Durham University

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Atoms, Lasers, and Qubits

Quantum Information and Computing

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Lecture 1 Introduction

1.1 What is a quantum computer?

It's like a classical computer, but we replace 'bits' (0s and 1s) with *qubits*. But what is a qubit? A qubit is a 2-level quantum system, with the quantum levels referred to $|0\rangle$ and $|1\rangle$.

Examples of 2-level systems:

- Spin-$\frac{1}{2}$ particle: 2 states are spin 'up' and spin 'down'.
- Photon: 2 polarisations, e.g. vertical and horizontal or left-circular and right-circular.
- Atoms, ions, molecules with many energy levels and we can select 2 as our qubit states.
- 'Artificial atoms' in solid state, e.g. quantum dots in semiconductors or LC resonator in a superconductor.

List five physical implementations of qubits and their problems:

- Two energy levels in an atom trapped by an optical tweezer - difficult to localise.
- Two energy levels in an ion trapped using electrodes - it's only in one dimension (scaling problem).
- Two energy levels of an impurity ion (spin) in a semi-conductor (e.g. phosphorous in Si) - interacts with surroundings, i.e. Silicon.
- Two energy levels of an LC circuit in a superconductor - very bulky and needs $10mK$ cryostat.
- Two polarisation modes of a photon - photons don't interact.

Lecture 2 DiVincenzo Criteria and Motivations

2.1 DiVincenzo Criteria

The DiVincenzo criteria are often used to frame discussions about the advantages and disadvantages of different quantum computing platforms. The five criteria are:

1. Initialisation (**state preparation**) - typically means the ability to prepare identical qubits through cooling and address each qubit independently (localisation).
2. A universal set of quantum **gates** - single- and two-qubit gates at minimum.
3. Measurement (**read out**) of $|0\rangle$ or $|1\rangle$.
4. Low **decoherence** - qubits isolated from environment (external world).
5. **Scalability** - the ability to scale up to say 100 or 1000 or more qubits.

2.2 Why Quantum Computing?

1. Moore's law - as transistor size is reduced, we approach atomic dimensions.
2. Energy efficiency - replace dissipative classical gates with reversible quantum gates.
3. Quantum 'advantage' - quantum computers can store more information and compute (certain problems) much faster.

A classical bit has states 0 and 1: N bits have $2N$ states; a qubit has states $0\rangle$ and $|0\rangle$: N qubits have 2^N states, i.e. exponential scaling. To see why, we need the **Qubit State Vector**.

$$|\psi\rangle = a|0\rangle + b|1\rangle, \tag{2.1}$$

where a and b are complex coefficients that may be time-dependent which obey the normalisation criterion.

Now we want a 2 qubit state vector, where our two qubits are A and B, as a normalised product state:

$$|\Psi\rangle_{AB} = (a|0\rangle_A + b|1\rangle_A) \otimes (c|0\rangle_B + d|1\rangle_B), \tag{2.2}$$
$$= ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle. \tag{2.3}$$

In addition to product, we can have **entangled states** that are not factorisable into products.

What about 3 qubits, A, B, C?

$$|\Psi\rangle_{ABC} = c_{000}|000\rangle + c_{001}|001\rangle + c_{010}|010\rangle + c_{011}|011\rangle + c_{100}|100\rangle + c_{101}|101\rangle + c_{110}|110\rangle + c_{111}|111\rangle, \tag{2.4}$$

which is $2^3 = 8$ states. For 4 qubits, we would have $2^4 = 16$ states; for N qubits, 2^N states. For 40 qubits, $2^{40} \approx 10^{12}$; for 100 qubits, $2^{100} \approx 10^{30}$.

Lecture 3 Two-Level Quantum Mechanics

Can think of the state vector similar to spin:

$$|\psi\rangle = a|0\rangle + b|1\rangle = a|\uparrow\rangle + b|\downarrow\rangle = \begin{pmatrix} a \\ b \end{pmatrix}. \tag{3.1}$$

How does the time-dependence appear? $0\rangle$ and $|1\rangle$ are solutions of a Schrodinger equation,

$$i\hbar\frac{\partial}{\partial t}|\alpha\rangle = H_0|\alpha\rangle, \tag{3.2}$$

with energies E_0 and E_1:

$$i\hbar\frac{\partial}{\partial t}|0\rangle = E_0|0\rangle,\ i\hbar\frac{\partial}{\partial t}|1\rangle = E_1|1\rangle, \tag{3.3}$$

$$|\psi(t=0)\rangle = a(0)|0\rangle \implies a(t) = a(0)e^{-iE_0t/\hbar}, \tag{3.4}$$

$$|\psi\rangle = ae^{-iE_0t/\hbar}|0\rangle + be^{-iE_1t/\hbar}|1\rangle, \tag{3.5}$$

$$= e^{-E_0t/\hbar}\left(a|0\rangle + be^{-i\omega_0t}|1\rangle\right). \tag{3.6}$$

ω_0 is the angular resonant frequency of the qubit, $\omega_0 = (E_1 - E_0)/\hbar$. We define

$$G \equiv e^{-iE_0t/\hbar} \text{ - global phase,} \qquad R \equiv e^{-i\omega_0t} \text{ - relative phase.} \tag{3.7}$$

$$|a|^2 + |b|^2 = 1 \tag{3.8}$$

$$|\psi\rangle = a|0\rangle + be^{i\phi}|1\rangle, \tag{3.9}$$

So only have two free parameters in the ratio $\frac{a}{b}$ and the phase ϕ.

$$|\psi\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} a \\ be^{i\phi} \end{pmatrix} \tag{3.10}$$

$\tan\frac{\theta}{2} = \frac{b}{a}$ and θ and ϕ are angles (from z down and x round respectively as spherical coordinates. So we can talk about our state vector as a Bloch vector, with all possible states of the qubit are points on the Bloch sphere.

3.1 Single-qubit gates

All single qubits are rotations on the Bloch sphere. These rotations are described by unitary operator, e.g. U,

$$|\psi_f\rangle = U|\psi_i\rangle, \tag{3.11}$$

where U is a 2×2 matrix. We can write U as a sum of Pauli spin matrices (and the identity matrix),

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \hat{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.12}$$

Example: Density operator
The density operator is defined as:

$$\hat{\rho} = |\psi\rangle\langle\psi|. \tag{3.13}$$

We will write this in the form of the Pauli matrices and identity, then as a single matrix:

$$\hat{\rho} = \frac{1}{2}\left(\hat{\sigma}_0 + u\hat{\sigma}_x + v\hat{\sigma}_y + w\hat{\sigma}_z\right) \tag{3.14}$$

$$= \frac{1}{2}\begin{pmatrix} 1+w & u-iv \\ u+iv & 1-w \end{pmatrix}. \tag{3.15}$$

u, v, w are the expectation values of $\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$ for the state $|\psi\rangle$.

Example: Short Question #7

Let's take a particular state,

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\phi}|1\rangle\right). \tag{3.16}$$

What is the value of θ?

$$\cos\frac{\theta}{2} = \sin\frac{\theta}{2} = \frac{1}{\sqrt{2}} \implies \frac{\theta}{2} = \frac{\pi}{4} \implies \theta = \frac{\pi}{2}. \tag{3.17}$$

Bloch vector is in equatorial plane. Now calculate expectation values of $\hat{\sigma}_i$:

$$\langle\psi|\hat{\sigma}_x|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & e^{-i\phi}\end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{i\phi}\end{pmatrix} \tag{3.18}$$

$$= \frac{1}{2}\left(e^{i\phi} + e^{-i\phi}\right) = \cos\phi. \tag{3.19}$$

$\phi = 0 \implies u = +1, \phi = \pi \implies u = -1.$

$$\langle\psi|\hat{\sigma}_y|\psi\rangle = \frac{1}{2}\begin{pmatrix} 1 & e^{-i\phi}\end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\begin{pmatrix} 1 \\ e^{i\phi}\end{pmatrix} \tag{3.20}$$

$$= \frac{1}{2i}\left(e^{i\phi} + e^{-i\phi}\right) = \sin\phi. \tag{3.21}$$

$$\langle\psi|\hat{\sigma}_z|\psi\rangle = \frac{1}{2}\begin{pmatrix} 1 & e^{-i\phi}\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix} 1 \\ e^{i\phi}\end{pmatrix} \tag{3.22}$$

$$= \frac{1}{2}(1-1) = 0. \tag{3.23}$$

So the Bloch vector is in equatorial plain as thought.

Lecture 4 The Density Matrix and Qubit Rotations

- Any qubit can be represented by a point on the Bloch sphere.
- We have two parameters: the polar angle, θ; and the azimuthal angle, ϕ.
- The qubit state vector is defined as

$$|\psi\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}. \tag{4.1}$$

In Cartesian coordinates, the qubit state is given by the Bloch vector $\underline{b} = (u, v, w)$, with $\sqrt{u^2+v^2+w^2} = 1$. This Cartesian description allows for

$$\sqrt{u^2+v^2+w^2} < 1, \tag{4.2}$$

i.e. point inside the sphere. In the real world, we have decoherence where the effect is to shorten the Bloch vector. We have two types of states:

- Pure states: $\sqrt{u^2+v^2+w^2} = 1$, on the Bloch sphere, represented by the wavefunction $|\psi\rangle$.
- Mixed states: $\sqrt{u^2+v^2+w^2} < 1$, inside the Bloch sphere.

We cannot describe mixed states by a wavefunction, so we introduce the density matrix.

4.1 Density matrix for a pure state of a qubit

We define the density matrix as:

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2} \end{pmatrix} \tag{4.3}$$

$$= \begin{pmatrix} \cos^2\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+\cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & 1-\cos\theta \end{pmatrix} \tag{4.4}$$

$$= \frac{1}{2}\begin{pmatrix} 1+\cos\theta & \sin\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\phi + i\sin\theta\sin\phi & 1-\cos\theta \end{pmatrix} \tag{4.5}$$

$$\rho = \frac{1}{2}\begin{pmatrix} 1+w & u-iv \\ u+iv & 1-w \end{pmatrix} = \frac{1}{2}\left(\hat{\sigma}_0 + u\hat{\sigma}_x + v\hat{\sigma}_y + w\hat{\sigma}_z\right). \tag{4.6}$$

What is the expectation value of ρ for our wavefunction? If we have a normalised wavefunction, $\langle\psi|\rho|\psi\rangle = \langle\psi|\psi\rangle\langle\psi|\psi\rangle = 1$.

$$\langle\psi|\rho|\psi\rangle = \frac{1}{2}\left(\langle\psi|\sigma_0|\psi\rangle + u\langle\psi|\sigma_x|\psi\rangle + \cdots\right) \tag{4.7}$$

$$= \frac{1}{2}\left(1+u^2+v^2+w^2\right) = 1 \tag{4.8}$$

For mixed states, $\sqrt{u^2+v^2+w^2} < 1$, so this will no longer be the case. We will explore that properly later.

4.2 Qubit Rotations

All unitary operations (that conserve the norm $\langle\psi|\psi\rangle$) can be described as rotations of the Bloch vector. A general rotation by an angle Θ about a unit vector $\underline{n} = (n_x, n_y, n_z)$. We define the Rotation operator,

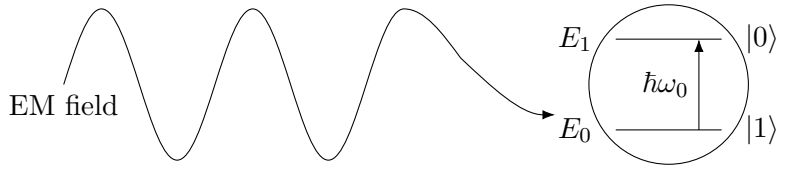
$$\hat{R} = \exp\left(-i\frac{1}{2}\underline{\hat{\sigma}}\cdot\underline{n}\Theta\right) \tag{4.9}$$

$$= \hat{\sigma}_0\cos\frac{\Theta}{2} - i\underline{\hat{\sigma}}\cdot\underline{n}\sin\frac{\Theta}{2} \tag{4.10}$$

$$= \begin{pmatrix} \cos\frac{\Theta}{2} - in_z\sin\frac{\Theta}{2} & (-in_x - ny)\sin\frac{\Theta}{2} \\ (-in_x + n_y)\sin\frac{\Theta}{2} & \cos\frac{\Theta}{2} + in_z\sin\frac{\Theta}{2} \end{pmatrix} \tag{4.11}$$

Lecture 5 Qubit Rotations on the Bloch Sphere

How do we implement rotation on the Bloch sphere? We need to drive the qubit with a resonant (or near-resonant) EM field.



The qubit is like an oscillator we can drive. EM field:

$$\mathcal{E} = \mathcal{E}_0 \cos(\phi_L - \omega t), \tag{5.1}$$

where ϕ_L is the laser phase. We could include any phase offset. We assume the qubit is at $\underline{r} = 0$, and $\phi = 0$ at $t = 0$, so $\phi_L = \underline{k} \cdot \underline{r}$. This then depends on the relative position of the qubit and field. Note that since there is always some δr uncertainty in position of the qubit, this gives rise to a range of phase ϕ_L which can influence the qubit behaviour, e.g. phonons are solid vibrating all the time → phase is changing. Hence inside the sphere for the qubit and not on the sphere. We can try to reduce this by lowering temperatures. Other notation:

$$\Delta = \omega - \omega_0, \tag{5.2}$$

where ω_0 is the qubit frequency, ω the field frequency, and Δ is defined as the qubit-field detuning. If the qubit is subject to an external field, the coefficients a and b may become functions of time, and we should write the qubit state vector in the form

$$|\psi(t)\rangle = a(t)|0\rangle e^{-iE_0 t/\hbar} + b(t)|1\rangle e^{-iE_1 t/\hbar}. \tag{5.3}$$

The time dependence of the coefficients a and b depends on the interaction between the qubit and the EM field. We can use the Schrodinger equation:

$$i\hbar \frac{\partial}{\partial t}|\psi\rangle = (\mathcal{H}_0 + \mathcal{H}')|\psi\rangle. \tag{5.4}$$

Previously we just had $\mathcal{H}_0$ which describes the qubit, but now we add $\mathcal{H}'$ which is the perturbation due to the EM field. Typically, this interaction is of the form

$$\mathcal{H}' = -\hat{\underline{d}} \cdot \underline{\mathcal{E}}, \qquad = -d\mathcal{E}_0 \cos(\phi_L - \omega t), \tag{5.5}$$

where $\underline{d} = -e\hat{\underline{r}}$ is the electric dipole operator, and we can also define the Rabi frequency in Hz as

$$\Omega = -\frac{d\mathcal{E}_0}{\hbar}, \tag{5.6}$$

$$\mathcal{H}' = \hbar\Omega \cos(\phi_L - \omega t). \tag{5.7}$$

The Rabi frequency is a measure of the coupling between the field and the qubit. If the field is put on and left on, then the state will oscillate between $|0\rangle$ and $|1\rangle$ at the Rabi frequency. Subbing the perturbation, we get two equations for a and b:

$$i\dot{a}(t) = \frac{1}{2}\Omega e^{-i\phi_L} e^{i\Delta \cdot t} b, \qquad i\dot{b}(t) = \frac{1}{2}\Omega e^{i\phi_L} e^{-i\Delta \cdot t} a. \tag{5.8}$$

These equations have an explicit time dependence. To deal with this, we can go into a rotating frame using a substitution:

$$\tilde{a} = a e^{-it\cdot\Delta/2}, \qquad \tilde{b} = b e^{it\cdot\Delta/2}. \tag{5.9}$$

This is so that we spin at the same frequency as the qubit and since we are interested in the relative phase of the field and qubit, this makes sense. The Schrodinger equation for the interaction between a qubit and an oscillatory EM field is

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \mathcal{H}_{int}|\psi\rangle, \tag{5.10}$$

where for the vector for of the state vector, the **interaction Hamiltonian** is a 2×2 matrix:

$$\mathcal{H}_{int} = \frac{\hbar}{2}\begin{pmatrix}\Delta & \Omega e^{-i\phi_L}\\ \Omega e^{i\phi_L} & -\Delta\end{pmatrix}, \tag{5.11}$$

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\tilde{a}\\ \tilde{b}\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}\Delta & \Omega e^{-i\phi_L}\\ \Omega e^{i\phi_L} & -\Delta\end{pmatrix}\begin{pmatrix}\tilde{a}\\ \tilde{b}\end{pmatrix}. \tag{5.12}$$

We can also write $\mathcal{H}_{int}$ in terms of the spin matrices. Note that we have the detuning on the diagonals (like $\hat{\sigma}_z$) and to get the exponential, we need cos and sin terms:

$$\mathcal{H}_{int} = \frac{\hbar}{2}\left[\Delta\hat{\sigma}_z + \Omega\left(\cos\phi_L\hat{\sigma}_x + \sin\phi_L\hat{\sigma}_y\right)\right], \tag{5.13}$$

where $\hat{\sigma}_y$ contains the imaginary values necessary for the exponents. Consider zero detuning, i.e. $\Delta = 0, \omega_0 = \omega$, and $\phi_L = 0$:

$$i\dot{a} = \frac{1}{2}\Omega b, \qquad\qquad i\dot{b} = \frac{1}{2}\Omega a. \tag{5.14}$$

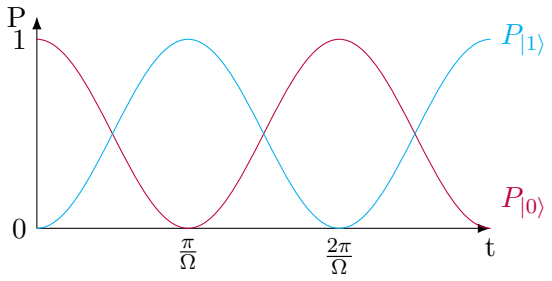
Taking time derivatives again:

$$i\ddot{a} = \frac{1}{2}\Omega\dot{b} = -\frac{i}{4}\Omega a \implies \ddot{a} = -\frac{1}{4}\Omega^2 a. \tag{5.15}$$

For $a(0) = 1, b(0) = 0$, $a = \cos\left(\frac{\Omega t}{2}\right)$. Getting probabilities:

$$P_{|0\rangle} = |a|^2 = \cos^2\left(\frac{\Omega t}{2}\right) = \frac{1}{2}(1+\cos\Omega t), \tag{5.16}$$

$$P_{|1\rangle} = |b|^2 = \sin^2\left(\frac{\Omega t}{2}\right) = \frac{1}{2}(1-\cos\Omega t), \tag{5.17}$$



$\theta = \frac{\Omega t}{2}$ is the rotation angle. The effect of the resonant EM field is to drive the qubit from $|0\rangle \to |1\rangle$ and back at Ω frequency.

The qubit follows a line of longitude. ϕ_L determines the direction in the xy plane about which we rotate, i.e. can change which longitude line we travel around. Derive rotation matrix $\hat{R}$ from the interaction Hamiltonian:

$$|\psi(t)\rangle = e^{-i\mathcal{H}_{int}t/\hbar}|\psi(0)\rangle = \hat{R}|\psi(0)\rangle. \tag{5.18}$$

To find the rotation matrix, we rewrite the interaction in the form:

$$\hat{R} = e^{-i\mathcal{H}_{int}t/\hbar} = e^{-(i/2)\hat{\sigma}\cdot\underline{\hat{n}}\Theta}, \quad \Theta = t\sqrt{\Omega^2 + \Delta^2}. \tag{5.19}$$

We had defined $\mathcal{H}_{int}$ as

$$\mathcal{H}_{int} = \frac{\hbar}{2}\left[\Delta\hat{\sigma}_z + \Omega\left(\cos\phi_L\hat{\sigma_x} + \sin\phi_L\hat{\sigma}_y\right)\right], \tag{5.20}$$

so using this in $\hat{R}$ above, we can determine $\underline{\hat{n}}$ to be

$$\underline{\hat{n}} = \frac{1}{\Theta}\begin{pmatrix}\Omega\cos\phi_L & \Omega\sin\phi_L & \Delta\end{pmatrix}. \tag{5.21}$$

Lecture 6 Single Qubit Rotations and Hadamard Operations

From normalisation we define $\Theta = t\sqrt{\Omega^2 + \Delta^2}$, with the quantity in brackets known as the *effective Rabi frequency.* Substituting the unit vector $\underline{\hat{n}}$ above into the rotation matrix:

$$\hat{R} = \begin{pmatrix} \cos\frac{\theta}{2} - in_z \sin\frac{\theta}{2} & (-in_x - n_y)\sin\frac{\theta}{2} \\ (-in_x + n + y)\sin\frac{\theta}{2} & \cos\frac{\theta}{2} + in_z \sin\frac{\theta}{2} \end{pmatrix} \tag{6.1}$$

$$= \begin{pmatrix} \cos\frac{\theta}{2} - i\frac{\Delta}{\Theta}\sin\frac{\theta}{2} & -i\frac{\Omega}{\Theta}e^{i\phi_L}\sin\frac{\theta}{2} \\ -i\frac{\Omega}{\Theta}e^{-i\phi_L}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} + i\frac{\Delta}{\Theta}\sin\frac{\theta}{2} \end{pmatrix}. \tag{6.2}$$

This is known as the **Rabi solution.** We consider the first of 3 special cases:
Case 1: $\Delta = 0$, i.e. on resonance. The Rabi solution reduces to

$$\hat{R} = \begin{pmatrix} \cos\frac{\Omega t}{2} & -ie^{i\phi_L}\sin\frac{\Omega t}{2} \\ -ie^{-i\phi_L}\sin\frac{\Omega t}{2} & \cos\frac{\Omega t}{2} \end{pmatrix}, \tag{6.3}$$

which corresponds to a rotation about a vector in the equatorial plane in the Bloch sphere. For $\Delta = 0$, we often use notation $\hat{R}(\theta, \phi_L)$ where $\theta = \frac{\Omega t}{2}$ is the rotation angle and ϕ_L determines the direction in the xy plane about which we rotate. For an atom initially in state $|0\rangle$, i.e.

$$|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle, \tag{6.4}$$

and an interaction of duration t, the Rabi solution gives

$$|\psi(t)\rangle = \hat{R}|\psi(0)\rangle = \begin{pmatrix} \cos\frac{\Omega t}{2} \\ -ie^{-i\phi_L}\sin\frac{\Omega t}{2} \end{pmatrix} = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}, \tag{6.5}$$

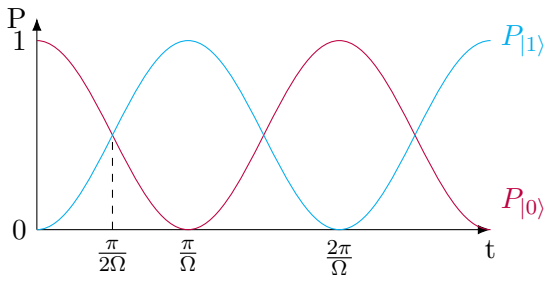
$$a(t) = \cos\frac{\Omega t}{2}, \qquad b(t) = -ie^{-i\phi_L}\sin\frac{\Omega t}{2}, \tag{6.6}$$

and we get the populations in states $|0\rangle, |1\rangle$ from

$$P_{|0\rangle} = |a(t)|^2 = \cos^2\frac{\Omega t}{2}, \tag{6.7}$$

$$P_{|1\rangle} = |b(t)|^2 = \sin^2\frac{\Omega t}{2}, \tag{6.8}$$

which are the **Rabi oscillations.** The population oscillates between the two states at the Rabi frequency, Ω. The Rabi frequency is proportional to the field amplitude, i.e. the square root of the field intensity.



Another interesting case is when there is no interaction ($\Omega = 0$) but the field and the qubit have a different frequency $\Delta \neq 0$, i.e. *free evolution rotations.* In this case, the rotation matrix is

$$\hat{R} = \begin{pmatrix} \exp\left(-i\frac{\Delta t}{2}\right) & 0 \\ 0 & \exp\left(i\frac{\Delta t}{2}\right) \end{pmatrix}, \tag{6.9}$$

which corresponds to a rotation about the z-axis in the Bloch sphere. We use this matrix to describe the free evolution in a *Ramsey interferometer.*

6.1 Single Qubit Rotations

The Rabi solution describes how a near-resonance EM field may be used to drive the qubit from any initial state to any final state. Refer to this operation as a **single-qubit rotation** or **single-qubit gate.** Three useful interactions come from the times you pulse, e.g. as above the $\frac{\pi}{2}$-, π-, and 2π-pulse. For all except the **Hadamard gate**, we use resonant driving $\Delta = 0$.

➤ $\frac{\pi}{2}$-pulse (particularly useful):
For a $\frac{\pi}{2}$ pulse, we choose the EM field intensity and the pulse duration such that $\Omega t = \frac{\pi}{2}$. This corresponds to a rotation of 90° on the Bloch sphere which could take us from pole to equator, i.e. a qubit initially in $|0\rangle$ or $|1\rangle$ is excited → an equal superposition on the equator (or start on the equator and rotate to a pole). Substituting $\Omega t = \frac{\pi}{2}$ in the Rabi solution, we obtain

$$|\psi(t_{\pi/2})\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -ie^{i\phi_L} \\ -ie^{-i\phi_L} & 1 \end{pmatrix} |\psi(0)\rangle. \tag{6.10}$$

To simplify the matrix, we can choose a particular laser phase, e.g. $\phi_L = \frac{\pi}{2}$, so:

$$\hat{R}(\theta, \phi_L) = \hat{R}\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \tag{6.11}$$

and acting on our initial state $|\psi(0)\rangle = |0\rangle$,

$$|\psi(t_{\pi/2})\rangle = \hat{R}\left(\frac{\pi}{2}, \frac{\pi}{2}\right) |\psi(0)\rangle \tag{6.12}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{6.13}$$

$$= \frac{1}{\sqrt{2}} \left(|0\rangle - |1\rangle\right), \tag{6.14}$$

which is a superposition between $|0\rangle$ and $|1\rangle$. This corresponds to a rotation around the y axis, ending up in the $-x$ direction, i.e. the expectation value $\langle\hat{\sigma}\rangle = -1$. If we apply an arbitrary phase ϕ_L to $|1\rangle$, we find

$$|1\rangle \to \frac{1}{\sqrt{2}} \left(-ie^{i\phi_L}|0\rangle + |1\rangle\right). \tag{6.15}$$

For $\phi_L = 0$, we would rotate around the x axis and end up along y. So the laser phase sets the vector in the xy plane that we rotate around. If we had started with a superposition, e.g. in Eq (6.14), it carries on to the $|1\rangle$ state if we apply the same $\hat{R}\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$:

$$\hat{R}\left(\frac{\pi}{2}, \frac{\pi}{2}\right) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = - \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{6.16}$$

So 2 $\frac{\pi}{2}$ pulses are equivalent to a π-pulse and takes us from $|0\rangle \to |1\rangle$. We shouldn't be concerned about the '-1' as when we take the modulus this global phase disappears.

➤ π-pulse: For $\Omega t_\pi = \pi$,

$$|\psi(t_\pi)\rangle = \begin{pmatrix} 0 & -ie^{i\phi_L} \\ -ie^{-i\phi_L} & 0 \end{pmatrix} |\psi(0)\rangle, \tag{6.17}$$

$$|0\rangle \to -ie^{-i\phi_L}|1\rangle, \qquad |1\rangle \to -ie^{i\phi_L}|0\rangle. \tag{6.18}$$

For $\phi_L = 0$ and $\phi_L = \frac{\pi}{2}$ respectively, the rotation matrix is

$$\hat{R}(\pi, 0) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}; \qquad \hat{R}\left(\pi, \frac{\pi}{2}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{6.19}$$

Except for the sign change, this is equivalent to a '*bit-flip*' or *NOT gate.*

➤ 2π-pulse: For $\Omega t_{2\pi} = 2\pi$, we have

$$\hat{R}\left(2\pi, \phi_L\right) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\hat{\sigma}_0, \tag{6.20}$$

so it effectively just multiplies the entire wavefunction by -1, which is a global phase factor (and property of spin-$\frac{1}{2}$ particles) that can be useful in gate operations. In the case of two qubits, if one of them picks up a -1, then it does become significant as they are entangled and we can now detect this.

6.2 Hadamard Gate

The application of two successive $\frac{\pi}{2}$-pulses inverts the state. However, it is also useful to have a pulse that takes us into a superposition and back to the same state. This is a **Hadamard gate** or **Hadamard transform**. The Hadamard operator is

$$\mathcal{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{6.21}$$

and the application of two of these leave the state unchanged, i.e. $\mathcal{H}^2 = \hat{\sigma}_0$ (as long as nothing happens in-between applying them). There are different ways to implement a Hadamard. One is to set $\Delta = \Omega$, in which case the Rabi solution becomes

$$\hat{R} = \begin{pmatrix} \cos\frac{\Omega}{\sqrt{2}}t - \frac{i}{\sqrt{2}}\sin\frac{\Omega}{\sqrt{2}}t & -\frac{i}{\sqrt{2}}e^{i\phi_L}\sin\frac{\Omega}{\sqrt{2}}t \\ -\frac{i}{\sqrt{2}}e^{-i\phi_L}\sin\frac{\Omega}{\sqrt{2}}t & \cos\frac{\Omega}{\sqrt{2}}t + \frac{i}{\sqrt{2}}\sin\frac{\Omega}{\sqrt{2}}t \end{pmatrix}, \tag{6.22}$$

and if we set $\frac{\Omega}{\sqrt{2}}t = \frac{\pi}{2}$ and $\phi_L = 0$, we obtain

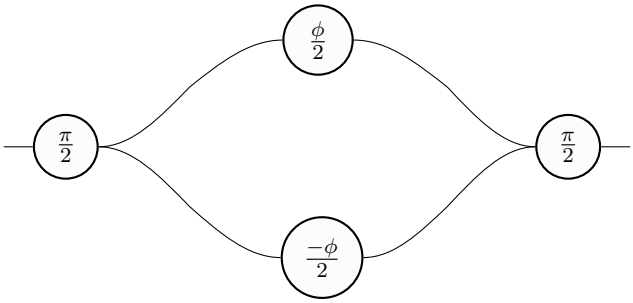
$$\hat{R} = -\frac{i}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{6.23}$$

so $\hat{R}$ is the Hadamard up to some global factor which can be ignored. The Hadamard is a rotation around some 45°; applying twice brings us back to the start.

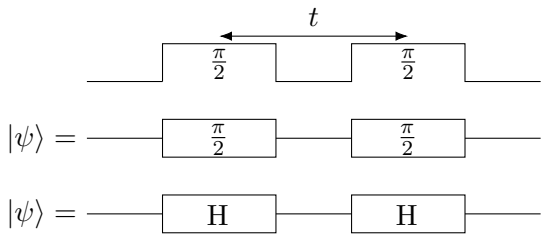
Lecture 7 Ramsey Interferometry and Two-Qubit Operations

7.1 Ramsey Interferometry

We have two $\frac{\pi}{2}$ pulses separated by time t (or Hadamards):



We can also schematically represent paths in Hilbert space:



There is a read-out of relative phase. This interval t is often called the *free evolution time,* but this could include an interaction. During this free evolution, the qubit Bloch vector rotates around the z-axis. We can describe this precession using a rotation matrix:

$$\hat{R}_z(\phi) = \begin{pmatrix} e^{i-\phi/2} & 0 \\ 0 & e^{-\phi/2} \end{pmatrix} = \begin{pmatrix} e^{-i\Delta t/2} & 0 \\ 0 & e^{i\Delta t/2} \end{pmatrix}, \tag{7.1}$$

so we can say that the Bloch vector precesses around the z-axis at a rate given by the difference between the field and qubit evolution, $\Delta = \omega - \omega_0$. The total rotation angle is $\phi = \Delta t$.

The complete Ramsey sequence is given by two $\frac{\pi}{2}$ pulses separated by free evolution. The $\frac{\pi}{2}$ pulse matrix is obtained from the Rabi solution for $\Delta = 0$.

$$\hat{R} = \begin{pmatrix} \cos\frac{\theta}{2} & -ie^{i\phi_L}\sin\frac{\theta}{2} \\ -ie^{-i\phi_L}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}, \tag{7.2}$$

where $\theta = \Omega t$. We choose $\phi_L = \frac{\pi}{2}$:

$$\hat{R}\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \hat{R}_y\left(\frac{\pi}{2}\right). \tag{7.3}$$

Our complete sequence is given by $\hat{R}\left(\frac{\pi}{2}\right)\hat{R}\left(\phi\right)\hat{R}_y\left(\frac{\pi}{2}\right)$, where we write our sequence of rotations in reverse order.

$$\hat{R}_y\left(\frac{\pi}{2}\right)\hat{R}(\phi)\hat{R}\left(\frac{\pi}{2}\right) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{7.4}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^{-i\phi/2} & e^{-i\phi_2} \\ -e^{-\phi/2} & e^{i\phi/2} \end{pmatrix} \tag{7.5}$$

$$= \begin{pmatrix} -i\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \\ -\cos\frac{\phi}{2} & i\sin\frac{\phi}{2} \end{pmatrix}. \tag{7.6}$$

Initial Condition (at the North Pole):

$$|\psi\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{7.7}$$

$$\hat{R}\left(\frac{\pi}{2}\right)\hat{R}_z(\phi)\hat{R}_y\left(\frac{\pi}{2}\right)|\psi\rangle = \begin{pmatrix} -\sin\frac{\phi}{2} \\ -\cos\frac{\phi}{2} \end{pmatrix} \tag{7.8}$$

Our output state has:

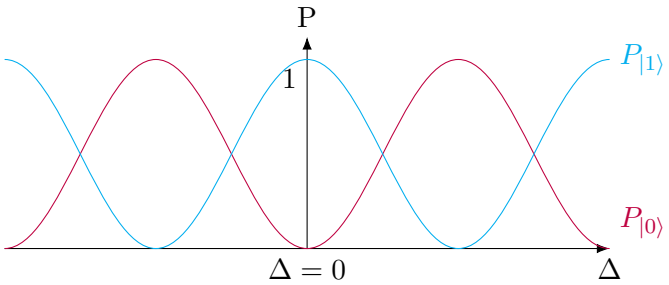
$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -\sin\frac{\phi}{2} \\ -\cos\frac{\phi}{2} \end{pmatrix} \tag{7.9}$$

The probability to be in state $|0\rangle$ at the output is then

$$P_{|0\rangle} = |a|^2 = \frac{1}{2}(1 - \cos\phi) \tag{7.10}$$

$$P_{|1\rangle} = |b|^2 = \frac{1}{2}(1 + \cos\phi) \tag{7.11}$$

For $\phi = \Delta t$,



At $\Delta = 0$, we see $P_{|0\rangle} \to 0$, because the second $\frac{\pi}{2}$ pulse completes the excitation to state $|1\rangle$. The fringes we see here are known as the Ramsey fringes.

7.2 Two qubits

We like to name our qubits, so we may have:

- A and B
- C (control) and T (target)

Now we need to write our two-qubit state vector to describe our two-qubit system:

$$|\psi\rangle_{AB} = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle. \tag{7.12}$$

Here, $|ij\rangle = |0\rangle \otimes |1\rangle$, i.e. it is an abbreviation for the tensor product. The state vector can also be represented as a column vector:

$$|\psi\rangle_{AB} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, \tag{7.13}$$

where it has four components, a, b, c, d. All two-qubit operators, therefore, are 4×4 matrices.

How do we construct two-qubit operators? As an example, we apply the interaction Hamiltonian $\mathcal{H}_{int}$ to both qubits A and B:

$$\mathcal{H}_2 = \mathcal{H}_{int} \otimes \hat{\sigma}_0 + \hat{\sigma}_0 \otimes \mathcal{H}_{int}, \tag{7.14}$$

where the first term takes our previous 2×2 interaction Hamiltonian and places it in the new 4×4 matrix so it interacts with A and does nothing to B, and the second term such that it interacts with B and not A. For $\Delta = 0$:

$$\mathcal{H}_2 = \frac{\hbar}{2} \begin{pmatrix} 0 & \Omega \\ \Omega & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{\hbar}{2} \begin{pmatrix} 0 & \Omega \\ \Omega & 0 \end{pmatrix} \tag{7.15}$$

$$= \frac{\hbar}{2} \begin{pmatrix} 0 & 0 & \Omega & 0 \\ 0 & 0 & 0 & \Omega \\ \Omega & 0 & 0 & 0 \\ 0 & \Omega & 0 & 0 \end{pmatrix} + \frac{\hbar}{2} \begin{pmatrix} 0 & \Omega & 0 & 0 \\ \Omega & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega \\ 0 & 0 & 0 & \Omega \end{pmatrix} \tag{7.16}$$

$$= \frac{\hbar}{2} \begin{pmatrix} 0 & \Omega & \Omega & 0 \\ \Omega & 0 & 0 & \Omega \\ \Omega & 0 & 0 & \Omega \\ 0 & \Omega & \Omega & 0 \end{pmatrix}. \tag{7.17}$$

It follows that the operators acting on three-qubit systems correspond to 8×8 matrices, and in general, operators acting on N-qubit systems correspond to $2^N \times 2^N$ matrices.

Lecture 8 Quantum Entanglement

8.1 Entanglement

Entanglement is what gives quantum computing its 'advantage'. It is essential in all forms of two-qubit gates, such as CNOT ('Control-NOT'). An entangled state cannot be separated into a product of individual qubit state vectors.

8.1.1 Examples of entangled states

The complete set of "maximally entangled" two-qubit states are the so-called *Bell states:*

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle \pm |11\rangle\right) \tag{8.1}$$

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle \pm |10\rangle\right) \tag{8.2}$$

We then can tell something about B if we know A and vice versa: for (8.1), if we measure A in 0, then we know B is 0 too regardless of where B is (non-local effect). We cannot transmit information though, so the limit of transferring is still less than the speed of light; we may know something about B if we observe A, but we cannot tell B that.

8.1.2 Measuring a part of a two-qubit state

This can be particularly interesting for entangled states. We introduce the idea of *projection operators.* If we have a state $|\psi\rangle_{AB}$, what is the probability that A is in the state $|\phi_A\rangle$ regardless of B or that both A and B are in the state $|\phi_{A,B}\rangle$?

We use a projection operator $\hat{P}$ that projects $|\psi_{AB}\rangle$ onto the state we want to see.

$$\text{Probability} = \langle\hat{P}\rangle = \text{Tr}(\hat{\rho}_{AB}\hat{P}), \text{ where } \hat{\rho} = |\psi\rangle_{AB}{}_{AB}\langle\psi| \tag{8.3}$$

is the density matrix and $\hat{P}$ is the projection operator. The trace Tr is the sum of diagonal elements.

Example:
Consider the entangled Bell state

$$|\Phi^{+}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right). \tag{8.4}$$

What is the probability of observing qubit A in state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$? We need to find a projection operator that

1. projects A on $|+\rangle$,
2. and leaves B unchanged.

This operator will read

$$\hat{P} = |+\rangle\langle+| \otimes \hat{\sigma}_0. \tag{8.5}$$

For a single qubit,

$$\hat{P} = \langle\psi|+\rangle\langle+|\psi\rangle = |\langle+|\psi\rangle|^2, \tag{8.6}$$

i.e. the probability overlap between $|\psi\rangle$ and $|+\rangle$. For two qubits, the operator reads

$$\hat{P} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \tag{8.7}$$

$$= \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{8.8}$$

$$= \frac{1}{2}\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\end{pmatrix}. \tag{8.9}$$

Now we define the density matrix as

$$\hat{\rho} = |\Phi^+\rangle\langle\Phi^+| \tag{8.10}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\\ 0\\ 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0 & 1\end{pmatrix} \tag{8.11}$$

$$= \frac{1}{2}\begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 1\end{pmatrix}. \tag{8.12}$$

The product of these is then

$$\hat{\rho}\hat{P} = \frac{1}{2}\begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 1\end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\end{pmatrix} \tag{8.13}$$

$$= \frac{1}{4}\begin{pmatrix}1 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 1 & 1 & 1\end{pmatrix} \tag{8.14}$$

$$P_{A+} = \mathrm{Tr}(\hat{\rho}\hat{P}) = \frac{1}{2}, \tag{8.15}$$

i.e. the probability that A is in state $|+\rangle$ is $\frac{1}{2}$. The probability that B is in state $|+\rangle$ is also $\frac{1}{2}$.

What is the probability that both A and B are in state $|+\rangle$? Classically, we would say that $P_{AB} = P_A P_B = \frac{1}{4}$; this is wrong. A and B now have a correlation between them in the quantum system, so we use the projection operators to find the probability that both A and B are in $|+\rangle$. The projection operator reads

$$\hat{P} = |+\rangle\langle+| \otimes |+\rangle\langle+| \tag{8.16}$$

$$= \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \otimes \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \tag{8.17}$$

$$= \frac{1}{4}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\end{pmatrix}. \tag{8.18}$$

Our product reads

$$\hat{\rho}\hat{P} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \tag{8.19}$$

$$= \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}. \tag{8.20}$$

$$\mathrm{Tr}(\hat{\rho}\hat{P}) = \frac{1}{2}, \tag{8.21}$$

i.e. the probability that both are in state $|+\rangle$ is $\frac{1}{2}$. Therefore, $P_{AB} \neq P_A P_B$, because A and B are **correlated** - whenever A is $|+\rangle$, then B is in $|+\rangle$.

Lecture 9 Entangled Correlation and Qubit Gates

Last time, we looked at entanglement in the Bell states - which are maximally entangled:

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle \pm |11\rangle\right). \tag{9.1}$$

We calculated the probability P_A that A is in the state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, which we found was $P_A = \frac{1}{2}$. The probability P_B that B is in the state $|+\rangle$ was also $P_B = \frac{1}{2}$. Then, we asked what the joint probability that both A and B are in the state $|+\rangle$. Classically, we would say that $P_{AB} = P_A P_B = \frac{1}{4}$, but here this is not the case as these states are correlated, i.e. whenever A is in the state $|+\rangle$, then B must also be in the state $|+\rangle$.
Entanglement implies correlation.

To see this correlation, we can rewrite the Bell state by defining the $|+\rangle, |-\rangle$ basis as

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle), \tag{9.2}$$

$$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \qquad |1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle). \tag{9.3}$$

Using these definitions, we simply rearrange the Bell state:

$$|\Phi^{+}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{9.4}$$

$$= \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \otimes \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) + \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) \otimes \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)\right] \tag{9.5}$$

$$= \frac{1}{\sqrt{2}}\left[\frac{1}{2}(|++\rangle + |-+\rangle + |+-\rangle + |--\rangle + |++\rangle - |-+\rangle - |+-\rangle + |--\rangle)\right] \tag{9.6}$$

$$= \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle). \tag{9.7}$$

This is an example of the rotational invariance of the Bell state. So the Bell state appears the same in this basis as before - we have simply 'rotated' the state vector around the Bloch sphere to align along the x-axis. In fact, the Bell state will appear the same regardless of the basis/rotation. The Bell state represents quantum entanglement and correlations which is invariant under rotations, *providing that we choose to measure A and B in the same basis.*

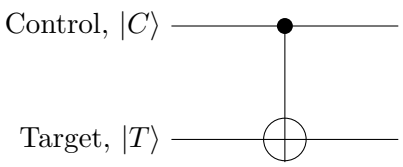
9.1 Qubit Gates

We have two basic types of gates:

- ➤ The NOT gate - a single qubit gate
- ➤ The Controlled NOT gate - two qubit gate that works on correlation for its output

All computations can be built up from these two basic gates.

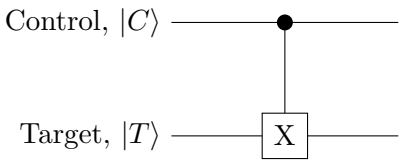
Let's consider the CNOT gate. This is a universal gate, i.e. having just CNOT gates + single qubit rotations is sufficient to build a universal quantum computer. We have:



What does it do? If C is in state $|1\rangle$, then NOT on T, otherwise we do nothing. This can be written as a matrix:

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \approx \begin{Bmatrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{Bmatrix}, \tag{9.8}$$

where the final term is in roughly what each line represents in the 4×4 matrix. For the first two lines, where A is in $|0\rangle$, we do nothing to B, hence the identity form; the latter two lines, where A is in $|1\rangle$, we perform NOT on B - this resembles the Pauli $\hat{\sigma}_x$ matrix. This is also called the Controlled-X (CX) gate, where X represents that Pauli $\hat{\sigma}_x$ matrix.



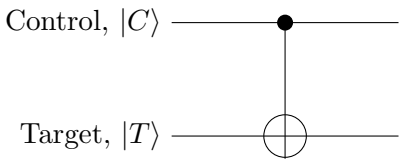
CNOT clearly involves entanglement then. We cannot separate CNOT into a product of single-qubit matrices, e.g.

$$\text{CNOT} \neq \hat{\sigma}_0 \otimes \hat{\sigma}_x. \tag{9.9}$$

So CNOT is an entangling operator.

Example: CNOT creates entanglement

Our Control and Target inputs are:



$$|C\rangle = |+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \tag{9.10}$$

$$|T\rangle = |0\rangle. \tag{9.11}$$

The full input state is then

$$|C\rangle \otimes |T\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes |0\rangle \tag{9.12}$$

$$|CT\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |10\rangle\right). \tag{9.13}$$

This is a **product state** - there is no entanglement between these yet. Now we pass this product state through the CNOT operator to give

$$\text{CNOT}|CT\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right), \tag{9.14}$$

where we not have the Bell state which is maximally entangled by definition. Therefore, we see *CNOT creates entanglement.*

Let's work through this again in matrix form.

$$|CT\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{9.15}$$

$$\text{CNOT}|CT\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{9.16}$$

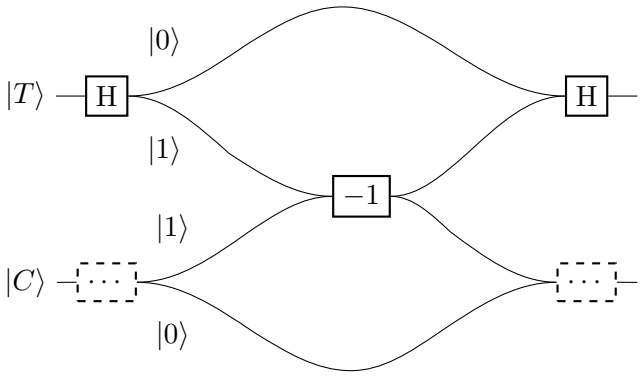
$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \approx \begin{Bmatrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{Bmatrix} \tag{9.17}$$

$$= \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) = |\Phi^+\rangle. \tag{9.18}$$

We get the same result as before.

9.1.1 How to realise CNOT?

We consider a conditional Ramsey interferometer:



If the Control is in state $|1\rangle$, then it applies a -1 factor to the state vector and this -1 factor flips the output of the Ramsey interferometer, i.e. perform the NOT.

9.2 Implementing Quantum Computing in real physical systems

We have a choice of our five possibilities:

1. Atoms
2. Ions
3. Semiconductors
4. Superconductors
5. Photons

We are going to use atoms from here, but the question is how to satisfy the 5 DiVincenzo criteria:

1. Initialisation: prepare qubits
2. Gates: single-qubit rotations and two-qubit gates (CNOT)
3. Read-out.
4. Low decoherence.
5. Scalability.

Lecture 10 Rydberg Atom Quantum Computer

1. Preparation: laser cooling (Nobel Prize 1997), and trapping with optical tweezers (Nobel Prize 2018) are used to assemble the atoms into our qubit states, all into state $|0\rangle$ for example to begin with.
2. Gates: qubit gates created using Rydberg states and dipole-dipole interactions.

Strike Impacts:

- ➤ Thursday Week 16 and possibly workshop Week 16
- ➤ Tuesday Week 17 and possibly workshop Week 17
- ➤ All lectures Week 18 and 19

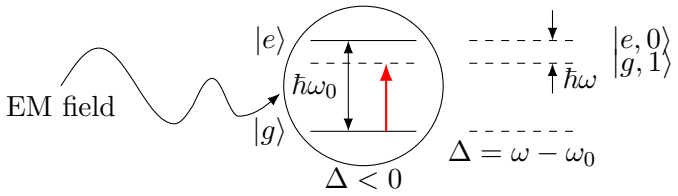
These are the possible impacts of striking, if Charles strikes. We need to believe in the many worlds interpretation of quantum mechanics and only by coming to the lectures and collapsing the wavefunction will we find out if we are in a universe where he is striking or not. The question of what universe Charles is in has been one considered by some of the greatest thinkers of modern times; the results are inconclusive.

10.1 Optical Tweezing

We use one focussed laser beam to move an atom and place it exactly where we want it. Light exerts a force on atoms and they are forced either to the ends of the beam or the focus of it. Two types of light force:

- ➤ Optical dipole force - cycles of absorption (stimulated emission and trapping)
- ➤ Spontaneous scattering force - cycles of absorption (spontaneous emission and cooling)

For our optical dipole potential, we recall the two state model, with $\hbar\omega_0$ between $|g\rangle$ and $|e\rangle$, leading to the detuning frequency Δ.



We can form our interaction operator as we have done previously when looking at the Rabi model, yielding

$$\mathcal{H}_{int} = \frac{\hbar}{2}\begin{pmatrix} \Delta & \Omega \\ \Omega & -\Delta \end{pmatrix}. \tag{10.1}$$

We do not consider between $|g\rangle$ and $|e\rangle$, but $|g,1\rangle$ (the ground state plus a photon) and $|e,0\rangle$ (the excited state with no photon). These states are only slightly separated by $\hbar(\omega_0 - \omega)$, i.e. $-\hbar\Delta$, from which we can see we have chosen negative detuning where $\Delta < 0$. This gives our bare states when $\Omega = 0$, and then we see that increasing Ω *dresses* the states, increasing the upper state and decreasing the lower state.

$$\mathcal{H}_{int} = \frac{\hbar}{2}\begin{pmatrix} \Delta & \Omega \\ \Omega & -\Delta \end{pmatrix}. \tag{10.2}$$

The energies of the states in the laser field are given by the eigenvalues of $\mathcal{H}_{int}$:

$$\left(\mathcal{H}_{int} - \frac{\hbar}{2}\lambda\hat{\sigma}_0\right)\begin{pmatrix} a \\ b \end{pmatrix} = 0 \tag{10.3}$$

$$\frac{\hbar}{2}\begin{vmatrix}\Delta-\lambda & \Omega \\ \Omega & -\Delta-\lambda\end{vmatrix}=0 \tag{10.4}$$

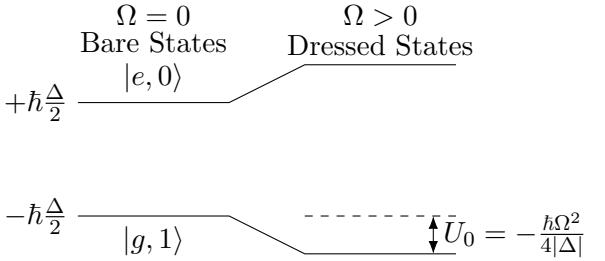
$$-\Delta^2+\lambda^2-\Omega^2=0 \tag{10.5}$$

$$\lambda^2=\Delta^2+\Omega^2 \tag{10.6}$$

$$\lambda=\pm\sqrt{\Delta^2+\Omega^2} \tag{10.7}$$

$$E_\pm=\pm\frac{\hbar}{2}\sqrt{\Delta^2+\Omega^2}, \tag{10.8}$$

where $E_\pm$ are our eigenenergies. We choose $\Delta<0$ (red detuning), and $|\Delta|\gg\omega$ (far detuned to avoid spontaneous emission). We then get



$$E_\pm=\pm\frac{\hbar}{2}|\Delta|\sqrt{1+\frac{\Omega^2}{|\Delta|^2}} \tag{10.9}$$

$$=\pm\frac{\hbar}{2}|\Delta|\left(1+\frac{1}{2}\frac{\Omega^2}{|\Delta|^2}\right) \tag{10.10}$$

$$=\pm\frac{\hbar}{2}|\Delta|\pm\frac{\hbar\Omega^2}{4|\Delta|}. \tag{10.11}$$

The shift of the lower state,

$$U_0=-\frac{\hbar\Omega^2}{4|\Delta|}, \tag{10.12}$$

is known as the **Light Shift** (or ac-Stark shift). This tells use the depth of the optical tweezer indent(handwriting?). We want to rewrite this in terms of laser intensity. Recall that

$$\Omega=-\frac{d\mathcal{E}}{\hbar}, \qquad I=\frac{1}{2}c\epsilon_0\mathcal{E}^2. \tag{10.13}$$

The dipole moment d is related to the spontaneous emission rate of the excited state $|e\rangle$, Γ, by

$$\Gamma=\frac{d^2}{3\pi\epsilon_0(\lambda_0/2\pi)^3}, \tag{10.14}$$

where λ_0 is the wavelength of the transition $|g\rangle\to|e\rangle$, $\omega_0=\frac{2\pi c}{\lambda_0}$. Using these,

$$\Omega=\Gamma\sqrt{\frac{I}{2I_s}},\ \text{where } I_s=\frac{\pi}{3}\frac{hc\Gamma}{\lambda_0^3} \tag{10.15}$$

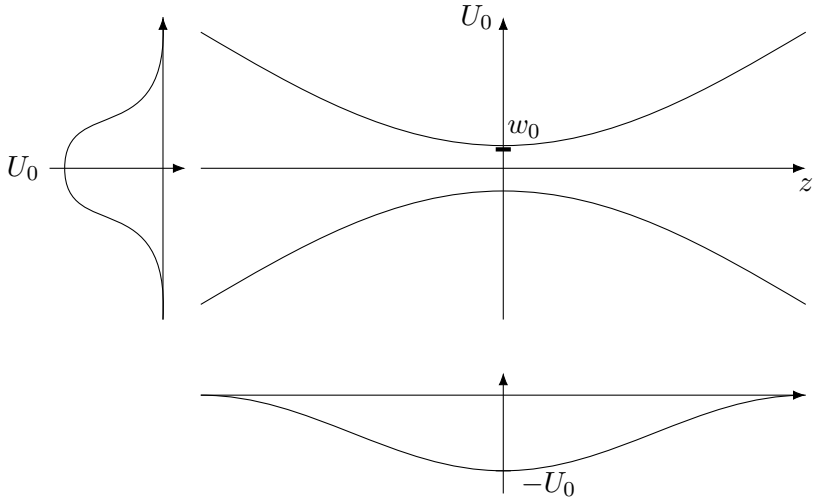
is the saturation intensity. Our optical tweezer trap depth is now

$$U_0=-\frac{\hbar\Omega^2}{4|\Delta|}=-\frac{\hbar\Gamma^2}{4|\Delta|}\frac{I}{2I_s}. \tag{10.16}$$

The trap depth U_0 is proportional to the laser intensity and inversely proportional to detuning, $|\Delta|$. We set the limit of large detuning to avoid spontaneous emission, but this impacts the trap depth. To get this back, however, we use a large laser intensity.
A laser beam is described by

$$I(x,y,z)=I_0\frac{w_0^2}{w^2}e^{-2(x^2+y^2)/w^2}, \qquad w=w_0\left(1+\frac{z^2}{z_R^2}\right)^{1/2}, \qquad z_R=\frac{\pi w_0^2}{\lambda}. \tag{10.17}$$

We can therefore plot what the potential U_0 looks like as $U_0(x, y, z)$:



The atom is now trapped at the focus in all 3 dimensions, giving us an optical tweezer. Consider

$$\int_{-\infty}^{\infty} I(z=0)\,dx\,dy = P \implies I_0 = \frac{2P}{\pi w_0^2}, \tag{10.18}$$

where P is the laser power. This tells us that we want to have as powerful a laser as possible, with a smaller beam waist size. However, we cannot focus the beam waist arbitrarily small because diffraction and other effects will take hold. Typically, $w_{0,min} \approx \lambda$.

Lecture 11 Optical Tweezing

Example: Workshop 2, Question 2: Ramsey Interferometry and Decoherence
The Hadamard is how we read out the 'phase' of the state vector. Why is that? We take a general state (as in part c of question),

$$|\psi\rangle = \begin{pmatrix} ae^{-i\Delta t/2} \\ be^{i\Delta t/2} \end{pmatrix}. \tag{11.1}$$

Then we must take the density matrix for this state - where it is a pure state, so it is completely coherent and has a well-defined state. To describe decoherence, which is due to interaction with the environment, we use the density matrix description because it allows for mixed states, i.e. states without perfectly defined phases.

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} |a|^2 & ab^* e^{-i\Delta t} \\ ba^* e^{i\Delta t} & |b|^2 \end{pmatrix}, \tag{11.2}$$

where the diagonal terms represent the probability to be in each state, and the off-diagonal terms are called **Coherences** which contain all the information about phase.
These may contain all the *wave* information, but not necessarily all the *quantum* information. If a state decoheres completely, i.e. coherence decay to zero:

$$\rho_{dec} = \begin{pmatrix} |a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix} \tag{11.3}$$

$$= |a|^2|0\rangle\langle 0| + b^2|1\rangle\langle 1|, \tag{11.4}$$

so we still have a superposition, but no phase information. This is called a **mixed state.**

For the read-out process, we go back to earlier on in the question now. We define a general density matrix as

$$\rho = \frac{1}{2}\begin{pmatrix} 1+w & u-iv \\ u+iv & 1-w \end{pmatrix}. \tag{11.5}$$

We now use the Hadamard to modify the density matrix, yielding

$$\rho' = \hat{H}\rho\hat{H}^\dagger = \frac{1}{2}\begin{pmatrix} 1+u & w+iv \\ w-iv & 1-u \end{pmatrix}. \tag{11.6}$$

The probability to be in $|0\rangle$ is then

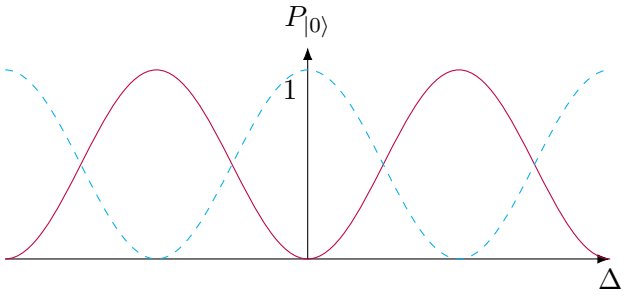
$$P_{|0\rangle} = \frac{1}{2}(1+u), \tag{11.7}$$

where we can now read out the real part of coherence. The Hadamard is used to measure the coherence of the qubit, by projecting the real part of the coherence u from the off-diagonals to the diagonals.

Now let's jump down to part f of the question because Charles says so. We transform $u \to ue^{-it/T_2}$, where T_2 is some time constant representing how the coherence decays. The probability will be modified by this, showing a decaying and oscillating amplitude:

$$P_{|0\rangle} = \frac{1}{2}(1+ue^{-t/T_2}), \quad u = \cos(\Delta t) \tag{11.8}$$

$$= \frac{1}{2}(1+\cos(\Delta t)e^{-t/T_2}). \tag{11.9}$$



These are the Ramsey fringes for $t \ll T_2$. Pure states are good because we have no decoherence, so quantum computing will work. Over time, the oscillations can get smaller, begin to vibrate around $\frac{1}{2}$, so the Ramsey fringe visibility is reduced, leading to decoherence. Working at low temperatures, i.e. mK, reduces likelihood of decoherence.

11.1 Optical Tweezing Ctd.

We used a laser to trap an atom qubit in its beam, with

$$U_0 = -\frac{\hbar\Omega}{4|\Delta|}, \qquad \Omega = \Gamma\sqrt{\frac{I}{2I_s}}. \tag{11.10}$$

The trap depth is proportional to the laser intensity (and power):

$$I = I_0 \frac{w_0^2}{w^2} e^{-2(x^2+y^2)/w_0^2}. \tag{11.11}$$

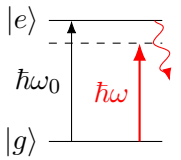
doodle The trap depth forms a harmonic potential, $\frac{1}{2}m\omega_z z^2$. This means we want to cool the atom to its Harmonic Oscillator ground state. *doodle* This is the minimum size of the atomic wave packet. We then send in our laser beam with an electric field,

$$\mathcal{E} = \mathcal{E}_0 \cos(\phi_L - \omega t), \quad \phi_L = \underline{k} \cdot \underline{r}. \tag{11.12}$$

There is now an error in the phase,

$$\Delta\phi_L = k\Delta x, \tag{11.13}$$

if the field propagates in the x-direction (switch out x for y or z depending on propagation direction). This error is a source of decoherence, so we need to try and minimise it. However, the scaling is very bad, on the order of needing 10^4 more laser power to decrease Δx by a factor of 10.



$$\Delta = \omega - \omega_0 < 0, \tag{11.14}$$

$$|\Delta| \gg \Omega. \tag{11.15}$$

We need $|\Delta|$ large to avoid the decay of our excited state $|e\rangle$. Calculating the probability that qubit scatters a photon, we have

- spontaneous decay rate of $|e\rangle$ is Γ
- the photon scattering rate is $R = \Gamma P_{|e\rangle}$
- $P_{|e\rangle}$ is the probability to be in state $|e\rangle$; for a two-level model, this is $|b|^2$

This model, however, did not include the possibility of decay, so we need to add decay to our equations of motion of the Bloch vector. From Workshop 1,

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} 0 & -\Delta & 0 \\ \Delta & 0 & -\Omega \\ 0 & \Omega & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}. \tag{11.16}$$

Where the RHS is Torque×Bloch vector. We now have

$$\dot{u} = -\frac{u}{T_2} - \Delta v, \tag{11.17}$$

$$\dot{v} = \Delta u - \frac{v}{T_2} - \Omega w, \tag{11.18}$$

$$\dot{w} = \Omega v - \frac{1}{T_1}(w - 1). \tag{11.19}$$

T_2 is the decay of coherence; T_1 is the decay of population. These are the **Optical Bloch Equations.**

Lecture 12 Spontaneous Photon Scattering

In our optical tweezer model described above, not all photons going into the qubit will hit the same way, so sometimes we will have scattering of photons off the photon. Last time we discussed spontaneous emission from the excited state, Γ, where the rate of spontaneous scattering can then be described as

$$R = \Gamma P_{|e\rangle}. \tag{12.1}$$

12.1 Optical Bloch Equations

If spontaneous emission is the only decoherence/decay mechanism, then

$$T_1 = \frac{1}{\Gamma}, \qquad T_2 = \frac{2}{\Gamma}, \tag{12.2}$$

$$\frac{1}{T_1} = \Gamma = \frac{1}{\tau}, \qquad \frac{1}{T_2} = \frac{\Gamma}{2}, \tag{12.3}$$

where τ is the lifetime of the excited state. We can then substitute these into the optical Bloch equations defined last lecture, yielding

$$\dot{u} = -\frac{\Gamma}{2}u - \Delta v, \tag{12.4}$$

$$\dot{v} = \Delta u - \frac{\Gamma}{2}v - \Omega w, \tag{12.5}$$

$$\dot{w} = \Omega v - \Gamma(w - 1). \tag{12.6}$$

From here, we solve for the steady state, $\dot{u} = \dot{v} = \dot{w} = 0$. Following lots of algebra, we find

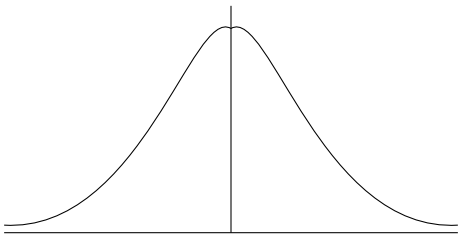
$$P_{|e\rangle} = \frac{1}{2}(1 - w_{ss}) = \frac{\Omega^2/4}{\Delta^2 + \Gamma^2/4 + \Omega^2/2}. \tag{12.7}$$

This is one of our Very Important Equations. We want to plot this for three cases:

1. Small Ω: $\Omega \ll |\Delta|$ and Γ. (NB: not the case for optical tweezer!)

$$P_{|e\rangle} \approx \frac{\Omega^2/4}{\Delta^2 + \Gamma^2/4}. \tag{12.8}$$

This corresponds to weak driving, where it will result in low intensity and low probability. This equation will look familiar as a Lorentzian, or a Cauchy distribution.



Our Full Width Half Maximum is then

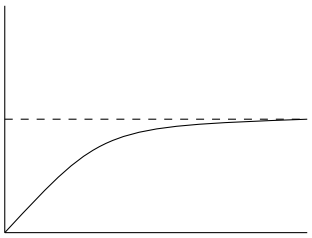
$$\text{FWHM} = \Gamma = \frac{1}{T_1} = \frac{1}{\tau}, \tag{12.9}$$

which is the natural width of $|g\rangle \to |e\rangle$ transition.

2. $\Delta = 0$, i.e. on resonance. Again, not like our tweezer where we are very far from resonance. Now, we have a probability of

$$R_{|e\rangle} = \frac{\Omega^2/4}{\Gamma^2/4 + \Omega^2/2}. \tag{12.10}$$

We can then plot the probability, now against Ω^2:



$$\frac{\Omega^2}{2} = \frac{\Gamma^2}{4} \implies P_{|e\rangle} = 0.25 \tag{12.11}$$

This is linear for $\Omega < \Gamma$. Then

$$\Omega^2 = \Gamma^2 \frac{I}{2I_S}, \qquad P_{|e\rangle} = 0.25 \text{ for } I = I_S, \tag{12.12}$$

where I_S is the saturation intensity.

3. $|\Delta| \gg \Gamma, \Omega$, i.e. the case for optical tweezing. $\Omega \gg \Gamma$ also. We have strong driving and strong detuning.

$$P_{|e\rangle} = \frac{\Omega^2/4}{\Delta^2} = \frac{\Omega^2}{4\Delta^2}. \tag{12.13}$$

Therefore, the photon scattering rate in an optical tweezer is

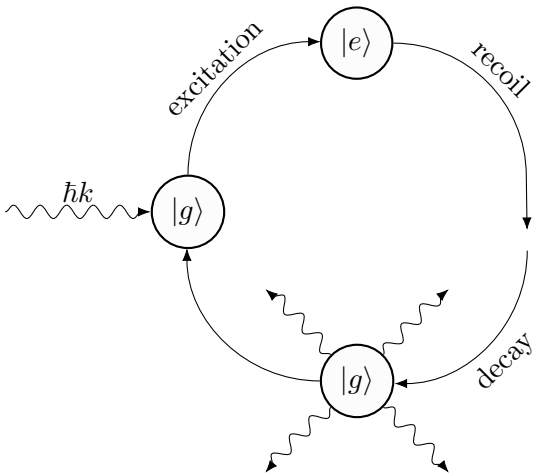
$$R = \Gamma P_{|e\rangle} = \Gamma \frac{\Omega^2}{4\Delta^2}. \tag{12.14}$$

Now to the trap depth,

$$U_0 = -\frac{\hbar\Omega^2}{4|\Delta|} \implies \frac{|U_0|}{R} = \frac{\hbar\Omega^2/4|\Delta|}{\Gamma\Omega^2/4\Delta^2} = \hbar\left(\frac{\Delta}{\Gamma}\right). \tag{12.15}$$

So we want to use $\Delta \gg \Gamma$ to minimise scattering for a given trap depth, i.e. *far detuning.*

We can use spontaneous scattering for cooling! Laser cooling works through a spontaneous scattering force. Each scattering event cause a recoil on the qubit or atom.



Cycles of excitation and decay transfer, on average, $\hbar\underline{k}$ of momentum to the atom per cycle. This exerts a force

$$\underline{F} = \hbar\underline{k}R = \hbar\underline{k}\Gamma P_{|e\rangle} \tag{12.16}$$

$$= \hbar\underline{k}\Gamma\frac{\Omega^2/4}{\Delta^2 + \Gamma^2/4 + \Omega^2/2}. \tag{12.17}$$

This force is called the **Spontaneous Scattering Force.** The properties of this force will then follow the previous discussions on the behaviour of the probability in different regimes.

How will we use this force for cooling? Use 2 counter-propagating lasers, both red detuned. *doodle* The force on the atom is then the sum of the force from laser 1 and laser 2:

$$F = F_+ + F_-. \tag{12.18}$$

If the atom is moving, however, there is a Doppler shift.

- ➤ Laser 1: it's moving away, so detuning, $\Delta \to \Delta - kv_{atom}$.
- ➤ Laser 2: it's moving towards, so detuning, $\Delta \to \Delta + kv_{atom}$.

$$F_\pm = \pm\hbar\underline{k}\Gamma\frac{\Omega^2/4}{\Gamma^2/4 + \Omega^2/2 + (\Delta \mp kv_{atom})^2}. \tag{12.19}$$

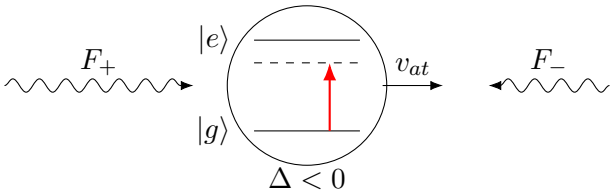
Lecture 13 Laser Cooling

We have cycles of absorption and spontaneous emission which give rise to the spontaneous force,

$$F = \hbar \underline{k} R, \text{ where } R = \Gamma P_{|e\rangle} = \frac{\Gamma \Omega^2/4}{\Omega^2/2 + \Gamma^2/4 + \Delta^2}. \tag{13.1}$$

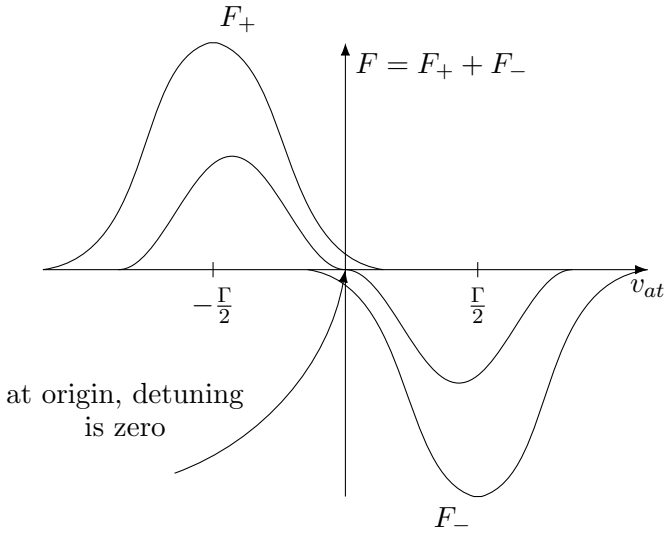
Note that $R_{max} = \frac{\Gamma}{2}$, for $\Omega \gg |\Delta|, \Gamma$.

Laser cooling, counter-propagating red detuned ($\Delta < 0$) laser beams.



$$F_\pm = \pm \frac{\Gamma}{2} \frac{I/I_r}{1 + 4(\Delta \mp kv)^2/\Gamma^2}, \text{ where } \Omega^2 = \Gamma^2 \frac{I}{2I_s}. \tag{13.2}$$

If $I < I_s$:



$$\text{for} |v_{at}| < \frac{\Gamma}{k}, \quad \underline{F} = -\alpha \underline{v}_{at} \tag{13.3}$$

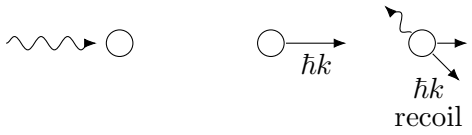
Defining the atomic momentum as p, we can use $F = \frac{dp}{dt} = -\alpha \frac{p}{m}$ to find the rate of change of kinetic energy:

$$\left(\frac{dE}{dt}\right) = \frac{d}{dt}\left(\frac{p^2}{2m}\right) = \frac{p}{m}\frac{dp}{dt} = -\alpha \left(\frac{p}{m}\right)^2, \tag{13.4}$$

where the negative means the rate is decreasing. We now have laser cooling!

13.1 How cold can we cool?

There is also a heating due to random recoils due to spontaneous emission.



➤ Heating Rate:

$$\left(\frac{dE}{dt}\right)_{heat} = \frac{d}{dt}\left(\frac{p^2}{2m}\right). \tag{13.5}$$

➤ Equilibrium temp (minimum temperature occurs at $\Delta = \frac{\Gamma}{2}$:

$$\left(\frac{dE}{dt}\right)_{cool} = -\left(\frac{dE}{dt}\right)_{heat}, \tag{13.6}$$

$$k_B T_{dopp} = m v_{at}^2 = \frac{\hbar\Gamma}{2}. \tag{13.7}$$

This is the Doppler Cooling Limit. Typically, $T_{dopp} \approx 1 - 100\,\mu$K - the coldest thing in the Universe.

We can use laser cooling to load cold atoms into optical tweezers. As with before, we have cycles of absorption and spontaneous emission, but this is only found in simple atoms as we need a **Closed Transition.**

Let's focus on a particular atom, e.g. Cs (single valence electron).

➤ Ground state $|g\rangle$, 6s $(n = 6, l = 0)$
➤ Excited state $|e\rangle$, 6p $(n = 6, l = 1)$

Laser cooling on $|g\rangle \to |e\rangle$ transition.

Cs atom has nuclear spin $I = \frac{7}{2}$ and electron spin $s = \pm\frac{1}{2}$. The total 'spin' (atomic angular momentum) is $F = I + S = 3, 4$. For each F, we have $2F + 1$ magnetic sub-levels labelled m_F. So the ground state has 16-fold degeneracy. We select 2 of these 16 as our qubit states $|0\rangle$ and $|1\rangle$:

$$|F = 3, m_F = 3\rangle = |0\rangle,$$
$$|F = 4, m_F = 4\rangle = |1\rangle.$$

As both are ground states, they do not decay (not spontaneous emission), i.e. low decoherence **(DiVincenzo #4).**

Lecture 14 Optical Pumping and Raman Transitions

Recall Cs in the ground state $F = 4, F = 3$ where we choose our qubits:

$$|F = 3, m_F = 3\rangle = |0\rangle,$$
$$|F = 4, m_F = 4\rangle = |1\rangle.$$

This gave us the 16 ground state levels, with $h\nu_{hfs}$ energy between $|0\rangle, |1\rangle$. $F = 3, 4$ are split by the *hyperfine interaction* between the electrons and nuclear spin. The frequency of splitting is $\nu_{hfs} = 9.2\,\text{GHz}$ (microwaves) - this is the transition used in atomic clocks.

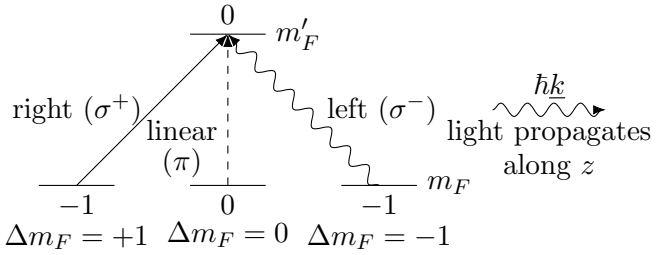
The qubit choice is not unique - we could use others, e.g. $m_F = 0$ states. The larger $|m_F|$ values are most sensitive to magnetic fields. If your computer has magnets in some way, then you might want to use $m_F = 0$ states which are not affected. Even the Earth's magnetic field is sufficient to perturb $4, 3$ states but in a lab setting we can shield against magnetic effects, i.e. compensate by making a magnetic-free environment. The final stage of preparation is to initialise the atom in one or other qubit level - our Cs atom could be in a superposition of any of the 16 states!

Initialisation is after laser cooling and trapping in an optical tweezer. We can do this by *optical pumping* the atom into either state $|0\rangle$ and $|1\rangle$, i.e. $|\psi(0)\rangle = |0\rangle$ or $|1\rangle$ - first step of DiVincenzo.

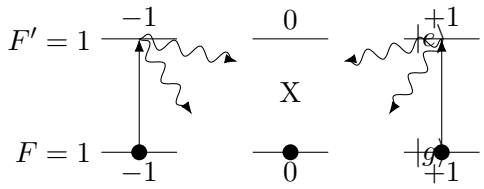
14.1 Optical Pumping

This is the transfer of atomic population using *polarisation selection rules.* If we apply light resonance with the $|g\rangle \to |e\rangle$ transition that for a particular polarisation then the atom is pumped into a particular magnetic sublevel. The selection rules for optical pumping are:

1. For excitations, we can select $\Delta m_F = -1, 0, +1$ transitions using *left-circular,linear, and right-circular* light, respectively, e.g.



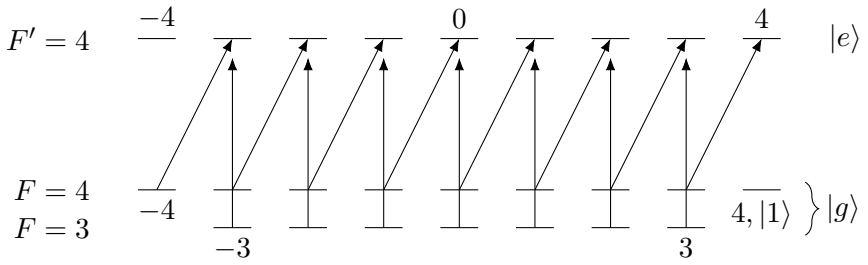
   The photon has some angular momentum which depends on the polarisation. The linear route doesn't have any angular momentum along this axis. The σ^+, π, σ^- are just labels/notation; the aren't related to the Pauli matrices.
2. In decay $|e\rangle \to |g\rangle, \Delta m_F = -1, 0, +1$ are all allowed. Which of these we observe depends on the polarisation of the emitted photon, i.e. if we detect a linearly polarised photon, we know we went $m'_F = 0 \to m'_F = 0$, and we can similarly tell for left- and right-circularly polarised light.
3. The exception to rules 1 and 2 is that $\Delta m_F = 0$ is not allowed for a transition where the initial and final total angular momentum F are the same, e.g. $F = F'$,



   The middle $0 \to 0$ transition is not allowed! The 0 is made up of a superposition of states with opposite signs, if we look at the *transition matrix element* for this particular transition then you

get zero by symmetry. We can also drive the population on either side of this which can then decay. These can decay back into the zero state so all population gets pumped into the middle state where it becomes trapped, i.e. all population pumped into $m_F = 0$ in $F = 1\,|g\rangle$ state.

Cs has $F = 3, 4$ and we want to pump into one state. How do we apply the above to Cs? Using $\Delta m_F = +1$ (right-circular):



We can see that the F' states can decay, and we cannot drive out the $F = 4, m_F = 4$ state. Using right-circular light, we can drive tendency to the right side of the population. As the excited states decay, they can fall to both $F = 4, 3$! We need to repump out of lower hyperfine state to excited. As long as we repump out of $F = 3$ state with right-circular light, all the population gets transferred to $|F = 4, m_F = +4\rangle$ state which was $|1\rangle$ in the qubit. We have initialised qubits to state $|1\rangle$.

We now need the second DiVincenzo criteria - Gates.

14.2 Gates

Single-qubit rotation, where we have a difference of 9.2 GHz$\rightarrow \lambda \approx 3$cm:

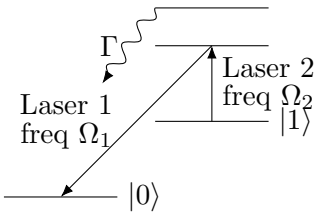
$$|F = 4, m_F = 4\rangle \rightarrow |1\rangle,$$
$$|F = 3, m_F = 3\rangle \rightarrow |0\rangle.$$

Problem! Ideally to address each qubit individually, we would like them to be separated by a distance of the order of the wavelength. The microwave wavelength is still too long. For microwaves, the qubits need to be ≈ 3cm apart; wires can be used to localise the microwave field, however, the large wavelength is an issue! To make the qubits interact, we also want them to be as close as possible. A solution - use the ground state hyperfine states, but drive transitions between them optically via an excited state $|e\rangle$.

14.3 Stimulated Raman Transitions

Such a two-photon transition is known as a *stimulated-Raman transition. Raman scattering* is any process where we scatter light but end up in a different state to where we started, e.g. absorb then emit a photon and come back down to a different state. This is opposed to *Rayleigh scattering* when you absorb and emit the photon returning to the original state. Laser cooling uses Rayleigh scattering as we always return to the same state; now we want Raman scattering.

Stimulated Raman transitions combine single-qubit addressability with low decoherence due to spontaneous emission. Going via our excited state to get from $|0\rangle \rightarrow |1\rangle$:



finish doodle $|e\rangle$ can decay by spontaneous emission but we don't want this as we instead want to do the calculation. Any emission of a photon will destroy it.

A resonant stimulated Raman transition is driven by how laser beams both detuned by Δ from the $|0\rangle \to |e\rangle$ and $|1\rangle \to |e\rangle$ transitions, where $|0\rangle$ and $|1\rangle$ are typically the lower and upper ground state hyperfine states. We choose large detuning Δ to avoid spontaneous emission similar to an optical tweezer. For $|\Delta| \gg \Omega_1, \Omega_2$, we can derive that the effective Rabi frequency from $|0\rangle \to |1\rangle$ is

$$\Omega_{eff} = \frac{\Omega_1 \Omega_2}{2|\Delta|}. \tag{14.1}$$

Note that since this is proportional to the individual Rabi frequencies, this is also proportional to the power of the laser beams. The spontaneous scattering rate from $|e\rangle$ for large $|\Delta|$ is

$$R = \frac{\Omega_{1,2}^2}{4|\Delta|^2}\Gamma. \tag{14.2}$$

For $\Omega_1 = \Omega_2$,

$$\Omega_{eff} = \frac{\Omega^2}{2|\Delta|} \implies R = \frac{\Omega^2}{4|\Delta|^2}\Gamma. \tag{14.3}$$

We can then work out the probability of a scattering event per Rabi angle. The average number of spontaneous emission events per Rabi angle is given by the Rabi period divided by the average time between spontaneous emissions, or inversely, the spontaneous emissions rate divided by the effective Rabi frequency, i.e.

$$N_{spont} = \frac{R}{\Omega_{eff}} = \frac{(\Omega^2/4|\Delta|^2)\Gamma}{\Omega^2/2|\Delta|} = \frac{\Gamma}{2\Delta}, \tag{14.4}$$

when $\Omega_1 \approx \Omega_2$. This shows we can reduce spontaneous emission by increasing $\Delta \to$ achieve low decoherence single qubit operations. In a typical experiment, we might have

$$\Gamma = 2\pi(6\,\text{MHz}),\ \Delta = 2\pi(80\,\text{GHz}) \implies N_{spont} \approx 10^{-4}. \tag{14.5}$$

This wouldn't limit us in quantum computation. However, the large detuning also reduces the *effective Rabi frequency* and hence the gate speed - there is still a trade-off between speed and decoherence.

Lecture 15 Rydberg Atoms

Let's look at the DiVincenzo criteria for alkali-atom QC!

1. Initialisation
   (a) Optical tweezer traps ✓
   (b) Laser cooling ✓
   (c) Optical pumping to initialise qubit state ($|0\rangle$ or $|1\rangle$) ✓
2. Gates
   (a) Single qubit gates ✓
   (b) Stimulated Raman transition ✓
   (c) Two-qubit gates ✘
3. Read-out
   (a) Fluorescence on $|g\rangle \to |e\rangle$ transition ✓ *doodle*
4. Low decoherence
   (a) $|0\rangle$ and $|1\rangle$ are both 'ground' state, i.e. hyperfine states with the $|g\rangle$ manifold ✓
5. Scaling
   (a) Optical tweezer array ✓

The only thing we have left to do is the two-qubit gates, i.e. we need interacting qubits.

15.1 Two-Qubit Gates

Atoms in states $|g\rangle$ and $|e\rangle$ do not interact unless they are very close ($\ll 100\,\text{nm}$ apart), but we will create our atom qubit array with spacings of order $5 - 10\,\mu\text{m}$. We need some way of introducing interactions at these scales.

To create interaction, we excite to **Highly-Excited Rydberg States,** i.e. states with principal quantum number $n \gtrapprox 50$.

15.2 Rydberg Scalings

If we consider the Bohr model, and take a state with principal quantum number n, then

$$mv_n r_n = n\hbar. \tag{15.1}$$

Equating the centripetal force with the Coulomb force, we find

$$r_n = n^2 a_0. \tag{15.2}$$

This tells us that the size of the electronic wavefunction is $\propto n^2$; for $n \approx 100$, the atom is $> 1\,\mu\text{m}$ across. The dipole D is also $\propto n^2$. The dipole-dipole $\propto D^2 \propto n^4$, so using high n we can induce strong interactions.

Using the Bohr model and virial theorem, the energy is $\frac{1}{2}$ the potential energy:

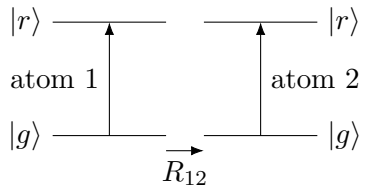
$$E_n = -\frac{e^2}{2(4\pi\epsilon_0)r_n} = -\frac{R_H}{n^2}, \tag{15.3}$$

where R_H is the Rydberg constant for Hydrogen. For Cs,

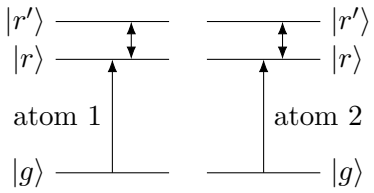
$$E_n = -\frac{R_{Cs}}{n^*}, \quad n^* = n - \delta, \tag{15.4}$$

where n^* is the reduced principal quantum number offset from n by δ.

15.3 Rydberg Interactions



finish doodle Exciting two atoms into states $|r\rangle$ allows them to strongly via dipole-dipole interactions (DDI). To calculate the DDI, we need to consider couplings to other Rydberg states, e.g. $|r'\rangle$.



The DDI mixes some of $|r'\rangle$ into the state $|r\rangle$ which gives rise to the Van Der Waals energy shift. This can be written as a matrix

$$\hat{\mathcal{H}} = \hbar \begin{pmatrix} 0 & 0 & 0 & V \\ 0 & \omega_{rr'} & V & 0 \\ 0 & V & \omega_{rr'} & 0 \\ V & 0 & 0 & 2\omega_{rr'} \end{pmatrix}, \tag{15.5}$$

where $\hbar\omega_{rr'}$ is the energy splitting between r and r', and

$$V = \frac{D^2/\hbar}{4\pi\epsilon_0 R_{12}^3}. \tag{15.6}$$

The basis for $\hat{\mathcal{H}}_{DDI}$ is $|rr\rangle, |rr'\rangle, |r'r\rangle, |r'r'\rangle$. Consider the sub-bases $|rr\rangle, |r'r'\rangle$:

$$\hat{\mathcal{H}} = \hbar \begin{pmatrix} 0 & V \\ V & 2\omega_{rr'} \end{pmatrix}. \tag{15.7}$$

This should be familiar from previous examples. This is analogous to the light shift $\frac{\Omega^2}{4\Delta}$. *doodle* This is the Van Der Waals Interaction.

$$V \propto \frac{1}{R_{12}^3}, \quad V_{vdW} \propto \frac{1}{R_{12}^6} \tag{15.8}$$

$$V \propto D^2, \quad D \propto n^2 \tag{15.9}$$

$$V \propto n^4 \tag{15.10}$$

$$E_n = -\frac{R_{Cs}}{n^2}, \quad E_{n+1} - N \propto \frac{1}{n^3},\ n \gg 1, \tag{15.11}$$

$$V_{vdW} \propto \frac{V^2}{\omega_{rr'}} \propto \frac{n^8}{n^{-3}} \propto n^{11}. \tag{15.12}$$

So van der Waals has n^{11} scaling! Tunability is strength of interactions.

Lecture 16 Rydberg Blockade and Entanglement

doodle Excitation linewidth $|g\rangle \to |r\rangle$ for $\Omega \gg \Gamma$ is of order Ω. If $V_{vdW} > \Omega$ (linewidth), the excitation is blocked. This is called **Rydberg Blockade.**
This occurs at distance R_{12}, where

$$V_{vdW} = \frac{C_6}{R_{12}^6} = \Omega \tag{16.1}$$

$$R_{12} = \left(\frac{C_6}{\Omega}\right)^{1/6} = R_b, \tag{16.2}$$

where R_b is the blockade radius, $R_b \approx 10\mu$m which is relatively large! *doodle* We cannot excite the second Rydberg atom within this sphere.

Now we consider starting with both atoms in the ground state, and drive the $|g\rangle \to |r\rangle$ transition on resonance. *doodle* We have two two-level systems, giving a four state basis: $|gg\rangle, |gr\rangle, |rg\rangle, |rr\rangle$. Recall with one atom and $\Delta = 0$, our interaction Hamiltonian was given as

$$\hat{\mathcal{H}}_{int} = \frac{\hbar}{2}\begin{pmatrix} 0 & \Omega \\ \Omega & 0 \end{pmatrix}. \tag{16.3}$$

Now for two atoms, our interaction Hamiltonian reads

$$\mathcal{H} = \mathcal{H}_{int} \otimes \sigma_0 + \sigma_0 \otimes \mathcal{H}_{int} \tag{16.4}$$

$$= \begin{pmatrix} 0 & \Omega \\ \Omega & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & \Omega \\ \Omega & 0 \end{pmatrix} \tag{16.5}$$

$$= \begin{pmatrix} 0 & 0 & \Omega & 0 \\ 0 & 0 & 0 & \Omega \\ \Omega & 0 & 0 & 0 \\ 0 & \Omega & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \Omega & 0 & 0 \\ \Omega & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega \\ 0 & 0 & \Omega & 0 \end{pmatrix} \tag{16.6}$$

$$= \begin{pmatrix} 0 & \Omega & \Omega & 0 \\ \Omega & 0 & 0 & \Omega \\ \Omega & 0 & 0 & \Omega \\ 0 & \Omega & \Omega & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & V_{vdW} \end{pmatrix} \tag{16.7}$$

$$\mathcal{H}_{|g\rangle\to|r\rangle} = \frac{\hbar}{2}\begin{pmatrix} 0 & \Omega & \Omega & 0 \\ \Omega & 0 & 0 & \Omega \\ \Omega & 0 & 0 & \Omega \\ 0 & \Omega & \Omega & 2V_{vdW} \end{pmatrix} \tag{16.8}$$

This is valid for $R_{12} < R_b, V_{vdW} \gg \Omega$ ($\frac{\hbar}{2}$ should be throughout it, but he missed it out and I'm lazy). The $|rr\rangle$ state is completely decoupled by the vdW shift, so we can neglect this and turn it into a 3 state basis as

$$\mathcal{H}_{|g\rangle\to|r\rangle} = \frac{\hbar}{2}\begin{pmatrix} 0 & \Omega & \Omega \\ \Omega & 0 & 0 \\ \Omega & 0 & 0 \end{pmatrix}. \tag{16.9}$$

We have the eigenvectors,

$$|gg\rangle, \quad \frac{1}{\sqrt{2}}\left(|gr\rangle + |rg\rangle\right), \quad \frac{1}{\sqrt{2}}\left(|gr\rangle - |rg\rangle\right).$$

If we then rewrite the interaction in the basis of these states, we get

$$\mathcal{H} = \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{2}\Omega & 0 \\ \sqrt{2}\Omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{16.10}$$

where we can now see that $|gg\rangle$ couples to $\frac{1}{\sqrt{2}}(|gr\rangle + |rg\rangle)$ with Rabi frequency $\sqrt{2}\Omega$, but it does not couple to $\frac{1}{\sqrt{2}}(|gr\rangle - |rg\rangle)$ at all. *doodle* The effect of the Rydberg blockade is that we excite an entangled Bell state where only one atom is in the Rydberg state but we do not know (and cannot know) which one.

Lecture 17 Rydberg Interactions

17.1 Review of last week

We have two atoms: *doodle* We have both our qubit states $|0\rangle, |1\rangle$ contained as part of hyperfine splitting in a initial state $|g\rangle$. In order to create entanglement, we excite them to the highly-excited Rydberg state, $|r\rangle, n > 50$. In $|r\rangle$, atoms interact strongly via dipole-dipole interactions (DDI) that couple to other states, e.g. $|r'\rangle$. Due to interactions, the energy levels of these states move. If we have a distance between two atoms $R_{12} < R_b$ (R_b the blockade radius), then excitation to $|r, r\rangle$ is blocked due to Van der Waals shift. Instead, we excited either or of

$$\frac{1}{\sqrt{2}}\left(|gr\rangle + |rg\rangle\right), \quad \frac{1}{\sqrt{2}}\left(|0r\rangle + |r0\rangle\right).$$

This will oscillate between $|00\rangle$ and $\frac{1}{\sqrt{2}}\left(|0r\rangle + |r0\rangle\right)$ with frequency $\sqrt{2}\Omega$. Alternatively: *doodle* We excite $|r, r'\rangle$, and we find that this couples only to $|r', r\rangle$ and our excitation energy oscillates back and forth with a frequency

$$V = \frac{D^2/\hbar^2}{4\pi\epsilon_0 R_{12}^2}. \tag{17.1}$$

This is called a flip flop interaction. Apparently. For example, if we excite one atom into the s series and the other into the corresponding p series, the coupled state can oscillate as one falls to p from s and the other rises to s from p then vice versa again.

17.2 The Start Of The Lecture, Time Stamped 15:35

The Werner state,

$$|W\rangle = \frac{1}{\sqrt{2}}\left(|0r\rangle + |r0\rangle\right). \tag{17.2}$$

With what fidelity have we created the state $|W\rangle$ experimentally? We need to create the density matrix for this state,

$$\rho_W = |W\rangle\langle W| = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{17.3}$$

But if we measure the experimental value of the density, we have

$$\rho_{expt} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{17.4}$$

This is an approximation, where the zero elements could in fact be very small numbers. Then, we define **Fidelity** as

$$\mathcal{F} = \langle W|\rho_{expt}|W\rangle. \tag{17.5}$$

To measure the components of the density matrix, we need to do perform a rotation on the state before measurement, allowing us access to off-diagonal elements. This is known as **Quantum State Tomography.** What we want to do is perturb one part of the superposition, thereby adding a phase.

doodle We want to shine on $|0\rangle \to |e\rangle$ transition on atom 1, $|1\rangle$, such that $|0\rangle$ on atom 1 accumulates a phase $e^{-i\phi}$ and state

$$|W'\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\phi}|0r\rangle + |r0\rangle\right). \tag{17.6}$$

Apply π pulse on $|W\rangle \to |0\rangle$ transition. This is given by the matrix

$$\hat{R} = \begin{pmatrix} 0 & \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{i}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{Bmatrix} |00\rangle \\ |0r\rangle \\ |r0\rangle \\ |rr\rangle \end{Bmatrix} \tag{17.7}$$

$$\hat{r}_{W'}\hat{R}^\dagger = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}}\left(1+e^{-i\phi}\right) & \frac{1}{2}(1-e^{-i\phi}) & -\frac{1}{2}(1-e^{-i\phi}) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{17.8}$$

doodle - Write extra line from notes. P_{00}, the probability to be in $|00\rangle$ is

$$P_{|00\rangle} = \frac{1}{2}(1+\cos\phi). \tag{17.9}$$

doodle THE SAME PLOT YET AGAIN Application of light to one changes the observation on the other - this only applies within the blockade sphere, $R < R_b$.

Lecture 18 Rydberg CNOT Gates

lots of doodles We want to realise a CNOT gate using Rydberg blockade.

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \approx \begin{Bmatrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{Bmatrix}. \tag{18.1}$$

We start is $|10\rangle$ (last column), and

$$\text{CNOT}|10\rangle = |11\rangle. \tag{18.2}$$

doodle

$$\text{CZ} = \begin{pmatrix} 1 & 0 & 0 &) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{18.3}$$

This is what we call a Conditional Ramsey Interferometer. *doodle*

1. $\frac{\pi}{2}$ pulse (or Hadamard) on $|0\rangle \to |1\rangle$ on target.

$$\hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{18.4}$$

   We want to write this in the $|CT\rangle$ basis. $\hat{H}$ acts on $|T\rangle$ only:

$$\hat{H}_{|CT\rangle} = \hat{\sigma}_0 \otimes \hat{H} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{18.5}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}. \tag{18.6}$$

   This is also for 5.

2. π pulse $|0\rangle \to |1\rangle$ on control.

$$U_{2,4} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \approx \begin{Bmatrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{Bmatrix}. \tag{18.7}$$

3. 2π pulse on $|0\rangle \to |1\rangle$ on target. It only happens if target is in $|0\rangle$ and control is in $|1\rangle$. If control were in $|0\rangle$, then it goes to $|r\rangle$ and blocks excitation of a 2nd Rydberg.

$$U_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \approx \begin{Bmatrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{Bmatrix}. \tag{18.8}$$

4. π pulse on $|0\rangle \to |1\rangle$ on control.
5. $\frac{\pi}{2}$ pulse (or Hadamard) on $|0\rangle \to |1\rangle$ on target.

Let's start putting this all together then:

$$U_2U_3U_4 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{18.9}$$

$$= -CZ = - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{18.10}$$

$$\hat{H}U_2U_3U_4\hat{H} = - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{18.11}$$

$$= -\text{CNOT}. \tag{18.12}$$

Another 2π pulse on either $|C\rangle$ or $|T\rangle$ could give CNOT.