IB Mathematics Higher Level Exploration on Statistics

Deriving an Alternative Data Approximation Method to Be Used in Non-typical Circumstances

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Contents

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1 Introduction

In both physics and mathematics practical work has as much value to humanity as theoretical does. I have come to notice that during labs, practicals and experiments that we do in school in any field, if the lab, practical or experiment is quantitative, the data gathered, almost always, has to be approximated to some kind of functional dependence to be of further use. This made me wonder how does one make such approximations; how does a calculator know what the best fit line is? Further more, I've also noticed that in most of these practicals, the method of approximation that my classmates and I are told to use (and the one that is most commonly used overall) is the so called Least Square Method (LSM). What does it imply? What logic does it use to approximate? When does it work best, and when does it not work at all? I've grown concerned about the validity of my practical results that I have used in the past. What if the approximation had large errors? How big could those errors be?

In this exploration I aim to suggest an alternative method/algorithm of approximation One that performs better in certain situations, in which the LSM does not. I will do so by deriving the formula for the optimal parameter(s) of a given functional dependence. Then I will use self-written software to (via the formula derived) calculate trials in which it is shown that the method that is suggested in this exploration, indeed works better in given circumstances when compared to LSM.

2 Mathematical Background

2.1 Least Square Method

The Least Square Method aims to minimize the squares of differences between a potential best fit line and the data point for all discrete values of x. Because it squares the difference, if the difference is large, it will be more influential on the final solution. In a sense if a data point is further away from the rest, it will have a greater effect on the final solution. The calculation of LSM is done later in this exploration, also with a more thorough explanation. The main point is that a data set which contains outliers, will not be well represented by LSM.

2.2 Vectors and m-dimensional space

During the IB Math HL program, students learn about vectors. 2D-Vectors usually have only magnitude and direction, however, as we know a vector can be 3D, or even 4D. Essentially, so long that a single value has multiple parameters (components) it is a vector in the same number of dimensions as there are parameters. In this exploration, all vectors will be denoted with the variable $\vec{\mathbf{c}}$. Note that this is just a notation that I will use only in this paper. In addition, I will be using herizontal vector notation like so: $\vec{\mathbf{c}} = (c_1, c_2, \dots c_m)$. Where the vector $\vec{\mathbf{c}}$ is in real coordinate (euclidean) space <1>; This allows several real variables to be treated as one, single variable. Note that when dealing with space in higher than three dimentions (when talking about any function with more than three parameters), the notation $\boldsymbol{R}^m$ will be used, where m is any positive integer. This notation describes the dimension of space. i.e. $\boldsymbol{R}^1$ is one-dimensional space, $\boldsymbol{R}^2$ two-dimensional, and so on. <2>

2.3 Partial Derivatives

When it comes to functions with multiple variables, their derivative is taken differently. The derivative of such functions is called a partial derivative. Denoted using a stylized symbol ∂, it is the derivative of a function with multiple variables, with respect to one of them, when others are held constant. <3, pages 45-48> <4, pages 878-891> The following example will explain this further:

Find the local minimum of function z with multiple variables:

$$z = f(x, y) = 5(2x - 3)^2 + 4(4y + 1)^2$$

Taking the squares $2x-3$ and $4y+1$ and equating them to zero, then solving for x and y, we see that the minimum of this function occurs when $x=\frac{3}{2}$,$y=-\frac{1}{4}$.

(Expanding this function gives)

$$z=20x^2+64y^2-60x+32y+49$$

Now according to the normal method of finding a minimum of a function, we equate, in this case, its partial derivatives (one with respect to x, the other to y) to zero and solve for x and y.

$$\begin{aligned}\frac{\partial f(x,y)}{\partial x} &= 40x-60=0\rightarrow x=\frac{60}{40}=\frac{3}{2}\\ \frac{\partial f(x,y)}{\partial y} &= 128y-32=0\rightarrow y=-\frac{32}{128}=-\frac{1}{4}\end{aligned}$$

Partial derivatives gave the answer as well.

2.4 Multiple Integrals

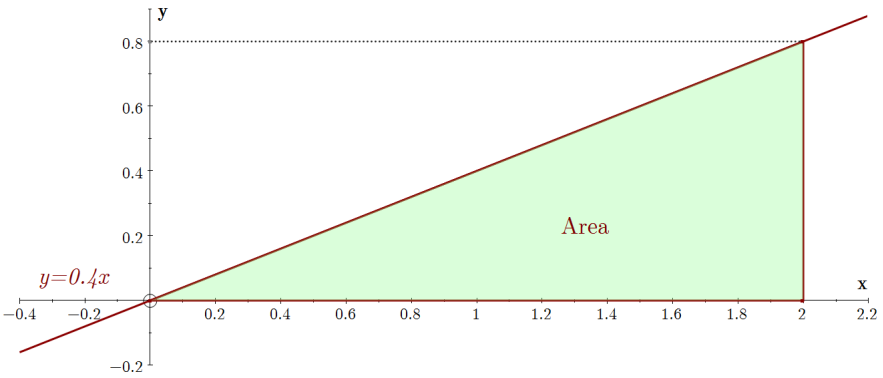
This exploration invokes the use of multiple integrals. These are basically integrals of other integrals, where any function of multiple variables is in question. <5> Here, each integral has a separate variable of integration, even though they stem from the same function. i.e. a double (two-multiple) integral of some two-variable function $g(x,y)$ is denoted as such:

$$\iint g(x,y)\mathrm{d}x\mathrm{d}y$$

Likewise, this can be applied to a function with m variables like so:

$$\int\cdots\int g(x,y,\ldots,m)\mathrm{d}x\mathrm{d}y\ldots\mathrm{d}m$$

Yet again let's employ an example:
Find the area of the shaded region



Obviously,

$$A=\frac{1}{2}\cdot 2\cdot 0.8=0.8\text{ units}^2$$

Granted, the following method is very unnecessary (and this problem doesn't even require a multiple integral), but for the sake of explanation, let's employ multiple integrals as well.

$$A=\underbrace{\int_0^2}_{\text{(against }x\text{)}}\ \underbrace{\int_0^{0.4x}}_{\text{(against }y\text{)}}\mathrm{d}x\mathrm{d}y=\int_0^2\mathrm{d}x\cdot[y]_0^{0.4x}=\int_0^2\mathrm{d}x\cdot[0.4x-0]=\int_0^2\mathrm{d}x\cdot 0.4x=0.4\int_0^2 x\mathrm{d}x=$$

$$0.4\cdot\left[\frac{x^2}{2}\right]_0^2=0.4\cdot\left[\frac{2^2}{2}-\frac{0^2}{2}\right]=0.4\cdot 2=0.8.$$

And, again, the answer is the same.

2.5 Lebesgue Measure

In mathematical analysis, a measure on a set, is the assignment of a number to all suitable sub-sets of that set. <6> Essentially, a measure of a set can be interpreted as its size, and is therefore a generalization of length, area and volume. In particular, the Lebesgue measure assigns length, area and volume (in a conventional sense) to suitable sub-sets of an m-dimensional euclidean space $\boldsymbol{R}^m$. <7> This is particularly useful when dealing with any set that is in more than tree dimensions, as just 'volume' of such a set can not be defined. Thus, a more general idea of 'volume' is used: the Lebesgue Measure.

If there is a set K in $\boldsymbol{R}^m$, then the Lebesgue measure of K will be denoted as $\mu(K)$. For example, if K is in $\boldsymbol{R}^1$, and is the interval $[0,1]$, its Lebesgue measure $\mu([0,1])$ is its length, i.e. 1. Similarly, if K is in $\boldsymbol{R}^3$, and is a cube with sides 2, its Lebesgue measure $\mu(K)$ is its volume, i.e. 2^3 or 8. This principal may be applied to any set in $\boldsymbol{R}^m$ euclidean space.

3 Formulation of the Problem

A function is one of the most known mathematical notions. An important task which has practical applications, is the approximation of a function or relationship based on some information known about the function or relationship in question. This information may either be determinate or statistical. An example of a determinate piece of information about function $f(x)$ is its range (or the possible values this function may have) on a given interval $[\alpha, \beta]$. Example of a statistical information may be the law of distribution of random errors ξ_i in approximate values $\tilde{y}_i = f(x_i) + \xi_i$ of the function, which in turn can describe a certain physical process (change of temperature over time, for example). In practice, n number of points x_i can be obtained as results of some kind of physical experiment. Where in this case, the approximation of function $f(x)$ only makes sense if this function is described by a finite number $m < n$ of parameters (coefficients) c_j, where the true values of said parameters will be denoted as $\dot{c}_j,\ j = 1, 2, \ldots, m$. For the sake of convenience, these parameters will be written as an m-dimensional vector; $\vec{\mathbf{c}}$ storing the possible parameters and $\dot{\vec{\mathbf{c}}}$ storing the true parameters.

This exploration will focus on the estimation of the parameters of a function

$$y = f(\dot{\vec{\mathbf{c}}}, x), \quad \dot{\vec{\mathbf{c}}} \in \boldsymbol{R}^m, \quad x \in [\alpha, \beta], \quad \dot{\vec{\mathbf{c}}} = (\dot{c}_1, \dot{c}_2, \ldots, \dot{c}_m) \tag{3.0.1}$$[1]

based on its approximate (i.e containing error) values

$$\tilde{y}_i = f(x_i) + \xi_i, \quad i = 1, 2, \ldots, n, \tag{3.0.2}$$

when additionally it is also known, that: (1) vector $\dot{\vec{\mathbf{c}}} = (\dot{c}_1, \dot{c}_2, \ldots, \dot{c}_m)$ belongs to a given limited set D, like for example a parallelotope[2] in $\boldsymbol{R}^m$; (2) ξ is a limited continuous random value; the median of which ($Med(\xi)$) is equal to zero.

Judging by references in scientific works that I read while researching for this exploration, the most popular linear model of a studied relationship is

$$f(\dot{\vec{\mathbf{c}}}, x) = \sum_{j=1}^{m} \dot{\vec{\mathbf{c}}}_j \phi_j(x), \tag{3.0.3}$$

specifically in polynomial form, when

$$\phi_j(x) = x^{j-1}, \quad j = 1, 2, \ldots, m; \quad \phi_1(x) = 1. \tag{3.0.4}$$

[1] Here, because the vector $\dot{\vec{\mathbf{c}}}$ has m parameters (components), it will also be in m-dimensional space.

[2] In geometry, a parallelepiped is a polyhedron in $\boldsymbol{R}^3$, with six faces, each being a parallelogram. It can be generalized to be in $\boldsymbol{R}^m$, then it would take on the name of Parallelotope <8>. I specifically refer to a Parallelotope as both any cube and any cuboid in $\boldsymbol{R}^m$ is just a special case of a Parallelotope, therefore the term Parallelotope is more general and more accurate in this context. To clarify, a Parallelotope in $\boldsymbol{R}^3$ is by definition a parallelepiped.

There are countless papers dedicated to the approximation of functions based on their approximate values (in practice - experimental data). Usually, in such papers the consensus is to use a certain condition. This condition is to assume that the mathematical expectancy of error is equal to zero. <9>

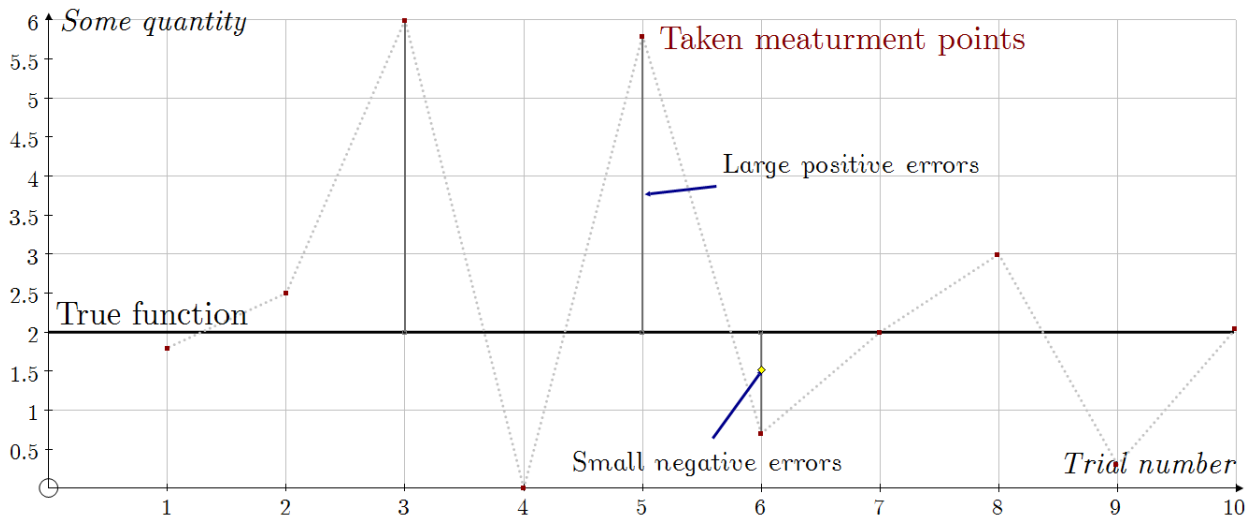
$$E(\xi) = 0 \tag{3.0.5}$$

However, in this exploration this condition will not be used. Here, instead of the condition of mathematical expectancy of error ξ being equal to zero, I will assume that the median of the same error ξ being equal to zero,

$$Med(\xi) = 0. \tag{3.0.6}$$

In a measurement containing an error ξ, that error ξ is considered positive if the measurement itself differs from the true function (like 3.0.1 on the preceding page) by a positive value, and negative if that measurement differs by a negative value. I justify my interest to the condition $Med(\xi) = 0$ by the case when the traditional condition $E(\xi) = 0$ is unachievable. This happens when measurements are taken close to one of the natural limits of the physical relationship being measured. An example of such natural limit is the inability of some magnitude, such as weight, to be negative. In this case, large errors ξ (relative to others) may only occur in such errors ξ that have the same sign, either positive or negative, i.e. in a given case, only positive errors ξ are large or only negative errors ξ are large, depending on the natural limit. Figure 1 shows an example of this graphically. Here the natural limit is 0 (measurements can not be negative), so large errors ξ only occur while they're positive.

Figure 1: Graphical representation of a case where $E(\xi) = 0$ does not work effectively.



I want to bring attention to the fact, that the argument that this kind of measurement could be withdrawn by human intervention, is invalid for 2 reasons: (1) Any such withdrawal usually leads to loss of information. (2) In cases where an experiment requires high-capacity data collection human intervention may not be possible.

What is meant is not only error that was produced by a faulty measurement, but also any error caused by some factor that was either omitted or unaccounted for in function $f(x)$. Even though both conditions $E(\xi) = 0$ and $Med(\xi) = 0$ are not special cases of each other, it could be argued that from a point of view of solving practical problems, the condition $Med(\xi) = 0$ is the more broad of the two (as in, it is easier to meet). The only requirement for meeting this condition is that the probability of ξ being positive is 50%. Hence the condition $Med(\xi) = 0$ allows for some comparatively large random values of error ξ to be on one side of the true function and not on the other, without the approximation to be significantly affected by those large values, unlike the condition $E(\xi) = 0$. With condition $Med(\xi) = 0$ the approximation can account for large peaks in values of ξ.

As mentioned before, the aim of this exploration is to create an algorithm, capable of estimating the parameters of a functional dependence whose structure is known, accurately, and to securely give an estimate to the accuracy of the calculated approximate values of the functional dependence.

It's clear that the quality of the solution of this problem is dependant of an array of factors, which include: (1) the ratio between the n number of measurements $\tilde{y}_i$ and the m number of estimated parameters c_j. (2) the intensity of error ξ_i. This means that to confirm my theoretical reasoning, quite an ambitions computational experiment is required. I will proceed with the necessary calculations using custom software.

4 Algorithm

Approximation of a functional dependence taking this new condition $Med(\xi) = 0$ in mind, has been looked at in mathematics. <10> It is believed that in this case it is necessary to minimize the sum of absolute values of deviations of the modelled dependence $f(\vec{\mathbf{c}}^*, x)$ from the unknown true function $f(\dot{\vec{\mathbf{c}}}, x)$, where $\vec{\mathbf{c}}^*$ is the found optimal value of vector $\vec{\mathbf{c}}$. This method is referred to as the Least Absolute Deviations (LAD). However, through my research I have found no methods of estimating LAD's accuracy. What value does an approximation method have if there is no way to determining the error it made?

It is clear, that in any separately taken case (run of an algorithm), the factual accuracy of the model solution (when the true function is known) cannot serve as either a comparative evaluation of two competing algorithms, nor criteria of effectiveness of any given algorithm. This is beacuse the intention of the algorithm itself is to be accurete in practice, i.e. when the true functionn is not known. One separately taken case when compared to the true function may give a good result however there is no way to tell if that result was accedental to that one separatly taken case. It is also clear, that if all, or close to all errors ξ_i have the same sign (the condition $Med(\xi) = 0$, and the condition $E(\xi) = 0$ as well, allow this, be it with a small probability), then neither method will give any good solutions. So, when estimating the effectiveness of a method, it is necessary to rely on average results of some number of random solutions. In conjunction with this, I will take as a measure of accuracy of constructed approximation $f(\vec{\mathbf{c}}^*, x)$ to the true function $f(\dot{\vec{\mathbf{c}}}, x)$, the mathematical expectation <11>

$$E(\rho(\vec{\mathbf{c}}^*, \dot{\vec{\mathbf{c}}})) = \int_D P(\vec{\mathbf{c}})\rho(\vec{\mathbf{c}}^*, \dot{\vec{\mathbf{c}}})\mathrm{d}c_1\mathrm{d}c_2 \ldots \mathrm{d}c_m, \quad \vec{\mathbf{c}} = (c_1, c_2, \ldots, c_m) \qquad (4.0.1)\ \text{<10>}$$

of proximity (distance) $\rho(\vec{\mathbf{c}}^*, \dot{\vec{\mathbf{c}}})$ of function $f(\vec{\mathbf{c}}^*, x)$ from $f(\dot{\vec{\mathbf{c}}}, x)$ where in the role of distance $\rho(\vec{\mathbf{c}}^*, \dot{\vec{\mathbf{c}}})$, one could take on of the functions

$$\rho_1(\vec{\mathbf{c}}^*, \vec{\mathbf{c}}) = \sum_{j=1}^{m} \left|c_j^* - c_j\right| \qquad (\text{ taken from LAD }) \qquad (4.0.2)$$

$$\rho_2(\vec{\mathbf{c}}^*, \vec{\mathbf{c}}) = \sum_{j=1}^{m} \left(c_j^* - c_j\right)^2 \qquad (\text{ taken from LSM }). \qquad (4.0.3)$$

During my research I've found that formula (4.0.1) was considered in the article *'Reggression-type Problems under Zero Median Noise'* by Peter Balk (referenced earlier as <10>). There, Balk used a proximity function that is not considered in this exploration. The reason for me using the proximity functions (4.0.2) and (4.0.3) instead of the one used in <10>, is that I will be comparing the method derived here to the already established methods: LSM and LAD. Using a different proximity function, to the ones in those methods, will lead to faulty results.

The probability density function $P(\vec{\mathbf{c}})$, ($\vec{\mathbf{c}} \in D$ where $\vec{\mathbf{c}}$ is any vector that could be the unknown true vector $\dot{\vec{\mathbf{c}}}$), which appears in the m-multiple integral (4.0.1), must be estimated to discrete values, as we have to calculate a discrete number of probabilities of the number r transitions of sign

between each two consecutive data points known.[3] This can be done via binomial distribution of a random value. In fact, let's say: $\vec{\mathbf{c}} \in D$ is one of the vectors which claims that it is the unknown true vector $\dot{\vec{\mathbf{c}}}$ from function (3.0.3 on page 4); q_i are the elements of the sequence

$$q_1(\vec{\mathbf{c}}) = \tilde{y}_1 - f(\vec{\mathbf{c}}, x_1),\ q_2(\vec{\mathbf{c}}) = \tilde{y}_2 - f(\vec{\mathbf{c}}, x_2), \ldots,\ q_n(\vec{\mathbf{c}}) = \tilde{y}_n - f(\vec{\mathbf{c}}, x_n); \tag{4.0.4}$$

where η is a discreet random variable, that can assume values

$$r = r(\vec{\mathbf{c}}) = \sum_{i=1}^{n-1} \delta_i(\vec{\mathbf{c}}), \tag{4.0.5}$$

where

$$\delta_i = \delta_i(\vec{\mathbf{c}}) = \begin{cases} 1, \text{if} & q_i(\vec{\mathbf{c}})q_{i+1}(\vec{\mathbf{c}}) < 0 \\ 0, \text{if} & q_i(\vec{\mathbf{c}})q_{i+1}(\vec{\mathbf{c}}) \geq 0 \end{cases} \tag{4.0.6}$$

In meaningful terms, δ_i is either 0 or 1 depending on if there has been a transition of sign between two consecutive elements of the sequence (4.0.5). (If the product of those two elements is negative, then one of those elements is negative; hence the transition of sign). Then the value of r is the number of transitions of sign of the elements of (4.0.4) where $r \in [0, n-1]$. If it truly happens that $\vec{\mathbf{c}} = \dot{\vec{\mathbf{c}}}$ (the case that is being looked for), then the values of q_i would be nothing but the errors ξ_i, and by the condition $Med(\xi) = 0$ the probabilities p_r of r transitions of sign (or the event $\eta = r$) could be written as the classical formula of binomial distribution, where the probability of success (that there has been a transition of sign) is 0.5 (this is because out of the 4 possibilities where two points might be (above or below the function given by $\vec{\mathbf{c}}$), 2 produce a transition), and number of trials is equal to the number of pairs of consecutive points n, i.e. $n-1$.

$$p_r = \binom{n-1}{r} \cdot \left(\frac{1}{2}\right)^r \cdot \left(\frac{1}{2}\right)^{n-1-r} \quad = \quad \binom{n-1}{r} \cdot \left(\frac{1}{2}\right)^{n-1}$$

$$p_r = \frac{\binom{n-1}{r}}{2^{n-1}}, \quad r = 0, 1, \ldots, n-1. \tag{4.0.7}$$ <10>

Lets move on from discussing the question of the transition of sign with some one vector $\vec{\mathbf{c}}$, to the analysis of this situation with regards to the whole set D. Say that there exists a partitioning (splitting) of set D into a family of sub-sets $D_1, D_2, \ldots, D_n$ such that the elements (in this case vectors $\vec{\mathbf{c}}$) of sub-set D_r provide the same number $r-1$ of transitions of sign of elements of (4.0.4). Note that here the sub-sets D_r will take the form of m-dimensional polyhedrons [4] as they are in $\boldsymbol{R}^m$. In this way, the *a priori* probability[5] that the unknown true vector $\dot{\vec{\mathbf{c}}} \in D_r$ in given by (4.0.7). However, in any separate case, some of the sub-sets D_r could be empty, meaning that with some vectors of parameters $\vec{\mathbf{c}} \in D$, the number of transitions of sign of (4.0.4) will not equal the given value of r. The a priori probabilities p_r that the unknown true vector $\dot{\vec{\mathbf{c}}}$ is in one of the non-empty sub-sets D_r can be recalculated to the *posteriori* probabilities[6] p_r^*. Let's define I as the set of the numbers of all the non-empty sub-sets D_r. Then

$$p_r^* = \begin{cases} \frac{p_r}{\sum\limits_{s \in I} p_s}, & r \in I. \\ 0, & r \in \{0, 1, \ldots, n-1\} \setminus I. \end{cases} \tag{4.0.8}$$[7]

To clarify, here the a priori probabilities suggested a chance that the true vector $\dot{\vec{\mathbf{c}}}$ could be in sub-sets D_r which are empty. This is obviously not possible. The equation (4.0.8) nullifies those cases, and

[3] This is because, the more transitions of sign there are, the closer is $Med(\xi)$ to 0

[4] In geometry, a polyhedron is a solid in three (in this case m) dimensions with flat polygonal faces, straight edges and sharp vertices. A polyhedron is said to be convex, if any two points within it can be connected by a line segment with this segment also being within the polyhedron. <12>

[5] An a priori probability is a probability that is derived purely by deductive reasoning. <13>

[6] The posteriori probabilities of an event is the ratio of the number of outcomes in which a specified event occurs to the total number of trials, not in a theoretical sample space but in an actual experiment. In a more general sense, empirical probability estimates probabilities from experience and observation. <13>

[7] $r \in \{0, 1, \ldots, n-1\} \setminus I$, is a set of such r whose D_r are empty.

redistributes probabilities which were assigned to empty sets proportionally across the experimentally possible cases.

Because in our case the vectors $\vec{\mathbf{c}}$, are continuous values, and not discreet ones, then to use formula (4.0.1 on page 6) we must modify estimates p_r^* of the probabilities of event $\dot{\vec{\mathbf{c}}} \in D_r$ to estimates $p_r^*(\vec{\mathbf{c}}), \vec{\mathbf{c}} \in D_r$, of a probability density function. As there is no information suggesting the distribution of vectors $\vec{\mathbf{c}}$ in the bounds of each sub-set D_r, it is logical to assume uniform distribution across each sub-set. Then for all $r \in I$

$$P_r^*(\vec{\mathbf{c}}) = \frac{p_r^*}{\mu(D_r)}, \quad \vec{\mathbf{c}} \in D_r \tag{4.0.9}$$

where $\mu(D_r)$ is the Lebesgue measure (explained in section 2.5 on page 4) of sub-set D_r in $\boldsymbol{R}^m$.

Therefore (4.0.1 on page 6) becomes

$$E(\rho(\vec{\mathbf{c}}^*, \dot{\vec{\mathbf{c}}})) = \sum_{r \in I} \frac{p_r^*(\vec{\mathbf{c}})}{\mu(D_r)} \int\limits_{D_r} \rho(\vec{\mathbf{c}}^*, \dot{\vec{\mathbf{c}}}) \mathrm{d}c_1 \mathrm{d}c_2 \dots \mathrm{d}c_m \tag{4.0.10}$$

(where $\int_{D_r} (\bullet)$ is a laconic notation of an m-multiple integral)

In the general case, such as this one, the limits of integration of each univariate integral depend on other variables, based on which the integration of the external integral relative to the given univariate integral is carried out. <5> (Refer to the example in section 2.4 on page 3). This significantly complicates the calculation of these multiple integrals.

4.1 Illustrative example

Let's consider an illustrative example. Here, I will be comparing the derived method to the LSM. In this example $m = 1$ (i.e. the vector $\dot{\vec{\mathbf{c}}}$ can be treated as a scalar value), $y = f(\vec{\mathbf{c}}; x)$ is a function with one parameter (i.e. a horizontal line), where $\dot{\vec{\mathbf{c}}}$ is a vector that has to be found. Let $n = 6$ and $D = \{\vec{\mathbf{c}} : 2 \leq \vec{\mathbf{c}} \leq 6\}$.

Table 1: Table of values for this example

i	x_i	$f(\dot{\vec{\mathbf{c}}}, x_i)$	ξ_i	$\tilde{y}_i = f(\dot{\vec{\mathbf{c}}}, x_i) + \xi_i$
1	1	4	1.0	5.0
2	2	4	-0.1	3.9
3	3	4	0.2	4.2
4	4	4	10	14.0
5	5	4	-0.15	3.85
6	6	4	0.3	4.3

For this example I will use proximity function (4.0.3 on page 6) also used in LSM. I am doing this as I will later compare the two methods.
As we already know, we can calculate the a priori probabilities using (4.0.7 on the previous page), where $r = 0, 1, 2, 3, 4, 5$ like so:

$$p_0 = \frac{\binom{5}{0}}{2^5} = \frac{1}{32};\ p_1 = \frac{\binom{5}{1}}{2^5} = \frac{5}{32};\ p_2 = \frac{\binom{5}{2}}{2^5} = \frac{10}{32};\ p_3 = \frac{\binom{5}{3}}{2^5} = \frac{10}{32};\ p_4 = \frac{\binom{5}{4}}{2^5} = \frac{5}{32};\ p_5 = \frac{\binom{5}{5}}{2^5} = \frac{1}{32}$$

But in this case there can only be certain values of r, namely the values of set $I = \{0, 2, 3, 4\}$. (This is because in this example there is no possible vector $\vec{\mathbf{c}}$ that would produce a line that would give either 1 or 5 transitions of sign). And ,so

$$\sum_{s \in I} p_s = p_0 + p_2 + p_3 + p_4 = \frac{1}{32} + \frac{10}{32} + \frac{10}{32} + \frac{5}{32} = \frac{26}{32}.$$

So, to calculate the posteriori probabilities using (4.0.9 on the preceding page)

$$\begin{aligned} p_0^* &= p_0 : \frac{26}{32} = \frac{1}{32} : \frac{26}{32} = \frac{1}{26} \\ p_1^* &= 0 \\ p_2^* &= p_2 : \frac{26}{32} = \frac{10}{32} : \frac{26}{32} = \frac{10}{26} \\ p_3^* &= p_3 : \frac{26}{32} = \frac{10}{26} \\ p_4^* &= p_4 : \frac{26}{32} = \frac{5}{32} : \frac{26}{32} = \frac{5}{26} \\ p_5^* &= 0 \end{aligned}$$

Calculating $E(\rho(\vec{\mathbf{c}}^*, \vec{\mathbf{c}}))$

$$E(\rho(\vec{\mathbf{c}}^*, \vec{\mathbf{c}})) = \int_2^{3.85} \underbrace{(\vec{\mathbf{c}}^* - \vec{\mathbf{c}})^2}_{\rho(\vec{\mathbf{c}}^*, \vec{\mathbf{c}})} \cdot \underbrace{\frac{\overbrace{\frac{1}{26}}^{p_0^*}}{3.85 - 2}}_{\mu(D_0)} \mathrm{d}\vec{\mathbf{c}} + \int_{3.85}^{3.9} (\vec{\mathbf{c}}^* - \vec{\mathbf{c}})^2 \cdot \frac{\frac{10}{26}}{1.05} \mathrm{d}\vec{\mathbf{c}} + \int_{3.9}^{4.3} (\vec{\mathbf{c}}^* - \vec{\mathbf{c}})^2 \cdot \frac{\frac{5}{26}}{4.3 - 3.9} \mathrm{d}\vec{\mathbf{c}}$$
$$+ \int_{4.3}^{5.0} (\vec{\mathbf{c}}^* - \vec{\mathbf{c}})^2 \cdot \frac{\frac{10}{26}}{5.0 - 4.3} \mathrm{d}\vec{\mathbf{c}} + \int_{5.0}^{6.0} (\vec{\mathbf{c}}^* - \vec{\mathbf{c}})^2 \cdot \frac{\frac{10}{26}}{1.05} \mathrm{d}\vec{\mathbf{c}}$$

$$\begin{aligned} E(\rho(\vec{\mathbf{c}}^*, \vec{\mathbf{c}})) = \; & 0.021 \cdot \left[(\vec{\mathbf{c}}^*)^2 \cdot \vec{\mathbf{c}} - \vec{\mathbf{c}}^* \cdot \vec{\mathbf{c}}^2 + \frac{1}{3} \vec{\mathbf{c}}^3 \right]_2^{3.85} \\ + \; & 0.366 \cdot \left[(\vec{\mathbf{c}}^*)^2 \cdot \vec{\mathbf{c}} - \vec{\mathbf{c}}^* \cdot \vec{\mathbf{c}}^2 + \frac{1}{3} \vec{\mathbf{c}}^3 \right]_{3.85}^{3.9} \\ + \; & 0.481 \cdot \left[(\vec{\mathbf{c}}^*)^2 \cdot \vec{\mathbf{c}} - \vec{\mathbf{c}}^* \cdot \vec{\mathbf{c}}^2 + \frac{1}{3} \vec{\mathbf{c}}^3 \right]_{3.9}^{4.3} \\ + \; & 0.549 \cdot \left[(\vec{\mathbf{c}}^*)^2 \cdot \vec{\mathbf{c}} - \vec{\mathbf{c}}^* \cdot \vec{\mathbf{c}}^2 + \frac{1}{3} \vec{\mathbf{c}}^3 \right]_{4.3}^{5.0} \\ + \; & 0.366 \cdot \left[(\vec{\mathbf{c}}^*)^2 \cdot \vec{\mathbf{c}} - \vec{\mathbf{c}}^* \cdot \vec{\mathbf{c}}^2 + \frac{1}{3} \vec{\mathbf{c}}^3 \right]_{5.0}^{6.0} \\ = \; & 1.000 \cdot (\vec{\mathbf{c}}^*)^2 - 9.55 \vec{\mathbf{c}}^* + 23.282 \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}E(\rho(\vec{\mathbf{c}}^*, \vec{\mathbf{c}}))}{\mathrm{d}\vec{\mathbf{c}}^*} &= 2\vec{\mathbf{c}}^* - 9.55 = 0 \\ \vec{\mathbf{c}}^* &= \frac{9.55}{2} = 4.773 \end{aligned}$$

Seeing that $\dot{\vec{\mathbf{c}}} = 4.0$, we finally calculate ρ given by the derived method

$$\rho = |4.773 - 4.0| = 0.773$$

Error made by this method is 0.431. Now let's see what kind of error will the LSM give:

Using Least Square Method

$$\phi(\vec{\mathbf{c}}^*) = \sum_{i=1}^{6}(\tilde{y}_i - \vec{\mathbf{c}}^*)^2 \longrightarrow min \qquad \text{(Sum of the squares of errors has to be minimised)}$$

$$\phi(\vec{\mathbf{c}}^*) = \sum_{i=1}^{6}(\tilde{y}_i^2 - 2\tilde{y}_i\vec{\mathbf{c}}^* + (\vec{\mathbf{c}}^*)^2) \qquad \text{(Expanding the polynomial)}$$

$$\frac{\mathrm{d}\phi(\vec{\mathbf{c}}^*)}{\mathrm{d}\vec{\mathbf{c}}^*} = \sum_{i=1}^{6}(0 - \cancel{2}\tilde{y}_i + \cancel{2}\vec{\mathbf{c}}^*) = 0 \qquad \text{(Finding the derivative and equating it to zero)}$$

$$\sum_{i=1}^{6}\vec{\mathbf{c}}^* = \sum_{i=1}^{6}\tilde{y}_i \qquad \text{(Re-arranging the result)}$$

$$6 \cdot \vec{\mathbf{c}}^* = \sum_{i=1}^{6}\tilde{y}_i = 35.25 \qquad \text{(35.25 here refers to the sum of all function values)}$$

$$\vec{\mathbf{c}}^* = \frac{35.25}{6} = 5.875 \qquad \text{(Solving for } \vec{\mathbf{c}}^*\text{)}$$

And so the error $\rho = 1.875$

LSM gives a bigger error because in it deviations are squared <9>, and therefore large derivations are weighted more heavily. Because this example shows, that in the case of significant non-compliance to condition $E(\xi) = 0$, my method is notably superior to LSM, and because of the limited format of this exploration, I will not include further solutions that used LSM. Back to the theory.

Let the proximity $\rho(\vec{\mathbf{c}}^*, \vec{\mathbf{c}})$ in (4.0.10 on page 8), be taken in the form (4.0.3 on page 6). This proximity is chosen because it is the one used in LSM. Then the following simplification takes place.

$$E(\rho(\vec{\mathbf{c}}^*, \dot{\vec{\mathbf{c}}})) = \sum_{r\in I}\frac{p_r^*}{\mu(D_r)}\int\limits_{D_r}\sum_{j=1}^{m}\left(c_j^* - c_j\right)^2 \mathrm{d}c_1\mathrm{d}c_2\ldots\mathrm{d}c_m$$

$$expanding\left(c_j^* - c_j\right)^2$$

$$= \sum_{r\in I}\frac{p_r^*}{\mu(D_r)}\sum_{j=1}^{m}\left[(c_j^*)^2\underbrace{\int\limits_{D_r}\mathrm{d}c_1\mathrm{d}c_2\ldots\mathrm{d}c_m}_{=\mu(D_r)} - 2c_j^*\int\limits_{D_r}c_j\ \mathrm{d}c_1\mathrm{d}c_2\ldots\mathrm{d}c_m + \int\limits_{D_r}c_j^2\ \mathrm{d}c_1\mathrm{d}c_2\ldots\mathrm{d}c_m\right]$$

$$= \sum_{j=1}^{m}\left(\sum_{r\in I}\frac{p_r^*}{\mu(D_r)}\left[(c_j^*)^2\mu(D_r) - 2c_j^*\int\limits_{D_r}c_j\ \mathrm{d}c_1\mathrm{d}c_2\ldots\mathrm{d}c_m + \int\limits_{D_r}c_j^2\ \mathrm{d}c_1\mathrm{d}c_2\ldots\mathrm{d}c_m\right]\right)$$

$$= \sum_{j=1}^{m}\left((c_j^*)^2\cancel{\mu(D_r)}\underbrace{\sum_{r\in I}\frac{p_r^*}{\cancel{\mu(D_r)}}}_{\sum_{r\in I}p_r^*=1} - 2c_j^*\sum_{r\in I}\frac{p_r^*}{\mu(D_r)}\int\limits_{D_r}c_j\ \mathrm{d}c_1\mathrm{d}c_2\ldots\mathrm{d}c_m + \sum_{r\in I}\frac{p_r^*}{\mu(D_r)}\int\limits_{D_r}c_j^2\ \mathrm{d}c_1\mathrm{d}c_2\ldots\mathrm{d}c_m\right) \tag{4.1.1}$$

And finally:

$$E(\rho(\vec{\mathbf{c}}^*,\dot{\vec{\mathbf{c}}})) = \sum_{j=1}^{m}\left((c_j^*)^2 - 2c_j^*\sum_{r\in I}p_r^*\frac{\int\limits_{D_r} c_j\ \mathrm{d}c_1\mathrm{d}c_2\ldots\mathrm{d}c_m}{\mu(D_r)} + \underbrace{\sum_{r\in I}p_r^*\frac{\int\limits_{D_r} c_j^2\ \mathrm{d}c_1\mathrm{d}c_2\ldots\mathrm{d}c_m}{\mu(D_r)}}_{\text{derivative of this with respect to } c_1^*,c_2^*,\ldots,c_m^*=0}\right) \tag{4.1.2}$$

In order to find the vector $\vec{\mathbf{c}}^*$, which provides the minimum of function (4.1.2) and which I will take as the optimal estimate of the unknown vector $\dot{\vec{\mathbf{c}}}$, it is necessary (like in the case with a simple one-parameter function) to take the first partial derivative $\frac{\partial E}{\partial c_j^*}$ of function (4.1.2) with respect to each variable c_j^* <4>. After that, equate these derivatives to zero and solve the resulting system of linear equations. In this case, this system splits into m separate linear equation with one variable:

$$2c_j^* - 2\sum_{r\in I}p_r^*\cdot\frac{\int\limits_{D_r} c_j\mathrm{d}c_1\mathrm{d}c_2\ldots\mathrm{d}c_m}{\mu(D_r)} = 0, \quad j=1,2,\ldots,m. \tag{4.1.3}$$

These equations have the following solutions:

$$c_j^* = \sum_{r\in I}p_r^*\bar{c}_{(j,r)}, \quad j=1,2,\ldots,m, \tag{4.1.4}$$

where

$$\bar{c}_{(j,r)} = \frac{\int\limits_{D_r} c_j\mathrm{d}c_1\mathrm{d}c_2\ldots\mathrm{d}c_m}{\mu(D_r)}. \tag{4.1.5}$$

After calculating the vector $\vec{\mathbf{c}}^*$, plugging it back into this equation will produce a prediction on the error that the method made in calculating $\vec{\mathbf{c}}^*$, the expected error. An indication on the factual error the method made.

5 Algorithm trials

Let us look at a model function, the parameters (coefficients) of which will have to be approximated by the method derived in this exploration and the known, conventional methods LAD and LSM. This function $f(\dot{\vec{\mathbf{c}}},x) = \dot{c}_1 + \dot{c}_2 x$ has two parameters which will be $\dot{c}_1 = 2$ and $\dot{c}_2 = 0.5$ respectively. The approximation of these parameters $\dot{c}_1$ and $\dot{c}_2$ will be carried out from $n = 20$ values (containing random errors ξ_i) $\tilde{y}_i = f(\dot{\vec{\mathbf{c}}},x)+\xi_i, \quad i=1,2,\ldots,n$. The errors ξ_i are random normally distributed values with mean 0 and standard deviation σ.

Due to a the technical requirement (described in Appendix A: section 7 on page 15) (and also theoretically explained in section 3 on page 4) it is also necessary that the true vector $\dot{\vec{\mathbf{c}}}$ be limited to a set D. Therefore we define this set D such that

$$\dot{\vec{\mathbf{c}}} = (\dot{c}_1,\dot{c}_2) \in D = \{(c_1,c_2): \quad 1\leqslant c_1\leqslant 3; \quad 0\leqslant c_2\leqslant 1\}. \tag{5.0.1}$$

All values were calculated using the technical description in Appendix A, with custom built software.

5.1 Trial 1: Comparison of derived method vs. LAD

Patterns in the derived algorithm will not show in individual solutions, because of that we will look at the averages of 5 solutions each with a different randomized set of errors ξ_i an therefore different values $\tilde{y}_i$.

We can notice a concrete pattern in these values (shown in Table 2 on the next page). For the

Table 2: Average results of 5 algorithm trials. Each group of 5 calculated with a increasing standard deviation σ of error ξ_i. Proximity function used (4.0.2 on page 6) is taken from LAD in order to compare the derived method with LAD.

Intensity of error	Derived method				LAD		
	c_1^*	c_2^*	Expected error	Factual error	c_1^*	c_2^*	Factual error
$\sigma = 0.1$	2.027	0.497	0.269	0.192	2.051	0.459	0.100
$\sigma = 0.3$	2.049	0.517	0.468	0.302	2.146	0.388	0.288
$\sigma = 1.0$	2.218	0.459	0.655	0.354	2.411	0.297	0.697

solutions of the derived method, as the standard deviation σ of error ξ increases (i.e. the error values spread to a larger range.) we also see an increase in both the expected error proposed by the derived method and the factual error it made. This is not surprising as this is the expected result of the increase of σ of ξ. There is also nothing special in that LAD is generally better than the derived method at giving the parameters of the function, other than the case where $\sigma = 1.0$, there (I presume) some values ξ were slanted to be large in the positive sign, skewing LAD's solution up. The reason why the derived method gives a better result there is because it is less responsive to such large peaks in ξ as it weighs them the same way as the rest of the values of ξ. This will be important in the second trial.

5.2 Trial 2: Comparison of derived method vs. LSM

Clearly, the derived method will have advantage over the LSM if the condition $E(\xi) = 0$ is not fulfilled. To show this, I calculated 5 algorithm trails using the same description as stated above with $\sigma = 0.3$ while using the proximity function used in LSM, but with one big difference. Two of the largest values of ξ are increased by a factor of 5. Table 3 illustrates the solutions given by the derived method and LSM with these conditions in mind.

The factual errors made by the derived method is tens times smaller than the ones made by LSM.

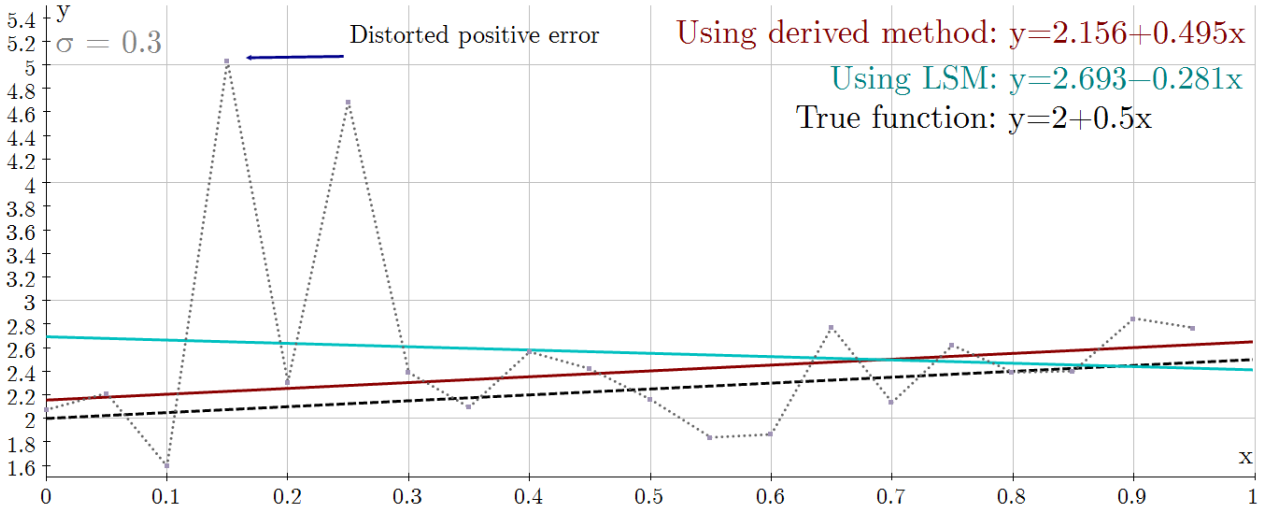
Table 3: One algorithm run with 5 trials (each trial has randomized errors ξ_i). Proximity function used (4.0.3 on page 6) is taken from LSM in order to compare the derived method to LSM.

Trial	Derived method				LSM		
	c_1^*	c_2^*	Expected error	Factual error	c_1^*	c_2^*	Factual error
1	2.065	0.404	0.199	0.013	2.593	-0.217	0.866
2	2.136	0.556	0.157	0.021	2.416	0.438	0.177
3	1.844	0.707	0.161	0.067	2.470	0.223	0.298
4	2.156	0.495	0.150	0.024	2.693	-0.281	1.091
5	2.156	0.343	0.161	0.049	2.550	0.075	0.483
Average	2.071	0.501	0.165	0.035	2.544	0.047	0.583

This is semi-misleading, as the LSM proximity (factual error in $c_1 = (\dot{c_1} - c_1)^2$ or the difference between true and approximated value all squared.) function was used to calculate the factual error for both methods, and as the difference between the true and approximated components by the derived method

are themselves quite small and less than 1, their squares will resemble the factual error displayed in the table.

Figure 2: Graph of trial number 4 from Table 3. LSM's solution is definitively worse.



To visually represent in what ways the derived method is superior to LSM, I graphed the 4th trail from the previous table in Figure 2. Here, due to the two enlarged values of errors ξ (which are weighted more by the LSM) LSM provides a solution linear function with a negative gradient, completely misrepresenting the true function that has a positive gradient.

5.3 Trial 3: Comparison of Solutions of Derived Method with Increasing n.

To show the pattern here, 4 series of algorithm runs were calculated using an increasing number of point n. Each series contains 25 individual trials, where each trial used differing a randomized set of normally distributed values of ξ_i with $\sigma = 0.3$. Table 4 contains the average values of the parameters c_1^* and c_2^*.

Table 4: Average values of c_1^* and c_2^* with an increasing number of points n used.

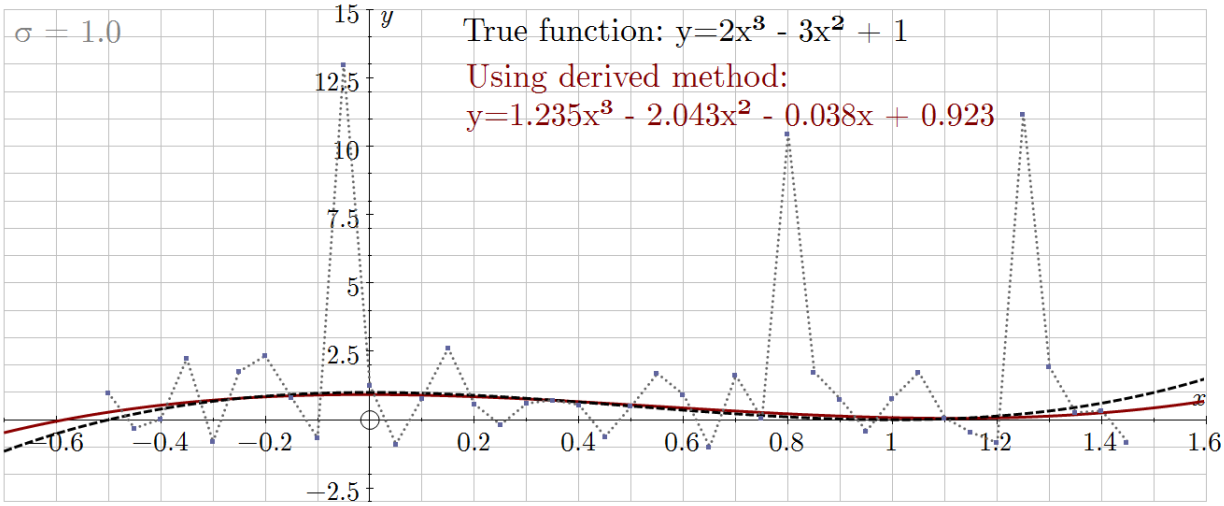
Average	$n = 6$	$n = 10$	$n = 20$	$n = 30$
c_1^*	2.089	2.025	1.999	1.993
c_2^*	0.3877	0.4759	0.4816	0.4824

Yet again there is a concrete pattern: with the increase of the number of point n (i.e. with the increase of information known about the true function), the accuracy of the method derived also increases.

5.4 Trial 4: Approximation of a Cubic Function with the Derived Method.

The Figure 3 on the next page shows the solutions calculated by the derived method when the true function is a 3rd degree polynomial. Figure 3 on the following page shows the solution when $\sigma = 1.0$. Also, similarly to Trial 2, 3 of the largest values of ξ_i have been increased by a factor of 5. The derived method proves to approximate the true 3rd degree function exceptionally well.

Figure 3: Graph of a 3rd degree polynomial and its approximation using the derived method with $\sigma = 0.1$



6 Conclusion

In this exploration I have successfully derived a reliable and predictable data approximation method based on the condition that errors must be assumed to have a median value equal to zero. During this exploration I have learned about multiple data approximation methods and their strengths and weaknesses. The most commonly used Least Squares Method has high accuracy and is easily implemented with a set of linear equations. Mostly it is a reliable way of approximation and modelling a set of discrete data points into a continuous functional representation. However, LSM is highly responsive to outliers, and therefore less accurate in cases with them. The Least Modulus Deviations method is conventionally used in such cases, however it too has a flaw. In neither method can one estimate the error made by the method itself, in contrast with the here derived method.

I have satisfied the aim of deriving a method that would perform better in cases where LSM lacks in accuracy. The values of errors are not regarded as valuable information in this method, only if they are above or below the approximated function. Meaning that the method derived will always give a better representation of a data set with outliers than LSM.

This is not to say that the method derived is perfect. It requires an estimate of all parameters of the looked for function to be determined before calculation, making room for systematic error. It is also quite complex to calculate, with increasing difficulty the more parameters have to be found. Even though this method is suited for situations with outliers, it is still not capable of giving a reliable solution in every case, say when all/most error values are of a single sign. In fact, a 'perfect method' is impossible to derive as any method must assume a set of conditions (just like the 3 methods mentioned here have) and this set of conditions can not conceivably encapsulate all possible cases.

Others could expand of the here derived method of approximation by removing the need to specify an estimated range for all functional parameters, effectively reducing the complexity of the calculation and removing the possibility of systematic error. This will also allow for the method to be used with functions other then polynomials, in fact all functions, as long as they depend on a finite set of parameters and there is determinate information about them, could be approximated this way (some not with great accuracy though).

7 Appendix A

7.1 Method Description of Computation of Theoretical Algorithm

Here I will make use of the product notation:

$$\prod_{i=m}^{n} x_i = x_m \cdot x_{m+1} \cdot \ \cdots \ \cdot x_{n-1} \cdot x_n.$$

This essentially works in much of the same way the familiar summation notation we use in class(using the Greek capital letter sigma), except that instead of summing values, it multiplies them (and uses Greek capital letter pi). <14>

From the point of view of practical realisation of my algorithm, that is based of formulae (4.1.4 on page 11),(4.1.5 on page 11), I can ask two questions: (1) Is it possible to derive a simple method of constructing the sets D_r? (2) Is it possible to derive a simple method of calculation the integrals of those sets D_r, which appear in the RHS of (4.1.5 on page 11). What a geometrical construction (Figure omitted) has shown me, is that even in the case where $m = 2$ and $n = 10$, the regions D_r have a relatively complicated geometry (some multi-peaked star-like shapes). However, this problem can be solved with the principal of the 'Gordian knot' - refuse to work with directly with sets D_r, but instead, to use their point (grid) approximation (even without describing these sets concretely) and use an analog of (4.1.4 on page 11).

The essence of this suggested method of thinking is as follows. Define $k(1), k(2), \ldots, k(m)$ as some quite large real values, and a m-dimensional Parallelotope

$$W = \{c : c_j^{(\min)} \le c_j \le c_j^{(\max)}, \quad j = 1, 2, \ldots, m\} \subset \boldsymbol{R}^m \tag{7.1.1}$$

which contains the set D (if the set D itself is a Parallelotope, then $W = D$). Cover this Parallelotope with a dense grid Γ with

$$L = \prod_{t=1}^{m} k(t) \tag{7.1.2}$$

number of nodes $\vec{\mathbf{c}}^{(l)} = (c_1^{(l)}, (c_2^{(l)}, \ldots, c_m^{(l)}), \quad l = 1, 2, \ldots, L$, where each coordinate $c_j^{(l)}, \quad j = 1, 2, \ldots, m$, could independently from all the other $m - 1$ coordinates (note that $m - 1$ does not refer to sets of coordinates) assume one of $k(j)$ of values

$$c_j = c_j^{(\min)} + \frac{c_j^{(\max)} - c_j^{(\min)}}{k(j)}(t - \frac{1}{2}), \quad t = 1, 2, \ldots, k(j). \tag{7.1.3}$$

Parallel to grid Γ implement the system A of Parallelotopes $W_l \subset \boldsymbol{R}^m, \quad l = 1, 2, \ldots, L$, the centres of which are their nodes with sides

$$h_j = \frac{(c_j^{(\max)} - c_j^{(\min)})}{k(j)} : \tag{7.1.4}$$

$$W_l = \{\vec{\mathbf{c}} = (c_1, c_2, \ldots, c_m) : c_j^{(l)} - \frac{1}{2}h_j \le c_j \le c_j^{(l)} + \frac{1}{2}h_j, \quad j = 1, 2, \ldots, m\}. \tag{7.1.5}$$

These Parallelotopes form the surface of the Parallelotope W. Now we have to chose from the defined L nodes those, the ones who belong to the set D. Let L_0 be the quantity of the chosen nodes. Renumber the chosen nodes, by assigning the first L_0 numbers l to them.

For each $l = 1, 2, \ldots, L_0$ find elements of $q_i(c^{(l)}), \quad i = 1, 2, \ldots, n$ of sequence (4.0.4 on page 7), and based on their values, using (4.0.5 on page 7), calculate the number r of transition of sign of elements of (4.0.4 on page 7). Let $J_r, \quad r = 0, 1, \ldots, n - 1$ be the set of numbers l of chosen nodes $c^{(l)}$ providing r transitions of sign, and $|J_r|$ - the number of nodes giving that many transitions. In

this way, without directly constructing the sub-sets D_r, I approximate each of them with the union. <15>

$$\tilde{D}_r = \bigcup_{l \in J_r} W_l. \tag{7.1.5}$$[8]

It is clear, that the assumption, that all vectors $\vec{\mathbf{c}}$, belonging to the Parallelotope W_l, will provide the same number of transitions of sign as the center $\vec{\mathbf{c}}^{(l)}$ of the Parallelotope, will only be incorrect for those Parallelotope W_l, and do not completely belong to some one sub-set D_r. But with a dense grid Γ, and respectively, quite small-sized Parallelotope W_l, the percent of such 'debatable' Parallelotope compared to their general number L_0 will be quite small (so, in figure (omitted), when $m = 2$, these 'debatable' Parallelotopes are positioned along the lines, and the 'non-debatable' - in the areas). In addition, the error resulting from classification of such 'debatable' Parallelotopes W_l completely to some set D_r, will get (mostly) redeemed/repaid. Summing up all these arguments, it can be said, that the approximations of sets D_r, with the union of relatively 'small' Parallelotopes will give a good approximation of the mathematical expectancy (4.0.10 on page 8) of error in estimation of parameters of a studies functional dependence:

$$E(\rho_2(\vec{\mathbf{c}}^*, \vec{\mathbf{c}})) \approx \sum_{r \in I} \frac{p_r^*}{\mu(\tilde{D}_r)} \int\limits_{\tilde{D}_r} \sum_{j=1}^{m} (c_j^* - c_j)^2 \mathrm{d}c_1 \mathrm{d}c_2 \dots \mathrm{d}c_m =$$

(integral with limit D_r can be split as the sum of integrals with limit W_l)

$$\sum_{r \in I} \frac{p_r^*}{\mu(\tilde{D}_r)} \sum_{l \in J_r} \int\limits_{W_l} \sum_{j=1}^{m} (c_j^* - c_j)^2 \mathrm{d}c_1 \mathrm{d}c_2 \dots \mathrm{d}c_m = \tag{7.1.6}$$

(the sum $\sum_{j=1}^{m}$ is facrotised out)

$$\sum_{j=1}^{m} \left[\sum_{r \in I} \frac{p_r^*}{\mu(\tilde{D}_r)} \left(\sum_{l \in J_r} \int\limits_{W_l} ((c_j^*)^2 - 2c_j^* c_j + c_j^2) \right) \mathrm{d}c_1 \mathrm{d}c_2 \dots \mathrm{d}c_m \right] =$$

(expanding $((c_j^*)^2 - 2c_j^* c_j + c_j^2)$)

$$\sum_{j=1}^{m} \left[\sum_{r \in I} \frac{p_r^*}{\mu(\tilde{D}_r)} \left(\sum_{l \in J_r} \left[(c_j^*)^2 \mu(W_{(l)}) - 2c_j^* \int\limits_{W_l} c_j \mathrm{d}c_1 \mathrm{d}c_2 \dots \mathrm{d}c_m + \int\limits_{W_l} c_j^2 \mathrm{d}c_1 \mathrm{d}c_2 \dots \mathrm{d}c_m \right] \right) \right]. \tag{7.1.7}$$

If in a multiple integral, the limits of integration for each variable are given by other variables, then this multiple integral is equal to the product of included in it, one-dimensional integrals <5>. Now, if we define coordinates of the edges of W_l as $c_j^{(l)} - 0.5h_j = a(j,l)$ and $c_j^{(l)} + 0.5h_j = b(j,l)$ then it is not hard to be sure that

$$\frac{1}{\mu(W_l)} \int\limits_{W_l} c_j \mathrm{d}c_1 \mathrm{d}c_2 \dots \mathrm{d}c_m = \frac{1}{\prod\limits_{s=1}^{m} h_s} \left(\int\limits_{b(j,l)}^{a(j,l)} c_j \mathrm{d}c_j \right) \prod_{\substack{s=1 \\ (s \neq j)}}^{m} \int\limits_{a(s,l)}^{b(s,l)} \mathrm{d}c_s = c_j^{(l)}. \tag{7.1.8}$$

If in addition we accept that

$$\sum_{r \in I} p_r^* = 1, \tag{7.1.9}$$

$$\sum_{l \in J_r} \mu(W_l) = \mu(\tilde{D}_r), \tag{7.1.10}$$

$$\mu(W_l) \equiv \prod_{j=1}^{m} h_j, \tag{7.1.11}$$

[8] Here, $\tilde{D}_r$ is the arbitrary union of all sets W_l such that l is an element of J_r.

then the simplification chain (7.1.7 on the previous page) could be continued

$$E(\rho_2(\vec{\mathbf{c}}^*,\vec{\mathbf{c}})) \approx \sum_{j=1}^{m}\left[\sum_{r\in I}\frac{p_r^*}{\mu(\tilde{D}_r)}\left(\sum_{l\in J_r}\left[(c_j^*)^2\mu(W_l)-2c_j^*\mu(W_l)c_j^{(l)}+\int\limits_{W_l}c_j^2\mathrm{d}c_1\mathrm{d}c_2\ldots\mathrm{d}c_m\right]\right)\right]=$$

$$\left(\text{ middle step: }\sum_{l\in J_r}^{m}(c_j^*)^2\mu(W_l)=(c_j^*)^2\underbrace{\sum_{l\in J_r}\mu(W_l)}_{=\mu(\tilde{D}_r)}=(c_j^*)^2\mu(\tilde{D}_r)\ \right)$$

$$\sum_{j=1}^{m}\left[(c_j^*)^2\sum_{r\in I}p_r^*-2c_j^*\sum_{r\in I}\frac{p_r^*}{\mu(\tilde{D}_r)}\sum_{l\in J_r}\mu(W_l)c_j^{(l)}+\sum_{r\in I}\frac{p_r^*}{\mu(\tilde{D}_r)}\sum_{l\in J_r}\int\limits_{W_l}c_j^2\mathrm{d}c_1\mathrm{d}c_2\ldots\mathrm{d}c_m\right]=$$

$$\left(\text{ middle step: }\sum_{r\in I}(c_j^*)^2\frac{p_r^*}{\cancel{\mu(\tilde{D}_r)}}\cancel{\mu(\tilde{D}_r)}=(c_j^*)^2\underbrace{\sum_{r\in I}p_j^*}_{=1}=(c_j^*)^2\ \right)$$

$$\sum_{j=1}^{m}\left[(c_j^*)^2-2c_j^*\sum_{r\in I}p_j^*\frac{1}{\sum\limits_{l\in J_r}\mu(W_l)}\sum_{l\in J_r}\mu(W_j)c_j^{(l)}+\sum_{r\in I}\frac{p_r^*}{\mu(\tilde{D}_r)}\sum_{l\in J_r}\int\limits_{W_l}c_j^2\mathrm{d}c_1\mathrm{d}c_2\ldots\mathrm{d}c_m\right]=$$

$$\sum_{j=1}^{m}\left[(c_j^*)^2-2c_j^*\sum_{r\in I}p_j^*\frac{1}{|J_r|\prod\limits_{s=1}^{m}h_s}\sum_{l\in J_r}\left(\prod_{s=1}^{m}h_s\right)c_j^{(l)}+\sum_{r\in I}\frac{p_r^*}{\mu(\tilde{D}_r)}\sum_{l\in J_r}\int\limits_{W_l}c_j^2\mathrm{d}c_1\mathrm{d}c_2\ldots\mathrm{d}c_m\right]=$$

$$\sum_{j=1}^{m}\left[c_j^*)^2-2c_j^*\sum_{r\in I}\frac{p_j^*}{|J_r|}\sum_{l\in J_r}c_j^{(l)}+\sum_{r\in I}\frac{p_r^*}{\mu(\tilde{D}_r)}\sum_{l\in J_r}\int\limits_{W_l}c_j^2\mathrm{d}c_1\mathrm{d}c_2\ldots\mathrm{d}c_m\right]. \tag{7.1.12}$$

In order to find vector $\vec{\mathbf{c}}^*$, which provides the minimum estimate mathematical expectation (7.1.12) of error in the approximation of unknown true parameters of the functional dependence in question, and therefore also of vector $\dot{\vec{\mathbf{c}}}$, we take the same steps as in the case when we calculated the minimum of function (4.1.2 on page 11). Taking into account, that the last, third, summand in (7.1.12) does not contain any variables c_j^* (which are being optimised), its derivative is again equal to zero, we find its partial derivatives:

$$\frac{\partial E(\rho_2(\vec{\mathbf{c}}^*,\vec{\mathbf{c}}))}{\partial c_j^*}\approx 2c_j^*-2\sum_{r\in I}\frac{p_r^*}{|J_r|}\sum_{l\in J_r}c_j^{(l)},\quad j=1,2,\ldots,m. \tag{7.1.13}$$

Equating these partial derivatives (7.1.13) to zero we get, and therefore conclude with:

$$c_j^*=\sum_{r\in I}\frac{p_r^*}{|J_r|}\sum_{l\in J_r}c_j^{(l)},\quad j=1,2,\ldots,m. \tag{7.1.14}$$

This equation (7.1.14) is the one used in calculations of approximate values of the optimised vector $\vec{\mathbf{c}}^*$ in my custom software, the trails of which are earlier in this document.

Now, let's compare previousy suggested in "Reggression-type Problems under Zero Median Noise" equations (4.1.4 on page 11) and (4.1.5 on page 11) with the suggested equation (7.1.14). The superiority of the later is shown in that: 1. it does not require the direct construction of sub-sets D_r , which in fact are quite complex convex polyhedrons in $\boldsymbol{R}^m$. Therefore 2. this equation does not require the expansive calculation of multiple integrals found in (4.1.4 on page 11) and (4.1.5 on page 11). In "Reggression-type Problems under Zero Median Noise" there is one more problem at

hand: even a polygon, not to mention a polyhedron, can not be defined only by its vertexes (those could be connected with line segments in not just one way). The algorithm proposed in this exploration to estimate the optimal parameters $\vec{\mathbf{c}}^*$, only requires the regular (evenly distributed) surface of set D, for the nodes of which the only thing left to do is evaluate the elements of the sequence (4.0.4 on page 7).

8 Bibliography

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