Mathematics Workshop

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Michaelmas Term 2018 - Epiphany Term 2019

Contents

Part I

Complex Analysis

Lecture 1 The Complex Plane

1.1 Basics

Course will use Riley, Hobson and Bence Chapters 3, 24, 25

Recall $i = \sqrt{-1}$ allows us to extend our notion of numbers as we go from a $\mathbb{R}$eal to the $\mathbb{C}$omplex plane.

$$\mathbb{C} = \{z = x + iy \mid x, y \in \mathbb{R}\} \tag{1.1}$$

We represent the complex plane with an argand diagram.

$$x = \mathrm{Re}(z) = \frac{z + \bar{z}}{2} \tag{1.2}$$

$$y = \mathrm{Im}(z) = \frac{z - \bar{z}}{2} \tag{1.3}$$

$$r = \mathrm{mod}(z) = \sqrt{x^2 + y^2} \tag{1.4}$$

$$\theta = \mathrm{Arg}(z) = \arctan\left(\frac{y}{x}\right) \tag{1.5}$$

Notice that $e^{i\theta} = e^{i\theta + 2i\pi}$, so θ is not uniquely defined, so choose a range for θ, e.g. $\theta \in [0, 2\pi), \theta \in (-\pi, \pi]$.

1.2 Complex Functions

- $\mathbb{C} \to \mathbb{C}$
- $z \to f(z) = u(x, y) + iv(x, y)$
- Like two functions of real variables, e.g. $f(z) = z^2 = (x + iy)^2 = (x^2 - y^2) + 2ixy$
  - $u(x, y) = x^2 - y^2$
  - $v(x, y) = 2xy$
- Exponential: $e^z, \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

1.3 Branch Cuts

For real numbers, $1^2 = (-1)^2 = 1$, you have two roots; but functions should be single valued.
$\sqrt{}$ is fine on the Real line - choose a root and stick to it, but for the Complex plane, say $\sqrt{1} = 1$, the disc around $z = 1$ means a loss of continuity.

$$z = re^{i\theta} \tag{1.6}$$

$$\sqrt{z} = \sqrt{r}e^{i\theta/2}, \theta \in (0, 2\pi) \tag{1.7}$$

We can choose this branch, but must cut the complex plane along the reals so that θ can't run higher than 2π, or could choose

$$\sqrt{z} = \sqrt{r}e^{i\theta/2 + i\pi} \tag{1.8}$$

but still need the same cut, so no ambiguity in definition of the function's square root. Here, the function's square root is double valued.

$$\log(re^{i\theta}) = \log(r) + i\theta \tag{1.9}$$

θ is not unique, $\theta + 2n\pi$ is also a legitimate answer. Again, cut the plane somewhere, decide on the branch of the log. Can cut on the positive reals or negatives. Branch choices appear around zeroes because polar coordinates are singular there: θ is not specified at $r = 0$. We say $z = 0$ is a branch point, indicated with a wavy line. Branch cuts from branch point either to infinity or another branch point.

Example:

$$f(z) = \sqrt{z^4+1} \tag{1.10}$$

Think of this through two steps: $z \to z^4+1 \to \sqrt{z^4+1}$. There is a branch cut around $z^4+1=0$. There are four branch points:

$$z^4 = -1 \tag{1.11}$$
$$z = e^{i\pi/4}... \tag{1.12}$$

Around a branch point:

$$z = e^{i\pi/4} + \epsilon e^{i\theta+3i\pi/4} \tag{1.13}$$
$$z^4 = -1 + 4\epsilon e^{i\theta+3i\pi/4} \tag{1.14}$$
$$\sqrt{z^4+1} = 2\sqrt{\epsilon}e^{i\theta/2+3i\pi/8} \tag{1.15}$$

This gives rise to the same problem as circling the origin for $\sqrt{z}$. This example has other choices for branch cuts, some may be able to limit ambiguities to limits.

1.4 Trig and Hyperbolic Functions

Trig functions can be generalised to include complex numbers, and can be expressed as exponentials in the usual way, similarly with hyperbolics. sinh and cosh have periodicity of $2\pi i$.

$$\cos(z) = \cosh(iz) \tag{1.16}$$
$$\sin(z) = i\sinh(iz) \tag{1.17}$$

Lecture 2 Complex Differentiation and Cauchy-Riemann

2.1 Continuity

Definition:

$$\lim_{z\to z_0} f(z) = w \in \mathbb{C} \iff \forall \epsilon > 0\, \exists\, \delta > 0\; s.t.\; |z - z_0| < \delta \implies |f(z) - w| < \epsilon \tag{2.1}$$

If you are close to a point z_0, then f(z) is close to $f(z_0)$. f is continuous at $z_0 \iff \lim_{z\to z_0} f(z) = f(z_0)$.

Note that the limit must be path independent, and the real and imaginary parts must be continuous.

2.2 Differentation

Recall

$$\frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \tag{2.2}$$

For $\mathbb{C}$ having two $\mathbb{R}$ dimensions, we have $\underline{\nabla}, \underline{\nabla}\cdot, \underline{\nabla}\times$.

- X Grad - acts on scalars ($\mathbb{R}$)
- X Div - acts on vectors ($\mathbb{C}$), gives scalar
- X Curl - in 3D

Try

$$\frac{df}{dz} = \lim_{\delta\to 0} \frac{f(z+\delta) - f(z)}{\delta}, \delta, z \in \mathbb{C} \tag{2.3}$$

Require limit independent of direction.

Example:
$f(z) = z^2$:

$$\lim_{\delta\to 0} \frac{(z+\delta)^2 - z^2}{\delta} \tag{2.4}$$

$$\implies \lim_{\delta\to 0} \frac{\cancel{z^2} + 2z\cancel{\delta} + \delta^{\cancel{2}} - \cancel{z^2}}{\cancel{\delta}} = 2z \tag{2.5}$$

What about $f(z) = \bar{z}$?

$$\frac{\bar{z} + \bar{\delta} - \bar{z}}{\delta} = \frac{\bar{\delta}}{\delta} = \exp(-2i\mathrm{Arg}(\delta)) \tag{2.6}$$

Not path independent, so no limit.

$\frac{df}{dz}$ is the complex derivative of f at z, and f is differentiable at z if this limit exists.

2.3 Analytic Functions

An analytic function is a complex function that is differentiable, at least in some region.

Definition: A neighbourhood of $z \in \mathbb{C}$ is an open set U such that $z \in U$.

Definition: $f : \mathbb{C} \to \mathbb{C}$ is analytic/holomorphic at $z_0 \in \mathbb{C}$ if $\exists$ a neighbourhood U of z_0 on which f is differentiable $\forall\, z \in U$.

Example:

$$|z|^2 = z\bar{z} \tag{2.7}$$

$$\lim_{\delta\to 0} \frac{(z+\delta)(\bar{z}+\bar{\delta})-z\bar{z}}{\delta} = \bar{z} + \frac{\bar{\delta}}{\delta} z \tag{2.8}$$

If $z \neq 0$, no limit, but $z = 0$ has limit, 0, $\implies$ differentiable at $z = 0$, but not analytic.

z^n is differentiable everywhere, so analytic on all $\mathbb{C}$.

2.4 Cauchy-Riemann Equations

We have

$$f(z) = f(x+iy) \tag{2.9}$$

$$= u(x,y) + iv(x,y) \tag{2.10}$$

Derivation:

$$\underbrace{\frac{f(z_0+\delta)-f(z_0)}{\delta}}_{z-z_0} = \frac{u(x_0+\delta x, y_0+\delta y)-u(x_0,y_0)}{\delta x + i\delta y} + i\frac{v(x_0+\delta x, y_0+\delta y)-v(x_0,y_0)}{\delta x + i\delta y} \tag{2.11}$$

$$= \frac{u(x_0,y_0) + \delta x \frac{\partial u}{\partial x}|_0 + \delta y \frac{\partial u}{\partial y}|_0 - u_0}{\delta x + i\delta y} + i\frac{v_0 + \delta x \frac{\partial v}{\partial x}|_0 + \delta y \frac{\partial v}{\partial y} - v_0}{\delta x + i\delta y} \tag{2.12}$$

$$= \frac{\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + i\left(\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}\right)\delta y}{\delta x + i\delta y} \tag{2.13}$$

For complex differentiation, limit must be independed of $\delta x + i\delta y$, so the numerator must factorise as $() \times (\delta x + i\delta y)$.

$$\implies \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \tag{2.14}$$

This gives the C-R relations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{2.15}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{2.16}$$

Check with $z^2 = (x^2-y^2) + 2ixy$:

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial y} = 2x \tag{2.17}$$

$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial x} = 2y \tag{2.18}$$

Note:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \implies \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y^2} \tag{2.19}$$

Or

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \nabla^2 u = 0 \tag{2.20}$$

Suppose we regard z and $\bar{z}$ as independed variables.

$$x = \frac{z+\bar{z}}{2}, \ y = \frac{z-\bar{z}}{2i} \tag{2.21}$$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial x}{\partial \bar{z}} \frac{\partial}{\partial \partial x} + \frac{\partial y}{\partial \bar{z}} \frac{\partial}{\partial y} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \tag{2.22}$$

$$\frac{\partial}{\partial z} = \frac{\partial x}{\partial z} \frac{\partial}{\partial x} + \frac{\partial y}{\partial z} \frac{\partial}{\partial y} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \tag{2.23}$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} + i \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} \right) \tag{2.24}$$

$$= \frac{1}{2} \left(\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) + i \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right) \tag{2.25}$$

$$= 0 \iff \text{CR satisfied} \tag{2.26}$$

Can express analytically/differentiable as $\frac{df}{d\bar{z}} = 0$.

Lecture 3 Complex Integration and Cauchy's Theorem

3.1 Subsets of the Plane - Curves and Domains

A continuous curve, γ, in $\mathbb{C}$ is a map.

$$\gamma : [a,b] \subset \mathbb{R} \to \mathbb{C} \tag{3.1}$$

$[a,b]$ is an interval, $\gamma(a)$ is the starting point, and $\gamma(b)$ is the final point. $-\gamma$ is the 'opposite' curve - it is the same path but opposite direction.

$$\gamma : x \in [0,2\pi] \to \mathbb{C}, \ z = e^{ix} \tag{3.2}$$

This is a unit circle anti-clockwise. e^{-ix} is $-\gamma$, a unit circle but clockwise.

An open set G is said to be split if

$$G = G_1 \cup G_2 \tag{3.3}$$

such that G_1, G_2 are open, not empty, and

$$G_1 \cap G_2 = \phi \tag{3.4}$$

G is connected if it does not split. This means that any two points in G can be connected by a curve lying in G.

$$z_1, z_2 \in G : \exists \ \gamma \ s.t. \ \gamma_i = z_1, \gamma_f = z_2, \gamma \subset G. \tag{3.5}$$

G is simply connected if any pair of curves connecting any pair of points in G can be continuously deformed into each other without leacing G.

3.2 Complex Integration

For real integration, there was no ambiguity in the path. For complex integration however, we have many paths.

We must specify the curve of integration,

$$\int_\gamma f(z)\,dz = \int_\gamma (u+iv)(dx+i\,dy) \tag{3.6}$$

$$= \int_\gamma (u\,dx - v\,dy) + i(v\,dx + u\,dy) \tag{3.7}$$

dx and dy are determined y γ.

$$dx = x'(t)\,dt \tag{3.8}$$

$$dy = y'(t)\,dt \tag{3.9}$$

$t \in [a,b]$, parameterising γ.

Example:
Find $\int_\gamma z^2\,dz$ for $\gamma : [0,\pi] \to \mathbb{C}, t \to e^{it}$.

$$x = \cos t \tag{3.10}$$

$$y = \sin t \tag{3.11}$$

$$u = \cos^2 t - \sin^2 t = \cos 2t \tag{3.12}$$

$$v = 2\sin t \cos t = \sin 2t \tag{3.13}$$

$$\implies dx = -\sin t\,dt \tag{3.14}$$

$$\implies dy = \cos t\,dt \tag{3.15}$$

$$\int_\gamma u\,dx - v\,dy = \int_0^\pi [-\cos 2t \sin t - \sin 2t \cos t]\,dt \tag{3.16}$$
$$= -\int_0^\pi \sin 3t\,dt \tag{3.17}$$
$$= \left[\frac{1}{3}\cos 3t\right]_0^\pi \tag{3.18}$$
$$= -\frac{2}{3} \tag{3.19}$$
$$\int_\gamma v\,dx + u\,dy = \int_0^\pi [-\sin 2t \sin t + \cos 2t \cos 2]\,dt \tag{3.20}$$
$$= \int_0^\pi \cos 3t\,dt \tag{3.21}$$
$$= 0 \tag{3.22}$$

Note: $\int_{-1}^1 x^2\,dx = [\frac{1}{3}x^3]_{-1}^1 = -\frac{2}{3}$. Also, $-\frac{2}{3}$ if γ is lower semi-circle - the result seems to be path independent.

3.3 Cauchy's Theorem

Let $D \subset \mathbb{C}$ be a simply connected open subset of $\mathbb{C}$, and if $f : D \to \mathbb{C}$ is analytic on D, then for any closed curve $C \subset D$:

$$\oint_C f(z)\,dz = 0. \tag{3.23}$$

C indicates a closed curve. $\oint$ indicates integral round closed curve. $\oint$ is a contour integral and C the contour of integration. Justify with real analysis.

Recall Green's theorem:

$$\oint_C [A\,dx + B\,dy] = \iint_S \left[\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right] dx\,dy \tag{3.24}$$

Then

$$\oint_C f(z)\,dz = \oint_C [u\,dx - v\,dy] + i[u\,dy + v\,dx] \tag{3.25}$$
$$= \iint_S \left[-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] - i\left[\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right]. \tag{3.26}$$

Here, we use simple-connectedness. But f is analytic in D, so analytic in S, hence by CR relations, this integral is 0. Note, if γ_1, γ_2 are 2 curves from $z_1 \to z_2$, $\gamma_1 - \gamma_2$ is a closed curve, hence

$$\int_{\gamma_1} f = \int_{\gamma_2} f, \text{ for analytic f.} \tag{3.27}$$

$C = \gamma_2 - \gamma_1$ closed.

The integral of analytic functions are independent of the path.

This also allows us to define an integral function as we now know it is path independent:

$$F(z_2) - F(z_1) = \int_\gamma f\,dz, \ \frac{dF}{dz} = f. \tag{3.28}$$

Finally, if C_1 and C_2 are closed contours enclosing surfaces D_1 and D_2, then if f is analytic on $D = D_2 - D_1$,

$$\oint_{C_1} f(z_1)\,dz = \oint_{C_2} f(z)\,dz. \tag{3.29}$$

Note that f doesn't need to be analytic on D_1 or D_2, e.g. $C_1 = e^{it}, C_2 = 2e^{it}, t \in [0, 2\pi], f(z) = \frac{1}{z^2}$. This is not analytic on D_1 or D_2, but it is on $D_2 - D_1$.

Lecture 4 Cauchy's Integral Formula and Taylor Series

Last time:

$$\oint_C f(z)\,dz = 0 \tag{4.1}$$

if f is analytic inside C, and

$$\oint_{C_1} = \oint_{C_2} \tag{4.2}$$

if f is analytic between C_1 and C_2.
Apply this to $f(z) = \frac{1}{z}$. Let C_R be a circular contour $|z| = R$.

$$C_R = \left\{z|z = Re^{i\theta}\right\} \tag{4.3}$$

$$\oint_{C_R} \frac{dz}{z} = \int_0^{2\pi} \frac{Re^{i\theta} i\, d\theta}{Re^{i\theta}} = 2\pi i \tag{4.4}$$

This result is independent of R.

4.1 Cauchy's Integral Formula

Let $f(z)$ be analytic on a simply-connected domain D, $C = \partial D$ is the contour formed by the boundary of D. Then for $z_0 \in D$,

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\,dz. \tag{4.5}$$

Note: this says that the value of a harmonic function in a domain is specified by the values on its boundary.

4.1.1 Proof

$$\frac{f(z)}{z - z_0} \tag{4.6}$$

is not analytic on D, but on $D - z_0$.

Define $D_\epsilon = \{z||z - z_0| \leq \epsilon\} \subset D$. $(D - D_\epsilon$ f is analytic on, hence

$$\oint_C \frac{f(z)}{z - z_0}\,dz = \oint_{C_\epsilon} \frac{f(z)}{z - z_0}\,dz \tag{4.7}$$

$$= \int_0^{2\pi} \frac{f(z_0 + \epsilon e^{i\theta}) i\epsilon e^{i\theta}}{\epsilon e^{i\theta}}\,d\theta \tag{4.8}$$

But f is analytic in D, so differentiable at z_0, hence continuous at z_0. By the definition of continuity,

$$|f(z_0 + \epsilon e^{i\theta}) - f(z_0)| < \delta_{(\epsilon)} \tag{4.9}$$

$$\implies f(z_0 + \epsilon e^{i\theta}) = f(z_0) + O(\epsilon) \tag{4.10}$$

$$\implies \oint_C \frac{f(z)}{z - z_0}\,dz = 2\pi i f(z_0) + O(\epsilon) \tag{4.11}$$

Taking $\epsilon \to 0$ gives result.

Example: Checking

$$f(z) = z + a, z_0 = 0 \tag{4.12}$$

$$\oint_{C_1} \frac{f(z)}{z}\,dz = \oint_{C_1} \left(1 + \frac{a}{z}\right) dz \tag{4.13}$$
$$= 2\pi i a = 2\pi i f(0) \tag{4.14}$$

What about f'?

$$f'(z_0) = \lim_{\delta\to 0} \frac{f(z_0+\delta) - f(z_0)}{\delta} \tag{4.15}$$
$$= \lim_{\delta\to 0} \frac{1}{2\pi i\delta} \oint_C \left(\frac{f(z)}{z-(z_0+\delta)} - \frac{f(z)}{z-z_0}\right) dz \tag{4.16}$$
$$= \lim_{\delta\to 0} \frac{1}{2\pi i\delta} \oint_C \frac{f(z)\,dz\,\delta}{(z-z_0)(z-z_0-\delta)} \tag{4.17}$$
$$= \lim_{\delta\to 0} \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-z_0)(z-z_0-\delta)} \tag{4.18}$$
$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^2}\,dz \tag{4.19}$$

Can repeat and prove by induction, that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \tag{4.20}$$

Every analytic function is infinitely differentiable.

Example:
Check formula for f', $f = e^{2z}, C = 2e^{i\theta}, z_0 = 1$:

$$\oint_{C_2} \frac{e^{2z}}{(z-1)^2}\,dz = \oint_{C_\epsilon} \frac{e^{2(w+1)}}{w^2}\,dw,\ w = z-1, C_2 \to C_\epsilon = 1 + \epsilon e^{i\theta} \tag{4.21}$$
$$= \int_0^{2\pi} \frac{i\epsilon e^{i\theta}\,d\theta \times e^{2+2\epsilon e^{i\theta}}}{(\epsilon e^{i\theta})^2} \tag{4.22}$$
$$e^{2\epsilon e^{i\theta}} = e^{2\epsilon(\cos\theta + i\sin\theta)} \tag{4.23}$$
$$= e^{2\epsilon\cos\theta(\cos(2\epsilon\sin\theta)+i\sin(2\epsilon\sin\theta))} \tag{4.24}$$
$$= (1 + 2\epsilon\cos\theta + \cdots)(1 + 2i\epsilon\sin\theta + O(\epsilon^2)) \tag{4.25}$$
$$= 1 + 2\epsilon\cos\theta + 2i\epsilon\sin\theta + \cdots \tag{4.26}$$
$$= 1 + 2\epsilon e^{i\theta} + O(\epsilon^2) \tag{4.27}$$
$$\oint_{C_2} \frac{e^{2z}}{(z-1)^2}\,dz = \int_0^{2\pi} \frac{e^2 i\,d\theta\,(1 + 2\epsilon e^{i\theta})}{\epsilon e^{i\theta}} \tag{4.28}$$
$$= 2\pi i \cdot \underbrace{2e^2}_{f'(1)} \tag{4.29}$$

4.2 Taylor Series

Let f be analytic on a simply-connected domain D. Then for every $z_0 \in D$, there is a radius r such that for $|z - z_0| < r$,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n. \tag{4.30}$$

4.2.1 Proof

Uses

$$\frac{1}{1-\epsilon} = \sum_{0}^{\infty} \epsilon^n \ \forall\, |\epsilon| < 1 \tag{4.31}$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = \sum_{0}^{\infty} \frac{1}{2\pi i} \oint_C \frac{f(\zeta)\,d\zeta}{(\zeta - z_0)^{n+1}}(z-z_0)^n \tag{4.32}$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \oint_C \frac{f(\zeta)}{\zeta - z_0} \left(\frac{z-z_0}{\zeta - z_0} \right)^n d\zeta \tag{4.33}$$

Make sure to take C such that $|\zeta - z_0| > |z - z_0|$. Swap $\oint$ and $\sum$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z_0} \frac{1}{\left(1 - \frac{z-z_0}{\zeta - z_0}\right)} d\zeta \tag{4.34}$$

$$= \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} d\zeta = f(z) \tag{4.35}$$

Note that while a finite polynomial in z is an entire function, an infinite series may not be.

If

$$\mathcal{P}(z) = \sum_{n=0}^{\infty} a_n z^n \tag{4.36}$$

then $\mathcal{P}(z)$ converges if

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} z \right| = |z| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1 \tag{4.37}$$

which gives us a radius of convergence

$$R = \frac{1}{\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|} \tag{4.38}$$

$$\mathcal{P}(z) = \sum_{0}^{\infty} z^n, \frac{a_{n+1}}{a_n} = 1, R = 1 \tag{4.39}$$

$$e^z = \sum_{0}^{\infty} \frac{z^n}{n!}, R = \infty \tag{4.40}$$

Lecture 5 Zeros and Poles

5.1 Liouville Theorem

Recall:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \qquad \text{(Taylor Series)}$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \qquad \text{(Cauchy Int Form)}$$

Liouville: *Every bounded entire function is constant.*

5.1.1 Proof

Consider

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_{C_R} \frac{f(z)}{(z-z_0)^{n+1}}\,dz \tag{5.1}$$

$$C_R = \{z \mid z = z_0 + Re^{i\theta}\}, R > |z_0| \tag{5.2}$$

Since f is bounded, $\exists M$ such that $|f(z)| < M$ on $\mathbb{C}$.

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi}\left|\oint_{C_R} \frac{f(z)}{(z-z_0)^{n+1}}\,dz\right| < \frac{n!}{2\pi}\oint_{C_R}\frac{|f|\,|dz|}{|z-z_0|^{n+1}} \tag{5.3}$$

$$< \frac{n!}{2\pi}\int_0^{2\pi} d\theta\,\frac{M}{R^n} = \frac{n!M}{R^n} \tag{5.4}$$

but RHS $\to 0$ as $R \to \infty$, hence

$$f^{(n)}(z_0) \equiv 0 \tag{5.5}$$

for all $n \geq 1$. The only Taylor series coefficient left is $n = 0$, i.e. f is constant.

5.2 Zeros and Singularities

An analytic function has a zero at z_0 if we can write (around z_0)

$$f(z) = (z-z_0)^n g(z) \tag{5.6}$$

where $g(z) \neq 0$, g is analytic in a region around z_0. This zero has multiplicity n, or "a zero of order n."

Zeros of an analytic function are isolated. Suppose there exists a subset around zero of f that is also zero ($f(z_0) = 0$).

$$f'(z_0) = \lim_{\delta\to 0}\frac{f(z_0+\delta) - f(z_0)}{\delta} \tag{5.7}$$

Choose to take δ along the path, so that $f(z_0+\delta) = f(z_0) = 0$. This gives $f'(z_0) = 0$ since the limit is path independent. By iteration, all derivatives, $f^{(n)}(z_0) = 0$. Therefore, f is a constant around z_0, i.e. 0. Either f is identically zero, or zeros are isolated.

Definition: Let f be analytic in a domain $D - \{z_0\}$, then f has a pole at z_0 if we can write

$$f(z) = \frac{g(z)}{(z-z_0)^m}, g(z_0) \neq 0 \tag{5.8}$$

g is analytic at z_0. The order of the pole is m.

A function is meromorphic if it is analytic/holomorphic except for isolated poles or singularities.

Note:

$$\oint_C \frac{dz}{z} = 2\pi i \tag{5.9}$$

$$\oint \frac{dz}{z^2} = 0 \tag{5.10}$$

Contour integration picks out the "$\frac{1}{z}$" piece. For analytic g:

$$g(z) = g(z_0) + g'(z_0)(z-z_0) + \frac{g''(z_0)}{2}(z-z_0)^2 + \cdots \tag{5.11}$$

Construct meromorphic f:

$$f(z) = \frac{g(z)}{(z-z_0)^2} = \frac{g(z_0)}{(z-z_0)^2} + \frac{g'(z_0)}{(z-z_0)} + \frac{g''(z_0)}{2} + \cdots \tag{5.12}$$

$$\oint_C f(z)\,dz = \oint_C \frac{g(z_0)}{(z-z_0)^2} + \frac{g'(z_0)}{(z-z_0)} + \cdots \tag{5.13}$$

This leads to a general theorem about contour integration. Suppose f has a simple pole at z_0 (order 1), then

$$\oint_C f(z)\,dz = 2\pi i \lim_{z\to z_0}(z-z_0)f(z) \tag{5.14}$$

$$f(z) = \frac{g(z)}{z-z_0} \tag{5.15}$$

For a simple pole, hence

$$(z-z_0)f = g \tag{5.16}$$

g is analytic, so

$$g(z) = g(z_0) + O(z-z_0) \tag{5.17}$$

$$\implies \oint_C f(z)\,dz = \oint_C \frac{g(z_0)+\cdots}{(z-z_0)}\,dz = 2\pi i g(z_0) \tag{5.18}$$

Example:

$$\oint_{C_2} \frac{dz}{z(z-1)} \tag{5.19}$$

$$\frac{1}{z(z-1)} = \frac{1}{z-1} - \frac{1}{z} \tag{5.20}$$

$$f_1(z) = \frac{1}{z-1} \to \oint_{C_2} \frac{dz}{z-1} = 2\pi i \tag{5.21}$$

$$\oint_{C_2} \frac{dz}{z} = 2\pi i \tag{5.22}$$

$$\implies \oint_{C_2} f\,dz = 0 \tag{5.23}$$

But for pole at $z=1$,

$$\lim_{z\to 1}(z-1)f = \lim_{z\to 1}\frac{1}{z} = 1 \tag{5.24}$$

and for pole at $z=0$,

$$\lim_{z\to 0} zf = \lim_{z\to 0}\frac{1}{z-1} = -1 \tag{5.25}$$

Sum of these is 0. Call this coefficient of $\frac{1}{z-z_0}$ the residue.

Example:

$$\oint_C \frac{e^{2z}}{z^2-4}\,dz = 2\pi i \times (z=2) \tag{5.26}$$

$$\lim_{z\to 2}(z-2)f(z) = \lim_{z\to 2}\frac{e^{2z}}{z+2} = \frac{e^4}{4} \tag{5.27}$$

Next time: Laurent Series

$$f(z) = \sum_{n\to 0}^{\infty} a_n (z-z_0)^n \tag{5.28}$$

where

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \tag{5.29}$$

Lecture 6 Meromorphic Functions and Laurent Series

Recall: A meromorphic function in a domain D is holomorphic except for isolated poles.

6.1 Laurent's Theorem

Let $f(z)$ be holomorphic in the annulus $\{|z-z_0| \in (r_1, r_2)\}, f(z)$ undefined at z_0. Then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \tag{6.1}$$

$$= \underbrace{\sum_{0}^{\infty} a_n (z-z_0)^n}_{\text{Taylor}} + \underbrace{\sum_{1}^{\infty} a_{-n} (z-z_0)^{-n}}_{\text{Principal part}} \tag{6.2}$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz \tag{6.3}$$

C is any simply-closed, noncontractible curve in the annulus.

6.1.1 Proof

Without loss of generality, take $z_0 = 0, c_1 = r_1 e^{i\theta}, c_2 = r_2 e^{i\theta}$. By Cauchy,

$$f(z) = \frac{1}{2\pi i} \oint_{C'} \frac{f(\zeta)}{\zeta - z} d\zeta \tag{6.4}$$

$$= \frac{1}{2\pi i} \left[\underbrace{\oint_{C_2} \frac{f(\zeta)}{\zeta - z} d\zeta}_{f_2} - \underbrace{\oint_{C_1} \frac{f(\zeta)}{\zeta - z} d\zeta}_{f_1} \right] \tag{6.5}$$

$$f_2(z) = \frac{1}{2\pi i} \oint_{C_2} \frac{f(\zeta)}{\zeta - z} d\zeta, \ (\zeta - z) = \zeta \left(1 - \frac{z}{\zeta} \right) \tag{6.6}$$

$$= \frac{1}{2\pi i} \oint_{C_2} \frac{f(\zeta)}{\zeta} \sum_{0}^{\infty} \left(\frac{z}{\zeta} \right)^n d\zeta \tag{6.7}$$

$$= \frac{1}{2\pi i} \oint_{C_2} \frac{f(\zeta)}{\zeta} \sum_{0}^{N} \left(\frac{z}{\zeta} \right)^n d\zeta + \frac{1}{2\pi i} \oint_{C_2} \frac{f(\zeta)}{\zeta} \sum_{N+1}^{\infty} \left(\frac{z}{\zeta} \right)^n d\zeta \tag{6.8}$$

$$= \sum_{0}^{N} a_n z^n + \frac{1}{2\pi i} \oint_{C_2} \left(\frac{z}{\zeta} \right)^{N+1} \frac{f(\zeta)}{\zeta - z} d\zeta \tag{6.9}$$

$$\left| \oint_{C_2} \left(\frac{z}{\zeta} \right)^{N+1} \frac{f(\zeta)}{\zeta - z} d\zeta \right| < \oint_{C_2} \left| \frac{z}{\zeta} \right|^{N+1} \left| \frac{f(\zeta)}{\zeta - z} \right| |d\zeta| \tag{6.10}$$

$$f_2(z) = \sum_{0}^{N} a_n z^n \tag{6.11}$$

$$f_1(z) = -\frac{1}{2\pi i} \oint_{C_1} \frac{f(\zeta)}{\zeta - z} d\zeta \tag{6.12}$$

$$= \frac{1}{2\pi i} \oint_{C_1} \frac{f(\zeta)}{z} \sum_{0}^{\infty} \left(\frac{\zeta}{z} \right)^n d\zeta \tag{6.13}$$

$$= \sum_{0}^{\infty} \frac{1}{2\pi i} \oint_{C_1} f(\zeta) \zeta^n z^{-(n+1)} d\zeta \tag{6.14}$$

$$= \sum_{m=1}^{\infty} \frac{1}{2\pi i} \oint_{C_1} \frac{f(\zeta)}{\zeta^{1-m}} d\zeta\, z^{-m} \tag{6.15}$$

$$= \sum_{m=1}^{\infty} a_m z^{-m} \tag{6.16}$$

$$f(z) = f_1(z) + f_2(z) = \sum_{-\infty}^{\infty} a_n z^n \tag{6.17}$$

Clearly, if $f(z)$ has a pole at z_0, the Laurent series terminates at some finite negative n, the other of the pole. If the series does not terminate at negative n, then f has an essential singularity at z_0.

Example:

$$f(z) = \exp\left(\frac{1}{z}\right) \tag{6.18}$$

$$= \sum_{0}^{\infty} \frac{1}{n! z^n}, \ (x \to 0^+, f \to \infty) \tag{6.19}$$

$$= \sum_{-\infty}^{0} \frac{z^n}{n!}, \ (x \to 0^-, f \to 0) \tag{6.20}$$

The singularity is "nasty" and f takes all values (except possibly one) in any neighbourhood of $z = 0$.

6.2 Integration of meromorphic functions

The poles or singularities of a meromorphic function are isolated. Can now use Laurent series to identify contour integrals.

$$\oint_C f(z)\, dz = \oint_C \left[-\sum_{-\infty}^{-1} a_n (z - z_0)^n + \underbrace{\sum_{0}^{\infty} a_n (z - z_0)^n}_{0,\text{ by Cauchy}} \right] \tag{6.21}$$

$$\oint_C (z - z_0)^n\, dz = 0, \text{ unless } n = -1 \tag{6.22}$$

$$\oint_C f(z)\, dz = 2\pi i a_{-1} \tag{6.23}$$

This is the definition of a_{-1}, the residue or $f(z)$ at z_0.

6.2.1 Calculating Residues

Techniques depend on the situation:

- Simple poles:

$$f(z) = \frac{g(z)}{z - z_0} \tag{6.24}$$

  g is holomorphic at z_0, so

$$\text{Res}(f, z_0) = g(z_0) = \lim_{z \to z_0} (z - z_0) f(z) \tag{6.25}$$

- Pole of order N

$$f(z) = \frac{g(z)}{(z - z_0)^N} \tag{6.26}$$

  g is holomorphic at z_0, and $\neq 0$. Need the $(N-1)^{th}$ coefficient of Taylor series of g.

$$\text{Res}(f, z_0) = \frac{1}{(N-1)!} g^{(N-1)}(z_0) \tag{6.27}$$

$$= \frac{1}{(N-1)!} \lim_{z \to z_0} \frac{d^{N-1}}{dz^{N-1}} \left[(z - z_0)^N f \right] \tag{6.28}$$

➤

$$f(z) = \frac{g(z)}{h(z)} \tag{6.29}$$

$$\mathrm{Res}(f, z_0) = \frac{g(z_0)}{h'(z_0)} \tag{6.30}$$

g, h are holomorphic, h has a simple zero at z_0.
If not a simple pole, safest to series expand.

Example:
Pole or order 2 at $z = 0$:

$$f(z) = \frac{1}{z \sin z} \tag{6.31}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \tag{6.32}$$

$$f(z) = \frac{1}{z^2(1 - \frac{z^2}{6} + \cdots)} = \frac{1 + \frac{z^2}{6} + \cdots}{z^2} \tag{6.33}$$

$$\mathrm{Res}(f, z_0) = 0 \tag{6.34}$$

6.3 Residue Theorem

Let f be meromorphic in a domain D, with poles at $\{z_i\}$. Then,

$$\oint_C f(z)\, dz = 2\pi i \sum \mathrm{Res}(f, z_i) \tag{6.35}$$

Lecture 7 Contour Integration

7.1 Examples

Example: 1

$$f(z) = \frac{\cos(\pi z)}{z^2(1-z)} \tag{7.1}$$

Poles at $z = 1$ (simple), and $z = 0$ (double).

- Look at $z = 1$:

$$\text{Res} = \lim_{z\to 1}(z-1)f(z) \tag{7.2}$$

$$= \lim_{z\to 1} -\frac{\cos(\pi z)}{z^2} = 1 \tag{7.3}$$

- Look at $z = 0$:

$$\text{Res} = \lim_{z\to 0}\frac{d}{dz}[z^2 f(z)] \tag{7.4}$$

$$= \lim_{z\to 0}\frac{d}{dz}\left[\frac{\cos(\pi z)}{(1-z)}\right] \tag{7.5}$$

$$= \lim_{z\to 0}\left[-\frac{\pi\sin(\pi z)}{1-z} + \frac{\cos(\pi z)}{(1-z)^2}\right] \tag{7.6}$$

$$= 1 \tag{7.7}$$

- Missing pole at $z = 1$:

$$I = \oint_{C_{1/2}} f(z)\,dz \tag{7.8}$$

$$= 2\pi i(\text{Res}(f,0)) = 2\pi i \tag{7.9}$$

- Including pole at $z = 1$:

$$I = \oint_{C_2} f(z)\,dz \tag{7.10}$$

$$= 2\pi i(\text{Res}(f,0) + \text{Res}(f,1)) = 4\pi i \tag{7.11}$$

Example: 2

$$f(z) = \frac{e^z}{z^2+z+1} \tag{7.12}$$

$$\tag{7.13}$$

The first job is to identify poles:

$$z^2 + z + 1 = 0 \tag{7.14}$$

$$\implies z = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1-4} \tag{7.15}$$

$$= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2} \tag{7.16}$$

$$= e^{\pm 2\pi i/3} \tag{7.17}$$

Now for the residues:

1.

$$\operatorname{Res}(f, e^{2\pi i/3}) = \lim_{z\to e^{2\pi i/3}} \frac{e^z}{\left(z+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)} \tag{7.18}$$

$$= \frac{e^{-1/2}e^{i\sqrt{3}/2}}{i\sqrt{3}} \tag{7.19}$$

2.

$$\operatorname{Res}(f, e^{-2\pi i/3}) = \lim_{z\to e^{-2\pi i/3}} \frac{e^z}{z+\frac{1}{2}-\frac{i\sqrt{3}}{2}} \tag{7.20}$$

$$= \frac{e^{-1/2}e^{-i\sqrt{3}/2}}{-i\sqrt{3}} \tag{7.21}$$

$$\oint_{C_2} f(z)\,dz = 2\pi i \times \left[\operatorname{Res}(f, e^{2\pi i/3} + \operatorname{Res}(f, e^{-2\pi i/3}\right] \tag{7.22}$$

$$= 2\pi i\left[\frac{e^{-1/2}e^{i\sqrt{3}/2}}{i\sqrt{3}} - \frac{e^{-1/2}e^{-i\sqrt{3}/2}}{i\sqrt{3}}\right] \tag{7.23}$$

$$= 4\pi\frac{e^{-1/2}}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}\right) \tag{7.24}$$

7.2 Zeros and Poles Theorem

Let f be a meromorphic function inside $\mathbb{C}$, then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = n_{zeros} - n_{poles} \tag{7.25}$$

Inside the contour, you count multiplicity for these.

7.2.1 Proof

If f has a pole at z_0, so does f' (Laurent expansion or expression for pole). Also, clearly if f has a zero at z_0, $\frac{1}{f}$ has a pole there, i.e. for a zero z_0,

$$f = (z-z_0)^n g(z) \tag{7.26}$$

$$f' = (z-z_0)^n g' + n(z-z_0)^{n-1} g \tag{7.27}$$

$$\implies \frac{f'}{f} = \underbrace{\frac{g'}{g}}_{\text{regular}} + \underbrace{\frac{n}{z-z_0}}_{\text{pole (simple), Res = n}} \tag{7.28}$$

Similarly, at a pole,

$$f = \frac{g}{(z-z_0)^m} \tag{7.29}$$

$$\frac{f'}{f} = \underbrace{\frac{g'}{g}}_{\text{regular}} - \underbrace{\frac{m}{z-z_0}}_{\text{simple pole, Res = m}} \tag{7.30}$$

7.3 Trigonometric Integrals

Consider an integral pf the form

$$I = \int_0^{2\pi} R(\cos(m\theta), \sin(n\theta))\, d\theta \tag{7.31}$$

R is a rational function - a ratio of polynomials. Can reinterpret as a complex integral on

$$C_1 : z = e^{i\theta} \implies d\theta = -i\frac{dz}{z} \tag{7.32}$$

$$\cos(m\theta) = \frac{e^{im\theta} + e^{-im\theta}}{2} = \frac{z^m + z^{-m}}{2} \tag{7.33}$$

$$\sin(n\theta) = \frac{z^n - z^{-n}}{2i} \tag{7.34}$$

Since R is a rational function, transforming to a complex integral in z, we see our integrand is a rational function of z, hence meromorphic.

$$I = -i\oint_C \frac{dz}{z} R\left(\frac{z^m + z^{-m}}{2}, \frac{z^n - z^{-n}}{2i}\right) \tag{7.35}$$

Example:

$$I = \int_0^{2\pi} \frac{d\theta}{2 - \cos\theta} \tag{7.36}$$

$$= -i\oint_{C_1} \frac{dz}{z}\frac{1}{2 - \frac{z+z^{-1}}{2}} \tag{7.37}$$

$$= \oint_{C_1} \frac{2i\, dz}{z^2 - 4z + 1} \tag{7.38}$$

Integrand has poles where $z^2 - 4z + 1 = 0$:

$$z = 2 \pm \sqrt{4-1} = 2 \pm \sqrt{3} \tag{7.39}$$

Pole at $2 - \sqrt{3}$ is inside unit circle, so calculate residue:

$$\text{Res} = \lim_{z \to 2-\sqrt{3}} \frac{\cancel{(z-2+\sqrt{3})} \cdot 2i}{\cancel{(z-2+\sqrt{3})}(z-2-\sqrt{3})} \tag{7.40}$$

$$= \frac{2i}{-2\sqrt{3}} = -\frac{i}{\sqrt{3}} \tag{7.41}$$

$$I = 2\pi i \text{Res}(f, 2-\sqrt{3}) = 2\pi i \times -\frac{i}{\sqrt{3}} = \frac{2\pi}{\sqrt{3}} \tag{7.42}$$

Example:

$$I = \int_0^{2\pi} \frac{\sin\theta\, d\theta}{(3+\cos\theta)(2+\sin\theta)} \tag{7.43}$$

$$= \oint_{C_1} -\frac{i\, dz}{z} \frac{\frac{z-z^{-1}}{2i}}{\left(3 + \frac{z+z^{-1}}{2}\right)\left(2 + \frac{z-z^{-1}}{2i}\right)} \tag{7.44}$$

$$= \oint_{C_1} \frac{-2i(z^2-1)\, dz}{(z^2+6x+1)(z^2+4iz-1)} \tag{7.45}$$

- $z^2 + 6z + 1 = 0 \implies z = -3 \pm \sqrt{8} = -3 \pm 2\sqrt{2}$. Pole inside C_1 at $z_1 = -3 + 2\sqrt{2}$.
- $z^2 + 4iz - 1 = 0 \implies z = -2i \pm \sqrt{-4+1} = -2i \pm i\sqrt{3}$. Pole inside C_1 at $z_2 = -2i + i\sqrt{3}$.

Now look at residues:

1.

$$\mathrm{Res}(f, z_1) = \lim_{z \to z_1} (z - z_1) f(z) \tag{7.46}$$

$$z_1^2 = -6z_1 - 1 \tag{7.47}$$

$$\implies z_1^2 - 1 = -2 - 6z_1 = -2 + 18 - 12\sqrt{2} \tag{7.48}$$

$$= 4\sqrt{2} z_1 \tag{7.49}$$

$$z_1^2 + 4iz - 1 = 4z_1(\sqrt{2} + i) \tag{7.50}$$

$$\mathrm{Res}(f, z_1) = \lim_{z \to -3+2\sqrt{2}} \frac{-2i(z^2 - 1)}{(z_1 + 3 + 2\sqrt{2})(z_1^2 + 4iz - 1)} \tag{7.51}$$

$$= \frac{-2i \times 4\sqrt{2} z_1}{4\sqrt{2} \times 4z_1(\sqrt{2} + i)} \frac{\sqrt{2} - i}{\sqrt{2} - i} \tag{7.52}$$

$$= \frac{-i(\sqrt{2} - i}{2 \times 3} = -\frac{i}{6}(\sqrt{2} - i) \tag{7.53}$$

Some reductions work for other residues.

2.

$$\mathrm{Res}(f, z_2) = \frac{i}{6}(\sqrt{3} - i) \tag{7.54}$$

Hence,

$$I = 2\pi i \times \frac{i}{6}(\sqrt{3} - i - (\sqrt{2} - i)) \tag{7.55}$$

$$= -\frac{\pi}{3}(\sqrt{3} - \sqrt{2}) \tag{7.56}$$

Lecture 8 Integrals over the real line

8.1 Real integrals

Sometimes a real integral can be more easily performed by analytic continuation to the $\mathbb{C}$-plane and using residues.

Example: 1

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} \tag{8.1}$$

➤ In real analysis: $x = \tan u$, $dx = \sec^2 u\, du$, $1 + x^2 = \sec^2 u$, $x = \pm\infty, u = \pm\pi/2$.

$$I = \int_{\pi/2}^{\pi/2} \frac{sec^2 u\; du}{\sec^2 u} = \pi \tag{8.2}$$

➤ In complex analysis:

$$f(z) = \frac{1}{1+z^2} \tag{8.3}$$

f has 2 simple poles at $z = \pm i$, so,

$$\operatorname{Res}(f, \pm i) = \lim_{z\to\pm i} (z \mp i) f(z) \tag{8.4}$$

$$= \lim_{z\to\pm i} \frac{1}{z \pm i} = \frac{1}{\pm 2i} \tag{8.5}$$

Let C be the contour

$$[-R, R] \cup \left\{ Re^{i\theta} | \theta \in [0, \pi] \right\} \tag{8.6}$$

For $|z| = R$, $R|f(z)| \to 0$ as $R \to \infty$. The integral around semi-circle goes to 0 as $R \to \infty$.

$$I = \lim_{R\to\infty} \oint_C f(z)\, dz \tag{8.7}$$

$$= 2\pi i \operatorname{Res}(f, i) = \pi \tag{8.8}$$

Example: 2

$$I_2 = \int_{-\infty}^{\infty} \frac{x+2}{(2x^2+3)^2}\, dx \tag{8.9}$$

$$f(z) = \frac{z+2}{(2z^2+3)^2} \tag{8.10}$$

Double poles at $2z^2 + 3 = 0$.

As before, $R|f(Re^{i\theta})| \to 0$ as $R \to \infty$, so take some contour as before. Integral around semicircle in upper $\frac{1}{2}\mathbb{C}$-plane $\to 0$.

$$I_2 = \lim_{R\to\infty} \oint_C f(z)\, dz \tag{8.11}$$

This encloses pole at $z = +i\sqrt{\frac{3}{2}}$. Note:

$$(2z^2+3)^2 = 4\left(z^2 + \frac{3}{2}\right)^2 \tag{8.12}$$

$$= \left(z + i\sqrt{\frac{3}{2}}\right)^2 \left(z - i\sqrt{\frac{3}{2}}\right)^2 \tag{8.13}$$

$$\mathrm{Res}\left(f, i\sqrt{\frac{3}{2}}\right) = \lim_{z\to i\sqrt{3/2}} \frac{d}{dz}\left[\left(z - i\sqrt{\frac{3}{2}}\right)^2 f(z)\right] \tag{8.14}$$

$$= \lim_{z\to i\sqrt{3/2}} \left[\frac{z+2}{4(z+i\sqrt{3/2})^2}\right] \tag{8.15}$$

$$= \left[\frac{1}{4(z+i\sqrt{3/2})^2} - \frac{2(z+2)}{4(z+i\sqrt{3/2})^3}\right] \times i\sqrt{\frac{3}{2}} \tag{8.16}$$

$$= \frac{1}{4(i\sqrt{6})^2} - \frac{2(i\sqrt{3/2}+2}{4(i\sqrt{6})^3} \tag{8.17}$$

$$= -\frac{1}{(i\sqrt{6})^3} = \frac{1}{i6\sqrt{6}} \tag{8.18}$$

$$\implies I_2 = 2\pi i \mathrm{Res}\left(f, i\sqrt{\frac{3}{2}}\right) \tag{8.19}$$

$$= \frac{\pi}{3\sqrt{6}} \tag{8.20}$$

The more complicated the rational function, the easier the complex method.

8.2 Branch cuts and contours

Sometimes we need to be careful about constructing a contour, as contours cannot cross branch cuts.

Example:

$$f(z) = z^{-\alpha} = \exp(-\alpha\log(z)) \tag{8.21}$$

Cannot integrate f around C_R if $\alpha \notin \mathbb{Z}$.

$$I = \int_0^\infty \frac{u^{-y}}{1+u}\,du \tag{8.22}$$

Analytically continue to $\mathbb{C}$:

$$f(z) = \frac{z^{-y}}{1+z} \tag{8.23}$$

$$z^{-y} = \exp(-y\log(z)) \tag{8.24}$$

$$(u+i\epsilon)^{-y} = u^{-y}\exp\left(-y\left(\frac{i\epsilon}{u}\right)\right) \tag{8.25}$$

Here, choosing standard branch for log, $\log(re^{i\theta}) = \log(r) + i\theta, \theta \in [0, 2\pi]$. Requires branch cut on $\mathbb{R}^+$. Since $\theta \in [0, 2\pi]$ on this branch of log, "$u - i\epsilon$" is really $\theta = +2\pi - O(\epsilon)$.

$$\log(u - i\epsilon) = \log(u) + \log\left(1 - \frac{i\epsilon}{u}\right) \tag{8.26}$$

$$= \log(u) + 2\pi i - \frac{i\epsilon}{u} \tag{8.27}$$

$$(u - i\epsilon)^{-y} = u^{-y}e^{-2\pi i y + i\epsilon y/u} \tag{8.28}$$

To avoid branch cut, must construct "Pacman"-style contour - circle going round at radius, R, then breaking near the positive reals to go back to the origin then curve round.

$$C = C_R' \cup I \cup C_\epsilon' \cup \gamma \tag{8.29}$$

- C_R':

$$\left|\frac{z^{-y}}{1+z}\right| R \to 0 \tag{8.30}$$

  for $|z| = R, R \to \infty$.

- C'_ϵ:

$$\left|\frac{z^{-y}}{1+}\right|\epsilon \to 0 \tag{8.31}$$

for $|z| = \epsilon, \epsilon \to 0$.

- γ_-:

$$z^{-y} = u^{-y}e^{-2\pi iy} \tag{8.32}$$

$$\int_\gamma f\,dz = \int_R^0 \frac{u^{-y}}{1+u}e^{-2\pi iy}\,du \tag{8.33}$$

$$= -e^{-2\pi iy}I \tag{8.34}$$

$$\oint_C f(z)\,dz = I - e^{-2\pi iy}I \tag{8.35}$$

$$= (1 - e^{-2\pi iy})I \tag{8.36}$$

R has a simple pole at $z = -1$.

$$\text{Res}(f, -1) = \lim_{z\to-1}(1+z)f(z) = (-1)^{-y} \tag{8.37}$$

$$= e^{-i\pi y} \tag{8.38}$$

$$I = \frac{2\pi i\text{Res}(f,-1)}{1-e^{-2\pi iy}} = \frac{2\pi ie^{-i\pi y}}{1-e^{-2\pi iy}} \tag{8.39}$$

$$= \pi \times \frac{2i}{e^{i\pi y} - e^{-\pi y}} = \frac{\pi}{\sin(\pi y)} \tag{8.40}$$

8.3 Closing contours

Example:

$$I = \int_{-\infty}^{\infty} \frac{x\sin(\pi x)}{x^2+1}\,dx \tag{8.41}$$

$$f(z) = \frac{z\sin(\pi z)}{1+z^2} \tag{8.42}$$

$$= \frac{z(e^{i\pi z} - e^{-i\pi z})}{(1+z^2)(2i)} \tag{8.43}$$

Looking to close contour in upper/lower half of plane. For semicircle in upper half plane, $R|e^{i\pi z}| \to 0$, but $R|e^{-i\pi z}| \to \infty$. Define:

$$f_{\pm} = \pm\frac{ze^{\pm i\pi z}}{2i(1+z^2} \tag{8.44}$$

Let $C_+ = [-R, R] \cup \left\{Re^{i\theta} \,||\theta \in [0,\pi]\right\}$.

$$\oint_{C_+} f_+\,dz = 2\pi i \text{Res}(f,i) \tag{8.45}$$

f_+ has simple pole at i.

$$\text{Res}(f_+, i) = \lim_{z\to i}(z-i)f_+ \tag{8.46}$$

$$= \lim_{z\to i}\frac{ze^{i\pi z}}{2i(z+i)} = \frac{ie^{-\pi}}{(2i)^2} \tag{8.47}$$

$$= \frac{e^{-\pi}}{4i} \tag{8.48}$$

$$\oint_{C_+} f_+\,dz = \frac{\pi e^{-\pi}}{2} \tag{8.49}$$

$$I = 2\text{Res}(f_+, i) = \pi e^{-\pi} \tag{8.50}$$

Lecture 9 Summation of Series

Cotour integrals can be used to good effect in evaluating infinite series.

9.1 Riemann-Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{9.1}$$

Look at $s \in \mathbb{Z}^+, (s > 1)$. Looks like we are summing over an infinite series of poles with residues, $\frac{1}{n^s}$.

Aim: Construct meromorphic function f_s such that f_s has poles at integer $z = n$, residues $\frac{1}{n^s}$, and $R|f| \to 0$ for $|z| = R \to \infty$. This last condition with lead to $\int_{CR} f \, dz = 0$. Note: $\sin(\pi z)$ has zeros at $z = n$. Around $z = n$:

$$\sin(n\pi + \pi(z-n)) = \cos(n\pi)\sin(\pi(z-n)) \tag{9.2}$$

$$= (01)^n \pi(z-n) + O(z-n)^3 \tag{9.3}$$

$$\pi\cot(\pi z) = \frac{1}{z-n} + O(z-n) \tag{9.4}$$

$$f_s(z) = \frac{\pi\cot(\pi z)}{z^s} \tag{9.5}$$

Simple poles at $z = n \neq 0$, residue $\frac{1}{n^s}$. At $z = 0$, pole of order $s+1$. If s is even, then

$$\frac{1}{(-n)^s} = \frac{1}{n^s}. \tag{9.6}$$

Poles on $\mathbb{Z}^+$ have same residue as poles on Z^-.

Consider

$$z = Re^{i\theta} = R\cos\theta + iR\sin\theta \tag{9.7}$$

$$e^{2i\pi z} = e^{2i\pi R\cos\theta} e^{-2\pi R\sin\theta} \tag{9.8}$$

$$\theta \in (0,\pi), \sin\theta > 0, e^{2i\pi z} \to 0, |z| = R \to \infty \tag{9.9}$$

$$\theta \in (\pi, 2\pi), \sin\theta < 0, e^{2i\pi z} \to 0, |z| = R \to \infty \tag{9.10}$$

$$|\cot(\pi z)| = \left|\frac{e^{2i\pi z}+1}{e^{2i\pi z}-1}\right| \to 1 \tag{9.11}$$

To cover $z \in \mathbb{R}$, take $z = \pm R, R = N + \frac{1}{2} \implies \cot(\pi(N+\frac{1}{2})) = 0$. So take the discrete series of contours, C_R, $R = N + \frac{1}{2}, N \in \mathbb{Z}^+ \to \infty$.

On C_R: $R_N|f(z)| \to 0$ as $N \to \infty$ and

$$\int_{C_R} f(z)\, dz = 0 \tag{9.12}$$

At finite N, $\oint_{C_R} f \, dz$ is small, poles run from $-N$ to N.

$$\sum_{res} \mathrm{Res}(f, z_i) = 0 \tag{9.13}$$

Poles at $z \in \mathbb{Z}$.

$$2\zeta(s) + \mathrm{Res}(f_s, 0) = 0 \tag{9.14}$$

➤ $s = 2$: At $z = 0$, has triple pole

$$f_2 = \frac{\pi\cot(\pi z)}{z^2} \tag{9.15}$$

$$\cos(\pi z) = 1 - \frac{(\pi z)^2}{2} + \frac{(\pi z)^4}{4!} + \cdots \tag{9.16}$$

$$\sin(\pi z) = \pi z - \frac{(\pi z)^3}{3!} + \frac{(\pi z)^5}{5!} + \cdots \tag{9.17}$$

$$\frac{\pi\cot(\pi z)}{z^2} = \pi\frac{1 - \frac{(\pi z)^2}{2} + \frac{(\pi z)^4}{4!}}{\pi z^3\left(1 - \frac{(\pi z)^2}{3!}\right)} \tag{9.18}$$

$$= \frac{1}{z^3}\left(1 - \frac{\pi^2 z^2}{3}\right) \tag{9.19}$$

$$\zeta(2) = -\frac{1}{2}\text{Res}(f_2, 0) = \frac{\pi^2}{6} \tag{9.20}$$

➤ $s = 4$:

$$\frac{\pi\cot(\pi z)}{z^4} = \frac{1}{z^5}\frac{1 - \frac{(\pi z)^2}{2} + \frac{(\pi z)^4}{4!}}{1 - \epsilon} \tag{9.21}$$

Let

$$\epsilon = \frac{(\pi z)^2}{3!} - \frac{(\pi z)^4}{4!} \tag{9.22}$$

$$\frac{1}{1-\epsilon} = 1 + \epsilon + \epsilon^2 + \cdots \tag{9.23}$$

$$\frac{\pi\cot(\pi z)}{z^4} = \frac{1}{z^5}\left(1 - \frac{(\pi z)^2}{2} + \frac{(\pi z)^4}{4!}\right)\left(1 + \frac{\pi z)^2}{3!} - \frac{(\pi z)^4}{5!} + \frac{(\pi z)^4}{36}\right) \tag{9.24}$$

$$\text{Residue} = -\frac{\pi^4}{45} \tag{9.25}$$

$$\zeta(4) = \frac{\pi^4}{90} \tag{9.26}$$

9.2 Other Series

What about

$$S = \sum_1^\infty \frac{(-1)^n}{n^2} \tag{9.27}$$

$\frac{\pi}{\sin(\pi z)}$ has alternating sign for residue.

$$f(z) = \frac{\pi}{z^2\sin(\pi z)} \tag{9.28}$$

Two copies of S on positive and negative real axis. As $z \to 0$:

$$f(z) = \frac{1}{z^3}\left(1 + \frac{(\pi z)^2}{3!}\right) \tag{9.29}$$

$$S = -\frac{1}{2}\text{Res}(f, 0) = -\frac{\pi^2}{12} \tag{9.30}$$

Finally, suppose the terms in the series are not symmetric around $z = 0$.

$$S = \sum_0^\infty \frac{1}{(n+1)(n+2)} \tag{9.31}$$

$$f(z) = \frac{\pi \cot(\pi z)}{(z+1)(z+2)} \tag{9.32}$$

This gives S from poles for $z = n \geq 0$. Consider $m = -n - \lambda \implies m + i = -(n + (\lambda - i))$, so if $\lambda = 3$, then $m + 1 = -(n + 2), m + 2 = -(n + 1)$.

So we have $(m+1)(m+2) = (n+1)(n+2)$. This gives two copies of S, but not symmetric around the origin.

$$\sum_{n=-3}^{-\infty} \frac{1}{(n+1)(n+2)} = \sum_{m=0}^{\infty} \frac{1}{(m+1)(m+2)} \tag{9.33}$$

Thus $2S + \mathrm{Res}(f, -1) + \mathrm{Res}(f, -2) = 0$. Poles at $z = -1, -2$ are double poles.

$$\pi \cot(\pi z) = \frac{1}{z-n}(1 + O(z-n)^2) \tag{9.34}$$

Both poles give residue of -1, hence $S = 1$.

Part II

Infinite Dimensional Vector Spaces

Lecture 1

Define vector space and scalars:

$$E = \{\underline{x}\} \tag{1.1}$$
$$k = (\mathbb{R}, \mathbb{C}) \tag{1.2}$$

Define '+' - the sum, and '·' - product by scalar.

The properties of the sum:

- commutative
- associative
- neutral element
- inverse element

The properties of the product:

- neutral element in k
- distributive

1.1 Bases and eigenvectors

Consider $\underline{x}$ in n-dimensional space, with orthonormal basis $\{\underline{e}_i\}, i = 1, \cdots, n$.

$$\underline{x} = \sum_{i=1}^{n} \underline{e}_i x^i = \underline{e}_1 x^1 + \cdots + \underline{e}_n x^n \tag{1.3}$$

$$\tag{1.4}$$

$(x^1 \cdots x^n)$ - x^i cooridnates of the vector in $\{\underline{e}_i\}$. In matrix form:

$$\underline{x} = \begin{bmatrix} \underline{e}_1 & \cdots & \underline{e}_n \end{bmatrix} \begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix} \tag{1.5}$$

$$\underline{x} = \sum_{i=1}^{n} \underline{e}'_i x^i = \underline{e}'_1 x^1 + \cdots + \underline{e}'_n x^n \tag{1.6}$$

Transformation of basis:

$$\underline{e}'_1 = \underline{e}_1 S^1_1 + \cdots + \underline{e}_n S^n_1 = \sum_i \underline{e}_i S^i_1 \tag{1.7}$$
$$\underline{e}'_n = \underline{e}_1 S^1_n + \cdots + \underline{e}_n S^n_n = \sum_i \underline{e}_i S^i_n \tag{1.8}$$

$$\underline{e}' = \begin{bmatrix} \underline{e}_1 \cdots \underline{e}_n \end{bmatrix} \begin{bmatrix} S^1_1 & S^1_2 & \cdots & S^1_n \\ \vdots & & & \\ S^n_1 & \cdots & \cdots & S^n_n \end{bmatrix} \tag{1.9}$$

$$\underline{x} = \sum_{i=1}^{n} \underline{e}'_i x'^i = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \underline{e}_j S^j_i \right) x'^i \tag{1.10}$$
$$= \sum_{j=1}^{n} \underline{e}_j \left(\sum_{i=1}^{n} S^j_i x'^i \right) = \sum_{j=1}^{n} \underline{e}_j x^j \tag{1.11}$$

$$x^j = \sum_{i=1}^{n} S^j_i x'^i \tag{1.12}$$

$$\begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{bmatrix} = \begin{bmatrix} S^1_1 & \cdots & \cdots & S^1_n \\ & \ddots & & \\ & & \ddots & \\ & & & S^n_n \end{bmatrix} \begin{bmatrix} x'^1 \\ x'^2 \\ \vdots \\ x'^n \end{bmatrix} \tag{1.13}$$

$$\begin{bmatrix} x'^1 \\ \vdots \\ x'^n \end{bmatrix} = \begin{bmatrix} & S^{-1} & \end{bmatrix} \begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix} \tag{1.14}$$

1.2 Einstein's Notation

$$m^2 = |p|^2 = \sum_{i=1}^{4} p_i p^i = p_i p^i \tag{1.15}$$

$$m^2 \cdot x^i = p_i p^i x^i \tag{1.16}$$

$$\delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{1.17}$$

$$\underline{x} = \underline{e}_i x^i = \underline{e}'_i x'^i \tag{1.18}$$

$$\underline{e}'_j = \underline{e}_i S^i_j \tag{1.19}$$

$$x^i = S^i_j x'^j \tag{1.20}$$

Can write product of matrices in the same way:

$$(AB)^i_j = A^i_k B^k_j \tag{1.21}$$

Example: 3D Rotations
Have basis of $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$

Consider the vector, $\underline{p} : (p^1, p^2, p^3)$, with a rotation of ϕ_3 around $\underline{e}_3$.

$$\underline{e}'_1 = \underline{e}_1 \cos(\phi) + \underline{e}_2 \sin(\phi) \tag{1.22}$$

$$\underline{e}'_2 = -\underline{e}_1 \sin(\phi) + \underline{e}_2 \cos(\phi) \tag{1.23}$$

$$\underline{e}'_3 = \underline{e}_3 \tag{1.24}$$

$$S(\underline{e}_3, \phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.25}$$

$$\begin{bmatrix} p'^1 \\ p'^2 \\ p'^3 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{S^{-1} = R(\underline{e}_3, \phi)} \begin{bmatrix} p^1 \\ p^2 \\ p^3 \end{bmatrix} \tag{1.26}$$

$$R(\underline{e}_2, \phi) = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \tag{1.27}$$

Now consider a consecutive rotation, $\phi_1 \to \phi_2, \underline{e}_2$:

$$R(\underline{e}_2, \phi_1 + \phi_2) = \begin{bmatrix} \cos(\phi_1 + \phi_2) & 0 & \sin(\phi_1 + \phi_2) \\ 0 & 1 & 0 \\ -\sin(\phi_1 + \phi_2) & 0 & \cos(\phi_1 + \phi_2) \end{bmatrix} \tag{1.28}$$

These rotations can lead to the matrix representation:

$$e^{i\phi_1} e^{i\phi_2} = e^{i(\phi_1 + \phi_2)} \tag{1.29}$$

- $R^{-1} = R^T \implies R^T R = 1$ - this is an example of an orthogonal matrix

$$S = \begin{bmatrix} -1 & & \\ & -1 & \\ & & -1 \end{bmatrix} \tag{1.30}$$

- When ϕ is small, R becomes $I +$ something else

Lecture 2

2.1 Linear Forms

$$E \to k \tag{2.1}$$
$$\underline{x} \to F(\underline{x}) \tag{2.2}$$

F is fully defined given:

$$F(\underline{e}_i) \equiv f_i \tag{2.3}$$
$$F(\underline{x}) = F\left(\sum \underline{e}_i x^i\right) = \sum F(\underline{e}_i) x^i = f_i x^i \tag{2.4}$$

2.2 Dual Space

Define dual space with respect to E - E^* is the set of all linear forms that can be defined on E.

$$E^* \text{ is a vector space: } \begin{cases} F + G \to & \text{linear functional} \\ \alpha F & \\ \text{zero functional} \to & O(\underline{x}) = 0 \\ \text{additive inverse element} & \end{cases} \tag{2.5}$$

Can also define a canonical basis on E^*:

$$E : \{\underline{e}_i\} \tag{2.6}$$
$$E^* : \{\underline{e}^{*i}\} \tag{2.7}$$
$$\underline{e}^{*i}(\underline{e}_j) = \delta^i_j \tag{2.8}$$
$$\underline{e}^{*i} \equiv \underline{e}^i \tag{2.9}$$
$$\underline{y} = \underline{e}_i y^i \tag{2.10}$$
$$y^i = \underline{e}^i(\underline{y}) = \underline{e}^i(\underline{e}_j y^j) = \delta^i_j y^j = y^i \tag{2.11}$$

Given $\underline{x}^*$ in in E^*:

$$\underline{x}^*(\underline{y}) = \underline{x}^*(\underline{e}_j y^j) = \overbrace{\underline{x}^*(\underline{e}_j)}^{x^*_j} y^j \tag{2.12}$$
$$= x^*_j \underline{e}^j(\underline{y}) \tag{2.13}$$
$$\underline{x}^* = x^*_j \underline{e}^j \tag{2.14}$$

Change of basis:

$$\underline{e}'_j = \underline{e}_i S^i_j \tag{2.15}$$
$$\implies x'^*_j = x_i S^i_j \tag{2.16}$$
$$\implies x^j = T^j_i x^i \tag{2.17}$$

Note: transforming under S is known as covariant, transforming under T is contravariant.

2.3 Tensors

Consider a tensor of the following form

$$A^{i1\cdots ir}_{j1\cdots js} \tag{2.18}$$

n-dimensional vector space, n^{r+s} components. It will transforms covariant for the subindices, and transforms contravariant for superindices:

$$A'^{i1\cdots ir}_{j1\cdots js} = T^{i1}_{m1}\cdots T^{ir}_{mr}A^{m1\cdots mr}_{n1\cdots nr}S^{n1}_{j1}\cdots S^{ns}_{js} \tag{2.19}$$

- $i1\cdots ir \rightarrow$ contravariant coordinates
- $j1\cdots js \rightarrow$ covariant coordinates

In orthonormal spaces, contravariant and covariant components are identical.

2.3.1 Examples

- Rank 0 Tensors:
  - Scalars

$$R : \phi \rightarrow \phi' = \phi \tag{2.20}$$
$$S : \phi \rightarrow \phi' = \phi \tag{2.21}$$

  - Pseudoscalars

$$R : \rho \rightarrow \rho' = \rho \tag{2.22}$$
$$S : \rho \rightarrow \rho' = -\rho \tag{2.23}$$
$$\rho = \underline{a}\cdot(\underline{x}\wedge\underline{y}) \tag{2.24}$$

- Rank 1 Tensors:
  - Vectors

$$R : v^i \rightarrow v'^i = R^i_j v^j \tag{2.25}$$
$$S : v^i \rightarrow v'^I = S^i_j v^j = -v^i \tag{2.26}$$

  - Pseudovectors (axial vectors)

$$R : \omega^i \rightarrow \omega'^i = R^i_j \omega^j \tag{2.27}$$
$$S : \omega^i \rightarrow \omega'^i = \det(S)\; S^i_j \omega^j = \omega^i \tag{2.28}$$

- Rank 2 Tensors
  - Moment of Inertia

$$I^{ij} = \sum_n m_n\left[|\underline{r}_n|^2\delta^{ij} - r^i_n r^j_n\right] \tag{2.29}$$

    This is a symmetric tensor, $\implies I^{ij} \rightarrow J^{ji} = J^{ij}$.

$$L^i = \omega_j I^{ji} = \delta_{jk}\omega^k I^{ji} \tag{2.30}$$

    where L^i is the angular momentum, ω_j is the angular velocity.
    Can choose a basis in which I' is diagonal:

$$I' = \begin{bmatrix} I_1 & & \\ & \ddots & \\ & & I_n \end{bmatrix} \tag{2.31}$$

    where $I_1 \rightarrow I_n$ are the principle moments of inertia.

- Surfaces and curvatures Consider $h(x,y)$ single valued and continuous.

$$\mathbb{R}^2 \to^h \mathbb{R} \tag{2.32}$$

Local minimum of $h(x,y)$:

$$h(x,y) = h(x_0,y_0) + \frac{1}{2}(\partial_x, \partial_y)\begin{pmatrix} h_{xx} & h_{xy} \\ h_{yx} & h_{yy} \end{pmatrix}\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \tag{2.33}$$

2x2 matrix is the curvature tensort, h. Gaussian curvature is det(h).

- Isotropic tensors - invariant under rotation.
  - All Rank 0 tensors are isotropic
  - No Rank 1 tensor is isotropic
  - The only Rank 2 tensor that is isotropic is the Kronecker delta, δ^{ij}:

$$\delta'_{mn} = \delta_{ij} R^i_m R^j_n = R_{jm} R^j_n \tag{2.34}$$
$$= R^T_{mj} R^j_n \tag{2.35}$$
$$= (R^T R)_{mn} \tag{2.36}$$
$$= \delta_{mn} \tag{2.37}$$

- Rank 3: Levi Civita:

$$\epsilon_{ijk} = \begin{cases} 1 & (1,2,3),(3,1,2),(2,3,1) \\ -1 & (1,3,2)\cdots \\ 0 & \text{repeated indices} \end{cases} \tag{2.38}$$

- Scalar product:

$$\underline{a} \cdot \underline{b} = a^i b^i \delta_{ij} \tag{2.39}$$

- Vector product:

$$(\underline{a} \wedge \underline{b})_i = \epsilon_{ijk} a^j b^k \tag{2.40}$$

Lecture 3 General Vector Spaces

3.1 Groups and Fields

We need a notion of both of these math objects to define a general VS.

3.1.1 Group

A group is a set G together with an operation $O : G \times G \to G$, with properties:

- Associativity, $(a \circ b) \circ c = a \circ (b \circ c)$
- Identity element, e: $a = e \circ a = a \circ e \; \forall a \in G$
- Inverse element $\forall a$, $a \circ a^{-1} = a^{-1} \circ a = e$

A few deductible properties:

- e is unique.
  Proof: Assume e, e' two identity elements:

$$e' = e \circ e' = e \implies e = e' \tag{3.1}$$

- a^{-1} is unique $\forall a$:

$$aa^{-1} = a \circ (a')^{-1} = e \tag{3.2}$$
$$a^{-1} \circ (a \circ a^{-1}) = a^{-1} \circ (a \circ (a')^{-1}) \tag{3.3}$$
$$(a^{-1} \circ a) \circ a^{-1} = (a^{-1} \circ a) \circ (a')^{-1} \tag{3.4}$$
$$e \circ a^{-1} = e \circ (a')^{-1} \tag{3.5}$$
$$a^{-1} = (a')^{-1} \tag{3.6}$$

- The inverse of $a \circ b$ is $b^{-1} \circ a^{-1}$:

$$(b^{-1} \circ a^{-1}) \circ (a \circ b) = b^{-1} \circ (a^{-1} \circ a) \circ b = b^{-1} \circ e \circ b = e \tag{3.7}$$

Examples:

- $\mathbb{Z}_2$ group has representation under matrix multiplication:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \tag{3.8}$$

- $\mathbb{Z}_n$ group has representation:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & e^{i2\pi k/n} \end{pmatrix}, k = 0 \to n-1 \right\} \tag{3.9}$$

- Groups can be infinite, e.g. $\mathbb{R}$
- These groups are commutative, $a \circ b = b \circ a \; \forall a$. Called Abelian groups (not true of all groups), e.g. rotation groups

3.1.2 Fields

A field is an extension of a group but with two operations, usually called 'addition' and 'multiplication' (+ and ·)

- $(F, +)$ is an abelian group with identity element 0
- $(F\ \{0\}, \cdot)$ is an abelian group with neutral element 1
- Distributive law ($\forall a, b, c \in F$):

$$a \cdot (b + c) = a \cdot b + a \cdot c \tag{3.10}$$
$$(a + b) \cdot c = a \cdot c + b \cdot c \tag{3.11}$$

Examples of Fields:

- $(\mathbb{R}, +, \cdot)$
- $(\mathbb{Q}, +, \cdot)$
- $(\mathbb{C}, +, \cdot)$
- *But not* $\mathbb{N}$

3.1.3 Definition of a Vector Space

A vector space over the field F is a set V together with the operations:

- '+':

$$V \times V \to V \tag{3.12}$$
$$(v, w) \to v + w \tag{3.13}$$

- '·':

$$F \times V \to V \tag{3.14}$$
$$(\lambda, v) \to \lambda v \tag{3.15}$$

For which, the following holds:

1. $(v + w) + z = v + (w + z)$
2. $\exists 0 \in V \mid v + 0 = 0 + v = v$
3. $\exists -v \in V \mid v + (-v) = (-v) + v = 0$
4. $v + w = w + v,\ \forall v, w \in V$
5. $(\alpha + \beta)v = \alpha v + \beta v$
6. $\alpha(v + w) = \alpha v + \alpha w$
7. $\alpha(\beta v) = (\alpha\beta)v$
8. $1v = v$

Using 1 and 4 above, can say $(V, +)$ is an abelian group, where 0 is the nullvector, and $v^{-1} = -v$. Note in 7, two different operations appear.

Common fields could be $\mathbb{C}$ or $\mathbb{R}$, i.e. $(\alpha, \beta \in \mathbb{R})$

NB:

- There is no scalar product in a general VS, i.e. this is an additional structure
- No mention of dimension (∞ dimension included)

Examples:

1. $\mathbb{R}^n$, n-tuples of real numbers is a real VS

$$\underline{p} = (p_1, \cdots, p_n) \tag{3.16}$$
$$\underline{q} = (q_1, \cdots, q_1) \tag{3.17}$$
$$\underline{p} + \underline{q} = (p_1 + q_1, \cdots, p_n + q_n) \tag{3.18}$$
$$\alpha \cdot \underline{p} = (\alpha p_1, \cdots, \alpha p_n) \tag{3.19}$$
$$\underline{0} = (0, \cdots, 0) \tag{3.20}$$
$$-\underline{p} = (-p_1, \cdots, -p_n) \tag{3.21}$$

2. $\mathbb{C}^n$, n-tuples of complex numbers is a complex vector space
3. $P^n(+)$ all (complex) polynomials of degree $\leq n$, $a_o + a_1 t + a_2 t^2 + \cdots a_n t^n$ is a complex VS (vector is a polynomial)
4. $M(n, m)$, the set of all complex matrices forms a complex VS $\implies$ tensors are also 'vectors' in this definition of a general VS
5. $C_{\mathbb{R}}[a, b]$, all continuous real valued functions defined on the interval $[a, b]$ is a real VS (vector = function)
6. $\mathcal{L}^2_{\mathbb{C}}[a, b]$, all complex valued integrable functions defined on the interval $[a, b]$ form a complex VS

$$f \in \mathcal{L}^2_{\mathbb{C}}[a, b] \implies \int_a^b |f(x)|^2 \, dx < \infty \tag{3.22}$$

7. $\forall$ linear forms on V form a vector space called the dual VS, $\tilde{V}$

$$l : V \to \mathbb{C}(\mathbb{R}) \tag{3.23}$$

$$l(\alpha \underline{p} + \beta \underline{q}) = \alpha l(\underline{p}) + \beta l(\underline{q}) \tag{3.24}$$

So a VS is a much more general concept than $\mathbb{R}^3$.

Lecture 4

4.1 Vector Spaces

$\underline{v}, \underline{w} \in V$ defined on a field $F \in \mathbb{R}/\mathbb{C}$.

Consider a linearly dependent set of vectors $\{\underline{v}_i\}$,

$$\sum_{i=0}^{k} \alpha_i \underline{v}_i = 0, \alpha_i \in \mathbb{R}/\mathbb{C} \tag{4.1}$$

With at least one $\alpha_i \neq 0$. If the only way to fulfil this is $\alpha_i = 0 \forall i$, then $\{\underline{v}_i\}$ is linearly independent.

Example: $\mathbb{C}^3$ on $\mathbb{C}$

$$\underline{v}_1 = (i, 1, 0) \tag{4.2}$$
$$\underline{v}_2 = (0, i, 0) \tag{4.3}$$
$$\underline{v}_3 = (k, 0, 0) \tag{4.4}$$
$$\implies i\underline{v}_1 + (-1)\underline{v}_2 + \frac{1}{k}\underline{v}_3 = 0 \tag{4.5}$$
$$\alpha_i \in \mathbb{C}, \neq 0 \tag{4.6}$$

A vector space is n-dimensional if there exists a subset of n linearly-independent vectors and if there does not exist any subset of $n+1$ linearly-independent vectors.

A Basis is defined as n linearly-independent vectors such that for $\underline{p} \in V$,

$$\underline{p} = \underline{e}_i \underbrace{p^i}_{\in F} \tag{4.7}$$

An n-dimensional space over $\mathbb{C}$ is equivalent 2n-dimensional over $\mathbb{R}$, e.g. $\mathbb{C}^3$ over $\mathbb{R}$ has is 6 dimensional.

4.1.1 Scalar Product

The scalar product of two vectors $\underline{p}, \underline{q} \in V$ returns a real or complex number.

- $$(\underline{p}, \alpha\underline{q} + \beta\underline{r}) = \alpha\underline{pq} + \beta\underline{pr} \tag{4.8}$$
- $$(\underline{p}, \underline{q}) = (\underline{q}, \underline{p})^* \tag{4.9}$$
- $$(\underline{p}, \underline{p}) \geq 0, \quad (\underline{p}, \underline{p}) = 0 \iff \underline{p} = 0 \tag{4.10}$$

$$(\alpha\underline{q} + \beta\underline{r}, \underline{p}) = \alpha^*(\underline{q}, \underline{p}) + \beta^*(\underline{r}, \underline{e}p) \tag{4.11}$$

Can be defined in terms of the basis: $B = \{\underline{e}_i\}$

$$\underline{x} = \underline{e}_i x^i \tag{4.12}$$
$$\underline{y} = \underline{e}_j y^j \tag{4.13}$$

$$\implies (\underline{x}, \underline{y}) = \underline{x} \cdot \underline{y} = \underline{e}_i \underline{e}_j x^i y^j \equiv g_{ij} x^i y^j \tag{4.14}$$

For an orthonormal basis, $g_{ij} = \delta_{ij}$.

- $g_{ij} = g_{ji} \implies G = G^T$, in real space
- $\det G = |G| \neq 0$

Norm of a vector:

$$||\underline{v}|| = \sqrt{\underline{v} \cdot \underline{c}} \tag{4.15}$$

- Positivity

$$||a\underline{v}|| = |a|||\underline{v}|| \geq 0 \tag{4.16}$$
$$||\underline{v}|| = 0 \iff \underline{v} = \underline{0} \tag{4.17}$$

- Symmetry

$$||\underline{v} - \underline{w}|| = ||\underline{w} - \underline{v}|| \tag{4.18}$$

- Triangle inequality

$$||\underline{v} + \underline{w}|| \leq ||\underline{v}|| + ||\underline{w}|| \tag{4.19}$$

4.2 Gram-Schmidt Orthogonalisation

Assume that we have a set of n linearly-independent vectors, $\{\underline{v}_i\}$ in n-dimensional vector space, V. Can we construct an orthonormal basis?

$$\{\underline{v}_i\} \to \underbrace{\{\underline{u}_i\}}_{\text{orthogonal}} \to \{\underline{e}_i\} \tag{4.20}$$

$$\underline{u}_1 = \underline{v}_1 \tag{4.21}$$
$$\text{Proj}_{u_1}(\underline{v}_2) = \frac{(\underline{v}_2, \underline{u}_1)}{(\underline{u}_1, \underline{u}_1)} \tag{4.22}$$
$$\underline{u}_2 = \underline{v}_2 - \text{Proj}_{u_1}(\underline{v}_2) \tag{4.23}$$
$$\underline{u}_3 = \underline{v}_3 - \text{Proj}_{u_1}(\underline{v}_3) - \text{Proj}_{u_2}(\underline{v}_3), etc \tag{4.24}$$
$$\underline{e}_i = \frac{\underline{u}_i}{||\underline{u}_i||} \tag{4.25}$$

4.3 Isomorphism of Finite Dimension Vector Spaces

Assume we have two vector spaces, V_1, V_2 if we can construct a linear map that relates the elements between the two vector spaces.

$$V_1 \to^L V_2 \tag{4.26}$$
$$L(\alpha \underline{p} + \beta \underline{q}) \to \alpha L(\underline{p}) + \beta L(\underline{q}) \tag{4.27}$$
$$\underline{r} = \alpha \underline{p} + \beta \underline{q} \to L(\underline{r}) = \alpha L(\underline{p}) + \beta L(\underline{q}) \tag{4.28}$$

Isomorphism is a linear map that is bijective. If V_1 and V_2 are isomorphic, $V_1 \approx V_2$.

> **Example:** $\mathbb{C}^n \approx \mathcal{P}^{n-1}$
>
> $$\underline{p} = (p_1, \cdots, p_n) \in \mathbb{C}^n \tag{4.29}$$
> $$L(\underline{p}) = p_1 + p_2 x + \cdots p_n x^{n-1} \tag{4.30}$$

Theorem: *Any n-dimensional real (or complex) vector space (where n is finite) is isomorphic to $\mathbb{R}^n$ (or $\mathbb{C}^n$).*

4.4 Tensor Product

Consider both V and W as vector spaces defined on F, with basis:

$$E_V : \{\underline{e}_n\} \tag{4.31}$$
$$G_W : \{\underline{g}_m\} \tag{4.32}$$

The tensor product $V \otimes W$ is a $n \cdot m$ dimensional vector space.

$$E \otimes G = \{(\underline{e}_i, \underline{g}_j)\} = \underline{e}_i \otimes \underline{g}_j \begin{cases} i = 1 \to n \\ j = 1 \to m \end{cases} \tag{4.33}$$

$$\underline{v} = \underline{e}_i v^i \tag{4.34}$$

$$\underline{w} = \underline{g}_j w^j \tag{4.35}$$

$$\underline{v} \otimes \underline{w} = \underline{e}_i \otimes \underline{g}_j v^i w^j \tag{4.36}$$

Lecture 5

5.1 Linear Operators

A linear operator A maps the vector $\underline{x}$ from a given domain in vector space X into its image $\underline{y} = A(\underline{x})$ of a target (or codomain) in vector space Y.

$$X \rightarrow^A Y \tag{5.1}$$
$$\underline{x} \rightarrow \underline{y} = A(\underline{x}) \tag{5.2}$$

We will often consider the case $X = Y$ (for example, the operators that act as $\mathbb{R}^n \rightarrow \mathbb{R}^n$ or $\mathbb{C}^n \rightarrow \mathbb{C}^n$). This map is linear and satisfies

$$A(\underline{x}_1 + \underline{x}_2) = A(\underline{x}_1) + A(\underline{x}_2),\ \forall \underline{x}_1, \underline{x}_2 \in X \tag{5.3}$$
$$A(\alpha \underline{x}) = \alpha A(\underline{x}),\ \forall \underline{x} \in X,\ \forall \alpha \in F \tag{5.4}$$

Notice that X and Y can in general have different dimensionality and we can choose different bases on each one.

Example:
Identity Operator:

$$O(\underline{x}) = \underline{x},\ \forall \underline{x} \in X \tag{5.5}$$

Null Operator:

$$E(\underline{x}) = \underline{0}_Y,\ \forall \underline{x} \in X \tag{5.6}$$

5.2 Matrix associated to a linear operator

Every linear operator A in $\mathbb{C}^n$ (or $\mathbb{R}^n$) can be represented by an $n \times n$ complex (real) matrix. Consider the linear operator A, such that

$$X \rightarrow^A Y \tag{5.7}$$
$$\underline{x} \rightarrow \underline{y} = A(\underline{x}) \tag{5.8}$$

where X is an n-dimensional vector space with basis $B_X = \underline{e}_i$ and Y is an m-dimensional vector space with basis $B_Y = \underline{e}_j$. Thus, a vector $\underline{x} inX$ can be expressed in terms of its components as

$$\underline{x} = \underline{e}_i x^i \tag{5.9}$$

and a vector $\underline{y} \in Y$ can be expressed in the basis B_Y as

$$\underline{y} = \underline{e}_i y^i. \tag{5.10}$$

Thus, we can do the same for the result of the linear operator A acting on $\underline{x}$

$$A(\underline{x}) = A(\underline{e}_j x^j) = A(\underline{e}_j) x^j = \underline{e}_i A^i_j x^j \tag{5.11}$$

Thus $A(\underline{x})$ has components $A^i_j x^j$ in the basis B_Y with $i = 1 \dots m$ and $j = 1 \dots n$. Notice that we can now write this as a matrix equation,

$$\underline{y} = A\underline{x}. \tag{5.12}$$

Notice that the matrix A in $m \times n$, it is not a square matric unless X and Y have the same dimension. Also, the matrix representation depends on the choice of basis B_X and B_Y. As mentioned above, we will often consider applications within the same vector space. In that case, we would identify $X = Y$, $B_X = B_Y$ and the matrix A would be square $n \times n$.

5.3 Change of basis

For simplicity, we are going to consider here the case where the domain and codomain coincide $X = Y$, and we can choose the original basis $B = B_X = B_Y = \underline{e}_i$. Consider an operator A, whcih in basis B has the components A^i_j. Consider now the change of basis, defined by

$$\underline{e}'_j = \underline{e}_i S^i_j \tag{5.13}$$

$$x^j = S^j_i x'^i \tag{5.14}$$

$$x'^j = \left(S^{-1}\right)^j_i x^i = T^j_i x^i \tag{5.15}$$

Then, given $\underline{y} = A\underline{x}$, in the old basis we have $y^j = A^j_i x^i$ and in the new basis $y'^j = A'^j_i x'^i$. Starting from the first expression and transforming the coordinates x^i and y^j, we have

$$y^j = A^j_i x^i \tag{5.16}$$

$$S^j_k y'^k = A^j_i S^i_l x'^l \tag{5.17}$$

$$y'^k = \left(S^{-1}\right)^k_j A^j_i S^i_l x'^l \tag{5.18}$$

whcih allows us to identify $A'^k_l = \left(S^{-1}\right) A^j_i S^i_l$,

$$A' = S^{-1} A S \tag{5.19}$$

or, equivalently,

$$A' = T A T^{-1} \tag{5.20}$$

Lecture 6

- Adjoint Operator:

$$(\underline{p}, A(\underline{q})) = (A^\dagger(\underline{p}), \underline{q}), \ \forall \underline{p}, \underline{q} \in V \tag{6.1}$$

- Hermitian operator (self-adjoint): $A^\dagger = A$
- Unitary operator on $\mathbb{C}^2$, preserve scalar product:

$$(A(\underline{p}), A(\underline{q})) = (\underline{p}, \underline{q}), \ \forall \underline{p}, \underline{q} \in V \tag{6.2}$$

$$A^\dagger A = I \tag{6.3}$$

$$A^\dagger = A^{-1} \tag{6.4}$$

6.1 Systems of Linear Equations

Consider a set of n linear equations with n unknowns $\{x^i\}$.

$$a_1^1 x^1 + a_2^1 x^2 + \cdots a_n^1 x^n = y^1 \tag{6.5}$$

$$a_1^2 x^1 + a_2^2 x^2 + \cdots a_n^2 x^n = y^2 \tag{6.6}$$

$$a_1^n y^1 + a_2^n x^n + \cdots a_n^n x^n = y^n \tag{6.7}$$

$$A\underline{x} = \underline{y} \tag{6.8}$$

$$A = \{a_j^i\}, i, j \in 1 \to n \tag{6.9}$$

$$\underline{x} = (A^{-1})\underline{y} \tag{6.10}$$

Given a linear operator A in $\mathbb{C}^n$. The following statements are equivalent:

- A is invertible
- A is injective (one-to-one)
- A is surjective
- $\det A \neq 0$

6.2 Eigenvectors and values

$$A(\underline{v}) = \alpha \underline{v} \tag{6.11}$$

For non-null vector, $\underline{v} \neq \underline{0}$, $\underline{v}$ is an eigenvector of A with eigenvalue α.

If $\underline{v}$ is an eigenvector of A, then so is $\beta \underline{v}$

$$A(\beta \underline{v}) = \beta A(\underline{v}) = \beta \alpha \underline{v} = \alpha(\beta \underline{v}) \tag{6.12}$$

Theorem: $\alpha \in \mathbb{C}$ is an eigenvalue of $A \iff \det(A - \alpha I) = 0$.

Proof: Assume α is an eigenvalue of A.

$$\exists \underline{v} \neq \underline{0} / A\underline{v} = \alpha \underline{v} \tag{6.13}$$

$$(A - \alpha I)\underline{v} = 0 \tag{6.14}$$

$$(A - \alpha I)\underline{0} = 0 \tag{6.15}$$

$\underline{v} \neq \underline{0}$, thus $A - \alpha I$ is not injective. Therefore,

$$\det(A - \alpha I) = 0 \tag{6.16}$$

$$\det(A - \alpha I) = 0 \tag{6.17}$$

So $A - \alpha I$ is not injective.

$$(A - \alpha I)\underline{v} = (A - \alpha I)\underline{y},\ \underline{z} = \underline{v} - \underline{y} \tag{6.18}$$
$$(A - \alpha I)\underline{z} = 0 \tag{6.19}$$
$$A\underline{z} = \alpha\underline{z} \tag{6.20}$$

Which then implies that $\underline{z}$ is an eigenvector with eigenvalue α.

6.2.1 Eigenvalue solutions

- $\det(A - \alpha I) = 0$ returns polynomial of degree n, implying n complex solutions, $\alpha_i \in \mathbb{C},\ i \in 1 \to n$.
- Now define the characteristic polynomial:

$$P^n(\alpha) = \det(A - \alpha I) \tag{6.21}$$

- If A is complex, there are always n solutions.
- Eigenvectors: $A\underline{v}_i = \alpha_i\underline{v}_i$
- We can find α_j that appears more than once (m times) - this eigenvalue is m-fold degenerate
- The characteristic polynomial is independent of the basis:

$$P(\alpha) = \det(A - \alpha I) = \det(A' - \alpha I) \tag{6.22}$$
$$\det(A' - \alpha I) = \det(T^{-1}AT - \alpha I) \tag{6.23}$$
$$= \det(T^{-1}AT - \alpha T^{-1}T) \tag{6.24}$$
$$= \det(T^{-1}(A - \alpha I)T) \tag{6.25}$$
$$= \det(T^{-1})\det(A - \alpha I)\det(T) \tag{6.26}$$
$$= \det(A - \alpha I) \tag{6.27}$$

- Choose the n eigenvectors $\{\underline{v}_i\}$ as a basis:

$$\underline{p} \in V, \underline{p} = \underline{v}_i p^i \tag{6.28}$$
$$A(\underline{p}) = A(\underline{v}_i p^i) = A(\underline{v}_i)p^i \tag{6.29}$$
$$= \sum_{i=1}^{n} \alpha_i \underline{v}_i p^i \tag{6.30}$$

The matrix representing A is diagonal:

$$A_{\underline{v}_i} = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix} \tag{6.31}$$

The new basis $\{\underline{v}_i\}$ is in general not orthonormal. However, if $A^\dagger = A^{-1}$ or $A^{dagger} = A$, then $\{\underline{v}_i\}$ are orthonormal.

6.3 Unitary Operators

$$(\underline{v}, \underline{w}) = (A(\underline{v}), A(\underline{w})) \tag{6.32}$$

$\underline{v}$ is an eigenvector of A:

$$(\underline{v}, \underline{v}) = (A(\underline{v}), A(\underline{v})) = (\alpha\underline{v}, \alpha\underline{v}) \tag{6.33}$$

$$= \alpha^* \alpha (\underline{v}, \underline{v}) \tag{6.34}$$
$$\implies |\alpha|^2 = 1 \tag{6.35}$$
$$\implies \alpha = e^{i\phi} \tag{6.36}$$

For $\underline{v}_1, \underline{v}_2$ eigenvectors of a unitary operator A:

$$(\underline{v}_1, \underline{v}_2) = (A(\underline{v}_1, A(\underline{v}_2)) \tag{6.37}$$
$$= \alpha_1^* \alpha_2 (\underline{v}_1, \underline{v}_2) \tag{6.38}$$
$$= \frac{\alpha_2}{\alpha_1} (\underline{v}_1, \underline{v}_2) \tag{6.39}$$
$$\alpha_1 = e^{+i\phi_1} \tag{6.40}$$
$$\alpha_1^* = e^{-i\phi_1} = \frac{1}{\alpha_1} (\underline{v}_1, \underline{v}_2) = 0 \iff \begin{cases} \alpha_1 \neq \alpha_2 \\ \frac{\alpha_2}{\alpha_1} \neq 1 \end{cases} \tag{6.41}$$

If A is a unitary or Hermitian operator, we can find an orthonormal basis $\{\underline{v}_i\}$ where all $\underline{v}_i$ are eigenvectors of A. In this basis

$$\underline{p} = (\underline{v}_i, \underline{p}) \underline{v}_i \tag{6.42}$$
$$A(\underline{v}_i) = \alpha_i \underline{v}_i \tag{6.43}$$
$$A(\underline{p}) = \sum_{i=1}^{n} (\underline{v}_i, \underline{p}) \alpha_i \underline{v}_i \tag{6.44}$$

Lecture 7

7.1 Hilbert Spaces

- For a finite n, find a basis (of n elements), $\{\underline{e}_i\}$

$$\underline{p} \in V \to \underline{p} = \underline{e}_i p^i \tag{7.1}$$

- Now we consider $n \to \infty$. This seems to imply that you can find infinitely many linearly-independent $\{\underline{e}_i\},\ i = i \to N,\ N \to \infty$.

$$\underline{p} \in H \to \underline{p} = \underline{e}_i p^i \tag{7.2}$$

  1. Does it converge?
  2. Does it converge to an element in the Hilbert Space?
- We need a concept of distance between two elements, $\underline{p}, \underline{q}$ in H.
- Distance: (scalar product)

$$\underline{f}, \underline{g} \in H \to (\underline{f}, \underline{g}) \in F \tag{7.3}$$

  The properties of the scalar product are:
  1. $(\underline{f}, \underline{g}) = (\underline{g}, \underline{f})$
  2. $(\underline{f}, \alpha\underline{g}) = \alpha(\underline{f}, \underline{g})$
  3. $(\alpha\underline{f}, \underline{g}) = \alpha^*(\underline{f}, \underline{g})$
  4. *missed some of these - he moves his notes on too fast*
-

$$\text{distance}(\underline{f}, \underline{g}) \equiv ||\underline{f} - \underline{g}|| \equiv \sqrt{(\underline{f} - \underline{g})(\underline{f} - \underline{g})} \tag{7.4}$$

  The properties are:
  1. $d(\underline{f}, \underline{g}) = d(\underline{g}, \underline{f})$
  2. $d(\underline{f}, \underline{g}) > 0,\ \forall \underline{f} \neq \underline{g}$
  3. $d(\underline{f}, \underline{g}) < d(\underline{f}, \underline{h}) + d(\underline{h}, \underline{g})$

7.2 Convergence of sequences in H

- Strong convergence criterion:
  Consider a sequence, $\{\underline{f}_i\} \in H$. This sequence converges to $\underline{f}$ if

$$||\underline{f} - \underline{f}_n|| \to 0,\ n \to \infty \tag{7.5}$$

$$||\underline{f} - \sum_{i=1}^{n} \underline{e}_i f^i||^2 \to 0,\ n \to \infty \tag{7.6}$$

- Weak convergence criterion:
  The sequence $\{\underline{f}_i\}$ converges weakly to $\underline{f}$ if

$$(\underline{g}, \underline{f}_i) \to (\underline{g}, \underline{f}),\ n \to \infty \tag{7.7}$$

- Strong convergence $\implies$ weak convergence
- We say that a Hilbert space is complete if all Cauchy sequences converge in H.

$$\text{Cauchy sequence} \iff ||\underline{f}_n - \underline{f}_m|| \to 0,\ n, m \to \infty \tag{7.8}$$

  H contains the limit of the sequences.

7.2.1 Examples of Hilbert Spaces

- $\underline{f} = \{a^1, a^2, a^3, \cdots, a^n, \cdots\}, a^i \in \mathbb{C}$ such that $|a^1|^2 + |a^2|^2 + \cdots$ converges - $\sum_{i=1}^{\infty} |a^i|^2 < \infty$.

$$\begin{pmatrix} a^1 \\ a^2 \\ \vdots \\ a^n \\ \vdots \\ \infty \end{pmatrix} \tag{7.9}$$

  All Hilbert spaces are isomorphic to this one, known as $l^2(\mathbb{C})$.
- Scalar product:

$$(f, g) = \sum_{i=1}^{\infty} |\bar{a}_i b^i| \tag{7.10}$$

- $L^2(a, b) = \left\{ f(x) / \int_a^b |f(x)|^2 \, dx < \infty \right\}$, so scalar product is

$$(\underline{f}, \underline{g}) = \int_a^b f(\bar{x}) g(x) \, dx \tag{7.11}$$

- Norm:

$$\|\underline{f}\| = \sqrt{\int_a^b f(x)^* f(x) \, dx} \tag{7.12}$$

7.3 Basis in H

- A complete set in H
- Contains all linear combinations and all elements in sequences
- Hilbert Space is separable - $\exists$ a sequence $\{\underline{f}_n\}$ such that $\underline{f}_n \rightarrow \underline{f}, \; n \rightarrow \infty$.
- All elements in H, $\underline{g}$ can be expressed as a linear combination:

$$\underline{g} = \sum_{i=1}^{\infty} \underline{f}_i c^i \tag{7.13}$$

- Define orthonormal basis:

$$(\underline{p}, \underline{q}) = \left(\sum_{i=1}^{\infty} \underline{f}_i p^i, \sum_{j=1}^{\infty} \underline{f}_j q^j \right), \; (\underline{f}_i, \underline{f}_j = \delta_{ij} \tag{7.14}$$

$$= p^i q^j \delta_{ij} = p^i q_i \tag{7.15}$$

7.4 QM Notation

- Go from vector notation to bra-ket notation
- Column vectors to kets, $|v\rangle$
- Row vectors to bras, $\langle v|$
- Scalar product to bra-ket, $\langle p|q\rangle$
- Operators shown with expectation values, $\langle p|A|q\rangle$
- Note: bra or ket is an operator acting on a state

$$(|p\rangle\langle q|)|v\rangle = |p\rangle\langle q|v\rangle \tag{7.16}$$

Lecture 8 Formulation of Quantum Mechanics

- Consider states in a QM system:
  - vectors in a Hilbert space
  - $|v\rangle$
  - norm - $||v|| = \langle v|v\rangle^{1/2} = 1$
- For observables (any physical, measurable quantity)
  - Represented by Hermitian operators, A
  - Measuring $\rightarrow$ obtain one of the eigenvalues of A, $\alpha_m \in \mathbb{R}$
  - if $|\psi\rangle$ is an eigenvector $|m\rangle$ of A,
$$A|m\rangle = \alpha_m|m\rangle \tag{8.1}$$
  - if $|\psi\rangle$ is not an eigenvector, $|\psi\rangle$ can be expressed in the basis of $|m\rangle$
  - Probability of measuring α_m:
$$P(\alpha_m)_\psi = \sum_{\alpha_m} |\langle m|\psi\rangle|^2 \tag{8.2}$$

    In most cases, α_m will be singular, but to account for degeneracy, we sum over all eigenvectors with eigenvalue α_m

$$|\psi\rangle = \psi_m|m\rangle \tag{8.3}$$
$$= \psi_1|1\rangle + \psi_2|2\rangle + \cdots + \psi_m|m\rangle \tag{8.4}$$

  - The probability of measuring any eigenvalue:

$$\sum_m |\langle m|\psi\rangle|^2 = \sum_m \langle\psi|m\rangle\langle m|\psi\rangle \tag{8.5}$$
$$= \langle\psi|\psi\rangle = 1 \tag{8.6}$$
$$\sum_m |m\rangle\langle m| = 1 \tag{8.7}$$

- Given A, what is the average result of a measurement of a given state, $|\psi\rangle$?

$$\langle A\rangle_\psi = \sum_i P(\alpha_i)\,di = \sum_i \alpha_i|\langle i|\psi\rangle|^2 \tag{8.8}$$
$$= \sum_i \alpha_i\langle\psi|i\rangle\langle i|\psi\rangle = \sum_i \langle\psi|A|i\rangle\langle i|\psi\rangle \tag{8.9}$$
$$= \langle\psi|A|\psi\rangle, \in \mathbb{R} \tag{8.10}$$

  This is called the ***expectation value*** of A.
- Spin-half states:
  1. What is the corresponding Hilbert space?
     - $\mathbb{C}^2$ - physcial states as vectors in $\mathbb{C}^2$ (spin-vectors/spinors)
  2. Choose basis
     -
$$|\uparrow\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.11}$$

     - For an arbitrary state, $|\psi\rangle$ and $v_1, v_2 \in \mathbb{C}$:

$$|\psi\rangle = v_1|\uparrow\rangle + v_2|\downarrow\rangle \tag{8.12}$$

     - Normalisation:
$$||\psi|| = 1 \implies |v_1|^2 + |v_2|^2 = 1 \tag{8.13}$$

3. Physical quantites as operators in that Hilbert space
   - Rotations:
     - z-component of spin
$$s_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{8.14}$$
     - x-component of spin
$$s_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{8.15}$$
     - y-component of spin
$$s_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{8.16}$$
     -
$$[S_i, S_j] = i\epsilon_{ijk} S_k \tag{8.17}$$
     - Note:
$$S_z|\uparrow\rangle = \frac{1}{2}|\uparrow\rangle \tag{8.18}$$
$$S_z|\downarrow\rangle = -\frac{1}{2}|\downarrow\rangle \tag{8.19}$$
     - S_+ measures the spin projection onto the z axis
   - Expection value for an arbitrary state, $|\psi\rangle$:
$$\langle S_z\rangle_\psi = \langle\psi|S_+|\psi\rangle = \begin{pmatrix} v_1^* & v_2^* \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{8.20}$$
$$= \frac{1}{2}(|v_1|^2 - |v_2|^2) \tag{8.21}$$

- Consider particle moving in ISW between $-\frac{L}{2}$ and $\frac{L}{2}$
  - Hilbert space, $L^2_{\mathbb{C}}[-\frac{L}{2}, \frac{L}{2}]$
  - States: vectors in Hilbert space, $\psi(x), |\psi\rangle$
$$\int_{-L/2}^{L/2} |\psi(x)|^2\, dx = 1 \tag{8.22}$$
$$|\langle\psi|\psi\rangle|^2 = 1 \tag{8.23}$$
  - Basis, n = $-\infty \to \infty$
$$|n\rangle = \frac{1}{\sqrt{L}} e^{i2\pi nx/L} \equiv \psi_n(x) \tag{8.24}$$
  - Physical quantities:
    - position, X:
$$X|\psi\rangle = x|\psi\rangle \tag{8.25}$$
    - momentum, $\underline{P}$:
$$\underline{P}|\psi\rangle = -i\frac{\partial}{\partial x}|\psi\rangle \tag{8.26}$$
    - energy,
$$\frac{\underline{P}^2}{2m} - V(x) \tag{8.27}$$
  - Note:
$$\underline{P}|n\rangle = -i\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{L}} e^{i2\pi nx/L}\right) = \frac{2\pi n}{L}|n\rangle \tag{8.28}$$

    $|n\rangle$ is an eigenstate of $\underline{P}$ with eigenvalue $\frac{2\pi n}{L}$

- Check $|n\rangle$ form an orthonormal basis
  - Given a state $|n\rangle$:

$$\langle X\rangle_n = 0 = \langle n|X|n\rangle \tag{8.29}$$

$$= \frac{1}{L}\int_{-L/2}^{L/2} e^{-i2\pi nx/L} x e^{i2\pi nx/L}\, dx \tag{8.30}$$

- What is the average energy?

$$\langle E\rangle_n = \frac{1}{2m}\left(\frac{2\pi n}{L}\right)^2 \tag{8.31}$$

- Time evolution of states (Schrodinger equation)

$$-i\hbar\frac{\partial}{\partial t}|\psi,t\rangle = H|\psi,t\rangle \tag{8.32}$$

  - $\hbar = 1$
  - $|\psi,t\rangle$ is state at time t
  - H is Hamiltonian
  - $U(t,t_0)$ is the time evolution operator

$$|\psi,t\rangle = U(t,t_0)|\psi,t_0\rangle \tag{8.33}$$

$$i\frac{\partial}{\partial t}(U(t,t_0))|\psi,t_0\rangle = i\frac{\partial}{\partial t}(U(t,t_0))\,U^{-1}(t,t_0)|\psi,t\rangle = H|\psi,t\rangle \tag{8.34}$$

$$i\frac{\partial}{\partial t}U(t,t_0) = HU(t,t_0) \tag{8.35}$$

  - The solution reads:

$$U(t,t_0) = e^{-iH(t-t_0)},\ U(t_0,t_0) = 1 \tag{8.36}$$

$$e^{-iH(t-t_0)} := 1 - iHt + \frac{(-iHt)^2}{2} + \cdots \tag{8.37}$$

  Expansion as a power series

$$|\psi,t\rangle = e^{-iH(t-t_0)}|\psi,t_0\rangle \tag{8.38}$$

$$= \sum_m e^{-iH(t-t_0)}|m\rangle\langle m|\psi,t_0\rangle \tag{8.39}$$

$$= \sum_m e^{-iE_m(t-t_0)}|m\rangle\langle m|\psi,t_0\rangle,\ \because H|m\rangle = E_m|m\rangle \tag{8.40}$$

Part III

Calculus of Variations and Infinite Series

Lecture 1 Infinite Series

For the most part, just use his notes, he's just reading from them

- Series converges to S: *A series with partial sums* $s_n = \sum_{i=0}^{n} u_i$ *converges to S iff*

$$S = \lim_{n\to\infty} \tag{1.1}$$

  More explicitly, this means that $\forall \epsilon \in \mathbb{R}$ with $\epsilon > 0$: $\exists N \in \mathbb{N}$ such that $|s_n - S| < \epsilon \forall n \leq N$.

Example: Geometric Series

$$1 + x + x^2 + \cdots \tag{1.2}$$

$$s_n = 1 + x + x^2 + \cdots \tag{1.3}$$

$$(1-x)s_n = 1 - x^{n+1} \tag{1.4}$$

$$s_n = \frac{1 - x^{n+1}}{1 - x} \tag{1.5}$$

Converges for $|x| < 1$.

Example: Harmonic Series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \tag{1.6}$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{\frac{1}{p+1}+\frac{1}{p+2}+\cdots+\frac{1}{2p} > \frac{p}{2p} = \frac{1}{2}} + \cdots \tag{1.7}$$

1.1 Convergence Tests

- Compare to another series
- Compare to integrals

Lecture 2

- Gauss' Test

$$\frac{u_n}{u_{n+1}} = 1 + \frac{h}{n} + \frac{B(n)}{n^2} \tag{2.1}$$

- Start with Kummer's test using $a_n = n\ln(n)$

$$n\ln(n)\left(1+\frac{1}{n}\right) - (n+1)\ln(n+1) = (n+1)\ln(n) - (n+1)\ln(n+1) \tag{2.2}$$

$$= (n+1)\left(\ln(n) - \ln(n) - \ln\left(1+\frac{1}{n}\right)\right) \tag{2.3}$$

$$\to -(n+1)\ln\left(1+\frac{1}{n}\right) \tag{2.4}$$

$$\to -1, n \to \infty, < 0 \tag{2.5}$$

- So the series diverges
- Test devised to examine the convergence of hyper-geometric series

$$F(a,b,c;x) = 1 + \frac{ab}{c}\frac{x}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!} + \cdots \tag{2.6}$$

- Using ratio of successive terms, this series converges for $|x| < 1$, or when $x = 1$ for $a + b - c < 0$.

2.1 Alternating Series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n \tag{2.7}$$

- Leibniz criterion: *An alternating series converges if* u_n *is monotonically decreasing and* $\lim_{n\to\infty} u_n = 0$.

2.2 Algebra of Series

- Series that are absolutely convergent may be re-ordered
- Their sum is independent of the order in which the terms are ordered
- A product of two such series, will also converge absolutely
- Taylor series - we know what these are by now, but he's giving us more explanation on this than anything else so far

Lecture 3 Special Functions

- Just normal differential equations that solve into sum of trigs

$$\nabla^2\psi = -k^2\psi \tag{3.1}$$
$$\nabla^2\psi = 4\pi G\rho \tag{3.2}$$
$$\nabla^2\psi = -k\rho \tag{3.3}$$

- Cylindrical symmetry

$$\psi = P(\rho)\Phi(\phi)Z(z) \tag{3.4}$$
$$\rho^2 P'' + \rho P' + (n^2\rho^2 - m^2)P = 0 \tag{3.5}$$

The resulting solutions for $P(\rho)$ are Bessel functions.

- Spherical symmetry

$$\psi = R(r)\Phi(\phi)\Theta(\theta) \tag{3.6}$$
$$(1-x^2)P'' - 2xP' - \frac{m^2}{1-x^2}P + l(l+1)P = 0 \tag{3.7}$$

Where P is $P(x), x \equiv \cos\theta$. Known as the Legendre polynomials.

- Gaussians and error functions

$$\mathcal{P}(x)\,dx = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx \tag{3.8}$$

Function normalised to unity.

- Error function

$$\text{erf}(x) \equiv \frac{2}{\sqrt{\pi}}\int_0^x \exp(-x'^2)\,dx',\ \lim_{x\to\infty} = 1 \tag{3.9}$$

To find an expression for this as $x \to \infty$, use *asymptotic expansion*. Use integration by parts using

$$e^{-t^2}\,dt = \frac{d}{dt}\left(\frac{e^{-t^2}}{2t}\right) - \frac{1}{2t^2}e^{-t^2}\,dt \tag{3.10}$$
$$1 - \text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \exp(-x'^2)\,dx \tag{3.11}$$
$$\text{erf}(x) = 1 - \frac{2}{\sqrt{\pi}}e^{-x^2}\left[\frac{1}{2x} - \frac{1}{2^2x^3} + \frac{1\cdot 3}{2^3x^5} - \cdots\right] + (-1)^n\frac{1\times3\times5\cdots(2n-1)}{2^n}\frac{2}{\sqrt{\pi}}\int_x^\infty \frac{e^{-t^2}}{t^{2n}}\,dt \tag{3.12}$$

Results in a series expansion in the notes.

- Gamma function defined as

$$\Gamma(z) \equiv \int_0^\infty e^{-t}t^{z-1}\,dt \tag{3.13}$$
$$\Gamma(n) = (n-1)!, n > 0 \tag{3.14}$$
$$\Gamma(z) = 2\int_0^\infty e^{-t^2}t^{2z-1}\,dt = \int_0^1 \left[\ln\left(\frac{1}{t}\right)\right]^{z-1}dt \tag{3.15}$$

- Beta function

$$\Gamma(m)\Gamma(n) = \int_0^\infty e^{-u}u^{m-1}du\int_0^\infty e^{-v}v^{n-1}dv \tag{3.16}$$
$$= 4\int_0^\infty e^{-x^2}x^{2m-1}dx\int_0^\infty e^{-y^2}y^{2n-1}dy \tag{3.17}$$

Making use of the substiution $u = x^2, v = y^2$, and now changing to polar coordinates.

$$\Gamma(m)\Gamma(n) = 4\int_0^\infty e^{-r^2} r^{2m+2n-1} dr \int_0^{\pi/2} \cos^{2m-1}(\theta)\sin^{2n-1}(\theta)\,d\theta \tag{3.18}$$

$$= 2\int_0^\infty e^{-t} t^{m+n-1} dt \int_0^{\pi/2} \cos^{2m-1}(\theta)\sin^{2n-1}(\theta)\,d\theta \tag{3.19}$$

Where $r^2 \to t$. Note that the first integral is $\Gamma(m+n)$, which then defines the Beta function.

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \tag{3.20}$$

$$= 2\int_0^{\pi/2} \cos^{2m-1}(\theta)\sin^{2n-1}(\theta)\,d\theta \tag{3.21}$$

➤ Legendre functions - probably just look at notes for Theoretical

Lecture 4

➤ Use a generating function, do stuff with its derivatives

$$g(t,x) = (1-2xt+t^2)^{-1/2} \equiv \sum_{n=0}^{\infty} P_n(x)t^n,\ |t|<1 \tag{4.1}$$

$$\frac{\partial}{\partial t} g(t,x) = \frac{x-t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} nP_n(x)t^{n-1} \tag{4.2}$$

$$(x-t)\sum P_n(x)t^n = (1-2xt+t^2)\sum nP_n(x)t^{n-1} \tag{4.3}$$

$$xP_nt^n - P_nt^{n+1} = nP_nt^{n-1} - 2xnP_nt^n + nP_nt^{n+1} \tag{4.4}$$

$$xP_nt^n - P_{n-1}t^n = (n+1)P_{n+1}t^n - 2xnP_nt^n + (n-1)P_{n-1}t^n \tag{4.5}$$

$$(1+2n)xP_n = (n+1)P_{n+1} + nP_{n-1} \tag{4.6}$$

$$\sum xP_nt^n = xP_0t^0 + xP_1t^1 + \cdots \tag{4.7}$$

$$\sum P_nt^{n+1} = P_0t^1 + P_1t^2 + \cdots \tag{4.8}$$

➤ Now do stuff with ∂x

$$\frac{\partial}{\partial x} g(t,x) = \frac{t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} P_n'(x)t^n \tag{4.9}$$

$$\frac{t}{(1-2xt+t^2)^{3/2}} = (1-2xt+t^2)\sum P_n't^n \tag{4.10}$$

$$t\sum P_nt^n = (1-2xt+t^2)\sum P_n't^n \tag{4.11}$$

$$P_{n-1}t^n = P_n't^n - 2xP_{n-1}'t^n + P_{n-2}'t^n \tag{4.12}$$

$$P_n = P_{n+1}' - 2xP_n' + P_{n-1}' \tag{4.13}$$

$$P_{n+1}' + P_{n-1}' = 2xP_n' + P_n \tag{4.14}$$

➤ You do some more maths that he isn't going to explain and you'll get something "interesting"

$$(1+2n)xP_n = (n+1)P_{n+1} + nP_{n-1} \tag{4.15}$$

$$P_{n+1}' + P_{n-1}' = 2xP_n' + P_n \tag{4.16}$$

Take $\partial/\partial x$ of (4.15) and $(2n+1)$ times (4.16):

$$(1+2n)P_n + (1+2n)xP_n' = (n+1)P_{n+1}' + nP_{n-1}' \tag{4.17}$$

$$(2n+1)P_{n+1}' + (2n+1)P_{n-1}' = 2x(2n+1)P_n' + (2n+1)P_n \tag{4.18}$$

Now do $(4.18) + 2(4.17)$.

$$(2n+1)P_n = P_{n+1}' + P_{n-1}' \tag{4.19}$$

➤ Anything that satisfies these recurrence relations, should satistfy

$$(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0 \tag{4.20}$$

➤ Legendre polynomials can be used to solve problems of electric potential.
- ➥ Look for the potential at point r from the origin, with a charge at a.
- ➥ Distance $d = (r^2 + a^2 - 2ar\cos\theta)^{1/2}$.

- For $r > a$

$$\phi = \frac{q}{4\pi\epsilon_0 d} \tag{4.21}$$

$$d = r\left(1 - 2\frac{a}{r}\cos\theta + \left(\frac{a}{r}\right)^2\right)^{1/2}, \frac{a}{r} = t \tag{4.22}$$

$$= r(1 - 2xt + t^2)^{1/2} \tag{4.23}$$

$$\phi = \frac{q}{4\pi\epsilon_0 r} g(t,x) \tag{4.24}$$

$$= \frac{q}{4\pi\epsilon_0 r} \sum_n P_n(x) t^n \tag{4.25}$$

- Or for $r < a$

$$\phi = \frac{q}{4\pi\epsilon_0 a} \sum_n P_n(x) t^n, t = \frac{r}{a} \tag{4.26}$$

- Legendre polynomials are orthogonal

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0,\ m \neq n \tag{4.27}$$

- Most functions in the interval of orthogonality can be written as a sum of the P_n

$$f(x) = \sum_n c_n P_n(x) \tag{4.28}$$

$$\int_{-1}^{1} f(x) P_m(x)\, dx = \int_{-1}^{1} \left[\sum_n c_n P_n(x)\right] P_m(x)\, dx \tag{4.29}$$

$$= \sum_n c_n \int_{-1}^{1} P_n(x) P_m(x)\, dx \tag{4.30}$$

$$= c_m g_m \tag{4.31}$$

- Spherical harmonics

$$Y_{lm} \propto P_l^m e^{im\phi} \tag{4.32}$$

- He's going on about some boring stuff now, just read the notes

Lecture 5 Integration

Who cares?

Lecture 6

Who cares again?

Part IV

Integral Transforms

Lecture 1

$$g(w) = \int_a^b K(w,x) f(x)\, dx \tag{1.1}$$

- ➤ $f(x), g(w)$: complex functions of a real variable
- ➤ $K(w,x)$: kernal propagator
- ➤ a, b can be (and every often are) $\pm\infty$
- ➤ $g(w) = Tf(w)$
- ➤ $F \rightarrow^T F$
- ➤ $f \rightarrow q = Tf$

The transformation is linear and invertible:

$$T[a_1 f_1 + a_2 f_2] = a_1 T[f_1] + a_2 T[f_2] \tag{1.2}$$

$$f(x) = T^{-1} g(w) \tag{1.3}$$

$$\begin{array}{ccc} \text{Original problem (complicated)} & \rightarrow & \text{Solution} \\ T \downarrow & & \uparrow T^{-1} \\ \text{Transformed problem} & \rightarrow_{\text{easier}} & \text{Transformed solution} \end{array}$$

Two main parts of the course:

- ➤ Fourier transforms
- ➤ Laplace transforms

1.1 Fourier Transforms

$$\mathcal{F}[f(x)](w) \equiv \tilde{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{iwx} dx \tag{1.4}$$

Requirements:

- ➤ $f(x)$ is square-integrable, i.e.

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx < \infty \tag{1.5}$$

- ➤ $f(x)$ is continuous (the transformation is invertible), i.e.

$$\mathcal{F}^{-1}[\tilde{f}(w)](x) = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(w) e^{-iwx} dw \tag{1.6}$$

- ➤ $\frac{1}{\sqrt{2\pi}}$ is used as a normalisation factor for $\mathcal{F}$ and $\mathcal{F}^{-1}$, but different normalisation factors may be used.

Generalisation to 3D:

$$\mathcal{F}[f(\underline{r})] \equiv \tilde{f}(\underline{k}) = \frac{1}{(2\pi)^{3/2}} \int f(\underline{r}) e^{i\underline{k}\cdot\underline{r}} d\underline{r} \tag{1.7}$$

1.2 Fourier Transform as a limit of Fourier series

Consider $f(x)$ periodic in $x \in [-L, L]$.

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] \tag{1.8}$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt \tag{1.9}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt \tag{1.10}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt \tag{1.11}$$

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt + \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(t)\left[\cos\left(\frac{n\pi t}{L}\right)\cos\left(\frac{n\pi x}{L}\right) + \sin\left(\frac{n\pi t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\right]dt \tag{1.12}$$

$$= \frac{1}{2L}\int_{-L}^{L} f(t)\,dt + \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi(t-x)}{L}\right)dt \tag{1.13}$$

$$= \frac{1}{2L}\int_{-L}^{L} f(t)\,dt + \frac{1}{2L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(t)\left[e^{in\pi(t-x)/L} + e^{-in\pi(t-x)/L}\right] \tag{1.14}$$

$$w_n = \frac{n\pi}{L} \implies \Delta w = w_{n+1} - w_n = \frac{\pi}{L} \tag{1.15}$$

$$L \to \infty,\ \Delta w \to dw \tag{1.16}$$

$$f(x) = \sum_{n=-\infty}^{\infty}\frac{1}{2L}\int_{-L}^{L} f(t)e^{in\pi(t-x)}L\,dt \tag{1.17}$$

$$\lim_{L\to\infty} f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dw \int_{-\infty}^{\infty} dt\, f(t)e^{iw(t-x)} \tag{1.18}$$

$$\frac{1}{L} = \frac{\Delta w}{\pi} \to \frac{dw}{\pi} \tag{1.19}$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-iwt}dt\right)e^{-iwx}dw \tag{1.20}$$

$$\implies \mathcal{F}^{-1}[\mathcal{F}[f(t)](w)] \tag{1.21}$$

1.3 Fourier transform of a Gaussian

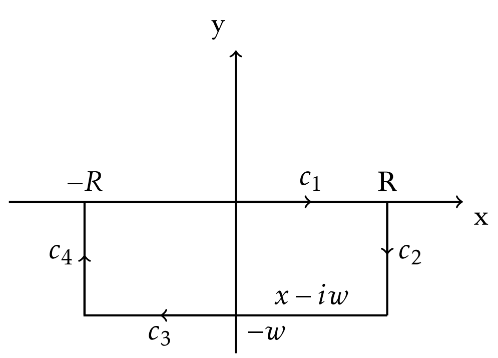
$$f(x) = e^{-x^2/2} \tag{1.22}$$

$$\mathcal{F}[f(x)](w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}e^{iwx}dx \tag{1.23}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x-iw)^2/2}e^{-w^2/2}dx \tag{1.24}$$

$$= \frac{e^{-w^2/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x-iw)^2/2}dx \tag{1.25}$$

$$\oint e^{-z^2/2}\,dx,\ z = x + iy \tag{1.26}$$



$$C = c_1 + c_2 + c_3 + c_4, \ \lim R \to \infty \tag{1.27}$$

$$\oint = \int_{c_1} + \int_{c_2} + \int_{c_3} + \int_{c_4} \tag{1.28}$$

$$\int_{c_1} e^{-z^2/2} dz = \int_{-R}^{R} e^{-x^2/2} dx = \sqrt{2\pi} \tag{1.29}$$

$$\lim_{R\to\infty}\left(\int_{c_2}\right) = \int_0^{-w} e^{-(x+iy)^2/2} dy = 0, \ \int_{c_4} = 0 \tag{1.30}$$

$$\int_{c_3} = \int_{R}^{-R} e^{-(x-iw)^2/2} dx \tag{1.31}$$

$$\int_{c_1} = -\int_{c_3} \tag{1.32}$$

$$\mathcal{F}[f(x)](w) = e^{-w^2/2} \tag{1.33}$$

$$\mathcal{F}[e^{-x^2/2}] = e^{-w^2/2} \tag{1.34}$$

Lecture 2

2.1 Fourier Transforms

$$\mathcal{F}[f(x)](w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iwx} f(x) dx \tag{2.1}$$

$$\mathcal{F}^{-1}[\hat{f}(w)](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iwx} \hat{f}(w) dw \tag{2.2}$$

1-dimensional string (wave equation):

$$\frac{\partial^2 Y(x,t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 Y(x,t)}{\partial t^2} \tag{2.3}$$

$$y''(x) - 4y(x) = e^{-x^2/8} \tag{2.4}$$

2.1.1 Fourier transform of a derivative

$$\mathcal{F}\left[\frac{df(x)}{dx}\right](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{df(x)}{dx} e^{iwx} dx \tag{2.5}$$

$$= \frac{1}{\sqrt{2\pi}} f(x) e^{iwx} \Big|_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) iw e^{iwx} dx \tag{2.6}$$

This implies that $f(x)$ vanishes at $-\infty$ and ∞.

$$\mathcal{F}\left[\frac{df(x)}{dx}\right](w) = -iw \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{iwx} dx \tag{2.7}$$

$$\mathcal{F}\left[\frac{df(x)}{dx}\right] = -iw\mathcal{F}[f(x)] \tag{2.8}$$

Can generalise to the n-th derivative:

$$\mathcal{F}\left[\frac{d^n f(x)}{dx^n}\right](w) = (-iw)^n \mathcal{F}[f(x)](w) \tag{2.9}$$

Differential equation in $x \to$ polynomial equation in w.

What about Fourier trnasform of $x^n f(x)$?

$$\mathcal{F}[xf(x)](w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overbrace{e^{iwx} x}^{\frac{1}{i}\frac{de^{iwx}}{dx}} f(x)\, dx \tag{2.10}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{i} \int_{-\infty}^{\infty} \frac{de^{iwx}}{dx} f(x)\, dx \tag{2.11}$$

$$= \frac{-i}{\sqrt{2\pi}} \frac{d}{dw} \int_{-\infty}^{\infty} e^{iwx} f(x)\, dx \tag{2.12}$$

$$= -i\frac{d}{dw}\left[\mathcal{F}[f(x)](w)\right] \tag{2.13}$$

And generally,

$$\mathcal{F}[x^n f(x)](w) = (-i)^n \frac{d^n}{dw^n}\left[\mathcal{F}[f(x)](w)\right] \tag{2.14}$$

2.1.2 Fourier transforms and scaling

$$\mathcal{F}[f(ax)](w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iwx} f(ax)\, dx,\ x' = ax \tag{2.15}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iwx'/a} f(x') \frac{dx'}{a} \tag{2.16}$$
$$= \frac{1}{a}\mathcal{F}[f(x)]\left(\frac{w}{a}\right) \tag{2.17}$$

2.1.3 Fourier transform of a translated function

$$\mathcal{F}[f(a+x)](w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iwx} f(a+x)\, dx,\ x' = a+x \tag{2.18}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iw(x'-a)} f(x')\, dx \tag{2.19}$$
$$= e^{-iwa}\mathcal{F}[f(x)](w) \tag{2.20}$$
$$\mathcal{F}[e^{iax} f(x)] = \mathcal{F}[f(x)](w+a) \tag{2.21}$$

2.1.4 1D String Wave Equation

$$\frac{\partial^2 Y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 Y(x,t)}{\partial t^2} \tag{2.22}$$
$$\mathcal{F}\left[\frac{\partial^2 Y(x,t)}{\partial x^2}\right] = \frac{1}{v^2}\mathcal{F}\left[\frac{\partial^2 Y(x,t)}{\partial t^2}\right],\ \mathcal{F}[Y(x,t)] \equiv \hat{Y}(w,t) \tag{2.23}$$
$$(-iw)^2\hat{Y} = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\hat{Y} \tag{2.24}$$
$$\frac{\partial^2}{\partial t^2}\hat{Y} = -(w^2v^2)\hat{Y} \tag{2.25}$$
$$\hat{Y}(w,t) = A(w)e^{iwvt} + B(w)e^{-iwvt} \tag{2.26}$$
$$Y(x,t) = \mathcal{F}^{-1}[\hat{Y}](x,t) \tag{2.27}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(A(w)e^{-iwx}e^{iwvt} + B(w)e^{-iwx}e^{-iwvt}\right) dw \tag{2.28}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(A(w)e^{-iw(x-vt)} + B(w)e^{-iw(x+vt)}\right) dw \tag{2.29}$$

Now consider

$$y'' - 4y = e^{-x^2/8} \tag{2.30}$$
$$\mathcal{F}[y'' - 4y] = \mathcal{F}\left[e^{-x^2/8}\right],\ \mathcal{F}[y] \equiv \hat{y} \tag{2.31}$$
$$(-iw)^2\hat{y} - 4\hat{y} = \mathcal{F}\left[e^{-x'2/2}\right],\ x' \equiv \frac{x}{2} \tag{2.32}$$
$$= 2e^{-2w^2} \tag{2.33}$$
$$\hat{y} = \frac{-2e^{-2w^2}}{4+w^2} \tag{2.34}$$
$$y = \mathcal{F}^{-1}[\hat{y}] \tag{2.35}$$

$$y(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{-2e^{-2w^2}}{4+w^2} \right) e^{-iwx} dw \tag{2.36}$$

Solution to the homogeneous equation: $y = y_h + y_p$ (above is particular).

$$y'' - 4y = 0 \tag{2.37}$$

$$y = A(x)e^{2x} + B(x)e^{-2x} \tag{2.38}$$

What about:

$$y'' - 4y = \cos(x) \tag{2.39}$$

$$\mathcal{F}[\cos(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iwx} \cos(x)\, dx \tag{2.40}$$

We have an issue here as $\cos(x)$ is not square integrable.

Lecture 3

Dirac δ function such that

$$\delta(x) = 0, x \neq 0 \tag{3.1}$$
$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1 \tag{3.2}$$
$$\int_{-\infty}^{\infty} \delta(x) f(x)\, dx = f(0) \tag{3.3}$$

Think of

$$f_n(x) \equiv \frac{n}{\sqrt{\pi}} e^{-n^2x^2} \tag{3.4}$$

Satisfies the properties of $f(x)$ where $\lim_{n\to\infty}$

$$\lim_{n\to\infty} \left(\int_{-\infty}^{\infty} \frac{n}{\sqrt{\pi}} e^{-n^2x^2}\, dx \right) = \lim_{n\to\infty} \frac{n}{\sqrt{\pi}} \frac{\sqrt{\pi}}{n} = 1 \tag{3.5}$$

Imagine graphically as Gaussians centered on $x = 0$ that slowly gets sharper and sharper towards delta spike.

$$g_n(x) = \frac{n}{\pi} \frac{1}{1 + n^2x^2} \tag{3.6}$$

Properties:

$$\delta(x) = \delta(-x) \tag{3.7}$$
$$\int_{-\infty}^{\infty} \delta(x-a) f(x) = \int_{-\infty}^{\infty} \delta(x-a) f(a) \tag{3.8}$$
$$= f(a) \int_{-\infty}^{\infty} \delta(x-a)\, dx = f(a) \tag{3.9}$$
$$\delta[g(x)] = \sum_{x_i,\text{ zeros of g}} \frac{\delta(x - x_i)}{|g'(x_i)|} \tag{3.10}$$

$g(x)$ has at most simple zeros.

$$\int_{-\infty}^{\infty} \delta[g(x)]\, dx = \int_{-\infty}^{\infty} \delta[(x - x_i) g'(x_i)]\, dx \tag{3.11}$$
$$x' = (x - x_i) g'(x_i) \tag{3.12}$$
$$dx = \frac{dx'}{|g'(x_i)|} \tag{3.13}$$
$$\delta(ax) = \frac{\delta(x)}{a} \tag{3.14}$$

δ is differentiable

$$\int_{-\infty}^{\infty} \delta'(x) f(x)\, dx = \cancel{f(x)\delta(x)\Big|_{-\infty}^{\infty}} - \int_{-\infty}^{\infty} \delta(x) f'(x)\, dx \tag{3.15}$$
$$= (-1) f'(0) \tag{3.16}$$

Generalising to n-th order:

$$\int_{-\infty}^{\infty} \delta^{(n)}(x) f(x)\, dx = (-1)^n f^{(n)}(0) \tag{3.17}$$

The Heaviside step function is the primitive of $\delta(x)$

$$\Theta(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{3.18}$$

$$\int_{-\infty}^{\infty} \Theta'(x) f(x)\, dx = \int_{-\infty}^{\infty} \delta(x) f(x)\, dx \tag{3.19}$$

3.1 Fourier Representation of delta function

$$\mathcal{F}[\delta(x)] \equiv \hat{\delta}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x) e^{iwx}\, dx \tag{3.20}$$

$$\hat{\delta}(w) = \frac{1}{\sqrt{2\pi}} \tag{3.21}$$

$$\mathcal{F}^{-1}[\hat{\delta}(w)] = \delta(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-iwx}\, dw \tag{3.22}$$

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iwx}\, dw \tag{3.23}$$

$$\delta(x) = \delta(-x) \tag{3.24}$$

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 1 \cdot e^{iwx}\, dw = \frac{1}{\sqrt{2\pi}} \mathcal{F}[1] \tag{3.25}$$

$$\mathcal{F}[1] = \sqrt{2\pi}\delta(x) \tag{3.26}$$

Also:

$$\mathcal{F}[e^{iax}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} e^{iwx}\, dx,\ |e^{iax}| = 1 \tag{3.27}$$

$$= \sqrt{2\pi}\delta(w + a) \tag{3.28}$$

3.2 Convolutions

$$f(x) \to \hat{f}(w) \tag{3.29}$$

$$g(x) \to \hat{g}(w) \tag{3.30}$$

$$f(x) * g(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y) f(x - y)\, dy \tag{3.31}$$

$$\tag{3.32}$$

This is the Convolution.

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y) f(x - y)\, dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{-iw(x-y)} dw \right) dy \tag{3.33}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) \left(\underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{iwy} dy}_{\hat{g}(w)} \right) e^{-iwx} dw \tag{3.34}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) \hat{g}(w) e^{-iwx}\, dw \tag{3.35}$$

$$f(x) * g(x) = \mathcal{F}^{-1}[\hat{f}(w)\hat{g}(w)] \tag{3.36}$$
$$\mathcal{F}[f(x) * g(x)] = \hat{f}(w)\hat{g}(w) \tag{3.37}$$

Example:

$$f''(x) - f(x) = e^{-x^2} \tag{3.38}$$
$$f(x) = f_h(x) + f_p(x) \tag{3.39}$$

Solution to the Homogeneous equation:

$$f''(x) - f(x) = 0 \tag{3.40}$$
$$f_h(x) = Ae^{-x} + Be^{x} \tag{3.41}$$
$$\tag{3.42}$$

Now for the particular:

$$\mathcal{F}[f''(x) - f(x)] = \mathcal{F}[e^{-x^2}] \tag{3.43}$$
$$(-iw)^2\hat{f}(w) - \hat{f}(w) = \mathcal{F}[e^{-x^2}] \tag{3.44}$$
$$\hat{f}(w) = \mathcal{F}[e^{-x^2}]\frac{-1}{1+w^2} \tag{3.45}$$
$$= -\mathcal{F}[e^{-x^2}]\mathcal{F}[something] \tag{3.46}$$
$$= -\mathcal{F}[e^{-x^2} * something] \tag{3.47}$$
$$f(x) = -e^{-x^2} * something \tag{3.48}$$
$$something = \mathcal{F}^{-1}\left[\frac{1}{1+w^2}\right] \tag{3.49}$$
$$\mathcal{F}^{-1}\left[\frac{1}{1+w^2}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{1+w^2}e^{-iwx}\,dw \tag{3.50}$$

Poles at $w = \pm i$. $x > 0$, $|e^{-iwx}| = |e^{-iw_{\mathbb{R}}x}e^{w_I x}| = e^{w_I x}$. If $x > 0$, integrating on lower half of the plane, so can apply residue theorem for $w = -i$.

$$\mathcal{F}^{-1}\left[\frac{1}{1+w^2}\right] = \frac{1}{\sqrt{2\pi}}2\pi i\left(-\text{Res}\left(\frac{e^{-iwx}}{1+w^2}\right)_{w=-i}\Bigg|_{x\geq 0} + \text{Res}\left(\frac{e^{-iwx}}{1+w^2}\right)_{w=i}\Bigg|_{x<0}\right) \tag{3.51}$$
$$= \sqrt{\frac{\pi}{2}}\left(e^{-x}\Big|_{x\geq 0} + e^{x}\Big|_{x<0}\right) = \sqrt{\frac{\pi}{2}}e^{-|x|} \tag{3.52}$$
$$\mathcal{F}^{-1}\left[\frac{1}{1+w^2}\right] = \sqrt{\frac{\pi}{2}}e^{-|x|} \tag{3.53}$$
$$f(x) = -e^{-x^2} * \sqrt{\frac{\pi}{2}}e^{-|x|} \tag{3.54}$$
$$= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-y^2}\sqrt{\frac{\pi}{2}}e^{-|x-y|}dy \tag{3.55}$$

Lecture 4

$$\mathcal{F}[f(x)](w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{iwx} dx \tag{4.1}$$

4.1 Parseval's Theorem

$$\int_{-\infty}^{\infty} f(x)^* g(x)\, dx = \int_{-\infty}^{\infty} \hat{f}(w)^* \hat{g}(w)\, dw \tag{4.2}$$

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |\hat{f}(w)|^2\, dw,\ f(x) = g(x) \tag{4.3}$$

This is useful in Quantum Mechanics to show the wavefunction of a particle makes sense in momentum-space as well.

$$\int_{-\infty}^{\infty} f(x)^* g(x)\, dx = \int_{-\infty}^{\infty} \left(\mathcal{F}^{-1}[\hat{f}(w)]\right)^* \mathcal{F}^{-1}[\hat{g}(w)]\, dx \tag{4.4}$$

$$= \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{-iwx}\, dw \right)^* \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{g}(w') e^{-iw'x}\, dw' \right) dx \tag{4.5}$$

$$= \frac{1}{2\pi} \int dw \int dw' \int_{-\infty}^{\infty} dx\, \hat{f}(w)^* \hat{g}(w') e^{-ix(w'-w)} \tag{4.6}$$

$$= \int dw \int dw' \hat{f}(w)^* \hat{g}(w') \delta(w'-w) \tag{4.7}$$

$$= \int_{-\infty}^{\infty} \hat{f}(w)^* \hat{g}(w)\, dw \tag{4.8}$$

Example:

$$I = \int_0^{\infty} \frac{dw}{(a^2+w^2)^2} \tag{4.9}$$

$$\mathcal{F}^{-1}\left[\frac{1}{1+k^2}\right] = \sqrt{\frac{\pi}{2}} e^{-|x|} \tag{4.10}$$

$$I = \frac{1}{2a^3} \int_{-\infty}^{\infty} \frac{dk}{(1+k^2)^2},\ k = \frac{w}{a} \tag{4.11}$$

$$= \frac{1}{2a^3} \int_{-\infty}^{\infty} \left(\sqrt{\frac{\pi}{2}} e^{-|x|}\right)^* \left(\sqrt{\frac{\pi}{2}} e^{-|x|}\right) dx \tag{4.12}$$

$$= \frac{1}{2a^3} \frac{\pi}{2} \int_{-\infty}^{\infty} \left(e^{-|x|}\right)^2 dx \tag{4.13}$$

$$= \frac{1}{2a^3} \frac{\pi}{2} \times 2 \int_0^{\infty} e^{-|x|}\, dx = \frac{\pi}{4a^3} \tag{4.14}$$

4.2 Fourier transform and integral equations

$$f(x) * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y) f(x-y)\, dy \tag{4.15}$$

Example:

$$h(x) = e^{i3x} + \int_{-\infty}^{\infty} e^{-|y|} h(x-y)\,dy \tag{4.16}$$

$$\hat{h}(w) = \sqrt{2\pi}\delta(w+3) + \sqrt{2\pi}\mathcal{F}[e^{-|x|}]\hat{h}(w) \tag{4.17}$$

$$= \sqrt{2\pi}\delta(w+3) + \frac{2}{1+w^2}\hat{h}(w) \tag{4.18}$$

$$= \sqrt{2\pi}\delta(w+3)\left(1 - \frac{2}{1+w^2}\right)^{-1} \tag{4.19}$$

$$h(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{2\pi}\delta(w+3)\left(1 - \frac{2}{1+w^2}\right)^{-1} e^{-iwx}\,dw \tag{4.20}$$

$$= \frac{5}{4}e^{i3x} \tag{4.21}$$

4.3 Discrete Fourier Transform

$$h(t) \to \hat{h}(w) \tag{4.22}$$

Frequency decomposition, period $h(t)$, T. Meausre $h(t)$ at $t_i = t_0 \cdots t_{2N}$. Discrete fourier transform:

$$\hat{h}(w_p) = \frac{1}{\sqrt{2N}} \sum_{j=0}^{2N-1} h_j e^{iw_p t_j} \tag{4.23}$$

$$t_j = \frac{T}{2N}j,\ w_p = \frac{2\pi}{T}p \tag{4.24}$$

This can be inverted to construct a continuous function of t:

$$h^{DFT}(t) = \frac{1}{\sqrt{2N}} \sum_{p=0}^{2N-1} \hat{h}(w_p) e^{-iw_p t} \tag{4.25}$$

$$h^{DFT}(t) \neq h(t) \tag{4.26}$$

They have the same periodic properties, h^{DFT} converges to $h(t)$ in the limit where $j \to \infty$.

4.3.1 Fourier Matrix

$$\begin{pmatrix} \hat{h}_0 \\ \vdots \\ \hat{h}_{2N-1} \end{pmatrix} = \left(\frac{e^{iw_p t_j}}{\sqrt{2N}}\right)_{pj} \begin{pmatrix} h_0 \\ \vdots \\ h_{2N-1} \end{pmatrix} \tag{4.27}$$

$$\left(\frac{e^{iw_p t_j}}{\sqrt{2N}}\right) = \left(\frac{e^{i\pi/Npj}}{\sqrt{2N}}\right) \tag{4.28}$$

Example:

$$h(t) = \cos(t),\ T = 2\pi \tag{4.29}$$

$$h(0) = 1,\ t_0 = 0 \tag{4.30}$$

$$h(t_1) = 0,\ t_1 = \frac{\pi}{2} \tag{4.31}$$

$$h(t_2) = -1,\ t_2 = \pi \tag{4.32}$$

$$h(t_3) = 0,\ t_3 = \frac{3\pi}{2} \tag{4.33}$$

$$w_p = \frac{2\pi}{T} p = p \tag{4.34}$$

$$w_0 = 0,\ w_1 = 1, \cdots,\ N = 2 \tag{4.35}$$

$$\begin{pmatrix} \hat{h}_0 \\ \hat{h}_1 \\ \hat{h}_2 \\ \hat{h}_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \tag{4.36}$$

$$h(t) = \frac{1}{2} \sum_{p=0}^{3} \hat{h}_p e^{-w_p t} = \frac{1}{2} \left(e^{-it} + e^{-3it} \right) \tag{4.37}$$

$$h^{DFT}(t) = \frac{1}{2} \left(\cos(t) + \cos(3t) \right) \neq h(t) = \cos(t) \tag{4.38}$$

Lecture 5

5.1 Laplace Transforms

$$\mathcal{L}[f(t)](s) = \bar{f}(s) = \int_0^\infty f(t)e^{-ts}\,dt \tag{5.1}$$

Convergence $\implies \bar{f}(s)$ not necessarily defined in the whole range of s. Typically, since $t > 0 \implies s > 0$.

Example:

$$\mathcal{L}[1] = \int_0^\infty e^{-ts}dt = -\frac{1}{s}e^{-ts}\Big|_0^\infty \tag{5.2}$$

$$= \frac{1}{s},\ s > 0 \tag{5.3}$$

Example:

$$f(t) = e^{at}, a > 0 \tag{5.4}$$

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{at}e^{-st}dt \tag{5.5}$$

$$= -\frac{1}{s-a}e^{-(s-a)t}\Big|_0^\infty \tag{5.6}$$

$$= \frac{1}{s-a},\ s > a \tag{5.7}$$

5.2 Relation between Laplace and Fourier transforms

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt,\ s = x + iy \tag{5.8}$$

$$= \int_0^\infty f(t)e^{-(x+iy)t}dt \tag{5.9}$$

$$= \int_0^\infty f(t)e^{-xt}e^{-iyt}dt \tag{5.10}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \sqrt{2\pi}f(t)e^{-xt}\Theta(t)e^{-iyt}dt \tag{5.11}$$

$$= \mathcal{F}^{-1}\left[\sqrt{2\pi}f(t)e^{-xt}\Theta(t)\right] \tag{5.12}$$

Consider

$$f(t) = \cosh(kt) = \frac{e^{kt} + e^{-kt}}{2} \tag{5.13}$$

$$\mathcal{L}[f(t)] = \frac{1}{2}\left(\underbrace{\frac{1}{s-k}}_{s>k} + \underbrace{\frac{1}{s+k}}_{s+k>0}\right) = \underbrace{\frac{s}{s^2-k^2}}_{s>|k|} \tag{5.14}$$

5.3 Laplace Transform and Derivatives

$$\mathcal{L}\left[\frac{df}{dt}\right] = \int_0^\infty \frac{df}{dt} e^{-st} dt \tag{5.15}$$

$$= f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty f(t)e^{-st} dt \tag{5.16}$$

$$= -f(0) + s\mathcal{L}[f(t)](s) \tag{5.17}$$

$$\mathcal{L}\left[\frac{d^2 f}{dt^2}\right] = \mathcal{L}\left[\frac{d}{dt}\left(\frac{df}{dt}\right)\right] \tag{5.18}$$

$$= -f'(0) + s\mathcal{L}\left[\frac{df}{dt}\right] \tag{5.19}$$

$$= -f'(0) - sf(0) + s^2\mathcal{L}[f(t)](s) \tag{5.20}$$

$$\mathcal{L}\left[\frac{d^n f}{dt^n}\right] = s^n\mathcal{L}[f(t)](s) - \sum_{k=0}^{n-1} s^{n-1-k} f^{(k)}(0) \tag{5.21}$$

Now consider:

$$\mathcal{L}[\sinh(kt)] = \frac{1}{k}\mathcal{L}\left[\frac{d}{dt}\cosh(kt)\right] \tag{5.22}$$

$$= \frac{1}{k}\left(s\mathcal{L}[\cosh(kt)] - \cosh(0)\right) \tag{5.23}$$

$$= \frac{1}{k}\left(s\underbrace{\frac{s}{s^2-k^2}}_{s>|k|} - 1\right) = \underbrace{\frac{k}{s^2-k^2}}_{s>|k|} \tag{5.24}$$

5.4 Laplace Transforms and Integrals

$$\mathcal{L}\left[\int_a^t f(t')dt'\right] = \int_0^\infty \int_a^t f(t')dt' e^{-st} dt,\ du = e^{-st}dt, u = \frac{-1}{s}e^{-st} \tag{5.25}$$

$$= -\frac{1}{s}\int_a^t f(t')dt' e^{-st}\Bigg|_0^\infty + \frac{1}{s}\int_0^\infty f(t)e^{-st}dt \tag{5.26}$$

$$= -\frac{1}{s}\int_0^a f(t')dt' + \frac{1}{s}\mathcal{L}[f(t)] \tag{5.27}$$

Example:

$$f(t) = \int_0^\infty \frac{\sin(tx)}{x} dx \tag{5.28}$$

$$\mathcal{L}[f(t)] = \mathcal{L}\left[\int_0^\infty \frac{\sin(tx)}{x} dx\right] \tag{5.29}$$

$$= \int_0^\infty \int_0^\infty \frac{\sin(tx)}{x} dx e^{-st} dt \tag{5.30}$$

$$t > 0 \implies \mathcal{L}[\sin(tx)] = \frac{x}{s^2+x^2},\ |s| > 0 \tag{5.31}$$

$$\mathcal{L}[f(t)] = \int_0^\infty \frac{\mathcal{L}[\sin(tx)]}{x} dx \tag{5.32}$$

$$= \int_0^\infty \frac{1}{s^2 + x^2} dx = \frac{\pi}{2s} \tag{5.33}$$

$$\mathcal{L}^{-1}\left[\frac{\pi}{2s}\right] = \frac{\pi}{2} = f(t) \tag{5.34}$$

$$t < 0 \implies \sin(tx) = -\sin(-tx) \tag{5.35}$$

$$\mathcal{L}^{-1}[\bar{f}(s)] = -\frac{\pi}{2} = f(t) \tag{5.36}$$

$$f(t) = \begin{cases} \frac{\pi}{2} & t > 0 \\ -\frac{\pi}{2} & t < 0 \end{cases} = \frac{\pi}{2}\operatorname{sign}(t) \tag{5.37}$$

Lecture 6

$$\mathcal{L}[f(t)](s) = \int_0^\infty f(t)e^{-ts}dt \tag{6.1}$$

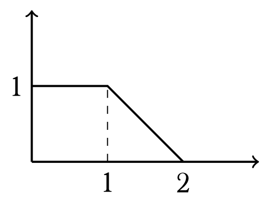
Converges for a given interval in s. Typically applied for $t > 0$, and typically $s > 0$ but not always.

Properties:

$$\mathcal{L}[1] = \frac{1}{s} \tag{6.2}$$
$$\mathcal{L}[t] = \frac{1}{s^2},\ s > 0 \tag{6.3}$$
$$\mathcal{L}[e^{-at}f(t)] = \int_0^\infty e^{-at}f(t)e^{-st}dt \tag{6.4}$$
$$= \int_0^\infty e^{-(s+a)t}f(t)\,dt \tag{6.5}$$
$$= \mathcal{L}[f(t)](s+a) \tag{6.6}$$
$$\mathcal{L}\left[f\left(\frac{t}{a}\right)\right] = a\mathcal{L}[f(t)](as) \tag{6.7}$$
$$\mathcal{L}[f(t-a)\Theta(t-a)] = e^{-as}\mathcal{L}[f(t)] \tag{6.8}$$

Consider:

$$f(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 2-t & 1 < t \le 2 \\ 0 & t > 2 \end{cases} \tag{6.9}$$



$$f(t) = \begin{cases} \Theta(t) - \Theta(t-1) \\ (2-t)\left[\Theta(t-1) - \Theta(t-2)\right] \\ 0 \end{cases} \tag{6.10}$$
$$f(t) = \Theta(t) + (1-t)\Theta(t-1) + (t-2)\Theta(t-2) \tag{6.11}$$
$$\mathcal{L}[f(t)] = \mathcal{L}[1] - \mathcal{L}[(t-1)\Theta(t-1)] + \mathcal{L}[(t-2)\Theta(t-2)] \tag{6.12}$$
$$= \mathcal{L}[1] - e^{-s}\mathcal{L}[t] + e^{-2s}\mathcal{L}[t] \tag{6.13}$$
$$= \frac{1}{s} - \frac{e^{-s}}{s^2} + \frac{e^{-2s}}{s^2} \tag{6.14}$$

Another useful trick:

$$\mathcal{L}[tf(t)] = \int_0^\infty tf(t)e^{-ts}dt \tag{6.15}$$

$$= -\int_0^\infty f(t)\frac{de^{-ts}}{ds}dt \tag{6.16}$$

$$= -\frac{d}{ds}\left(\int_0^\infty f(t)e^{-ts}dt\right) \tag{6.17}$$

$$= -\frac{d}{ds}\Big(\mathcal{L}[f(t)]\Big) \tag{6.18}$$

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}\Big(\mathcal{L}[f(t)]\Big) \tag{6.19}$$

$$\mathcal{L}[1] = \frac{1}{s} \tag{6.20}$$

$$\mathcal{L}[t] = \frac{1}{s^2} = -1\frac{d}{ds}\Big(\mathcal{L}[1]\Big) = \frac{1}{s^2} \tag{6.21}$$

6.1 Periodic Functions

Consider $f(t)$, periodic with period p.

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-ts}dt \tag{6.22}$$

$$= \int_0^p f(t)e^{-ts}dt + \int_p^{2p} f(t)e^{-ts}dt + \cdots + \int_{np}^{(n+1)p} f(t)e^{-ts}dt + \cdots \tag{6.23}$$

Shift, $t = x + np$, $x = 0 \to t = np,\ x = p \to t = (n+1)p.$

$$\int_0^p f(x+np)e^{-(x+np)s}dx = e^{-snp}\int_0^p f(x)e^{-xs}dx \tag{6.24}$$

$$\mathcal{L}[f(t) = \left(\int_0^p f(x)e^{-xs}dx\right)[1 + e^{-ps} + \cdots + e^{-nps} + \cdots] \tag{6.25}$$

$$= \frac{1}{1-e^{-ps}}\int_0^p f(x)e^{-xs}dx \tag{6.26}$$

Example:
Consider

$$f(t) = \begin{cases} 1 & n+0 < t < n+1 \\ 0 & n+1 < t < n+2 \end{cases} \tag{6.27}$$

Period, $p = 2$.

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-2s}}\int_0^2 f(t)e^{-st}dt \tag{6.28}$$

$$= \frac{1}{1-e^{-2s}}\int_0^1 e^{-st}dt \tag{6.29}$$

$$= \frac{1-e^{-s}}{s(1-e^{-2s})} \tag{6.30}$$

6.2 Laplace transform of delta function

$$\mathcal{L}[\delta(t-t')] = e^{-t's},\ t' > 0, s > 0 \tag{6.31}$$

$$\mathcal{L}[\delta(t)] = 1 \tag{6.32}$$

Consider Newton's second law, applied to an instantaneous impulse:

$$m\ddot{x} = \underline{p}\delta(t),\ x(0) = x_0, \dot{x}(0) = v_0 \tag{6.33}$$

$$\mathcal{L}[m\ddot{x}] = \mathcal{L}[p\delta(t)] \tag{6.34}$$

$$m\Big(s^2\mathcal{L}[x] - sx(0) - \dot{x}(0)\Big) = \underline{p} \tag{6.35}$$

$$\mathcal{L}[x] = \left(\frac{p}{m} + v_0\right)\frac{1}{s^2} + \frac{x_0}{s} \tag{6.36}$$

$$x(t) = \left(\frac{p}{m} + v_0\right)t + x_0 \tag{6.37}$$

6.3 Convolution Theorem

$$(f \times g) = \int_0^t f(x)g(t-x)\,dx \tag{6.38}$$

$$\mathcal{L}[f \times g] = \mathcal{L}[f]\mathcal{L}[g] \tag{6.39}$$

$$\mathcal{L}[f \times g] = \int_0^\infty \left(\int_0^t f(x)g(t-x)\,dx\right)e^{-st}dt \tag{6.40}$$

$$= \int_0^\infty \left(\int_x^\infty f(x)g(t-x)\,e^{-ts}dt\right)dx,\ y = t-x,\ dy = dt \tag{6.41}$$

$$= \int_0^\infty \left(\int_0^\infty f(x)g(y)e^{-(y+x)s}dy\right)dx \tag{6.42}$$

$$= \left(\int_0^\infty f(x)e^{-xs}dx\right)\cdot\left(\int_0^\infty g(y)e^{-ys}dy\right) = \mathcal{L}[f]\cdot\mathcal{L}[g] \tag{6.43}$$

A useful tool for integrals:

$$I(x) = \int_0^x \cos\Big(b(x-u)\Big)e^{au}\,du \tag{6.44}$$

Lecture 7

7.1 Solving Differential Equations with Laplace Transforms

$$\mathcal{L}\left[\frac{d^n}{dt^n}f(t)\right] = s^n\mathcal{L}[f(t)] - \sum_{k=0}^{n-1} s^{n-1-k}f^{(k)}(0) \tag{7.1}$$

$$\mathcal{L}[f(t-a)\Theta(t-a)] = e^{-as}\mathcal{L}[f(t)] \tag{7.2}$$

Example:
Consider:

$$y'' + 3y' + 2y = 2 - 2\Theta(t-1) \tag{7.3}$$

$$y(0) = 0,\ y'(0) = 2 \tag{7.4}$$

$$\mathcal{L}[y''] = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} - 2 \tag{7.5}$$

$$\mathcal{L}[y'] = s\bar{y} - y(0) = s\bar{y} \tag{7.6}$$

$$\mathcal{L}[2] = \frac{2}{s} \tag{7.7}$$

$$\mathcal{L}[\Theta(t-1)] = \frac{e^{-s}}{s} \tag{7.8}$$

$$s^2\bar{y} - 2 + 3s\bar{y} + 2\bar{y} = \frac{2}{s} - \frac{e^{-s}}{s} \tag{7.9}$$

$$\bar{y}(s^2 + 3s + 2) = 2\left(1 + \frac{1}{s} - \frac{2e^{-s}}{s}\right) \tag{7.10}$$

$$\bar{y}(s+1)(s+2) = 2\left(1 + \frac{1}{s} - \frac{2e^{-s}}{s}\right) \tag{7.11}$$

$$\bar{y} = \frac{2}{(s+1)(s+2)}\left(\frac{s+1}{s} - \frac{2e^{-s}}{s}\right) \tag{7.12}$$

$$= \frac{2}{s(s+2)} - \frac{2e^{-s}}{s(s+1)(s+2)} \tag{7.13}$$

$$= \frac{1}{s} - \frac{1}{s+1} - e^{-s}\left(\frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2}\right) \tag{7.14}$$

$$y(t) = 1 - e^{-2t} - \Theta(t-1)\left(1 - 2e^{-(t-1)} + e^{2(t-1)}\right) \tag{7.15}$$

7.2 Coupled Differential Equations

Example: Electrical circuits

$$L\ddot{q}_1 + M\ddot{q}_2 + \frac{1}{c}q_1 = 0 \tag{7.16}$$

$$M\ddot{q}_1 + L\ddot{q}_2 + \frac{1}{c}q_2 = 0 \tag{7.17}$$

$$q_1(0) = \dot{q}_1(0) = \ddot{q}_2(0) = 0 \tag{7.18}$$

$$q_2(0) = V_0C \tag{7.19}$$

$$\mathcal{L}[\ddot{q}_1] = s^2\mathcal{L}[q_1] - \dot{q}_1(0) - sq_1(0) = s^2\mathcal{L}[q_1] \tag{7.20}$$

$$\mathcal{L}[\ddot{q}_2] = s^2\mathcal{L}[q_2]V_0Cs \tag{7.21}$$

$$\mathcal{L}[q_1]\left(Ls^2 + \frac{1}{c}\right) + \mathcal{L}[q_2]s^2 M = sMV_0C \tag{7.22}$$

$$\mathcal{L}[q_1]s^2 M + \mathcal{L}[q_2]\left(Ls^2 + \frac{1}{c}\right) = sLV_0C \tag{7.23}$$

$$\mathcal{L}[q_1] = \frac{V_0C}{2}\left[\frac{(L+M)s}{(L+M)s^2 + \frac{1}{c^2}} - \frac{(L-M)s}{(L-M)s^2 + \frac{1}{c^2}}\right], \approx \frac{s}{s^2+a^2} \tag{7.24}$$

$$q_1(t) = \frac{V_0C}{2}\left[\cos\left(\frac{t}{\sqrt{(L+M)c^2}}\right) - \cos\left(\frac{t}{\sqrt{(L-M)c^2}}\right)\right] \tag{7.25}$$

Theorem: Suppose that $\mathcal{L}[f](s)$ admits a series expansion of the form:

$$\mathcal{L}[f](s) = \sum_{n=0}^{\infty} a_n s^{-n-1},\ |s| > f \tag{7.26}$$

Then, for $t > 0$:

$$f(t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n \tag{7.27}$$

Example:

$$ty'' + y' + ty - 0 \tag{7.28}$$

$$y(0) = 1,\ y'(0) = 0 \tag{7.29}$$

$$\mathcal{L}[ty] = -\frac{d}{ds}\bar{y} \tag{7.30}$$

$$\mathcal{L}[y'] = s\bar{y} - y(0) = s\bar{y} - 1 \tag{7.31}$$

$$\mathcal{L}[ty''] = -\frac{d}{ds}\left(\mathcal{L}[y'']\right) = -\frac{d}{ds}\left(s^2\bar{y} - sy(0) - y'(0)\right) \tag{7.32}$$

$$= -\left(2s\bar{y} + s^2\frac{d}{ds}\bar{y} - 1\right) \tag{7.33}$$

$$-\left(2s\bar{y} + s^2\frac{d}{ds}\bar{y} - 1\right) + s\bar{y} - 1 - \frac{d}{ds}\bar{y} = 0 \tag{7.34}$$

$$-\frac{d}{ds}\bar{y}(s^2+1) - s\bar{y} = 0 \tag{7.35}$$

$$\frac{d}{ds}\bar{y} = \frac{-s}{s^2+1}\bar{y} \tag{7.36}$$

$$\int \frac{d\bar{y}}{\bar{y}} = \int \frac{-s\,ds}{s^2+1} \tag{7.37}$$

$$\log\bar{y} = C - \frac{1}{2}\log\left(s^2+1\right) \tag{7.38}$$

$$\bar{y} = \frac{C}{\sqrt{s^2+1}},\ |s| > 1 \tag{7.39}$$

$$\bar{y} = \frac{C}{s\sqrt{1+\frac{1}{s^2}}} = \sum_{n=0}^{\infty} \frac{(-1)^n(2n)!}{(n!)^2 2^{2n}} \frac{1}{s^{2n+1}} \tag{7.40}$$

$$y(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \frac{t^{2n}}{2^{2n}} = J_0(t) \tag{7.41}$$

Lecture 8

8.1 Inverting the Laplace Transform

Inspirations:

- ➤ Inspection
- ➤ Convolution Theorem

Perspiration:

- ➤ Inverse Laplace Transform - Bromwich integral

8.1.1 Inspection

$$\mathcal{L}[1] = \frac{1}{s} \tag{8.1}$$

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}} \tag{8.2}$$

$$\mathcal{L}[e^{at}] = \frac{1}{s-a} \tag{8.3}$$

Example:

$$\mathcal{L}[f](s) = \frac{4}{s(s+2)} = \frac{2}{s} - \frac{2}{s+2} \tag{8.4}$$

$$= 2\mathcal{L}[1] - 2\mathcal{L}[e^{-2t}] \tag{8.5}$$

$$f(t) = 2 - 2e^{-2t} \tag{8.6}$$

8.1.2 Convolution

$$(f \times g) = \int_0^t f(x)g(t-x)\,dx \tag{8.7}$$

$$\mathcal{L}[f \times g] = \mathcal{L}[f] \cdot \mathcal{L}[g] \tag{8.8}$$

Example:

$$\mathcal{L}[f](s) = \frac{4}{s^2(s+2)^2} \tag{8.9}$$

$$\frac{1}{s^2} = \mathcal{L}[t](s) \tag{8.10}$$

$$\frac{1}{(s+2)^2} = -\frac{d}{ds}\left(\frac{1}{s+2}\right) = -\frac{d}{ds}\left(\mathcal{L}[e^{-2t}]\right) \tag{8.11}$$

$$= \mathcal{L}[te^{-2t}](s) \tag{8.12}$$

$$\mathcal{L}[f](s) = 4\mathcal{L}[t] \cdot \mathcal{L}[te^{-2t}] = 4\mathcal{L}[t \times te^{-2t}] \tag{8.13}$$

$$f(t) = 4\left(t \times te^{-2t}\right) \tag{8.14}$$

$$= 4\int_0^t x(t-x)e^{-2(t-x)}dx = (1+t)e^{-2t} + t - 1 \tag{8.15}$$

This could have been done in other ways:

$$\mathcal{L}[f](s) = \frac{4}{s^2(s+2)^2} \tag{8.16}$$

$$= \frac{1}{4}\frac{4}{s(s+2)} \cdot \frac{4}{s(s+2)} \tag{8.17}$$

$$= \frac{1}{4}\mathcal{L}[2-2e^{-2t}] \cdot \mathcal{L}[2-2e^{-2t}] \tag{8.18}$$

$$= \frac{1}{4}\mathcal{L}[(2-2e^{-2t} \times (2-2e^{-2t})] \tag{8.19}$$

$$f(t) = \frac{1}{4}\Big((2-2e^{-2t}) \times (2-2e^{-2t})\Big) \tag{8.20}$$

8.1.3 Inversion Theorem

Consider a function $f(t)$ which is piece smooth in $[0,\infty]$ and whose Laplace transform $\mathcal{L}[f(t)](s)$ exists for $\mathrm{Re}(s) > 0$, then

$$f(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{L}[f](s)e^{st}ds. \tag{8.21}$$

Proof:
First consider $s = \sigma + ix$. Now recall

$$\mathcal{L}[f](s) = \sqrt{2\pi}\mathcal{F}^{-1}[f(t)\Theta(t)e^{-\sigma t}](x) \tag{8.22}$$

$$f(t)\Theta(t)e^{-\sigma t} = \frac{1}{\sqrt{2\pi}}\mathcal{F}[\mathcal{L}[f](s)](t) \tag{8.23}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{L}[f](\sigma+ix)e^{ixt}dx \tag{8.24}$$

$$f(t)\Theta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{L}[f](\sigma+ix)e^{(\sigma+ix)t}dx \tag{8.25}$$

$$= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{L}[f](s)\,e^{st}ds \tag{8.26}$$

If $\mathcal{L}[f]$ is a meromorphic function with a finite number of poles $\{a_i\}, i = \{1\ldots,n\}$. The key thing is that the singularities are finite, therefore $\exists M \in \mathbb{R} \,/\, |\mathcal{L}[f](s)| \leq M|s|^{-k}$. We choose $t > 0, \sigma > \mathrm{Re}(a_i)$, and due to the positive exponential, we choose the left handside of the line from $\sigma - i\infty \to \sigma + i\infty$ to close the contour.

$$f(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{L}[f](s)\,e^{st}ds \tag{8.27}$$

$$= \sum_{i=1}^{n} \mathrm{Res}\Big[\mathcal{L}[f](s)\,e^{st}\Big]\Big|_{a_i} \tag{8.28}$$

The integral along the curve, C_R, vanishes:

$$\lim_{R\to\infty}\left|\int_{C_R}\right| = 0 \tag{8.29}$$

Parameterisation, $C_R:\ \sigma + Re^{i\theta},\ \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

$$|s| = |\sigma + Re^{i\theta}| \geq \big||\sigma| - |Re^{i\theta}|\big| \tag{8.30}$$

$$= |\sigma - R| = R - \sigma \tag{8.31}$$

Now show, using the same definition of s:

$$\left|\frac{1}{2\pi i}\int_{C_R} \mathcal{L}[f](s)\,e^{st}ds\right| = \left|\frac{1}{2\pi i}\int_{\pi/2}^{3\pi/2} \mathcal{L}[f](\sigma+Re^{i\theta})\,e^{(\sigma+Re^{i\theta})t}iRe^{i\theta}\,d\theta\right| \tag{8.32}$$

$$\leq \frac{1}{2\pi}\int_{\pi/2}^{3\pi/2}\left|\mathcal{L}[f](\sigma+Re^{i\theta})\right|\left|e^{(\sigma+Re^{i\theta})t}iRe^{i\theta}\right|d\theta \tag{8.33}$$

$$\left|\mathcal{L}[f](\sigma+Re^{i\theta})\right| \leq M|s|^{-k} \tag{8.34}$$

$$\leq M(R-\sigma)^{-k} \tag{8.35}$$

$$\left|e^{t(\sigma+Re^{i\theta})}\right| = \left|e^{t(\sigma+R\cos\theta)}e^{itR\sin\theta}\right| \tag{8.36}$$

$$= e^{t(\sigma+R\cos\theta)} \tag{8.37}$$

$$\left|\frac{1}{2\pi i}\int_{C_R}\mathcal{L}[f](s)e^{st}ds\right| < \frac{1}{2\pi}\int_{\pi/2}^{3\pi/2} M(R-\sigma)^{-k}\,e^{t(\sigma+R\cos\theta)}R\,d\theta \tag{8.38}$$

$$\lim_{R\to\infty} \to 0 \tag{8.39}$$

If instead you want to calculate for $-t$, you choose $\sigma < \mathrm{Re}(a_i)$ and integrate over a contour to the right.

Example:

$$\mathcal{L}[f](s) = \frac{4}{s^2(s+2)^2} \tag{8.40}$$

We have double poles at 0 and -2. Use above method, integrate to the right of these, and close the circle on the left enclosing the pooles.

$$f(t)_{t>0} = \mathrm{Res}\left[\frac{4}{s^2(s+2)^2}e^{st}\right]_{s=0} + \mathrm{Res}\left[\frac{4}{s^2(s+2)^2}e^{st}\right]_{s=-2} \tag{8.41}$$

For a pole of order m,

$$\mathrm{Res}[f(z)]_{z=z_0} = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]_{z=z_0} \tag{8.42}$$

So, (8.41) simplifies to

$$f(t) = t - 1 + e^{-2t}(1+t) \tag{8.43}$$

Lecture 9 Fourier Transforms and Quantum Mechanics

Note: Calculating residues for second and third order poles will be on the exam.
$f(x)$ is square-integrable:

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty \tag{9.1}$$

The set of all square-integrable functions is denoted by the $\mathbb{L}^2$ Hilbert space with with the inner product,

$$\langle f|g\rangle = \int_{-\infty}^{\infty} f(x)^* g(x) \, dx, \tag{9.2}$$

and the norm,

$$||f||^2 = \langle f|f\rangle = \int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty \tag{9.3}$$

Consider a linear operator, $\hat{F}[f(x)]$.

$$\hat{F}[f(x)](k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \tag{9.4}$$

$$= \mathcal{F}^{-1}[f(x)](k) \tag{9.5}$$

$$\hat{F}^{-1}[\bar{f}(k)](x) = \mathcal{F}[\bar{f}(k)](x) \tag{9.6}$$

$$\hat{F} f = \bar{f} \tag{9.7}$$

Here, x is position space, and k is momentum space.

9.1 Parseval Function

$$\langle f|g\rangle = \langle \hat{F} f | \hat{F} g\rangle \tag{9.8}$$

The Fourier transform preserves the scalar transform in $\mathbb{L}^2$. $\hat{F}$ is a unitary operator, $\hat{F}^{-1} = \hat{F}^{\dagger}$. The norm of a function is invariant under the transform.

9.2 The Position and Momentum Operators

Wave function,

$$\psi(x) \in \mathbb{R}^2 \tag{9.9}$$

$$||\psi(x)|| = 1 \tag{9.10}$$

$|\psi(x)|^2$ is the probability density of finding a particle in position x.

What is the meaning of $\hat{F}\psi = \bar{\psi}$?
The momentum operator,

$$\hat{p} = -i \frac{\partial}{\partial x}, \tag{9.11}$$

has eigenvalues $\phi \in \mathbb{R}$, and eigenvectors,

$$\Phi_p(x) = \frac{1}{\sqrt{2\pi}} e^{\phi x}. \tag{9.12}$$

$$\hat{p}\Phi_p(x) = p_0 \Phi_p(x) \tag{9.13}$$

$$|\langle \phi_p|\psi\rangle|^2 = \left| \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} \psi(x) e^{-ip_0 x} \right|^2 \tag{9.14}$$

$$= |\hat{F}\psi(x)|^2 \tag{9.15}$$
$$= |\bar{\psi}(p_0)|^2 \tag{9.16}$$

This is the wavefunction in momentum space.

$$\|\psi\| = \|\bar{\psi}\| = 1 \tag{9.17}$$
$$\hat{p}\bar{\psi}(p) = \hat{F}[\hat{p}\psi(p^*)] \tag{9.18}$$
$$= \hat{F}[-i\frac{\partial}{\partial x}\psi(x)] \tag{9.19}$$
$$= -iip\bar{\psi}(p) \tag{9.20}$$
$$= \underline{p}\bar{\psi}(p) \tag{9.21}$$
$$\bar{\Phi}_p(p) = \hat{F}[\Phi_p(x)] \tag{9.22}$$
$$= \hat{F}\left[\frac{1}{\sqrt{2\pi}}e^{ipx}\right] \tag{9.23}$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-i(p-p_0)x}dx \tag{9.24}$$
$$= \delta(p-p_0) \tag{9.25}$$
$$\hat{p}\Phi_p(p) = p\bar{\Phi}_p(p) \tag{9.26}$$
$$= p\delta(p-p_0) \tag{9.27}$$
$$= p_0\bar{\Phi}_p(p) \tag{9.28}$$

9.3 Position operator

Defined in position space as

$$\hat{x}\psi(x) = x\psi(x). \tag{9.29}$$

Its eigenfunctions are delta functions, $\chi_{x_0} = \delta(x-x_0)$. So in momentum space,

$$\hat{x}\bar{\psi}(p) = \hat{F}[\hat{x}\psi(x)] = i\frac{\partial}{\partial p}\bar{\psi}(p) \tag{9.30}$$
$$\bar{\chi}_{x_0} = \hat{F}(\delta(x-x_0)) \tag{9.31}$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x-x_0)e^{-ipx}dx = \frac{1}{\sqrt{2\pi}}e^{-ipx_0} \tag{9.32}$$

Configuration Space	Momentum Space
$\psi(x)$	$\bar{\psi}(x) = \hat{F}[\psi(x)](p)$
$\hat{x} = x$	$i\frac{\partial}{\partial p}$
$\hat{p} = -i\frac{\partial}{\partial x}$	p
$\Phi_{p_0}(x) = \frac{1}{\sqrt{2\pi}}e^{ip_0x}$	$\bar{\Phi}_{p_0}(p) = \delta(p-p_0)$
$\chi_{x_0} = \delta(x-x_0)$	$\bar{\chi}_{x_0}(p) = \frac{1}{\sqrt{2\pi}}e^{-ipx_0}$

Table of Laplace Transforms

$f(t)$	$\bar{f}(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin(wt)$	$\frac{w}{s^2+w^2}$
$\cos(wt)$	$\frac{s}{s^2+w^2}$
$t^n g(t)$	$(-1)^n \frac{d^n G(s)}{ds^n}$
$t\sin(wt)$	$\frac{2ws}{(s^2+w^2)^2}$
$t\cos(wt)$	$\frac{s^2-w^2}{(s^2+w^2)^2}$
$g(at)$	$\frac{1}{a}G\left(\frac{s}{a}\right)$
$e^{at}g(t)$	$G(s-a)$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
te^{-t}	$\frac{1}{(s+1)^2}$
$1-e^{-t/T}$	$\frac{1}{s(1+Ts)}$
$e^{at}\sin(wt)$	$\frac{w}{(s-a)^2+w^2}$
$e^{at}\cos(wt)$	$\frac{s-a}{(s-a)^2+w^2}$
$g'(t)$	$sG(s)-g(0)$
$g''(t)$	$s^2G(s)-sg(0)-g'(0)$
$g^{(n)}(t)$	$s^nG(s)-\sum_{k=0}^{n-1} s^{n-1-k}g^{(k)}(0)$
$\int_0^t g(t)\,dt$	$\frac{G(s)}{s}$
$\int g(t)\,dt$	$\frac{G(s)}{s}+\frac{1}{s}\left\{\int g(t)\,dt\right\}_{t=0}$