Stars and Galaxies

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Contents

1 Stars **3**

Lecture 1 . . . 3

Lecture 2 . . . 3

Excitation Energies . . . 3

Ratios of Excitation Levels . . . 4

Ionisation Energies . . . 4

Lecture 3 . . . 4

Binary Star Systems . . . 4

Visual Binary Systems . . . 5

Normal Example . . . 5

Inclination Example . . . 5

Spectroscopic Binaries . . . 5

Special Case: Eclipsing Spectroscopic Binaries . . . 6

Lecture 4 . . . 6

Lecture 5 . . . 7

Virial Theorem . . . 7

Energy from Gravitational Collapse . . . 8

Lecture 6 . . . 8

Binding Energies of Fusion . . . 8

Coulomb Barrier . . . 8

Probability of Nuclear Reactions . . . 9

Lecture 7 . . . 9

Nuclear Conservation Rules . . . 9

Proton-Proton Chains . . . 10

CNO Cycle . . . 10

Lecture 8 . . . 10

Energy produced in Stars . . . 10

Slide 5 diagram . . . 10

Energy Seen on Earth . . . 10

Mean Free Paths . . . 11

Radiation . . . 11

Lecture 9 . . . 12

Opacity . . . 12

Different sources of Opacity . . . 12

Lecture 10 . . . 12

Schwarzchild Criterion for Convection . . . 12

Convection in the Sun . . . 13

Mixing length . . . 14

Lecture 12 . . . 14

Cepheid Variables . . . 14

Stellar Pulsation . . . 15

Lecture 13 . . . 15

Jeans Mass 15
Free-fall gravitational collapse 16

Chapter 1

Stars

see DUO for pdf slides

Lecture 1

- Black body emission curve
  - LHS from peak lambda is Rayleigh Jeans tail
  - RHS from peak is Wien tail

$$\lambda_{max} = \frac{2.9 \times 10^{-3}}{T}\, m$$
$$\lambda_{max.\,Bet} = 8.3 \times 10^{-7} m \implies T \approx 3500\, K$$
$$\lambda_{max,Sun} = 5.5 \times 10^{-7} m \implies T \approx 5300\, K$$
$$\lambda_{max,Bel} = 3.0 \times 10^{-7} m \implies T \approx 9400\, K$$

Lecture 2

Excitation Energies

- Bohr model
- page 8 on slides
- n denotes the orbitals/electron shells
- $n = 1$ is the ground state

$$E = E_{high} - E_{low} = \frac{hc}{\lambda} = -13.6\Big(\frac{1}{n_{high}^2} - \frac{1}{n_{low}^2}\Big)$$
$$n = 2 \rightarrow 4$$
$$E = 2.55\, eV \implies \lambda = 486.1\, nm \implies H\beta$$

- this was absorption
- $H\beta$ is shorthand for Balmer series β
  - Optical light

$$n = 2 \to 1$$

$$E = 10.2\,eV \implies \lambda = 121.6\,nm \implies Ly\alpha$$

- this was emission
- $Ly\alpha$ is shorthand for Lyman series α
  - UV light
- Photons emitted from de-excitation in random direction
  - statistics means we probably won't see this

Ratios of Excitation Levels

$$n = 2 \to 1$$

$$\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-\frac{(E_2 - E_1)}{kT}}$$

$$g_1 = 2\ ;\ g_2 = 8\ ;\ T = 5800\,K$$

$$\frac{N_2}{N_1} = 5.1 \times 10^{-9}$$

- 1 billionth of H atoms in first excited state, negligible

Ionisation Energies

- χ is the ionisation energy

$$\frac{N_{i+1}}{N_i} = \frac{2Z_{i+1}}{n_e Z_i} \Big(\frac{2\pi m_e kT}{h^2}\Big)^{\frac{3}{2}} e^{-\frac{\chi}{kT}}$$

$$E > -13.6\Big(\frac{1}{\infty^2} - \frac{1}{n_{low}^2}\Big)\,eV$$

$$n = 1 \to \infty \implies E > 13.6\,eV$$

$$n = 2 \to \infty \implies E > 3.4\,eV$$

Lecture 3

Binary Star Systems

- slide 8, binary system
- look at the semi-major axes of the orbits of the two stars around the centre of mass of the system
  - a_1 and a_2 for m_1 and m_2

$$P^2 = \frac{4\pi^2 a^3}{G(m_1 + m_2)}$$

$$a = a_1 + a_2$$

- Smaller semi-major axis means larger mass
- similar to see-saw

$$m_1 a_1 = m_2 a_2 \implies \frac{m_1}{m_2} = \frac{a_2}{a_1}$$

- ratio of the semi-major axes gives ratio of masses
- actually measure α, angle of separation:
  - for d, distance from us

$$\alpha_n = \frac{a_n}{d} \implies \frac{m_1}{m_2} = \frac{\alpha_2}{\alpha_1}$$

Visual Binary Systems

Normal Example

- $d = 10\,pc$; $P = 200$ days
- $\alpha_1 = 0.02$" ; $\alpha_2 = 0.08$"

$$a_1 = \alpha_1 d = 0.2\,Au \text{ ; } a_2 = a_2 = \alpha_2 d = 0.8\,Au$$
$$a = a_1 + a_2 = 1\,Au$$
$$m_1 + m_2 = \frac{4\pi^2 a^3}{GP^2} = 3.4M_\odot = M_{tot}$$
$$\frac{m_1}{m_2} = \frac{\alpha_2}{\alpha_1} = \frac{a_2}{a_1} = 4.0 = M_{rot}$$
$$m_1 = \left[\frac{M_{rot}}{1 + M_{rot}}\right] M_{tot} = 2.72M_\odot$$
$$m_2 = \left[\frac{1}{1 + M_{rot}}\right] M_{tot} = 0.68M_\odot$$

Inclination Example

- For angled systems that aren't flat against our observations:

$$\hat{\alpha}_n = \alpha_n \cos i$$
$$m_1 + m_2 = \frac{4\pi^2}{G}\left(\frac{d}{\cos i}\right)\frac{\hat{\alpha}^3}{P^2}$$
$$\hat{\alpha} = \hat{\alpha}_1 + \hat{\alpha}_2$$

- Has no effect on mass ratios observed - cos cancels
- Above equation means the actual masses will be affected by the inclination

Spectroscopic Binaries

- Correcting for inclination:

$$v_{nr}^{max} = v_n \sin i$$

- Assume $e << 1$

$$v_n = \frac{2\pi a_n}{P}$$
$$\frac{m_1}{m_2} = \frac{v_2}{v_1}$$

- Same sort of stuff as visual binaries, but sin instead of cos basically

Special Case: Eclipsing Spectroscopic Binaries

- $i \approx 90°$
- don't need any corrections etc

Lecture 4

$$P = \underbrace{\frac{\rho k T}{\mu m_H}}_{\text{ideal gas law}} + \frac{1}{3} a T^4$$

- Hydrostatic Equilibrium:
  - Pressure force = Gravitational force

$$\begin{aligned}
P\,on\,dA &= [P(r+dr) - P(r)]dA \\
&= dP\,dA \\
Gravitational &= g\,\underbrace{\underbrace{dA\,dr}_{volume}\,\rho}_{mass},\;\; g = \frac{GM_r}{r^2} \\
dP\,dA &= -g\rho\,dA\,dr \\
\frac{dP}{dr} &= -\frac{GM_r\rho}{r^2} \\
\frac{dM_r}{dr} &= 4\pi r^2\rho \\
M_r &= \frac{4}{3}\pi r^2\rho \\
\frac{dP}{dr} &= -G\frac{4}{3}\pi r\rho^2 \\
\int_{P_s}^{P_c} dP &= -\frac{4}{3}\pi G\rho^2 \int_R^0 r\,dr \\
P_c &= \frac{2}{3}\pi G\rho^2 r^2,\;\; P_s = 0\,at\,r = R \\
&= \frac{2}{3}\pi G r^2\left[\frac{3}{4}\frac{M}{\pi r^3}\right]^2 \\
&= \frac{3}{8\pi}\frac{GM^2}{R^4}
\end{aligned}$$

- Example for our sun:

$$M = 2\times 10^{30}kg\;;\;\; R \approx 7\times 10^8 m$$

$$P_c \approx 10^{14} N\,m^{-2}$$

$$P_{c,\,true} \approx 2\times 10^{16} N\,m^{-2}$$

- out as assumed uniform density

Lecture 5

Virial Theorem

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2} \times V = \frac{4}{3}\pi r^3$$
$$V\frac{dP}{dr} = -\frac{GM\rho}{r^2}\frac{4}{3}\pi r^3$$
$$-\text{ plug in } \frac{dm}{dr} = 4\pi r^2 \rho$$
$$V\frac{dP}{dr} = \frac{1}{3}\frac{GM}{r}\frac{dm}{dr}$$
$$\int_0^{P(R)} V\,dP = -\frac{1}{3}\underbrace{\int_0^M \frac{GM}{r}dm}_{\text{Total GPE}=U}$$
$$LHS: \int U\,dV = UV - \int V\,dU$$
$$\int_0^{P(R)} V\,dP = \underbrace{[PV]_0^{R_0}}_{=0} - \int_0^{V(R)} P\,dV = -\frac{1}{3}U$$
$$-3\int_0^{V(R)} P\,dV = U,\ dV = \frac{dm}{\rho} \implies$$
$$-3\int_0^M \frac{P}{\rho}dm = U \quad \text{- generalised form of Virial Theorem}$$
$$\text{Ideal Gas: } P = nkT = \frac{\rho kT}{\mu m_H}$$
$$\text{Average KE: } = \frac{3}{2}kT$$
$$\text{KE per kilo: } = \frac{3}{2}\frac{kT}{\mu m_H}$$
$$E_{KE} = \frac{3}{2}\frac{kT}{\mu m_H} = \frac{3}{2}\frac{P}{\rho}$$
$$-3\int_0^M \frac{P}{\rho}dm = U,\ \frac{P}{\rho} = \frac{2}{3}E_{KE}$$
$$\underbrace{\int_0^M E_{KE}\,dm}_{\text{Total KE, assume ideal gas}} = -\frac{1}{2}U$$
$$\implies K = -\frac{1}{2}U$$

Energy from Gravitational Collapse

$$dU_{g,i} = -\frac{GM_r dm_i}{r} \text{ - GPE of point mass}$$

Consider shells of material

$$dm = 4\pi r^2 \rho dr$$

$$dU_g = -\frac{GM_r 4\pi r^2 \rho}{r} dr \text{ - GPE of a shell}$$

$$U_g = -4\pi G \int_0^R M_r \rho_r dr$$

$$M_r = \frac{4}{3}\pi r^3 \bar{\rho} \text{ - avg density isn't too bad here}$$

$$U_g = -\frac{16}{3}\pi^2 G \bar{\rho}^2 \int_0^R r^4 dr$$

$$= -\frac{16}{15}\pi^2 G \bar{\rho}^2 R^5$$

Convert back to mass

$$U = -\frac{9}{15}\frac{GM^2}{R} \text{ - GPE of the star}$$

$$K = -\frac{1}{2}U$$

$$\implies E = \frac{3}{10}\frac{GM^2}{R}$$

$$E \approx \frac{3}{10} GM^2 \Big[\frac{1}{R} - \frac{1}{R_{initial}}\Big]$$

$$= \frac{3}{10}\frac{GM^2}{R} \iff R << R_{initial}$$

Lecture 6

Binding Energies of Fusion

$$E_b(Z,N) = \Delta mc^2 = [Zm_p + Nm_n - m(Z,N)]c^2$$

$$E_b(4,0) = [4m_p - m_{He,4}]c^2 = 26.731\, MeV$$

$$\frac{4m_p}{m_{He,4}} = 1.007 \implies e = 0.7\%$$

$$E_\odot = (0.1 \times M_\odot) \times 0.007 \times c^2$$

$$= 1.3 \times 10^{44} J$$

$$t \approx \frac{E_\odot}{L_\odot} = 10^{10} yr$$

Coulomb Barrier

- looking at probability that two particles are close enough for nuclear force to be important
- see figure on page 7 of slides
- using classical physics, we get

$$
\begin{aligned}
E &= \frac{1}{2}mv^2 = \frac{3}{2}kT = \frac{1}{4\pi\epsilon_0}\frac{Z_1Z_2e^2}{r} \\
T &= \frac{1}{6\pi\epsilon_0}\frac{Z_1Z_2e^2}{rk} = \underbrace{1.1\times 10^{10}K}_{r=10^{-15}m\ ;\ Z_1=Z_2=1}
\end{aligned}
$$

- too high for our Sun
- use deBroglie wavelength and consider quantum effects

$$
\begin{aligned}
\lambda &= \frac{h}{p},\ p = mv\ [m=\mu_m] \\
E &= \frac{1}{2}mv^2\ ;\ v^2 = \frac{p^2}{m^2} \\
E &= \frac{p^2}{2m} \\
p^2 &= \left(\frac{h}{\lambda}\right)^2 \\
E &= \frac{(\frac{h}{\lambda})^2}{2m} = \frac{h^2}{\lambda^2}\frac{1}{2m} \\
&= \frac{1}{4\pi\epsilon_0}\frac{Z_1Z_2e^2}{\lambda} = \frac{h^2}{\lambda^2}\frac{1}{2m} \\
\frac{1}{\lambda} &= \frac{2}{4\pi\epsilon_0}\frac{Z_1Z_2e^2m}{h^2} \\
\text{replace } \frac{1}{r} &\text{ with } \frac{1}{\lambda} \\
T &= \frac{1}{12\pi^2\epsilon_0^2}\frac{Z_1^2Z_2^2e^4m}{kh^2} = 9.8\times 10^6K
\end{aligned}
$$

- this happens due to quantum tunneling

Probability of Nuclear Reactions

- see graph on page 13 of slides
- nuclear reaction probability is the product of Maxwell-Boltzmann and Tunneling Probability

Lecture 7

Nuclear Conservation Rules

1. electric charge must be conserved
2. nucleon umber must be conserved
    - $p, n = +1$
3. lepton number must be conserved
    - $e^{\mp} = \pm 1$
    - $\nu_e^{\mp} = \pm 1$

$$
{}^A_ZX
$$

- A - atomic number for element X (nucleon number)
- Z - number of protons (electric charge)

Proton-Proton Chains

$$\begin{aligned} {}^1_1H + {}^1_1H &\rightarrow {}^2_1H + e^+ + \nu_e \\ {}^2_1H + {}^1_1H &\rightarrow {}^3_2He + \gamma \\ {}^3_2He + {}^3_2He &\rightarrow {}^4_2He + {}^1_1H + {}^1_1H \\ \implies 4{}^1_1H &\rightarrow {}^4_2He + \underbrace{2e^+ + 2\nu_e + 2\gamma}_{26.7\,MeV} \end{aligned}$$

CNO Cycle

$$\begin{aligned} {}^{12}_6C + {}^1_1H &\rightarrow {}^{13}_7N + \gamma \\ {}^{13}_7N &\rightarrow \underbrace{{}^{13}_6C + e^+ + \nu_e}_{\beta \text{ decay}} \\ {}^{13}_6C + {}^1_1H &\rightarrow {}^{14}_7N + \gamma \\ {}^{14}_7N + {}^1_1H &\rightarrow {}^{15}_8O + \gamma \\ {}^{15}_8O &\rightarrow \underbrace{{}^{15}_7N + e^+\nu_e}_{\beta \text{ decay}} \\ {}^{15}_7N + {}^1_1H &\rightarrow {}^{12}_6C + {}^4_2He \\ \text{Total: } 4{}^1_1H &\rightarrow {}^4_2He + \underbrace{2e^+ + 2\nu_e + 3\gamma}_{E=26.7\,MeV} \end{aligned}$$

Lecture 8

Energy produced in Stars

$$\begin{aligned} dL &= \epsilon\, dm \quad [W] \\ \epsilon_{i,X} &= \epsilon_0 X_i X_X \rho^\alpha T^\beta \quad [W\, kg^{-1}] \\ dm &= 4\pi r^2 \rho\, dr \\ \implies \frac{dL}{dr} &= 4\pi r^2 \rho \epsilon \end{aligned}$$

Slide 5 diagram

- Solid line just to do with fusion then no fusion
- Dashed line has that shape as volume increase so dL/dr does but then temperature starts falling so fusion decreases

Energy Seen on Earth

- Electrons lose energy travelling through sun

$$\frac{\lambda_{surface}}{\lambda_{core}} \approx 3 \times 10^6$$

Mean Free Paths

- vt - distance travelled
- n - particles per unit volume
- nvt - particle per unit area
- $n\sigma vt$ - number of interactions

$$\begin{aligned} l &= \frac{vt}{n\sigma vt} \\ &= \frac{1}{n\sigma} \end{aligned}$$

- This is the mean distance before a collision

$$\begin{aligned} d &= \sum_i l_i \\ d^2 &= d \cdot d \\ &= \sum_j \sum_i l_i \cdot l_j \end{aligned}$$

- When $i \neq j$, $l_i \cdot l_j = 0$

$$\begin{aligned} d^2 &= Nl^2 \\ \implies N &= \left(\frac{d}{l}\right)^2 \end{aligned}$$

- Use ideal gas law to help in questions of this

$$\begin{aligned} t_{total} &= t_{travel} + Nt_{scatter} \\ &= \frac{Nl}{c} + N \times 10^8 \\ &= 5700\ yrs + \cdots = 10^6\ yrs \end{aligned}$$

Radiation

$$\begin{aligned} P &= \frac{1}{3}aT^4 \\ \frac{dP}{P}dr &= \frac{dP}{dT}\frac{dT}{dr} \\ \frac{dP}{dr} &= \frac{4}{3}aT^3\frac{dT}{dr} \\ \frac{dP}{dr} &= -\frac{\kappa\rho}{c}F_{rad} \\ \kappa rho &= n\sigma \\ \frac{dT}{dr} &= -\frac{3}{4ac}\frac{\kappa\rho F_{rad}}{T^3} \\ L &= 4\pi r^2 F_{rad} \\ \frac{dT}{dr} &= -\frac{3}{16\pi ac}\frac{\kappa\rho L_r}{T^3 r^2} \end{aligned}$$

Lecture 9

Opacity

$$
\begin{aligned}
dI_\lambda &= -\kappa_\lambda \rho I_\lambda ds \\
\int_{I_{\lambda,0}}^{I_{\lambda,f}} \frac{dI_\lambda}{I_\lambda} &= -\int \kappa_\lambda \rho ds \\
\implies I_{\lambda,f} &= I_{\lambda,0} e^{-\int_0^s \kappa_\lambda \rho ds} \\
I_{\lambda,f} &= I_{\lambda,0} \underbrace{e^{-\kappa_\lambda \rho s}}_{\text{optical depth, } \tau} \\
&= I_{\lambda,0} e^{-\tau},\ \tau = \kappa_\lambda \rho s
\end{aligned}
$$

- $\tau < 1$ - optically thin
- $\tau > 1$ - optically thick

Different sources of Opacity

- Two classes of opacity:
    1. Absorption - photon energy lost of KE of gas or degraded
    2. Scattering - photon reemitted at different direction, sometimes degraded

1. Bound-Bound transitions
    - typical temperature roughly $\leq 10^5$K
    - most effective for neutral gas
    - scattering and absorption
2. Bound-free transitions
    - typical temperature of $10^4 \to 10^6$K
    - partially ionised gas
    - absorption
3. Free-free emission
    - typical temperature of $10^4 \to 10^6$K
    - partially ionised gas
    - absorption
4. Electron scattering
    - dominant at roughly $\geq 10^6$K
    - fully ionised gas
    - scattering

Lecture 10

Schwarzchild Criterion for Convection

- slide 4 - 9

$$\gamma = \frac{C_p}{C_V} = \frac{s+2}{s}$$

- s is degrees of freedom

$$P = k_a \rho^\gamma$$
$$\frac{dP}{P} = \frac{\gamma d\rho}{\rho}$$
$$\gamma = \frac{\rho}{P}\frac{dP}{d\rho}$$
Surrounding gas
$$P = nkT = \frac{\rho kT}{\mu m_H}$$
$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T}$$
$$\frac{d\rho}{\rho} = \frac{dP}{P} - \frac{dT}{T}$$
$$\frac{dP}{d\rho}_{sur} > \frac{dP}{d\rho}_{adiab} \Big[\times \frac{\rho}{P}$$
$$\frac{\rho}{P}\frac{dP}{d\rho}_{sur} > \frac{\rho}{P}\frac{dP}{d\rho}_{adiab}$$
$$\frac{\rho}{P}\frac{dP}{d\rho}_{sur} > \gamma_{ad}$$
$$\frac{P}{dP}\Big(\frac{dP}{P} - \frac{dT}{T}\Big)_{sur} < \frac{1}{\gamma_{adiab}}$$
$$\frac{P}{dP}\frac{dP}{P} - \frac{P}{dP}\frac{dT}{T} < \frac{1}{\gamma_{adiab}}$$
$$1 - \Big(\frac{P}{dP}\frac{dT}{T}\Big)_{sur} < \frac{1}{\gamma_{adiab}}$$
$$\frac{T}{P}\Big(\frac{dP}{dT}\Big)_{sur} < \frac{\gamma_{adiab}}{\gamma_{adiab} - 1}$$
$$\Big|\frac{dT}{dr}\Big|_{sur} > \Big(\frac{\gamma_{adiab} - 1}{\gamma_{adiab}}\Big)\frac{T}{P}\Big|\frac{dP}{dr}\Big|_{sur}$$

Convection in the Sun

For the sun:

$$-\frac{3}{16\pi ac}\frac{k\rho L_r}{T^3 r^2} > \Big(\frac{\gamma - 1}{\gamma}\Big)\frac{T}{P}\frac{dP}{dr}$$
$$\frac{dP}{dr} = -\frac{GM_r\rho}{r^2}$$
$$\frac{L_r}{M_r} > \frac{16\pi acG}{\kappa\rho}\frac{aT^4}{3}\frac{\gamma - 1}{\gamma}$$
$$> \frac{16\pi acG}{\kappa\rho}P_{rad}\frac{\gamma - 1}{\gamma}$$
$$> 1.9 \times 10^{-3}\, W\, kg^{-1}$$

Mixing length

$$l = \alpha Hp$$
$$\frac{dP}{dr} = -\frac{GM_r\rho}{r^2} \implies \frac{1}{Hp} = -\frac{1}{P}\frac{dP}{dr}$$
$$Hp = \frac{Pr^2}{GM_r\rho}$$
$$l = \frac{\alpha Pr^2}{GM_r\rho}$$

Lecture 12

Cepheid Variables

$$\log\left(\frac{L}{L_\odot}\right) = 1.15\log_{10}\Pi^d + 2.47$$
$$\Pi^d = 10\,\text{days} \implies L = 4200\,L_\odot$$
$$\text{observed} < f > = 10^{-15} W\, m^{-2}$$
$$L = 4\pi d^2 < f >$$
$$d = \sqrt{\frac{L}{4\pi < f >}}$$

Stellar Pulsation

$$V_s = \sqrt{\frac{\gamma P}{\rho}},\ \gamma = \frac{C_p}{C_V}$$

$$\Pi = 2\int_0^R \frac{dr}{V_s}$$

$$\frac{dP}{dr} = -\frac{GM_r\rho}{r^2}$$

$$\text{const } p \implies \mu = \frac{4}{3}\pi r^3\rho$$

$$\frac{dP}{dr} = -\frac{4}{3}G\pi r\rho^2$$

$$dP = -\frac{4}{3}G\pi\rho^2\int_0^R r\,dr$$

$$P(r) = -\frac{4}{3}G\pi\rho^2\left[\frac{R^2}{2} - \frac{r^2}{2}\right]$$

$$\begin{aligned}\Pi &= 2\int_0^R \frac{dr}{V_s} \\ &= 2\int_0^R \frac{dr}{\sqrt{\frac{2}{3}\gamma G\rho(R^2 - r^2)}} \\ &= 2\sqrt{\frac{3}{2\gamma\pi G\rho}}\left[\sin^{-1}\left(\frac{r}{R}\right)\right]_0^R \\ &= \sqrt{\frac{3\pi}{2G\rho\gamma}}\end{aligned}$$

Lecture 13

Jeans Mass

- For the gravitational collapse of a gas cloud:

$$
\begin{aligned}
GE = U &= -\frac{3}{5}\frac{GM^2}{R} \\
KE = K &= \frac{3}{2}NkT \\
&= \frac{3}{2}\frac{M_c}{\mu m_H}kT \\
2K &< |U| \\
2\left(\frac{3}{2}\frac{M_c kT}{\mu m_H}\right) &< \frac{3}{5}\frac{GM_c^2}{R_c} \\
R_c &= \left(\frac{3}{4}\frac{M_c}{\pi\rho_0}\right)^{\frac{1}{3}} \\
2\left(\frac{3}{2}\frac{M_c kT}{\mu m_H}\right) &< \frac{3}{5}GM_c^2\left(\frac{4}{3}\frac{\pi\rho_0}{M_c}\right)^{\frac{1}{3}} \\
\frac{5M_c kT}{\mu m H G} &< M_c^2\left(\frac{4}{3}\frac{\pi\rho_0}{M_c}\right)^{\frac{1}{3}} \\
M_c &< M_J \\
M_J &\approx \left(\frac{5kT}{G\mu m_H}\right)^{\frac{3}{2}}\left(\frac{3}{4\pi\rho_0}\right)^{\frac{1}{2}}
\end{aligned}
$$

Free-fall gravitational collapse

1. $M_c > M_J$
    - free fall collapse
    - optically thin
    - pressure increase
    - temperature constant
2. Fragmentation
    - optically thin
    - individual regions exceed local M_J
3. M_J minimised: Protostar
    - optically thick
    - pressure increase
    - temperature increase
    - Slow contraction (Kelvin-Helmholtz timescale)