Relativistic Electrodynamics

The Brief Summary

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Syllabus

The Syllabus and Learning Outcomes

https://www.dur.ac.uk/faculty.handbook/module_description/?year=2017&module_code=PHYS3661

Content

- The syllabus contains:
- Relativistic Electrodynamics: Einstein's postulates, the geometry of relativity, Lorentz transformations, structure of space-time, proper time and proper velocity, relativistic energy and momentum, relativistic kinematics, relativistic dynamics, magnetism as a relativistic phenomenon, how the fields transform, the field tensor, electrodynamics in tensor notation, relativistic potentials, scalar and vector potentials, gauge transformations, Coulomb gauge, retarded potentials, fields of a moving point charge, dipole radiation, radiation from point charges.

Learning Outcomes

Subject-specific Knowledge:

- Having studied this module, students will have developed a working knowledge of tensor calculus, and be able to apply their understanding to relativistic electromagnetism.

Reading List

http://www.dur.ac.uk/physics/modules/2017/phys3661/
Sections in “Introduction to Electrodynamics”, D. J. Griffiths.

1. Einsteins postulates [12.1]
2. The geometry of relativity [12.1]
3. Lorentz transformations [12.1]
4. Structure of space-time [12.1]
5. Proper time and proper velocity [12.2]
6. Relativistic energy and momentum [12.2]
7. Relativistic Kinematics [12.2]
8. Relativistic Dynamics [12.2]
9. Magnetism as a relativistic phenomena [12.3]
10. How the Fields transform [12.3]
11. The Field Tensor [12.3]
12. Electrodynamics in Tensor notation [12.3]
13. Relativistic potentials [12.3]
14. Scalar and Vector potentials [10.1]
15. Gauge transformations [10.1]
16. Coulomb gauge [10.1]
17. Retarded potentials [10.2]
18. Fields of a moving point charge [10.3]
19. Dipole radiation [11.1]
20. Radiation from point charges [11.2]

Einstein’s Postulates

Inertial frames (IF)
A Euclidean frame in which a body not acted upon by external forces moves with constant velocity.

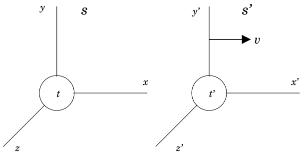
Relativity principle
The laws of physics have the same form in all IF.

Invariance of c
The speed of light in vacuum, c, is independent of the motion of its source.

Lorentz transformation (LT)

Assume the IF's S and S' are in the standard configuration

For $t = t' = 0$
S and S' coincide



$$\boxed{\begin{array}{ll} ct' = \gamma\, ct - \gamma\beta\, x & ct = \gamma\, ct' + \gamma\beta\, x' \\ x' = \gamma\, x - \gamma\beta\, ct & x = \gamma\, x' + \gamma\beta\, ct' \\ y' = y & y = y' \\ z' = z & z = z' \end{array}}$$

$$\gamma \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \beta \equiv \frac{v}{c}$$

Lorentz invariant: $(ct)^2 - x^2 - y^2 - z^2 = (ct')^2 - x'^2 - y'^2 - z'^2$

One-dimensional addition of velocities: $v_{AC} = \dfrac{v_{AB} + v_{BC}}{1 + v_{AB}v_{BC}/c^2}$

Basics of Relativity

Lorentz invariant classification of 4-vectors

consider a 4-vector $s^\mu = (s^0, \vec{s})$ and it's (Minkowski) square $s^2 \equiv s^\mu s_\mu$.

$s^2 = 0$	lightlike	
$s^2 < 0$	spacelike	there is an IF S' s.t. $s'^0 = 0$
$s^2 > 0$	timelike	there is an IF S'' s.t. $\vec{s}'' = \vec{0}$

- The distance of any two points on the worldline of a particle is timelike (or lightlike if the particle travels at the speed of light).
- The worldline is globally and locally within the lightcone
- Spacelike separated events cannot be causually connected

Basics of Relativity

Time dilation

τ : The proper time of the object, i.e. the time as measured in the rest frame of the object

t : The time as measured in an IF moving with velocity v with respect to the object

$$\boxed{d\tau = \frac{dt}{\gamma(v)}}$$

Length contraction

l_0 : The proper length of the object, i.e. the length as measured in the rest frame of the object

l' : The length as measured in an IF moving with velocity v with respect to the object

$$\boxed{l_0 = \gamma(v)\, l'}$$

Tensor Calculus

scalar: A single quantity that is the same in all IF

4-vector: Any quantity that transforms like

$x^\mu = (x^0, x^1, x^2, x^3) \stackrel{\text{old not}}{=} (ct, x, y, z) = (ct, \vec{x}\,)$

Transformation of x^μ

$$\boxed{x'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu x^\nu \stackrel{\Sigma\text{conv}}{\equiv} \Lambda^\mu{}_\nu x^\nu = \frac{\partial x'^\mu}{\partial x^\nu} x^\nu}$$

or written explicitly for the standard configuration of frames

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

x^μ is a contravariant 4-vector.

Tensor Calculus

The corresponding covariant 4-vector is defined by

$$\boxed{x_\mu = g_{\mu\nu} x^\nu \overset{\text{old not}}{=} (ct, -\vec{x}\,)}$$

where $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the metric. It transforms as

$$\boxed{x'_\mu = \Lambda_\mu{}^\nu x_\nu = \frac{\partial x^\nu}{\partial x'^\mu} x_\nu \quad \text{with} \quad \Lambda^\mu{}_\rho \Lambda_\mu{}^\nu = g_\rho{}^\nu = \delta^\nu_\rho}$$

The Minkowski scalar product of two 4-vectors
$x^\mu y_\mu = x_\mu y^\mu = x_\mu y_\nu g^{\mu\nu} = x^0 y^0 - \vec{x} \cdot \vec{y} = \dots$
is invariant under LT

Tensor of rank 2: Any quantity $F^{\mu\nu}$ that transforms as follows:

$$\boxed{F'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma}}$$

Important 4-vectors

4-velocity: $\boxed{u^\mu \equiv \frac{dx^\mu}{d\tau} = \gamma(v)\,(c, \vec{v})}$ Note: $u^\mu u_\mu = c^2$

4-momentum: $\boxed{p^\mu \equiv m_0 u^\mu = m_0 \gamma(v)\,(c, \vec{v}) \equiv m_\gamma\,(c, \vec{v})}$

m_0 : The rest mass of the object, i.e. the mass as measured in its rest frame.

m_γ : The inertial mass. This mass is not Lorentz invariant and is hardly ever used: $m_\gamma \equiv m_0 \gamma(v)$

E_0 : The rest energy of the object: $E_0 = m_0\, c^2$

E : The total relativistic energy of the object: $E = E_0 + E_{\rm kin}$

$\vec{p}$: The relativistic momentum: $\vec{p} = \gamma m_0 \vec{v}$

$\boxed{p^\mu = \left(\frac{E}{c}, \vec{p}\right)}$ and $\boxed{p^2 \equiv p^\mu p_\mu = \frac{E^2}{c^2} - \vec{p}^2 = m_0^2 c^2}$

Important 4-vectors

4-acceleration: $\boxed{a^\mu \equiv \frac{d^2x^\mu}{d\tau^2} = \frac{du^\mu}{d\tau}}$

- proper acceleration: the acceleration as measured in the rest frame
- uniformly accelerated: accelaration with constant proper acceleration

Note: $a^\mu = \gamma\left(\dot{\gamma}c, \dot{\gamma}\vec{v} + \gamma\vec{a}\right)$ where $\dot{\gamma} \equiv d\gamma/dt$. Also, $a^\mu u_\mu = 0$.

4-force: $\boxed{f^\mu \equiv \frac{dp^\mu}{d\tau}}$ $f^\mu = \left(\frac{\gamma}{c}\frac{dE}{dt}, \gamma\vec{F}\right)$ where $\vec{F}$ is normal 3-force.

$\boxed{\vec{F} = \gamma m_0\left(\vec{a} + \frac{\vec{v}(\vec{v}\cdot\vec{a})}{c^2 - \vec{v}^2}\right)}$ Only if $\vec{v}\cdot\vec{a} = 0(\dot{\gamma} = 0)$ is $\vec{F} = \gamma m_0\vec{a}$.

Electromagnetism

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$$

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

Electromagnetic 4-Vectors

4-current density: $\boxed{j^\mu \equiv \rho_0\gamma\,(c,\vec{v}) = \rho(c,\vec{v}) = (\rho c,\vec{j})}$

ρ_0: charge density in rest frame; ρ: charge density;

Continuity equation: $\boxed{\partial_\mu j^\mu = 0}$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right)$$

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right)$$

4-potential: $\boxed{A^\mu \equiv (\Phi, c\vec{A})}$ Φ: scalar potential, $\vec{A}$: vector potential

The relation of the potential to the $\vec{E}$ and $\vec{B}$ field is as follows:

$$\boxed{\vec{E} = -\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}; \qquad \vec{B} = \vec{\nabla}\times\vec{A};}$$

These relations entail $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ and $\vec{\nabla}\cdot\vec{B} = 0$

The 4-Potential

Gauge transformation: For any “reasonable” function $\Psi(t, \vec{x})$, the transformation

$$\boxed{\Phi \to \Phi - \frac{\partial \Psi}{\partial t};\ \vec{A} \to \vec{A} + \vec{\nabla}\Psi \quad \therefore \quad A^\mu \to A^\mu - c\,\partial^\mu \Psi}$$

leaves $\vec{E}$ and $\vec{B}$ unchanged. This allows fixing $\vec{\nabla} \cdot \vec{A}$ (chosing a gauge). The Lorenz gauge is convenient in electrodynamics:

$$\boxed{\vec{\nabla} \cdot \vec{A} = -\frac{1}{c^2}\frac{\partial \Phi}{\partial t}} \quad \text{or} \quad \boxed{\partial_\mu A^\mu = 0}$$

Wave equation:

The two remaining Maxwell equations $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ and $\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$ lead to (using Lorenz Gauge):

$$\boxed{\partial_\mu \partial^\mu A^\nu = \frac{1}{c\epsilon_0} j^\nu} \qquad \text{wave equation}$$

In vacuum, $\partial_\mu \partial^\mu A^\nu = 0$ and the solutions are plane waves: $e^{-ik^\nu x_\nu} = e^{-i\omega t + i\vec{k}\vec{x}}$ where $k^\nu = (\frac{\omega}{c}, \vec{k})$ is the wave 4-vector.

The Field Strength Tensor

The Electromagnetic Field Strength Tensor is defined from the potential as

$$\boxed{F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu} \qquad \text{dual:} \qquad \boxed{\widetilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}}$$

with $\epsilon^{\mu\nu\rho\sigma}$ the totally antisymmetric 4-dimensional Levi-Civita tensor: $\epsilon^{\mu\nu\rho\sigma} = -\epsilon_{\mu\nu\rho\sigma}$ and $\epsilon_{0123} = +1$
The explicit form of $F^{\mu\nu}$ in terms of the fields is

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix} \qquad \widetilde{F}^{\mu\nu} = \begin{pmatrix} 0 & cB_x & cB_y & cB_z \\ -cB_x & 0 & -E_z & E_y \\ -cB_y & E_z & 0 & -E_x \\ -cB_z & -E_y & E_x & 0 \end{pmatrix}$$

Maxwell Equations and Transformation of the Fields

Maxwells equations in covariant form ($\mu_0\epsilon_0 = c^{-1/2}$)

$$\left.\begin{array}{r}\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}\\ \vec{\nabla}\times\vec{B} = \mu_0\vec{j} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}\end{array}\right\} \rightarrow \boxed{\partial_\mu F^{\mu\nu} = \frac{1}{c\epsilon_0}j^\nu}$$

$$\left.\begin{array}{r}\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}\\ \vec{\nabla}\cdot\vec{B} = 0\end{array}\right\} \rightarrow \boxed{\partial_\mu \widetilde{F}^{\mu\nu} = 0}$$

Assume S and S' are in standard configuration:

$$\boxed{\begin{array}{ll} E'_x = E_x & B'_x = B_x \\ E'_y = \gamma(E_y - vB_z) & B'_y = \gamma(\frac{v}{c^2}E_z + B_y) \\ E'_z = \gamma(E_z + vB_y) & B'_z = \gamma(-\frac{v}{c^2}E_y + B_z) \end{array}}$$

Lorentz Invariants

$$\left.\begin{array}{ll}\vec{E}\cdot\vec{B} & = \frac{1}{4c}F_{\mu\nu}\widetilde{F}^{\mu\nu}\\ \vec{E}^2 - c^2\vec{B}^2 & = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu}\end{array}\right\} \quad \text{Invariant under LT}$$

Fields of a moving point charge

The 4-potential (Lorenz gauge) at the point $\vec{r}$ at time t of a point charge in arbitrary motion

$$A^{\mu}(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{u^{\mu}}{u_{\nu} R^{\nu}}$$

(u^{μ} is the 4-velocity of the particle at the retarded time t_r).
The corresponding electromagnetic fields are given by

$$\vec{E}(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{R}{(\vec{R} \cdot \vec{u})^3} [\overbrace{(c^2 - v^2)\vec{u}}^{\text{velocity field}} + \overbrace{\vec{R} \times (\vec{u} \times \vec{a})}^{\text{acceleration field}}]$$

$$\vec{B}(\vec{r}, t) = \frac{1}{c} \hat{\vec{R}} \times \vec{E}(\vec{r}, t)$$

where $\vec{R}$ is the vector between the point charge and the observer, $\vec{v}$ is the velocity of the point charge, $\vec{u} = c\hat{\vec{R}} - \vec{v}$, and $\vec{a}$ is the acceleration of the point charge. $\vec{R}$, $\vec{u}$, $\vec{v}$, and $\vec{a}$ must be evaluated at the retarded time $t_r = t - R/c$.

Power radiated

The Poynting vector is

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$$

The Power radiated by an accelerated point charge

$$P = \int \left(\frac{\vec{R}\cdot\vec{u}}{Rc}\right) |\vec{S}| R^2 d\Omega = -\frac{\mu_0 q^2}{6\pi c}\frac{1}{(mc)^2}\frac{dp^\mu}{d\tau}\frac{dp_\mu}{d\tau}$$

$$= \frac{\mu_0 q^2\gamma^2}{6\pi m^2 c}\left[\left|\frac{d\underline{p}}{dt}\right|^2 - \beta^2\left(\frac{dp}{dt}\right)^2\right]$$

$$= \frac{\mu_0 q^2\gamma^6}{6\pi c}\left(\vec{a}^2 - \left|\frac{\vec{v}\times\vec{a}}{c}\right|^2\right)$$