# Mathematical Methods in Physics

# Warming up exercises

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## 1 Lecture 1

### 1.1

Show that the vectors $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ and $\mathbf{c} = 2\mathbf{i} + \mathbf{j} - 4\mathbf{k}$ are parallel to the side of a certain right-angled triangle.

### 1.2

Show that the points $A : (1, 2, 2)$, $B : (3, 4, 5)$, $C : (-1, 0, -1)$ - calculated with respect to a fixed origin - lie on the same plane.

### 1.3

Assume that all indices can have the values 1, 2, 3.

a) Write the following expression in full: $a_{jk}x_k$.

b) Write the following expression using Einstein summation convention: $a_1x_1x_3 + a_2x_2x_3 + a_3x_3x_3$.

c) Simplify the expression $\delta_{ij}\delta_{jk}$.

## 2 Lecture 2

### 2.1

Show that the planes $2x + 2y - z = 10$ and $3x - 2y + 2z = 0$ are perpendicular.

### 2.2

Determine if the following sets are vector spaces. If any of them fails to be a vector space, state an axiom that fails to hold

a) $V_1 = \{\text{the set of vectors } (x_1, x_2, 0)^T, x_, x_2 \in \mathbb{R}\}$ and the operations as in $\mathbb{R}^3$.

b) $V_2 = \{\text{the set of vectors } (x_1, x_2, 1)^T, x_, x_2 \in \mathbb{R}\}$ and the operations as in $\mathbb{R}^3$.

# 3 Lecture 3

## 3.1

Verify whether the following sets of elements are linearly dependent or independent

a) $\{(-1,1,1,1),(1,-1,1,1),(1,1,-1,1),(1,1,1,-1)\}$;

b) $\left\{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\right\}$.

## 3.2

Verify that the inner product in $\mathbb{C}^3$ defined as

$$\langle \mathbf{v}|\mathbf{w}\rangle = \mathbf{v}^\dagger \cdot \mathbf{w} = v_1^* w_1 + v_2^* w_2 + v_3^* w_3 \equiv v_i^* w_i,$$

where $\mathbf{v} = (v_1, v_2, v_3)^T$ and $\mathbf{w} = (w_1, w_2, w_3)^T$, satisfies the three conditions that an inner product needs to fulfill.

# 4 Lecture 4

## 4.1

Calculate the determinant of the following matrices and show explicitly your calculations.

a) $\begin{pmatrix} 5 & 1 & 8 \\ 15 & 3 & 6 \\ 10 & 4 & 2 \end{pmatrix}$.

b) $\begin{pmatrix} 16 & 22 & 4 \\ 4 & -3 & 2 \\ 12 & 25 & 2 \end{pmatrix}$.

c) $\begin{pmatrix} 1 & -1 & -3 & 2 \\ 0 & -2 & 0 & 0 \\ 4 & 0 & -3 & 3 \\ 4 & -2 & -2 & -2 \end{pmatrix}$.

## 4.2

Use the Gauss-Jordan method to calculate the inverse of the matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3/17 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

### 4.3

Calculate the inverse of the matrix $C = \begin{pmatrix} -5 & 3 & 7 \\ -2 & -1 & -8 \\ 7 & 3 & -8 \end{pmatrix}$.

# 5 Lecture 5

### 5.1

Find the eigenvalues and eigenvectors of the following matrices

a) $\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$.

b) $\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$.

# 6 Lecture 6

### 6.1

Find a transformation that diagonalise the matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

Note that eigenvalues and eigenvectors of this matrix were obtained in Lecture 5, exercise 1.

# 7 Lecture 7

### 7.1

If the following functions are defined over the interval $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$, state whether or not each function can be represented by a Fourier series:

$$\text{a) } f(x) = x^3, \quad \text{b) } f(x) = 4x - 5, \quad \text{c) } f(x) = \frac{2}{x},$$
$$\text{d) } f(x) = \frac{1}{x-5}, \quad \text{e) } f(x) = \tan x, \quad \text{f) } f(x) = y, \text{ where } x^2 + y^2 = \pi^2.$$

### 7.2

What is the value at $x = 4$ of the Fourier series for the function $f(x) = f(x + 2\pi)$ defined by

$$f(x) = \begin{cases} 2 & 0 \le x < 2 \\ 4 & 2 \le x < 4 \\ -2 & 4 \le x < 2\pi \end{cases}.$$

## 8 Lecture 8

### 8.1

A wave is described by the $2\pi$ periodic function

$$f(x) = \begin{cases} 0 & -\pi < x \le 0 \\ \sin x & 0 < x \le \pi \end{cases}.$$

Find the Fourier series of $f(x)$.

## 9 Lecture 9

### 9.1

Find the Fourier transform of $f(t) = \begin{cases} 0 & t < 0 \\ e^{-\alpha t} & t \ge 0 \quad (\alpha > 0). \end{cases}$

### 9.2

The Fourier transform of $f(t)$ is $\hat{f}(\omega)$. Using the properties of the Fourier transforms calculate the Fourier transform of

$$\text{a) } f(at + 2), \quad \text{b) } f(t + 4) + f(t - 4).$$

## 10 Lecture 10

### 10.1

Calculate the following integrals

$$\text{a) } \int_{-\infty}^{\infty} \delta(x - 3) f(x + 5) dx, \quad \text{b) } \int_{-\infty}^{\infty} \delta(x - 1) e^{i\pi x} dx, \quad \text{c) } \int_{-2}^{2} \delta(x - \pi) \cos x \, dx.$$

### 10.2

The differential equation for unforced and undamped harmonic motion is of the form $mf''(t) + kf(t) = 0$. Find the permitted frequencies of oscillations by taking the Fourier transform of this equation.

# 11 Lecture 11

### 11.1

Simplify the following Dirac $\delta$-functions

$$a)\,\delta(x^2-9), \quad b)\,\delta(2\,x), \quad c)\,\delta((x+1)x)).$$

### 11.2

Rewrite the following functions by means of the Heaviside step function

$$\text{a) } f_1(x) = \begin{cases} 1 & t \leq 2 \\ 0 & \text{otherwise} \end{cases}, \quad \text{b) } f_2(x) = \begin{cases} \cos t & 0 < t \leq \pi \\ 0 & \text{otherwise} \end{cases}, \quad \text{c) } f_3(x) = \begin{cases} t & 0 < t \leq 1 \\ 2-t & 0 < 1 \leq 2 \\ 0 & \text{otherwise} \end{cases}.$$

# 12 Lecture 12

### 12.1

Calculate the Laplace transforms of the following functions

$$a)\,f_1(t) = H(t-3), \quad b)\,f_2(t) = t^3\delta(t-2), \quad c)\,f_3(t) = 4\sinh 3t.$$

### 12.2

Use the Laplace transform properties and the table in Riley (page 455) to find the inverse Laplace transforms of the following functions

$$a)\,\bar{f}_1(s) = \frac{1}{(s-3)^2}, \quad b)\,\bar{f}_2(s) = \frac{s}{s^2+25}, \quad c)\,\bar{f}_3(s) = \frac{3}{(s+1)^2+1} - \frac{3}{s-1}.$$

# 13 Lecture 13

### 13.1

Given the vector function $\mathbf{a}(u,v) = u^5\,\mathbf{i} + v\,e^{4u}\,\mathbf{j}$ find the differential $d\mathbf{a}$.

### 13.2

a) Find a parametric representation for the curve $C$ described by the functions $x+2z=1$, and $y=2$ with $-1 \leq z \leq 3$.

b) Find a parametric representation for a straight line with end points $A=(0,0,0)$ and $B=(1,2,3)$.

## 14 Lecture 14

### 14.1

Given the curve $C$ represented parametrically by

$$\mathbf{r}(u) = 2\cos u\,\mathbf{i} + 2\sin u\,\mathbf{j},$$

find $d\mathbf{r}/ds$, $d^2\mathbf{r}/ds^2$ and the radius of curvature $\rho$.

### 14.2

The surface of a sphere of radius $a$ is parametrised by the following expression

$$\mathbf{r}(\theta,\phi) = a\cos\phi\sin\theta\,\mathbf{i} + a\sin\phi\sin\theta\,\mathbf{j} + a\cos\theta\,\mathbf{k}.$$

Find the scalar and the vector area elements.

## 15 Lecture 15

### 15.1

Compute the following quantities:

a) the divergence of $\mathbf{a}(x,y,z) = xz\,\mathbf{i} + (y^2+x)\,\mathbf{j} + xyz\,\mathbf{k}$,

b) the curl of $\mathbf{b}(x,y,z) = (x/y^2)\,\mathbf{i} - (yz/x)\,\mathbf{j} + (1/z)\,\mathbf{k}$,

c) the Laplacian of $\phi(x,y,z) = (x^2z^3)/y^4$.

### 15.2

Show that $\nabla \times (\nabla\phi) = 0$ and $\nabla \cdot (\nabla \times \mathbf{a}) = 0$ for any scalar field $\phi$ and any vector field $\mathbf{a}$.

### 15.3

Evaluate the line integral $I = \int_C \mathbf{a} \cdot d\mathbf{r}$ where $\mathbf{a} = x^2\,\mathbf{i} + 2yz\,\mathbf{j} + y\,\mathbf{k}$ along a straight line with endpoints $A=(1,0,1)$ $B=(2,4,-2)$.

# 16 Lecture 16

## 16.1

Check, by calculating its curl, that the vector field $\mathbf{a} = 2xz\,\mathbf{i} + 2yz^2\,\mathbf{j} + (x^2 + 2y^2z - 1)\,\mathbf{k}$ is conservative. By finding its potential, or otherwise, calculate the integral $I = \int_C \mathbf{a} \cdot d\mathbf{r}$, where $C$ is the curve $\mathbf{r} = u\,\mathbf{i} + \mathbf{j} + u^2\,\mathbf{k}$, with $-1 \leq u \leq 1$.

## 16.2

Evaluate the integral $I = \int_C (xdy - ydx)$ where $C$ is a semicircle lying in the first and fourth quadrant of the plane. Do so by using two different parametrisations for $C$:
a) $x = \cos u, \quad y = \sin u$, for $-\pi/2 \leq u \leq \pi/2$;
b) $x = (1-t)^{1/2}, \quad y = t$, for $-1 \leq t \leq 1$.

# 17 Lecture 17

## 17.1

Calculate the surface integral $I = \int_S \mathbf{a} \cdot d\mathbf{S}$ where $\mathbf{a} = 2y\,\mathbf{j} + z\,\mathbf{k}$ and $S$ is the surface $x^2 + y^2 = 4$ in the first two octants bounded by the plane $z = 0$, $z = 5$ and $y = 0$.

## 17.2

Calculate the volume elements $dV$ for cylindrical polar coordinates given by $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$, then calculate the three vector area elements when $\rho$ is constant, $\phi$ is constant and $z$ is constant, respectively.

# 18 Lecture 18

## 18.1

A surface consists of that part of the cylinder $x^2 + y^2 = 9$ between $z = 0$ and $z = 4$ for $y \geq 0$ and the two semicircles of radius 3 in the plane $z = 0$ and $z = 4$. What is the boundary of this surface and its orientation when $d\mathbf{S}$ points out of the surface?

## 18.2

Verify the Stokes' theorem for the vector function $\mathbf{a} = xz\,\mathbf{j}$ and the surface $S$ defined by $\mathbf{r} = a\sin\theta\cos\phi\,\mathbf{i} + a\sin\theta\sin\phi\,\mathbf{j} + r\cos\theta\,\mathbf{k}$, with $0 \leq \phi \leq 2\pi$ and $0 \leq \theta \leq \alpha$.

# 19 Lecture 19

## 19.1

For spherical polar coordinates $\mathbf{r} = r\,\cos\phi\sin\theta\,\mathbf{i} + r\,\sin\phi\sin\theta\,\mathbf{j} + r\,\cos\theta\,\mathbf{k}$. Find the unit vectors $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\phi$, $\hat{\mathbf{e}}_\theta$ and the scale factors. Then calculate the curl of $\mathbf{a} = (\tan\theta/2)/r\,\hat{\mathbf{e}}_\phi$ $(\theta \neq \pi)$.

## 19.2

Paraboloid coordinates $u$, $v$ and $\phi$ are defined in terms of Cartesian coordinates by

$$x = uv\cos\phi, \qquad y = uv\sin\phi, \qquad z = \frac{1}{2}\left(u^2 - v^2\right).$$

Find the scale factors and the unit vectors $\hat{\mathbf{e}}_u$, $\hat{\mathbf{e}}_v$ and $\hat{\mathbf{e}}_\phi$, then show that the system of coordinates is orthogonal.