Atoms, Lasers, and Qubits

Dr Weatheril and Prof Adams

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Contents

Part I

LASER PHYSICS

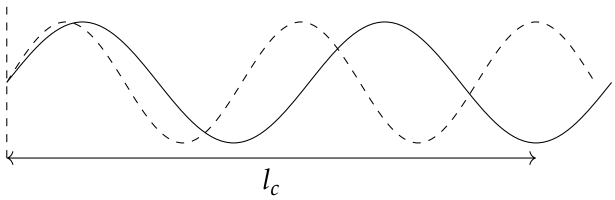
Lecture 1 Introduction

- **Note**: The course will be more reading based than math based. Read the references on each summary sheet.
- Need light oscillation, not just amplification.
- 1 in 10^{18} atomic clock accuracy.
- LD: laser diode
- Non-linear crystals allow different wavelengths
- Laser transitions based on E group of materials
- Learn the unit conversions
- Magneto-optical trap to cool atoms
- Optical frequency comb → accurate measurement of wavelength of light
- Sodium atoms in upper atmosphere which we fluoresce for AO
- **LEARN!** Q on paper always - contents of a laser
  1. More in excited state than ground
  2. Pump gets energy in
  3. Mirrors to make light bounce back and forward

1.1 Introduction to lasers

Lasers - coherence → 2 types - longitudinal and transverse

- Lasers are highly coherent, both transversely and longitudinally. Longitudinal and temporal coherence is related to linewidth, and will be discussed. Coherence length l_c and coherence time τ_c are the distance and time over which a coherent wave maintains a specified degree of coherence, i.e. when its phase is predictable.



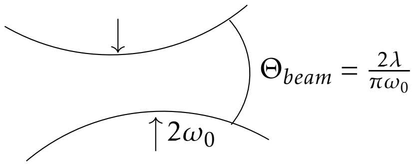
Coherence length and time:

$$l_c = \frac{2\pi c}{\delta\omega}, \quad \tau_c = \frac{2\pi}{\delta\omega} \tag{1.1}$$

- Can't have an infinitely narrow spectrum. Monochromaticity - laser has a spectral linewidth $\delta\omega$, this is much smaller than the actual carrier/centre frequency. $\delta\omega \ll \omega_0$ for a laser where ω_0 is the centre frequency. From mHz to GHz in range.
- Highly directional beam → energy contained in one region.

Directionality:

- Lasers have highly directional beams that diverge due to diffraction
- Beam will be larger
- Waist of beam, $2\omega_0$.



- All in a very low frequency range → all energy oscillating in small aarea in small frequency range - useful applications.

Brightness:

- Lasers are spectrally bright

- Definition of brightness - amount of power in particular area (solid angle) of the beam:

$$B_\omega = \frac{P}{A\Delta\Omega\Delta\omega}, \tag{1.2}$$

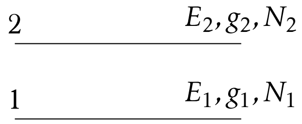
where A is the area, $\Delta\Omega$ is the solid angle, and $\Delta\omega$ is the linewidth.

Electromagnetic Field Modes - not examinable:

- 1st Chapter of 'Laser Physics' book
- Planck's radiation law
- Each unique solution of field is EM mode
- L^3 factored out when divided by volume
- Modes exist with or without energy

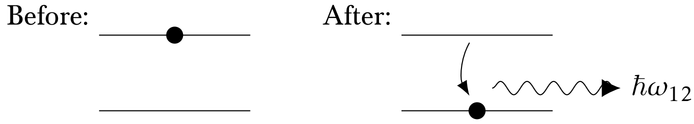
Lecture 2 Einstein's rate equations

A laser requires amplification due to stimulated emission of radiation.

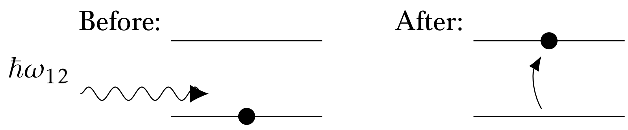


$$E_2 - E_1 = \hbar\omega_{12} \tag{2.1}$$

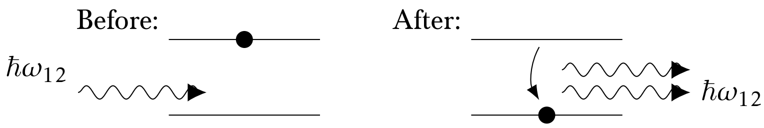
1. Spontaneous emission: Atom in some excited state, until some time later where it spontaneously decays into a lower state, with a photon emitted with energy shown in Eq (2.1). Rates: A_{21} per N_i atoms, or $A_{21}N_2$ per m^3.



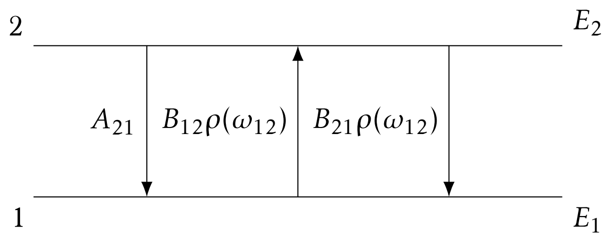
2. Absorption: Excite into excited state using energy of photon. Rates: $B_{12}\rho(\omega_{12})$, $B_{12}\rho(\omega_{12})N_1$.



3. Stimulated emission: At the initial time we have an atom in an excited state which we then apply a radiated field (photon) to. Later, the atom will decay into a lower state and there will be two outgoing photons - they have been emitted into the same mode. Rates: $B_{21}\rho(\omega_{12})$, $B_{21}\rho(\omega_{12})N_2$. Note:
   - Here, $\rho(\omega_{12})$ is the spectral energy density - the energy density, per unit angular frequency range at ω, with units of $J\,m^{-3}\,s$.
   - Generally, the light being used here is broad-band.



2.1 Apply the 3 processess



Conservation of atom number:

$$\frac{dN_2}{dt} = -\frac{dN_1}{dt} \tag{2.2}$$

$$N_1 + N_2 = N = \text{const} \tag{2.3}$$

$$N_1 B_{12}\rho(\omega_{12}) = N_2 A_{21} + N_2 B_{21}\rho(\omega_{12}) \tag{2.4}$$

Rearrange for the spectral energy density:

$$\rho(\omega_{12}) = \frac{N_2 A_{21}}{N_1 B_{12} - N_2 B_{21}} = \frac{\frac{A_{21}}{B_{21}}}{\frac{N_1}{N_2}\frac{B_{12}}{B_{21}} - 1} \tag{2.5}$$

Substitute using Boltzmann Law:

$$\frac{N_2}{N_1} = \frac{g_2}{g_1}\exp\left(-\frac{\hbar\omega_{12}}{k_B T}\right) \tag{2.6}$$

$$\implies \rho(\omega_{12}) = \frac{\frac{A_{21}}{B_{21}}}{\frac{g_1}{g_2}\frac{B_{12}}{B_{21}}\exp\left(\frac{\hbar\omega_{12}}{k_B T}\right) - 1} \tag{2.7}$$

Now look at Planck's Law:

$$\rho(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3}\frac{1}{\exp\left(\frac{\hbar\omega}{k_B T}\right) - 1} \tag{2.8}$$

Einstein realised there must be an extra condition to switch between these two forms. This reveals:

$$g_1 B_{12} = g_2 B_{21} \tag{2.9}$$

$$A_{21} = \frac{\hbar\omega_{12}^3}{\pi^2 c^3} B_{21} \tag{2.10}$$

Notes:

- Effectively only 1 coefficient as if we know A, we know B.
- A_{21} is the radiative decay rate,

$$A_{21} = \frac{1}{\tau_2} \tag{2.11}$$

- B has units $m^3 J^{-1} s^{-2}$.
- A and B are constants for a particular atom.
- From the ω^3 term, we can see that an infrared transition may decay very fast, but a microwave transition may decay very slow.
- Ratio of $A/B \propto \omega^3$ - lasers at high frequency harder to achieve.
- The principle of detailed balance states that **in equilibrium, the total number of particles entering a quantum state by a particular rate per unit time is the same as the number leaving by the same rate.**

2.2 Steady State Solution

For simplicity, we will assume that $g_1 = g_2$ such that the B coefficients are the same.

$$N_1 B_{12}\rho(\omega_{12}) = N_2 A_{21} + N_2 B_{12}\rho(\omega_{12}) \tag{2.12}$$

$$N_1 = N - N_2 \tag{2.13}$$

Now we can rearrange to eliminate N_1.

$$\frac{N_2}{N} = \frac{B_{12}\rho(\omega_{12})}{A_{21} + 2B_{12}\rho(\omega_{12})} \tag{2.14}$$

Now consider the form of the above as the spectral energy density tends to infinity. What we see is that $N_2 \to \frac{N}{2}$. This tells us that for at least a two-level atom we cannot get population inversion, which is required for lasers, i.e. steady state inversion impossible.

2.3 Number of photons per mode

$\bar{n}$ can be thought of as the mean number of photons per mode, and $g(\omega)\,d\omega$ as the mode density.

$$\rho(\omega)\,d\omega = \bar{n} \times g(\omega)\,d\omega \times \hbar\omega \tag{2.15}$$

Standard result:

$$g(\omega)\,d\omega = \frac{\omega^2}{\pi^2 c^3}\,d\omega \tag{2.16}$$

$$\implies \bar{n} = \frac{\rho(\omega)}{\hbar\omega g(\omega)} = \frac{\pi^2 c^3}{\hbar\omega^3}\rho(\omega) \tag{2.17}$$

$$\bar{n} = \frac{B_{21}\rho(\omega)}{A_{21}} = \frac{\text{rate of stimulated emission}}{\text{rate of spontaneous emission}} \tag{2.18}$$

It follows that:

- $\bar{n} > 1$ - stimulated emission dominates $\implies$ LASERS
- $\bar{n} < 1$ - spontaneous emission dominates $\implies$ classical light source

For a black body:

$$\bar{n} = \frac{1}{\exp\left(\frac{\hbar\omega_{12}}{k_B T}\right) - 1} = \frac{B_{12}\rho(\omega)}{A_{21}} \tag{2.19}$$

These rates are equal when

$$\frac{\hbar\omega_{12}}{k_B T} = \ln(2) \tag{2.20}$$

For $\lambda = 500\,nm$, $T = 41400\,K$. So for most black bodies, stimulated emission is negligible.

Lecture 3 Linewidths and Lineshapes

2 types of broadening:

- Homogeneous - all atoms in sample affected the same
- Inhomogeneous - atoms in sample affected differently

3.1 Homogeneous Broadening

Every atom/molecule exhibits this type of broadening to varying magnitudes.
Radiative (natural) broadening:

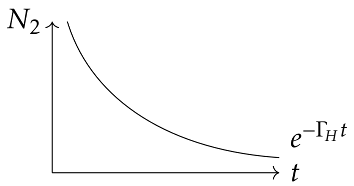
- Consider some excited state population, N_2. $e^{-\Gamma_H t}$ Leads to some spread in maximum.

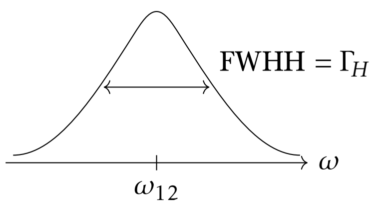
$$FWHM = \Gamma_H \tag{3.1}$$

  This broadening follows from the Heisenberg uncertainty principle: Finite lifetime $\implies$ spread in energy, $\Delta E \Delta t \approx \hbar/2$, then a spread in energy $\implies$ a spread in frequency,

$$\Delta E = \hbar \Delta\omega \implies \Delta\omega = \frac{1}{\tau_2} = A_{21} = \Gamma_H \tag{3.2}$$

  This is only for a 2 level atom.



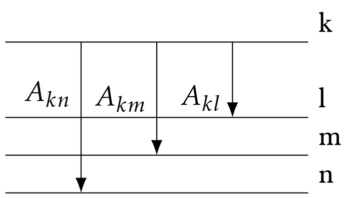


- Define the normalised lineshape function:

$$L_H(\omega) = \frac{\Gamma_H/2\pi}{(\omega - \omega_{12})^2 + \frac{\Gamma_H^2}{4}} \tag{3.3}$$

$$\int_0^\infty L_H(\omega)\, d\omega = 1 \tag{3.4}$$

- For multiple decay paths:

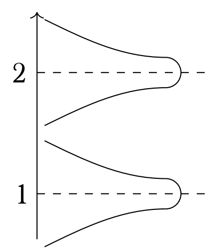


For level k

$$\frac{dN_k}{dt} = -N_k A_{kl} - N_k A_{km} - N_k A_{kn} \tag{3.5}$$

$$\tau_k = \frac{1}{\sum A_{ki}} = \frac{1}{\Gamma_H} \tag{3.6}$$

- When both levels decay:



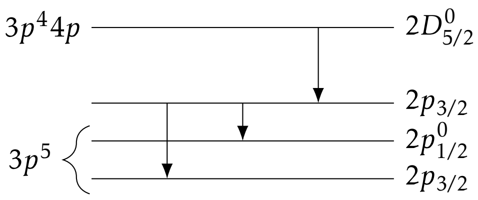
$$\Gamma_2 = \sum A_{2i} = \frac{1}{\tau_2} \tag{3.7}$$

$$\Gamma_1 = \sum A_{1i} = \frac{1}{\tau_1} \tag{3.8}$$

$$\Gamma_{21} = \Gamma_2 + \Gamma_1 \tag{3.9}$$

Γ_{21} is the emission linewidth.

Example: Argon ion laser



$\lambda = 488\,nm$, $A = 7.8 \times 10^7\ s^{-1}$, $1 : 73,1\ nm;\ A = 4.5 \times 10^8\ s^{-1}$, $2 : 72.3\ nm;\ A = 23 \times 10^8\ s^{-1}$.
Emission linewidth:

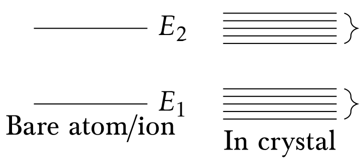
$$\sum A = 2.8 \times 10^9 = (2\pi)450\,MHz \tag{3.10}$$

Collisional/pressure broadening:

- Collisions between atoms - de-excite the atoms $\longrightarrow$ reduce excited state lifetime $\longrightarrow$ broader transition
- Depends on pressure - important for gas lasers

Phonon broadening:

- In a crystal, two distinct groups of energy levels packed together, quantised vibrational modes $\Longrightarrow$ phonons.
- Occurs in solid state lasers, and is temperature dependent
- Dominant broadening at room temperature



3.2 Inhomogeneous Broadening

Doppler Broadening:

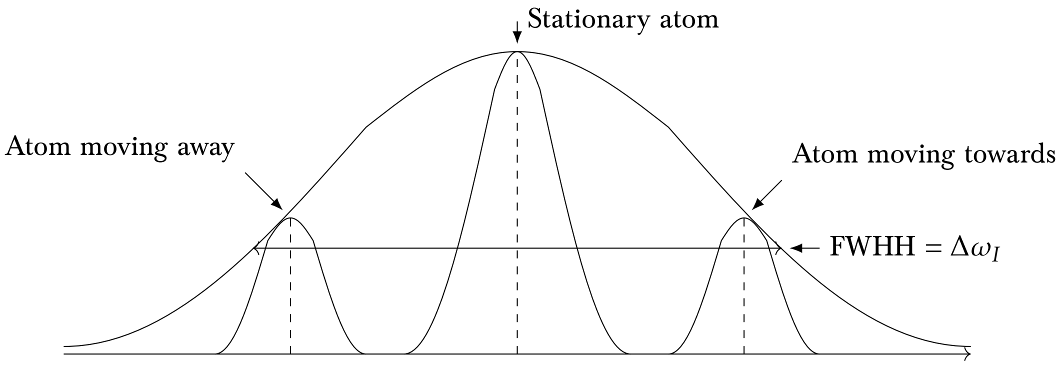
- Arises due to motion of atoms - when a moving atom emits, there is a Doppler shift dependent on the component of velocity along the direction of the emitted photon.

$$\omega = \omega_{12}\left(1 \pm \frac{v_z}{c}\right) \tag{3.11}$$

  $\pm$ for blue/red shift - blue as velocity towards observer, red away.
- Broadening arises due to Maxwell-Boltzmann distribution of velocities,

$$P(v_z)\,dv_z = \left(\frac{M}{2\pi k_B T}\right)^{1/2} \exp\left(-\frac{M v_z^2}{2k_B T}\right) dv_z \tag{3.12}$$

  Use correspondence between v_z and ω to get



$$P(\omega)\,d\omega = \frac{c}{\omega_{12}}\left(\frac{M}{2\pi k_B T}\right)^{1/2} \exp\left[-\frac{Mc^2}{2k_B T}\frac{(\omega-\omega_{12})^2}{\omega_{12}^2}\right] \tag{3.13}$$

➤ For Doppler broadening,

$$\Delta\omega_I = \frac{2\omega_{12}}{c}\left(\frac{2k_B T}{M}\ln 2\right)^{1/2} \tag{3.14}$$

$$= 7.16\times 10^{-7}\omega_{12}\left(\frac{T}{M_A}\right)^{1/2} \tag{3.15}$$

$\Delta\omega_I$ is known as the Doppler width, M_A is using the mass in atomic units instead of kg as used so far for M.

Example: Argon ion laser 2
$M_A = 40$, $\lambda = 488\,nm$, Discharge temperature $\approx 1200°C$.

$$\Delta\omega_I = (2\pi)2.7\,GHz \tag{3.16}$$

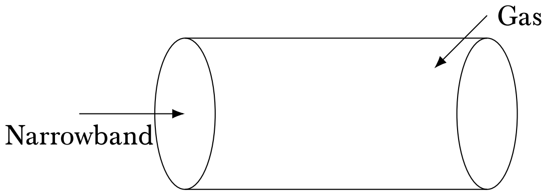
This is many times larger than the natural broadening.

Amorphous crystal broadening:

➤ Occurs in glass materials
➤ Inhomogeneities are Gaussian (normal), the emission is also Gaussian

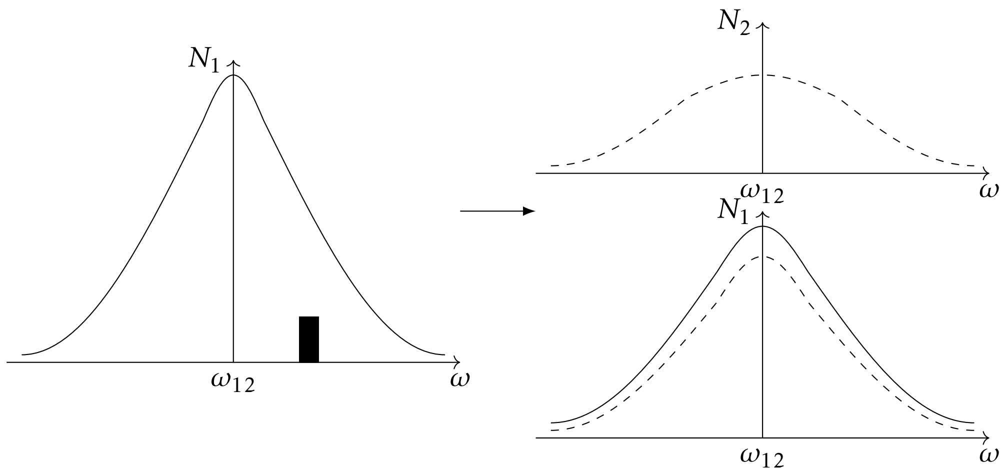
Lecture 4 Amplification by Stimulated Emission

Example:



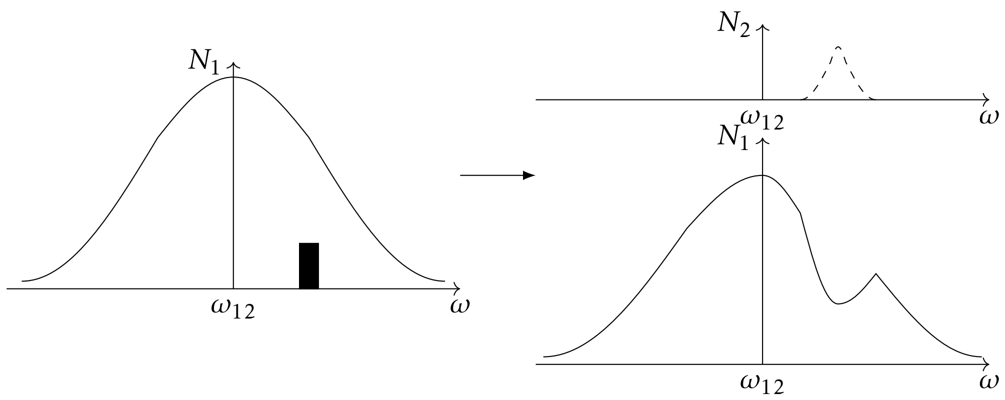
If the light beam is resonant with the atoms in the gas, we would expect some interaction. But how is the medium excited? It depends on the broadening.

1. Homogeneous broadening (Lorentzian):



Turn on radiation within natural linewidth. All of these distributions will be the same - all atoms are affected equally, and the population of N_1 is reduced slightly as it moves into N_2.

2. Inhomogeneous broadening (Gaussian):



Distributions are now very different - atoms are affected differently, and only a subset of the atoms interact. For example, with Doppler broadening, only particles with a specific velocity would be excited strongly by the light beam. The width of the N2 Gaussian is determined by natural broadening.

Reminder:

$$A_{21}N_2 \equiv \text{rate of spontaneous emission per } m^3 \tag{4.1}$$
$$B_{12}\rho(\omega_{12})N_1 \equiv \text{rate of absorption per } m^3 \tag{4.2}$$
$$B_{21}\rho(\omega_{12})N_2 \equiv \text{rate of stimulated emission per } m^3 \tag{4.3}$$

4.1 Homogeneous broadening

All atoms are affected the same $\implies$ simply define a new constant:

$$a_{21}(\omega) = A_{21}L_H(\omega) \qquad b_{12}(\omega) = B_{12}L_H(\omega) \qquad b_{21}(\omega) = B_{21}L_H(\omega) \tag{4.4}$$

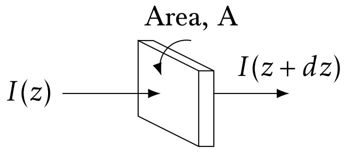
Then if we wanted to know the rate of emission now:

$$a_{21}(\omega)\,d\omega = \text{rate of emission in range } \omega \to \omega + d\omega \text{ per atom} \tag{4.5}$$

We have to multiply by the $d\omega$ because $L_H(\omega)$ has units $\frac{1}{\omega}$. Integrating recovers the result,

$$\int L_H(\omega)\,d\omega = 1 \tag{4.6}$$

Inhomogeneous broadening is more difficult. We want to do this, but we can use the results from homogeneous broadening in certain circumstances, i.e. for narrowband light and $\Gamma_H \ll \Delta\omega_I$.



Consider a weak narrowband beam of light, frequency ω and bandwidth $d\omega$.
Assume:

- No spontaneous emission
- Homogeneous broadening
- Steady state

Find change in power of beam.

$$(I(z+dz) - I(z))A = [N_2B_{21}L_H(\omega)\rho(\omega_{12})\,d\omega - N_1B_{12}L_H(\omega)\rho(\omega_{12})\,d\omega] \times \hbar\omega \times A\,dz \tag{4.7}$$

The term inside the square brackets above is the net rate of photons added to the field (m^{-3}).
For a beam, the intensity is described as

$$I(z) = c\rho(\omega)\,d\omega \tag{4.8}$$

Substitute this into Eq (4.7) and rearrange, also using the expressions for A and B coefficients found in lecture 2:

$$\frac{1}{I}\frac{dI}{dz} = \frac{\hbar\omega}{c}L_H(\omega)[N_2B_{21} - N_1B_{12}] \tag{4.9}$$
$$= \frac{\pi^2c^2}{\omega_{12}^2}A_{21}L_H(\omega)\left[N_2 - \frac{g_2}{g_1}N_1\right] \tag{4.10}$$

This uses the assumption that we are close to resonance.
The prefactor can be defined as the optical gain cross section,

$$\sigma(\omega) = \frac{\pi^2c^2}{\omega_{12}^2}A_{21}L_H(\omega). \tag{4.11}$$

Solve equation:

$$I(z) = I(0)\exp\left[\sigma(\omega) \times \left(N_2 - \frac{g_2}{g_1}N_1\right)z\right] \tag{4.12}$$

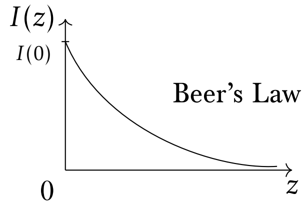
There are two cases:

1. Absorption:

$$N_2 < \frac{g_2}{g_1}N_1 \tag{4.13}$$

$$\implies I(z) = I(0)e^{-\kappa(\omega)z},\ \kappa(\omega) = \sigma(\omega)\left(\frac{g_2}{g_1}N_1 - N_2\right) \tag{4.14}$$

   κ is the absorption coefficient.

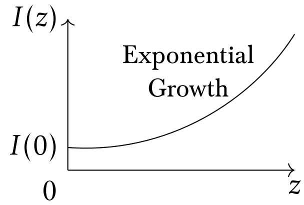


2. Amplification.

$$N_2 > \frac{g_2}{g_1}N_1 \tag{4.15}$$

$$I(z) = I(0)e^{g(\omega)z},\ g(\omega) = \sigma(\omega)\left(N_2 - \frac{g_2}{g_1}N_1\right) \tag{4.16}$$

   $g(\omega)$ is the gain coefficient, $e^{g(\omega)z}$ is the gain.



The condition for gain is

$$N_2 > \frac{g_2}{g_1}N_1. \tag{4.17}$$

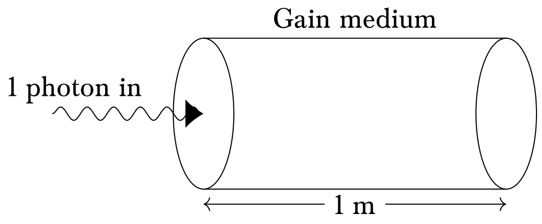
This is known as Population Inversion. Frequency dependence of the gain:

$$g(\omega) = \sigma(\omega) \times N^*,\ N^* = N_2 - \frac{g_2}{g_1}N_1 \tag{4.18}$$

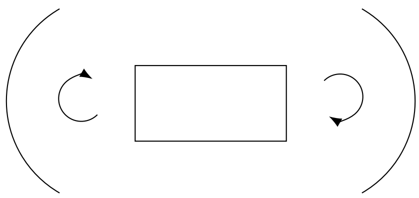
$$= \frac{\pi^2 c^2}{\omega_{12}^2} A_{21} L_H(\omega) \times \left[N_2 - \frac{g_2}{g_1}N_1\right] \tag{4.19}$$

We call N^* the population inversion density. **Note**: Frequency dependence is in the optical gain cross section via $L(\omega)$. $\sigma(\omega)$ is determined by the atom, we cannot control it. Broadening spreads gain over a range of frequencies.

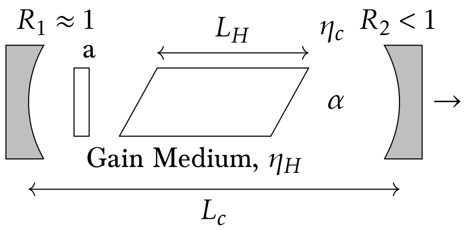
Lecture 5 The Laser Oscillator: Cavity Basics and Threshold



Imagine we have a sample which is our laser gain sample, and we will send one photon in. For most lasers, the peak gain, $g(\omega_{12}) \approx 0.01\ cm^{-1}$. So one photon in to our 1m long sample, means $e^1 = 2.7$ photons out. We must add a cavity to recirculate the light, and now using mirrors either side of the gain medium, we will pass through the gain medium 40 times, $e^{40} \approx 10^{17}$ photons.



5.1 General Cavity Design



- R_1 is mirror 1 and must have a reflectivity as close to 1 as possible
- R_2 is the output coupler and must have a lower reflectivity so that the laser can eventually escape through this
- L_c is the mirror separation, the length of the cavity
- α is the distributed loss over the length of the gain medium
- a is the intracavity element - it can make pulsed lasers or change the frequency, or induce a fixed loss
- The gain cell is angled, cut at Brewster's angle to minimise loss.
- The cavity imposes spectrum and characteristics to the laser.
- Distributed loss throughout cavity of α per meter.

5.2 Longitudinal Cavity Modes

The field in the cavity forms standing waves, i.e.

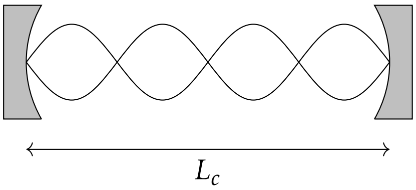
$$n\frac{\lambda_n}{2} = L_c \tag{5.1}$$

n is called the mode order. Mode separation,

$$\Delta\nu_{fsr} = \nu_{n+1} - \nu_n = \frac{c}{2L_c} \tag{5.2}$$

This is the Free Spectral Range.

Example: Mode order of 10cm cavity at $\lambda = 500\,nm$



$$n = \frac{2L_c}{\lambda} = 4 \times 10^5 \tag{5.3}$$

$$\Delta\nu_{fsr} = \frac{c}{2L_c} = 1.5\ Ghz \tag{5.4}$$

5.3 Cavity Losses

Trace intensity around the cavity:

$$I_0 \implies I_0 R_1(1-a)R_2(1-a)e^{-2\alpha L_c} \tag{5.5}$$

- Pass through the cavity twice, so $(1-a)$ twice.
- $e^{-2\alpha L_c}$ is the distributed loss over length, $2L_c$.

$$I_0 \implies I_0 R_1 R_2 (1-a)^2 e^{-2\alpha L_c} \tag{5.6}$$

Express as round trip loss,

$$I_0 \implies I_0 e^{-\delta_c} \tag{5.7}$$

$$\delta_c = \ln\left(\frac{1}{R_1 R_2 (1-a)^2}\right) + 2\alpha L_c \tag{5.8}$$

$$\frac{I_0 - I_0 e^{-\delta_c}}{I_0} = 1 - e^{-\delta_c} \approx 1 - (1-\delta_c) = \delta_c \tag{5.9}$$

This is the Fractional Round Trip Loss.

5.4 Cavity Lifetime

The lifetime of a photon in a cavity is a useful concept. Define cavity lifetime, τ_c.

$$I(t) = I_0 e^{-t/\tau_c} \tag{5.10}$$

$$= I_0 e^{-N\delta_c} = I_0 e^{-t\delta_c/T_{RT}} \tag{5.11}$$

$$\tau_c = \frac{2L_c}{c\delta_c} = \frac{T_{RT}}{\delta_c} \tag{5.12}$$

T_{RT} is called the Round Trip Time. Cavity has a quality factor,

$$Q = \frac{\omega}{\Delta\omega_c} = \omega\tau_c \tag{5.13}$$

So we can then think of cavities as having a linewidth, $\Delta\omega_c = \frac{1}{\tau_c}$.

5.5 Threshold Condition

$$\text{Round Trip Gain} \times \text{Round Trip Loss} = 1 \tag{5.14}$$

- Gain x loss > 1, intensity grows $\implies$ laser oscillation
- Gain x loss < 1, intensity decays $\implies$ no laser :(

Substitute for gain and loss:

$$e^{2g(\omega_{12}L_m} \times e^{-\delta_c} = 1 \tag{5.15}$$

$$g_{th}(\omega_{12}) = \frac{\delta_c}{2L_m} \tag{5.16}$$

Example: Helium Neon Laser

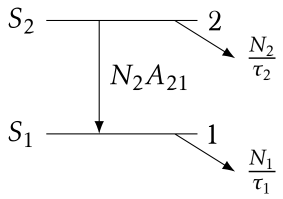
- Gain cell length = $L_m = L_c = 50\,cm$.
- Two mirrors with $R_1 = 0.998$ and $R_2 = 0.98$.
- Distributed losses are $\alpha = 0.02\,m^{-1}$.
- $A_{21} = 3.4 \times 10^6\,s^{-1},\ \lambda = 632.8\,nm.$
- Doppler width is dominant as it is a gas (could be pressure broadening, but assume not for this) - $\Delta\omega_I = 2\pi \cdot 1500\,MHz.$

So what is the population inversion density, N^*, required for laser oscillation?

- Considering round trip losses
- Fractional round trip loss
- Atomic properties for expression of gain
- Only thing left to then find is N^*

Lecture 6 The Laser Oscillator: Oscillation and Gain Saturation

6.1 Gain Saturation



Develop rate equations for N_1 and N_2.

$$\frac{dN_2}{dt} = S_2 - \frac{N_2}{\tau_2} \tag{6.1}$$

$$\frac{dN_1}{dt} = S_1 + N_2A_{21} - \frac{N_1}{\tau_1} \tag{6.2}$$

In the steady state,

$$\frac{dN_2}{dt} = \frac{dN_1}{dt} = 0 \tag{6.3}$$

$$N_2 = S_2\tau_2 \tag{6.4}$$

$$N_1 = S_1\tau_1 + A_{21}\tau_1 N_2 \tag{6.5}$$

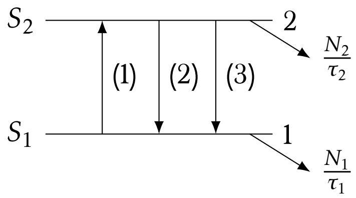
$$= S_1\tau_1 + A_{21}\tau_1 S_2\tau_2 \tag{6.6}$$

Population inversion density,

$$N_0^* = N_2 - \frac{g_2}{g_1}N_1 \tag{6.7}$$

$$= S_2\tau_2\left(1 - \frac{g_2}{g_1}A_{21}\tau_1\right) - \frac{g_2}{g_1}S_1\tau_1 \tag{6.8}$$

'0' denotes small signal coefficient (in the absence of field).



$$(1)\quad \frac{g_2}{g_1}N_1\sigma(\omega)\frac{I}{\hbar\omega} \tag{6.9}$$

$$(2)\quad N_2\sigma(\omega)\frac{I}{\hbar\omega} \tag{6.10}$$

$$(3)\quad N_2A_{21} \tag{6.11}$$

We have assumed $L(\omega) < \Gamma_{21}$.

$$\frac{dN_2}{dt} = S_2 - \frac{N_2}{\tau_2} - N^*\sigma(\omega)\frac{I}{\hbar\omega} \tag{6.12}$$

$$\frac{dN_1}{dt} = S_1 - \frac{N_1}{\tau_1} + N^*\sigma(\omega)\frac{I}{\hbar\omega} + A_{21}N_2 \tag{6.13}$$

$$N_2 = S_2\tau_2 - N^*\left(\frac{\sigma I}{\hbar\omega}\right)\tau_2 \tag{6.14}$$

$$N_1 = S_1\tau_1 + N^*\left(\frac{\sigma I}{\hbar\omega}\right)\tau_1 + A_{21}\tau_1 N_2 \tag{6.15}$$

$$N^* = \frac{S_2\tau_2\left(1 - \frac{g_2}{g_1}A_{21}\tau_1\right) - \frac{g_2}{g_1}S_1\tau_1}{1 + \left(\frac{\sigma I}{\hbar\omega}\right)\left[\tau_2 + \frac{g_2}{g_1}(1 - A_{21}\tau_2)\tau_1\right]} \tag{6.16}$$

Numerator is N_0^*:

$$N^* = \frac{N_0^*}{1 + \frac{I}{I_s}} \tag{6.17}$$

We can now define Saturation Intensity:

$$I_s(\omega) = \frac{\hbar\omega}{\sigma(\omega)} \frac{1}{\tau_2 + \frac{g_2}{g_1}\tau_1(1 - A_{21}\tau_2)} \tag{6.18}$$

Gain is saturated too, since

$$g(\omega) = \sigma(\omega)N^* = \frac{g_0(\omega)}{1 + \frac{I}{I_s(\omega)}} \tag{6.19}$$

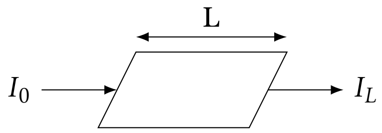
Comments:

- At $I = I_s(\omega) \to g(\omega) = \frac{g_0(\omega)}{2}$.
- $I_s(\omega)$ is frequency dependent through $\sigma(\omega)$, so will happen quicker at line centre.
- For an efficient laser, expect large population inversion $\implies \tau_2 \gg \tau_1$.
- All decays from state 2 into state 1 $\implies A_{21} \approx \frac{1}{\tau_2}$.

$$I_s(\omega) = \frac{\hbar\omega}{\sigma(\omega)\tau_2} \tag{6.20}$$

This is usually an excellent approximation.

Example: Linear Amplifier



$$\frac{dI}{dz} = g(\omega)I = \frac{g_0(\omega)I}{1 + \frac{I}{I_s(\omega)}} \tag{6.21}$$

We just integrate this, after length L:

$$\int_{I(0)}^{I(L)} \frac{1 + \frac{I}{I_s(\omega)}}{I}\, dI = \int_0^L g_0(\omega)\, dz \tag{6.22}$$

$$\ln\left(\frac{I_L}{I_0}\right) + \frac{I_L - I_0}{I_s(\omega)} = g_0(\omega)L \tag{6.23}$$

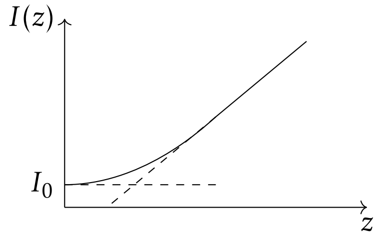
Two limiting cases:

1. $I/I_s(\omega) \ll 1 \implies$ exponential growth,

$$I_L = I_0 e^{g_0(\omega)L} \tag{6.24}$$

2. $I/I_s(\omega) \gg 1 \implies$ linear growth,

$$I_L = I_0 + g_0(\omega)I_s(\omega)L \tag{6.25}$$


I(z)
I0
z

Lecture 7 Multimode lasing and output power

Last time:

$$\frac{dN_2}{dt} = S_2 - \frac{N_2}{\tau_2} - \frac{N^*\sigma(\omega)I}{\hbar\omega} \tag{7.1}$$

$$\frac{dN_1}{dt} = S_1 - \frac{N_1}{\tau_1} + \frac{N^*\sigma(\omega)I}{\hbar\omega} + N_2A_{21} \tag{7.2}$$

We need a third equation to describe the system. Develop equation governing the the intensity build up. Consider photon density, n_ϕ.

$$\frac{dn_\phi}{dt} = \left(N^*\sigma(\omega)\frac{I}{\hbar\omega}\right)\left(\frac{L_m}{L_c}\right) - \frac{n_\phi}{\tau_c} \tag{7.3}$$

Intensity of beam,

$$I = \text{energy density} \times \text{velocity} \tag{7.4}$$

$$= n_\phi\hbar\omega \times c \tag{7.5}$$

$$n_\phi = \frac{I}{\hbar\omega c} \tag{7.6}$$

$$dn_\phi = \frac{I}{\hbar\omega c}dI \tag{7.7}$$

$$\frac{dI}{dt} = N^*\sigma(\omega)cI\left(\frac{L_m}{L_c}\right) - \frac{I}{\tau_c} \tag{7.8}$$

We know that

$$\tau_c = \frac{2L_c}{c\delta_c} \qquad\qquad g_{th} = \frac{\delta_c}{2L_m}. \tag{7.9}$$

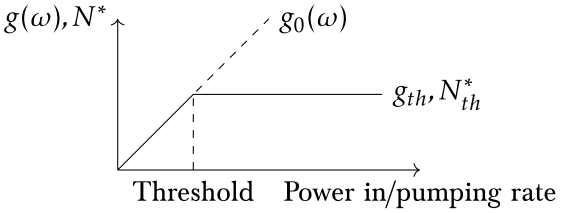
We then find the following useful for the final substitution:

$$\frac{1}{\tau_c} = \frac{c\delta_c}{2L_c} = c\left(\frac{L_m}{L_c}\right)g_{th} \tag{7.10}$$

$$g(\omega) = N^*\sigma(\omega) \tag{7.11}$$

$$\frac{dI}{dt} = c\left(\frac{L_m}{L_c}\right)[g(\omega) - g_{th}]I(t) \tag{7.12}$$

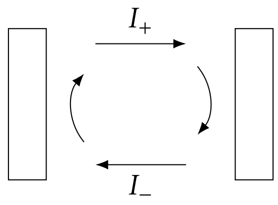
In the steady state, this should be zero which is found either if $I(t)$ is zero which is meaningless solution, or if $g_{ss}(\omega) = g_{th}$. This highlights the principle of gain saturation. We can take the expression found for gain last time and apply the above relation.



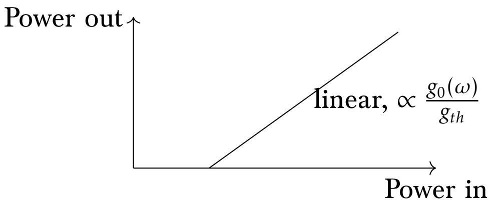
$$g_{ss}(\omega) = \sigma(\omega)N^* = \frac{g_0(\omega)}{1 + I_{ss}/I_s(\omega)} = g_{th} \tag{7.13}$$

$$I_{ss} = I_s(\omega)\left[\frac{g_0(\omega)}{g_{th}} - 1\right] \tag{7.14}$$

Consider simple cavity,



Mirror transmission, $T = (I_+ - R)$, $I_{out} = T \times I_+$. Assume low output coupling-uniform field approximation, i.e. $I_+ \approx I_-$.



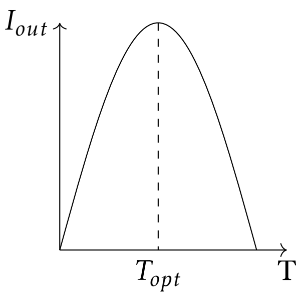
$$I_{ss} = I_+ + I_- \approx 2I_+ \tag{7.15}$$

$$I_{out} = T \times \frac{I_{ss}}{2} = T\frac{I_s(\omega)}{2}\left[\frac{g_0(\omega)}{g_{th}} - 1\right] \tag{7.16}$$

To get output power, we need to know mode area, A_{mode}.

$$P_{out} = A_{mode}I_{out} = \frac{1}{2}I_s(\omega)A_{mode}T\left[\frac{g_0(\omega)}{g_{th}} - 1\right] \tag{7.17}$$

7.1 Optimum Output Coupling



$$I_{out} = T\frac{I_s(\omega)}{2}\left[\frac{g_0(\omega)}{g_{th}} - 1\right] \tag{7.18}$$

$$g_{th} = \frac{\delta_c}{2L_m}, \; \delta_c = T + \delta_{loss} \tag{7.19}$$

$$I_{out} = T\frac{I_s(\omega)}{2}\left[\frac{2g_0(\omega)L}{T + \delta_{loss}} - 1\right] \tag{7.20}$$

Differentiate expression for I_{out} with respect to T and set to zero:

$$\frac{dI_{out}}{dT} = \frac{I_s(\omega)}{2}\left[\frac{2g_0(\omega)L}{T + \delta_{loss}} - 1\right] - \frac{TI_s(\omega)}{2}\frac{2g_0(\omega)L}{(T + \delta_{loss})^2} \tag{7.21}$$

$$2g_0(\omega)L(T + \delta_{loss}) - (T + \delta_{loss})^2 = 2g_0(\omega)LT \tag{7.22}$$

$$(T + \delta_{loss})^2 = 2g_0(\omega)L\delta_{loss} \tag{7.23}$$

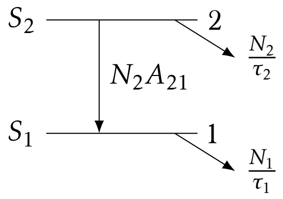
$$T_{opt} = \sqrt{2g_0(\omega)L\delta_{loss}} - \delta_{loss} \tag{7.24}$$

$$I_{opt} = \left[1 - \left(\frac{\delta_{loss}}{2g_0(\omega)L}\right)^{1/2}\right]^2 g_0(\omega)LI_s(\omega) \tag{7.25}$$

$$I_{max} = g_0(\omega)LI_s(\omega) \tag{7.26}$$

Lecture 8 Requirements for Population Inversion

8.1 Condition for Steady State Population Inversion



Develop rate equations for N_1 and N_2.

$$\frac{dN_2}{dt} = S_2 - \frac{N_2}{\tau_2} \tag{8.1}$$

$$\frac{dN_1}{dt} = S_1 + N_2 A_{21} - \frac{N_1}{\tau_1} \tag{8.2}$$

In steady state,

$$N_2 = \tau_2 S_2 \tag{8.3}$$

$$N_1 = \tau_1 S_1 + \tau_2 S_2 A_{21} \tau_1 \tag{8.4}$$

For population inversion, $N^* > 0$, or $\frac{N_2}{g_2} > \frac{N_1}{g_1}$.

$$\frac{\tau_2 S_2}{g_2} > \frac{\tau_1 S_1}{g_1} + \frac{\tau_2 S_2 A_{21} \tau_1}{g_1} \tag{8.5}$$

$$1 < \frac{g_1}{g_2} \frac{S_2 \tau_2}{S_1 \tau_1} \left[1 - \frac{g_2}{g_1} A_{21} \tau_1 \right] \tag{8.6}$$

We want:

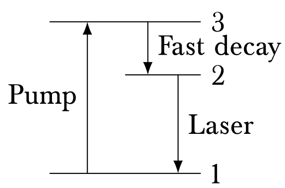
- $S_2 > S_1$ - selective pumping
- $\tau_2 > \tau_1$ - favourable lifetime ratio
- $g_1 > g_2$ - favourable degeneracy ratio
- The term in the square brackets only depends on the atomic properties, and we require

$$A_{21} < \frac{g_1}{g_2} \frac{1}{\tau_1} \tag{8.7}$$

to be able to get laser oscillation. This is the **minimum** requirement for steady state inversion - alone, it may not be a sufficient condition, but it is necessary. The lower laser level must decay faster than it is being filled by spontaneous emission.

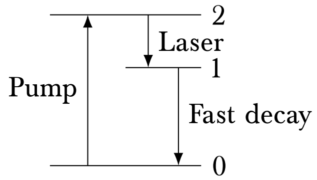
8.2 Three and Four Level Lasers

8.2.1 Traditional Solid State Three-level Laser



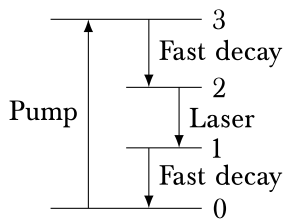
This is known as the traditional solid state three-level laser, e.g. Ruby.

8.2.2 Gas lasers



This is common for gas lasers, e.g. Ar+.

8.2.3 Solid state and dye lasers



This is most solid state lasers and dye lasers, e.g. Nd:YAG.

8.2.4 General comments

- We only show three/four levels here but each one of these levels may be a tight collection of thousands of levels which may be close enough to be treated as a single big level.
- For three-level schemes, one transition must be non-radiative to conserve parity.
  - For three-level solid state lasers, there is fast phonon decay from 3 to 2.
  - For gas lasers, both downward transitions are radiative but pumping is achieved by collisions.
- Traditional sold state laser will have a high pumping threshold because we need to transfer $\frac{N}{2}$ out of state 1 to get inversion, so these are not very efficient.
- Gas lasers have a low *quantum efficiency*.

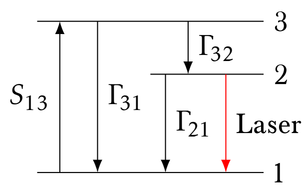
$$Q.E. = \frac{E_2 - E_1}{E_2 - E_0} = \frac{\text{laser photon energy}}{\text{pump 'photon' energy}} = \frac{\hbar\omega_{12}}{\hbar\omega_{20}} \tag{8.8}$$

  because spontaneous emission rate $A_{21} \propto \omega_{12}^3$, hence to achieve $\tau_2 > \tau_1$ requires $\omega_{10} > \omega_{12}$.
- Four-level lasers are best, *cause bigger is better, baybeeeee.*

$$\text{Gain} = e^{g(\omega)L} = e^{\sigma(\omega)N^* L_m} \tag{8.9}$$

$\sigma(\omega)$ and L_m are fixed (or practically so), so for us to maximise our gain, we have to maximise N^* through pumping.

8.3 Three-Level Traditional Solid State Lasers



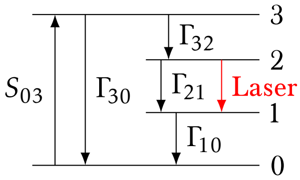
To study the steady state small signal population inversion, we ignore the effects of the laser field on the populations. The rate equations are:

$$\frac{dN_3}{dt} = S_{13}N_1 - (\Gamma_{31} + \Gamma_{32})N_3 \tag{8.10}$$

$$\frac{dN_2}{dt} = \Gamma_{32}N_3 - \Gamma_{21}N_2 \tag{8.11}$$

$$\frac{dN_1}{dt} = -S_{13}N_1 + \Gamma_{31}N_3 + \Gamma_{21}N_2 \tag{8.12}$$

Lecture 9 Population Inversion in Four-level lasers



Γ_{32} is a fast, non-radiative decay; Γ_{10} likewise.

$$\frac{dN_3}{dt} = N_0 S_{03} - (\Gamma_{30} + \Gamma_{32}) N_3 \tag{9.1}$$

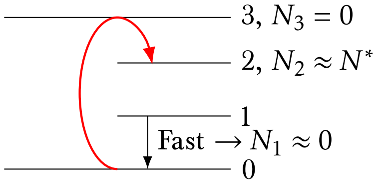
$$\frac{dN_2}{dt} = N_3 \Gamma_{32} - N_2 \Gamma_{21} \tag{9.2}$$

$$\frac{dN_1}{dt} = N_2 \Gamma_{21} - N_1 \Gamma_{10} \tag{9.3}$$

$$\frac{dN_0}{dt} = -N_0 S_{03} + N_3 \Gamma_{30} + N_1 \Gamma_{10} \tag{9.4}$$

In the ideal case, $\Gamma_{32} \gg \Gamma_{30}$, and $N_3 \approx 0$, i.e. decay is faster than pumping.

9.1 Idealised Four-level laser



The idealised four-level laser equations:

$$N_0 \approx N \qquad N_2 \approx \frac{S_{03}}{\Gamma_{21}} N_0 \qquad N_3 = 0 \tag{9.5}$$

9.2 Comparison of three- and four-level laser schemes

For a three-level scheme, we found that,

$$S_{13}^{th} = \frac{N + N_{th}^*}{N - N_{th}^*} \Gamma_{21} \approx \Gamma_{21}. \tag{9.6}$$

For four-level schemes, we found that

$$S_{03}^{th} = \frac{N_{th}^*}{N - N_{th}} \Gamma_{21} \approx \frac{N_{th}^*}{N} \Gamma_{21} \tag{9.7}$$

If we assume equal Γ_{21}s,

$$\frac{\text{four-level}}{\text{three-level}} = \frac{S_{03}^{th}}{S_{13}^{th}} = \frac{N_{th}^*}{N + N_{th}^*} \approx \frac{N_{th}^*}{N} \ll 1 \tag{9.8}$$

This reiterates that four-level lasers are much more efficient than three-level ones. Comparing power per unit volume absorbed at threshold:

$$\left(\frac{P}{V}\right)_{th,3} = \hbar\omega_{13}\Gamma_{21}\frac{N}{2} \qquad \left(\frac{P}{V}\right)_{th,4} = \hbar\omega_{30}N^*\Gamma_{21} \tag{9.9}$$

Example: Ruby vs Nd:YAG
Ruby doped at 1% by weight.

$$N_0 = 3.3 \times 10^{26}\, m^{-3} \qquad \Gamma_{21} = 3.3 \times 10^2\, s^{-1} \tag{9.10}$$

Assume pumping at $\lambda = 505\,nm$. Laser rod, diameter $6\,mm$, length $10\,cm$.

$$P_{th} = \left(\frac{hc}{\lambda}\right)\frac{N_0}{2}\Gamma_{21} \times V \approx 5.4\,kW \tag{9.11}$$

Now for the ND:YAG,

$$\Gamma_{21} = 1.8 \times 10^3\, s^{-1} \qquad \sigma = 9 \times 10^{-19}\, cm^2 \tag{9.12}$$

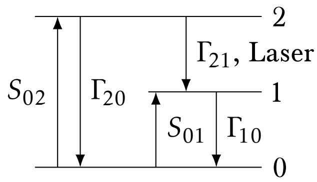
Assume cavity loss of $\delta_c = 0.05$, with diameter $4\,mm$, and length $5\,cm$. Pumped at $808\,nm$.

$$N^*_{th} = \frac{\delta_c}{2\sigma L_m} = 5.6 \times 10^{21}\, m^{-3} \tag{9.13}$$

$$P_{th} = \hbar\omega_{30} N^*_{th} \Gamma_{21} V \approx 1.66\, W \tag{9.14}$$

So Ruby has a massive energy cost compared to Nd:YAG, Nd:YAG much preferable.

Lecture 10 Population Inversion in Three-level Gas Lasers



Develop rate equations in absence of field:

$$\frac{dN_2}{dt} = S_{02}N_0 - (\Gamma_{20} + \Gamma_{21})N_2 \tag{10.1}$$

$$\frac{dN_1}{dt} = S_{01}N_0 + \Gamma_{21}N_2 - \Gamma_{10}N_1 \tag{10.2}$$

$$\frac{dN_0}{dt} = -(S_{02} + S_{01})N_0 + \Gamma_{20}N_2 + \Gamma_{10}N_1 \tag{10.3}$$

Using the steady state definition, $\frac{dN_i}{dt} = 0$:

$$N_2 = \frac{S_{02}}{\Gamma_{20} + \Gamma_{21}}N_0 \tag{10.4}$$

$$S_{01}N_0 + \Gamma_{21}\left[\frac{S_{02}}{\Gamma_{20} + \Gamma_{21}}N_0\right] - \Gamma_{10}N_1 = 0 \tag{10.5}$$

$$N_1 = \frac{S_{01}(\Gamma_{20} + \Gamma_{21}) + \Gamma_{21}S_{02}}{\Gamma_{10}(\Gamma_{20} + \Gamma_{21})}N_0 \tag{10.6}$$

For population inversion, we need

$$\frac{N_2}{N_1} > \frac{g_2}{g_1} \tag{10.7}$$

$$N^* = N_2 - \frac{g_2}{g_1}N_1 > 0 \tag{10.8}$$

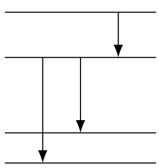
$$\frac{N_2}{N_1} = \frac{S_{02}\Gamma_{10}}{S_{01}(\Gamma_{20} + \Gamma_{21}) + \Gamma_{21}S_{02}} > \frac{g_2}{g_1} \tag{10.9}$$

We have to find another pumping technique as we cannot use light. Both down transitions are optical (radiative), so S_{02} cannot be optical due to parity conservation, and Γ_{20} is negligible.

$$\frac{N_2}{N_1} = \frac{S_{02}\Gamma_{10}}{(S_{01} + S_{02})\Gamma_{21}} > \frac{g_2}{g_1} \tag{10.10}$$

$$\frac{S_{02}}{S_{01}} > \left(\frac{g_1\Gamma_{10}}{g_2\Gamma_{21}} - 1\right)^{-1} \tag{10.11}$$

Example: Argon ion laser



$\lambda = 488.0nm$ with $A = 7.8 \times 10^7 s^{-1}$, decays): $73.1nm$, $A = 4.5 \times 10^8 s^{-1}$; $72.3nm$, $A = 23{\times}10^8 s^{-1}$. We have a degeneracy of $g = 2J+1$, so $g_1 = 4, g_2 = 6$. $\Gamma_{21} = 7.8 \times 10^7\, s^{-1}$, $\Gamma_{10} = 27.5 \times 10^8\, s^{-1}$.
From this, we get a ratio of 0.044. For this particular system, we can pump into the lower level 22.5 times as quickly as into the upper one and still maintain population inversion due to fast natural decay rates. **Striking result!**

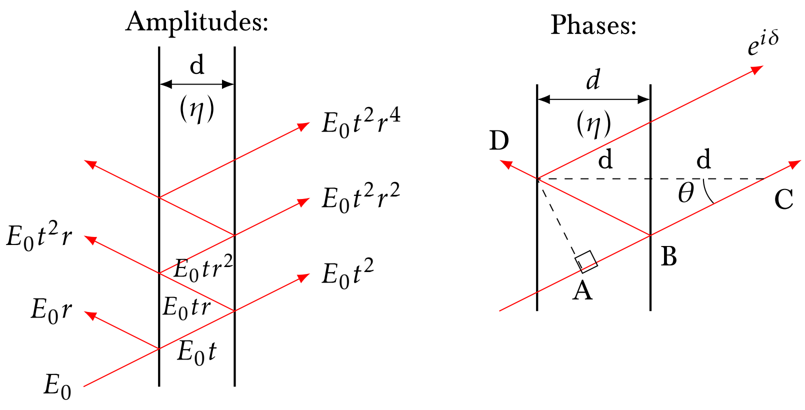
> This can happen because $A \propto \omega^3$ so massively different for the higher frequencies of the lower level decay. The downside is low quantum efficiency.

Gas lasers cannot be optically pumped, so they use collisions, or **particle pumping**.

Lecture 11 The Fabry-Perot Etalon and Laser Cavity Modes

11.1 The Fabry-Perot Elaton

Consider two identical semi-reflecting parallel mirrors:



t and r are the amplitude transmission and reflection coefficients respectively.

11.1.1 Path different between succesive rays

$$P.D. = AB + BD = AB + BC = AC \tag{11.1}$$
$$= 2d\cos\theta \tag{11.2}$$
$$\delta = 2kd\cos\theta = \frac{4\pi d}{\lambda}\cos\theta \tag{11.3}$$

11.1.2 Transmitted amplitude

$$E_T = E_0t^2\left(1 + r^2e^{i\delta} + r^4e^{2i\delta} + r^6e^{3i\delta} + \dots\right) \text{ - Geometric Progression} \tag{11.4}$$
$$= \sum_0^\infty r^{2n}e^{in\delta} = \frac{1}{1 - r^2e^{i\delta}} \tag{11.5}$$
$$= \frac{E_0T}{1 - Re^{i\delta}},\ T = |t|^2,\ R = |r|^2 \tag{11.6}$$

11.1.3 Transmitted Intensity

$$I_T = |E_T|^2 = \frac{I_0T^2}{|1 - Re^{i\delta}|^2} \tag{11.7}$$

One can easily show that

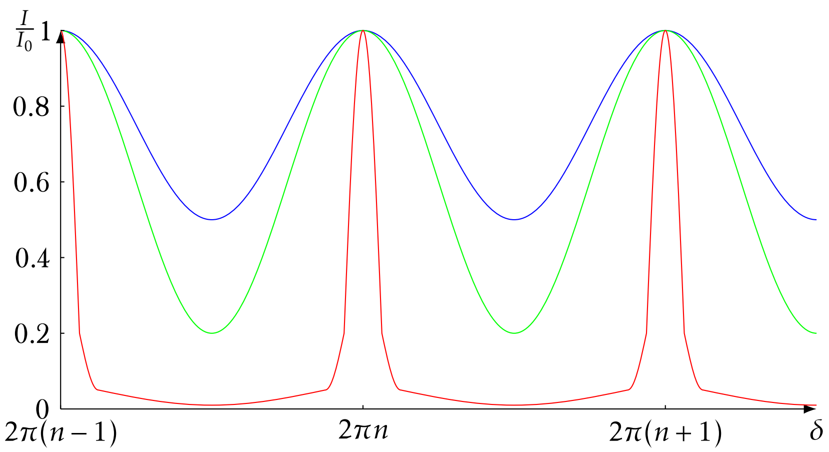
$$|1 - Re^{i\delta}|^2 = (1-R)^2\left[1 + \frac{4R}{(1-R)^2}\sin^2\left(\frac{\delta}{2}\right)\right] \tag{11.8}$$

Hence, we have

$$\frac{I_T}{I_0} = \frac{T^2}{(1-R)^2}\frac{1}{1+\frac{4R}{(1-R)^2}\sin^2\frac{\delta}{2}} = \frac{T^2}{(1-R)^2}\frac{1}{1+\frac{4F^2}{\pi^2}\sin^2\frac{\delta}{2}} \tag{11.9}$$

Where we can define **Finesse:**

$$F = \frac{\pi\sqrt{R}}{1-R} \tag{11.10}$$



11.1.4 Points to note

- For mirrors, $T+R+A=1$, absorption $A \approx 0$, so $T = 1-R$.

$$\left.\frac{I_T}{I_0}\right|_{max} = 1 \tag{11.11}$$

  So all light trnasmitted at cavity resonance.
- Phase difference:

$$\delta = \frac{4\pi d}{\lambda}\cos\theta \tag{11.12}$$

  Dependent on angle, separation, and wavelength.
- For bright fringes, $\sin\frac{\delta}{2} = 0$ so $\delta = 2n\pi$, or $2d\cos\theta = n\lambda$.
- Free spectral range, $\Delta\nu_{fsr}$. At maxima,

$$\lambda = \frac{2d\cos\theta}{n} \implies \nu = \frac{c}{\lambda} = \frac{nc}{\cos\theta} \tag{11.13}$$

$$\Delta\nu_{fsr} = \nu_{n+1} - \nu_n = \frac{c}{2d\cos\theta} \tag{11.14}$$

  Then, at normal incidence,

$$\Delta\nu_{fsr} = \frac{c}{2d} \tag{11.15}$$

- Fringe width. What is the full width at half maximum (FWHM) of fringes? Assume $\frac{I_T}{I_0} = \frac{1}{2}$ when $\delta = 2\pi n \pm \delta_{1/2}$.

$$\frac{1}{1+\frac{4F^2}{\pi^2}\sin^2(\frac{\delta_{1/2}}{2})} = \frac{1}{2} \tag{11.16}$$

$$\sin^2\left(\frac{\delta_{1/2}}{2}\right) = \frac{\pi^2}{4F^2} \tag{11.17}$$

$$\delta_{1/2} = 2\sin^{-1}\left(\frac{\pi}{2F}\right) \tag{11.18}$$

$$\delta_{1/2} \approx \frac{\pi}{F},\ F >> 1 \tag{11.19}$$

$$\text{FWHM} = 2\delta_{1/2} = \frac{2\pi}{F} \tag{11.20}$$

Separation of adjacent peaks (in radians) = 2π. Hence, finesse,

$$F = \frac{\Delta\nu_{fsr}}{\Delta\nu_{FWHM}}. \tag{11.21}$$

So high finesse → sharp fringes, and vice versa. Before,

$$F = \frac{\pi\sqrt{R}}{1-R} \implies \Delta\nu_{FWHM} = \frac{c}{2d}\left(\frac{1-R}{\pi\sqrt{R}}\right). \tag{11.22}$$

Example:
30cm long etalon with mirrors, $R = 0.99$.

$$\Delta\nu_{fsr} = \frac{c}{2d} = 500\,MHz \tag{11.23}$$

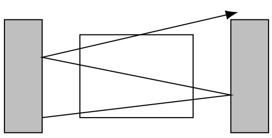
$$F = \frac{\pi\sqrt{R}}{1-R} \approx 313 \tag{11.24}$$

$$\Delta\nu_{FWHM} = \frac{\Delta\nu_{fsr}}{F} = 1.6\,MHz \tag{11.25}$$

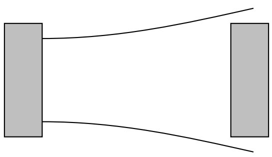
In visible range, say $\lambda = 550\,nm,\ \nu = 545\,THz$. We can use this etalon to reolve to 1 part in 10^8.

11.2 Laser Cavities

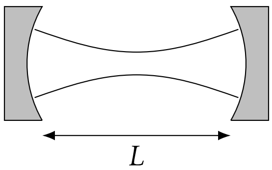
The Fabry-Perot etalon is not a good laser cavity. We need to place a gain medium between mirrors.



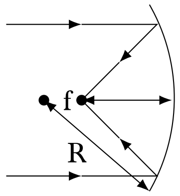
This cavity however is difficult to align. More fundamentally, diffraction will cause losses.



Diffraction leads to large losses. The solution is to use curved mirrors:

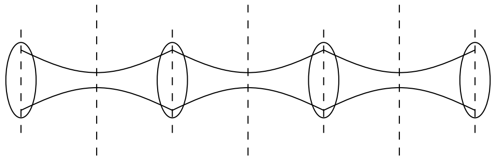


Each curved mirror acts as a lens,



$$f = \frac{R}{2}. \tag{11.26}$$

where here R is the radius of curvature. Cavity acts as a sequence of lenses.



Related by Fourier transform, so essentially we want a function that is its own Fourier transform, we will see Gauss-Hermite.

Lecture 12 Gaussian Beams and Cavity Stability

The electric field distribution needs to reproduce after a round trip, so it needs to be its own Fourier transform, i.e. Gause-Hermite modes.

$$U = \frac{U_0}{q} e^{i(kz-\omega t)} e^{ikr^2/2q} H_l\left(\frac{\sqrt{2}x}{w}\right) H_m\left(\frac{\sqrt{2}y}{w}\right) e^{-i(l+m)\alpha} \tag{12.1}$$

$$q = z - z_0 - iZ_R,\ r^2 = x^2 + y^2,\ \tan\alpha = \frac{z}{Z_R} \tag{12.2}$$

Where Z_R is the Rayleigh range, and z_0 is the location of beam waist. Gaussian beam (TEM_{00}):

$$U = U_0 e^{i(kz-\omega t)} \frac{e^{ikr^2/2q}}{q} \tag{12.3}$$

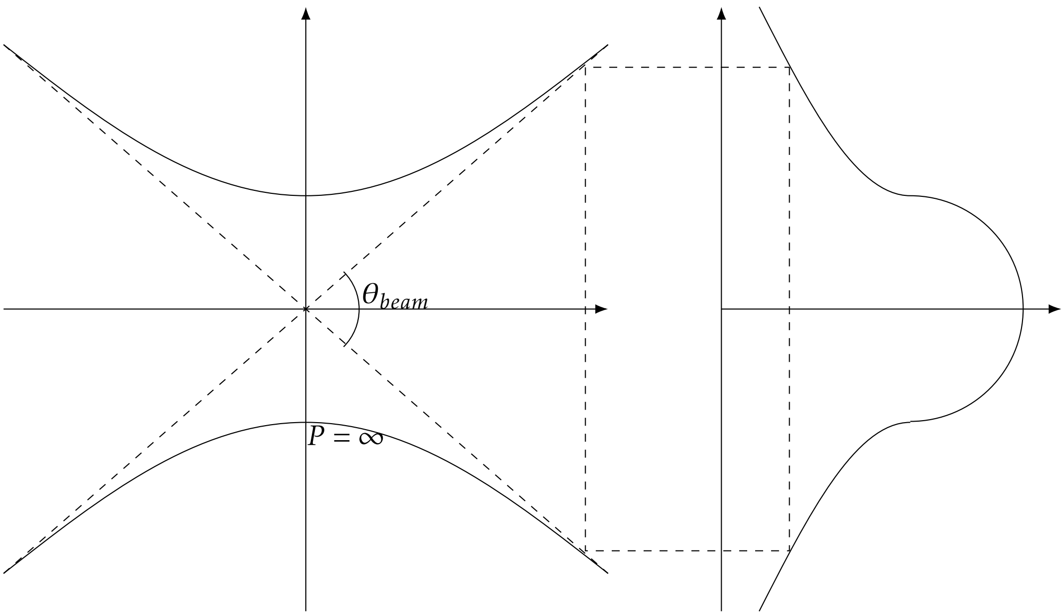
take some notes from the powerpoint slides

12.1 Propagation of Gaussian beams

$$w(z) = w_0\left(1 + \left(\frac{z}{z_R}\right)^2\right)^{1/2} \implies \frac{w(z)^2}{w_0^2} - \frac{z^2}{z_R^2} = 1 \tag{12.4}$$

This is the equation of hyperbola with asymptotes at

$$w(z) = \pm\frac{w_0 z}{z_R} \tag{12.5}$$



$$\theta_{beam} = \frac{2w_0}{z_R} = \frac{2\lambda}{\pi w_0} = 1.27\frac{\lambda}{w_0} \tag{12.6}$$

So a Gaussian beam is diffraction limited, as we get a very similar result here as you would for a circular aperture.

$$I = \frac{2P}{\pi w(z)^2} e^{-2r^2/w(z)^2} \tag{12.7}$$

doodle cavity again What values of L_c, R_1, and R_2 lead to a stable solution? Some things won't work. *more doodles* The stability condition is

$$0 < \left(1 - \frac{L_c}{R_1}\right)\left(1 - \frac{L_c}{R_2}\right) < 1 \tag{12.8}$$

This can often be represented graphically: *doodle from notes*

1. Symmetric concentric: *doodle*
2. Symmetric confocal: *doodle*
3. Plane parallel: *doodle*

We want to design cavity to be in the shaded regions of the graph, and this requires satisfying two conditions in the inequality:

- $L_c < R_1 + R_2$
- The centres of curvature must **both** be inside or outside the cavity. *doodles*

Example:
A laser has mirrors with $R_1 = 1.0m$, and $R_2 = 0.35m$. What ranges of L_c are stable?

From the second requirement, we know there are two ranges.

- Both inside $\rightarrow 1.0 < L_c < 1.35$
- Both outside $\rightarrow 0 < L_c < 0.35$

12.2 Finding the mode shape for a cavity

Simple case, **Symmetric**:
The waist must be at the centre, so $R_1 = R_2 = R$. *doodle* At each mirror, the radius of curvature of the beam is the same as the mirror. Recall

$$R(z) = \frac{z^2 + z_R^2}{z} \tag{12.9}$$

$$R\left(\frac{L_c}{2}\right) = R = \frac{\frac{L_c^2}{4} + z_R^2}{\frac{L_c}{2}} \tag{12.10}$$

$$z_R^2 = \frac{L_c}{4}(2R - L_c) \equiv \left(\frac{\pi w_0^2}{\lambda}\right)^2 \tag{12.11}$$

$$w_0^2 = \frac{\lambda}{2\pi}\left[L_c(2R - L_c)\right]^{1/2} \tag{12.12}$$

We know the beam wasist inside a symmetric cavity from R and L_c.

12.3 Mode Volume

How do we choose what cavity parameters to use? Try to match mode volume to Gain medium. *doodle*

$$V \leq T w_0^2 L_c \tag{12.13}$$

This assume the beam is not spreading, i.e. $Z_R \gg L_c$.

Lecture 13 Cavity Effects: Single Frequency Operation

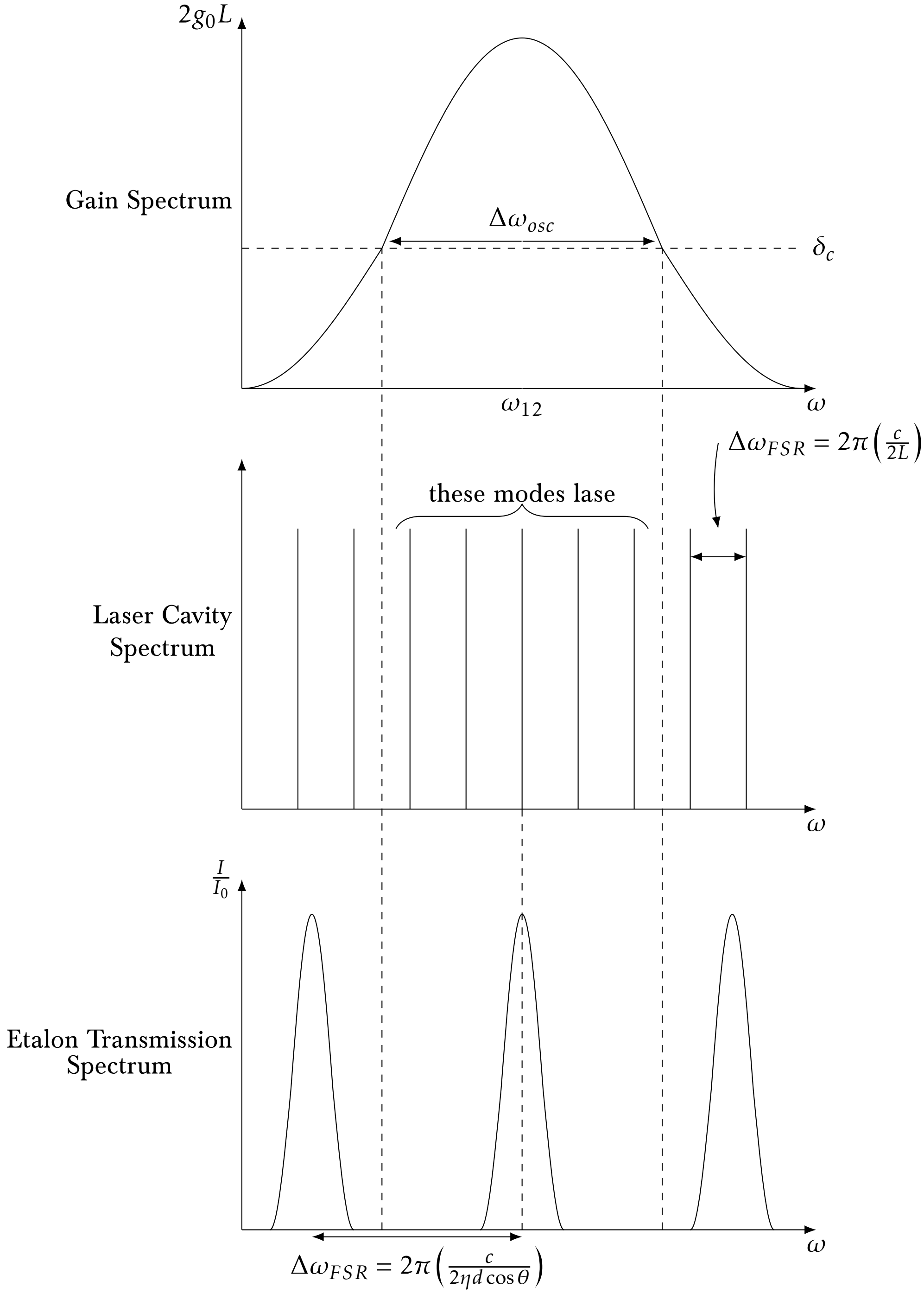
Spectral and spatial hole burning occur for homegeneous and inhomogeneous systems → multimode lasing:

- Homogeneous - **spatial** hole burning
- Inhomogeneous - **spectral** hole burning

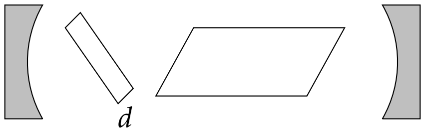
This multimode (longitudinal) behaviour can be eliminated but to do so in each case is different.

13.1 Inhomogeneous - spectral hole burning

In a Doppler broadened gain medium, it is likely the laser will oscillate simultaneously on several longitudinal modes if the cavity mode spacing is less than the range of frequencies over which there is net gain. $\Delta\omega_{osc}$ = range over which the laser can lase on; $\Delta\omega_{FSR}$ = the mode separation.



A common solution to multimode lasing is to introduce an **intracavity etalon** (Fabry-Perot Etalon), whose thickness is such that only one longitudinal cavity mode lases.

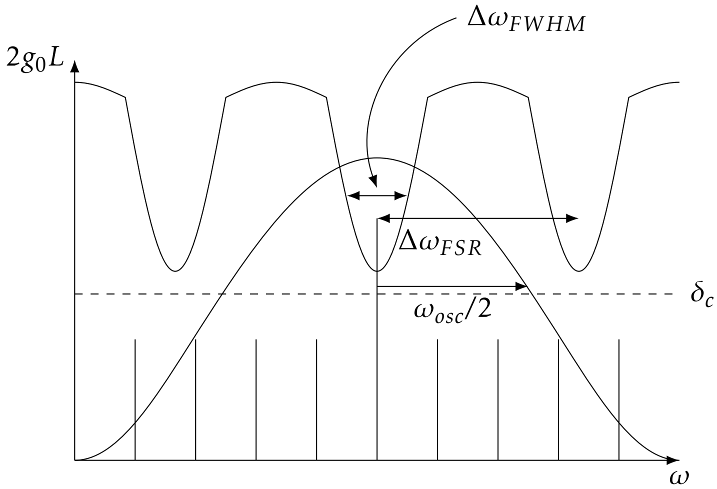


As you decrease d, the free spectral range decreases.
Etalon is usually solid and mounted at a small angle. This may allow for turning of the etalon transmission, since

$$\Delta\omega_{FSR} = 2\pi\left(\frac{c}{2\eta d\cos\theta}\right) \tag{13.1}$$

This can be a frequency control (moves transition peak around).

There are now two cavities to consider: cavity length, L; and etalon length, d. It is instructive to think of the etalon as altering the round trip lenses; the etalon loss is high when light is reflected from the etalon, but then drops on its resonance. So we end up with a single mode to lase on.



The etalon must satisfy two conditions:

1. Etalon free spectral range must be greater than half the cavity spectral range → to prevent modes in wings from lasing.

$$\Delta\omega_{FSR} = 2\pi\left(\frac{c}{2\eta d\cos\theta}\right) > \frac{\Delta\omega_{osc}}{2} \tag{13.2}$$

2. To select only one mode, less than the mode FSR:

$$\Delta\omega_{FWHM} < 2\pi\left(\frac{c}{2L_c}\right) \tag{13.3}$$

Since

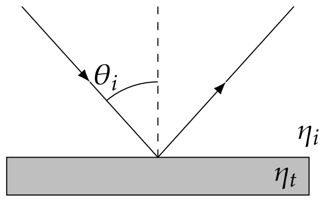
$$\Delta\omega_{FSR} = \frac{2\pi c}{2\eta d\cos\theta}, \qquad F = \frac{\Delta\omega_{FSR}}{\Delta_{FWHM}}, \tag{13.4}$$

the above conditions imply conditions on d and R → etalon thickness and reflectivity.

13.2 Homogeneous - Spatial Hole Burning

For a homogeneously broadened gain medium, we may get **multimode operation** due to spatial hole burning. We have a standing wave in the cavity of $\cos^2$ nature which depletes gain in some regions, but leaves unused gain in other regions. This is bad because the laser generally finds a way to use this gain by running on additional modes. Solution: eliminate the standing wave to remove spatial hole burning. We can do this by using a **ring cavity** (a square or ring), in which the laser mode only propagates in one direction using an optical diode.

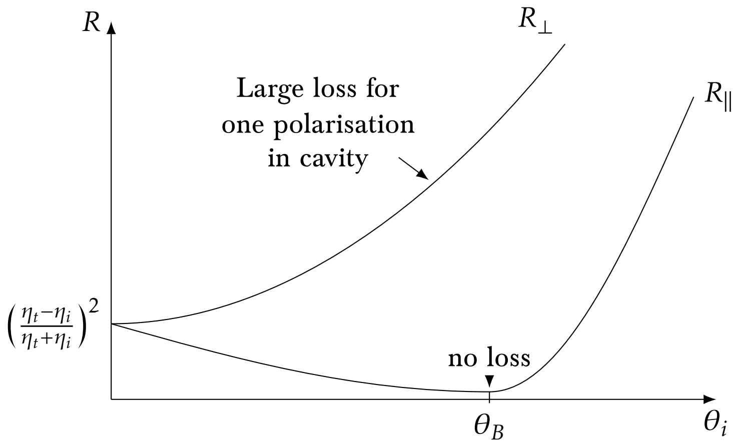
1. Mirror (M1) transmits the pump light (but reflects laser light) and allows pump beam into cavity. The beam is focused to a similar waist to the cavity mode within the crystal. Pump is Ar^+ or doubled Nd:YAG.
2. Crystal



Cut such that it is inserted into the cavity as Brewster's angle. Typically $\approx 2cm$ long and $5mm$ diameter. Brewster's angle means we have zero reflection at the air-crystal boundary for one polarisation. Boundary conditions $\implies$ Fresnal coefficients.

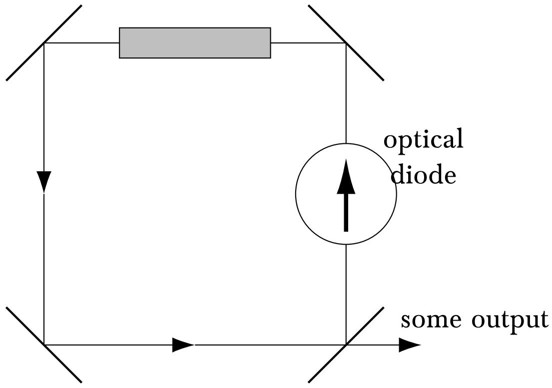
$$\tan\theta_B = \frac{\eta_t}{\eta_i} \tag{13.5}$$

For Ti:Sapphire: $\eta = 1.76 \implies \theta_B \approx 60°$. Note: for one population in the cavity, there is no loss at the crystal; for the other, there is a very large reflection loss.



3. Optical Diode: Consists of 2 elements that rotate the polarisation of the light. One is the standard waveplate, $\frac{\lambda}{2}$. Other is a Faraday Rotator; this is a device based on the Faraday effect - for some materials placed in a magnetic field, the light polarisation is rotated. The direction of rotation depends on the direction of light propagation with respect to the direction of the $\underline{\mathbf{B}}$-field.
   There is no time reversal symmetry $\rightarrow$ it imposes directionality on the cavity such that if it is the wrong way, then there is complete loss. So the laser is forced to lase in a single direction; elimates spatial hole burning and makes it easier to achieve single frequency oscillation.

A simple ring cavity might look something like:

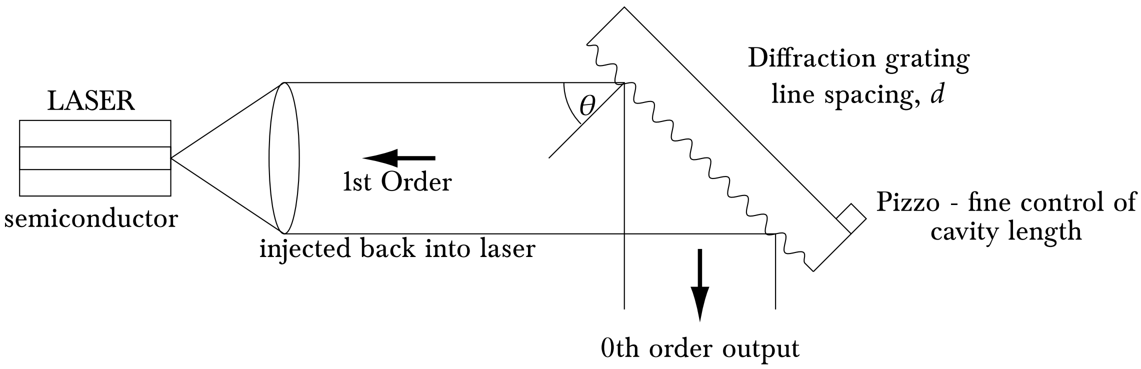


Instead of two mirrors, we now have multiple.

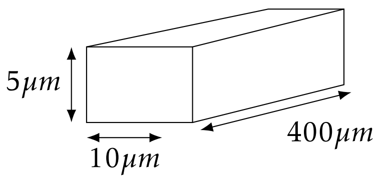
How much is single frequency output, single frequency?

13.3 External Cavity Diode Laser

Another example of using the elements of a cavity to achieve tunable single frequency.



Semiconductor emits light at recombination across pn junction; lens to focus diffracted light.
A typical semiconductor looks like:



Usually high refractive index, $\eta = 3.6$.

$$\Delta\nu_{FSR} = \frac{c}{2\eta d} \approx 100\,GHz \tag{13.6}$$

Relectivity of uncoated ends:

$$R = \left(\frac{\eta - 1}{\eta + 1}\right)^2 \approx 0.32 \tag{13.7}$$

I.e. expect 32% reflection $\rightarrow$ just from interface of semiconductor and air.

We define Finesse:

$$F = \frac{\pi\sqrt{R}}{1 - R} = 2.6 \text{ (low)} \tag{13.8}$$

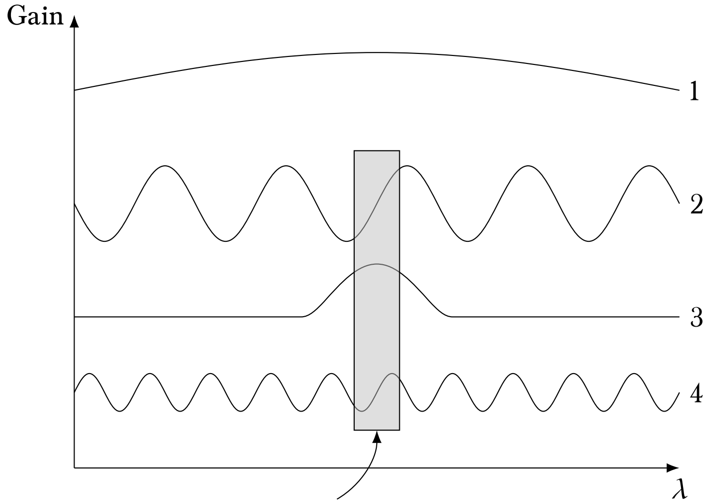
$$\Delta\nu_{FWHM} = \frac{\Delta\nu_{FSR}}{F} \approx 40\,GHz \tag{13.9}$$

Diffraction grating does two things:

- Selects λ via angle: $2d\sin\theta = n\lambda$. For 1800 lines mm^{-1}, $\lambda = 780nm \implies \theta = 45°$. Selects a range of frequencies by resolving power:

$$\frac{\nu}{\Delta\nu} = N = \text{total no. of lines illuminated} \tag{13.10}$$

- Forms an extended cavity
  1. Broad gain of laser from semiconductor itself.
  2. Internal cavity modes from semiconductor $\implies$ expect low finesse oscillation from low F cavity.
  3. Diffraction grating feedback - from a secondary cavity as some light is reflected straight back to initial cavity, which results in a longer cavity and extra spectra structure.
  4. External Cavity modes.

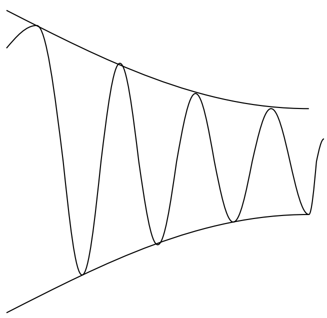


We want to tune all these such that lasing occurs where gain is large enough relative to loss. We can have a very narrow frequency output from these systems but it cannot be infinitely small $\implies$ there is a limit on size range.

From Lecture 5, we saw **cavity linewidth**,

$$\Delta\omega_c = \frac{1}{\tau_c} \tag{13.11}$$

It depends on the cavity lifetime - if we had a cavity with some light in it, the light would escape and the field in the cavity would decay away exponentially with some time constant, τ_c.



A Fourier transform gives $\Delta\omega$ spread in frequency. Here we now have an oscillating field that doesn't die away. The τ_c is infinitely long lifetime and the gain exactly balances the loss and there is no decay of the field in the cavity. We get a perfect single frequency out of the laser! So what have we missed..? $\Longrightarrow$ **Spontaneous Emission**.

Spontaneous Emission can occur instead of an atom being stimulated by the laser field; stimulated emission adds coherently to the cavity field whereas spontaneous emission does not. It has no phase relationship to the cavity field and adds incoherently. It also has an inherently Lorentzian distribution of frequencies. Note: spontaneous emission has little effect on amplitude/intensity of laser output - on a plot of Power vs time, the power would appear very stable and constant, with only small oscillations about the central value.

$$P_{em} = \frac{\hbar\omega_{12}^3}{\pi^2 c^3}(1+\bar{n})B_{21},\ \bar{n} \geq 10^6 \tag{13.12}$$

Where $(1+\bar{n})$ represents the spontaneous and stimulated effects respectively, combined; and $\bar{n}$ is the number of photons per mode. Remember:

$$A_{21} = \frac{\hbar\omega_{12}^3}{\pi^2 c^3}B_{21} \tag{13.13}$$

However, spontaneous emission causes phase of the electric field to wander randomly - noticeable as the variation in phase leads to some linewidth.

$$\omega \propto \frac{d\phi}{dt} \tag{13.14}$$

A full quantum optics treatment leads to the **Schawlow-Townes formula**:

$$\Delta\omega_{laser} = \frac{\hbar\omega}{2P_{out}}\Delta\omega_c^2 \tag{13.15}$$

This is the 'Quantum limit to laser linewidth' - can't get any better than this.

Example: HeNe Laser
$1mW$ - low output laser, $\lambda = 632.8nm$. Cavity: $L_c = 1m,\ \delta_c = 1\%$.

$$\Delta\omega_c = \frac{c\delta_c}{2L_c} = 2\pi \times 0.24MHz \text{ - Cavity Linewidth} \tag{13.16}$$

$$\Delta\omega_{laser} = \frac{\hbar\omega}{2P_{out}}\Delta\omega_c^2 = 2\pi \times 0.006MHz \text{ - Laser linewidth} \tag{13.17}$$

The laser linewidth is the frequency spread of output.

Such low linewidths are very difficult to achieve as they require phenomenably stable cavities.

13.4 Practical Limit

Recall modes:

$$\omega_n = 2\pi\left(\frac{c}{2L_c}\right)n \tag{13.18}$$

$$\frac{\Delta\omega}{\omega} = -\frac{\Delta L_c}{L_c} \tag{13.19}$$

So what limits the laser linewidth? Consider the HeNe example - to reach the fundamental limit:

$$\Delta\omega_{laser} = 2\pi \times 0.006MHz \Longrightarrow \frac{\Delta\omega_{laser}}{\omega} \approx 10^{-19} \Longrightarrow \Delta L \approx 10^{-19}M \tag{13.20}$$

This is a tiny fraction of the size of the atom (better than LIGO!). If we stabilise the cavity to 0.1% of the optical wavelength:

$$\frac{\Delta L}{L} \approx 6 \times 10^{-10} \Longrightarrow \Delta\omega_{laser} \approx 2\pi \times 0.3MHz \tag{13.21}$$

Lecture 14 Pulsed Lasers

We will consider two types of pulsed operation:

1. Mode Locking
2. Q-Switching

14.1 Mode Locking

get some notes from the slides doodle Modes - each with its own frequency and phase. If the phsaes are 'locked', the intensity output of the laser is

$$I(t) = \frac{I_0 \sin^2\left(\frac{1}{2}N\Delta\omega t\right)}{\sin^2\left(\frac{1}{2}\Delta\omega t\right)} \tag{14.1}$$

doodle There are $N-2$ intermediate peaks, where N is the number of modes lasing. What is the time separation between these peaks, τ_{rep}?

14.1.1 Repetition Rate

When numerator and denominator are simultaneously zero, i.e. when $\frac{1}{2}\Delta\omega t = m\pi$. Hence,

$$\tau_{rep} = \frac{2\pi}{\Delta\omega} = \frac{2L}{c} \tag{14.2}$$

This is just the cavity round trip time - every time the beam in the cavity hits the output coupler, some beam gets emitted again. *doodle*

14.1.2 Pulse Duration

Time separation between peak pulse and first minima, $\Delta\tau$. This will occur when numerator is zero, i.e.

$$\sin\frac{1}{2}N\Delta\omega t = 0 \tag{14.3}$$

$$\frac{1}{2}N\Delta\omega\Delta\tau = \pi \tag{14.4}$$

$$\Delta\tau = \frac{2\pi}{N\Delta\omega} = \frac{\tau_{rep}}{N} \tag{14.5}$$

$$\Delta\tau = \frac{2\pi}{\Delta\omega_{osc}} \approx \frac{1}{\text{Gain Bandwidth}} \tag{14.6}$$

The final approximation is accurate when assuming very large N.

14.1.3 Maximum Intensity

$$I(t) = N^2 I_0 \tag{14.7}$$

This can be huge. N can be > 1000. You can deduce $\Delta\omega_{osc}$ from gain and loss: *doodle*

14.1.4 Examples

- HeNe laser:
  $L = 0.5m$, $\lambda = 632.8nm$, Doppler width - $\Delta\nu_D = 1.5GHz$.

$$\tau_{rep} = \frac{2L}{c} \approx 3ns \qquad \Delta\tau = \frac{1}{\Delta\nu_D} \approx 0.7ns \tag{14.8}$$

There are four modes a lasing, 3 nanoseconds, 2 lectures left, and a partidge in a pear treeeeeee.

- Ti:Sapphire:
  $L = 2m$, Gain from $700nm - 1000nm \rightarrow 120THz$.

$$\tau_{rep} = \frac{2L}{c} \approx 12ns \qquad\qquad \Delta\tau = \frac{1}{\Delta\nu} \approx 10fs \tag{14.9}$$

  This is roughly equivalent to 1.6 million modes lasing.

lots more from powerpoints

Mode locking can be achieved using active or passive methods. *doodle - actively switch the cavity loss using AOM or EOM doodle - Passively switch the loss using the optical Kerr effect*

14.2 Q Switching

Reminder - cavity linewidth $\Delta\omega_c = \frac{1}{\tau_c}$, and a Quality factor, $Q = \frac{\omega}{\Delta\omega_c} = \omega\tau_c$. The cavity Q factor is switched dynamically to control the build up of gain. *doodle - giant pulse operation* Historically achieved using rotating mirrors, while modern lasers use the AOM or EOM methods - pulses limited to $\approx 100ps$. *doodle*

Lecture 15 Nonlinear Optics

We have net non-linear optical effects as **saturation**, **frequency doubling**, and **Keir lensing**; here we consider the physics of optical non-linearities, and how non-linearities are used in laser physics and fundamental research.
Nonlinear optics - *"the modification of the optical properties of a medium by the presence of light."* Typically, only laser light is intense enough to do this.

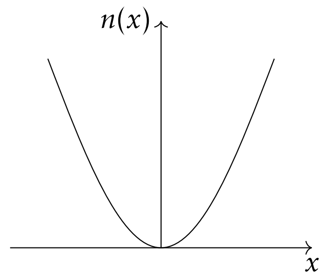
Consider the **dipole moment per unit volumen**, or **polarisation**:

$$P(t) = \epsilon_0 \chi E(t) \tag{15.1}$$

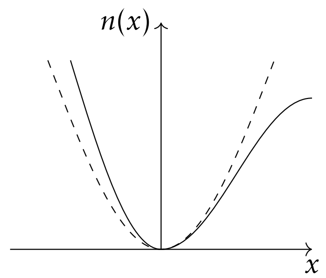
Where our variables are: ϵ_0 - permittivity of free space; $E(t)$ - electric field; χ - electric susceptibility, which is related to refractive index η, by

$$\eta = \sqrt{1+\chi}. \tag{15.2}$$

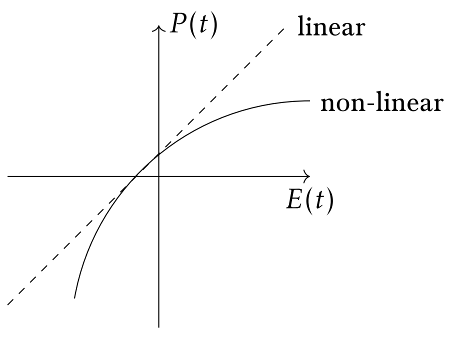
- Electron moving in a parabolic (harmonic) potential:
  Charge radiated with the same frequency as the driving field.

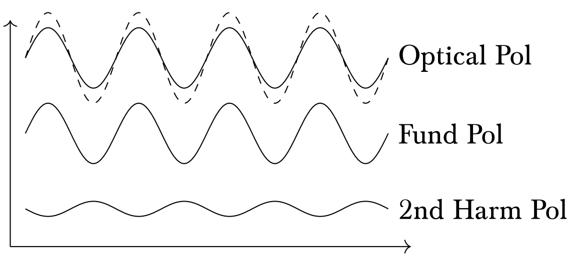


- If the potential is **anharmonic**, the picture changes:
  This is what the potential might look like for a non centro-symmetric material.



For low amplitude driving, the potential looks harmonic - high intensity lasers are needed to observe non-linear effects.





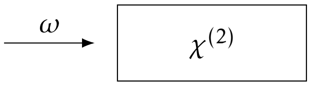
Nonlinear optics result in frequencies which pick up other harmonics, not just the fundamental frequency. For an anharmonic potential, express the polarisation as a power series in field strength:

$$P(t) = \epsilon_0 \left[\chi^{(1)} E(t) + \chi^{(2)} E^2(t) + \chi^{(3)} E^3(t) + \dots \right] \tag{15.3}$$

$$P(t) = P^{(1)}(t) + P^{(2)}(t) + P^{(3)}(t) + \dots \tag{15.4}$$

The (i) numbers are just notation differentiating $\chi^{(i)}$ as the different order susceptibilities. We take these as scalar quantities for simplicity ($P^{(1)}$ etc). In general, χ are frequency dependent but we neglect that here. Also assume the response to driving is instantaneous.

15.1 Second Order Effects



Consider the time dependence of the electric field.

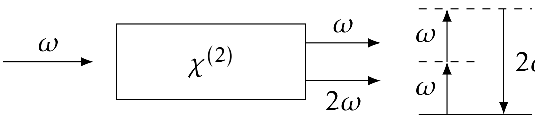
$$E(t) = E_0 e^{-i\omega t} + E_0^* e^{i\omega t} \tag{15.5}$$

Second Order polarisation:

$$\begin{aligned} P^{(2)}(t) &= \epsilon_0 \chi^{(2)} E(t)^2 && (15.6)\\ &= \epsilon_0 \chi^{(2)} \left[E_0^* E_0 + E_0 E_0^* + E_0^2 e^{-i2\omega t} + E_0^{*2} e^{-2\omega t} \right] && (15.7)\\ &= 2\epsilon_0 \chi^{(2)} E_0 E_0^* + \epsilon_0 \chi^{(2)} E_0^2 e^{-i2\omega t} + \epsilon_0 \chi^{(2)} E_0^{*2} e^{i2\omega t} && (15.8) \end{aligned}$$

The first term describes the time-dependent DC field referred to as **optical rectification**; the latter terms which contain frequencies at twice the original frequency.

15.1.1 Second Harmonic Generation



- ➤ Not all light converted but can be highly efficient.
- ➤ For every two photons with angular momentum ω, we get one out with 2ω.
- ➤ Note: spin is not conserved here (but not of concern).
- ➤ Initial path is shining a laser onto a crystal.
- ➤ This is used for **frequency doubling**, e.g. Nd:YAG at $1064nm \rightarrow 532mm$.

15.2 Third Order Polarisation

$$P^{(3)}(t) = \epsilon_0 \chi^{(3)} E(t)^3 \tag{15.9}$$

Consider applied field,

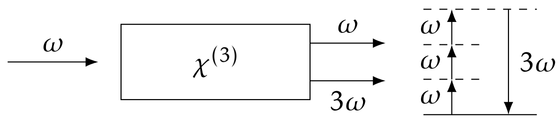
$$E(t) = E_0 \cos(\omega t) \tag{15.10}$$

Using the trig identity:

$$\begin{aligned} \cos^3(\omega t) &= \frac{1}{4}\cos(3\omega t) + \frac{3}{4}\cos(\omega t) && (15.11)\\ \implies P^{(3)}(t) &= \frac{1}{4}\epsilon_0 \chi^{(3)} E_0^3 \cos(3\omega t) + \frac{3}{4}\epsilon_0 \chi^{(3)} E_0^3 \cos(\omega t) && (15.12)\\ &= \chi^{(3)} \epsilon_0 E_0^3 \left[\frac{1}{4}\cos(3\omega t) + \frac{3}{4}\cos(\omega t) \right] && (15.13) \end{aligned}$$

Similar to the second order case, we can achieve Third Harmonic Generation.

15.2.1 Third Harmonic Generation



Sum and difference generation can also be achieved; typically we will have $\chi^{(2)} \approx 10^{-11}[mV^{-1}]$, $\chi^{(3)} \approx 10^{-22}[m^2V^{-2}]$.

15.3 Intensity Dependent Refractive Index

Consider a component of the field at frequency ω:

$$P = \epsilon_0 \left(\chi^{(1)} + \frac{3}{4}\chi^{(3)}E_0^2 \right) E_0 \cos(\omega t) \tag{15.14}$$

This looks like a linear susceptibility with an additional non-linear term:

$$\chi = \chi_{linear} + \chi_{non-linear} = \chi^{(1)} + \frac{3}{4}\chi^{(3)}E_0^2 \tag{15.15}$$

and since the refractive index

$$\eta = \sqrt{1+\chi} \tag{15.16}$$

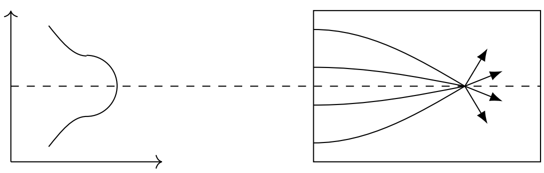
$$= \sqrt{1+\chi_L+\chi_{NL}},\ \eta_0 = \sqrt{1+\chi_L} \tag{15.17}$$

$$= \eta_0 \sqrt{1 + \frac{\chi_{NL}}{2\eta_0^2}} \tag{15.18}$$

Using a Taylor expansion, since $\chi_{NL} \ll \eta_0^2$:

$$\eta = \eta_0 + \frac{3}{8}\frac{\chi^{(3)}}{\eta_0}E_0^2 = \eta_0 + \eta_2 I \tag{15.19}$$

η_2 is the **Second Order Non-linear Refractive Index**. It is very small for most materials, e.g. $10^{-20}m^2/W$ for glass. This is often called the optical (or AC) Kerr effect, and leads to **self-focusing**:



- Care is required as it can lead to damage due to 'runaway' effect.
- Used to create ultrafast pulses in mode-locked lasers.
- The intensity profile of the beam creates equivalent to graded refractive index (GRIN) lens.
- Also interesting physics, e.g. optical solitons.

Part II

Quantum Information and Computing

Lecture 1

1.1 What is a quantum computer?

It's like a classical computer, but we replace 'bits' (0s and 1s) with ***qubits***. But what is a qubit? A qubit is a 2-level quantum system, with the quantum levels referred to $|0\rangle$ and $|1\rangle$. *copy from notes.*

Examples of 2-level systems:

- Spin-$\frac{1}{2}$ particle: 2 states are spin 'up' and spin 'down'.
- Photon: 2 polarisations, e.g. vertical and horizontal or left-circular and right-circular.
- Atoms, ions, molecules with many energy levels and we can select 2 as our qubit states.
- 'Artifical atoms' in solid state, e.g. quantum dots in semiconductors or LC resonator in a superconductor.

List five physical implementations of qubits and their problems:

- Two energy levels in an atom trapped by an optical tweezer - difficult to localise.
- Two energy levels in an ion trapped using electrodes - it's only in one dimension (scaling problem).
- Two energy levels of an impurity ion (spin) in a semi-conductor (e.g. phosphorous in Si) - interacts with surroundings, i.e. Silicon.
- Two energy levels of an LC circuit in a superconductor - very bulky and needs $10mK$ cryostat.
- Two polarisation modes of a photon - photons don't interact.

Lecture 2

2.1 DiVincenzo Criteria

The DiVincenzo criteria are often used to frame discussions about the advantages and disadvantages of different quantum computing platforms. The five criteria are:

1. Initialisation (**state preparation**) - typically means the ability to prepare identical qubits (cooling) and address each qubit independently (localisation).
2. A universal set of quantum **gates** - single- and two-qubit gates at minimum.
3. Measurement (**read out**) of $|0\rangle$ or $|1\rangle$.
4. Low **decoherence** - qubits isolated from environment (external world).
5. **Scalability** - the ability to scale up to say 100 or 1000 or more qubits.

2.2 Why Quantum Computing?

1. Moore's law - as transistor size is reduced, we approach atomic dimensions.
2. Energy efficiency - replace dissipative classical gates with reversible quantum gates.
3. Quantum 'advantage' - quantum computers can store more information and compute (certain problems) much faster.

A classical bit has states 0 and 1: N bits have $2N$ states; a qubit has states $0\rangle$ and $|0\rangle$: N qubits have 2^N states, i.e. exponential scaling. To see why, we need the **Qubit State Vector**.

$$|\psi\rangle = a|0\rangle + b|1\rangle, \tag{2.1}$$

where a and b are complex coefficients that may be time-dependent which obey the normalisation criterion.

Now we want a 2 qubit state vector, where our two qubits are A and B, as a normalised product state:

$$\begin{aligned}|\Psi\rangle_{AB} &= (a|0\rangle_A + b|1\rangle_A) \otimes (c|0\rangle_B + d|1\rangle_B), &(2.2)\\ &= ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle. &(2.3)\end{aligned}$$

In addition to product, we can have **entangled states** that are not factorisable into products.

What about 3 qubits, A, B, C?

$$|\Psi\rangle_{ABC} = c_{000}|000\rangle + c_{001}|001\rangle + c_{010}|010\rangle + c_{011}|011\rangle + c_{100}|100\rangle + c_{101}|101\rangle + c_{110}|110\rangle + c_{111}|111\rangle, \tag{2.4}$$

which is $2^3 = 8$ states. For 4 qubits, we would have $2^4 = 16$ states; for N qubits, 2^N states. For 40 qubits, $2^{40} \approx 10^{12}$; for 100 qubits, $2^{100} \approx 10^{30}$.

Lecture 3 Two-level quantum mechanics

Can think of the state vector similar to spin:

$$|\psi\rangle = a|0\rangle + b|1\rangle = a|\uparrow\rangle + b|\downarrow\rangle = \begin{pmatrix} a \\ b \end{pmatrix}. \tag{3.1}$$

How does the time-dependence appear? $0\rangle$ and $|1\rangle$ are solutions of a Schrodinger equation,

$$i\hbar\frac{\partial}{\partial t}|\alpha\rangle = H_0|\alpha\rangle, \tag{3.2}$$

with energies E_0 and E_1:

$$i\hbar\frac{\partial}{\partial t}|0\rangle = E_0|0\rangle,\ i\hbar\frac{\partial}{\partial t}|1\rangle = E_1|1\rangle, \tag{3.3}$$

$$|\psi(t=0)\rangle = a(0)|0\rangle \implies a(t) = a(0)e^{-iE_0t/\hbar}, \tag{3.4}$$

$$|\psi\rangle = ae^{-iE_0t/\hbar}|0\rangle + be^{-iE_1t/\hbar}|1\rangle, \tag{3.5}$$

$$= e^{-E_0t/\hbar}\left(a|0\rangle + be^{-i\omega_0 t}|1\rangle\right). \tag{3.6}$$

ω_0 is the angular resonant frequency of the qubit, $\omega_0 = (E_1 - E_0)/\hbar$. We define

$$G \equiv e^{-iE_0t/\hbar} \text{ - global phase,} \qquad R \equiv e^{-i\omega_0 t} \text{ - relative phase.} \tag{3.7}$$

$$|a|^2 + |b|^2 = 1 \tag{3.8}$$

$$|\psi\rangle = a|0\rangle + be^{i\phi}|1\rangle, \tag{3.9}$$

So only have two free parameters in the ratio $\frac{a}{b}$ and the phase ϕ.

$$|\psi\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} a \\ be^{i\phi} \end{pmatrix} \tag{3.10}$$

$\tan\frac{\theta}{2} = \frac{b}{a}$ and θ and ϕ are angles (from z down and x round respectively as spherical coordinates. So we can talk about our state vector as a Bloch vector, with all possible states of the qubit are points on the Bloch sphere.

3.1 Single-qubit gates

All single qubits are rotations on the Bloch sphere. These rotations are described by unitary operator, e.g. U,

$$|\psi_f\rangle = U|\psi_i\rangle, \tag{3.11}$$

where U is a 2×2 matrix. We can write U as a sum of Pauli spin matrices (and the identity matrix),

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \hat{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.12}$$

Example: Density operator

The density operator is defined as:

$$\hat{\rho} = |\psi\rangle\langle\psi|. \tag{3.13}$$

We will write this in the form of the Pauli matrices and identity, then as a single matrix:

$$\hat{\rho} = \frac{1}{2}\left(\hat{\sigma}_0 + u\hat{\sigma}_x + v\hat{\sigma}_y + w\hat{\sigma}_z\right) \tag{3.14}$$

$$= \frac{1}{2}\begin{pmatrix} 1+w & u-iv \\ u+iv & 1-w \end{pmatrix}. \tag{3.15}$$

u, v, w are the expectation values of $\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$ for the state $|\psi\rangle$.

Example: Short Question #7

Let's take a particular state,

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\phi}|1\rangle\right). \tag{3.16}$$