Theoretical Physics 2

Classical Mechanics

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Lecture 1

➤ DoF
➤ Constraints

Lecture 2

➤ Differential Formulation of Lagrangian Mechanics

2.1 Generalised Coordinates

➤ Page 6
➤ Only need as many coordinates q_k as there are DoFs ($N = 3M - j$)
➤ Generalised velocities are $\frac{dq_k}{dt}$
➤ Regard q_k and $\dot{q}_k$ as independent variables
  ➥ $\frac{\partial q_k}{\partial \dot{q}_k} = \frac{\partial \dot{q}_k}{\partial q_k} = 0$

2.1.1 Lagrangian

➤ Page 7
➤ Assume:
  1. Holonomic constraints
  2. Constraining forces do no work
  3. Applied forces are conservative, such that a scalar potential energy function exists
     ➥ Potential function may change with time
➤ Generalised Coordinates don't necessarily have to be in metres - Force is not in Newtons, it is $\frac{\partial W}{\partial q_k}$

Lecture 3

➤ Ignorable coords
  ➥ if it doesn't appear in the Lagrangian, it can be ignored
  ➥ Canonically conjugate momentum to the coord
     ➥ $p_k = \frac{\partial L}{\partial \dot{q}_k}$
  ➥ simplifies eqn of motion
  ➥ example:
     ➥ $L = \frac{1}{2}m\dot{x}^2 \implies p = \frac{\partial L}{\partial \dot{q}_k} = m\dot{x}$ is constant
➤ E.g. Ladder sliding down a wall

➥ A ladder of length 2a and uniform mass/length $= \lambda$ leans against a wall. It is initially stationary before sliding down the wall without friction. Find $\ddot{\theta}$ where θ is the ladder's angle to the horizontal, while it remains in contact with the wall.

➥ We need to define the Lagrangian in terms of the one generalised coord, θ

➥ Define s as the distance along the ladder from the end in contact with the floor. Then $dm(s) = \lambda \, ds$. Also, we can write $\underline{r}(s) = -2a\cos\theta + s\cos\theta = [(s - 2a)\cos\theta, \sin\theta]$

➥ Differentiating wrt time: $\underline{\dot{r}}(s) = [-(s - 2a)\dot{\theta}\sin\theta, s\dot{\theta}\cos\theta]$

➥ $\therefore \underline{\dot{r}}^2(s) = \dot{\theta}^2[s^2 + (4a^2 - 4as)\sin^2\theta]$

➥ Combining this into T gives:

$$T = \int_0^{2a} \frac{1}{2}\underline{\dot{r}}^2(s)\,dm(s)$$

$$T = \int_0^{2a} \frac{\lambda\dot{\theta}^2}{2}[s^2 + (4a^2 - 4as)\sin^2\theta]\,ds$$

$$T = \frac{\lambda\dot{\theta}^2}{2}[\frac{s^3}{3} + \sin^2\theta(4a^2s - 2as^2)]_0^{2a}$$

$$T = \frac{4\lambda\dot{\theta}^2 a_1^3}{3}$$

$$V = (\lambda 2a)g(a\sin\theta) \implies$$

$$L = \frac{4\lambda\dot{\theta}^2 a^3}{3} - \lambda 2a^2 g\sin\theta = 2\lambda a^2(\frac{2a\dot{\theta}^2}{3} - g\sin\theta)$$

E-L:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}(\frac{4a\dot{\theta}}{3}) + g\cos\theta = 0$$

$$\therefore \ddot{\theta} = -\frac{3g}{4a}\cos\theta$$

3.1 An example of an integral principle: Fermat's Principle

➤ *A light ray takes the quickest available path between two points*
➤ Consider a boundary between two materials
  ➥ Refractive indices of n1 and n2
  ➥ Going from A in n1 to B in n2
  ➥ Travelling x1 in A and x2 in B
  ➥ Distance l between A and B in y dirn
  ➥ travels distance y in A, so $l - y$ in B
  ➥ paths of s1 and s2 in A and B
  ➥ θ_1 and θ_2 are angles from sn to xn

$$T = \int_A^B dt = \int_A^B \underline{ds}$$

$$T = \frac{1}{c}\int_A^B n\,ds$$

$$cT = n_1 s_1 + n_2 s_2$$

$$s_1 = \sqrt{x_1^2 + y^2}$$

$$s_2 = \sqrt{x_2^2 + (l - y)^2}$$

$$cT = n_1\sqrt{x_1^2 + y^2} + n_2\sqrt{x_2^2 + (l - y)^2}$$

$$\frac{dT}{dy} = 0 \implies$$

$$\frac{n_1 y}{\sqrt{x_1^2 + y^2}} - \frac{n_2(l - y)}{\sqrt{x_2^2 + (l - y)^2}} = 0$$

$$n_1 \sin\theta_1 - n_2 \sin\theta_2 = 0 \quad \text{i.e. Snell's Law}$$

3.2 Variational Calculus

➤ $I[y] = \int_{x_1}^{x_2} f(y, y', x)\, dx$
  ➥ I is a functional
  ➥ for a whole path
  ➥ solution will be in form of E-L
➤ For any perturbation we make to the path, the value of the functional will not change
  ➥ like how looking for dy/dx = 0
  ➥ essentially same but extra dimension
➤ ϵ is small number, constant, and independent of x
  ➥ how far away from x
➤ η tells us above or below x
  ➥ $\eta(x_1) = \eta(x_2) = 0$
➤ $\left(\frac{dI}{d\epsilon}\right)_{\epsilon=0}$
➤ For $\frac{\partial f}{\partial x} = 0$:

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx}$$

$$\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0$$

➤ **Alternate form of E-L which can be useful**

Lecture 4

4.1 Hamilton's Principle

$$S[\underline{q}(t)] = \int_{t_1}^{t_2} L(\underline{q}(t), \dot{q}(t), t)\, dt$$

$$\frac{\delta L}{\delta q} \equiv \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$$

4.2 Brachistochrome

➤ What is the quickest route for mass m to fall under gravity from A to B?
  ➥ assuming no friction
➤ Conservation of energy to start with for speed:

$$\frac{1}{2}mv^2 + mgy = 0$$

$$\implies v = \sqrt{-2gy}$$

➤ Total time taken:

$$t = \int_A^B \frac{ds}{v} \; ; \; (ds)^2 = (dx)^2 + (dy)^2$$

$$ds = \sqrt{1 + (\frac{dy}{dx})^2} \cdot dx$$

$$\implies t = \int_{x_A}^{x_B} \frac{\sqrt{1 + y'^2} \cdot dx}{\sqrt{-2gy}}$$

➤ Calculus of Variations problem
  ➥ $f = \sqrt{\frac{1+y'^2}{-2gy}}$
  ➥ no explicit dependence on x
    ➥ from eq. 9 on notes, we get $f - y'\frac{\partial f}{\partial y'} = c$

$$\sqrt{\frac{1 + y'^2}{-2gy}} - \frac{y' \cdot y'}{\sqrt{-2gy(1 + y'^2)}} = c$$

$$\frac{1}{\sqrt{-2gy(1 + y'^2)}} = c$$

$$y(1 + y'^2) = A$$

$$\frac{dy}{dx} = \sqrt{\frac{A - y}{y}}$$

$$\int \sqrt{\frac{y}{A - y}} \cdot dy = \int dx = x$$

➤ Try substitution:
  ➥ $y = \frac{A}{2}(1 - \cos\theta) = A\sin^2(\frac{\theta}{2})$
  ➥ from $1 - \cos\theta = 2\sin^2(\frac{\theta}{2})$

$$dy = A\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$x = \int \sqrt{\frac{A\sin^2\frac{\theta}{2}}{A\cos^2\frac{\theta}{2}}} \cdot A\sin\frac{\theta}{2}\cos\frac{\theta}{2} d\theta$$

$$x = A\int \frac{1}{2}(1 - \cos\theta) d\theta = \frac{A}{2}(\theta - \sin\theta) + B$$

$$x_A = y_A = 0 \; ; \; \theta = 0 \; ; \; \theta = 0 \therefore$$

$$x = \frac{A}{2}(\theta - \sin\theta)$$

$$y = \frac{A}{2}(1 - \cos\theta)$$

➤ x and y final solns give a cycloid
➤ see on DUO for python code for showing particle motion

4.3 Lagrange Multiplier

➤ If you have a constraint, and want to eliminate number of variables to solve, use Lagrange Multipliers

➤ Consider a pendulum of length, l
  ➥ The natural coordinate to choose is θ
  ➥ If you want to use x and y then,

$$\delta S = \int \left[\frac{\delta L}{\delta x} \cdot \delta x + \frac{\delta L}{\delta y} \cdot \delta y \right] dt$$

➤ Hamilton's principle says $\delta S = 0$ but non-independence of x and y means can't say each variational deriv is zero

➤ Use λ as an arbitrary Lagrange multiplier

$$\delta S = \int \left[\left(\frac{\delta L}{\delta x} - \lambda \frac{\partial G}{\partial x} \right) \delta x + \left(\frac{\delta L}{\delta y} - \lambda \frac{\partial G}{\partial y} \right) \delta y \right] dt$$

$$\delta(F - \lambda G) = 0$$

4.4 Linear Oscillators

➤ Mech system is at equilibrium at rest

➤ This occurs at points in space where all forces, F_k vanish

➤ For conservative systems, this is when the potential energy is stationary
  ➥ its first derivs wrt to q_k vanish

➤ Near Static Equilibrium
  ➥ define equilibrium psn and velocity as zero
  ➥ $\frac{\delta V}{\delta q} = 0$ at equilibrium point
  ➥ expand out and only go up to quadratic terms as near equilibrium
  ➥ Sub into E-L eqn:

$$L = \frac{m}{2} \left(\dot{q}^2 + \frac{D}{F} q^2 \right)$$

➤ Normal EoM for particle

➤ SHO:
  ➥ equilibrium psn is minimum of pot energy
  ➥ close to equilibrium, we have SHM
  ➥ $\omega = \sqrt{-\frac{D}{F}}$
  ➥ if $\frac{D}{F} > 0$, it is unstable
  ➥ $\ddot{q} + \omega^2 q = 0$ is linear, homogeneous diff eqn

$$q(t) = q(0) \cos(\omega t) + \frac{\dot{q}(0)}{\omega} \sin(\omega t)$$

➤ Damping force
  ➥ frictional force to suppress motion
  ➥ opposes motion and vanishes when motion ceases
  ➥ $F_d = -\gamma \dot{q} = -(\frac{m\omega}{Q})\dot{q}$
  ➥ Q is dimensionless quality factor
  ➥ higher quality system leads to small Q and motion lasts longer

4.5   Damped Oscillators

➤ For Damped SHO, E-L:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = F_d$$

$$\ddot{q} + \frac{\omega}{Q}\dot{q} + \omega^2 q = 0$$

$$q = e^{\lambda t} \implies \lambda^2 + \frac{\omega}{Q}\lambda + \omega^2 = 0$$

$$\lambda = -\frac{\omega}{2Q} \pm \sqrt{(\frac{\omega}{2Q})^2 - \omega^2}$$

$$\lambda = -\frac{\omega}{2Q} + \underbrace{\frac{\omega}{2Q}\sqrt{1 - 4Q^2}}_{\omega'}$$

➤ The sqrt becomes positive, zero, or negative depending on Q
➤ Overdamped: $Q < \frac{1}{2} \implies \sqrt{} \in \mathbb{R}$
   ➥ Increasing Q but keeping it under $\frac{1}{2}$ makes it tend to equilibrium faster

$$q(t) = e^{-\frac{\omega t}{2Q}}\left(A\cosh(\omega't) + B\sinh(\omega't)\right)$$

$$\dot{q}(t) = e^{-\frac{\omega t}{2Q}}\left(A\omega'\cosh(\omega't) + B\omega'\sinh(\omega't)\right) \cdot -\frac{\omega}{2Q}q$$

$$\dot{q}(0) = B\omega' - \frac{\omega}{2Q}q(0)$$

$$B = \frac{1}{\omega'}\left(\dot{q}(0) + \frac{\omega}{2Q}q(0)\right)$$

$$q(0) = A$$

Lecture 5

5.1   Damped Oscillators Ctd

➤ Critically Damped: $Q = \frac{1}{2} \implies$ 2 equal roots for $\lambda = -\frac{omega}{2Q}$
   ➥ Fastest return to equilibrium without overshooting

$$q(t) = e^{-\frac{\omega t}{2Q}}(\underbrace{A}_{q(0)} + \underbrace{B}_{\dot{q}(0)+\frac{\omega}{2Q}q(0)} t)$$

➤ Underdamped: $Q > \frac{1}{2} \implies \lambda = -\frac{\omega}{2Q} \pm i\omega'$; $\omega = \frac{\omega}{2Q}\sqrt{4Q^2 - 1}$
   ➥ Solution is like overdamped case but with cosh replaced by cos and sinh replaced by sin
   ➥ Sin/cos oscillation with exponential decay envelope
   ➥ Larger quality factor, Q, implies slower decay, and more oscillations

5.2   Driven Oscillators

5.2.1   **Oscillators Driven by an External Force**

➤ Consider an undamped oscillator, driven by time dep external force, F(t)
➤ Driving force is not part of the system
  ➥ do not consider force back on driver by system

$$L = \underbrace{\frac{m\dot{q}^2}{2}}_{T} - \underbrace{\left[\frac{m\omega^2 q^2}{2} - F(t)q\right]}_{V}$$

$$\ddot{q} + \omega^2 q = \frac{F(t)}{m}$$

➤ Yield a second order linear, inhomogeneous differential eqn
➤ Split into infinite number of impulsive forces and sum up displacements caused by each
➤ Method is analogous to Poisson's eqn in electrostatics
  ➥ $\nabla^2 \Phi(x) = -\frac{\rho(r)}{\epsilon}$

5.3   Dirac delta-fn

➤ Zero-width limit of a function/distribution of t, sharply peaked at $t = t'$
➤ If t has dimension of time, δ has dimension of inverse time, i.e. frequency

$$f(t') = \int_{-\infty}^{\infty} \delta(t - t')f(t)\,dt$$

➤ Use this to find impulsive force at time t' as $F(t) = K\delta(t - t')$

5.4   Response of oscillator to an indep impulsive force

$$\int_{t'-\epsilon}^{t+\epsilon} (\ddot{q} + \omega^2 q)dt$$

$$\implies \dot{q}(t' + \epsilon) - \dot{q}(t' - \epsilon) + \underbrace{\omega^2 \int_{t'-\epsilon}^{t+\epsilon} q(t)dt}_{\to 0 \text{ as } \epsilon \to 0, \text{ unless } q(t') \to \infty} = \frac{K}{m}$$

➤ Instantaneous change in velocity of $\frac{K}{m}$ from this force but no change in psn as no change in time

Evolution of SHO

1. $t < t'$ - normal SHO motion
2. $t = t'$ - Instantaneous jump in velocity as above
3. $t > t'$ - no more driving force, back to SHO

$$q(t) = q(t')\cos(\omega[t - t']) + \frac{\dot{q}(t') + \frac{K}{m}}{\omega}\sin(\omega[t - t'])$$

5.5   Green's Function

➤ Solution to differential equation:

$$\ddot{G}(t - t') + \omega^2 G(t - t') = \delta(t - t')$$

$$G(t - t') = \frac{1}{\omega}\sin(\omega[t - t']), \ t \geq t'$$

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➤ Using defn of δ-fn, the eqn of motion for driven oscillator can be written as

$$\ddot{q} + \omega^2 q = \frac{1}{m} \int_{-\infty}^{\infty} dt'\, F(t')\delta(t - t')$$

➤ Just splitting up driving force into infinite number of impulsive forces
➤ eliminate δ-fn by using Green's fn

$$\ddot{q} + \omega^2 q = \frac{1}{m} \int_{-\infty}^{\infty} dt'\, F(t')[\ddot{G}(t - t') + \omega^2 G(t - t')]$$

$$q(t) = \frac{1}{m\omega} \int_{-\infty}^{t} F(t')G(t - t')dt' = \frac{1}{m\omega} \int_{-\infty}^{t} F(t')\sin(\omega[t - t'])dt'$$

Example:
➤ Consider driving force of the form
  ➥ $F(t) = F_0 \sin(\omega_0 t)$ for $t > 0$
  ➥ $F(t) = 0$ for $t \leq 0$

$$q(t) = \frac{F_0}{m\omega} \int_0^t dt' \sin(\omega_0 t') \sin(\omega(t - t'))$$

➤ Use $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$

$$q(t) = \frac{F_0}{m\omega} \int_0^t \frac{1}{2}\Big[\cos[t'(\omega_0 + \omega) - \omega t] - \cos[t'(\omega_0 - \omega) + \omega t]\Big] dt$$

➤ For $\omega_0 \neq \omega$, one observes oscillating behaviour with

$$q(t) = \frac{F_0}{m\omega}\Big[\frac{\sin(\omega_0 t) + \sin(\omega t)}{2(\omega_0 + \omega)} - \frac{\sin(\omega_0 t) - \sin(\omega t)}{2(\omega_0 - \omega)}\Big]$$

➤ In resonance, (when $\omega_0 = \omega$), q(t) grows without limit

$$q(t) = \frac{F_0}{m\omega}\Big[\underbrace{\frac{\sin(\omega t)}{2\omega}}_{\text{oscillates}} - \underbrace{\frac{t\cos(\omega t)}{2}}_{\text{grows}}\Big]$$

➤ If we consider an underdamped SHO subject to the same driving force as above, then EoM can be solved to give

$$q(t) = q_s(t) + q_t(t)$$

➤ Solution is split into the steady-state solution, q_s, and the transient solution, q_t.
  ➥ In this case, the Green's function for the damped oscillator is

$$G(t - t') = \frac{e^{-\hat{\omega}(t - t')}}{\omega'} \sin[\omega'(t - t')], \quad t \geq t'$$

$$w' = \hat{\omega}\sqrt{4Q^2 - 1} \ ; \ \hat{\omega} = \frac{\omega}{2Q}$$

Lecture 6

6.1 Recap

➤ Green's function

➤ response of an oscillator to an impulsive force
➤ Split force up into lots of tiny impulsive forces

$$q(t) = \frac{1}{m\omega} \int_{-\infty}^{t} F(t')G(t-t')dt'$$

6.2 Underdamped Driven Oscillator Ctd

➤ Displacement at time, t:

$$q(t) = \frac{1}{m} \int_{-\infty}^{t} F_0 \sin(\omega_0 t') \frac{e^{-\hat{\omega}(t-t')}}{\omega'} \sin[\omega'(t-t')]dt'$$

➤ To solve this, just write sins as $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$ and integrate
➤ The steady-state soln is:

$$q_s(t) = \frac{F_0}{2m\omega'} \left\{ \left[\frac{\hat{\omega}}{\hat{\omega}^2 + (\omega_0 + \omega')^2} - \frac{\hat{\omega}}{\hat{\omega}^2 + (\omega_0 - \omega')^2} \right] \cos(\omega_0 t) \right.$$

$$\left. + \left[\frac{\omega_0 + \omega'}{\hat{\omega}^2 + (\omega_0 + \omega')^2} - \frac{\omega_0 - \omega'}{\hat{\omega}^2 + (\omega_0 - \omega')^2} \right] \sin(\omega_0 t) \right\}$$

➤ The transient state is:

$$q_t(t) = \frac{F_0 e^{-\hat{\omega}t}}{2m\omega'} \left\{ \left[\frac{\omega_0 + \omega'}{\hat{\omega}^2 + (\omega_0 + \omega')^2} + \frac{\omega_0 + \omega'}{\hat{\omega}^2 - (\omega_0 - \omega')^2} \right] \sin(\omega' t) \right.$$

$$\left. - \left[\frac{\hat{\omega}}{\hat{\omega}^2 + (\omega_0 + \omega')^2} - \frac{\hat{\omega}}{\hat{\omega}^2 + (\omega_0 - \omega')^2} \right] \cos(\omega' t) \right\}$$

$$q(t) = q_s(t) + q_t(t)$$

➤ The transient soln oscillates with the intrinsic oscillator frequency, ω', but dies away with time
➤ The steady-state soln oscillates with driving frequency, and remains at large t
➤ After the transient has died away, the dynamical behaviour is like that of SHO, but at the driving frequency
➤ Note that there is no singularity at resonance
  ➥ The effect of resonance has been softened by the presence of damping

6.3 Coupled Small Oscillations

➤ look at page 20 of notes
➤ Consider system with two pendulums of length, l, connected half way down with a spring
➤ Use θs as generalised coords
➤ Use Taylor expansion of cos up to deg 2 to get potential expression

$$T = \frac{ml^2}{2}(\dot{\theta}_1^2 + \dot{\theta}_2^2)$$

$$V = \frac{mgl}{2}(\theta_1^2 + \theta_2^2)$$

$$V_{coupling} = \frac{k}{2}\left(\frac{l}{2}\right)^2(\theta_2 - \theta_1)^2$$

$$L = \frac{ml^2}{2}(\dot{\theta}_1^2 + \dot{\theta}_2^2) - \frac{1}{2}\left(mgl(\theta_1^2 + \theta_2^2) + k\left(\frac{l}{2}\right)^2(\theta_2 - \theta_1)^2 \right)$$

➤ $\theta_1\theta_2$ makes things awkward, so transform to centre of mass, $\theta_c = \frac{\theta_1+\theta_2}{2}$ and relative, $\theta_r = \theta_2 - \theta_1$

➤ Also define $\omega_0 = \sqrt{\frac{g}{l}}$ and constant, $\eta = \frac{kl}{4mg}$

$$L = ml^2\left[(\dot{\theta}_c^2 - \omega_0^2\theta_c^2) + \frac{1}{4}(\dot{\theta}_r^2 - \omega_0^2\left(1 + 2\eta)\theta_r^2\right)\right]$$

$$\ddot{\theta}_c + \omega_c^2\theta_c = 0$$

$$\ddot{\theta}_r + \omega_0^2(1 + 2\eta)\theta_r = 0$$

➤ η is essentially the relative strengths of the spring force to gravity

➤ Total motion is linear combination of two EoMs above

6.4 General Method

➤ N coupled oscillators gives N generalised coords, yielding N homogeneous eqns

➤ If undergoing SHM, *normal coordinates*

1. Find equilibrium config, values of generalised coords where generalised forces are 0
2. Taylor expand Lagrangian to second order in generalised coords, around values for the generalised coords given by equilibrium config
3. For N DoF, the Taylor expansion must in principal be in 2N variables
   ➤ In practice, fewer variables may be adequate

➤ Use Matrices
   ➥ page 21
   ➥ generalised coordinates are column vectors

$$L = \underbrace{\dot{\underline{q}}^T\hat{\tau}\dot{\underline{q}}}_{T} - \underbrace{\underline{q}^T\hat{v}\underline{q}}_{V} + c$$

➤ Elements of these matrices:

$$\tau_{jk} = \frac{1}{2}\frac{\partial^2 T}{\partial\dot{q}_j\partial\dot{q}_k}\bigg|_{\dot{q}_j,\dot{q}_k=0} \quad ; \quad v_{jk} = \frac{1}{2}\frac{\partial^2 V}{\partial\dot{q}_j\partial\dot{q}_k}\bigg|_{q_j,q_k=0}$$

Lecture 7

7.1 Recap from Last time

$$T = \dot{\underline{x}}\hat{\tau}\dot{\underline{x}}$$

$$V = \underline{x}^T\hat{v}\underline{x}$$

$$L = T - V = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}k(x_1^2 + x_2^2 - 2x_1x_2)$$

➤ E-L: (x_1)

$$m\ddot{x}_1 - k(x_1 - x_2) = 0$$

➤ E-L: (x_2)

$$m\ddot{x}_2 + k(x_2 - x_1) = 0$$

➤ Other coordinates in each E-L is the coupling

➤ Equivalent to

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$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} k & -k \\ -k & k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underline{0}$$

$$2(\hat{\tau}\underline{\ddot{x}} + \hat{v}\underline{x})$$

➤ Equation for coupled oscillators:

$$\hat{\tau}\underline{\ddot{q}} + \hat{v}\underline{q} = 0$$

7.2   Normal Modes

➤ Usual thing to do is guess a solution
  ➥ try $\underline{q} = \underline{b}e^{i\omega t}$
➤ This yields

$$(\hat{v} - \omega^2\hat{\tau})\underline{b}$$

➤ Eigenvalue problem
  ➥ det must be zero
  ➥ $\hat{\tau}$ must be a multiple of the identity matrix
➤ In the pendulum example, $T = \frac{mgl}{2\omega_0^2}(\dot{\theta}_1^2 + \dot{\theta}_2^2)$

$$\tau_{ij} = \frac{1}{2}\frac{\partial^2 T}{\partial\dot{\theta}_i\partial\dot{\theta}_j}$$

$$\tau_{11} = \frac{1}{2}\frac{\partial^2 T}{\partial\dot{\theta}_1^2} = \frac{mgl}{2\omega_0^2} \ ; \ \tau_{12} = \frac{1}{2}\frac{\partial^2 T}{\partial\dot{\theta}_1\partial\dot{\theta}_2} = 0 = \tau_{21}$$

$$\tau_{22} = \tau_{11}$$

$$\hat{\tau} = \frac{mgl}{2\omega_0^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\omega^2\hat{\tau} = \frac{mgl}{2}\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \ ; \ \lambda = \left(\frac{\omega}{\omega_0}\right)^2$$

➤ page 21
➤ find eigenvalues from det of matrix above
➤ Gives

$$\frac{b_{(j)2}}{b_{(j)1}} = \frac{1 + \eta - \lambda_j}{\eta}$$

7.3   Normal Coordinates

$$r_j = \sum_{k=1}^{N} b_{(j)k}q_k$$

➤ Note that for coupled pendulum example $r_1 \propto (\theta_1 + \theta_2)$ and $r_2 \propto (\theta_2 - \theta_1)$
➤ Any general motion is a superposition of the normal modes
  ➥ similar to eigenfunctions in Quantum

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7.4 Mode Orthogonality

$$\underline{b}_i^T \hat{\tau} \underline{b}_j = 0 \; ; \; i \neq j$$

1. If ω_is are all different, then b_is are unique
2. If some ω_is are the same then there is a choice of b_is but can always find orthogonal ones

Example: Free Vibrations of a linear triatomic molecule
➤ Central atom of mass, M and outer atoms have masses, m
  ➡ constrained to move on an axis
  ➡ connected together via springs of constant, k and natural length, l
  ➡ point masses
  ➡ Consider x_1, x_2, x_3

$$T = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_3^2) + \frac{M}{2}\dot{x}_2^2$$

$$\hat{\tau} = \frac{1}{2}\begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix}$$

$$V = \frac{k}{2}\left[(x_2 - x_1 - l)^2 + (x_3 - x_2 - l)^2\right]$$

$$\hat{v} = \frac{1}{2}\frac{\partial^2 V}{\partial x_i \partial x_j} = \frac{k}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\hat{\tau}\ddot{\underline{q}} + \hat{v}\underline{q} = \underline{0}$$

➤ Trial solution: $\underline{q} = \underline{b}e^{i\omega t}$

$$(\hat{v} - \omega^2 \hat{\tau})\underline{b} = 0$$

$$\begin{pmatrix} k - \omega^2 m & -k & 0 \\ -k & 2k - \omega^2 M & -k \\ 0 & -k & k - \omega^2 m \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = 0$$

➤ The normal modes will oscillate with frequencies satisfying

$$\begin{vmatrix} k - \omega^2 m & -k & 0 \\ -k & 2k - \omega^2 M & -k \\ 0 & -k & k - \omega^2 m \end{vmatrix} = 0$$

➤ Do some algebra and you get

$$\omega^2(k - \omega^2 m)[k(M + 2m) - \omega^2 Mm] = 0$$

➤ Therefore, the normal modes oscillate with frequencies
  ➡ $\omega_{(1)} = 0$
  ➡ $\omega_{(2)} = \sqrt{\frac{k}{m}}$
  ➡ $\omega_{(3)} = \sqrt{\frac{k}{m}\left(1 + \frac{2m}{M}\right)}$
➤ The corresponding, unnormalised mode vectors are found by substituting these eigenvalues into the matrix above
➤ Doing this, you find that they are:

$$b_{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \;\; ; \;\; b_{(2)} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \;\; ; \;\; b_{(3)} = \begin{pmatrix} 1 \\ -\frac{2m}{M} \\ 1 \end{pmatrix}$$

➤ Three different states:
1. Just a translation
   ➥ could have constrained the centre of mass to remain at origin and solve in 2 DoFs
2. The two outer atoms are oscillating in anti-phase with big M static
3. The m atoms are in phase with one another, and anti-phase with M atom, which has a relative amplitude of $\frac{2m}{M}$ so as to keep the centre of mass fixed

Lecture 8

8.1 Central Forces

8.1.1 Two Interacting Bodies

➤ Consider two point masses with positions $\underline{r}_1$ and $\underline{r}_2$
➤ Connected via central force between them
   ➥ Interaction potential
   ➥ Potential depends on distance between bodies in straight line
   ➥ $V_12 = V_{12}(\underline{r}_1, \underline{r}_2) = V_{12}(|\underline{r}_1 - \underline{r}_2|) = V(r)$

$$L = \frac{1}{2}(m_1 \dot{\underline{r}}_1^2 + m_2 \dot{\underline{r}}_2^2) - \underbrace{V_1(\underline{r}_1) - V_2(\underline{r}_2)}_{\text{external potentials}} - \underbrace{V_{12}(|\underline{r}_1 - \underline{r}_2|)}_{\text{interaction potential}}$$

➤ Consider an infinitesimal translation $\epsilon \underline{a}$ and Taylor expand Lagrangian to first order in ϵ
   ➥ page 25 for the maths
   ➥ canonically conjugate momentum is first expression of E-L equation
      ➥ in this case, the linear momentum
   ➥ this comes from the translational invariance of the Lagrangian

8.2 Ignorable CoM

➤ Conservation of total momentum implies that CoM ($\underline{R} = \frac{m_1 \underline{r}_1 + m_2 \underline{r}_2}{(m_1 + m_2)}$) moves like a free particle
   ➥ rewrite Lagrangian in terms of $\underline{R}$ and the relative coordinate, $\underline{r} = \underline{r}_1 - \underline{r}_2$

$$L = \frac{1}{2}M\dot{\underline{R}}^2 + \frac{1}{2}\mu \dot{\underline{r}}^2 - V_{12}(r)$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$M = m_1 + m_2$$

➤ Check that

$$\frac{1}{2}M\dot{\underline{R}}^2 + \frac{1}{2}\mu\dot{\underline{r}}^2 = \frac{1}{2}(m_1 \dot{\underline{r}}_1^2 + m_2 \dot{\underline{r}}_2^2)$$

$$LHS = \frac{1}{2}\left[(m_1 + m_2)\left(\frac{m_1 \dot{\underline{r}}_1 + m_2 \dot{\underline{r}}_2}{m_1 + m_2}\right)^2 + \frac{m_1 m_2}{m_1 + m_2}(\dot{\underline{r}}_1 = \dot{\underline{r}}_2)^2\right]$$

$$LHS = \frac{1}{2(m_1 + m_2)}\left[m_1^2 \dot{\underline{r}}_1^2 + m_2^2 \dot{\underline{r}}_2^2 + 2m_1 m_2 \dot{\underline{r}}_1 \dot{\underline{r}}_2 + m_1 m_2(\dot{\underline{r}}_1^2 + \dot{\underline{r}}_2^2 - 2\dot{\underline{r}}_1 \dot{\underline{r}}_2)\right]$$

$$LHS = \frac{1}{2(m_1 + m_2)} \left[m_1 \dot{\underline{r}}_1^2 (m_1 + m_2) + m_2 \dot{\underline{r}}_2^2 (m_1 + m_2) \right]$$

$$LHS = \frac{1}{2}(m_1 \dot{\underline{r}}_1^2 + m_2 \dot{\underline{r}}_2^2) = RHS$$

8.3 Rotational Invariance

➤ Transform to the centre of mass frame and remove first expression from Lagrangian
  ➥ potential on depends on distance so we get rotational invariance

$$\dot{\underline{r}}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

$$\dot{\underline{r}}^2 = \dot{r}^2 + r^2\dot{\theta}^2 + (r\sin\theta)^2\dot{\phi}^2$$

$$L = \frac{1}{2}\mu\dot{\underline{r}}^2 - V_{12}(r)$$

$$L = \frac{1}{2}\mu[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta] - V_{12}(r)$$

➤ ϕ is ignorable, as it does not appear in L, therefore:

$$p_\phi \equiv \frac{\partial L}{\partial \dot{\phi}} = \mu r^2 \dot{\phi} \sin^2\theta = \text{constant} \equiv \underbrace{J_z}_{\text{z component of } \underline{J} = \underline{r} \times \mu \dot{\underline{r}}}$$

➤ Can choose coordinate system such that total angular momentum, $\underline{J}$, lies along the z-axis
  ➥ Therefore, $\underline{J}$ is always constant

$$\underline{J} = \underline{r} \times \mu\dot{\underline{r}} \ ; \ \underline{r} = \begin{pmatrix} r\cos\phi\sin\theta \\ r\sin\phi\sin\theta \\ r\cos\theta \end{pmatrix}$$

$$\dot{\underline{r}} = \begin{pmatrix} \dot{r}\cos\phi\sin\theta - r\dot{\phi}\sin\phi\sin\theta + r\dot{\theta}\cos\phi\cos\theta \\ \dot{r}\sin\phi\sin\theta + r\dot{\phi}\cos\phi\sin\theta + r\dot{\theta}\sin\phi\cos\theta \\ \dot{r}\cos\theta - r\dot{\theta}\sin\theta \end{pmatrix}$$

$$\underline{J} = \mu \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ r_i & r_j & r_k \\ \dot{r}_i & \dot{r}_j & \dot{r}_k \end{vmatrix} = \mu \begin{pmatrix} r_j\dot{r}_k - r_k\dot{r}_j \\ r_k\dot{r}_i - r_i\dot{r}_k \\ r_i\dot{r}_j - r_j\dot{r}_i \end{pmatrix}$$

➤ Plug in $\dot{\underline{r}}$:

$$J_z = \mu r\sin\theta[r\dot{\phi}\sin\theta] = \mu r^2\dot{\phi}\sin^2\theta$$

➤ $\underline{J} = \underline{r} \times \mu\dot{\underline{r}}$, so $\underline{p}$ and $\underline{r}$ must be perpendicular to $\underline{J}$
➤ Hence, these must be in a plane perpendicular to $\underline{J}$, and remain in that plain as $\underline{J}$ is constant
➤ Choose $\underline{J}$ to lie along z direction, so that angle of elevation,

$$\theta = \frac{\pi}{2}$$

$$\dot{\theta} = 0$$

$$J = |\underline{J}| = J_z = \mu r^2\dot{\phi}$$

Lecture 9

9.1   Kepler's Second Law

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\phi} = \frac{J}{2\mu}$$

➤ J is conserved, so the rate at which area is swept out is constant
➤ Kepler's second law comes from this
  ➥ nothing to do much with gravity specifically

9.2   Equivalent 1D Problem

➤ In this case, remove θ and ϕ from Lagrangian

$$E = T + V = \frac{1}{2}\mu[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta] + V_{12}(r)$$

$$\dot{\theta} = 0 \; ; \; \theta = \frac{\pi}{2} \; ; \; \dot{\phi} = \frac{J}{\mu r^2}$$

$$E = \frac{1}{2}\mu\dot{r}^2 + \underbrace{\frac{J^2}{2\mu r^2} + V_{12}(r)}_{V_{eff}(r)}$$

➤ Effective potential combines interaction potential with angular momentum barrier
➤ total energy E is the same as that corresponding to an effective Lagrangian
  ➥ $L_{eff} = \frac{1}{2}\mu\dot{r}^2 - V_{eff}$ yields:

$$\mu\ddot{r} = \frac{J^2}{\mu r^3} - \frac{\partial V_{12}}{\partial r}$$

Example: Central Forces
  ➤ Find the orbit shape for the following potential:

$$V(r) = \begin{cases} -V_0 & r \le R \\ 0 & r > R \end{cases} \implies V_{eff}(r) = \begin{cases} \frac{J^2}{2\mu r^2} - V_0 & r \le R \\ \frac{J}{2\mu r^2} & r > R \end{cases}$$

➤ If $\frac{J^2}{2\mu R^2} - V_0 < E < \frac{J^2}{2\mu R^2}$, then we have a bound orbit
  ➥ r_t there for E
  ➥ $r_t \le r \le R$

$$r_t^2 = \frac{J^2}{2\mu(E + V_0)}$$

➤ Set the azimuthal angle, $\phi = 0$, and $r = r_t$ at $t = 0$, and find the shape of the orbit:

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{J^2}{2\mu r^2} - V_0, \; r \le R \quad (1)$$

➤ Define $\mu = \frac{1}{r}$, then $\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt}$

$$\dot{\phi} = \frac{d\phi}{dt} = \frac{J}{\mu r^2} = \frac{Ju^2}{\mu}$$

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{Ju^2}{\mu}\frac{du}{d\phi}$$

$$\to (1) : \implies$$

$$E + V_0 - \frac{J^2 u^2}{2\mu} = \frac{\mu}{2}\frac{J^2}{\mu^2}\left(\frac{du}{d\phi}\right)^2$$

$$\frac{du}{d\phi} = \pm\sqrt{\underbrace{\frac{2\mu}{J^2}(E+V_0)-u^2}_{\frac{1}{r_t^2}=u_t^2}}$$

$$\int_{U_0}^{U} -\frac{du}{\sqrt{u_t^2 - u^2}} = \int_{\phi_0}^{\phi} d\phi = \phi$$

➤ Use substitution: $u = u_t\cos\alpha,\, du = -u_t\sin\alpha\, d\alpha$

$$\phi = \alpha + \alpha_0, \quad t = 0 \implies \alpha_0 = \cos^{-1}\left(\frac{u_t}{u_t}\right) = 0$$

$$\phi = +\cos^{-1}\left(\frac{r_t}{r}\right)$$

➤ Sign is set by sign of angular momentum, J

$$\cos\phi = \frac{r_t}{r} \implies r = r_t\sec\phi, \ \phi < \Phi, \ R = r_t\sec\Phi$$

9.3 Gravitational Attraction

➤ For gravity, we have the differential to be solved as:

$$\mu\ddot{r} = \frac{J^2}{\mu r^3} - \frac{k}{r^2}, \ k = Gm_1 m_2$$

➤ solving this to determine orbits of bodies

9.4 The $u = \frac{1}{r}$ transformation

➤ See last example for this transformation or page 28

$$\frac{d^2 u}{d\phi^2} + u = \frac{\mu k}{J^2}$$

$$\implies \frac{1}{r} \equiv u = \frac{\mu k}{J^2} + B\cos\phi$$

9.5 Elliptical Orbits

➤ Define $p \equiv \frac{J^2}{\mu k}$ and $\epsilon \equiv pB \implies$

$$E = \frac{\mu k^2}{2J^2}(\epsilon^2 - 1) = \frac{k}{2p}(\epsilon^2 - 1)$$

➤ For $0 \le \epsilon < 1$, bound elliptical orbit
➤ page 29
➤ ϵ is the eccentricity of the elliptical orbit:
  ➥ $\epsilon = 0 \implies$ circular orbit
  ➥ $\epsilon = 1 \implies$ parabolic trajectory
  ➥ $\epsilon > 1 \implies$ hyberbola
➤ elliptical orbits come from defining the $\frac{1}{r^2}$ potential

9.6 Third Law

$$T = 2\pi\sqrt{\frac{1}{G(m_1 + m_2)}}a^{3/2} \implies T^2 \propto a^2$$

➤ Same proportionality constant for all planets
  ➥ This is approximately true as the mass of the sun largely cancels out the mass of the planet

See page 29 and 30 for Bertrand's Theorem and Quadrature

Example: Kepler's Equation
  ➤ Given a set of observations of positions of a solar system body, how would one infer the orbit of that body?
  ➤ From the equation for an ellipse in the notes (Eq. 22), we can use the identity $\cos^2\psi + \sin^2\psi = 1$ to introduce the parameterisation:

$$x + \frac{\epsilon p}{1 - \epsilon^2} = a\cos\psi \implies x = a(\cos\psi - \epsilon)$$

$$y = b\sin\psi \implies y = a\sqrt{1 - \epsilon^2}\sin\psi$$

➤ It follows that:

$$r = \sqrt{x^2 + y^2} = a(1 - \epsilon\cos\psi)$$

Lecture 10

10.1 Kepler's Equation Continued

➤ Relationship of the eccentric anomaly, ψ, to the true anomaly, ϕ, for an effective particle placed a distance r and azimuthal angle ϕ from the centre of mass
  ➥ The elliptical orbit, with semi-major axis a, has been placed inside an imaginary circle of radius a
➤ We can solve the effective 1D problem by quadrature
  ➥ Eq 17 in notes, page 27, implies:

$$\int dt' = \sqrt{\frac{\mu}{2}}\int\frac{dr'}{\sqrt{E + \frac{k}{r'} - \frac{J^2}{2\mu r'^2}}}$$

➤ For elliptical orbits, $E = -\frac{k}{2a}$
  ➥ Also use $\frac{J^2}{\mu k} \equiv p$
  ➥ $p = a(1 - \epsilon^2)$

$$\int dt' = \sqrt{\frac{\mu}{2k}}\int\frac{r'\,dr'}{\sqrt{-\frac{r'^2}{2a} + r' - \frac{a}{2}(1 - \epsilon^2)}}$$

➤ Finally, use $r = a(1 - \epsilon\cos\psi)$
  ➥ $dr = a\epsilon\sin\psi\,d\psi$

$$\int dt' = \sqrt{\frac{\mu}{2k}}\int\frac{a^2\epsilon\sin\psi'(1 - \epsilon\cos\psi')\,d\psi'}{-\frac{a}{2}(1 - \epsilon\cos\psi')^2 + a(1 - \epsilon\cos\psi') - \frac{a}{2}(1 - \epsilon^2)}$$

$$\int dt' = \sqrt{\frac{\mu}{2k}}\int\frac{a^2\epsilon\sin\psi'(1 - \epsilon\cos\psi')\,d\psi'}{\sqrt{\frac{a}{2}}(\epsilon\sin\psi')}$$

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$$\int dt' = \sqrt{\frac{\mu a^3}{k}} \int (1 - \epsilon \cos \psi') \, d\psi'$$

➤ This integral measures the time elapsed between initial and final value of this eccentric anomaly, ψ'
  ➥ Customary to choose $t' = 0$ for when $\psi' = \phi' = 0$
    ➡ when the orbiting body is at its perihelion
    ➡ i.e. closest to the sun
➤ Hence, if $t(\psi)$ is the time elapsed from perihelion to when the eccentric anomaly takes the values ψ, then we have Kepler's equation

$$t(\psi) = \sqrt{\frac{\mu a^3}{k}} (\psi - \epsilon \sin \psi)$$

➤ If we measure elapsed times and radii since perihelion, then Kepler's equation and the definition of an ellipse $(r = a(1 - \epsilon \cos \psi))$ can be solved to give the semi-major axis, a, and the eccentricity, ϵ

10.2 Hamiltonian Mechanics

Example: Swinging Atwood Machine
➤ Atwood machine with slight difference
  ➥ One mass moves like a normal pulley, other like a pendulum
  ➥ $DoFs = 2$

10.3 Noether's Theorem and Invariance of L

"If the Lagrangian is invariant under a continuous symmetry transformation, then there are conserved quantities associated with that symmetry, one for each parameter of the transformation. These can be found by differentiating each coordinate wrt to the parameters of the transformation in the immediate neighbourhood of the identity transformation, multiplying by the conjugate momentum, and summing over the degrees of theorem."

➤ page 31 for maths
➤ point mass moving in free space

$$L = \frac{1}{2} m \dot{q}^2$$
$$Q = q + s$$
$$\left. \frac{dQ}{ds} \right|_{s=0} = 1 \implies I(q) = m\dot{q}$$

➤ Consider phase space approach of 6D instead of just normal space
  ➥ normal dimensions plus the canonically conjugate momenta

10.4 Legendre Transformations

➤ $A(x, y)$, intro third variable z
  ➥ $B(x, y, z) \equiv yz - A(x, y)$
  ➥ use chain rule
  ➥ page 32
  ➥ y can be written in terms of x and z
  ➥ $B = B(x, y(x, z), z) = B(x, z) = yz - A(x, y)$

$$\left. \frac{\partial B}{\partial z} \right|_x = y \; ; \; \left. \frac{\partial B}{\partial x} \right|_z = - \left. \frac{\partial A}{\partial x} \right|_y$$

➤ Similar to thermodynamics:
- ➥ Enthalpy, X, such that
- ➥ Use for isentropic and isobaric processes

$$dX = T\,dS + V\,dP$$

➤ However for isothermal and isobaric processes, better to use the Gibbs energy:

$$\underbrace{G = X - TS}_{\text{Legendre transformation}} \quad \rightarrow dG = S\,dT + V\,dp$$

10.5 The Hamiltonian

$$H(q, p) \equiv p\dot{q} - L(q, \dot{q}) \;;\; p \equiv \left.\frac{\partial L}{\partial \dot{q}}\right|_{constant\,q}$$

$$\left.\frac{\partial H}{\partial p}\right|_q = \dot{q} \;;\; \left.\frac{\partial H}{\partial q}\right|_p = -\left.\frac{\partial L}{\partial q}\right|_{\dot{q}} = -\frac{d}{dt}\left.\frac{\partial L}{\partial \dot{q}}\right|_q = -\dot{p}$$

Lecture 11

11.1 Hamilton's Principle Continued

➤ $S = \int L\,dt \implies \delta S = S[q + \delta q]$
- ➥ $S[q] = \int \delta L\,dt$

➤ page 33 for maths from the following:

$$\delta L = \dot{q}\delta p + p\delta\dot{q} - \underbrace{\left(\frac{\partial H}{\partial q}\delta q + \frac{\partial H}{\partial q}\delta p\right)}_{\delta H}$$

➤ $\int \delta L\,dt = 0$

➤ independent infinitesimal variations in $\delta q, \delta p$ from physical space in coordinate-momentum space (phase space) do not change the action, S

11.2 Hamilton's Equations of Motion

➤ For N DoF, Hamiltonian is defined using the Legendre transformation:

$$H \equiv \left(\sum_{k=1}^{N} p_k \dot{q}_k\right) - L$$

➤ L may now be explicitly time-dependent

➤ Include a time differential to give

$$dH = \sum_{k=1}^{N} \left(\dot{q}_k dp_k - \dot{p}_k dq_k\right) - \frac{\partial L}{\partial t}dt$$

➤ page 33 for more

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \;;\; \dot{q}_k = -\frac{\partial H}{\partial q_k} \;;\; \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

11.3 Examples in Hamiltonians

Example: 1

➤ particle of mass, m, moving vertically in a uniform gravitational field

$$T = \frac{1}{2}m\dot{z}^2 \; ; \; V = mgz$$

$$L = T - V = \frac{1}{2}m\dot{z}^2 - mgz$$

➤ The canonically conjugate momentum to coordinate z,

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

➤ Legendre transformation of $L(z, \dot{z})$ to give $H(z, p_z)$:

$$H(z, p_z) = p_z \dot{z} - L(z, \dot{z})$$

$$H(z, p_z) = p_z \cdot \frac{p_z}{m} - \frac{m}{2}\left(\frac{p_z}{m}\right)^2 + mgz$$

$$H(z, p_z) = \frac{p_z^2}{2m} + mgz = T + V = E_T s$$

➤ EoM:

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \; ; \; \dot{p}_z = -\frac{\partial H}{\partial z} = -mg$$

➤ Integrate $\dot{p}_z$ first to find p_z as a function of time and then integrate $\dot{z}$ using this to find z as a function of time

Example: 2

➤ particle of mass m moves in gravitational field, along the spiral of the form, $z = k\theta$ and $r = cst$, where k is a constant and z is the vertical direction
➤ Find the Hamiltonian, $H(z, p)$, and the equation of motion
➤ Show in the limit $r \to 0$, that $\ddot{z} \to -g$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \; ; \; x = r\cos\theta, \; y = r\sin\theta, \; z = k\theta$$

$$\dot{x} = -r\dot{\theta}\sin\theta \; ; \; \dot{y} = r\dot{\theta}\cos\theta \; ; \; \dot{z} = k\dot{\theta}$$

$$T = \frac{1}{2}m(r^2\dot{\theta}^2 + \dot{z}^2) = \frac{1}{2}m\dot{z}^2\left(1 + \frac{r^2}{k^2}\right) \; ; \; V = mgz$$

$$L = \frac{1}{2}m\dot{z}^2\left(1 + \frac{r^2}{k^2}\right) - mgz$$

➤ Canonically conjugate momentum to z:

$$p = \frac{\partial L}{\partial \dot{z}} = m\dot{z}\left(1 + \frac{r^2}{k^2}\right) \implies \dot{z} = \frac{p}{m\left(1 + \frac{r^2}{k^2}\right)}$$

$$H = p\dot{z} - L = \frac{p^2}{m\left(1 + \frac{r^2}{k^2}\right)} - \frac{p^2}{2m\left(1 + \frac{r^2}{k^2}\right)} + mgz$$

$$H(z, p) = \frac{p^2}{2m\left(1 + \frac{r^2}{k^2}\right)} + mgz = T + V = E_T$$

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➤ EoM:

$$\dot{z} = \frac{\partial H}{\partial p} = \frac{p}{m\left(1 + \frac{r^2}{k^2}\right)} \; ; \; \dot{p} = -\frac{\partial H}{\partial z} = -mg$$

➤ Integrating:

$$p(t) = p(0) - mgt$$

➤ Put this into the other equation:

$$z = \int_0^t \frac{p(0) - mgt'}{m\left(1 + \frac{r^2}{k^2}\right)} dt' = z(0) + \frac{p(0)t}{m\left(1 + \frac{r^2}{k^2}\right)} - \frac{gt^2}{2\left(1 + \frac{r^2}{k^2}\right)}$$

➤ Find expression for $\ddot{z}$:

$$\ddot{z} = \frac{\dot{p}}{m\left(1 + \frac{r^2}{k^2}\right)} = \frac{-g}{\left(1 + \frac{r^2}{k^2}\right)}$$

➤ As $r \to 0$, $\ddot{z} \to -g$ as denominator tends to 1

Example: 3

➤ Stretched Spring attached to a uniformly moving cart
➤ passes $x = 0$ at $t = 0$
➤ cart moving at speed, v_0
➤ unstretched length of the spring is negligible

$$L(x, \dot{x}, t) = T - V = \frac{m\dot{x}^2}{2} - \frac{k}{2}(x - v_0 t)^2$$
$$E - L \implies m\ddot{x} = -k(x - v_0 t), \; [x' = x - v_0 t]$$
$$\ddot{x}' = \ddot{x} \; ; \; m\ddot{x}' = -kx'$$

➤ SHM in the cart's frame of reference

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$H(x, p_x) = p_x\dot{x} - \frac{m\dot{x}^2}{2} + \frac{k}{2}(x - v_0 t)^2 = \frac{p_x^2}{2m} + \frac{k}{2}(x - v_0 t)^2 = T + V = E_T$$

➤ H explicitly depends on t, therefore it is not conserved
➤ To keep the cart moving uniformly despite the oscillating mass, energy flows in and out of the system
➤ However, in terms of $x' = x - v_0 t$, $\dot{x}' = \dot{x} = v_0$:

$$L(x', \dot{x}') = \frac{m\dot{x}'^2}{2} + m\dot{x}'v_0 + \frac{mv_0^2}{2} - \frac{kx'^2}{2}$$
$$p_{x'} = \frac{\partial L}{\partial \dot{x}'} = m\dot{x}' + mv_0$$
$$H'(x', p_{x'}) = p_{x'}\dot{x}' - L(x', \dot{x}')$$
$$H'(x', p_{x'}) = m\dot{x}'^2 + m\dot{x}'v_0 - \frac{m\dot{x}'^2}{2} - m\dot{x}'v_0 - \frac{mv_0^2}{2} + \frac{kx'^2}{2}$$
$$H'(x', p_{x'}) = \frac{m\dot{x}'^2}{2} + \frac{kx'^2}{2} - \frac{mv_0^2}{2}$$
$$H'(x', p_{x'}) = \frac{(p_{x'} - mv_0)^2}{2m} + \frac{kx'^2}{2} - \frac{mv_0^2}{2} \neq E_T$$

➤ $H'(x', p_{x'})$ has no explicit time dependence so it is conserved but it is no longer the total energy

Lecture 12

12.1 Hamilton's Principle

➤ **A mechanical system follows the path which minimises action, where action is $\int L\,dt$**

12.2 Canonical Transformations

➤ A transformation is by definition *canonical*, if it preserves the structure of Hamilton's equations for all dynamical systems
➤ In phase space, can consider new coords Q and momenta P as functions of the original coords and momenta:

$$Q = Q(q, p, t) \; ; \; P = P(q, p, t)$$

➤ Will need to define a new Hamiltonian, H' for new Q and P
➤ page 34 for Lagrangian equivalency

12.3 The Generating Function

➤ $F(q, Q, t)$
➤ $\frac{dF}{dt} = \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial Q}\dot{Q} + \frac{\partial F}{\partial t}$
➤ see page 35

$$\frac{\partial L}{\partial \dot{q}} - \frac{\partial F}{\partial q} \implies p = \frac{\partial F}{\partial q}$$

$$\frac{\partial L'}{\partial \dot{Q}} = -\frac{\partial F}{\partial Q} \implies P = -\frac{\partial F}{\partial Q}$$

12.4 Transformed Hamiltonian

$$H'(Q, P, t) = H(q, p, t) + \frac{\partial F(q, Q, t)}{\partial t}$$

Example:
➤ SHO
➤ For a mass m moving horizontally on a frictionless surface, attached to a spring of spring constant k, the Lagrangian can be written as:

$$L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2 = \frac{m}{2}(\dot{q}^2 - \omega^2 q^2), \; \omega^2 = \frac{k}{m}$$

➤ The canonically conjugate momentum to coordinate q is

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}$$

➤ Therefore, the Hamiltonian is:

$$H = p\dot{q} - L = \frac{p^2}{m} - \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}$$
$$= \frac{1}{2}\left(\frac{p^2}{m} + m\omega^2 q^2\right)$$

➤ Now consider the Generating Function:

$$F(q, Q) = \frac{1}{2} m\omega q^2 \cot(2\pi Q)$$

➤ This yields:

$$P = -\frac{\partial F}{\partial Q} = \frac{m\pi\omega q^2}{\sin^2(2\pi Q)}$$

$$p = \frac{\partial F}{\partial q} = m\omega q \cot(2\pi Q)$$

➤ Eliminating q gives

$$p(P, Q) = \sqrt{\frac{m\omega P}{\pi}} \cos(2\pi Q)$$

➤ This can then be used to give

$$q(P, Q) = \sqrt{\frac{P}{m\omega\pi}} \sin(2\pi Q)$$

➤ Substituting these back into Hamiltonian:

$$H(q, p) = \frac{1}{2} \left[\frac{\omega P}{\pi} \cos^2(2\pi Q) + \frac{\omega P}{\pi} \sin^2(2\pi Q) \right] = \frac{\omega P}{2\pi}$$

$$\frac{\partial F}{\partial t} = 0 \implies H'(Q, P) = H(q, p) = \frac{\omega P}{2\pi}$$

➤ Q is an ignorable coordinate ($\frac{\partial H}{\partial Q} = 0$) and hence P is a constant of the motion
➤ Hamilton's equations give:

$$\dot{Q} = \frac{\partial H'}{\partial P} = \frac{\omega}{2\pi} \implies Q(t) = \frac{\omega t}{2\pi}$$

➤ Transforming back to (q, p) yields the familiar solution:

$$q = \sqrt{\frac{P}{\pi m\omega}} \sin(\omega t)$$

$$p = \sqrt{\frac{m\omega P}{\pi}} \cos(\omega t)$$

$$\sqrt{P} \propto \text{amplitude of oscillation}$$

$$P \propto \text{energy of oscillator}$$

$$Q \propto \text{phase of oscillator}$$

12.5   Forms of Generating Functions

➤ forms are shown on page 36

12.6   Poisson Brackets

➤ page 36 again
➤ For N DoFs, Poisson bracket is defined as:

$$\{F, G\} \equiv \sum_{k=1}^{N} \left(\frac{\partial F}{\partial q_k} \frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k} \frac{\partial G}{\partial q_k} \right)$$

➤ Can rewrite Hamilton's equations of motion:

$$\dot{q} = \{q, H\} \ ; \ \dot{p} = \{p, H\}$$

$$F(q, p, t) \to \dot{F} = \{F, H\} + \frac{\partial F}{\partial t}$$

Lecture 13

13.1 Rotating Reference Frames

13.1.1 Accelerating Reference Frames

➤ page 38
➤ changing from an inertial frame of reference, S, to a linearly accelerating one, B:

$$m\underline{\ddot{r}}_B = m(\underline{\ddot{r}}_S - \underline{\ddot{R}}) = \underbrace{\underline{F}}_{\text{true force}} - \underbrace{m\underline{\ddot{R}}}_{\text{fictitious force}}$$

13.1.2 Rotated Reference Frames in 2D

1. In frame S, position vector has positive x and y components A new fixed reference frame, B, is defined wrt an original reference frame by an anticlockwise rotation through angle θ
2. In frame B, $\underline{r}$ has a positive x component and a negative y component
3. An observer who changes their observation frame from S to B finds that $\underline{r}$ appears to have rotated clockwise through an angle θ, from $\underline{r}_S$ to $\underline{r}_B$

➤ 2D rotation matrix:

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$R \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \ ; \ R \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$$R^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

13.2 Rotations in 3D

➤ rotational operations in 3D **do not commute**
➤ page 39 has derivation leading to the rotation formula:

$$\underline{r}_B = \underline{r}_S \cos\theta + (\underline{n} \cdot \underline{r}_S)\underline{n}[1 - \cos\theta] + (\underline{r}_S \times \underline{n})\sin\theta$$

13.2.1 Infinitesimal Rotations

➤ page 40
➤ fixed in frame S
➤ take the small angle limit for $d\theta$, and rotation formula becomes:

$$\underline{r} + d\underline{r} = \underline{r} + (\underline{r} \times \underline{n})d\theta$$

➤ velocity in B relative to S

$$\left[\frac{dr}{dt}\right]_{in\,B} = -\left(\underline{n}\frac{d\theta}{dt}\right) \times \underline{r} = -\underbrace{\underline{\omega}}_{\text{angular velocity}} \times \underline{r}$$

Example: Pseudo-vectors
➤ Consider vectors in a particular coordinate system:
  ➥ $\underline{r}_1 = (x_1, y_1, z_1)$
  ➥ $\underline{r}_2 = (x_2, y_2, z_2)$
  ➥ $\underline{u} = \underline{r}_1 \times \underline{r}_2$
➤ If the coordinate system is reflected such that x' is in the direction of $-x$ and similarly with the other coordinates, then in this reflected coordinate system, vector $\underline{r}_1$ will have components:
  ➥ $\underline{r}_1' = (-x_1, -y_1, -z_1)$
➤ However, $\underline{u}' = \underline{r}_1' \times \underline{r}_2'$, has components that are the same as $\underline{u}$
  ➥ it has not changed under reflection
➤ $\underline{u}$ is a pseudo-vector
  ➥ it behaves like a vector under rotation but is invariant under reflection

---

13.3 Velocity and Acceleration

➤ must account for time-dependence in frame S:

$$\underbrace{\left[\frac{dr}{dt}\right]_{in\,B}}_{\text{velocity in B}} = \underbrace{\left[\frac{dr}{dt}\right]_{in\,S}}_{\text{velocity in S}} - \underbrace{\underline{\omega} \times \underline{r}}_{\text{motion due to motion of B relative to S}}$$

➤ Use $\left[\frac{d}{dt}\right]_{in\,B}$ as operator

$$\left[\frac{d}{dt}\right]_{in\,B} = \left(\left[\frac{d}{dt}\right]_{in\,S} - \underline{\omega} \times\right)$$

➤ Use this to find the acceleration:

$$\left[\frac{dv_B}{dt}\right]_{in\,B} = \underline{a}_S - 2\underline{\omega} \times \underline{v}_B - \underline{\omega} \times (\underline{\omega} \times \underline{r}) - \dot{\underline{\omega}} \times \underline{r}$$

13.4 Inertial Forces in a Rotating Frame

$$m\ddot{\underline{r}} = \underline{F} - \underbrace{2m\underline{\omega} \times \dot{\underline{r}}}_{\text{coriolis force}} - \underbrace{m\underline{\omega} \times (\underline{\omega} \times \underline{r})}_{\text{centrifugal force}} - \underbrace{m\dot{\underline{\omega}} \times \underline{r}}_{\text{Euler force}}$$

Tying back into Central Forces

➤ Comparing to Central Forces:

$$E = \overbrace{\underbrace{\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2}_{T}}^{T_{eff}} + \underbrace{V_{int}(r)}_{V}$$

➤ Angular momentum barrier, $\frac{1}{2}mr^2\dot{\phi}^2$ is

$$-\nabla(\frac{1}{2}mr^2\dot{\phi}^2) = -m\dot{\phi}^2 r$$

➤ This is just the centrifugal force
➤ Can consider the effective 1D problem as having been set in a rotating, non-inertial frame

Lecture 14

> **Example: Homework**
>   ➤ Sketch
>
> $$V_{eff}(r) = \frac{J^2}{2mr^2} - \frac{k}{r}e^{-\frac{r}{a}}$$
>
>   ➤ Sketch parts independently then add together
>   ↪ exponential not as significant so more similar to first term

14.1   Inertial Forces on Earth

14.1.1   Local Coordinate System

➤ page 41

1. Model Earth as perfect solid sphere and require an appropriate coordinate system to describe motion near a point on its surface
    ➤ Start with Cartesian coordinates with z running through poles
2. Rotate coordinate system around x axis so that z emerges at a chose point on Earth's surface
    ➤ In new coordinates, looks like a rotation of $-\theta$ about x
    ➤ in local frame, ω has y and z components with trig resolutions

$$\underline{\omega} = \begin{pmatrix} 0 \\ \omega\sin\theta \\ \omega\cos\theta \end{pmatrix}$$

3. Displace origin of coordinate system by $\underline{R}$ to the point where the local z emerges from the Earth's surface

14.2   Inertial Forces

➤ need to substitute in $\underline{r} = \underline{r}' + \underline{R}$

14.3   Forces

14.3.1   Centrifugal Forces

➤ $-m\underline{\omega} \times (\underline{\omega} \times \underline{r})$
➤ must be perpendicular to all terms above

- ➤ points directly away from the axis of rotation
- ➤ page 42 for diagrams
- ➤ largest at equator, decreases smoothly as a trig function towards poles
- ➤ adding gravity and centrifugal together gives almost standard gravity map but decreased
    - ➥ centrifugal is not acting radially
    - ➥ effect of gravity is shifted slightly from exactly in the middle

14.3.2   The Deviation of a Plumb Bob

- ➤ Set $\underline{r}' = (0, 0, 0)$ and $\underline{R} = (0, 0, R)$, where R is radius of Earth
- ➤ use

$$\underline{\omega} = \begin{pmatrix} 0 \\ \omega \sin \theta \\ \omega \cos \theta \end{pmatrix}$$

- ➤ obtain centrifugal force on plumb bob of mass m:

$$
\begin{aligned}
\underline{F}_c &= -m\underline{\omega} \times [\underline{\omega} \times (\underline{r}' + \underline{R})] \\
&= -m\omega\underline{\omega} \times \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & \sin\theta & \cos\theta \\ 0 & 0 & R \end{vmatrix} \\
&= -m\omega^2 \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & \sin\theta & \cos\theta \\ R\sin\theta & 0 & 0 \end{vmatrix} \\
&= -m\omega^2 R(0, \sin\theta\cos\theta, -\sin^2\theta)
\end{aligned}
$$

- ➤ Incorporating the effect of gravity, one obtains

$$m\underline{\ddot{r}}' = -m(0, \omega^2 R \sin\theta\cos\theta, g - \omega^2 R \sin^2\theta)$$

- ➤ Now consider the deflection angle, ϕ_d, defined through

$$\tan\phi_d = \frac{\omega^2 R \sin\theta\cos\theta}{g - \omega^2 R \sin^2\theta} \approx \underbrace{\frac{\omega^2 R \sin\theta\cos\theta}{g}}_{g >> \omega^2 R \sin^2\theta}$$

- ➤ For $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$, ϕ_d is maximised at 1.7 milliradians
- ➤ Buildings at colatitude $\frac{\pi}{4}$ are therefore tilted by this amount from the true vertical if aligned with a plumb bob, or a spirit level
- ➤ Note that $\omega = \frac{2\pi}{86400}$rad s^{-1}, $R \approx 6400\,km \implies \frac{\omega^2 r}{g} = 0.003$

14.3.3   Coriolis Force

- ➤ page 43
- ➤ Coriolis force is given by

$$-2m\underline{\omega} \times \underline{\dot{r}}' = -2m\omega(\dot{z}\sin\theta - \dot{y}\cos\theta, \dot{x}\cos\theta, -\dot{x}\sin\theta)$$

- ➤ only affects moving systems, depends on velocities
- ➤ often considered to be constrained in a plane, so ignore z dimension and simplify equation

Example: Coriolis Force
- ➤ particle with mass, m, moves with constant relative speed, v, around the rim of a wheel of radius, b
- ➤ wheel rolls along a fixed straight line with uniform velocity, u
- ➤ taking wheel as frame of reference, find centrifugal and Coriolis forces
- ➤ relative to the inertial frame, moving right with speed, u, and has its origin at O, the rotating wheel frame has angular velocity, $\underline{\omega}$:

$$\underline{\omega} = \frac{u}{b} \text{ - into the page}$$

$$\underline{F}_c = -m\underline{\omega} \times (\underline{\omega} \times \underline{r})$$
$$= m\omega^2 b\hat{\underline{r}}$$
$$= \frac{mu^2}{b}\hat{r}$$

- ➤ Coriolis force:

$$\underline{F}_{cor} = -2m\underline{\omega} \times \overbrace{\dot{\underline{r}}}^{=\underline{v}}$$
$$= 2m\frac{uv}{b}\hat{\underline{r}}$$
$$= \frac{2v}{u}\underline{F}_c$$

14.3.4 Loose Ends

- ➤ Euler force is not zero on Earth, but tends to be negligible
- ➤ In rotating reference frame that is also accelerating translationally, $\ddot{\underline{R}}$, general EoM given by:

$$m\ddot{\underline{r}} = \underline{F} - 2m\underline{\omega} \times \dot{\underline{r}} - m\underline{\omega} \times (\underline{\omega} \times \underline{r}) - m\dot{\underline{\omega}} \times \underline{r} - m\ddot{\underline{R}}$$

Lecture 15

15.1 Examples in the Coriolis Force

$$\underline{F}_{cor} = -2m\underline{\omega} \times \dot{\underline{r}}$$

15.1.1 Weather Systems

- ➤ Storm rotating clockwise is a cyclone, in the southern hemisphere
  - ➥ Coriolis force pushes to the left
- ➤ $\underline{\omega} \times \dot{\underline{r}}$ is West, so $-\underline{\omega} \times \dot{\underline{r}}$ is East
- ➤ As pressure gradient pushes air north towards the centre of low pressure, the coriolis force deflects it East.
- ➤ In the northern hemisphere, coriolis force acts to the right of the motion; in the southern, it acts to the left of the motion.

15.1.2 Foucault's Pendulum

- ➤ Local coordinate frame:

- ➥ $\underline{i}$ corresponding to east,
- ➥ $\underline{j}$ is north,
- ➥ $\underline{k}$ is radially outwards
- ➤ In local frame,

$$\underline{\omega} = \omega \sin\theta \underline{j} + \omega \cos\theta \underline{k}$$

- ➤ θ is the colatitude
- ➤ $\dot{\underline{\omega}} \approx 0$ for Earth and

$$\omega^2 \ll \omega_0^2 = \frac{g}{l}$$

- ➤ Will neglect the Euler and centrifugal forces respectively
- ➤ For the coriolis force:

$$\underline{F}_{cor} = -2m\underline{\omega} \times \underline{\dot{r}}$$

$$\underline{\omega} \times \underline{\dot{r}} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & \omega\sin\theta & \omega\cos\theta \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix}$$

$$= \omega \begin{pmatrix} \dot{z}\sin\theta - \dot{y}\cos\theta \\ \dot{x}\cos\theta \\ -\dot{x}\sin\theta \end{pmatrix}$$

- ➤ Motion is largely in $\underline{i} - \underline{j}$ plane for small angle oscillations, $\therefore \dot{z} \approx 0$
- ➤ Include force due to gravity:

$$\underline{F} = -m\omega_0^2(x\underline{i} + y\underline{j})$$

- ➤ Then, ignoring vertical motion,

$$m\underline{\ddot{r}} = -m\omega_0^2(x\underline{i} + y\underline{j}) - 2m\omega(-\dot{y}\cos\theta\underline{i} - \dot{x}\cos\theta\underline{j})$$

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} 2\omega\dot{y}\cos\theta - \omega_0^2 x \\ -2\omega\dot{x}\cos\theta - \omega_0^2 y \end{pmatrix}$$

- ➤ Set $\zeta(t) = x(t) + iy(t)$, then

$$\begin{aligned} \ddot{\zeta} &= \ddot{x} + i\ddot{y} \\ &= 2\omega\cos\theta \underbrace{(-i\dot{x} + \dot{y})}_{-i\dot{\zeta}} - \omega_0^2 \underbrace{(x + iy)}_{\zeta} \\ &= -2i\omega\cos\theta\dot{\zeta} - \omega_0^2\zeta \\ \ddot{\zeta} &+ 2i\omega\cos\theta\dot{\zeta} + \omega_0^2\zeta = 0 \end{aligned}$$

- ➤ This looks like a damped oscillator with complex damping.
- ➤ Try a solution, $\zeta = e^{\lambda t}$

$$\lambda^2 + 2i\omega \cos\theta \lambda + \omega_0^2 = 0$$

$$\implies \lambda = -i\omega \cos\theta \pm \sqrt{-\omega^2 \cos^2\theta - \omega_0^2}$$

$$= i\left(-\omega \cos\theta \pm \sqrt{\omega^2 \cos^2\theta + \omega_0^2}\right)$$

$$= i\left(-\hat{\omega} \pm \sqrt{\hat{\omega} + \omega_0^2}\right), \ \hat{\omega} = \omega \cos\theta \ll \omega_0$$

$$\approx i(-\hat{\omega} \pm \omega_0)$$

$$\therefore \zeta = x + iy = \underbrace{e^{-i\hat{\omega}t}}_{\substack{\text{precession of the} \\ \text{plane of oscillation}}} \underbrace{(\alpha e^{i\omega_0 t} + \beta e^{-i\omega_0 t})}_{\text{normal pendulum motion}}$$

$$\text{Precession rate} = \hat{\omega} = \omega \cos\theta = \omega \underbrace{\sin\lambda}_{\text{latitude}}$$

15.2 The Euler Force

➤ The moon moves away by about 4 cm a year by taking orbital angular momentum from the spin angular momentum of the earth
  ➥ Earth's days are slowing down due to this
  ➥ In 100 years, days on Earth will be about 2 ms longer
  ➥ $\underline{\dot{\omega}} < 0$ - difference between two ω vectors, points out the south pole

Lecture 16

16.1 Rotational Inertia, Angular Momentum, and Kinetic Energy

➤ See pages 44-46

$$I = \sum_{k=1}^{N} m_k |(\underline{r}_k \cdot \underline{n})\underline{n} - \underline{r}_k|^2 = \sum_{k=1}^{\infty} m_k [r_k^2 - (\underline{r}_k \cdot \underline{n})^2]$$

16.1.1 Defining the Inertia Tensor, $\hat{I}$

$$\underline{J} = \sum_{k=1}^{N} m_k [r_k^2 \underline{\omega} - (\underline{r}_k \cdot \underline{\omega})\underline{r}_k]$$

$$J_x = \sum_{k=1}^{N} m_k (r_k^2 - x_k^2)\omega_x - m_k x_k y_k \omega_y - m_k x_k z_k \omega_z$$

$$J_x = \sum_{k=1}^{N} m_k (r_k^2 - y_k^2)\omega_y - m_k x_k y_k \omega_x - m_k y_k z_k \omega_z$$

$$J_x = \sum_{k=1}^{N} m_k (r_k^2 - z_k^2)\omega_z - m_k x_k z_k \omega_x - m_k y_k z_k \omega_y$$

$$\underbrace{\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix}}_{\underline{J}} = \underbrace{\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}}_{\hat{I}} \underbrace{\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}}_{\underline{\omega}}$$

$$I_{\alpha\beta} = \sum_{k}^{N} m_k (r_k^2 \delta_{\alpha\beta} - r_{k\alpha} r_{k\beta})$$

- ➤ A ratio of vectors is called a Tensor
- ➤ The Inertial tensor is the ratio of $\underline{J}$ to $\underline{\omega}$
- ➤ For continuous rigid bodies:

$$I_{\alpha\beta} = \int_V dx\, dy\, dz\, \rho(x,y,z)(r^2\delta_{\alpha\beta} - r_\alpha r_\beta)$$

Example: Example of Inertia Tensor
- ➤ Consider the Inertia tensor for a uniform cube of length a, mass m, with origin at the centre of mass
- ➤ This is a uniform cube so it has a density

$$\rho = \frac{M}{a^3}$$

$$
\begin{aligned}
I_{xx} &= \int_{-\frac{a}{2}}^{\frac{a}{2}} dx \int_{-\frac{a}{2}}^{\frac{a}{2}} dy \int_{-\frac{a}{2}}^{\frac{a}{2}} dz\, \rho \underbrace{(r^2 \cdot 1 - x^2)}_{y^2 + z^2} \\
&= \frac{M}{a^3} a = \int_{-\frac{a}{2}}^{\frac{a}{2}} dy \left[y^2 z + \frac{z^3}{3} \right]_{-\frac{a}{2}}^{\frac{a}{2}} \\
&= \frac{M}{a^2} \int_{-\frac{a}{2}}^{\frac{a}{2}} dy \cdot 2 \left(\frac{y^2 a}{2} + \frac{a^3}{24} \right) \\
&= \frac{M}{a^2} \left[\frac{y^3 a}{3} + \frac{a^3 y}{12} \right]_{-\frac{a}{2}}^{\frac{a}{2}} \\
&= \frac{2M}{a^3} \left[\frac{a^4}{24} + \frac{a^4}{24} \right] = \frac{Ma^4}{6}
\end{aligned}
$$

- ➤ From symmetry, $I_{xx} = I_{yy} = I_{zz}$

$$
\begin{aligned}
I_{xy} &= \int_{-\frac{a}{2}}^{\frac{a}{2}} dx \int_{-\frac{a}{2}}^{\frac{a}{2}} dy \int_{-\frac{a}{2}}^{\frac{a}{2}} dz\, \rho(r^2 \cdot 0 - xy) \\
&= -\rho a \int_{-\frac{a}{2}}^{\frac{a}{2}} dx \underbrace{\left[\frac{xy^2}{2} \right]_{-\frac{a}{2}}^{\frac{a}{2}}}_{\text{even fn of y}} = 0
\end{aligned}
$$

- ➤ From symmetry, all off-diagonal elements are zero.
- ➤ Hence,

$$\hat{I} = \frac{Ma^2}{6} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- ➤ If origin of coordinates is at one corner of the cube, then

$$I_{xx} = I_{yy} = I_{zz} = \rho \int_0^a dx \int_0^a dy \int_0^a dz \, (y^2 + z^2)$$

$$= \frac{M}{a^2} \int_0^a dy \left[y^2 a + \frac{a^3}{3} \right]$$

$$= \frac{M}{a^2} \left[\frac{y^3 a}{3} + \frac{a^3 y}{3} \right]_0^a = \frac{2Ma^2}{3}$$

$$I_{xy} = -\rho \int_0^a dx \int_0^a dy \int_0^a dz \cdot xy$$

$$= -\rho a \int_0^a dx \left[\frac{xa^2}{2} \right] = -\frac{M}{a^2} \frac{a^2}{2} \frac{a^2}{2}$$

$$= -\frac{Ma^2}{2}$$

➤ Hence,

$$\hat{I} = \frac{Ma^2}{12} \begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix}$$

➤ The inertia tensor of an object is with respect to rotations about a particular point
➤ Inertia tensors will always be symmetric

16.2   Rotational Kinetic Energy

$$T = \frac{1}{2} \underline{\omega} \cdot \underbrace{\underline{J}}_{\hat{I}\underline{\omega}}$$

$$T = \frac{\omega^2}{2} \underline{n}^T \hat{I} \underline{n}$$

➤ $\underline{n}^T \hat{I} \underline{n}$ is the moment of the inertia, I about the axis defined by $\underline{n}$

Lecture 17

17.1   Angular Momentum and KE

➤ page 46 - 47
➤ Define CoM position and time derivative
➤ Total Momentum $P = \sum_{k=1}^N m_k \dot{\underline{r}}_k$
➤ Use

$$\underline{J} = \sum_{k=1}^N m_k (\underline{r}_k \times \dot{\underline{r}}_k)$$

➤ follow derivations in notes

17.2   Parallel and Principal Axis Theorems

17.2.1   Displaced Axis Theorem

➤ Can convert Inertia Tensor into other form using CoM position:

$$I_{\alpha\beta} = \sum_{k=1}^{N} m_k(r_k'^2 \delta_{\alpha\beta} - r_{k,\alpha}' r_{k,\beta}') + M(R_C^2 \delta_{\alpha\beta} - R_{C,\alpha} R_{C,\beta})$$

➤ full derivation on page 48

$$\hat{I} = \hat{I}_{CoM} + M\hat{A}$$

➤ $\hat{A}$ can be represented as a matrix, the elements of which are determined by the elements of the CoM position vector

$$A_{\alpha\beta} = R_C^2 \delta_{\alpha\beta} - R_{C,\alpha} R_{C,\beta}$$

Example: 1

$$A_{xx} = R_C^2 - x_C^2 = y_C^2 + z_C^2$$
$$A_{xy} = -x_C y_C$$

$$\hat{A} = \begin{pmatrix} y_C^2 + z_C^2 & -y_C x_C & -z_C x_C \\ -x_C y_C & z_C^2 + x_C^2 & -z_C y_C \\ -x_C z_C & -y_C z_C & x_C^2 + y_C^2 \end{pmatrix}$$

➤ The moment of inertia for rotations about the z axis through the centre of mass:

$$I_z = \underline{\hat{z}}^T \hat{I}_{CoM} \underline{\hat{z}}$$

➤ $\underline{\hat{z}}$ is a unit vector in the z direction
➤ For a parallel axis is a distance $d = \sqrt{x_C^2 + y_C^2}$ away, the moment of inertia is I_d, where

$$\underline{\hat{z}}^T \hat{I}_d \underline{\hat{z}} = \underline{\hat{z}}^T \hat{I}_{CoM} \underline{\hat{z}} + M \underline{\hat{z}}^T \hat{A} \underline{\hat{z}}$$
$$I_d = I_z + M \underline{\hat{z}}^T \hat{A} \underline{\hat{z}}$$
$$= I_z + M \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} & \hat{A} & \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
$$= I_z + M \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -z_c x_C \\ -z_C y_C \\ x_C^2 + y_C^2 \end{pmatrix}$$
$$= I_z + M(x_C^2 + y_C^2)$$
$$= I_z + Md^2$$

Example: 2
➤ Calculate $\hat{I}$ about the centre of mass for the 3 mass point system shown.
➤ Triangular system, two masses of mass m_1 and one of m_2
➤ First fine $\hat{I}$ for the two masses at m_1 about their CoM, a point X
➤ Choice of axes takes advantage of symmetry, hence

$$I_{xx} = \sum_{k=1}^{2} m_k(x_k^2 - x_k^2) = 0$$

$$I_{yy} = \sum_{k=1}^{2} m_k(\overbrace{x_k^2}^{\pm a/2} - \overbrace{y_k^2}^{0}) = \frac{m_1 a^2}{2}$$

$$\implies I_{zz} = I_{yy}$$

➤ All of-diagonal elements will be 0
  ↪ they involve the product of two different components and only x_k is non-zero

$$\hat{I}_{2m_1,X} = \frac{m_1 a^2}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

➤ CoM of triangle is at $x = 0, z = 0, y = \frac{m_2 h}{m_2 + 2m_1}$

$$\underline{R}_C = \begin{pmatrix} 0 \\ \frac{m_2 h}{m_2 + 2m_1} \\ 0 \end{pmatrix}$$

➤ Therefore, the inertia tensor defined wrt CoM of the triangle for these 2 masses is

$$\hat{I}_{2m_1} = \frac{m_1 a^2}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 2m_1 \begin{pmatrix} R_C^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & R_C^2 \end{pmatrix}$$

$$= m_2 \begin{pmatrix} 2R_C^2 & 0 & 0 \\ 0 & \frac{a^2}{2} & 0 \\ 0 & 0 & \frac{a^2}{2} + 2R_C^2 \end{pmatrix}$$

➤ Inertia Tensor wrt CoM of triangle:

$$\hat{I}_{m_2} = m_2 \begin{pmatrix} (h - R_C)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (h - R_C)^2 \end{pmatrix}$$

$$h - R_C = \frac{h(m_2 + 2m_1) - m_2 h}{m_2 + m_1} = \frac{2m_1 h}{m_2 + 2m_1}$$

$$R_C^2 = \frac{(m_1 h)^2}{(m_2 + 2m_1)^2}$$

$$\therefore \hat{I} = \hat{I}_{2m_1} + \hat{I}_{m_2} = \begin{pmatrix} 2m_1 R_C^2 + m_2(h - R_C)^2 & 0 & 0 \\ 0 & \frac{m_1 a^2}{2} & 0 \\ 0 & 0 & \frac{m_1 a^2}{2} + 2m_1 R_C^2 + m_2(h - R_C)^2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2m_1 m_2 h^2}{m_2 + 2m_1} & 0 & 0 \\ 0 & \frac{m_1 a^2}{2} & 0 \\ 0 & 0 & \frac{2m_1 m_2 h^2}{m_2 + 2m_1} + \frac{m_1 a^2}{2} \end{pmatrix}$$

Lecture 18

18.1 Symmetry of Inertia Tensor

➤ $I_{\alpha\beta} = I_{\beta\alpha}$

18.2 Orthogonal Matrices

➤ page 49
➤ Orthogonal matrices have the transpose as their inverse:
   ➥ $A^{-1} = A^T$
➤ 2D rotation matrix is orthogonal

18.3 Principal Axis Theorem

➤ $\hat{I}$ is represented by a 3x3 matrix so has three eigenvalues.
   ➥ These are the principal moments of inertia
➤ The eigenvectors from this form the principal axes

18.3.1 Transforming $\hat{I}$ under rotations

➤ The rotational KE is invariant under rotations
   ➥ prime indicates rotated coordinates

$$T_{rot} = \frac{1}{2}\underline{\omega}^T \hat{I}\underline{\omega} = \frac{1}{2}\underline{\omega}'^T \hat{I}'\underline{\omega}'$$

$$\underline{\omega}' = \hat{U}\underline{\omega}$$

➤ $\hat{U}$ is an orthogonal matrix representing a general rotation

$$\underline{\omega}'^T = \underline{\omega}^T \hat{U}^T$$

$$T_{rot} = \frac{1}{2}(\underline{\omega}^T \hat{U}^T)\hat{I}'(\hat{U}\underline{\omega})$$

$$= \frac{1}{2}\underline{\omega}^T (\hat{U}^T \hat{I}' \hat{U})\underline{\omega}$$

$$\implies \hat{I} = \hat{U}^T \hat{I}' \hat{U}$$

$$\implies \hat{I}' = \hat{U} \hat{I} \hat{U}^T$$

18.4 Inertia Tensor of a Dumbbell

➤ Find the inertia tensor for a dumbbell
   ➥ two uniform spheres of mass m_1 and m_2 with radii a and b respectively, separated by a massless rigid rod of length R
➤ From the symmetry of the dumbbell, one of the principal axes lies along the line connecting the masses
   ➥ choose one axis lying along this direction
➤ Place origin at CoM
➤ Then the centres of m_1 and m_2 respectively are at

$$\frac{m_2 R}{m_1 + m_2}\hat{k} \; ; \; \frac{-m_1 R}{m_1 + m_2}\hat{k}$$

➤ First calculate the inertia tensor for a sphere of mass $m_1 = \frac{4}{3}\pi a^3 \rho$, which by symmetry has $I_1 = I_2 = I_3$

➤ take u to be distance to axis of rotation of mass dm

$$dI_{3,a} = u^2 \, dm$$
$$dm = \rho \, dV = \rho(r^2 \sin\theta \, dr \, d\theta \, d\phi)$$
$$u = r \sin\theta$$
$$I_{3,a} = \int_0^{2\pi} d\phi \int_0^\pi \int_0^a r^4 \sin\theta (1 - \cos^2\theta)\rho \, d\theta \, dr$$
$$= \frac{8}{15}\rho\pi a^5 = \frac{2}{5}m_1 a^2$$

➤ Similarly for the other sphere,

$$I_{3,b} = \frac{2}{5}m_2 b^2$$

➤ To find the inertia tensor fore the dumbbell, use the displaced axis theorem:

$$I_1 = I_2 = \frac{2}{5}m_1 a^2 + m_1 \frac{m_2 R}{m_1 + m_2} + \frac{2}{5}m_2 b^2 + m_2 \frac{m_1 R}{m_1 + m_2}$$
$$I_3 = \frac{2}{5}m_1 a^2 + \frac{2}{5}m_2 b^2$$

➤ Now that we have the inertia tensor for the principal axis frame, we can find it for any other rotated frame by applying the transformation $\hat{I}' = \hat{P}\hat{I}\hat{P}^T$
➤ For instance, the inertia tensor for a dumbbell in a set of axes rotated through an angle θ about the 1 axis can be found using the rotation matrix

$$\hat{P}_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$\hat{I}' = \hat{P}_x \hat{I} \hat{P}^T = \hat{P}_x \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$= \cdots = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_1\cos^2\theta + I_3\sin^2\theta & (I_3 - I_1)\sin\theta\cos\theta \\ 0 & (I_3 - I_1)\sin\theta\cos\theta & I_1\sin^2\theta + I_3\cos^2\theta \end{pmatrix}$$

Lecture 19

19.1 Rigid Body Dynamics

19.1.1 Euler's Equations of Motion

➤ page 51
➤ take time derivative of angular momentum, $\underline{J}$
↪ gives torque, $\underline{N}$

$$\left(\frac{d\underline{J}}{dt}\right)_B + \underline{\omega} \times \underline{J} = \underline{N}$$
$$I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) = N_1$$
$$I_2\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) = N_2$$
$$I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) = N_3$$

19.1.2 Torque-Free Motion

➤ If there are no external torques, then set Ns to 0
  ➥ no torques when force is all through centre of rotation
➤ Diagram on page 52
  ➥ contour lines map kinetic energy
  ➥ away from 1-2 equator, higher KE; closer to, smaller KE

19.1.3 Motion of a Torque-free symmetric top

➤ Consider a symmetric top with $I_1 = I_2 \neq I_3$
  ➥ Isolated from external torques
  ➥ Spinning freely
➤ As $\underline{N} = 0$, the total angular momentum and rotational KE are conserved as viewed from an inertial frame.
➤ However, in the frame fixed with the principal axes of the top,

$$I_1\dot{\omega}_1 = (I_1 - I_3)\omega_2\omega_3$$
$$I_1\dot{\omega}_2 = (I_3 - I_1)\omega_1\omega_3$$
$$I_3\dot{\omega}_3 = 0$$

➤ Evidently, ω_3 is a constant
➤ Differentiating the other equations wrt time leads to

$$I_1\ddot{\omega}_1 = (I_1 - I_3)\omega_3\dot{\omega}_2$$
$$\implies I_1\ddot{\omega}_1 = -\frac{(I_1 - I_3)^2}{I_1}\omega_3^2\omega_1$$
$$\therefore \ddot{\omega}_1 = -\left[\frac{I_1 - I_3}{I_1}\omega_3\right]^2\omega_1$$

➤ Therefore, $\omega_{1/2}$ undergoes SHM with an angular frequency

$$\Omega_b = \frac{|I_1 - I_3|}{I_1}\omega_3$$

➤ Thus, in the non-inertial frame attached to the principal axes of the top, the angular velocity, $\underline{\omega}$, traces out a circular path around the 3 axis, at fixed ω_3
➤ The rate of this precession of $\underline{\omega}$ is given by Ω_b
➤ In an inertial ('space'-coordinates) frame, without loss of generality, can assume that the angular momentum, $\underline{J}$, lies in the plane of body axes 2 and 3.

Example:
  ➤ Now assume a prolate symmetric top such that $I_1, I_2 > I_3$
  ➤ Choice of axes implies that $\phi_1(t = 0) = 0$
  ➤ Hence, Euler's equations imply

$$I_1\dot{\omega}_1 = (I_1 - I_3)\omega_2\omega_3$$
$$I_1\dot{\omega}_2 = 0$$

  ➤ The instantaneous change in $\underline{\omega}$ is entirely in one direction
  ➤ If we consider $d\underline{\omega}(t = 0) = d\omega_1$ in the plane perpendicular to $\underline{J}$
    ➥ i.e. looking down the $\underline{J}$ axis
    ➥ call the angle between the ω and $\underline{J}$ directions θ_S

$$\Omega_S dt = \frac{d\omega_1}{\omega \sin \theta_S}$$

➤ Therefore, as viewed in the inertial frame, the 3 axis precesses around the angular momentum vector, $\underline{J}$, with angular frequency

$$\Omega_S = \frac{\dot{\omega}_1}{\left(\frac{|\underline{\omega} \times \underline{J}|}{|\underline{J}|}\right)}$$

$$\underline{\omega} \times \underline{J} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \omega_2 & \omega_3 \\ 0 & I_1\omega_2 & I_3\omega_3 \end{vmatrix} = \begin{pmatrix} I_3\omega_2\omega_3 - I_1\omega_2\omega_3 \\ 0 \\ 0 \end{pmatrix}$$

$$|\underline{\omega} \times \underline{J}| = (I_1 - I_3)\omega_2\omega_3, \ (I_1 > I_3)$$

$$\implies \Omega_S = \frac{I_1 - I_3}{I_1}\omega_2\omega_3 \bigg/ \left[\frac{(I_1 - I_3)\omega_2\omega_3}{J}\right]$$

$$= \frac{J}{I_1}$$