgs scale, the self-coupling constant is obtained as:

$$\lambda_{\Lambda_{\text{eff}}} \equiv R_h^2\,\check{m}^2.$$

In natural units ($\hbar = c = 1$), where $\check{m} \neq 1$ is explicitly retained as the mass scale, using $R_h \approx 1.13 \times 10^{-18}\,\text{m} \approx 5.75 \times 10^{-3}\,\text{GeV}^{-1}$ and $\check{m} = \dfrac{m_h}{2} \approx 62.5\,\text{GeV}$, we obtain

$$\lambda_{\Lambda_{\text{eff}}} \approx (5.75 \times 10^{-3}\,\text{GeV}^{-1})^2 \cdot (62.5\,\text{GeV})^2 \approx 0.129,$$

which coincides with the Standard Model self-coupling constant $\lambda \approx 0.129$.

These results in this section indicate that mass generation is driven by the geometric repulsive response.

Is Double-Higgs (hh) Production Possible?

The production of a single Higgs boson (h) occurs when two half-mass quantum states $|m_h/2\rangle_{1,2}$ coherently superpose within the squared region of the characteristic R_h-scale:

$$\frac{1}{\sqrt{2}}\Big(|m_h/2\rangle_1 + |m_h/2\rangle_2\Big) \;\Rightarrow\; |h\rangle.$$

This coherent-overlap condition is reflected in the relation

$$\lambda_{\Lambda_{\text{eff}}} = R_h^2\left(\frac{m_h}{2}\right)^2.$$

In contrast, the production of a double-Higgs (hh) consists of two independent single-Higgs superposition states,

$$|hh\rangle = |h\rangle \otimes |h\rangle = \left(\frac{1}{\sqrt{2}}\right)^2 \Big(|m_h/2\rangle_1 + |m_h/2\rangle_2\Big) \otimes \Big(|m_h/2\rangle_3 + |m_h/2\rangle_4\Big).$$

However, since the characteristic length scale $R_h \approx 1.13 \times 10^{-18}$ m corresponds to an energy scale $1/R_h \approx 174$ GeV, localizing four half-mass components with total energy ≈ 250 GeV within the same R_h-scale region would exceed this limit:

$$E_{hh} \equiv 4 \cdot \left(\frac{m_h}{2}\right) \approx 250 \text{ GeV} \quad > \quad E_{R_h} \equiv \frac{1}{R_h} \approx 174 \text{ GeV}.$$

Consequently, the four-point overlap — i.e., double-Higgs production — is energetically prohibited by the R_h-scale confinement limit.

11 Discussion and Conclusion

In this study, we derived the following scale-dependent effective cosmological term from the scalar field $\phi(r)$, which is formulated as a quasi-solution that preserves the characteristic scale R while remaining consistent with the variational principle:

$$\Lambda_{\text{eff}}(r) = \frac{4r^2}{(R^2 + r^2)^2},$$

functions effectively in the following three aspects:

1. **Consistency with cosmology:**
   The effective term Λ_{eff} reaches its maximum value at $r = R$, where

   $$\Lambda_{\text{eff}}^{\max} = \frac{1}{R^2}.$$

   By identifying this with the observed cosmological constant, we obtain $R \approx 10^{26}$ m, which coincides with the cosmological scale, $\sim 10^{26}$ m.

2. **Reproduction of galactic rotation curves:**
   Using the repulsive term based on Λ_{eff}, we constructed the following composite formula for rotational velocity:

   $$v_{\text{total}}(r) = \sqrt{v_{\text{grav}}^2(r) + v_{\text{rep}}^2(r)}, \qquad v_{\text{grav}}^2(r) = \frac{4}{3}\pi G \rho \, r^2, \quad v_{\text{rep}}^2(r) = v_0^2 \cdot \frac{2r^2}{R^2 + r^2}.$$

   Numerical evaluation under a realistic galactic-scale configuration and representative parameters confirmed that this structure successfully reproduces the observed flat rotation velocity profiles of galaxies. As a result, the geometric repulsive term associated with Λ_{eff} was found to be dominant over the gravitational contribution throughout the galaxy.

   Therefore, the galactic rotation curves are systematically characterized by the re-

pulsive component:

$$v_{\mathrm{rep}}(r) = v_0 \cdot \sqrt{\frac{2r^2}{R^2 + r^2}}.$$

3. **Critical radius and singularity avoidance**

   By deriving the condition under which the repulsive energy equals the mass–energy of the black hole, we obtained a critical radius $r_{\mathrm{crit}} = \frac{1}{2} r_s$ at which collapse is doubly locked by energy conservation and the speed-of-light limit. This indicates that singularities can be naturally avoided within the internal dynamics of the geometric structure itself, without requiring any external cutoff.

Consequently, the contribution of Λ_{eff} manifests as a geometrical effect opposite in direction to conventional gravity, playing a unifying role across scales—from the accelerated expansion of the universe and the flattening of galactic rotation curves to the prevention of black hole singularities.

Continued theoretical advancement of the model, along with its observational and empirical verification, remains a central task. As a first step in this direction, we have confirmed its consistency with the Higgs mechanism within the scalar potential framework of the fΛ theory.

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Ethical Statement

This study involved no human or animal subjects; ethical approval was not applicable.

Conflict of Interest Statement

The author declares no conflict of interest.

References

[1] V. C. Rubin and W. K. Ford. Rotation of the Andromeda Nebula from a Spectroscopic Survey of Emission Regions. *The Astrophysical Journal*, 159:379, 1980.

[2] Y. Sofue and V. Rubin. Rotation Curves of Spiral Galaxies. *Annual Review of Astronomy and Astrophysics*, 39:137–174, 2001.

[3] Roger Penrose. Gravitational Collapse and Space-Time Singularities. *Physical Review Letters*, 14(3):57–59, 1965.

[4] Planck Collaboration. Planck 2018 Results. VI. Cosmological Parameters. *Astronomy & Astrophysics*, 641:A6, 2020.

[5] T. Padmanabhan. Cosmological Constant—The Weight of the Vacuum. *Physics Reports*, 380:235–320, 2003.