General Relativity

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Just intro stuff

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2.1 Intro to Tensor Notation

Consider the cartesian definition for $\underline{r}$:

$$\underline{r} = x\underline{i} + y\underline{j} + \underline{z}. \tag{2.1}$$

We have the basis vector $\{\underline{i}, \underline{j}, \underline{k}\}$ and coordinate values $\{x, y, z\}$. We can write this in a different form as

$$\underline{r} = x^1\underline{e}_1 + x^2\underline{e}_2 + x^3\underline{e}_3. \tag{2.2}$$

Note: $x^2 \neq x * x$. The 2 is an index, not a power. If we want to square something, we will write $(x^1)^2 = x^1x^1$. We can rewrite the above again as

$$\underline{r} = \sum_{i=1}^{3} x^i\underline{e}_i. \tag{2.3}$$

We can then simplify this further using the **Einstein summation convention**:

$$\underline{r} = x^i\underline{e}_i, \tag{2.4}$$

i.e. whenever there is a repeated index, we sum over them. Different letters will imply different things:

- Roman letters $i, j, \ldots$ - summing over 3D space
- Roman letters $a, b, c, \ldots$ - summing over ND space
- Roman letters $A, B, \ldots$ - summing over 2D space
- Greek letters $\alpha, \beta, \mu, \nu, \ldots$ - summing over 4D space-time $\{x^0, x^1, x^2, x^3\}$, starting from 0 as time is different slightly, so $\{ct, x^i\}$

2.2 Coordinate Transformation

You may be used to

$$x' = \gamma\left(x - \frac{vct}{c}\right), \tag{2.5}$$

where the extra c factor to make time space-like. This notation can get confusing so instead we use:

$$x^{\bar{1}} = \gamma\left(x^1 - \frac{v}{c}x^0\right), \tag{2.6}$$

where the 'bar' indicates new coordinate system.

For a minute vector difference between points P and Q $d\underline{r}$ in two coordinate systems, we can define $\underline{e}_a$:

$$\underline{r}(P) = \underline{e}_a x^a \qquad \underline{r}(P) = \underline{e}_{\bar{b}} x^{\bar{b}} \tag{2.7}$$

$$d\underline{r} = dx^a\underline{e}_a \tag{2.8}$$

$$\frac{\partial\underline{r}}{\partial x^a} = \underline{e}_a \qquad \frac{\partial\underline{r}}{\partial x^{\bar{b}}} = \underline{e}_{\bar{b}} \tag{2.9}$$

But what is the relationship between these two coordinate systems? Start with $x^{\bar{b}} = x^{\bar{b}}(x^a)$, and consider a general function

$$f = f(x^1, x^2, x^3) \tag{2.10}$$
$$\Delta f = \frac{\partial f}{\partial x^1}\Delta x' + \frac{\partial f}{\partial x^2}\Delta x^2 + \frac{\partial f}{\partial x^2}\Delta x^3 = \frac{\partial f}{\partial x^a}\Delta x^a \tag{2.11}$$

How do we get a small change in $x^{\bar{b}}$?

$$\Delta x^{\bar{b}} = \frac{\partial x^{\bar{b}}}{\partial x^a}\Delta x^a \tag{2.12}$$
$$dx^{\bar{b}} = \frac{\partial x^{\bar{b}}}{\partial x^a}dx^a \tag{2.13}$$
$$dx^{\bar{a}} = \frac{\partial x^{\bar{a}}}{\partial x^b}dx^b \tag{2.14}$$

Notice how we can simply just switch round the indices - **these are all dummy variables and as long as the index notation is consistent, it is completely arbitrary which letter is used,** i.e. the letters themselves mean nothing.

2.3 Tensors

Any quantity which transforms as

$$A^{\bar{b}} = \frac{\partial x^{\bar{b}}}{\partial x^a}A^a \tag{2.15}$$

is a Rank (1,0) or order 1 contravariant tensor. What about $\underline{e}_a$?

$$\underline{r} = x^a \underline{e}_a = x^{\bar{b}} \underline{e}_{\bar{b}} \tag{2.16}$$
$$\underline{e}_{\bar{b}} = \frac{\partial \underline{r}}{\partial x^{\bar{b}}} = \frac{\partial \underline{r}}{\partial x^a}\frac{\partial x^a}{\partial x^{\bar{B}}} = \frac{\partial x^a}{\partial x^{\bar{b}}}\underline{e}_a \tag{2.17}$$

So now we have reversed the position of the indices in Eq (2.15).

How do we define scalars?

$$\nabla\phi = \frac{\partial \phi}{\partial x^i}\underline{e}_i \tag{2.18}$$
$$\frac{\partial \phi}{\partial x^{\bar{j}}} = \frac{\partial x^i}{\partial x^{\bar{j}}}\frac{\partial \phi}{\partial x^i} \tag{2.19}$$

In general, we have

$$A_{\bar{j}} = \frac{\partial x^i}{\partial x^{\bar{j}}}A_i, \tag{2.20}$$

which we call a Rank (0,1) or order 1 covariant tensor.

Lecture 3

3.1 Higher order tensors

Consider

$$T^{ab} = A^a B^b, \tag{3.1}$$

$$T^{\bar{c}\bar{d}} = A^{\bar{c}} B^{\bar{d}} = \left(\frac{\partial x^{\bar{c}}}{\partial x^a} A^a\right)\left(\frac{\partial x^{\bar{d}}}{\partial x^b} B^b\right) = \frac{\partial x^{\bar{c}}}{\partial x^a}\frac{\partial x^{\bar{d}}}{\partial x^b} A^a B^b = \frac{\partial x^{\bar{c}}}{\partial x^a}\frac{\partial x^{\bar{d}}}{\partial x^b} T^{ab}. \tag{3.2}$$

This is the definition of a second order contravariant tensor.

3.2 Tensor Equations

We can write a basic tensor equation,

$$T^a = k(A^a + B^a), \tag{3.3}$$

and wonder how this would look in a transformed coordinate system?

$$T^{\bar{b}} = \frac{\partial x^{\bar{b}}}{\partial x^a} T^a = k\left(\frac{\partial x^{\bar{b}}}{\partial x^a} A^a + \frac{\partial x^{\bar{b}}}{\partial x^a} B^a\right) \tag{3.4}$$

$$= k(A^{\bar{b}} + B^{\bar{b}}). \tag{3.5}$$

So if a tensor equation is true, it is true in all coordinate systems.

3.3 The metric tensor

What is the metric? *The metric is a measure of space.* We define the metric tensor,

$$g_{ab} = \underline{e}_a \cdot \underline{e}_b = g_{ba}, \tag{3.6}$$

so it is symmetric. We can use this when calculating spacetime distances:

$$ds^2 = \underline{dr} \cdot \underline{dr} = (dx^a \underline{e}_a) \cdot \left(dx^b \underline{e}_b\right) \tag{3.7}$$

$$= (\underline{e}_a \cdot \underline{e}_b) dx^a\, dx^b = g_{ab} dx^a\, dx^b. \tag{3.8}$$

Is it a tensor?

$$g_{\bar{a}\bar{b}} = (\underline{e}_{\bar{a}} \cdot \underline{e}_{\bar{b}}) = \left(\frac{\partial x^c}{\partial x^{\bar{a}}} \underline{e}_c\right) \cdot \left(\frac{\partial x^d}{\partial x^{\bar{b}}} \underline{e}_d\right) \tag{3.9}$$

$$= \frac{\partial x^c}{\partial x^{\bar{a}}}\frac{\partial x^d}{\partial x^{\bar{b}}} (\underline{e}_c \cdot \underline{e}_d) = \frac{\partial x^c}{\partial x^{\bar{a}}}\frac{\partial x^d}{\partial x^{\bar{b}}} g_{cd}, \tag{3.10}$$

so it transforms as a tensor; a second order covariant tensor.

3.4 Kronecker Delta

We can write an arbitrary vector as

$$\underline{A} = A^a \underline{e}_a = A^{\bar{b}} \underline{e}_{\bar{b}} = \left(\frac{\partial x^{\bar{b}}}{\partial x^a} A^a\right)\left(\frac{\partial x^d}{\partial x^{\bar{b}}} \underline{e}_d\right) \tag{3.11}$$

$$= \left(\frac{\partial x^{\bar{b}}}{\partial x^a}\frac{\partial x^d}{\partial x^{\bar{b}}}\right) A^a \underline{e}_d = \left(\frac{\partial x^d}{\partial x^a}\right) A^a \underline{e}_d \tag{3.12}$$

$$= \delta_a{}^d A^a \underline{e}_d = A^d \underline{e}_d = A^a \underline{e}_a \tag{3.13}$$

Lecture 4

Lecture 5

Lecture 6

Asbolute Derivative:

$$\frac{D\lambda^a}{ds} = \frac{d\lambda^a}{ds} + \Gamma^a{}_{bc}\lambda^b \frac{dx^c}{ds} \tag{6.1}$$

Covariant Derivative:

$$\lambda^a{}_{;c} = \frac{\partial \lambda^a}{\partial x^c} + \Gamma^a{}_{bc}\lambda^b \tag{6.2}$$

Christoffel Symbols:

$$\Gamma^c{}_{ab}\underline{e}_c = \frac{\partial \underline{e}_a}{\partial x^b}, \quad \Gamma^c{}_{ab} = \Gamma^c{}_{ba} \tag{6.3}$$

Other stuff:

$$\frac{\partial g_{ab}}{\partial x^c} = \Gamma^d{}_{ac}g_{bd} + \Gamma^d{}_{bc}g_{ad} \tag{6.4}$$

$$\frac{\partial g_{bc}}{\partial x^a} = \Gamma^d{}_{ba}g_{cd} + \Gamma^d{}_{ca}g_{bd} \tag{6.5}$$

$$\frac{\partial g_{ca}}{\partial x^b} = \Gamma^d{}_{cd}g_{ad} + \Gamma^d{}_{ab}g_{cd} \tag{6.6}$$

$$2\Gamma^d{}_{ac}g_{bd} = \frac{\partial g_{ab}}{\partial x^c} + \frac{\partial g_{bc}}{\partial x^a} - \frac{\partial g_{ca}}{\partial x^b} \tag{6.7}$$

$$\Gamma^f{}_{ac} = \frac{1}{2}g^{fb}\left(\frac{\partial g_{bc}}{\partial x^a} - \frac{\partial g_{ca}}{\partial x^b} + \frac{\partial g_{ab}}{\partial x^c}\right) \tag{6.8}$$

$$= \frac{1}{2}g^{fb}\left(\partial_a g_{bc} - \partial_b g_{ca} + \partial_c g_{ab}\right) \tag{6.9}$$

We multiplied lefthandside of (6.7) by $\delta^f{}_d$.

> **Example: 2D flat space**
> $x^A = \{x, y\}$:
>
> $$g_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{diag}(1,1) \tag{6.10}$$
>
> $$\Gamma^A{}_{BC} = 0 \tag{6.11}$$
>
> So we don't have to deal with these in Cartesian coordinates. What about polar coordinates? $x^A = \{r, \theta\}$:
>
> $$ds^2 = dr^2 + r^2\,d\theta^2 \tag{6.12}$$
>
> $$g_{AB} = \mathrm{diag}(1, r^2) \tag{6.13}$$
>
> $$\Gamma^A{}_{BC} \neq 0 \tag{6.14}$$
>
> So we can still get non-zero Christoffel symbols even for flat space, but it is still "boring" really.

Let's consider something more interesting, i.e. curved. For 3D space, we have

$$ds^2 = dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2 \tag{6.15}$$

But we want to use just the surface of a sphere, so fixed $r = a$:

$$ds^2 = a^2\,d\theta^2 + a^2\sin^2\theta\,d\phi^2 = g_{AB}dx^A\,dx^B \tag{6.16}$$

$$g_{AB} = \text{diag}(a^2, a^2 \sin^2\theta) \tag{6.17}$$

We have g_{AB}, but we want g^{AB}. Recall

$$g^{AB} g_{BC} = \delta^A{}_C. \tag{6.18}$$

So we have a set of 4 simultaneous equations:

$$g^{A1} g_{1C} + g^{A2} g_{2C} = \delta^A{}_C. \tag{6.19}$$

For diagonal g_{AB} **ONLY**:

$$g^{AB} g_{BA} = g^{AA} g_{AA} = 1 \implies g^{AA} = \frac{1}{g_{AA}} \tag{6.20}$$

$$g^{AB} = \text{diag}\left(\frac{1}{a^2}, \frac{1}{a^2 \sin^2\theta}\right) \tag{6.21}$$

So now we want to calculate

$$\Gamma^\theta{}_{\theta\theta} = \frac{1}{2} g^{\theta B} \left(\partial_\theta g_{B\theta} - \partial_B g_{\theta\theta} + \partial_\theta g_{\theta B}\right), \quad g^{\theta B} = 0, B \neq \theta \tag{6.22}$$

$$= \frac{1}{2}\frac{1}{a^2} \left(\partial_\theta g_{\theta\theta} - \partial_\theta g_{\theta\theta} + \partial_\theta g_{\theta\theta}\right) = 0 \tag{6.23}$$

$$\Gamma^\theta{}_{\phi\theta} = \Gamma^\theta{}_{\theta\phi} = \frac{1}{2} g^{\theta B} \left(\partial_\theta g_{B\phi} - \partial_B g_{\phi\theta} + \partial_\phi g_{\theta B}\right) \tag{6.24}$$

$$= \frac{1}{2} g^{\theta\theta} \left(\partial_\theta g_{\theta\phi} - \partial_\theta g_{\phi\theta} + \partial_\phi g_{\theta\theta}\right) = 0 \tag{6.25}$$

$$\Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta \tag{6.26}$$

$$\Gamma^\phi{}_{\theta\phi} = \cot\theta \tag{6.27}$$

The rest of the Christoffel symbols for this example are 0 (there are $2^3 = 8$ in total?).

6.1 Geodesic Equations

The velocity is a tensor,

$$\underline{v} = v^\alpha \underline{e}_\alpha = \frac{\partial x^\alpha}{\partial \tau} \underline{e}_\alpha \tag{6.28}$$

If there's no force, then there's no change in the velocity vector doesn't change, but its components might change. No force means the absolute derivative of the components:

$$\frac{Dv^\alpha}{d\tau} = 0 \tag{6.29}$$

By an affine parameter, we mean a linear function of path length $u = A + Bs$, such as the proper time τ.

$$\frac{dv^\alpha}{d\tau} + \Gamma^\alpha{}_{\beta\gamma} v^\beta \frac{dx^\gamma}{d\tau} = 0 \tag{6.30}$$

$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha{}_{\beta\gamma} \frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0 \tag{6.31}$$

$$\ddot{x}^\alpha + \Gamma^\alpha{}_{\beta\gamma} \dot{x}^\beta \dot{x}^\gamma = 0 \tag{6.32}$$

Let's guess and make a solution for the sphere, $s = a\theta$, so we are just going around the circumference of the sphere at constant ϕ. For θ:

$$\frac{d^2\theta}{ds^2} + \Gamma^\theta{}_{BC} \frac{dx^B}{ds}\frac{dx^c}{ds} = 0 + \Gamma^\theta{}_{\phi\phi} \frac{d\phi}{ds}\frac{d\phi}{ds} = 0 \tag{6.33}$$

We get a big tick and a gold star! For ϕ:

$$\frac{d^2\phi}{ds^2} + \Gamma^\phi{}_{BC} \frac{dx^B}{ds}\frac{dx^C}{ds} = 0 + \Gamma^\phi{}_{\theta\phi} \frac{d\theta}{ds}\frac{d\phi}{ds} + \Gamma^\phi{}_{\phi\theta} \frac{d\phi}{ds}\frac{d\theta}{ds} = 0 \tag{6.34}$$

So it's a geodesic path! Yayyyyyy!