General Relativity

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Just intro stuff

Lecture 2 Introduction to Tensors

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2.1 Intro to Tensor Notation

Consider the cartesian definition for $\underline{r}$:

$$\underline{r} = x\underline{i} + y\underline{j} + \underline{z}. \tag{2.1}$$

We have the basis vector $\{\underline{i}, \underline{j}, \underline{k}\}$ and coordinate values $\{x, y, z\}$. We can write this in a different form as

$$\underline{r} = x^1\underline{e}_1 + x^2\underline{e}_2 + x^3\underline{e}_3. \tag{2.2}$$

Note: $x^2 \neq x * x$. The 2 is an index, not a power. If we want to square something, we will write $(x^1)^2 = x^1x^1$. We can rewrite the above again as

$$\underline{r} = \sum_{i=1}^{3} x^i\underline{e}_i. \tag{2.3}$$

We can then simplify this further using the **Einstein summation convention**:

$$\underline{r} = x^i\underline{e}_i, \tag{2.4}$$

i.e. whenever there is a repeated index, we sum over them. Different letters will imply different things:

- Roman letters $i, j, \ldots$ - summing over 3D space
- Roman letters $a, b, c, \ldots$ - summing over ND space
- Roman letters $A, B, \ldots$ - summing over 2D space
- Greek letters $\alpha, \beta, \mu, \nu, \ldots$ - summing over 4D space-time $\{x^0, x^1, x^2, x^3\}$, starting from 0 as time is different slightly, so $\{ct, x^i\}$

2.2 Coordinate Transformation

You may be used to

$$x' = \gamma\left(x - \frac{vct}{c}\right), \tag{2.5}$$

where the extra c factor to make time space-like. This notation can get confusing so instead we use:

$$x^{\bar{1}} = \gamma\left(x^1 - \frac{v}{c}x^0\right), \tag{2.6}$$

where the 'bar' indicates new coordinate system.

For a minute vector difference between points P and Q $d\underline{r}$ in two coordinate systems, we can define $\underline{e}_a$:

$$\underline{r}(P) = \underline{e}_a x^a \qquad \underline{r}(P) = \underline{e}_{\bar{b}} x^{\bar{b}} \tag{2.7}$$

$$d\underline{r} = dx^a\underline{e}_a \tag{2.8}$$

$$\frac{\partial\underline{r}}{\partial x^a} = \underline{e}_a \qquad \frac{\partial\underline{r}}{\partial x^{\bar{b}}} = \underline{e}_{\bar{b}} \tag{2.9}$$

But what is the relationship between these two coordinate systems? Start with $x^{\bar{b}} = x^{\bar{b}}(x^a)$, and consider a general function

$$f = f(x^1, x^2, x^3) \tag{2.10}$$
$$\Delta f = \frac{\partial f}{\partial x^1}\Delta x' + \frac{\partial f}{\partial x^2}\Delta x^2 + \frac{\partial f}{\partial x^2}\Delta x^3 = \frac{\partial f}{\partial x^a}\Delta x^a \tag{2.11}$$

How do we get a small change in $x^{\bar{b}}$?

$$\Delta x^{\bar{b}} = \frac{\partial x^{\bar{b}}}{\partial x^a}\Delta x^a \tag{2.12}$$
$$dx^{\bar{b}} = \frac{\partial x^{\bar{b}}}{\partial x^a}dx^a \tag{2.13}$$
$$dx^{\bar{a}} = \frac{\partial x^{\bar{a}}}{\partial x^b}dx^b \tag{2.14}$$

Notice how we can simply just switch round the indices - **these are all dummy variables and as long as the index notation is consistent, it is completely arbitrary which letter is used,** i.e. the letters themselves mean nothing.

2.3 Tensors

Any quantity which transforms as

$$A^{\bar{b}} = \frac{\partial x^{\bar{b}}}{\partial x^a}A^a \tag{2.15}$$

is a Rank (1,0) or order 1 contravariant tensor. What about $\underline{e}_a$?

$$\underline{r} = x^a \underline{e}_a = x^{\bar{b}} \underline{e}_{\bar{b}} \tag{2.16}$$
$$\underline{e}_{\bar{b}} = \frac{\partial \underline{r}}{\partial x^{\bar{b}}} = \frac{\partial \underline{r}}{\partial x^a}\frac{\partial x^a}{\partial x^{\bar{B}}} = \frac{\partial x^a}{\partial x^{\bar{b}}}\underline{e}_a \tag{2.17}$$

So now we have reversed the position of the indices in Eq (2.15).

How do we define scalars?

$$\nabla\phi = \frac{\partial \phi}{\partial x^i}\underline{e}_i \tag{2.18}$$
$$\frac{\partial \phi}{\partial x^{\bar{j}}} = \frac{\partial x^i}{\partial x^{\bar{j}}}\frac{\partial \phi}{\partial x^i} \tag{2.19}$$

In general, we have

$$A_{\bar{j}} = \frac{\partial x^i}{\partial x^{\bar{j}}}A_i, \tag{2.20}$$

which we call a Rank (0,1) or order 1 covariant tensor.

Lecture 3

3.1 Higher order tensors

Consider

$$T^{ab} = A^a B^b, \tag{3.1}$$

$$T^{\bar{c}\bar{d}} = A^{\bar{c}} B^{\bar{d}} = \left(\frac{\partial x^{\bar{c}}}{\partial x^a} A^a\right)\left(\frac{\partial x^{\bar{d}}}{\partial x^b} B^b\right) = \frac{\partial x^{\bar{c}}}{\partial x^a}\frac{\partial x^{\bar{d}}}{\partial x^b} A^a B^b = \frac{\partial x^{\bar{c}}}{\partial x^a}\frac{\partial x^{\bar{d}}}{\partial x^b} T^{ab}. \tag{3.2}$$

This is the definition of a second order contravariant tensor.

3.2 Tensor Equations

We can write a basic tensor equation,

$$T^a = k(A^a + B^a), \tag{3.3}$$

and wonder how this would look in a transformed coordinate system?

$$T^{\bar{b}} = \frac{\partial x^{\bar{b}}}{\partial x^a} T^a = k\left(\frac{\partial x^{\bar{b}}}{\partial x^a} A^a + \frac{\partial x^{\bar{b}}}{\partial x^a} B^a\right) \tag{3.4}$$

$$= k(A^{\bar{b}} + B^{\bar{b}}). \tag{3.5}$$

So if a tensor equation is true, it is true in all coordinate systems.

3.3 The metric tensor

What is the metric? *The metric is a measure of space.* We define the metric tensor,

$$g_{ab} = \underline{e}_a \cdot \underline{e}_b = g_{ba}, \tag{3.6}$$

so it is symmetric. We can use this when calculating spacetime distances:

$$ds^2 = \underline{dr} \cdot \underline{dr} = (dx^a \underline{e}_a) \cdot \left(dx^b \underline{e}_b\right) \tag{3.7}$$

$$= (\underline{e}_a \cdot \underline{e}_b) dx^a\, dx^b = g_{ab} dx^a\, dx^b. \tag{3.8}$$

Is it a tensor?

$$g_{\bar{a}\bar{b}} = (\underline{e}_{\bar{a}} \cdot \underline{e}_{\bar{b}}) = \left(\frac{\partial x^c}{\partial x^{\bar{a}}} \underline{e}_c\right) \cdot \left(\frac{\partial x^d}{\partial x^{\bar{b}}} \underline{e}_d\right) \tag{3.9}$$

$$= \frac{\partial x^c}{\partial x^{\bar{a}}}\frac{\partial x^d}{\partial x^{\bar{b}}} (\underline{e}_c \cdot \underline{e}_d) = \frac{\partial x^c}{\partial x^{\bar{a}}}\frac{\partial x^d}{\partial x^{\bar{b}}} g_{cd}, \tag{3.10}$$

so it transforms as a tensor; a second order covariant tensor.

3.4 Kronecker Delta

We can write an arbitrary vector as

$$\underline{A} = A^a \underline{e}_a = A^{\bar{b}} \underline{e}_{\bar{b}} = \left(\frac{\partial x^{\bar{b}}}{\partial x^a} A^a\right)\left(\frac{\partial x^d}{\partial x^{\bar{b}}} \underline{e}_d\right) \tag{3.11}$$

$$= \left(\frac{\partial x^{\bar{b}}}{\partial x^a}\frac{\partial x^d}{\partial x^{\bar{b}}}\right) A^a \underline{e}_d = \left(\frac{\partial x^d}{\partial x^a}\right) A^a \underline{e}_d \tag{3.12}$$

$$= \delta_a{}^d A^a \underline{e}_d = A^d \underline{e}_d = A^a \underline{e}_a \tag{3.13}$$

Lecture 4

Lecture 5

Lecture 6

Asbolute Derivative:

$$\frac{D\lambda^a}{ds} = \frac{d\lambda^a}{ds} + \Gamma^a{}_{bc}\lambda^b \frac{dx^c}{ds} \tag{6.1}$$

Covariant Derivative:

$$\lambda^a{}_{;c} = \frac{\partial \lambda^a}{\partial x^c} + \Gamma^a{}_{bc}\lambda^b \tag{6.2}$$

Christoffel Symbols:

$$\Gamma^c{}_{ab}\underline{e}_c = \frac{\partial \underline{e}_a}{\partial x^b}, \quad \Gamma^c{}_{ab} = \Gamma^c{}_{ba} \tag{6.3}$$

Other stuff:

$$\frac{\partial g_{ab}}{\partial x^c} = \Gamma^d{}_{ac}g_{bd} + \Gamma^d{}_{bc}g_{ad} \tag{6.4}$$
$$\frac{\partial g_{bc}}{\partial x^a} = \Gamma^d{}_{ba}g_{cd} + \Gamma^d{}_{ca}g_{bd} \tag{6.5}$$
$$\frac{\partial g_{ca}}{\partial x^b} = \Gamma^d{}_{cd}g_{ad} + \Gamma^d{}_{ab}g_{cd} \tag{6.6}$$
$$2\Gamma^d{}_{ac}g_{bd} = \frac{\partial g_{ab}}{\partial x^c} + \frac{\partial g_{bc}}{\partial x^a} - \frac{\partial g_{ca}}{\partial x^b} \tag{6.7}$$
$$\Gamma^f{}_{ac} = \frac{1}{2}g^{fb}\left(\frac{\partial g_{bc}}{\partial x^a} - \frac{\partial g_{ca}}{\partial x^b} + \frac{\partial g_{ab}}{\partial x^c}\right) \tag{6.8}$$
$$= \frac{1}{2}g^{fb}\left(\partial_a g_{bc} - \partial_b g_{ca} + \partial_c g_{ab}\right) \tag{6.9}$$

We multiplied lefthandside of (6.7) by $\delta^f{}_d$.

Example: 2D flat space
$x^A = \{x, y\}$:

$$g_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \text{diag}(1,1) \tag{6.10}$$
$$\Gamma^A{}_{BC} = 0 \tag{6.11}$$

So we don't have to deal with these in Cartesian coordinates. What about polar coordinates? $x^A = \{r, \theta\}$:

$$ds^2 = dr^2 + r^2\, d\theta^2 \tag{6.12}$$
$$g_{AB} = \text{diag}(1, r^2) \tag{6.13}$$
$$\Gamma^A{}_{BC} \neq 0 \tag{6.14}$$

So we can still get non-zero Christoffel symbols even for flat space, but it is still "boring" really.

Let's consider something more interesting, i.e. curved. For 3D space, we have

$$ds^2 = dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \tag{6.15}$$

But we want to use just the surface of a sphere, so fixed $r = a$:

$$ds^2 = a^2\, d\theta^2 + a^2 \sin^2\theta\, d\phi^2 = g_{AB}dx^A\, dx^B \tag{6.16}$$

$$g_{AB} = \text{diag}(a^2, a^2 \sin^2\theta) \tag{6.17}$$

We have g_{AB}, but we want g^{AB}. Recall

$$g^{AB} g_{BC} = \delta^A{}_C. \tag{6.18}$$

So we have a set of 4 simultaneous equations:

$$g^{A1} g_{1C} + g^{A2} g_{2C} = \delta^A{}_C. \tag{6.19}$$

For diagonal g_{AB} **ONLY**:

$$g^{AB} g_{BA} = g^{AA} g_{AA} = 1 \implies g^{AA} = \frac{1}{g_{AA}} \tag{6.20}$$

$$g^{AB} = \text{diag}\left(\frac{1}{a^2}, \frac{1}{a^2 \sin^2\theta}\right) \tag{6.21}$$

So now we want to calculate

$$\Gamma^\theta{}_{\theta\theta} = \frac{1}{2} g^{\theta B}\left(\partial_\theta g_{B\theta} - \partial_B g_{\theta\theta} + \partial_\theta g_{\theta B}\right), \quad g^{\theta B} = 0, B \neq \theta \tag{6.22}$$

$$= \frac{1}{2}\frac{1}{a^2}\left(\partial_\theta g_{\theta\theta} - \partial_\theta g_{\theta\theta} + \partial_\theta g_{\theta\theta}\right) = 0 \tag{6.23}$$

$$\Gamma^\theta{}_{\phi\theta} = \Gamma^\theta{}_{\theta\phi} = \frac{1}{2} g^{\theta B}\left(\partial_\theta g_{B\phi} - \partial_B g_{\phi\theta} + \partial_\phi g_{\theta B}\right) \tag{6.24}$$

$$= \frac{1}{2} g^{\theta\theta}\left(\partial_\theta g_{\theta\phi} - \partial_\theta g_{\phi\theta} + \partial_\phi g_{\theta\theta}\right) = 0 \tag{6.25}$$

$$\Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta \tag{6.26}$$

$$\Gamma^\phi{}_{\theta\phi} = \cot\theta \tag{6.27}$$

The rest of the Christoffel symbols for this example are 0 (there are $2^3 = 8$ in total?).

6.1 Geodesic Equations

The velocity is a tensor,

$$\underline{v} = v^\alpha \underline{e}_\alpha = \frac{\partial x^\alpha}{\partial \tau}\underline{e}_\alpha \tag{6.28}$$

If there's no force, then there's no change in the velocity vector doesn't change, but its components might change. No force means the absolute derivative of the components:

$$\frac{Dv^\alpha}{d\tau} = 0 \tag{6.29}$$

By an affine parameter, we mean a linear function of path length $u = A + Bs$, such as the proper time τ.

$$\frac{dv^\alpha}{d\tau} + \Gamma^\alpha{}_{\beta\gamma} v^\beta \frac{dx^\gamma}{d\tau} = 0 \tag{6.30}$$

$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0 \tag{6.31}$$

$$\ddot{x}^\alpha + \Gamma^\alpha{}_{\beta\gamma}\dot{x}^\beta \dot{x}^\gamma = 0 \tag{6.32}$$

Let's guess and make a solution for the sphere, $s = a\theta$, so we are just going around the circumference of the sphere at constant ϕ. For θ:

$$\frac{d^2\theta}{ds^2} + \Gamma^\theta{}_{BC}\frac{dx^B}{ds}\frac{dx^c}{ds} = 0 + \Gamma^\theta{}_{\phi\phi}\frac{d\phi}{ds}\frac{d\phi}{ds} = 0 \tag{6.33}$$

We get a big tick and a gold star! For ϕ:

$$\frac{d^2\phi}{ds^2} + \Gamma^\phi{}_{BC}\frac{dx^B}{ds}\frac{dx^C}{ds} = 0 + \Gamma^\phi{}_{\theta\phi}\frac{d\theta}{ds}\frac{d\phi}{ds} + \Gamma^\phi{}_{\phi\theta}\frac{d\phi}{ds}\frac{d\theta}{ds} = 0 \tag{6.34}$$

So it's a geodesic path! Yayyyyyy!

Lecture 7

Lecture 8

Last lecture:

- ➤ Euler-Lagrange equations
- ➤ 'easier' way to find ${\Gamma^a}_{bc}$
- ➤ how to find Geodesic paths

This lecture:

- ➤ more tensor derivatives

Recall absolute derivative again:

$$\frac{D\lambda^a}{ds} = \frac{d\lambda^a}{ds} + {\Gamma^a}_{bc}\lambda^b\frac{dx^c}{ds}. \tag{8.1}$$

The first term above is the total change, and then the second is to "subtract off the change due to the coordinate system". So for parallel transport, this means there is no physical change, i.e.

$$\frac{D\lambda^a}{ds} = 0. \tag{8.2}$$

The absolute derivative obeys normal rules for derivatives.

- ➤ Linear operator -

$$\frac{D}{ds}\left(\lambda^a + k\mu^a\right) = \frac{D\lambda^a}{ds} + k\frac{D\mu^a}{ds}. \tag{8.3}$$

- ➤ The (Leibniz) chain rule -

$$\frac{D}{ds}\left(\lambda^a\mu^b\right) = \mu^b\frac{D\lambda^a}{ds} + \lambda^a\frac{D\mu^b}{ds}. \tag{8.4}$$

What is the absolute derivative of a scalar, ϕ? ϕ does not depend on the coordinates as tensors do, so it would just be the normal derivative, i.e.

$$\frac{D\phi}{ds} = \frac{d\phi}{ds}. \tag{8.5}$$

We have defined the absolute derivative of a contravariant tensor, but now what about a covariant tensor μ_a? We can write a scalar as $\phi = \lambda^a\mu_a$, so we can write

$$\frac{D\phi}{ds} = \frac{D}{ds}(\lambda^a\mu_a) = \mu_a\frac{D\lambda^a}{ds} + \lambda^a\frac{D\mu_a}{ds}, \tag{8.6}$$

$$\frac{d\phi}{ds} = \mu_a\left(\frac{d\lambda^a}{ds} + {\Gamma^a}_{bc}\lambda^b\frac{dx^c}{ds}\right) + \lambda^a\frac{D\mu_a}{ds}, \tag{8.7}$$

$$\lambda^a\frac{d\mu_a}{ds} = \mu_a{\Gamma^a}_{bc}\lambda^b\frac{dx^c}{ds} + \lambda^a\frac{D\mu_a}{ds}, \tag{8.8}$$

$$= \mu_b{\Gamma^b}_{ac}\lambda^a\frac{dx^c}{ds} + \lambda^a\frac{D\mu_a}{ds}, \tag{8.9}$$

where λ^a is any tensor, so if this is true, it must be true for any λ^a. We can then 'cancel' λ^a through unity, as the remaining equation must also be true:

$$\frac{d\mu_a}{ds} = {\Gamma_{ac}}^b\mu_b\frac{dx^c}{ds} + \frac{D\mu_a}{ds}, \tag{8.10}$$

$$\frac{D\mu_a}{ds} = \frac{d\mu_a}{ds} - {\Gamma_{ac}}^b\mu_b\frac{dx^c}{ds}. \tag{8.11}$$

This is the absolute derivative of a convariant tensor.

What is the absolute derivative of a rank (1,1) tensor $\tau^a{}_b = \lambda^a \mu_b$?

$$\frac{D\tau^a{}_b}{ds} = \frac{D(\lambda^a \mu_b}{ds} = \mu_b \frac{D\lambda^a}{ds} + \lambda^a \frac{D\mu_b}{ds}, \tag{8.12}$$

$$= \mu_b \left[\frac{d\lambda^a}{ds} + \Gamma^a{}_{dc} \lambda^d \frac{dx^c}{ds} \right] + \lambda^a \left[\frac{d\mu_b}{ds} - \Gamma_{bc}{}^d \mu_d \frac{dx^c}{ds} \right], \tag{8.13}$$

$$\frac{D\tau^a{}_b}{ds} = \frac{d}{ds}(\lambda^a \mu_b) + \Gamma^a{}_{dc} \tau^d{}_b \frac{dx^c}{ds} - \Gamma_{bc}{}^d \tau_d{}^a \frac{dx^c}{ds}. \tag{8.14}$$

This is the absolute derivative for a rank (1,1) tensor.

What about the covariant derivative? It is defined as:

$$\lambda^a{}_{;c} = \frac{\partial \lambda^a}{\partial x^c} + \Gamma^a{}_{bc} \lambda^b \tag{8.15}$$

$$\frac{D\lambda^a}{ds} = \lambda^a{}_{;c} \frac{dx^c}{ds} \tag{8.16}$$

$$= \frac{\partial \lambda^a}{\partial x^c} \frac{dx^c}{ds} + \Gamma^a{}_{bc} \lambda^b \frac{dx^c}{ds} = \frac{d\lambda^a}{ds} + \dots. \tag{8.17}$$

All the rules still apply, so we can write out the covariant derivative for a scalar, a covariant, and a higher order tensor:

$$\phi_{;c} = \frac{\partial \phi}{\partial x^c}, \tag{8.18}$$

$$\mu_{a;c} = \frac{\partial \mu + a}{\partial x^c} - \Gamma_{ac}{}^b \mu_b, \tag{8.19}$$

$$\lambda_{ab;c} = \frac{\partial \lambda_{ab}}{\partial x^c} - \Gamma_{ac}{}^d \lambda_{db} - \Gamma_{bc}{}^d \lambda_{ad}. \tag{8.20}$$

We can also consider the metric:

$$g_{ab;c} = ? \tag{8.21}$$

$$\frac{\partial g_{ab}}{\partial x^c} = \frac{\partial}{\partial x^c}(\underline{e}_a \cdot \underline{e}_b), \tag{8.22}$$

$$= \frac{\partial \underline{e}_a}{\partial x^c} \cdot \underline{e}_b + \underline{e}_a \cdot \frac{\partial \underline{e}_b}{\partial x^c}, \tag{8.23}$$

$$= \left(\Gamma_{ac}{}^d \underline{e}_d \right) \cdot \underline{e}_b + \underline{e}_a \cdot \left(\Gamma_{bc}{}^d \underline{e}_d \right), \tag{8.24}$$

$$= \Gamma_{ac}{}^d g_{db} + \Gamma_{bc}{}^d g_{ad}, \tag{8.25}$$

$$g_{ab;c} = \frac{\partial g_{ab}}{\partial x^c} - \Gamma_{ac}{}^d g_{db} - \Gamma_{bc}{}^d g_{ad} = 0, \tag{8.26}$$

where we used the previous definitions of the covariant derivative to find this definition. Why is this important? The metric allows us to switch coordinate systems as

$$R_a = g_{ab} R^b. \tag{8.27}$$

Now suppose we want to find the covariant derivative:

$$R_{a;c} = \cancel{g_{ab;c} R^b} + g_{ab} R^b{}_{;c} \tag{8.28}$$

$$= g_{ab} R^b{}_{;c}. \tag{8.29}$$

Lecture 9

9.1 The Riemann Curvature Tensor

- how do we know if space is curved?
- second derivatives of the metric
- 'space-sing' around a loop
- convergence of geodesic paths

The Riemann curvature tensor tells us *how much the direction of a vector changes as it goes round a loop*, or the *tidal forces of gravity.*

$$\left(\lambda^a{}_{;b}\right)_{;c} = \frac{\partial \lambda^a{}_{;b}}{\partial x^c} + \Gamma^a{}_{ec}\lambda^e{}_{;b} - \Gamma_{bc}{}^f \lambda_{;f}{}^a \tag{9.1}$$

$$= \frac{\partial}{\partial x^c}\left(\frac{\partial \lambda^a}{\partial x^b} + \Gamma^a{}_{bd}\lambda^d\right) + \cdots \tag{9.2}$$

$$\lambda^a{}_{;b;c} = \frac{\partial^2 \lambda^a}{\partial x^c \partial x^b} + \Gamma^d{}_{bd}\frac{\partial \lambda^d}{\partial x^c} + \lambda^d \frac{\partial \Gamma^a{}_{bd}}{\partial x^c} + \Gamma^a{}_{ec}\lambda^a{}_{;b} - \Gamma_{bc}{}^f \lambda^a{}_{;f} \tag{9.3}$$

In flat space, the Christoffel symbols go to zero, so we can sway $;b$ and $;c$ indices, but **not in curved space.**

$$\lambda^a{}_{;c;b} = \frac{\partial^2 \lambda^a}{\partial x^b \partial x^c} + \Gamma^a{}_{cd}\frac{\partial \lambda^d}{\partial x^b} + \lambda^d \frac{\partial \Gamma^a{}_{cd}}{\partial x^b} + \Gamma^a{}_{eb}\lambda^e{}_{;c} - \Gamma_{cb}{}^f \lambda^a{}_{;f} \tag{9.4}$$

$$\lambda^a{}_{;c;b} - \lambda^a{}_{;b;c} = \left(\Gamma^a{}_{cd}\frac{\partial \lambda^d}{\partial x^b} - \Gamma^a{}_{bd}\frac{\partial \lambda^d}{\partial x^c}\right) + \lambda^d\left(\frac{\partial \Gamma_{cd}{}^a}{\partial x^b} - \frac{\partial \Gamma_{bd}{}^a}{\partial x^c}\right) + \left(\Gamma^a{}_{eb}\lambda^e{}_{;c} - \Gamma^a{}_{ec}\Gamma^e{}_{;b}\right) \tag{9.5}$$

$$= \Gamma^a{}_{cd}\left(\frac{\partial \lambda^a}{\partial x^b} - \lambda^d{}_{;b}\right) + \Gamma^a{}_{bd}\left(\lambda^d{}_{;c} - \frac{\partial \lambda^d}{\partial x^c}\right) + \lambda^d\left(\frac{\partial \Gamma^a{}_{cd}}{\partial x^b} - \frac{\partial \Gamma^a{}_{bd}}{\partial x^c}\right) \tag{9.6}$$

$$= \left(\Gamma^a{}_{be}\Gamma^e{}_{cd} - \Gamma^a{}_{ce}\Gamma^e{}_{bd} + \frac{\partial \Gamma^a{}_{cd}}{\partial x^b} - \frac{\partial \Gamma^a{}_{bd}}{\partial x^c}\right)\lambda^d = R^a{}_{dbc}\lambda^d \tag{9.7}$$

We have arrived at the Riemann curvature tensor. Consider: *doodle diagram*

$$\lambda^a(B^1) = \lambda^a(A) + \lambda^a{}_{;b}\delta x^b + \lambda^a{}_{;c}\delta y^c + \lambda^a{}_{;b;c}\delta x^b \delta y^c \tag{9.8}$$

$$\lambda^a(B^2) = \lambda^a(A) + \lambda^a{}_{;c}\delta y^c + \lambda^a{}_{;b}\delta x^b + \lambda^a{}_{;c;b}\delta y^c \delta x^b \tag{9.9}$$

$$\Delta\lambda^a = \lambda^a(B^2) - \lambda^a(B^1) = \left(\lambda^a{}_{;c;b} - \lambda^a{}_{;b;c}\right)\delta x^b \delta y^c \tag{9.10}$$

$$= R^a{}_{dbc}\lambda^a \cdot \text{ area of loop} \tag{9.11}$$

In flat space, any two lines have a separation that increases linearly with distance s, e.g. $\partial_s^2 = 0$. But in curved space, our two lines can converge or diverge as they travel from initial parallel conditions. Consider two lines $x^a(s)$ and $\tilde{x}^a(s)$ with a separation $\zeta^a = \tilde{x}^a - x^a$. We can write our geodesic equation for this as

$$\frac{d^2 x^a}{ds^2} + \Gamma^a{}_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds} = 0 \tag{9.12}$$

$$\frac{d^2 \tilde{x}^a}{ds^2} + \tilde{\Gamma}^a{}_{bc}\frac{d\tilde{x}^b}{ds}\frac{d\tilde{x}^c}{ds} = 0 \tag{9.13}$$

$$\ddot{\zeta}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{\zeta}^c + \Gamma^a{}_{bc}\dot{\zeta}^b\dot{x}^c + \frac{\partial \Gamma^a{}_{bc}}{\partial x^d}\zeta^d \dot{x}^b \dot{x}^c = 0 \tag{9.14}$$

$$\frac{D^2 \zeta^a}{ds^2} = \frac{D}{ds}\left(\dot{\zeta}^a + \Gamma^a{}_{bc}\zeta^b\dot{x}^c\right) \tag{9.15}$$

$$= \ddot{\zeta}^a + \Gamma^a{}_{bc}\zeta^b\dot{x}^c + \frac{d}{ds}\left(\Gamma^a{}_{bc}\zeta^b\dot{x}^c\right) + \Gamma^a{}_{ef}\left(\Gamma^e{}_{bc}\zeta^b\dot{x}^c\right)\dot{x}^f \tag{9.16}$$