Mathematical Methods in Physics (Part I)

Summary of Lectures[1]

[1] *Despite the effort to eliminate all typographic errors, some of them could still be present. Hence be careful. Note that this summary is intended as a guideline for the materials covered in lectures and it is not supposed to replace the textbook.*

Vector Algebra (Chapter 7)

- **Orthonormal vectors:** The vectors $\mathbf{a_1}, \mathbf{a_2}, \cdots, \mathbf{a_n}$ are said to be orthonormal if they are unit vectors mutually orthogonal (Example: the three dimensional vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$).
- **(Einstein) summation convention:** An index that appears twice in a given term is understood to be summed over all values that the index can take. (Example: $\sum_{i=1}^{n} a_i b_i \equiv a_i b_i$.)

Consider the three dimensional space with basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ (Standard basis).

- **Kronecker delta:**

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

- **Levi-Civita symbol:**

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) = (1,2,3) = (2,3,1) = (3,1,2), \\ -1 & \text{if } (i,j,k) = (1,3,2) = (3,2,1) = (2,1,3), \\ 0 & \text{otherwise.} \end{cases}$$

- **Scalar (or dot) product**:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta = a_1 b_1 + a_2 b_2 + a_3 b_3 = a_i b_i.$$

  where θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$.

  i) $|\mathbf{b}| \cos\theta = (\mathbf{a} \cdot \mathbf{b})/|\mathbf{a}|$ is the modulus of the orthogonal projection of $\mathbf{b}$ onto the direction of $\mathbf{a}$.

  ii) $|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a}$.

  iii) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.

  iv) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$, $\quad \mathbf{a} \cdot (\beta\, \mathbf{b}) = \beta\, \mathbf{a} \cdot \mathbf{b}$, where β is a scalar.

- **Vector (or cross) product**:

$$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2)\, \mathbf{i} + (a_3 b_1 - a_1 b_3)\, \mathbf{j} + (a_1 b_2 - a_2 b_1)\, \mathbf{k}, \quad |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta,$$
$$(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk} a_j b_k.$$

  where θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$.

  i) $|\mathbf{a} \times \mathbf{b}|$ is the area of the parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$.

  ii) $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.

iii) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}, \quad \mathbf{a} \times (\beta\, \mathbf{b}) = \beta\, \mathbf{a} \times \mathbf{b}$.

iv) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

- **Scalar triple product:**

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \epsilon_{ijk} a_i b_j c_k.$$

  i) $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$ is the volume of the parallelepiped with sides $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

  ii) $[\alpha\, \mathbf{a} + \beta\, \mathbf{b}, \mathbf{c}, \mathbf{d}] = \alpha\, [\mathbf{a}, \mathbf{c}, \mathbf{d}] + \beta\, [\mathbf{b}, \mathbf{c}, \mathbf{d}]$.

  iii) $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}] = -[\mathbf{a}, \mathbf{c}, \mathbf{b}] = -[\mathbf{c}, \mathbf{b}, \mathbf{a}] = -[\mathbf{b}, \mathbf{a}, \mathbf{c}]$

- **Equation of a line.**

  A point R on a line passing through the point A and having a direction $\hat{\mathbf{b}}$ is:

$$\mathbf{r} = \mathbf{a} + \lambda \hat{\mathbf{b}}, \qquad (\mathbf{r} - \mathbf{a}) \times \hat{\mathbf{b}} = 0$$

  where λ is a scalar and $\mathbf{a}$, $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ are the position vectors of A, R, respectively.

- **Equation of a plane.**

  i) A point R on a plane perpendicular to the unit vector $\hat{\mathbf{n}}$ and passing through the point A is

$$(\mathbf{r} - \mathbf{a}) \cdot \hat{\mathbf{n}} = 0,$$

  where $\mathbf{a}$ and $\mathbf{r}$ are the position vectors of A and R, respectively.

  ii) A point R on a plane passing through the point A, B and C is

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a}),$$

  where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{r}$ are the position vectors of A, B, C and R, respectively.

Vector Spaces (Chapter 8)

Vector space. A set of objects called *vectors* forms a vector space V if there are two operations defined on the elements of the set called *addition* and *multiplication by scalars* and if the following properties are satisfied ($\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are vectors and α, β are scalars):

i) If $\mathbf{u}$ and $\mathbf{v}$ are in V then $\mathbf{u}+\mathbf{v}$ (addition) is in V.

   If $\mathbf{v}$ is in V then $\alpha\mathbf{v}$ (scalar multiplication) is in V.

ii) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$, $\quad (\alpha\,\beta)\mathbf{v}=\alpha(\beta\,\mathbf{v})$.

iii) There exists a *neutral element* $\mathbf{0}$ such that $\mathbf{v}+\mathbf{0}=\mathbf{0}$ for all $\mathbf{v}$.

iv) There exists an *inverse element* $-\mathbf{v}$ such that $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$ for all $\mathbf{v}$.

v) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.

vi) $\alpha(\mathbf{u}+\mathbf{v})=\alpha\,\mathbf{u}+\alpha\,\mathbf{v}$, $\quad (\alpha+\beta)\mathbf{v}=\alpha\,\mathbf{v}+\beta\,\mathbf{v}$.

vii) $1\,\mathbf{v}=\mathbf{v}$ for all $\mathbf{v}$.

If the scalars α are real V is called a real vector space, otherwise V is a complex vector space.

- **Linear combinations:**

  $$\alpha_1\,\mathbf{v}_1+\alpha_2\,\mathbf{v}_2+\cdots\alpha_k\,\mathbf{v}_k=\alpha_i\,\mathbf{v}_i,$$

  where $\mathbf{v}_1$, $\mathbf{v}_2, \cdots \mathbf{v}_k$, are k vectors in V.

  The set of all linear combinations of $\mathbf{v}_1$, $\mathbf{v}_2, \cdots \mathbf{v}_k$, is called a span of $\mathbf{v}_1$, $\mathbf{v}_2, \cdots \mathbf{v}_k$, and denoted $\mathrm{Span}(\mathbf{v}_1,\, \mathbf{v}_2, \cdots \mathbf{v}_k)$.

- **Linearly independent vectors:** k vectors $\mathbf{v_1}$, $\mathbf{v_2}, \cdots, \mathbf{v_k}$ are said to be *linearly independent* if the equation

  $$\alpha_1\mathbf{v_1}+\alpha_2\mathbf{v_2}+\cdots+\alpha_k\mathbf{v_k}=0$$

  is satisfied if and only if all $\alpha_i=0$. Otherwise, the vectors are said to be *linearly dependent.*

- **Basis:** A set of vectors $\mathbf{v}_1$, $\mathbf{v}_2, \cdots \mathbf{v}_k$, in V is called a basis of V if and only if $\mathbf{v}_1$, $\mathbf{v}_2, \cdots \mathbf{v}_k$, are linearly independent and $V=\mathrm{Span}(\mathbf{v}_1,\, \mathbf{v}_2, \cdots \mathbf{v}_k)$.

  The numbers of vector in a basis is called the dimension of the space V ($\dim V$).

- **Inner (or scalar) product:** Consider a vector space V. The inner product between two elements of V is a scalar function denoted $\langle \mathbf{u}|\mathbf{v}\rangle = (\mathbf{u}, \mathbf{v})$ that satisfies the following properties

  i) $\langle \mathbf{u}|\mathbf{v}\rangle = \langle \mathbf{v}|\mathbf{u}\rangle^*$.

  ii) $\langle \mathbf{u}|(\lambda\mathbf{v} + \mu\mathbf{w})\rangle = \lambda\langle \mathbf{u}|\mathbf{v}\rangle + \mu\langle \mathbf{u}|\mathbf{w}\rangle, \qquad \lambda, \mu$ scalars.

  iii) $\langle \mathbf{u}|\mathbf{u}\rangle > 0$ if $\mathbf{v} \neq 0$.

  The length of a vector (*norm*) is $|\mathbf{u}| = \sqrt{\langle \mathbf{u}|\mathbf{u}\rangle}$.

  two vectors are *orthogonal* if $\langle \mathbf{u}|\mathbf{w}\rangle = 0$.

Matrices (Chapter 8)

- **Linear operators:** An object $\mathcal{A}$ is called a linear operator if its action on vectors $\mathbf{u}$ and $\mathbf{v}$ is as follows
$$\mathcal{A}\left(\alpha\mathbf{u}+\beta\mathbf{v}\right)=\alpha\mathcal{A}\,\mathbf{u}+\beta\mathcal{A}\,\mathbf{v},$$
where α and β are scalars. Matrices are examples of operators.

- **Matrix operations:**
  - i) Matrix addition: $(A+B)_{ij}=A_{ij}+B_{ij}$.
  - ii) Multiplication by a scalar: $(\alpha A)_{ij}=\alpha A_{ij}$.
  - iii) Multiplication of matrices: $(AB)_{ij}=A_{ik}B_{kj}$, with $AB\neq BA$.
  - iv) Transposition: $(A^T)_{ij}=A_{ji}$, with $(ABC\ldots F)^T=F^T\ldots C^TB^TA^T$.
  - v) Complex conjugation: $(A^*)_{ij}=(A_{ij})^*$.
  - vi) Hermitian conjugation (adjoint): $(A^\dagger)_{ij}=(A_{ji})^*$, with $(ABC\ldots F)^\dagger=F^\dagger\ldots C^\dagger B^\dagger A^\dagger$.
  - vii) Trace of a square matrix: $\mathrm{Tr}\,A=\sum_k A_{kk}$, with $\mathrm{Tr}(AB)=\mathrm{Tr}(BA)$, $\mathrm{Tr}\,A^T=\mathrm{Tr}\,A$; $\mathrm{Tr}\,A^\dagger=(\mathrm{Tr}\,A)^*$.

- **The determinant of a square matrix:**
$$|A|=A_{jk}C_{jk},\quad\text{for any row } j,\qquad |A|=A_{kj}C_{kj},\quad\text{for any column } j,$$
where $C_{mn}=(-1)^{m+n}|A_{mn}|$ is the *cofactor* associated to the matrix element A_{mn}. In turn, $|A_{mn}|$ is the *minor* associated to the matrix element A_{mn}. The minor is the determinant of the matrix obtained by removing the m-th row and n-th column from the matrix A.

  Properties:
  - i) $|AB\ldots F|=|A||B|\ldots|F|$.
  - ii) $|A^T|=|A|,\quad |A^*|=|A|^*,\quad |A^\dagger|=|A|^*,\quad |A^{-1}|=|A|^{-1}$.
  - iii) If the rows (or the columns) are linearly dependent, then $|A|=0$.
  - iv) If B is obtained from A by multiplying the elements of any row (or column) by a factor α, then $|B|=\alpha\,|A$.
  - v) If B is obtained from A by interchanging two rows (or columns), then $|B|=-|A|$.
  - vi) If B is obtained from A by adding k times one row (or column) to the other row (or column), then $|A|=|B|$.

- **Elementary row operations (on matrices):**

  i) Multiply any row by a non zero constant.

  ii) Interchange any two rows.

  iii) Add some multiple of one row to any other row.

- **The inverse of a square matrix:**

$$A^{-1} = \frac{C^T}{|A|}, \quad \text{that is} \quad A^{-1}_{ij} = \frac{C_{ji}}{|A|}, \qquad A^{-1}A = AA^{-1} = I,$$

where C is the cofactor matrix and I the identity matrix ($I_{ij} = \delta_{ij}$). If $|A| = 0$ the inverse does not exist and the matrix A is said to be *singular*.

Note that in order to find the inverse of a matrix, you can also use the Gauss-Jordan method shown in the lectures, which makes use of the elementary row operations.

Properties:

  i) $(AB \ldots F)^{-1} = F^{-1} \ldots B^{-1}A^{-1}$.

  ii) $(A^T)^{-1} = (A^{-1})^T, \quad (A^\dagger)^{-1} = (A^{-1})^\dagger$.

- **Eigenvalue problem:**

  i) Eigenvalue equation: $A\mathbf{x} = \lambda\mathbf{x}$. The vector $\mathbf{x}$ is called *eigenvector* and λ is called the corresponding *eigenvalue*

  ii) Characteristic equation (or characteristic polynomial): $|A - \lambda I| = 0$.

  iii) The eigenvalues of the matrix A are the roots of the characteristic polynomial.

  iv) The eigenvectors associated to the eigenvalue μ are the vectors $\mathbf{x}$ such that $(A - \mu I)\mathbf{x} = 0$.

  v) Eigenvectors associated to different eigenvalues are linearly independent.

  vi) If a $n \times n$ matrix A has n distinct eigenvalues, then the set of corresponding eigenvectors represents a basis in the vector space on which the matrix acts. If the eigenvalues are not all distinct, it may or it may not exist a basis of eigenvectors.

- **Special types of square matrices:**

   i) *Hermitian matrix:* $A^\dagger = A$ (real matrix $A^T = A$ - *symmetric matrix*)
   The eigenvalues are real.

   ii) *Anti-Hermitian matrix:* $A^\dagger = -A$ (real matrix $A^T = -A$ - *antisymmetric matrix*)
   The eigenvalues are purely imaginary ore zero.

   iii) *Unitary matrix:* $A^\dagger = A^{-1}$ (real matrix $A^T = A^{-1}$ - *orthogonal matrix*)
   The eigenvalues have absolute value equal to one.

  The eigenvectors of symmetric, hermitian, anti-symmetric, anti-hermitian, orthogonal, unitary matrices are linearly independent. They can always be chosen in such a way that they form a mutually orthogonal set.

- **Similar matrices:** Two matrices A and B are said to be similar if it exist a matrix S such that
$$B = S^{-1}AS.$$
  Then, the matrices A and B have the same determinant, the same trace and the same set of eigenvalues.

- **Diagonalisation of a matrix:** If a $(n \times n)$ matrix A has a set of eigenvectors that forms a basis, then
$$D = S^{-1}AS,$$
  where D is a diagonal matrix with the eigenvalues of A as entries of the main diagonal and S is a matrix whose columns are the eigenvectors of the matrix A.

   i) $|A| = \prod_i \lambda_i, \quad \mathrm{Tr}A = \sum_i \lambda_i.$

   ii) If A is a special matrix, then A is diagonalisable and S is unitary, i.e.
$$D = S^\dagger AS,$$
   where the columns of S are the **orthonormal** eigenvectors of A.

Fourier Series (Chapter 12)

The Fourier series of a function $f(x)$ is a representation of the function $f(x)$ as an infinite series of cosine and sine functions

$$f(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty} \left[a_r \cos\left(\frac{2\pi r x}{L}\right) + b_r \sin\left(\frac{2\pi r x}{L}\right) \right].$$

The Fourier coefficients a_0, a_r and b_r are:

$$\begin{aligned}
a_0 &= \frac{2}{L} \int_{x_0}^{x_0+L} f(x)\, dx, \\
a_r &= \frac{2}{L} \int_{x_0}^{x_0+L} f(x) \cos\left(\frac{2\pi r x}{L}\right) dx \qquad r = 1, 2, 3, \ldots, \\
b_r &= \frac{2}{L} \int_{x_0}^{x_0+L} f(x) \sin\left(\frac{2\pi r x}{L}\right) dx \qquad r = 1, 2, 3, \ldots,
\end{aligned}$$

where x_0 is an arbitrary point along the x-axis. For a function $f(x)$ to be represented as a Fourier series the following conditions must be satisfied:

- The function $f(x)$ is periodic, that is $f(x) = f(x + L)$ where L is the period.
- The function is single-valued.
- The function $f(x)$ has a finite number of extreme points (maxima and minima) in the interval L.
- The function $f(x)$ has a finite number of finite discontinuities in the interval L.

Properties of the Fourier series:

- If the function $f(x)$ is even, that is $f(-x) = f(x)$, $b_r = 0$ for $r = 1, 2, 3, \ldots$
- If the function $f(x)$ is odd, that is $f(-x) = -f(x)$, $a_r = 0$ for $r = 0, 1, 2, 3, \ldots$
- If x_1 is a point of discontinuity for the function $f(x)$ in the interval L, then the value of the Fourier series at that point is:

  $$f(x_1) = \frac{f(x_1^-) + f(x_1^+)}{2},$$

  where $f(x_1^-)$ and $f(x_1^+)$ are the left and right limits of the function at x_1, respectively.

- The set of all periodic functions on the interval L that can be represented as Fourier series forms an infinite dimensional vector space:

  i) Orthogonal basis:

$$\cos\left(\frac{2\pi rx}{L}\right), \quad r = 0, 1, 2, 3, \ldots; \qquad \sin\left(\frac{2\pi rx}{L}\right), \quad r = 1, 2, 3, \ldots$$

  ii) Inner product:

$$\frac{2}{L}\int_0^L f(x)g(x)dx = \langle f|g\rangle.$$

Complex form of the Fourier series:

$$f(x) = \sum_{r=-\infty}^{\infty} c_r\, e^{2\pi irx/L}, \qquad c_r = \frac{1}{L}\int\limits_{x_0}^{x_0+L} f(x)\, e^{-2\pi irx/L}\, dx.$$

Integral Transforms (Chapter 13)

An integral transform is a function g that can be expressed as an integral of another function f in the form

$$\mathcal{I}[f(x)](y) \equiv g(y) = \int_{-\infty}^{\infty} K(x,y) f(x) dx,$$

where $K(x,y)$ is called the kernel of the transform.

- $\mathcal{I}$ is a linear operator, that is:

$$\mathcal{I}[c_1 f_1 + c_2 f_2] = c_1 \mathcal{I}[f_1] + c_2 \mathcal{I}[f_2], \qquad c_1, c_2 \quad \text{constants.}$$

- Given $\mathcal{I}$ such that $\mathcal{I}[f] = g$, the inverse operator $\mathcal{I}^{-1}$ is also a linear operator and $\mathcal{I}^{-1}[g] = f$.

The Fourier Transform of the function $f(t)$ is:

$$\mathcal{F}[f(t)](\omega) \equiv \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} dt.$$

The integral exists if:

- The function $f(x)$ has a finite number of finite discontinuities.
- $\int_{-\infty}^{\infty} |f(t)| dt$ is finite.

Its f is continuous the inverse is:

$$\mathcal{F}^{-1}[\hat{f}(\omega)](t) = f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega t} dw.$$

Properties of the Fourier transform

- **Scaling:**
$$\mathcal{F}[f(at)](\omega) = \frac{1}{a}\hat{f}\left(\frac{\omega}{a}\right), \qquad a \neq 0 \quad \text{constant.}$$

- **Translation:**
$$\mathcal{F}[f(t+a)](\omega) = e^{ia\omega}\hat{f}(\omega), \qquad a \quad \text{constant.}$$

- **Exponential multiplication:**
$$\mathcal{F}[e^{\alpha t}f(t)](\omega) = \hat{f}(\omega + i\alpha), \qquad \alpha \quad \text{constant.}$$

- **The Fourier transform of a derivative:**
$$\mathcal{F}[f^{(n)}(t)](\omega) = (i\omega)^n \hat{f}(\omega).$$
where $f^{(n)}$ is the n^{th} derivative of the function f.

The convolution of two functions f and g is a function h defined as follows:

$$h(y) = \int_{-\infty}^{\infty} f(x)g(y-x)\,dx \equiv (f*g)(y) = (g*f)(y).$$

The convolution theorem for Fourier transforms.
The Fourier transform of the convolution $h(y)$ is:

$$\mathcal{F}[h(z)](\omega) = \sqrt{2\pi}\hat{f}(\omega)\hat{g}(\omega).$$

The Laplace transform of the function $f(t)$ is:

$$\mathcal{L}[f(t)](s) \equiv \bar{f}(s) = \int_{0}^{\infty} f(t)\,e^{-st}dt,$$

where s is taken to be real. Note that, sometimes a constrain on the variable s should be imposed in order for the integral to exist.

Properties of the Laplace transform.

- **Delay rule:**

$$\mathcal{L}[H(t-a)f(t-a)](s) = e^{-sa}\bar{f}(s), \qquad a \quad \text{constant.}$$

- **Exponential multiplication:**

$$\mathcal{L}[e^{at}f(t)](s) = \bar{f}(s-a), \qquad a \quad \text{constant.}$$

- **Scaling:**

$$\mathcal{L}[f(at)](s) = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right), \qquad a \neq 0 \quad \text{constant.}$$

- **Polynomial multiplication:**

$$\mathcal{L}[t^n f(t)](s) = (-1)^n \frac{d^n \bar{f}(s)}{ds^n}, \quad n = 1, 2, 3 \ldots .$$

- **The Laplace transform of a derivative:**

$$\mathcal{L}[f^{(n)}(t)](s) = s^n \bar{f}(s) - s^{n-1}f(0) - s^{n-2}f^{(1)}(0) - \cdots - f^{(n-1)}(0), \quad s > 0,$$

  where $f^{(n)}$ is the n^{th} derivative of the function f.

- **The Laplace transform for integration:**

$$\mathcal{L}\left[\int_0^t f(u)du\right] = \frac{\bar{f}(s)}{s}.$$

The convolution theorem for a Laplace transform:
If the functions f and g have Laplace transforms $\bar{f}$ and $\bar{g}$, then:

$$\mathcal{L}[(f * g)](s) = \mathcal{L}[(g * f)](s) = \mathcal{L}\left[\int_0^t f(u)g(t-u)\,du\right](s) = \bar{f}(s)\bar{g}(s).$$

The inverse of a Laplace transform: $\mathcal{L}^{-1}[\bar{f}(s)] = f(t)$.

- Complex Analysis.
- Inspection, that is partial fraction decomposition together with Laplace transform properties and tables of known Laplace transforms.
- Convolution theorem.

The Dirac delta function (Chapter 13)

The Dirac delta function $\delta(x-a)$ - with a a constant - is a generalised function (or distribution) and it is defined as the limit of a sequence of functions. Its defining properties are:

$$\delta(x-a) = 0 \ \text{ for } \ x \neq a, \qquad \int\limits_{\alpha}^{\beta} f(x)\delta(x-a)\,dx = \begin{cases} f(a) & \alpha < a < \beta \\ 0 & \text{otherwise} \end{cases}$$

where f is a well behaved function.

Integral representation of the Dirac delta function:

$$\delta(t-x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} e^{i\omega(t-x)} d\omega$$

Properties of the Dirac delta function.

- $$\delta(x) = \delta(-x).$$
- $$x\delta(x) = 0.$$
- $$\delta(g(x)) = \sum_{a;\, g(a)=0;\, g'(a)\neq 0} \frac{\delta(x-a)}{|g'(a)|}.$$
- $$\int\limits_{-\infty}^{\infty} f(x)\delta'(x-a)dx = -f'(a).$$
- $H'(x) = \delta(x)$ where $H(x)$ is the Heaviside step function defined as follows

$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} .$$

Vector Calculus (Chapter 10)

- **Differentiation of vector functions:**

$$\frac{d\mathbf{a}}{du} \equiv a'(u) = \frac{da_x}{du}\,\mathbf{i} + \frac{da_y}{du}\,\mathbf{j} + \frac{da_z}{du}\,\mathbf{k}.$$

Rules:

i)

$$\frac{d}{du}(\phi\mathbf{a}) = \frac{d\phi}{du}\mathbf{a} + \phi\frac{d\mathbf{a}}{du}, \qquad \frac{d}{du}(\phi(u)) = \frac{d\mathbf{a}}{d\phi}\frac{d\phi}{du},$$

ii)

$$\frac{d}{du}(\mathbf{a}\cdot\mathbf{b}) = \frac{d\mathbf{a}}{du}\cdot\mathbf{b} + \mathbf{a}\cdot\frac{d\mathbf{b}}{du}, \qquad \frac{d}{du}(\mathbf{a}\times\mathbf{b}) = \frac{d\mathbf{a}}{du}\times\mathbf{b} + \mathbf{a}\times\frac{d\mathbf{b}}{du},$$

where **a**, **b** are vector functions and ϕ, ϕ are scalar functions.

- **Differential of a vector function:**

$$\begin{aligned} d\mathbf{a} &= \frac{d\mathbf{a}}{du}\,du \qquad \text{for } \mathbf{a} = \mathbf{a}(u), \\ d\mathbf{a} &= \frac{\partial\mathbf{a}}{\partial u}\,du + \frac{\partial\mathbf{a}}{\partial v}\,dv + \cdots \qquad \text{for } \mathbf{a} = \mathbf{a}(u, v\cdots). \end{aligned}$$

Gradient, divergence and curl in cartesian coordinates.

- **The vector differential operator del (or nabla):**

$$\nabla = \frac{\partial}{\partial x}\,\mathbf{i} + \frac{\partial}{\partial y}\,\mathbf{j} + \frac{\partial}{\partial z}\,\mathbf{k}$$

- **The scalar differential operator Laplacian:**

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

- **The gradient** of a scalar field ϕ:

$$\operatorname{grad}\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\,\mathbf{i} + \frac{\partial\phi}{\partial y}\,\mathbf{j} + \frac{\partial\phi}{\partial z}\,\mathbf{k}$$

Properties:

i) $\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$

ii) $\nabla(\phi\psi) = \psi\nabla\phi + \phi\nabla\psi$

iii) $\nabla(\psi(\phi)) = \psi'(\phi)\nabla\phi$

iv) Special cases: $\nabla r = \mathbf{r}/r$, $\nabla(1/r) = -\mathbf{r}/r^3$, $\nabla\phi(r) = \phi'\mathbf{r}/r$, where r is the modulus of the position vector $\mathbf{r}$, i.e. $r = \sqrt{x^2 + y^2 + z^2}$.

- **The divergence** of a vector field $\mathbf{a}$:

$$\operatorname{div}\mathbf{a} = \nabla \cdot \mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

i) $\nabla \cdot (\mathbf{a} + \mathbf{b}) = \nabla \cdot \mathbf{a} + \nabla \cdot \mathbf{b}$

ii) $\nabla \cdot (\phi\,\mathbf{a}) = \nabla\phi \cdot \mathbf{a} + \phi(\nabla \cdot \mathbf{a})$, $\quad \nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$

iii) Special case: $\nabla \cdot \mathbf{r} = 3$

- **The Laplacian** of a scalar field ϕ:

$$\nabla \cdot (\nabla\phi) = \nabla^2\phi = \frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\partial z}$$

- **The curl** of a vector field $\mathbf{a}$:

$$\operatorname{curl}\mathbf{a} = \nabla \times \mathbf{a} = \mathbf{i}\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)$$

i) $\nabla \times (\mathbf{a} + \mathbf{b}) = \nabla \times \mathbf{a} + \nabla \times \mathbf{b}$

ii) $\nabla \times (\phi\,\mathbf{a}) = (\nabla\phi) \times \mathbf{a} + \phi(\nabla \times \mathbf{a})$,
$\nabla \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \cdot \nabla)\mathbf{a} - (\nabla \cdot \mathbf{a})\mathbf{b} - (\mathbf{a} \cdot \nabla)\mathbf{b} + (\nabla \cdot \mathbf{b})\mathbf{a}$

iii) Special case: $\nabla \times \mathbf{r} = 0$

i) If a vector field $\mathbf{a}$ can be written as $\mathbf{a} = \nabla\phi$, the vector field $\mathbf{a}$ is said to be *irrotational* or *conservative* and the scalar field ϕ is called *potential*. Since $\nabla \times (\nabla\phi) = 0$ it follows that $\nabla \times \mathbf{a} = 0$.

- **Curves:** A curve C can be represented by a vector function $\mathbf{r}$ that depends on one parameter u (*parametric representation*)

$$\mathbf{r}(u) = x(u)\,\mathbf{i} + y(u)\,\mathbf{j} + z(u)\,\mathbf{k}$$

  i) The derivative $\mathbf{r}'(u) \equiv \mathbf{t}(u)$ is a vector tangent to the curve at each point.

  ii) The *arch length* s measured along the curve satisfies:

$$\left(\frac{ds}{du}\right)^2 = \frac{d\mathbf{r}}{du} \cdot \frac{d\mathbf{r}}{du}, \quad ds = \pm\sqrt{\frac{d\mathbf{r}}{du} \cdot \frac{d\mathbf{r}}{du}}\,du,$$

  where the sign fixes the direction of measuring s, for increasing or decreasing u.

  iii) *Unit tangent vector* to the curve:

$$\hat{\mathbf{t}} = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{du}\frac{du}{ds}.$$

  iv) *Principal normal* to the curve:

$$\frac{\hat{\mathbf{n}}}{\rho} = \frac{d^2\mathbf{r}}{ds^2}.$$

  where ρ is the *radius of curvature.*

- **Surfaces:** A surface S can be represented by a vector function $\mathbf{r}$ that depends on two parameters u and v (*parametric representation*)

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$$

  i) The vectors $\partial\mathbf{r}/\partial u$, $\partial\mathbf{r}/\partial v$ are linear independent and tangent to curves on S with v and u constant, respectively.

  ii) A vector normal to the surface is:

$$\mathbf{n} = \frac{\partial\mathbf{r}}{\partial u} \times \frac{\partial\mathbf{r}}{\partial v}.$$

  iii) Vector area element:

$$d\mathbf{S} = \left(\frac{\partial\mathbf{r}}{\partial u} \times \frac{\partial\mathbf{r}}{\partial v}\right) du\,dv = \mathbf{n}\,du\,dv.$$

  iv) Scalar area element:

$$dS = \left|\frac{\partial\mathbf{r}}{\partial u} \times \frac{\partial\mathbf{r}}{\partial v}\right| du\,dv = |\mathbf{n}|\,du\,dv.$$

  v) A surface can be represented by the equation $\phi(\mathbf{r}) = c$, where c is a constant. Then $\nabla\phi$ is perpendicular to the surface.

  vi) The orientation of the surface S is determined by the sign of $\mathbf{n}$.

  vii) A surface S is orientable if the vector $\mathbf{n}$ can be determined everywhere by a choice of sign.

Change of variables: orthogonal curvilinear coordinates

Given the position vector $\mathbf{r}$ expressed in cartesian coordinates x, y, z we can use a change of variable to express this vector in terms of a new set of coordinates u, v, w

$$\mathbf{r}(u,v,w) = x(u,v,w)\,\mathbf{i} + y(u,v,w)\,\mathbf{j} + z(u,v,w)\,\mathbf{k}$$

- **Line element:**
$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u}\,du + \frac{\partial \mathbf{r}}{\partial v}\,dv + \frac{\partial \mathbf{r}}{\partial w}\,dw.$$

- **New basis:**
$$\frac{\partial \mathbf{r}}{\partial u} = h_u\,\hat{\mathbf{e}}_u, \quad \frac{\partial \mathbf{r}}{\partial v} = h_v\,\hat{\mathbf{e}}_v, \quad \frac{\partial \mathbf{r}}{\partial w} = h_w\,\hat{\mathbf{e}}_w,$$
where h_u, h_v h_w are called *scale factors*. In an orthogonal curvilinear coordinate system these vectors are orthogonal and $\hat{\mathbf{e}}_u$, $\hat{\mathbf{e}}_v$, $\hat{\mathbf{e}}w$ form an orthonormal basis of the three dimensional vector space $\mathbb{R}^3$.

- **New line element:**
$$d\mathbf{r} = h_u\,\hat{\mathbf{e}}_u\,du + h_v\,\hat{\mathbf{e}}_v\,dv + h_w\,\hat{\mathbf{e}}_w\,dw.$$

- **Arc length:**
$$ds^2 = d\mathbf{r}\cdot d\mathbf{r} = h_1^2\,(du_1)^2 + h_2^2\,(du_2)^2 + h_3^2\,(du_3)^2$$

- **Vector area element (surface of constant w):**
$$d\mathbf{S} = \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right)du\,dv = h_u h_v\,dudv.$$

- **Volume element:**
$$dV = \left|\frac{\partial \mathbf{r}}{\partial u}\cdot\left(\frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial w}\right)\right| du\,dv\,dw = h_u h_v h_w\,dudvdw.$$

- **Del operator:**
$$\nabla = \frac{\hat{\mathbf{e}}_u}{h_u}\frac{\partial}{\partial u} + \frac{\hat{\mathbf{e}}_v}{h_v}\frac{\partial}{\partial v} + \frac{\hat{\mathbf{e}}_w}{h_w}\frac{\partial}{\partial w}.$$

- **Gradient:**
$$\nabla\phi = \frac{\hat{\mathbf{e}}_u}{h_u}\frac{\partial \phi}{\partial u} + \frac{\hat{\mathbf{e}}_v}{h_v}\frac{\partial \phi}{\partial v} + \frac{\hat{\mathbf{e}}_w}{h_w}\frac{\partial \phi}{\partial w}.$$

- **Divergence:**
$$\nabla\cdot\mathbf{a} = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}(h_v h_w a_u) + \frac{\partial}{\partial v}(h_w h_u a_v) + \frac{\partial}{\partial w}(h_u h_v a_w)\right],$$
where $\mathbf{a} = a_u\,\hat{\mathbf{e}}_u + a_v\,\hat{\mathbf{e}}_v + a_w\,\hat{\mathbf{e}}_w$.

- **Curl:**

$$\nabla \times \mathbf{a} = \frac{1}{h_u h_v h_w} \begin{vmatrix} h_u\,\hat{\mathbf{e}}_u & h_v\,\hat{\mathbf{e}}_v & h_w\,\hat{\mathbf{e}}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u\,a_u & h_v\,a_v & h_w\,a_w. \end{vmatrix}$$

- **Laplacian:**

$$\nabla^2 \phi = \frac{1}{h_u h_v h_w} \left[\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u} \frac{\partial \phi}{\partial u} \right) + \frac{\partial}{\partial v}\left(\frac{h_w h_u}{h_v} \frac{\partial \phi}{\partial v} \right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w} \frac{\partial \phi}{\partial w} \right) \right].$$

Examples of orthogonal curvilinear coordinates:

- **Cylindrical polar coordinates:**

$$\mathbf{r}(\rho, \phi, z) = \rho \cos\phi\,\mathbf{i} + \rho \sin\phi\,\mathbf{j} + z\,\mathbf{k}, \qquad \rho \geq 0, \quad 0 \leq \phi < 2\pi, \quad -\infty < z < \infty$$

$$\text{i)} \quad \begin{array}{ll} h_\rho = 1, & \hat{\mathbf{e}}_\rho = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}, \\ h_\phi = \rho, & \hat{\mathbf{e}}_\phi = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}, \\ h_z = 1, & \hat{\mathbf{e}}_z = \mathbf{k}. \end{array}$$

$$\text{ii)} \;\; d\mathbf{S} = \begin{cases} \hat{\mathbf{e}}_\rho\,\rho\,d\phi dz & (\rho = \text{const}) \\ \hat{\mathbf{e}}_\phi\,d\rho dz & (\phi = \text{const}) \\ \hat{\mathbf{e}}_z\,\rho\,d\rho d\phi & (z = \text{const}). \end{cases}$$

$$\text{iii)} \;\; dV = \rho\,d\rho d\phi dz.$$

- **Spherical polar coordinates:**

$$\mathbf{r}(r, \theta, \phi) = r \sin\theta \cos\phi\,\mathbf{i} + r \sin\theta \sin\phi\,\mathbf{j} + r \cos\theta\,\mathbf{k}, \qquad r \geq 0, \quad 0 \leq \phi < 2\pi, \quad 0 \leq \theta \leq \pi$$

$$\text{i)} \quad \begin{array}{ll} h_r = 1, & \hat{\mathbf{e}}_r = \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k}, \\ h_\theta = r, & \hat{\mathbf{e}}_\theta = \cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k}, \\ h_\phi = r\sin\theta, & \hat{\mathbf{e}}_\phi = \sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}. \end{array}$$

$$\text{ii)} \;\; d\mathbf{S} = \begin{cases} \hat{\mathbf{e}}_r\,r^2 \sin\theta\,d\theta d\phi & (r = \text{const}) \\ \hat{\mathbf{e}}_\theta\,r \sin\theta\,dr d\phi & (\theta = \text{const}) \\ \hat{\mathbf{e}}_\phi\,r\,dr d\theta & (\phi = \text{const}). \end{cases}$$

$$\text{iii)} \;\; dV = r^2 \sin\theta\,dr d\theta d\phi.$$

Integrals (Chapter 11)

- **The line integral (or path integral)** of a vector field $\mathbf{a}(\mathbf{r})$ along the curve C is:

$$\int_C \mathbf{a}(\mathbf{r}) \cdot d\mathbf{r} = \int_{u_{min}}^{u_{max}} \mathbf{a}(\mathbf{r}(u)) \cdot \frac{d\mathbf{r}}{du}\, du,$$

where C is a smooth *oriented* (a direction along C must be specified) curve defined by the equation $\mathbf{r}(u)$ with endpoints $A = \mathbf{r}(u_{min})$ and $B = \mathbf{r}(u_{max})$.

Properties/Observations.

i) In general the integral depends on the endpoints and the path C.

ii) $\int_C \mathbf{a} \cdot d\mathbf{r} = -\int_{-C} \mathbf{a} \cdot d\mathbf{r}$, where C is a curve with orientation $A \longrightarrow B$ and $-C$ is a curve with orientation $B \longrightarrow A$.

iii) If $C = C_1 + C_2 + \cdots + C_n$ then $\int_C \mathbf{a} \cdot d\mathbf{r} = \int_{C_1} \mathbf{a} \cdot d\mathbf{r} + \int_{C_2} \mathbf{a} \cdot d\mathbf{r}, \cdots + \int_{C_n} \mathbf{a} \cdot d\mathbf{r}$.

iv) Other kinds of line integrals are possible. For instance:

$$\int_C \phi d\mathbf{r}, \qquad \int_C \mathbf{a} \times d\mathbf{r}, \qquad \int_C \phi\, ds, \qquad \int_C \mathbf{a}\, ds.$$

- **Simply connected region.** A region D is simply connected if every closed path within D can be shrunk to a point without leaving the region.

- **Theorem:** Consider the integral $I = \int_C \mathbf{a} \cdot d\mathbf{r}$, where the path C is in a simply connected region D. Then, the following statements are equivalent:

i) The line integral I is independent of the path C. It only depends on the endpoints of the path C.

ii) It exists a scalar function ϕ (a potential) such that $\mathbf{a} = \nabla\phi$.

iii) $\nabla \times \mathbf{a} = 0$.

The vector field $\mathbf{a}$ is said to be *conservative* (or *irrotational*) and ϕ is its *potential*. In addition:

i) $I = \phi(B) - \phi(A)$ where A and B are the endpoints of the path C.

ii) The line integral I along any closed path C in D is zero.

- **The surface integrals** of vector functions $\mathbf{a}(\mathbf{r})$ over a smooth surface S, defined by $\mathbf{r}(u, v)$ with orientation given by the normal $\hat{\mathbf{n}}$, is:

$$\int_S \mathbf{a}(\mathbf{r}) \cdot d\mathbf{S} = \int_S \mathbf{a}(\mathbf{r}) \cdot \hat{\mathbf{n}} dS = \int_{u_{min}}^{u_{max}} \int_{v_{min}}^{v_{max}} \mathbf{a}(\mathbf{r}(u.v)) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) du\, dv.$$

Observation.

i) The integral depends on the orientation of the surface S.

ii) If the surface is closed, by convention, the vector $\mathbf{n}$ is pointed outwards the volume enclosed.

iii) Other kinds of integrals are possible. For instance:

$$\int_S \phi\, d\mathbf{S}, \qquad \int_S \mathbf{a} \times d\mathbf{S}, \qquad \int_S \phi\, dS, \qquad \int_S \mathbf{a}\, dS.$$

- **Volume integrals** of a function $\phi(\mathbf{r})$ (or $\mathbf{a}(\mathbf{r})$) over a volume V described by $\mathbf{r}(u, v, w)$ is:

$$\int_V \phi(\mathbf{r})\, dV = \int_{u_{min}}^{u_{max}} \int_{v_{min}}^{v_{max}} \int_{w_{min}}^{w_{max}} \phi(\mathbf{r}(u, v, w)) \left| \frac{\partial \mathbf{r}}{\partial u} \cdot \left(\frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial w} \right) \right| dudvdw.$$

- **Divergence theorem (Gauss' theorem):**

$$\iiint_V (\nabla \cdot \mathbf{a})\, dV = \iint_S \mathbf{a} \cdot d\mathbf{S},$$

where V is a bounded volume with boundary $\partial V = S$ and S is a closed surface with normal pointing onwards.

- **Stokes' theorem:**

$$\iint_S (\nabla \times \mathbf{a}) \cdot d\mathbf{S} = \int_C \mathbf{a} \cdot d\mathbf{r},$$

where S is a bounded smooth surface with boundary $\partial S = C$ and S is a piecewise smooth curve. C and S must have compatible orientation.