Advanced Theoretical Physics

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Part I

Quantum Optics

Lecture 1

Syllabus:

- ➤ quantization of light
- ➤ creation and annihilation operators
- ➤ Hamiltonian of the E field
- ➤ number states
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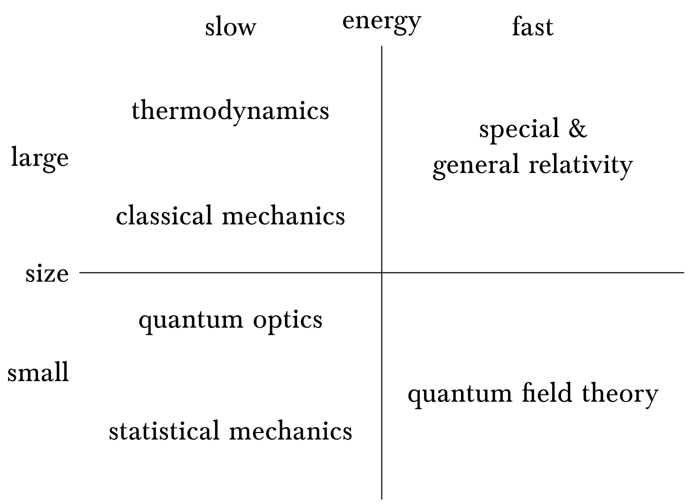
- ➤ squeezed states
- ➤ photon bunching and anti-bunching
- ➤ density operator
- ➤ pure states, mixed sates, entangled states
- ➤ decoherence
- ➤ atom-light interactions
- ➤ applications



Ingredients:

- ➤ harmonic oscillators
- ➤ Gaussian integrals
- ➤ Hamiltonian mechanics (canonical variables q and p)
- ➤ maths of operators - adjoint, self-adjoint, Hermitian, commutation relations
- ➤ QM in both Schrodinger and Heisenberg pictures
- ➤ density matrices
- ➤ classical EM - Maxwell's equations in Coulomb gauge - especially plane waves and dipoles

Hanbury Brown and Tiss:

$$G(\tau) = I_A(t)I_B(t+\tau) \tag{1.1}$$

Lecture 2

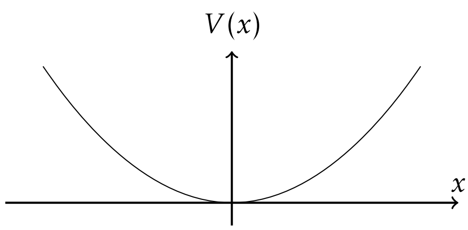
2.1 Learning Outcomes

To be able to state, explain and apply the operator formalism of the quantum harmonic oscillator, including:

- the Hamiltonian in terms of the creation and annihilation operators
- the number operator and number states, eigenstates of the Hamiltonian
- definition of the creation and annihilation operators, commutation relations, adjoint and self-adjoint operators
- mathematical properties of the number states, completeness
- systems of two or more independent oscillators

2.2 Quantum Harmonic Oscillator

$$
\begin{aligned}
F &= ma = m\ddot{x} && (2.1)\\
&= -kx && (2.2)\\
x(t) &= x_0 \sin\omega t && (2.3)\\
p_x(t) &= p_0 \cos\omega t && (2.4)\\
V(x) &= \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2 && (2.5)\\
\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} &+ V(x)\psi(x) = E\psi && (2.6)\\
E_n &= \left(n + \frac{1}{2}\right)\hbar\omega && (2.7)
\end{aligned}
$$



Start with writing the Hamiltonian, then turn everything into operators

$$H = \frac{p^2}{2m} + \underbrace{\frac{1}{2}m\omega^2 x^2}_{V(x)} \tag{2.8}$$

$$
\begin{aligned}
p \rightarrow \hat{p} &= -i\hbar\frac{d}{dx},\ x \rightarrow \hat{x} && (2.9)\\
[\hat{x}, \hat{p}] &= i\hbar && (2.10)\\
H &= \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 && (2.11)\\
\hat{a} &= \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\hat{x} + i\hat{p}) && (2.12)\\
\hat{a}^\dagger &= \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\hat{x} - i\hat{p}) && (2.13)\\
\hat{x} &= \left(\frac{\hbar}{2m\omega}\right)^{1/2}(\hat{a} + \hat{a}^\dagger) && (2.14)
\end{aligned}
$$

$$\hat{p} = -i\left(\frac{m\hbar\omega}{2}\right)^{1/2}(\hat{a} - \hat{a}^{\dagger}) \tag{2.15}$$

$$[\hat{a}, \hat{a}^{\dagger}] = \hat{a}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a} = 1 \tag{2.16}$$

$$\hat{H} = \hbar\omega(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}) \tag{2.17}$$

$$\hat{a}^{\dagger}\hat{a} = \hat{n},\ \hat{n}|n\rangle = n|n\rangle \tag{2.18}$$

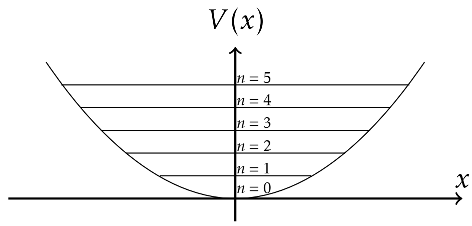
$$\hat{H}|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle = E_n|n\rangle \tag{2.19}$$

How do the annihilation and creation operators, $\hat{a}$ and $\hat{a}^{\dagger}$ interact with the number states, $|n\rangle$?

$$\hat{a}^{\dagger}|n\rangle = \sqrt{n+1}|n+1\rangle \tag{2.20}$$

$$\hat{a}|n\rangle = \sqrt{}|n-1\rangle \tag{2.21}$$

Together, the creation and annihilation operators are known as the *ladder operators*. Ladder operators move the system up or down the energy levels of the harmonic potential.



We now have a partly new mathematical representation. Notice that the potential still remains positive, it does not go negative. Therefore we must have:

$$\hat{a}|0\rangle = 0, \tag{2.22}$$

$$\hat{n} = \hat{a}^{\dagger}\hat{a}|0\rangle = 0. \tag{2.23}$$

$$\implies \hat{H}|0\rangle = E_0|0\rangle = \frac{1}{2}\hbar\omega|0\rangle \tag{2.24}$$

So the ground state is labelled '0' but does not have $E = 0$.
Now we introduce $\hat{O}^{\dagger}$ as the adjoint of $\hat{O}$ if

$$\langle\psi|\hat{O}|\phi\rangle = \langle\phi|\hat{O}^{\dagger}|\psi\rangle^{*}\ \forall\psi,\phi \tag{2.25}$$

A self-adjoint operator is equivalent to a Hermitian operator, i.e. $\hat{n}, \hat{H}$.
For adjoint operators:

$$(\hat{A} + \hat{B})^{\dagger} = \hat{A}^{\dagger} + \hat{B}^{\dagger} \tag{2.26}$$

$$(\hat{A}\hat{B})^{\dagger} = \hat{B}^{\dagger}\hat{A}^{\dagger} \tag{2.27}$$

$$(c\hat{A})^{\dagger} + c^{*}\hat{A}^{\dagger} \tag{2.28}$$

$$(\hat{A}^{\dagger})^{\dagger} = \hat{A} \tag{2.29}$$

More on the number states:

- ➤ they are orthogonal

$$\langle n|n\rangle = 1 \tag{2.30}$$

$$\langle n|m\rangle = 0,\ n \neq m \tag{2.31}$$

$$\langle n|m\rangle = \delta_{n,m} \tag{2.32}$$

$$\tag{2.33}$$

➤ they form a basis (note: not mathematically a Hilbert space, but a Banah(?) space)

$$|\psi\rangle = \sum_n c_n |n\rangle \tag{2.34}$$

$$0 \leq n \leq \infty \tag{2.35}$$

2.3 Two Oscillators - independent

$$|\psi_0\rangle = \sum_n c_n |n\rangle_0 \tag{2.36}$$

$$|\psi_1\rangle = \sum_m c_m |m\rangle_1 \tag{2.37}$$

$$|\psi_{01}\rangle = \sum_{n,m} c_{n,m} |n\rangle_0 |m\rangle_1 \tag{2.38}$$

What we are doing is "tensoring" the Hilbert spaces: $\mathcal{H}_0 \otimes \mathcal{H}_1$:

$$|n\rangle_0 |m\rangle_1 \equiv |n\rangle_0 \otimes |m\rangle_1. \tag{2.39}$$

Now we have the operators, $\hat{a}_0, \hat{a}_0^\dagger, \hat{a}_1, \hat{a}_1^\dagger$:

$$\hat{a}_0 \otimes \mathbb{I}_1,\ \mathbb{I}_0 \otimes \hat{a}_1, \ldots \tag{2.40}$$

$$[\hat{a}_0, \hat{a}_1] = [\hat{a}_0, \hat{a}_1^\dagger] = 0 \tag{2.41}$$

$$\hat{H} = \hat{H}_0 \otimes \mathbb{I}_1 + \mathbb{I}_1 \otimes \hat{H}_1 \tag{2.42}$$

Note this is for non-interacting oscillators. For interacting,

$$\hat{H} = \hat{H}_0 \otimes \mathbb{I}_1 + \mathbb{I}_1 \otimes \hat{H}_1 + \mathcal{H}_{int}. \tag{2.43}$$

Lecture 3

3.1 Learning Outcomes

To be familiar with the route to quantisation of the electromagnetic field, in particular to:

- Explain and state the description of the electromagnetic field in terms of modes, including polarization
- Be familiar with the equivalence between a mode of the field and a quantum harmonic oscillator
- To explain the form of (but not derive) expressions for the Hamiltonian of the electromagnetic field, and the electric and magnetic fields in terms of the creation and annihilation operatoes
- To recognise and explain the concepts of the Schrodinger and Heisenberg representations, and to explain which is being applied
- To explain and apply the concepts of adjoint and self-adjoint operators and their matrix elements

3.2 Quantising the EM field

Consider an EM scalar potential, $\phi = 0$ (no free charges), and a vector potential, $\underline{A}$.

$$\underline{\mathbf{E}}(\underline{r},t) = \frac{\partial}{\partial t}\underline{A} \qquad \underline{\mathbf{B}}(\underline{r},t) = \nabla \times \underline{A}(\underline{r},t) \tag{3.1}$$

$$\vec{\nabla}\left[\vec{\nabla}\cdot\underline{A}\right] - \nabla^2\underline{A} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\underline{A} = 0 \tag{3.2}$$

Coulomb gauge, $\nabla\cdot\underline{A} = 0$.

$$\underline{A} = \sum_{\underline{k}}\left\{\underline{A_k}\exp\left[i(\underline{k}\cdot\underline{r} - \omega_k t)\right] + \underline{A_k^*}\exp\left[-i(\underline{k}\cdot\underline{r} - \omega_k t)\right]\right\} \tag{3.3}$$

$$\omega_k = c|k|,\ \underline{k}\cdot\underline{A}_k = 0 \tag{3.4}$$

Polarisation vectors, $\underline{e}_{k1}, \underline{e}_{k2}$ - orthonormal vectors perpendicular to $\underline{k}$.

$$\underline{A_k} = A_{k1}\underline{e}_{k1} + A_{k2}\underline{e}_{k2} \tag{3.5}$$

$$\underline{A} = \sum_{\underline{k},s} A_{\underline{k},s}\underline{e}_{\underline{k},s}\exp\left\{i(\underline{k}\cdot\underline{r} - \omega_k t)\right\} + A^*_{\underline{k},s}\underline{e}_{\underline{k},s}\exp\left\{-i(\underline{k}\cdot\underline{r} - \omega_k t)\right\} \tag{3.6}$$

The labels of the modes are $\underline{k}, s$, $s \in 1,2$. They gives us the: direction; wavelength, $\frac{2\pi}{|\underline{k}|}$; and polarisation, s. To quantise this classically:

$$H = \frac{1}{2}\epsilon_0\int\left(\underline{\mathbf{E}}\cdot\underline{\mathbf{E}} + c^2\underline{\mathbf{B}}\cdot\underline{\mathbf{B}}\right)dV \tag{3.7}$$

$$= 2\epsilon_0 V\sum_{\underline{k},s}\omega_k^2 A_{\underline{k},s}A^*_{\underline{k},s} \tag{3.8}$$

$$A_{\underline{k},s} = \frac{1}{2\omega_k\sqrt{\epsilon_0 V}}\left\{\omega_k q_{\underline{k},s} + ip_{\underline{k},s}\right\} \tag{3.9}$$

$$A^*_{\underline{k},s} = \frac{1}{2\omega_k\sqrt{\epsilon_0 V}}\left\{\omega_k q_{\underline{k},s} - ip_{\underline{k},s}\right\} \tag{3.10}$$

$q_{\underline{k},s}, p_{\underline{k},s}$ canonical coordinates (x,p).

$$H_{\underline{k},s} = \frac{1}{2}\left(p^2_{\underline{k},s} + \omega_k^2 q_{\underline{k},s}\right) \tag{3.11}$$

Harmonic oscillator $m = 1$, $x \leftrightarrow p$. To transfer this from classical to quantum, you simply convert everything to its operator form. For a single mode:

$$\hat{H}_{\underline{k},s} = \left(\hat{a}^\dagger_{\underline{k},s}\hat{a}_{\underline{k},s} + \frac{1}{2}\right)\hbar\omega_k \tag{3.12}$$

$$[\hat{a}_{\underline{k},s}, \hat{a}^\dagger_{\underline{k},s}] = 1 \tag{3.13}$$

$$\hat{a}^\dagger_{\underline{k},s}\hat{a}_{\underline{k},s} = \hat{n}_{\underline{k},s} \tag{3.14}$$

Now we have eigenstates, $|n\rangle_{\underline{k},s}$. Note: modes are not always equal to photons, but you can have photons spread over several modes.
Going back on the substitution:

$$\hat{A}_{\underline{k},s} = \sqrt{\frac{\hbar}{2\omega_k\epsilon_0 V}}\hat{a}_{\underline{k},s} \qquad \hat{A}^\dagger_{\underline{k},s} = \sqrt{\frac{\hbar}{2\omega_k\epsilon_0 V}}\hat{a}^\dagger_{\underline{k},s} \tag{3.15}$$

From these, we can find the quantised electric and magnetic field expressions. We will mostly be concerned with the electric field throughout this course as it has a much stronger interaction with matter than the magnetic.

$$\underline{\hat{\mathbf{E}}}_{\underline{k},s}(\underline{r},t) = i\left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right)^{1/2} \underline{e}_{\underline{k},s}\left[\hat{a}_{\underline{k},s}\exp\{i(\underline{k}\cdot\underline{r}-\omega_k t)\} - \hat{a}^\dagger_{\underline{k},s}\exp\{-i(\underline{k}\cdot\underline{r}-\omega_k t)\}\right] \tag{3.16}$$

3.3 Multimode Fields

$$\hat{H}_{\underline{k},s} = \sum_{\underline{k},s}\hbar\omega_k\left(\hat{a}^\dagger_{\underline{k},s}\hat{a}_{\underline{k},s} + \frac{1}{2}\right) \tag{3.17}$$

So the modes are independent of each other, but will interact through matter. We have a basis of

$$|n_1 n_2 n_3 \ldots\rangle \equiv |n_1\rangle_{\underline{k}1,s} \otimes |n_2\rangle_{\underline{k}2,s} \otimes \ldots \tag{3.18}$$

Now we can write the electric field operator:

$$\underline{\hat{\mathbf{E}}}(\underline{r},t) = \sum_{\underline{k},s}\underline{\hat{\mathbf{E}}}_{\underline{k},s}(\underline{r},t) \tag{3.19}$$

$$= \sum_{\underline{k},s} i\left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right)^{1/2} \underline{e}_{\underline{k},s}\left\{\hat{a}_{\underline{k},s}\exp[i(\underline{k}\cdot\underline{r}-\omega_k t)] + \hat{a}^\dagger_{\underline{k},s}\exp[-i(\underline{k}\cdot\underline{r}-\omega_k t)]\right\} \tag{3.20}$$

This is written in the Heisenberg representation. Now if we look at the expectation value, for one mode of the electric field

$${}_{\underline{k},s}\langle n|\underline{\hat{\mathbf{E}}}(\underline{r},t)|n'\rangle_{\underline{k},s} \tag{3.21}$$

This is time dependent as seen by the field operator and will oscillate in time through some means. As a reminder, consider an operator in the Heisenberg picture:

$$\hat{O}_H(t) = \hat{U}^\dagger(t,t_0)\hat{O}\hat{U}(t,t_0) \tag{3.22}$$

$$\hat{U}(t,t_0) = \exp\left[-i\frac{\hat{H}(t-t_0)}{\hbar}\right] \tag{3.23}$$