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General Relativity

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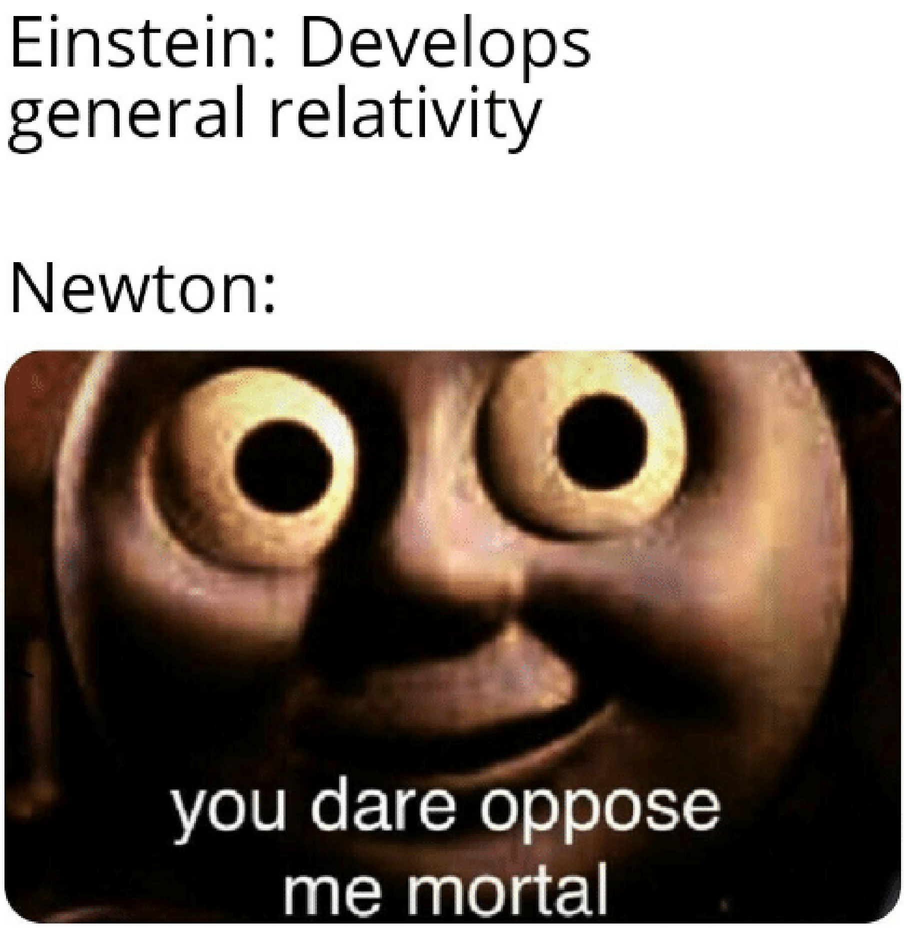

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Lecture 1 Introduction to GR: The Easy Way

1.1 Special Relativity

Observers see space and time differently, but the *spacetime interval* is the same for all observers. It is a *conserved quantity* and in 3D+time, it is

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2. \tag{1.1}$$

However, special relativity only works in an inertial frame - it doesn't handle acceleration. You will need to revise special relativity and be comfortable using it.

1.2 The Principle of Equivalence

What is the difference between an accelerated frame without a gravitational field and a stationary frame within a gravitational field? **Nothing!** This explains why inertial and gravitational masses are the same.

- **Inertial mass, m_i,** gives the constant of proportionality between force and acceleration,

$$F = m_i a. \tag{1.2}$$

- **Gravitational mass, m_g,** determines how the object is affected by the force of gravity,

$$F\frac{GMm_g}{r^2} = m_g g. \tag{1.3}$$

So when an object falls under a gravitational field, we have $a = \frac{m_g}{m_i}g$ and the only way for objects to fall the same way in a gravitational field is if $\frac{m_g}{m_i}$ =const. In electromagnetism, a particle with charge q and inertial mass m_i in a field $\mathcal{E}$ experiences a force $F = q\mathcal{E}$, and acceleration $m_i a$, so $a = \frac{q}{m_i}\mathcal{E}$ and different particles have different values of $\frac{m_g}{m_i}$. Gravity needs to be the same as being accelerated, i.e. the **Principle of Equivalence.** This means **Acceleration = Gravity** and we should no longer consider gravity a force.

1.3 Acceleration

Circular motion is a nice way to get an accelerated frame since the speed is constant. Consider a roundabout:

- If stationary, a person measuring the circumference and radius with a ruler gets a ratio of $\frac{c}{r} = 2\pi$.
- If spinning, if the rule is small enough then it lies along the direction of motion, its length is then shortened and more rules are needed to get round the circumference. Along the radius, the ruler is unaffected as it is perpendicular to the motion, so now $\frac{c}{r} > 2\pi$.

This is not possible in flat space, but it can be in curved space.

- Zero curvature $\implies C = 2\pi r$.
- Positive curvature (surface falls away from us) $\implies C < 2\pi r$.
- Negative curvature (curves away in one direction and together in the other) $\implies C > 2\pi r$. In this curvature, the angles of a triangle sum to $< 180°$.

We must use curved space for accelerating frames!

1.4 Curved Space

On a flat surface, we get a straight line path but on a *curved surface*, we trace out a curved path when we walk in a straight line from our perspective - we may move closer together as if there was an attractive force. **Gravity = acceleration $\implies$ gravity = curvature** - there is no force. The 'walking normally' paths are constant velocity frames, i.e. inertial frames but in geometric language they are *geodesic paths*

- shortest distance between two paths. Natural paths (no forces = inertial frame) are 'straight lines', geodesics of a curved surface. Locally, *geodesics* appear straight but over more extended regions of spacetime then geodesics originally receding from each other begin to approach at a rate governed by the curvature of spacetime. Warping of spacetime comes from matter/energy: **Space tells matter how to move, matter tells space how to curve.** In summary, **Gravity = Acceleration (EP); Acceleration = Curvature (SR); Gravity = Curvature (GR).**

1.5 Implications for Matter

Mass curves spacetime so energy curves spacetime as $E = mc^2$. Mass and energy are equivalent - any form of energy adds to the curvature of spacetime which is gravity. Kinetic energy adds to the response of the particle to gravity, i.e. adds to its mass, so when kinetic energy is a substantial fraction of the rest mass, then additional acceleration will increase the response to gravity of the particles, i.e. increase its mass, which stops the velocity increasing above c.

1.6 Implications on Light

Curved space affects everything, even *massless particles* such as light. This is not obvious in *Newtonian gravity* as here gravity affects things with mass through

$$F = \frac{GMm}{r^2}, \text{ but } a = \frac{F}{m} = \frac{GM}{r^2}, \tag{1.4}$$

so we could argue that in Newtonian gravity, light has no mass so it is not affected, or that gravitational acceleration is independent of mass so it affects everything. Light speed is constant so we would think it is unaffected by gravity. In general relativity, light is clearly affected as it travels across curved spacetime so its path will be curved. This was seen in first experimental tests of general relativity with light from distant stars which has a project line of sight which lies close to the sun. These paths are curved - there are different apparent positions 6 months later when the star is over the other side from the sun (Eddington *solar eclipse*). The measured deviation of this agrees with general relativity!

We can use this deflected light to determine the mass of a galaxy - this can be direct evidence for **Dark Matter.** If we understand the distortion, we can *reconstruct the galaxy* behind a lens and investigate it. The object will also be much brighter than without the lens.

1.7 Implications for Matter and Light

We need gravity to affect light, otherwise we could produce an infinite energy machine - *Pounds-Rebka -Synder experiment.* Particle dropped from h has energy of rest mass plus mgh at the bottom and converting this to a photon and sending it back up the tower. If gravity doesn't affect light, it arrives at the top with $h\nu = m_0c^2 + m_0gh$, then converting all energy to mass, we get a particle of mass $m_1c^2 = m_0c^2 + m_0gh, m_1 > m_0$. We could do this an infinite amount of times and yield infinite energy! If gravity can affect light, however, then the photon loses the same amount of energy on the way up as the particle gains on the way down - this is *gravitational redshift* which is measurable and general relativity correctly predicts this.

1.8 Gravity Waves

Space takes a finite time to respond to changes in the distribution of mass and energy. In general relativity, gravity does not instantaneously affect everything around it. If we have an oscillating distribution of mass, this leads to something like an electromagnetic wave travelling out from the source - a ripple in the structure of spacetime. Spacetime is rather stuff, and you need a huge mass to generate a significant deflection. We have now detected these waves thanks to LIGO!

1.9 Implications: Black Holes

What happens if gravity is really strong? So strong that even a photon can't escape? We have an **event horizon** at which point nothing can escape and a singularity is at the centre of this.

1.10 The Way Ahead

1. Understand how to describe curvature, using tensors and *metrics.*
2. Figure out how mass/energy curves spacetime, using *Einstein's equations.*
3. How to describe 'straight line' *geodesic paths*, using *Lagrangian mechanics.*

1.11 Mathematical Toolkit

We will need complicated maths for this. Also a way to characterise the curvature, or shape, of a surface - this will in the *metric.* The distance between points tells us the shape of the surface, e.g. the *metric for special relativity:*

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 = \begin{pmatrix} c\,dt & dx & dy & dz \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} c\,dt \\ dx \\ dy \\ dz \end{pmatrix}, \tag{1.5}$$

or this can be written as a direct sum:

$$ds^2 = \sum_{\alpha=0}^{3} \sum_{\beta=0}^{3} dx(\alpha)\eta(\alpha,\beta)dx(\beta), \tag{1.6}$$

where $dx(0) = c\,dt,\ dx(1) = dx,\ dx(2) = dy,\ dx(3) = dz$, and $\eta(\alpha,\beta) = \mathrm{diag}(1,-1,-1,-1)$. We also use the **Einstein summation convention**: wherever an expression contains one index as a superscript and the same index as a subscript, then the summation is implied. So for the above, we just write

$$ds^2 = \eta_{\alpha\beta}dx^\alpha dx^\beta. \tag{1.7}$$

Simpler but also $\eta_{\alpha\beta}$ contains a lot of new mathematics. Next is Tensors, a way of describing physical quantities independent of the coordinate system used to locate them in spacetime.

Lecture 2 Introduction to Tensors

- Notation
- Coordinate transforms
- Contravariant tensors
- Covariant tensors

We need **tensors** and compact notation to do physics in curved spacetime.

2.1 Notation

Consider the standard vectors in *Cartesian coordinates* and the definition for $\underline{r}$:

$$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}. \tag{2.1}$$

We have the basis vector $\{\underline{i}, \underline{j}, \underline{k}\}$ and coordinate values $\{x, y, z\}$. Instead, we can write axes as labelled by indices, i.e. $\{x, y, z\} = \{x^1, x^2, x^3\}$. Note: $x^2 \neq x * x$. The 2 is an index, not a power. If we want to square something, we will write $(x^1)^2 = x^1 x^1$. The *basis vectors* are now $\underline{e}_1, \underline{e}_2, \underline{e}_3$. We use the shorthand notation that $\{x^i\}$ means all of the set $\{x^1, x^2, x^3\}$. We can then simplify this further using the **Einstein summation convention**: wherever an expression contains one index as a superscript and the same index as a subscript, then the summation is implied, so

$$\underline{r} = x^1\underline{e}_1 + x^2\underline{e}_2 + x^3\underline{e}_3 = \sum_{i=1}^{3} x^i \underline{e}_i = x^i \underline{e}_i. \tag{2.2}$$

Different letters will imply different things:
- Roman letters $i, j, \ldots$ - summing over 3D space
- Roman letters $a, b, c, \ldots$ - summing over ND space
- Roman letters $A, B, \ldots$ - summing over 2D space
- Greek letters $\alpha, \beta, \mu, \nu, \ldots$ - summing over 4D spacetime $\{x^0, x^1, x^2, x^3\}$, starting from 0 as time is different slightly, so $\{ct, x^i\}$

2.2 Coordinate Transformation

We might want to do various coordinate transformations. Up till now, we have often see this denoted as the primed coordinate frame, using **Lorentz transformations** such as

$$x' = \gamma \left(x - \frac{vct}{c} \right), \tag{2.3}$$

where the extra c factor to make time space-like. Since the prime is similar to a 1, we now use a *bar on the index* to denote a *transformation to a different frame*, so we have

$$x^{\bar{1}} = \gamma \left(x^1 - \frac{vx^0}{c} \right), \tag{2.4}$$

where the 'bar' indicates new coordinate system.
Now thinking about more general spaces with arbitrary curvature. Define a point P in some space, and another point a little further called Q; these points have nothing to do with the coordinate system, they exist irrespective of labels. Assume they are defined on coordinates x^a with basis vectors $\underline{e}_a$:

$$\underline{r}(P) = x^a(P)\underline{e}_a, \qquad \underline{r}(Q) = x^a(Q)\underline{e}_a, \tag{2.5}$$

and the displacement vector which points from P to Q is

$$d\underline{r} = [x^a(Q) - x^a(P)]\, \underline{e}_a = dx^a \underline{e}_a. \tag{2.6}$$

Tensor representations in different coordinate systems depend only on the relative orientations and scales on the coordinate axes at that point and not the absolute values of coordinates - $d\underline{r}$ will be the same in all coordinate systems, though its components will be different depending on our choice of coordinates. An arbitrary position vector

$$\underline{r} = x^1\underline{e}_1 + x^2\underline{e}_2 + \cdots = x^a\underline{e}_a \tag{2.7}$$

has components x^a along whatever basis vectors $\underline{e}_a = \frac{\partial \underline{r}}{\partial x^a}$ we are using. We could use this definition to work our what a basis vector is. Note: they don't have to be orthogonal or unit vectors - or constant, they could be functions of time. Transforming to a different coordinate system $x^{\bar{b}}$, the vector is the same vector so

$$\underline{r} = x^a\underline{e}_a = x^{\bar{b}}\underline{e}_{\bar{b}}. \tag{2.8}$$

We again get basis vectors from partial derivatives, $\underline{e}_{\bar{b}} = \frac{\partial \underline{r}}{\partial x^{\bar{b}}}$, and we can transform between these, writing the *new coordinates* as a function of the *old ones*:

$$x^{\bar{1}} = x^{\bar{1}}(x^1, x^2, \ldots, x^N), \quad x^{\bar{2}} = x^{\bar{2}}(x^1, x^2, \ldots, x^N), \cdots \implies x^{\bar{b}} = x^{\bar{b}}(x^a). \tag{2.9}$$

If this was about a function $f = f(x^1, x^2, \ldots, x^N)$, then we would instantly know how to do a total differential in terms of the partials, i.e.

$$\Delta f = \frac{\partial f}{\partial x^1}\Delta x' + \frac{\partial f}{\partial x^2}\Delta x^2 + \frac{\partial f}{\partial x^2}\Delta x^3 = \frac{\partial f}{\partial x^a}\Delta x^a. \tag{2.10}$$

We can then write our coordinate transformations as (the N equations):

$$\Delta x^{\bar{b}} = \frac{\partial x^{\bar{b}}}{\partial x^a}\Delta x^a, \tag{2.11}$$

where there are N^2 separate $\frac{\partial x^{\bar{b}}}{\partial x^a}$. In the limit of infinitesimals, we have

$$dx^{\bar{b}} = \frac{\partial x^{\bar{b}}}{\partial x^a}dx^a, \qquad dx^{\bar{a}} = \frac{\partial x^{\bar{a}}}{\partial x^b}dx^b. \tag{2.12}$$

Notice how we can simply just switch round the indices - **these are all dummy variables and as long as the index notation is consistent, it is completely arbitrary which letter is used,** i.e. the letters themselves mean nothing.

2.3 Contravariant Tensors of 1st Order (4-vectors)

Entities which transform like the coordinate differences are called *contravariant tensors of first order,* or Rank (1,0) tensors. They are defined by their transformation properties. If $\underline{A} = A^a\underline{e}_a$ has components which transform as

$$A^{\bar{b}} = \frac{\partial x^{\bar{b}}}{\partial x^a}A^a, \tag{2.13}$$

then it is a *contravariant tensor.* Things like 4-velocity, -momentum, -force, etc all transform like this. For A^{ab}, this is an order 2, Rank (2,0) tensor. What about $\underline{e}_a$? The position vector is

$$\underline{r} = x^a\underline{e}_a = x^{\bar{b}}\underline{e}_{\bar{b}}. \tag{2.14}$$

The old basis vectors are $\underline{e}_a = \frac{\partial \underline{r}}{\partial x^a}$; new ones are again the tangents to the coordinate curves so:

$$\underline{e}_{\bar{b}} = \frac{\partial \underline{r}}{\partial x^{\bar{b}}} = \frac{\partial \underline{r}}{\partial x^a}\frac{\partial x^a}{\partial x^{\bar{B}}} = \frac{\partial x^a}{\partial x^{\bar{b}}}\underline{e}_a \tag{2.15}$$

This is the opposite to $dx^{\bar{a}}$. Think - superscript cancels subscript on the top and bringing up the $\bar{b}$ superscript then cancels the LHS → check equations visually this way. Our basis vectors don't transform like the coordinate differences, i.e. like *contravariant tensors* with the bar above on the numerator - they are the other way around!

Example: Basis Vectors for Spherical Polar Coordinates
We know how to do coordinate transforms; we can transform to spherical polars form Cartesian as we know.:

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi,, \qquad z = \cos\theta, \tag{2.16}$$

with $x^i = \{x, y, z\}$ and $\underline{e}_i = \{\hat{i}, \hat{j}, \hat{k}\}$. This is showing the old coordinates written as functions of the new $x^{\bar{j}} = \{r, \theta, \phi\}$. The new basis vectors along the new coordinate directions are

$$\underline{e}_j = \frac{\partial x^i}{\partial x^{\bar{j}}}\underline{e}_i. \tag{2.17}$$

For the radial coordinate, $r = x^{\bar{1}}$,

$$\begin{aligned}\underline{e}_{\bar{1}} = \underline{e}_r = \frac{\partial x^i}{\partial r}\underline{e}_i &= \frac{\partial x}{\partial r}\hat{i} + \frac{\partial y}{\partial r}\hat{j} + \frac{\partial z}{\partial r}\hat{k} \\ &= \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}.\end{aligned} \tag{2.18}$$

We can work our the other basis vectors in the same way:

$$\underline{e}_{\bar{2}} = \underline{e}_\theta = \frac{\partial x^i}{\partial\theta}\underline{e}_i = r\cos\theta\cos\phi\hat{i} + r\cos\theta\sin\phi\hat{j} + r\sin\theta\hat{k} \tag{2.19}$$

$$\underline{e}_{\bar{3}} = \underline{e}_\phi = \frac{\partial x^i}{\partial\phi}\underline{e}_i = -r\sin\theta\sin\phi\hat{i} + r\sin\theta\cos\phi\hat{j}. \tag{2.20}$$

2.4 Covariant Tensors of First Order

Scalar fields are just numbers at a given point - whatever coordinates you give the point makes no change. Its gradient will change depending on the coordinate system. Its gradient is

$$\nabla\phi(x') = \frac{\partial\phi}{\partial x^i}\underline{e}_i. \tag{2.21}$$

Transforming the frame, we have

$$\frac{\partial\phi}{\partial x^{\bar{j}}} = \frac{\partial\phi}{\partial x^i}\frac{\partial x^i}{\partial x^{\bar{j}}}, \tag{2.22}$$

so the gradient isn't a contravariant tensor. Gradients have components which transform the other way to contravariant transforms - like basis vectors. Things that transform like this are *covariant vectors*, also called Rank (0,1) tensors, or *covariant tensors of the first order*. We denote by a lower index on components and the transformation laws for the components are

$$A_{\bar{b}} = \frac{\partial x^a}{\partial x^{\bar{b}}}A_a. \tag{2.23}$$

2.5 Tensors Summary

- **Contravariant** (coordinates):

$$A^{\bar{b}} = \frac{\partial x^{\bar{b}}}{\partial x^a}A^a. \tag{2.24}$$

- **Covariant** (basis vectors):

$$A_{\bar{b}} = \frac{\partial x^a}{\partial x^{\bar{b}}}A_a. \tag{2.25}$$

The aim of all this is to write equations such that they can be used regardless of the coordinate system.

2.6 Higher-Order Tensors

What is $T^{ab} = A^a B^b$?

$$T^{\bar{c}\bar{d}} = A^{\bar{c}} B^{\bar{d}} = \left(\frac{\partial x^{\bar{c}}}{\partial x^a} A^a\right)\left(\frac{\partial x^{\bar{d}}}{\partial x^b} B^b\right) = \frac{\partial x^{\bar{c}}}{\partial x^a}\frac{\partial x^{\bar{d}}}{\partial x^b} A^a B^b = \frac{\partial x^{\bar{c}}}{\partial x^a}\frac{\partial x^{\bar{d}}}{\partial x^b} T^{ab}. \tag{2.26}$$

We have summed over both 'a' and 'b' indices. Therefore, second-order tensors transform with two transformations; this is a second-ord contravariant, or Rank (2,0), tensor. Getting higher-order tensors from lower-order ones in this way is called an *outer product*. We can also do this for covariant tensors, or even a *mixed tensor*, e.g. $T^a{}_b$, as long as they transform this way, it's a vector.

2.7 Tensor Algebra

We can relate tensors in equations, e.g.

$$T^a = k(A^a + B^a), \tag{2.27}$$

How would this look in a transformed coordinate system?

$$\begin{aligned} T^{\bar{b}} &= k(A^{\bar{b}} + B^{\bar{b}}) = k\left(\frac{\partial x^{\bar{b}}}{\partial x^a} A^a + \frac{\partial x^{\bar{b}}}{\partial x^a} B^a\right) \\ &= \frac{\partial x^{\bar{b}}}{\partial x^a} T^a. \end{aligned} \tag{2.28}$$

So tensors are linear (we can add them, multiply by constants, and it leaves their nature unchanged - still tensors of the same order). This has significance - different equations due to coordinate transformations will work if the initial expression does.

Lecture 3 Tensors Continued

We saw before that we have tensor relations which have different numbers but still hold true.

3.1 The Metric Tensor

We need a way of characterising the structure of space-time. We measure the distance between two nearby points and examine how this changes as we move across space.

$$d\underline{r} = dx^a \underline{e}_a \tag{3.1}$$

tells us the distance between the two points. We can get the length of this vector via the dot product:

$$\begin{aligned} ds^2 &= \underline{dr} \cdot \underline{dr} = (dx^a \underline{e}_a) \cdot \left(dx^b \underline{e}_b\right) \\ &= (\underline{e}_a \cdot \underline{e}_b) dx^a\, dx^b = g_{ab} dx^a\, dx^b, \end{aligned} \tag{3.2}$$

where we define the metric tensor as $g_{ab} = \underline{e}_a \cdot \underline{e}_b = g_{ba}$, which we can see is symmetric. This will be used frequently in solving metric equations. The metric tensor's transformation under coordinate change can be seen as we derived the basis vector transformations:

$$\begin{aligned} g_{\bar{a}\bar{b}} &= (\underline{e}_{\bar{a}} \cdot \underline{e}_{\bar{b}}) = \left(\frac{\partial x^c}{\partial x^{\bar{a}}} \underline{e}_c\right) \cdot \left(\frac{\partial x^d}{\partial x^{\bar{b}}} \underline{e}_d\right) \\ &= \frac{\partial x^c}{\partial x^{\bar{a}}} \frac{\partial x^d}{\partial x^{\bar{b}}} (\underline{e}_c \cdot \underline{e}_d) = \frac{\partial x^c}{\partial x^{\bar{a}}} \frac{\partial x^d}{\partial x^{\bar{b}}} g_{cd}, \end{aligned} \tag{3.3}$$

This proves that it is a second-order covariant tensor, or Rank (0,2) tensor. If we have a curved spacetime (or a perverse coordinate system), g_{ab} will be a function of position, and will vary from place to place.

Example: 3D Euclidean Space

The position vector between two points close together in flat 3D space is

$$d\underline{r} = dx\, \hat{i} + dy\, \hat{j} + dz\, \hat{k} = x^i \underline{e}_i. \tag{3.4}$$

What is the metric in this space? We have

$$g_{11} = g_{xx} = \underline{e}_x \cdot \underline{e}_x = \hat{i} \cdot \hat{i} = 1, \tag{3.5}$$

$$g_{22} = g_{yy} = \underline{e}_y \cdot \underline{e}_y = \hat{j} \cdot \hat{j} = 1, \tag{3.6}$$

$$g_{33} = g_{zz} = \underline{e}_z \cdot \underline{e}_z = \hat{k} \cdot \hat{k} = 1. \tag{3.7}$$

All the cross terms are zero as the basis is orthogonal, i.e. $g_{12} = \hat{i} \cdot \hat{j} = 0 \implies g_{ij} = \delta_{ij}$. We can write the metric in matrix form as

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.8}$$

It's not necessarily immediately apparent from the components of the metric tensor which ones will allow coordinate transformations to get us to the unit matrix. Derivatives of the metric can help.

3.2 Kronecker Delta amd Invariance of Tensor Equations

We saw that basis vectors transform as

$$\underline{e}_{\bar{b}} = \frac{\partial x^a}{\partial x^{\bar{b}}} \underline{e}_a. \tag{3.9}$$

This means that any quantity $\underline{A} = A^a \underline{e}_a$ in another frame:

$$
\begin{aligned}
A^{\bar{b}} \underline{e}_{\bar{b}} &= \left(\frac{\partial x^{\bar{b}}}{\partial x^a} A^a \right) \left(\frac{\partial x^d}{\partial x^{\bar{b}}} \underline{e}_d \right) \\
&= \left(\frac{\partial x^{\bar{b}}}{\partial x^a} \frac{\partial x^d}{\partial x^{\bar{b}}} \right) A^a \underline{e}_d = \left(\frac{\partial x^d}{\partial x^a} \right) A^a \underline{e}_d \\
&= \delta^d{}_a A^a \underline{e}_d = A^d \underline{e}_d = A^a \underline{e}_a
\end{aligned}
\tag{3.10}
$$

Note that indices become the same and the letter chosen is of no significance. So, if we have something that transforms as the coordinate differences, then this means its tensor equation looks the same in any frame. From this, we'll be able to write down physics equations that are independent of the coordinate system.

Lecture 4 Tensor Transformations

4.1 Basis Vectors for Covariant Components

Covariant componenets ome from $\nabla\phi$ but this in Cartesian coordinates is just

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{\imath} + \frac{\partial\phi}{\partial y}\hat{\jmath} + \frac{\partial\phi}{\partial z}\hat{k} = A_i\underline{e}^i,\ \text{where } A^i = \frac{\partial\phi}{\partial x^i} \tag{4.1}$$

are the components, and since we know that these are covariant, the basis vectors must have the high index. When $\hat{\imath}, \hat{\jmath}, \hat{k}$ are orthogonal, $\underline{e}_i = \underline{e}^i$, but when they are not orthogonal (as in general relativity's curved space), $\underline{e}_i \neq \underline{e}^i$. Let's see what happens if we go to a new coordinate frame $x^{\bar{j}}$:

$$\nabla\phi = \frac{\partial\phi}{\partial x^i}\underline{e}^i = \frac{\partial\phi}{\partial x^{\bar{j}}}\left(\frac{\partial x^{\bar{j}}}{\partial x^i}\underline{e}^i\right) = \frac{\partial\phi}{\partial x^{\bar{j}}}\nabla x^{\bar{j}} = \frac{\partial\phi}{\partial x^{\bar{j}}}\underline{e}^{\bar{j}}, \tag{4.2}$$

which tells us the *contravariant basis* transforms as

$$\underline{e}^{\bar{j}} = \nabla x^{\bar{j}} = \frac{\partial x^{\bar{j}}}{\partial x^i}\underline{e}^i. \tag{4.3}$$

Remember the *covariant basis* was

$$\underline{e}_i = \frac{\partial\underline{r}}{\partial x^i}. \tag{4.4}$$

Now $\underline{a} = A_a\underline{e}^a$ will transform to another frame as

$$A_{\bar{b}}\underline{e}^{\bar{b}} = \frac{\partial x^a}{\partial x^{\bar{b}}}A_a\frac{\partial x^{\bar{b}}}{\partial x^c}\underline{e}^c = \frac{\partial x^a}{\partial x^c}A_a\underline{e}^c = \delta_c{}^a A_a\underline{e}^c = A_a\underline{e}^a. \tag{4.5}$$

Note that the two sets of basis vectors themselves look identical if we have an *orthonormal set of coordinates*, but they are not identical if the coordinates are not orthogonal.

- In Cartesian Coordinates

$$\underline{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \tag{4.6}$$

  Covariant basis vectors are

$$\underline{e}_1 = \underline{e}_x = \frac{\partial\underline{r}}{\partial x} = \hat{\imath}. \tag{4.7}$$

  Contravariant basis vectors are

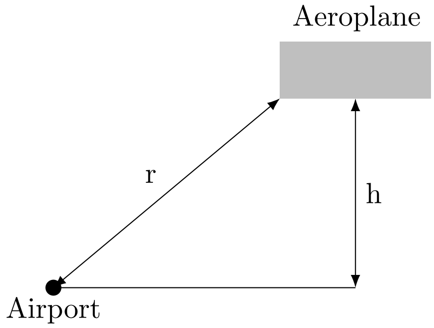
$$\underline{e}^1 = \underline{e}^x = \nabla x = \frac{\partial x}{\partial x}\hat{\imath} + \frac{\partial x}{\partial y}\hat{\jmath} + \frac{\partial x}{\partial z}\hat{k} = \hat{\imath}. \tag{4.8}$$

  This is a special example as written above.
- In 3D Spherical polars

$$r = x^2 + y^2 + z^2, \tag{4.9}$$

$$\begin{aligned}\underline{e}^r = \nabla r &= \frac{\partial r}{\partial x}\hat{\imath} + \frac{\partial r}{\partial y}\hat{\jmath} + \frac{\partial r}{\partial z}\hat{k} = \frac{x}{r}\hat{\imath} + \frac{y}{r}\hat{\jmath} + \frac{z}{r}\hat{k},\\ &= \sin\theta\cos\phi\hat{\imath} + \sin\theta\sin\phi\hat{\jmath} + \cos\theta\hat{k}.\end{aligned} \tag{4.10}$$

4.2 Height-Distance Coordinates



This is not an orthogonal system:

$$\underline{r} = (r^2 - h^2)^2\hat{\imath} + h\hat{\jmath}. \tag{4.11}$$

Work out the basis vectors in this coordinate system, i.e. what are $\underline{e}_i$?

$$\underline{e}_r = \frac{\partial \underline{r}}{\partial r} = \frac{1}{2}2r(r^2-h^2)^{-1/2}\hat{\imath} = \left(1 - \frac{h^2}{r^2}\right)^{-1/2}\hat{\imath}, \tag{4.12}$$

$$\underline{e}_h = \frac{\partial \underline{r}}{\partial h} = -\frac{1}{2}2h(r^2-h^2)^{-1/2}\hat{\imath} + \hat{\jmath} = \frac{-h}{r}\left(1 - \frac{h^2}{r^2}\right)^{-1/2}\hat{\imath} + \hat{\jmath}. \tag{4.13}$$

What are the contravariant basis vectors, $\underline{e}_i = \nabla x^i$? We have $r^2 = x^2 + y^2$, $2r\frac{\partial r}{\partial x} = 2x$, $h = y$, and then

$$\begin{aligned} \underline{e}^r = \nabla r &= \frac{\partial r}{\partial x}\hat{\imath} + \frac{\partial r}{\partial y}\hat{\jmath} = \frac{x}{r}\hat{\imath} + \frac{y}{r}\hat{\jmath} \\ &= \frac{\sqrt{r^2-h^2}}{r}\hat{\imath} + \frac{h}{r}\hat{\jmath} = \left(1-\frac{h^2}{r^2}\right)^{-1/2}\hat{\imath} + \frac{h}{r}\hat{\jmath}, \end{aligned} \tag{4.14}$$

$$\underline{e}^h = \nabla h = \frac{\partial h}{\partial x}\hat{\imath} + \frac{\partial h}{\partial y}\hat{\jmath} = \hat{\jmath}. \tag{4.15}$$

4.3 Covariant and Contravariant: The Metric

We can write vectors in either the old basis in the tangent space or the new basis in the cotangent space, i.e. $\underline{\mu} = \mu^a\underline{e}_a = \mu_a\underline{e}^a$. If the basis vectors are the same, i.e. orthonormal bases, then the contravariant and covariant components are identical, but in general this is not the case. Take another vector, $\underline{\lambda} = \lambda^a\underline{e}_a = \lambda_b\underline{e}^b$. The dot product of these is

$$\begin{aligned} \underline{\lambda}\cdot\underline{\mu} &= (\lambda^a\underline{e}_a)\cdot(\mu^b\underline{e}_b) = \lambda^a\mu^b(\underline{e}_a\cdot\underline{e}_b) = \lambda^a\mu^b g_{ab} \\ &= |\underline{\lambda}||\underline{\mu}|\cos\chi \end{aligned} \tag{4.16}$$

Here, we use the covariant form of the metric to express the dot product of the vectors with contravariant components. We could also express this with the covariant components, using the contravariant metric:

$$\underline{\lambda}\cdot\underline{\mu} = (\lambda_a\underline{e}^a)\cdot(\mu_b\underline{e}^b) = \lambda_a\mu_b(\underline{e}^a\cdot\underline{e}^b) = \lambda_a\mu_b g^{ab}. \tag{4.17}$$

What about one with covariant and one with contravariant components?

$$\underline{\lambda}\cdot\underline{\mu} = (\lambda^a\underline{e}_a)\cdot(\mu_b\underline{e}^b) = \lambda^a\mu_b(\underline{e}_a\cdot\underline{e}^b) \tag{4.18}$$

What is this dot product? Let's do it in 3D:

$$\begin{aligned} \underline{e}_i\cdot\underline{e}^j &= \left(\frac{\partial x^i}{\partial x^i}\hat{\imath} + \frac{\partial x^2}{\partial x^i}\hat{\jmath} + \frac{\partial x^3}{\partial x^i}\hat{k}\right)\cdot\left(\frac{\partial x^j}{\partial x^1}\hat{\imath} + \frac{\partial x^j}{\partial x^2}\hat{\jmath} + \frac{\partial x^j}{\partial x^3}\hat{k}\right) \\ &= \frac{\partial x^1}{\partial x^i}\frac{\partial x^j}{\partial x^1} + \frac{\partial x^2}{\partial x^i}\frac{\partial x^j}{\partial x^2} + \frac{\partial x^3}{\partial x^i}\frac{\partial x^j}{\partial x^3} = \frac{\partial x^j}{\partial x^i} = \delta_i{}^j. \end{aligned} \tag{4.19}$$

Now because of the summation convention, $\delta_i{}^i = \delta_1{}^1 + \delta_2{}^2 + \delta_3{}^3 = 3$, so we have

$$(\lambda^a \underline{e}_a) \cdot (\mu_b \underline{e}^b) = \lambda^a \mu_b (\underline{e}_a \cdot \underline{e}^b) = \lambda^a \mu_b \delta^b{}_a = \lambda^a \mu_a, \tag{4.20}$$

where we have contracted $\mu_b \delta^b{}_a$ and are now left with the repeated index to be summed over. By symmetry of the dot product and metric, then

$$\lambda^a \mu_a = \lambda_a \mu^a = g_{ab} \lambda^a \mu^b = g^{ab} \lambda_a \mu_b. \tag{4.21}$$

This gives us an easy way of swapping between contravariant and covariant components - we use the *metric*:

$$g_{ab} \lambda^a \mu^b = \lambda_a \mu^a, \qquad g^{ab} \lambda_a \mu_b = \lambda^b \mu_b, \tag{4.22}$$

$$\lambda^a = g^{ab} \lambda_b, \qquad \lambda_a = g_{ab} \lambda^b. \tag{4.23}$$

Tensors derived from other tensors by raising or lowering the indices via the metric are called *associated tensors.* We also get another useful metric relation in that

$$\lambda_a = g_{ab} \lambda^b = g_{ab} g^{bc} \lambda_c \implies g_{ab} g^{bc} = \delta_a{}^c. \tag{4.24}$$

Lecture 5 Derivatives in Curved Space

We want to write normal differential equations in curved space. We will need:

- **Parallel Transport** - derivatives in curved space are not the same as in flat space.
- **Absolute Derivative** - in flat space this is the same as the normal derivative, but this is not the case in curved space.
- **Covariant Derivative** - doesn't depend on the path.

5.1 The Metric and Curvature of Spacetime

Assume we deal with a space which is *continuous* and *differentiable*, i.e. **differentiable manifold**; also assume it has a metric. If the metric is positive definite (inner products are positive), then this is called a **Riemannian manifold**; if not, **pseudo-Riemannian.** In General Relativity, there is no 'force-at-a-distance' gravity anf the paths of freely-falling objects curve only because they are following the shortest path in curved spacetime (which is curved because of mass). Any paths which follow the local curvature of spacetime (free-fall) are inertial frames, and we know how to do physics in inertial frames so we need to find what these paths are.

5.2 Parallel Transport

In flat space, we can take a vector from a point and it keeps its direction - this is called **parallel transport.** This is important as when we do differentials, we are comparing a vector at some point with a vector at some other point. In curved space, the direction the vector points at the end of the path depends on the *path* as well as the start and end points. Consider a vector $\underline{\lambda} = \lambda^a(s)\underline{e}_a$ defined along some curve given by coordinates $x^a(s)$, where s is some point on the path. If we define the derivative in the 'obvious' way:

$$\frac{d\lambda^a}{ds} = \lim_{\delta s \to 0}\left[\frac{\lambda^a(s+\delta s) - \lambda^a(s)}{\delta s}\right]. \tag{5.1}$$

This does not transform, therefore it is **NOT** a tensor. We need a tensor version to be able to use our tensor algebra. We want derivatives of coordinate values and how they are in the new coordinate frame:

$$\begin{aligned}\frac{d\lambda^{\bar{a}}}{ds} &= \frac{d}{ds}\left(\frac{\partial x^{\bar{a}}}{\partial x^b}\lambda^b\right) = \frac{\partial x^{\bar{a}}}{\partial x^b}\frac{d\lambda^b}{ds} + \lambda^b\frac{d}{ds}\left(\frac{\partial x^{\bar{a}}}{\partial x^b}\right), \quad \frac{d}{ds} = \frac{\partial}{\partial x^c}\frac{dx^c}{ds}, \\ &= \underbrace{\frac{\partial a^{\bar{a}}}{\partial x^b}\frac{d\lambda^b}{ds}}_{\text{tensor-like}} + \lambda^b \underbrace{\left(\frac{\partial^2 x^{\bar{a}}}{\partial x^b \partial x^c}\right)}_{\substack{\text{change in} \\ \text{coord transforms}}} \frac{dx^c}{ds}.\end{aligned} \tag{5.2}$$

In general, the coordinate transforms such that $\frac{\partial x^{\bar{a}}}{\partial x^b}$ evaluated as $s+\delta s$ is not the same as $\frac{\partial x^{\bar{a}}}{\partial x^b}$ at s as they depend on position. For differentiation to give a vector (tensor), we must take component differences at the same point. In flat space, we slide one of the vectors to the other by moving one of the vectors parallel to itself, i.e. *parallel transport.* We mean that when we move the vector to its new position, it doesn't change, so

$$\frac{d\lambda}{ds} = 0. \tag{5.3}$$

What does this parallel transport look like for a tensor?

$$\frac{\lambda}{ds} = 0 = \frac{d}{ds}(\lambda^a\underline{e}_a) = \frac{d\lambda^a}{ds}\underline{e}_a + \lambda^a\frac{d\underline{e}_a}{ds} = \frac{d\lambda^a}{ds}\underline{e}_a + \lambda^a\frac{\partial \underline{e}_a}{\partial x^b}\frac{dx^b}{ds}. \tag{5.4}$$

We need to understand $\frac{\partial \underline{e}_a}{\partial x^b}$ but since this is the derivative of a vector then it is itself a vector so we can write it as a linear combination of basis vectors:

$$\frac{\partial \underline{e}_a}{\partial x^b} = \Gamma_{ab}{}^{c}\underline{e}_c, \tag{5.5}$$

where $\Gamma_{ab}{}^{c}$ are the **Christoffel Symbols,** or *connection coefficients.* We have written them in tensor notation, but they do not transform as tensors in general. You can't work out the Christoffel symbols in one coordinate system and transform them into another - you have to start from the basis vectors each time. Christoffel symbols are a property of the spacetime in which vectors are embedded.

$$\begin{aligned} \frac{d\lambda}{ds} = 0 &= \frac{d\lambda^a}{ds}\underline{e}_a + \lambda^a \Gamma_{ab}{}^{c}\underline{e}_c \frac{dx^b}{ds} \\ &= \frac{d\lambda^a}{ds}\underline{e}_a + \lambda^b \Gamma_{bc}{}^{a}\frac{dx^c}{ds}\underline{e}_a, \end{aligned} \tag{5.6}$$

where we relabelled indices $a \to b$, $b \to c$, $c \to a$ for the second line. Now we have $\underline{e}_a$ on both terms so this is all the same vector component and we can see that, for parallel transport, we have

$$\frac{d\lambda^a}{ds} + \lambda^b \Gamma_{bc}{}^{a}\frac{dx^c}{ds} = 0. \tag{5.7}$$

So if we parallelly-transport a vector λ at s to $s + \delta s$, we have

$$\frac{\delta\lambda^a}{\delta s} + \lambda^b \Gamma_{bc}{}^{a}\frac{\delta x^c}{\delta s} = 0 \implies \delta\lambda^a = -\lambda^b \Gamma_{bc}{}^{a}\frac{\delta x^c}{\delta s}\delta s \tag{5.8}$$

$$\implies \delta\lambda^a = -\lambda^b \Gamma_{bc}{}^{a}\delta x^c = \delta\lambda_{\parallel}{}^{a}. \tag{5.9}$$

So the parallelly-transported vector has components

$$\lambda^a_{\parallel}(s + \delta s) = \lambda^a(s) - \lambda^b \Gamma_{bc}{}^{a}\delta x^c, \tag{5.10}$$

and we can compare this with the vector $\underline{\lambda}(s + \delta s)$ to define the **absolute derivative.**

5.3 Absolute Derivative

The point is that $\underline{\lambda}(s + \delta s)$ contains two differences with respect to $\underline{\lambda}(s)$:

- Firstly, real physical differences - e.g. acceleration, deceleration, etc if the contravariant tensor is telling us about velocity.
- There can also be changes between these two vectors just because the *space is curved.*

We want to remove these 'space changes' which are what the parallelly-transported vector tells us about, so that we can see the real physical changes. We define the absolute derivative:

$$\begin{aligned} \frac{D\lambda^a}{ds} &= \lim_{\delta s \to 0}\left[\frac{\lambda^a(s+\delta s) - \lambda^a_{\parallel}(s+\delta s)}{\delta s}\right] \\ &= \frac{\left(\lambda^a(s) + \frac{d\lambda^a}{ds}\delta s\right) - \left(\lambda^a(s) + \delta\lambda^a_{\parallel}\right)}{\delta s} \\ &= \frac{\left(\lambda^a(s) + \frac{d\lambda^a}{ds}\delta s\right) - \left(\lambda^a(s) - \lambda^b \Gamma_{bc}{}^{a}\frac{dx^c}{ds}\delta x^c\right)}{\delta s} \end{aligned} \tag{5.11}$$

$$\frac{D\lambda^a}{ds} = \frac{d\lambda^a}{ds} + \lambda^b \Gamma_{bc}{}^{a}\frac{dx^c}{ds}. \tag{5.12}$$

Although $\frac{d\lambda^a}{ds}$ is not a tensor and $\Gamma_{bc}{}^{a}$ is not either, the two non-tensor bits cancel each other out such that $\frac{D\lambda^a}{ds}$ does transform as a tensor. So to define the physical, or **absolute derivative**, we use parallel transport to take out the space changes. We are left with a value which is *independent of coordinate system* but does depend on the path taken, i.e. **path dependent.**

5.4 Covariant Derivative

We can do this in terms of coordinates as opposed to a single parameter, and remove the path dependence. Writing the absolute derivative with a partial in the first term:

$$\frac{D\lambda^a}{ds} = \frac{\partial\lambda^a}{\partial x^c}\frac{dx^c}{ds} + \lambda^b\Gamma_{bc}{}^a\frac{dx^c}{ds} = \left(\frac{\partial\lambda^a}{\partial x^c} + \Gamma^a{}_{bc}\lambda^b\right)\frac{dx^c}{ds} = \lambda^a{}_{;c}\frac{dx^c}{ds} \tag{5.13}$$

So the components of the **covariant derivative** are

$$\lambda^a{}_{;c} \equiv \nabla_c\lambda^a = \frac{\partial\lambda^a}{\partial x^c} + \Gamma^a{}_{bc}\lambda^b \tag{5.14}$$

You can check this notation; note this is a derivative. This is closely related to the absolute derivative, which is a Rank (0,1) tensor, so this is a Rank (1,1) tensor.

5.5 The Christoffel Symbols

What are these $\Gamma^a{}_{bc}$? They come from the derivatives of the tangent basis vectors, but these are defined $\underline{e}_a = \frac{\partial \underline{r}}{\partial x^a}$, so

$$\Gamma_{ab}{}^c\underline{e}_c = \frac{\partial\underline{e}_a}{\partial x^b} = \frac{\partial^2\underline{r}}{\partial x^a\partial x^b} = \frac{\partial^2\underline{r}}{\partial x^b\partial x^a} = \frac{\partial\underline{e}_b}{\partial x^a} = \Gamma_{ba}{}^c\underline{e}_c. \tag{5.15}$$

They are symmetric, (changing the order of integration assumes that the second partial derivatives are continuous) so we only need half as many Christoffel symbols as initially thought since $\Gamma_{ab}{}^c = \Gamma_{ba}{}^c$. We can also look at how the metric changes as a function of the coordinates:

$$\begin{aligned}\frac{\partial}{\partial x^c}(g_{ab}) &= \frac{\partial}{\partial x}(\underline{e}_a\cdot\underline{e}_b) + \frac{\partial\underline{e}_a}{\partial x^c}\cdot\underline{e}_b + \underline{e}_a\cdot\frac{\partial\underline{e}_b}{\partial x^c}\\ &= (\Gamma_{ac}{}^d\underline{e}_d)\cdot\underline{e}_b + \underline{e}_a\cdot(\Gamma_{bc}{}^d\underline{e}_d),\end{aligned} \tag{5.16}$$

$$\frac{\partial}{\partial x^c}(g_{ab}) = \Gamma_{ac}{}^d g_{db} + \Gamma_{bc}{}^d g_{ad}, \tag{5.17}$$

$$\frac{\partial}{\partial x^a}(g_{bc}) = \Gamma_{ba}{}^d g_{dc} + \Gamma_{ca}{}^d g_{bd}, \tag{5.18}$$

$$\frac{\partial}{\partial x^b}(g_{ca}) = \Gamma_{cb}{}^d g_{da} + \Gamma_{ab}{}^d g_{cd}, \tag{5.19}$$

If we add the first two and subtract the third, we get an expression for the Christoffel symbols in terms of the derivatives of the metric:

$$\frac{\partial g_{ab}}{\partial x^c} + \frac{\partial g_{bc}}{\partial x^a} - \frac{\partial g_{ca}}{\partial x^b} = 2\Gamma_{ca}{}^d g_{db}. \tag{5.20}$$

Multiplying by $\frac{1}{2}g^{fb}$ and remembering that $g^{fb}g_{db} = \delta^f{}_d$, we get

$$\Gamma_{ca}{}^f = \frac{1}{2}g^{fb}\left(\frac{\partial g_{bc}}{\partial x^a} - \frac{\partial g_{ca}}{\partial x^b} + \frac{\partial g_{ab}}{\partial x^c}\right). \tag{5.21}$$

Using the notation $\frac{\partial}{\partial x^c} \equiv \partial_c$ and the symmetry $\Gamma_{ca}{}^f = \Gamma_{ac}{}^f$ yields

$$\Gamma_{ac}{}^f = \frac{1}{2}g^{fb}\left(\partial_a g_{bc} - \partial_b g_{ca} + \partial_c g_{ab}\right). \tag{5.22}$$

So if we have a metric (and time), we can calculate these, since we can calculate derivatives of the metric.

Notation:

- **Partial Derivative:** $\frac{\partial A^a}{\partial x^c} = \partial_c A^a = A^a{}_{;c}$
- **Covariant Derivative:** $\nabla_c A^a = A^a{}_{;c}$
- **Absolute Derivative:** $\nabla_{\bar{u}} A^a = \frac{DA^a}{ds}$
- **Christoffel Symbols:** $\Gamma_{bc}{}^a$ or $\{{a \atop bc}\}$ or $\{a, bc\}$

Lecture 6 Calculating Christoffel Symbols

Example: 2D flat space

The metric for flat space in Cartesian coordinates $x^A = \{x, y\}$, $g_{AB} = \mathrm{diag}(1,1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ doesn't depend on position so the partial derivatives of the metric are zero, i.e. ${\Gamma^A}_{BC} = 0$. This is not true if we use *polars*:

$$ds^2 = dr^2 + r^2\,d\theta^2 \tag{6.1}$$

$$g_{AB} = \mathrm{diag}(1, r^2) \tag{6.2}$$

$${\Gamma^A}_{BC} \neq 0 \tag{6.3}$$

So the Christoffel symbols tell us about the curvature, but also about the coordinate system we have chosen, i.e. non-zero in flat 2D from changing coordinate system.

Example: Surface of a Sphere

In 3D space, we have

$$ds^2 = dr^2 + r^2\,d\theta^2 + r^2 \sin^2\theta\,d\phi^2 \tag{6.4}$$

and we can limit this at a fixed radius $r = a$ so $dr = 0$ for 2D, so

$$ds^2 = a^2\,d\theta^2 + a^2 \sin^2\theta\,d\phi^2 = g_{AB}dx^A\,dx^B \tag{6.5}$$

$$g_{AB} = \mathrm{diag}(a^2, a^2\sin^2\theta) \tag{6.6}$$

The Christoffel symbols are defined as

$${\Gamma_{AC}}^F = \frac{1}{2} g^{FB}\left(\partial_A g_{BC} - \partial_B g_{CA} + \partial_C g_{AB}\right), \tag{6.7}$$

so we also need the *covariant metric components* g^{AB} from

$$g^{AB} g_{BC} = {\delta^A}_C. \tag{6.8}$$

In general, this leads to a set of simultaneous equations to solve:

$$g^{AB} g_{BC} = g^{A1} g_{1C} + g^{A2} g_{2C} = {\delta^A}_C. \tag{6.9}$$

For diagonal **only**, we don't have to do a matrix inverse to solve these as all cross-terms are zero, so we get

$$g^{AB} g_{BA} = g^{AA} g_{AA} = 1 \implies g^{AA} = \frac{1}{g_{AA}}. \tag{6.10}$$

If the covariant form of the metric is diagonal, then so is the *contravariant metric*. So we have

$$g^{AB} = \mathrm{diag}\left(\frac{1}{a^2}, \frac{1}{a^2 \sin^2\theta}\right) \tag{6.11}$$

So in our 2D space, we have

$${\Gamma^\theta}_{\theta\theta} = \frac{1}{2} g^{\theta B}\left(\partial_\theta g_{B\theta} - \partial_B g_{\theta\theta} + \partial_\theta g_{\theta B}\right), \quad g^{\theta B} = 0, B \neq \theta, \tag{6.12}$$

and as $g^{\theta B} = 0$ except for when $B = \theta$, this then collapses to

$${\Gamma^\theta}_{\theta\theta} = \frac{1}{2} g^{\theta\theta}\left(\partial_\theta \cancel{g_{\theta\theta}} - \partial_\theta \cancel{g_{\theta\theta}} + \partial_\theta \cancel{g_{\theta\theta}}\right) = \frac{1}{2} \cdot \frac{1}{a^2} \cdot 0 = 0. \tag{6.13}$$

The bracket terms are all zero as $g_{\theta\theta} = a^2$ is not proportional to θ and so the partial with repsect to θ is zero.

$$\Gamma^{\theta}{}_{\phi\theta} = \Gamma^{\theta}{}_{\theta\phi} = \frac{1}{2} g^{\theta B} \left(\partial_\theta g_{B\phi} - \partial_B g_{\phi\theta} + \partial_\phi g_{\theta B} \right), \tag{6.14}$$

where again $g^{\theta B}$ is only non-zero for $B = \theta$ since the metric is diagonal, so:

$$\Gamma^{\theta}{}_{\phi\theta} = \Gamma^{\theta}{}_{\theta\phi} = \frac{1}{2} g^{\theta\theta} \left(\partial_\theta \cancel{g_{\theta\phi}} - \partial_\theta \cancel{g_{\phi\theta}} + \partial_\phi \cancel{g_{\theta\theta}} \right) = \frac{1}{2} \cdot \frac{1}{a^2} \cdot 0 = 0, \tag{6.15}$$

$$\Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta, \tag{6.16}$$

$$\Gamma^{\phi}{}_{\theta\phi} = \cot\theta. \tag{6.17}$$

The rest of the Christoffel symbols for this example are 0 (there are $2^3 = 8$ in total).

6.1 Geodesic Equations

We have everything we need to get **geodesic paths** (inertial frames). In maths, they define geodesics as the shortest distance between two points. In physics, these are inertial frames and in an inertial frame the velocity doesn't change. There are *no forces to produce an acceleration.* We define velocity as a tensor as

$$\underline{v} = v^\alpha \underline{e}_\alpha = \frac{\partial x^\alpha}{\partial \tau} \underline{e}_\alpha. \tag{6.18}$$

If there is no change in this, then its derivative is zero, but we also saw that there can be swings in a vector which arise from curved space. We say its the *absolute derivative* which is zero:

$$\frac{Dv^\alpha}{d\tau} = 0 \tag{6.19}$$

By an affine parameter, we mean a linear function of path length $u = A + Bs$, such as the proper time τ, so the components change like

$$\frac{dv^\alpha}{d\tau} + \Gamma^{\alpha}{}_{\beta\gamma} v^\beta \frac{dx^\gamma}{d\tau} = 0, \tag{6.20}$$

$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma^{\alpha}{}_{\beta\gamma} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} = 0, \tag{6.21}$$

and we can use the notation where a dot means derivative with respect to time to write this as

$$\ddot{x}^\alpha + \Gamma^{\alpha}{}_{\beta\gamma} \dot{x}^\beta \dot{x}^\gamma = 0. \tag{6.22}$$

This works for any parameter linearly-related to path length s, i.e. for $u = A + Bs$ or $\tau = \frac{1}{c}s$, our *affine parameters.* This is the *affinely-parameterised geodesic equation* in an N-dimensional manifold. Although this is a second-order differential equation, we cannot just integrate due to the summations.

Example: Geodesics in Flat Space

In flat space, all $\Gamma^{i}{}_{jk} = 0$ so the geodesic equations become $\ddot{x}^a = 0$, where dot denotes derivatives with respect to path length s. We integrate to get $\dot{x}^a = A$ and $x^a = As + B$, so geodesics in flat space have constant velocity and direction - *Newtonian inertial frames.*

Example: Geodesics on a Sphere - Paths in θ

In general, it is too difficult to do geodesics in full generality - often, we just choose a path and see if its a geodesic. A natural choice is a path defined by only one of the parameters, called a *parameter curve.* We know that for a sphere the only non-zero Christoffel symbols are

$$\Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta, \qquad \Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \cot\theta. \tag{6.23}$$

Geodesic paths satisfy the equation

$$\frac{d^2x^A}{ds^2} + \Gamma^A{}_{BC}\frac{dx^B}{ds}\frac{dx^c}{ds} = 0. \tag{6.24}$$

Suppose the path s is just a change in θ, then $s = a\theta$, so there is no dependence on ϕ, i.e. $x^2 = \phi =$ constant $\implies \frac{d\phi}{ds} = 0$, while for θ we have

$$\theta = \frac{s}{a}, \quad \frac{d\theta}{ds} = \frac{1}{a}, \quad \frac{d^2\theta}{ds^2} = 0, \tag{6.25}$$

$$\frac{d^2\theta}{ds^2} + \Gamma^\theta{}_{BC}\frac{dx^B}{ds}\frac{dx^c}{ds} = \frac{d^\theta}{ds^2} + \Gamma^\theta{}_{\phi\phi}\frac{d\phi}{ds}\frac{d\phi}{ds} = 0. \tag{6.26}$$

This satisfies the geodesic equation in θ, so we get a big tick and a gold star! We should check that this also holds in ϕ:

$$\frac{d^2\phi}{ds^2} + \Gamma^\phi{}_{BC}\frac{dx^B}{ds}\frac{dx^C}{ds} = \frac{d^2\phi}{ds^2} + \Gamma^\phi{}_{\theta\phi}\frac{d\theta}{ds}\frac{d\phi}{ds} + \Gamma^\phi{}_{\phi\theta}\frac{d\phi}{ds}\frac{d\theta}{ds} = 0 \tag{6.27}$$

So it's a geodesic path! Yayyyyyy!

Example: Geodesics on a Sphere - Paths in ϕ

Suppose the path s is just a change in ϕ, then $s = a\phi$ so there is no dependence on θ, i.e.

$$x^\theta = \theta_0, \qquad \frac{d\theta}{ds} = 0, \qquad \frac{d^2\theta}{ds^2} = 0, \tag{6.28}$$

$$\phi = \frac{s}{a}, \qquad \frac{d\phi}{ds} = \frac{1}{a}, \qquad \frac{d^2\phi}{ds^2} = 0. \tag{6.29}$$

The LHS of the geodesic equation in ϕ is then

$$\frac{d^\phi}{ds^2} + \Gamma^\phi{}_{BC}\frac{dx^B}{ds}\frac{dx^C}{ds} = 0 + \Gamma^\phi{}_{\theta\phi}\frac{d\theta}{ds}\frac{d\phi}{ds} + \Gamma^\phi{}_{\phi\theta}\frac{d\phi}{ds}\frac{d\theta}{ds} = 0, \tag{6.30}$$

and the LHS of the geodesic equatoin in θ is

$$\frac{d^2\theta}{ds^2} + \Gamma^\theta{}_{BC}\frac{dx^B}{ds}\frac{dx^C}{ds} = \frac{d^2\theta}{ds^2} + \Gamma^\theta{}_{\phi\phi}\frac{d\phi}{ds}\frac{d\phi}{ds} = 0 - \sin\theta_0\cos\theta_0\frac{1}{a}. \tag{6.31}$$

This is only equal to zero (i.e. only a geodesic) for the special case of $\sin\theta_0\cos\theta_0 = 0 \to \theta_0 = 0, \frac{\pi}{2}, \pi$. $0, \pi$ are the N and S poles respectively where a path in ϕ is just a point. The only geodesics which involve any distance is the equator.

Lecture 7 Easier Christoffel Symbols and Geodesics Paths

7.1 Euler-Lagrange Equations

An easier way to find Christoffel symbols and geodesic paths is to use the *Euler-Lagrange equations.* We could have solved for the *geodesics* by saying that these are the paths which give the shortest distance between two points, i.e. we are looking for the external path which has

$$\delta \left[\int ds \right] = 0. \tag{7.1}$$

The shortest path is a minimum. This is often what we do in *classical mechanics* where we look for the minimum energy path by getting the *Lagrangian* $L = T - V$, as the sum of the kinetic T and potential V energies, and then finding the minimum energy path by integrating this over time, i.e.

$$\delta \left[\int \mathcal{L}\, dt \right] = 0. \tag{7.2}$$

For *freely-moving particles* (i.e. those on geodesics), there are no potential energy terms as $V = 0$, so we are only looking at the kinetic energy:

$$\delta \left[\int \mathcal{L}\, dt \right] = \delta \left[\int T\, dt \right] = 0. \tag{7.3}$$

In terms of per unit mass,

$$T = \frac{1}{2} \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 \right], \tag{7.4}$$

or in terms of the *metric* (in Euclidean 3D non-relativistic flat space),

$$T = \frac{1}{2} g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j = \frac{1}{2} \left(\frac{ds}{dt} \right)^2. \tag{7.5}$$

So the *minimum energy condition* (Hamilton's principle in classical mechanics) gives

$$\delta \left[\int T\, dt \right] = \delta \left[\int \left(\frac{ds}{dt} \right)^2 dt \right] = 0, \tag{7.6}$$

i.e. this is basically the same as the minimum path requirement which defines our geodesic path which is

$$\delta \left[\int \left(\frac{ds}{dt} \right) dt \right] = 0. \tag{7.7}$$

We know that in classical mechanics the solution with the minimum energy satisfies the *Euler-Lagrange equations,* i.e.

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^i} \right) - \frac{\partial \mathcal{L}}{\partial x^i} = 0, \qquad \mathcal{L} = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j. \tag{7.8}$$

There are some technical problems using this in 4D spacetime, where the metric is indefinite (it can be positive, negative, or zero); if you do it, then it comes out to be the same except we now take our derivative with respect to some *affine parameter*, linearly related to the *invariant path length, s* rather than to t. So the dots then stand for the derivative with respect to the affine parameter, e.g. *geodesic paths* are now

$$\ddot{x}^\alpha + \Gamma^\alpha{}_{\beta\gamma} \dot{x}^\beta \dot{x}^\gamma = 0. \tag{7.9}$$

Classically, we used $\frac{d}{dt}$, now in relativity we use $\frac{d}{ds}$ (which is pretty much equivalent to $c\frac{d}{dt}$) and/or $\frac{d}{d\tau}$. We could use any affine parameter, but we will use $\tau = \frac{s}{c}$ here. So we now have the *Euler-Lagrange equation* as

$$\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^b}\right) - \frac{\partial \mathcal{L}}{\partial x^b} = 0, \qquad \mathcal{L} = \frac{1}{2} g_{ab} \dot{x}^a \dot{x}^b. \tag{7.10}$$

From these, we can return an equation similar to the geodesic equation from which we can get the Christoffel symbols.

7.2 Equivalence of the Geodesic and Euler-Lagrange Equations

We frequently change the letters as we want here, so **be careful.** We write the *geodesic equation* as

$$\ddot{x}^f + \Gamma^f{}_{ac} \dot{x}^a \dot{x}^c = 0. \tag{7.11}$$

This will be helpful for later. The Christoffel symbols are defined as usual as

$$\Gamma^f{}_{ac} = \frac{1}{2} g^{fb} \left(\partial_a g_{bc} - \partial_b g_{ca} + \partial_c g_{av}\right). \tag{7.12}$$

Using the Euler-Lagrange equation and the Lagrangian as

$$\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^c}\right) - \frac{\partial \mathcal{L}}{\partial x^c} = 0, \qquad \mathcal{L} = \frac{1}{2} g_{ac} \dot{x}^a \dot{x}^c, \tag{7.13}$$

noting notation and that g_{ab} is a metric which only depends on position, i.e. x^c, and not velocity, i.e. $\dot{x}^c$. Doing the substitution, we get

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^b} = \frac{\partial}{\partial \dot{x}^b}\left(\frac{1}{2} g_{ab} \dot{x}^a \dot{x}^c\right) = \frac{1}{2} \dot{x}^a \dot{x}^c \frac{\partial g_{ac}}{\partial \dot{x}^b} + \frac{1}{2} g_{ac} \overbrace{\frac{\partial \dot{x}^a}{\partial \dot{x}^b}}^{\delta^a_b} \dot{x}^a + \frac{1}{2} g_{ac} \dot{x}^a \overbrace{\frac{\partial \dot{x}^c}{\partial \dot{x}^b}}^{\delta^c_b}, \tag{7.14}$$

and as mentioned above $\frac{\partial}{\partial \dot{x}^b} g_{ac} = 0$, so

$$0 + \frac{1}{2} g_{ac} \dot{x}^c \delta^a{}_b + \frac{1}{2} g_{ac} \dot{x}^a \delta^c{}_b = \frac{1}{2} g_{ab} \dot{x}^a + \frac{1}{2} g_{bc} \dot{x}^c, \tag{7.15}$$

and changing $c \to a$ and $g_{ab} = g_{ba}$, and then $a \to e$,

$$\frac{1}{2} g_{ab} \dot{x}^a + \frac{1}{2} g_{ab} \dot{x}^a = g_{be} \dot{x}^e. \tag{7.16}$$

Lecture 8

Last lecture:

- Euler-Lagrange equations
- 'easier' way to find $\Gamma^a{}_{bc}$
- how to find Geodesic paths

This lecture:

- more tensor derivatives

Recall absolute derivative again:

$$\frac{D\lambda^a}{ds} = \frac{d\lambda^a}{ds} + \Gamma^a{}_{bc}\lambda^b \frac{dx^c}{ds}. \tag{8.1}$$

The first term above is the total change, and then the second is to "subtract off the change due to the coordinate system". So for parallel transport, this means there is no physical change, i.e.

$$\frac{D\lambda^a}{ds} = 0. \tag{8.2}$$

The absolute derivative obeys normal rules for derivatives.

- Linear operator -

$$\frac{D}{ds}\left(\lambda^a + k\mu^a\right) = \frac{D\lambda^a}{ds} + k\frac{D\mu^a}{ds}. \tag{8.3}$$

- The (Leibniz) chain rule -

$$\frac{D}{ds}\left(\lambda^a\mu^b\right) = \mu^b\frac{D\lambda^a}{ds} + \lambda^a\frac{D\mu^b}{ds}. \tag{8.4}$$

What is the absolute derivative of a scalar, ϕ? ϕ does not depend on the coordinates as tensors do, so it would just be the normal derivative, i.e.

$$\frac{D\phi}{ds} = \frac{d\phi}{ds}. \tag{8.5}$$

We have defined the absolute derivative of a contravariant tensor, but now what about a covariant tensor μ_a? We can write a scalar as $\phi = \lambda^a\mu_a$, so we can write

$$\frac{D\phi}{ds} = \frac{D}{ds}(\lambda^a\mu_a) = \mu_a\frac{D\lambda^a}{ds} + \lambda^a\frac{D\mu_a}{ds}, \tag{8.6}$$

$$\frac{d\phi}{ds} = \mu_a\left(\frac{d\lambda^a}{ds} + \Gamma^a{}_{bc}\lambda^b\frac{dx^c}{ds}\right) + \lambda^a\frac{D\mu_a}{ds}, \tag{8.7}$$

$$\lambda^a\frac{d\mu_a}{ds} = \mu_a\Gamma^a{}_{bc}\lambda^b\frac{dx^c}{ds} + \lambda^a\frac{D\mu_a}{ds}, \tag{8.8}$$

$$= \mu_b\Gamma^b{}_{ac}\lambda^a\frac{dx^c}{ds} + \lambda^a\frac{D\mu_a}{ds}, \tag{8.9}$$

where λ^a is any tensor, so if this is true, it must be true for any λ^a. We can then 'cancel' λ^a through unity, as the remaining equation must also be true:

$$\frac{d\mu_a}{ds} = \Gamma_{ac}{}^b\mu_b\frac{dx^c}{ds} + \frac{D\mu_a}{ds}, \tag{8.10}$$

$$\frac{D\mu_a}{ds} = \frac{d\mu_a}{ds} - \Gamma_{ac}{}^b\mu_b\frac{dx^c}{ds}. \tag{8.11}$$

This is the absolute derivative of a contravariant tensor.

What is the absolute derivative of a rank (1,1) tensor $\tau^a{}_b = \lambda^a \mu_b$?

$$\frac{D\tau^a{}_b}{ds} = \frac{D(\lambda^a \mu_b}{ds} = \mu_b \frac{D\lambda^a}{ds} + \lambda^a \frac{D\mu_b}{ds}, \tag{8.12}$$

$$= \mu_b \left[\frac{d\lambda^a}{ds} + \Gamma^a{}_{dc} \lambda^d \frac{dx^c}{ds} \right] + \lambda^a \left[\frac{d\mu_b}{ds} - \Gamma_{bc}{}^d \mu_d \frac{dx^c}{ds} \right], \tag{8.13}$$

$$\frac{D\tau^a{}_b}{ds} = \frac{d}{ds}(\lambda^a \mu_b) + \Gamma^a{}_{dc} \tau^d{}_b \frac{dx^c}{ds} - \Gamma_{bc}{}^d \tau_d{}^a \frac{dx^c}{ds}. \tag{8.14}$$

This is the absolute derivative for a rank (1,1) tensor.

What about the covariant derivative? It is defined as:

$$\lambda^a{}_{;c} = \frac{\partial \lambda^a}{\partial x^c} + \Gamma^a{}_{bc} \lambda^b \tag{8.15}$$

$$\frac{D\lambda^a}{ds} = \lambda^a{}_{;c} \frac{dx^c}{ds} \tag{8.16}$$

$$= \frac{\partial \lambda^a}{\partial x^c} \frac{dx^c}{ds} + \Gamma^a{}_{bc} \lambda^b \frac{dx^c}{ds} = \frac{d\lambda^a}{ds} + \dots . \tag{8.17}$$

All the rules still apply, so we can write out the covariant derivative for a scalar, a covariant, and a higher order tensor:

$$\phi_{;c} = \frac{\partial \phi}{\partial x^c}, \tag{8.18}$$

$$\mu_{a;c} = \frac{\partial \mu + a}{\partial x^c} - \Gamma_{ac}{}^b \mu_b, \tag{8.19}$$

$$\lambda_{ab;c} = \frac{\partial \lambda_{ab}}{\partial x^c} - \Gamma_{ac}{}^d \lambda_{db} - \Gamma_{bc}{}^d \lambda_{ad}. \tag{8.20}$$

We can also consider the metric:

$$g_{ab;c} = ? \tag{8.21}$$

$$\frac{\partial g_{ab}}{\partial x^c} = \frac{\partial}{\partial x^c} (\underline{e}_a \cdot \underline{e}_b), \tag{8.22}$$

$$= \frac{\partial \underline{e}_a}{\partial x^c} \cdot \underline{e}_b + \underline{e}_a \cdot \frac{\partial \underline{e}_b}{\partial x^c}, \tag{8.23}$$

$$= \left(\Gamma_{ac}{}^d \underline{e}_d \right) \cdot \underline{e}_b + \underline{e}_a \cdot \left(\Gamma_{bc}{}^d \underline{e}_d \right), \tag{8.24}$$

$$= \Gamma_{ac}{}^d g_{db} + \Gamma_{bc}{}^d g_{ad}, \tag{8.25}$$

$$g_{ab;c} = \frac{\partial g_{ab}}{\partial x^c} - \Gamma_{ac}{}^d g_{db} - \Gamma_{bc}{}^d g_{ad} = 0, \tag{8.26}$$

where we used the previous definitions of the covariant derivative to find this definition. Why is this important? The metric allows us to switch coordinate systems as

$$R_a = g_{ab} R^b. \tag{8.27}$$

Now suppose we want to find the covariant derivative:

$$R_{a;c} = \cancel{g_{ab;c} R^b} + g_{ab} R^b{}_{;c} \tag{8.28}$$

$$= g_{ab} R^b{}_{;c}. \tag{8.29}$$

Lecture 9

9.1 The Riemann Curvature Tensor

- ➤ how do we know if space is curved?
- ➤ second derivatives of the metric
- ➤ 'space-sing' around a loop
- ➤ convergence of geodesic paths

The Riemann curvature tensor tells us *how much the direction of a vector changes as it goes round a loop*, or the *tidal forces of gravity.*

$$\left(\lambda^a{}_{;b}\right)_{;c} = \frac{\partial \lambda^a{}_{;b}}{\partial x^c} + \Gamma^a{}_{ec}\lambda^e{}_{;b} - \Gamma_{bc}{}^f \lambda_{;f}{}^a \tag{9.1}$$

$$= \frac{\partial}{\partial x^c}\left(\frac{\partial \lambda^a}{\partial x^b} + \Gamma^a{}_{bd}\lambda^d\right) + \cdots \tag{9.2}$$

$$\lambda^a{}_{;b;c} = \frac{\partial^2 \lambda^a}{\partial x^c \partial x^b} + \Gamma^d{}_{bd}\frac{\partial \lambda^d}{\partial x^c} + \lambda^d \frac{\partial \Gamma^a{}_{bd}}{\partial x^c} + \Gamma^a{}_{ec}\lambda^a{}_{;b} - \Gamma_{bc}{}^f \lambda^a{}_{;f} \tag{9.3}$$

In flat space, the Christoffel symbols go to zero, so we can sway $;b$ and $;c$ indices, but **not in curved space.**

$$\lambda^a{}_{;c;b} = \frac{\partial^2 \lambda^a}{\partial x^b \partial x^c} + \Gamma^a{}_{cd}\frac{\partial \lambda^d}{\partial x^b} + \lambda^d \frac{\partial \Gamma^a{}_{cd}}{\partial x^b} + \Gamma^a{}_{eb}\lambda^e{}_{;c} - \Gamma_{cb}{}^f \lambda^a{}_{;f} \tag{9.4}$$

$$\lambda^a{}_{;c;b} - \lambda^a{}_{;b;c} = \left(\Gamma^a{}_{cd}\frac{\partial \lambda^d}{\partial x^b} - \Gamma^a{}_{bd}\frac{\partial \lambda^d}{\partial x^c}\right) + \lambda^d\left(\frac{\partial \Gamma_{cd}{}^a}{\partial x^b} - \frac{\partial \Gamma_{bd}{}^a}{\partial x^c}\right) + \left(\Gamma^a{}_{eb}\lambda^e{}_{;c} - \Gamma^a{}_{ec}\Gamma^e{}_{;b}\right) \tag{9.5}$$

$$= \Gamma^a{}_{cd}\left(\frac{\partial \lambda^a}{\partial x^b} - \lambda^d{}_{;b}\right) + \Gamma^a{}_{bd}\left(\lambda^d{}_{;c} - \frac{\partial \lambda^d}{\partial x^c}\right) + \lambda^d\left(\frac{\partial \Gamma^a{}_{cd}}{\partial x^b} - \frac{\partial \Gamma^a{}_{bd}}{\partial x^c}\right) \tag{9.6}$$

$$= \left(\Gamma^a{}_{be}\Gamma^e{}_{cd} - \Gamma^a{}_{ce}\Gamma^e{}_{bd} + \frac{\partial \Gamma^a{}_{cd}}{\partial x^b} - \frac{\partial \Gamma^a{}_{bd}}{\partial x^c}\right)\lambda^d = R^a{}_{dbc}\lambda^d \tag{9.7}$$

We have arrived at the Riemann curvature tensor. Consider: *doodle diagram*

$$\lambda^a(B^1) = \lambda^a(A) + \lambda^a{}_{;b}\delta x^b + \lambda^a{}_{;c}\delta y^c + \lambda^a{}_{;b;c}\delta x^b \delta y^c \tag{9.8}$$

$$\lambda^a(B^2) = \lambda^a(A) + \lambda^a{}_{;c}\delta y^c + \lambda^a{}_{;b}\delta x^b + \lambda^a{}_{;c;b}\delta y^c \delta x^b \tag{9.9}$$

$$\Delta\lambda^a = \lambda^a(B^2) - \lambda^a(B^1) = \left(\lambda^a{}_{;c;b} - \lambda^a{}_{;b;c}\right)\delta x^b \delta y^c \tag{9.10}$$

$$= R^a{}_{dbc}\lambda^a \cdot \text{ area of loop} \tag{9.11}$$

In flat space, any two lines have a separation that increases linearly with distance s, e.g. $\partial_s^2 = 0$. But in curved space, our two lines can converge or diverge as they travel from initial parallel conditions. Consider two lines $x^a(s)$ and $\tilde{x}^a(s)$ with a separation $\zeta^a = \tilde{x}^a - x^a$. We can write our geodesic equation for this as

$$\frac{d^2 x^a}{ds^2} + \Gamma^a{}_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds} = 0 \tag{9.12}$$

$$\frac{d^2 \tilde{x}^a}{ds^2} + \tilde{\Gamma}^a{}_{bc}\frac{d\tilde{x}^b}{ds}\frac{d\tilde{x}^c}{ds} = 0 \tag{9.13}$$

$$\ddot{\zeta}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{\zeta}^c + \Gamma^a{}_{bc}\dot{\zeta}^b\dot{x}^c + \frac{\partial \Gamma^a{}_{bc}}{\partial x^d}\zeta^d\dot{x}^b\dot{x}^c = 0 \tag{9.14}$$

$$\frac{D^2\zeta^a}{ds^2} = \frac{D}{ds}\left(\dot{\zeta}^a + \Gamma^a{}_{bc}\zeta^b\dot{x}^c\right) \tag{9.15}$$

$$= \ddot{\zeta}^a + \Gamma^a{}_{bc}\zeta^b\dot{x}^c + \frac{d}{ds}\left(\Gamma^a{}_{bc}\zeta^b\dot{x}^c\right) + \Gamma^a{}_{ef}\left(\Gamma^e{}_{bc}\zeta^b\dot{x}^c\right)\dot{x}^f \tag{9.16}$$

Lecture 10

Things are about to get a little more physics! - Richard Bower.
In this lecture:

- ➤ Geodesic convergence.
- ➤ Symmetry of the Riemann tensor,

$$R^a{}_{dbc} = \Gamma^a{}_{bc}\Gamma^e{}_{cd} - \Gamma^a{}_{ce}\Gamma^e{}_{bd} + \partial_b\Gamma^a{}_{cd} - \partial_c\Gamma^a{}_{bd}. \tag{10.1}$$

- ➤ We will look towards Einstein's equations by combining our knowledge of movement in curved space with gravity.
- ➤ We will do this through the stress-energy tensor, $T^{\mu\nu}$. (Not as simple as it could be, R has four indices, this has two, but we'll get to that.)
- ➤ This will lead towards to conservation laws in GR.

At the end of last lecture, we considered divergent lines with a spatially-dependent separation ζ:

$$\ddot{x}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{x}^c = 0 \tag{10.2}$$
$$\ddot{\tilde{x}}^a + \tilde{\Gamma}^a{}_{bc}\dot{x}^b\dot{x}^c = 0 \tag{10.3}$$
$$\ddot{\zeta} + \Gamma^a{}_{bc}\dot{x}^b\dot{\zeta}^c + \Gamma^a{}_{bc}\dot{\zeta}^b\dot{x}^c + \frac{\partial\Gamma^a{}_{bc}}{\partial x^d}\zeta^d\dot{x}^b\dot{x}^c = 0. \tag{10.4}$$

We want to find the Tensor equation from this!

$$\frac{D^2\zeta^a}{ds^2} = \ddot{\zeta}^a + \Gamma^a{}_{bc}\dot{\zeta}^b\dot{x}^c + \frac{d}{ds}\left(\Gamma^a{}_{bc}\zeta^b\dot{x}^c\right) + \Gamma^a{}_{ef}\left(\Gamma^a{}_{bc}\zeta^b\dot{x}^c\right)\dot{x}^f. \tag{10.5}$$
$$\frac{d}{ds}\left(\Gamma^a{}_{bc}\zeta^b\dot{x}^c\right) = \Gamma^a{}_{be}\zeta^b\ddot{x}^e + \Gamma^a{}_{bc}\dot{\zeta}^b\dot{x}^c + \zeta^b\dot{x}^c\left(\frac{\partial\Gamma^a{}_{bc}}{\partial x^d}\dot{x}^d\right). \tag{10.6}$$

So we expanded out the derivative to get rid of something we didn't want, and now combine for the final result (as well as using the definition of a geodesic path, $\ddot{x}^e = -\Gamma^e{}_{cd}\dot{x}^c\dot{x}^d$, in Eq. (10.5)):

$$\frac{D^s\zeta^a}{ds^2} + \left(\Gamma^a{}_{be}\Gamma^e{}_{cd} - \Gamma^a{}_{ce}\Gamma^e{}_{bd} + \partial_b\Gamma^a{}_{cd} - \partial_c\Gamma^a{}_{bd}\right)\zeta^b\dot{x}^c\dot{x}^d = 0. \tag{10.7}$$

This looks very confused as it is, but we can simplify by now defining the Riemann tensor:

$$\frac{D^2\zeta}{ds^2} + R^a{}_{dbc}\zeta^b\dot{x}^c\dot{x}^d = 0, \tag{10.8}$$
$$R^a{}_{dbc} = \Gamma^a{}_{bc}\Gamma^e{}_{cd} - \Gamma^a{}_{ce}\Gamma^e{}_{bd} + \partial_b\Gamma^a{}_{cd} - \partial_c\Gamma^a{}_{bd}. \tag{10.9}$$

From this definition, we can quickly infer the symmetry relation

$$R^a{}_{dcb} = -R^a{}_{dbc}. \tag{10.10}$$

10.1 Symmetries of the Riemann tensor

In 4D spacetime, the Riemann tensor contains $4 \times 4 \times 4 \times 4 = 256$ numbers. This isn't very fun unless you are a computer, says Richard. However, using the symmetry of the Riemann tensor, there are only 20 independent elements, which can be a lot easier. How do we demonstrate this though? We can use mathematical tricks for tensors, whereby if we prove something in one coordinate system, it must be true in all coordinate systems.

10.1.1 Local Geodesic Coordinates

Local geodesic coordinates describe a flat space at a local point in curved space. We know, therefore, that we can write the metric at this point as

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{10.11}$$

which is known as the **Minkowski metric** for flat space. This is familiar from special relativity, where it represents the usual Minkowski product,

$$ds^2 = c^2\, d\tau^2 = c\, dt^2 - dx^2 - dy^2 - dz^2. \tag{10.12}$$

So the Christoffel symbol at this local point is

$$\Gamma^{\alpha}{}_{\beta\gamma} = 0, \tag{10.13}$$

where we have only chosen a point to set this locally to zero. However, the derivatives will not be zero, as if we move away from this local point, space will become curved again:

$$\frac{\partial \Gamma^{\alpha}{}_{\beta\gamma}}{\partial x^{\delta}} = 0. \tag{10.14}$$

Now this will allow us to simplify the Riemann tensor from its definition in Eq. (10.9), where it can now be written as

$$R^{a}{}_{dbc} = \partial_b \Gamma^{a}{}_{cd} - \partial_c \Gamma^{a}{}_{bd}. \tag{10.15}$$

What about the symmetry relations of this simplified Riemann tensor?

$$R^{a}{}_{cdb} = \partial_d \Gamma^{a}{}_{bc} - \partial_b \Gamma^{a}{}_{dc}, \tag{10.16}$$
$$R^{a}{}_{bcd} = \partial_c \Gamma^{a}{}_{db} - \partial_d \Gamma^{a}{}_{cb}. \tag{10.17}$$

What about the sum of these three tensors? Due to the symmetry of the Christoffel symbols' lower indices, we can see that there will be lots of cancellations in this sum, leading to

$$R^{a}{}_{dbc} + R^{a}{}_{cdb} + R^{a}{}_{bcd} = 0, \tag{10.18}$$

which is known as the cyclic relation of the Riemann tensor. The Riemann tensor can also be written in its covariant form:

$$R_{abcd} = g_{ae} R^{e}{}_{bcd}. \tag{10.19}$$

In its covariant form, we can also go through similar steps to above to prove other symmetry relations, including:

$$R_{abcd} = -R_{bacd}, \qquad R_{abcd} = -R_{abdc}, \qquad R_{abcd} = R_{cdab}. \tag{10.20}$$

From these relations, we see that repeated indices can lead to zero, such as

$$R_{aacd} = -R_{aacd} \implies R_{aacd} = 0, \tag{10.21}$$
$$R_{abcc} = -R_{abcc} \implies R_{abcc} = 0. \tag{10.22}$$

This allows us to neglect many of the 256 numbers as they will be zero, reducing the total of free elements of the Riemann tensor to 20.

10.2 How can we relate this to gravity?

Let's consider Eq. (10.4) for low speeds, i.e. non-relativistic.

$$\zeta^\alpha = \begin{pmatrix} 0, & 0, & y, & 0 \end{pmatrix}, \tag{10.23}$$

i.e. we have a simple separation in the y-direction between the two geodesic paths. We would then get something like

$$\frac{D^y}{ds^2} = -R^2{}_{020} y c^2, \tag{10.24}$$

where we only consider the indices relating to y and time. This should look somewhat familiar. What would we have written for this in Newtonian physics, if we have these two objects both moving towards a large mass M with a separation y and a distance from the massive object r?

$$\frac{d^2 y}{dt^2} = -\frac{GM}{r^2} \cdot \frac{y}{r}. \tag{10.25}$$

Important note: the massive object is the Sun or a molecular cloud or something, according to Richard. **A black hole wouldn't be happy, you wouldn't be able to tell.**

Using $ds = c\,dt$:

$$c^2 \frac{d^2 y}{ds^2} = -\frac{GM}{r^3} y, \tag{10.26}$$

$$-\frac{GM}{r^3} = -R^2{}_{020}. \tag{10.27}$$

So we now have an equivalence of curved space and Newtonian gravitational force. We can see that the Newtonian expression in Eq. (10.27) is the energy density of space, if mass and energy can be equivalent. This is what lead Einstein to his famous equations.

10.3 What is energy?

Energy is special relativity was defined in

$$E = \gamma m_0 c^2, \tag{10.28}$$

where we can now relate the mass and energy of objects. We also have a number density, n_0, for the number of particles "in a box" or in the universe or whatever you're considering. Recall from special relativity the four-velocity of a system:

$$u^\alpha = \frac{dx^\alpha}{dt} = \gamma \begin{pmatrix} c, & -\frac{dx^i}{dt} \end{pmatrix}. \tag{10.29}$$

There is also the four-momentum of the system:

$$p^\alpha = m_0 u^\alpha = \gamma m_0 \begin{pmatrix} c, & -\frac{dx^i}{dt} \end{pmatrix}. \tag{10.30}$$

The number per unit volume is γn_0. We also have a flux through the surface,

$$N^\rho = n_0 u^\rho. \tag{10.31}$$

Using all this, we can write down an expression for the energy-momentum density:

$$T^{\mu\nu} = N^\mu p^\nu = n_0 u^\mu m_0 u^\nu = \rho_0 u^\mu u^\nu. \tag{10.32}$$

In the rest frame, these two velocities will each be c, so the T^{00} term will get us back to Eq. (10.28) as $\rho_0 c^2$.

Lecture 11

Lecture 12

Lecture 13

Lecture 14

Lecture 15

Lecture 16 Time and Black Holes

- Last lecture - orbits around black holes
- This lecture - time depends on where you are and how you move; falling into a black hole

16.1 The Schwarzschild Metric

Defined as

$$ds^2 = c^2 d\tau^2 = c^2 \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\phi^2. \tag{16.1}$$

An observer experiences the proper time, τ, whereas t is the time coordinate. *doodle* For the coordinate time seen from the distant observer,

$$t_{orb} = 2\pi \left(\frac{r^3}{mc^2}\right)^{1/2}. \tag{16.2}$$

What does S_0 measure?

$$d\tau_0 = \left(1 - \frac{3m}{r}\right)^{1/2} dt, \tag{16.3}$$

$$\tau_{orb} = \left(1 - \frac{3m}{r}\right)^{1/2} t_{orb} \tag{16.4}$$

which tells us that there are no orbits possible with $r < 3m$. For the rocket S_1, we arrange such that $d\phi = 0, dr = 0$, i.e. the rocket is being held stationary. We then get from the metric

$$c^2 d\tau_1^2 = c^2 \left(1 - \frac{2m}{r_1}\right) dt^2, \tag{16.5}$$

$$\tau_{hov} = \left(1 - \frac{2m}{r_1}\right)^{1/2} t_{orb}, \tag{16.6}$$

$$\tau_{orb} \neq \tau_{hov}. \tag{16.7}$$

Consider the example where $r = r = 6m$. We will get

$$\tau_{orb} = \left(\frac{1}{2}\right)^{1/2} t_{orb}, \qquad \tau_{hov} = \left(\frac{2}{3}\right)^{1/2} t_{orb}, \qquad \frac{\tau_{orb}}{\tau_{hov}} = \frac{\sqrt{3}}{2} = 0.86. \tag{16.8}$$

So S_1 perceives themselves as older than S_0.

16.2 Radial Geodesics

Similar to before, but simpler form:

$$c^2 d\tau^2 = c^2 \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2. \tag{16.9}$$

$$\left(1 - \frac{2m}{r}\right) \dot{t} = E, \tag{16.10}$$

where we have solved the Euler-Lagrange equation in t, and $\dot{t}$ is $\frac{dt}{d\tau}$, and E is a constant. Dividing by $d\tau^2$, we rewrite as

$$c^2 = c^2 \left(1 - \frac{2m}{r}\right) \dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-1} \dot{r}^2, \tag{16.11}$$

$$c^2 = c^2 \left(1 - \frac{2m}{r}\right)^{-1} E^2 - \left(1 - \frac{2m}{r}\right)^{-1} \dot{r}^2, \tag{16.12}$$

$$\dot{r}^2 - c^2 E^2 + c^2 \left(1 - \frac{2m}{r}\right) = 0. \tag{16.13}$$

We need to find constant, so we use the condition $\dot{r} = 0$ at $r \gg 2m$:

$$-c^2 E^2 + c^2 = 0, \quad E = 1, \tag{16.14}$$

$$\dot{r}^2 - c^2 + c^2 \left(1 - \frac{2m}{r}\right) = 0, \tag{16.15}$$

$$\dot{r}^2 = \frac{2mc^2}{r} \implies \dot{r}^2 = -c\sqrt{\frac{2m}{r}}, \tag{16.16}$$

where we have chosen the negative solution for when we are approaching the black hole. This looks pretty Newtonian, but remember that $\dot{r} = \frac{dr}{d\tau}$, where τ is the proper time. So this would be the speed measured by the falling observer.

How long does it take to reach $r = 0$? We can work out the proper time as observed by the person falling into the black hole, as

$$\tau = \int_{r_1}^{r_2} \frac{d\tau}{dr} dr = \int_{r_1}^{r_2} \frac{1}{\dot{r}} dr = \frac{1}{c\sqrt{2m}} \int_{r_2}^{r_1} r^{1/2}\, dr = \frac{2}{3c\sqrt{2m}} \left(r_1^{3/2} - r_2^{3/2}\right). \tag{16.17}$$

We have a perfectly finite τ for $r_2 = 0$, e.g. $r_1 = 10m$, $r_2 = 0$ results in $\tau = 14.9\frac{m}{c}$.

If we consider the perspective of the person a long distance away, then

$$\frac{dr}{dt} = \frac{dr}{d\tau}\frac{d\tau}{dt} = -c\sqrt{\frac{2m}{r}} \left(1 - \frac{2m}{r}\right). \tag{16.18}$$

At $r = 2m$, $\frac{dr}{dt} \to 0$. From a distance, we will observe them fall in to the black hole, but slow down and "stop" at the event horizon; the person falling into the black hole will feel themselves falling in faster and faster. We have a gravitational redshift,

$$\frac{\lambda_R}{\lambda_E} = \sqrt{\frac{1 - \frac{2m}{r_2}}{1 - \frac{2m}{r_E}}}, \tag{16.19}$$

so the wavelength received $\lambda_R \to \infty$. It isn't quite that we see it come to rest, but rather its signal slowly fades out and gets weaker and weaker until we don't observe anything.

Now let's look at dropping off at some general radius. Drop the satellite at $r = r_0, \dot{r} = 0$ at $r = 0$.

$$E\sqrt{1 - \frac{2m}{r_0}}, \tag{16.20}$$

$$\dot{r}^2 = c^2 \left[\left(1 - \frac{2m}{r_0}\right) - \left(1 - \frac{2m}{r}\right)\right] \tag{16.21}$$

$$= c^2 \left(\frac{2m}{r} - \frac{2m}{r_0}\right), \tag{16.22}$$

$$\dot{t} = \left(1 - \frac{2m}{r}\right)^{-1} \sqrt{1 - \frac{2m}{r_0}}, \tag{16.23}$$

$$\frac{dr}{dt} = \frac{\dot{r}}{\dot{t}} = c\sqrt{\frac{2m}{r} - \frac{2m}{r_0}} \frac{\left(1 - \frac{2m}{r}\right)}{\sqrt{1 - \frac{2m}{r_0}}}. \tag{16.24}$$

We can also consider what happens for the hovering observer:

$$dR_h = \sqrt{1 - \frac{2m}{r}}\, dr, \quad d\tau_h = \sqrt{1 - \frac{2m}{r}}\, dr, \tag{16.25}$$

$$\frac{dR_h}{d\tau_h} = \frac{dR_h}{dr}\frac{dr}{dt}\frac{dt}{d\tau_h} = \left(1 - \frac{2m}{r}\right)^{-1}\frac{dr}{dt} = -c\sqrt{\frac{\frac{2m}{r} - \frac{2m}{r_0}}{1 - \frac{2m}{r_0}}}. \tag{16.26}$$

If we hover at $r_0 = 2m, r = 2m$, then $\frac{dR_h}{d\tau_h} = c$. This is quite strange. Let's check:

$$\frac{d^2R_h}{d\tau_h^2} = \frac{d}{d\tau_h}\left(\frac{dR_h}{d\tau_h}\right) = -\frac{mc^2}{r^2}\frac{\sqrt{1 - \frac{2m}{r}}}{\left(1 - \frac{2m}{r_0}\right)}. \tag{16.27}$$

If $r \to \infty \implies$ Newtonian physics is back. If $r_0 \to 2m$, then $\frac{d^2R_h}{d\tau_h} \to \infty$. However, hovering at $r = 2m$ requires an infinitely powerful rocket - you can't hover at $r \leq 2m$, i.e.

Lecture 17

Falling into a black hole! We have better ways of describing spacetime. For radial paths, we have the metric and the E-L equations:

$$c^2 d\tau^2 = c^2 \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 \tag{17.1}$$

$$\dot{t} = \left(1 - \frac{2m}{r}\right)^{-1}, \quad \dot{r} = \sqrt{\frac{2m}{r}} \tag{17.2}$$

$\dot{t}$ is dropped as $\to \infty$. We will define a new time coordinate and it can rewritten as

$$c\,dT = cd\tau = c\left(1 - \frac{2m}{r}\right) dt, \tag{17.3}$$

$$c\,dT = c\,dt - \frac{2mc}{r} dt, \tag{17.4}$$

$$dt = \frac{\dot{t}}{\dot{r}} = \left(\frac{-1}{1 - \frac{2m}{r}}\right) \sqrt{\frac{2m}{r}} \frac{1}{c} dr, \tag{17.5}$$

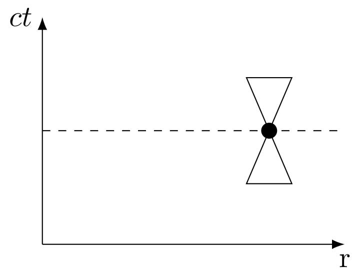
$$c\,dT = c, dt + \sqrt{\frac{2m}{r}} \left(1 - \frac{2m}{r}\right)^{-1} dr, \tag{17.6}$$

$$c^2 d\tau^2 = \left(1 - \frac{2m}{r}\right) \left[c\,dT - \sqrt{\frac{2m}{r}} \left(1 - \frac{2m}{r}\right) dr\right]^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2, \tag{17.7}$$

$$c^2 d\tau^2 = \left(1 - \frac{2m}{r}\right) c^2 dT^2 - 2\sqrt{\frac{2m}{r}} c\,dT\,dr - dr^2. \tag{17.8}$$

We now have a rewritten metric in terms of a time coordinate of someone falling into a black hole. τ is the proper time on the radial path, T is the proper time of the falling observer. So we can now solve in the falling observer's perspective rather than the orbiting observer.

What do we need to consider causally around the centre of the Black Hole?



finish doodle For photons, we have the null geodesic:

$$0 = c^2 \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 \tag{17.9}$$

$$c^2 dt^2 = \left(1 - \frac{2m}{r}\right)^{-2} dr^2 \tag{17.10}$$

$$c\frac{dt}{dr} = \pm \left(1 - \frac{2m}{r}\right)^{-1} \tag{17.11}$$

This breaks down as we approach the Schwarzchild radius. We need to think of a new coordinate system to describe the universe.

17.1 Eddington-Finkelstein Coordinates

If we define

$$dr_* = \left(1 - \frac{2m}{r}\right)^{-1} dr \implies \frac{dt}{dr_*} = \pm c \tag{17.12}$$

photons will not allows propagate at 45 degrees in spacetime diagrams, to the expense of our radial coordinate. If we integrate this,

$$r_* = r + 2m \log\left|\frac{r}{2m} - 1\right|, \tag{17.13}$$

$$ct = \pm r_* + v. \tag{17.14}$$

So we have a choice of two equations for defining our time variable, but we must choose one. They chose the inward-going light rays for

$$v = r_* + ct, \tag{17.15}$$

$$\frac{dv}{dr} = \frac{dr_*}{dr} + c\frac{dt}{dr}, \tag{17.16}$$

$$dv = \left(1 - \frac{2m}{r}\right)^{-1} dr + cdt. \tag{17.17}$$

Again, we can substitute this in to the metric, and find a new form:

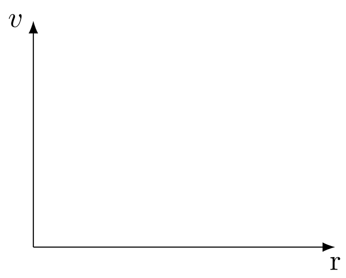
$$d\tau^2 = \left(1 - \frac{2m}{r}\right) dv^2 - 2\, dv\, dr. \tag{17.18}$$

When v is constant and we set $d\tau = 0$ we have the path taken by a photon. We can solve for the photon paths:

1. Inwards propagating, so $dv = 0$
2.

$$\frac{dv}{dr} = \frac{2}{\left(1 - \frac{2m}{r}\right)} \tag{17.19}$$

Plotting this, for



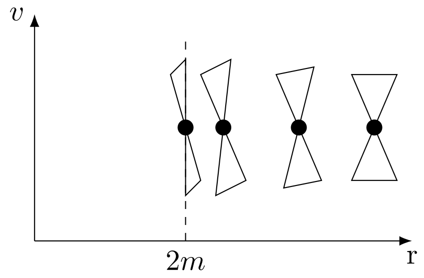
finish doodle Light ray which is always horizontal - would want both at 45 degrees. Light cone tips over at $r = 2m$. Transform variables again for

$$ct_* = v - r \tag{17.20}$$

$$\frac{cdt_*}{dr} = -1 \tag{17.21}$$

$$\frac{cdt_*}{dt} = \frac{\left(1 + \frac{2m}{r}\right)}{\left(1 - \frac{2m}{r}\right)} \tag{17.22}$$

another doodle



At large r, it always travels at 45 degrees. The cones of influence are tipping over as the outward ray moves less and less, until we reach $r = 2m$ and the outward propagating ray becomes purely time-like and cannot propagate outwards. Once we're inside this radius, both light rays are then tipping over and both move towards the centre of the black hole as our term in Eq. (17.22) goes negative for both. The zones of influence are aimed inwards to the Black Hole.

Lecture 18

18.1 Initial Overview

- We had the **Principle of Equivalence** - all bodies feel the same gravity
- Mass $\implies$ curved space $\implies$ curved paths *Yoda meme here*
- In Special Relativity, we had

$$\begin{aligned} c^2 d\tau^2 &= c^2 dt^2 - dx^2 - dy^2 - dz^2 \\ &= \eta_{\alpha\beta} dx^\alpha dx^\beta, \end{aligned} \tag{18.1}$$

  where we can express this using the Minkowski metric $\eta_{\alpha\beta}$.
- Einstein realised we can express this in a more general metric in General Relativity:

$$c^2 d\tau^2 = g_{\alpha\beta} dx^\alpha dx^\beta, \quad g_{\alpha_\beta} = (\underline{e}_\alpha \cdot \underline{e}_\beta). \tag{18.2}$$

- $\frac{\partial \lambda^\beta}{\partial x^\alpha} \implies \lambda^\beta{}_{j\alpha}$.

18.2 Tensor Maths Toolkit

We now need to figure out ways to represent everything in tensors. This follows from defining contravariant and covariant tensors respectively:

$$A^{\bar{b}} = \frac{\partial x^{\bar{b}}}{\partial x^a} A^a, \qquad A_{\bar{b}} = \frac{\partial x^a}{\partial x^{\bar{b}}} A_a. \tag{18.3}$$

We can convert between these two forms using

$$\lambda_b = g_{ab} \lambda^a, \tag{18.4}$$

where we need to remember we are using the Einstein Summation Convention over repeated indices.

18.3 Geodesic paths and curved space

When you move a tensor around in curved space, you get changes due to how the coordinate space changes. To accommodate for this, we introduce the Total Derivative:

$$\frac{D\lambda^a}{ds} = \frac{d\lambda^a}{ds} + \Gamma^a{}_{bc} \lambda^b \frac{dx^c}{ds} = \lambda^a{}_{;c} \frac{dx^c}{ds}, \tag{18.5}$$

$$\lambda^a{}_{;c} = \frac{\partial \lambda^a}{\partial x^c} + \Gamma^a{}_{bc} \lambda^b, \tag{18.6}$$

$$\Gamma^f{}_{bc} = \frac{1}{2} g^{bf} \left(\partial_a g_{bc} - \partial_b g_{ca} + \partial_c g_{ab} \right). \tag{18.7}$$

We then introduce the concept of *parallel transport*, which led us to geodesic paths:

$$\ddot{x}^a + \Gamma^a{}_{bc} \dot{x}^b \dot{x}^c = 0. \tag{18.8}$$

For working in equations of motion, we turn to the Euler-Lagrange equations. So we express our Lagrangian and then write the E-L equation:

$$\mathcal{L} = \frac{1}{2} g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta, \qquad \frac{d}{d\tau} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^a} \right) - \frac{\partial \mathcal{L}}{\partial x^\alpha} = 0. \tag{18.9}$$

The definition of geodesic paths and Euler-Lagrange equation are equivalent, although generally we will find the E-L equation easier to work with. This is all well and good, but then we realised that the Riemann curvature tensor $R^a{}_{dbc}$ was like the 'second derivative' and can tell us a lot about space. But we can't equate this with our other tensors yet.

18.4 Einstein's Equations

We want to describe how mass and the shape of space are related. This is derived in the **Stress-Energy Tensor**:

$$T^{\mu\nu} = \left(\rho_0 + \frac{p}{c^2}\right) u^\mu u^\nu - P g^{\mu\nu}, \tag{18.10}$$

so the '00' element describes energy, and the other elements describe momentum. In the last term above, we subtract by pressure to get the right form. All of physics then can be written as

$$T^{\mu\nu}{}_{;\mu} = 0. \tag{18.11}$$

To help equate different quantities, we introduce the Ricci tensor, which is a *compressed* form of the Riemann tensor:

$$R_{\mu\nu} = R^\alpha{}_{\mu\nu\alpha}. \tag{18.12}$$

We can then compress the Ricci tensor by performing its trace, yielding the **Curvature Scalar**:

$$\mathcal{R} = R^\alpha{}_\alpha. \tag{18.13}$$

Einstein then used all this to define his 'guess' as a way of writing down a relation between the curvature of space and the stress-energy:

$$\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} \mathcal{R}\right) = \kappa T^{\mu\nu}. \tag{18.14}$$

Turns out this wasn't quite right - there is also a small *cosmological constant* that needs to be added, which is famous for being Einstein's big blunder. The metric is then useful to be redefined once we have reached this point, so we now introduce the Schwarzchild Metric, defining $m = \frac{GM}{c^2}$:

$$c^2 d\tau^2 = c^2 \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2. \tag{18.15}$$

The rest of the lecture series was pretty much focused on how we solve this metric in different regimes.

- Particles in weak gravity - here, we seemed to recover essentially Newtonian physics, with slight corrections. This resolved the perihelion of Mercury and was the first big success of GR.
- Photons in weak gravity - leading us to gravitational lensing and photons' movement.
- Orbits in strong gravity - the metric terms are no longer small as in weak gravity, more complex to solve.
- Radial paths in strong gravity.
- Alternative ways of writing the metric - looking at causality and where the metric should fail and how we fix that (i.e. inside $r = 2m$).