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Stars and Galaxies

Stars

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Lecture 1

see DUO for slides Black body emission curve:

- LHS from peak lambda is Rayleigh Jeans tail
- RHS from peak is Wien tail

$$\lambda_{max} = \frac{2.9 \times 10^{-3}}{T} m \tag{1}$$
$$\lambda_{max.\,Bet} = 8.3 \times 10^{-7} m \implies T \approx 3500\,K \tag{2}$$
$$\lambda_{max,Sun} = 5.5 \times 10^{-7} m \implies T \approx 5300\,K \tag{3}$$
$$\lambda_{max,Bel} = 3.0 \times 10^{-7} m \implies T \approx 9400\,K \tag{4}$$
$$\tag{5}$$

Lecture 2

2.1 Excitation Energies

- Bohr model
- page 8 on slides
- n denotes the orbitals/electron shells
- $n = 1$ is the ground state

$$E = E_{high} - E_{low} = \frac{hc}{\lambda} = -13.6\Big(\frac{1}{n_{high}^2} - \frac{1}{n_{low}^2}\Big) \tag{6}$$
$$n = 2 \to 4 \tag{7}$$
$$E = 2.55\,eV \implies \lambda = 486.1\,nm \implies H\beta \tag{8}$$

- this was absorption
- $H\beta$ is shorthand for Balmer series β
- Optical light

$$n = 2 \to 1 \tag{9}$$
$$E = 10.2\,eV \implies \lambda = 121.6\,nm \implies Ly\alpha \tag{10}$$

- this was emission
- $Ly\alpha$ is shorthand for Lyman series α
    - UV light
- Photons emitted from de-excitation in random direction
    - statistics means we probably won't see this

<u>**Lecture 3**</u> Ratios of Excitation Levels

$$n = 2 \rightarrow 1 \tag{11}$$
$$\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-\frac{(E_2 - E_1)}{kT}} \tag{12}$$
$$g_1 = 2 \; ; \; g_2 = 8 \; ; \; T = 5800\,K \tag{13}$$
$$\frac{N_2}{N_1} = 5.1 \times 10^{-9} \tag{14}$$

- 1 billionth of H atoms in first excited state, negligible

3.1 Ionisation Energies

- χ is the ionisation energy

$$\frac{N_{i+1}}{N_i} = \frac{2Z_{i+1}}{n_e Z_i}\left(\frac{2\pi m_e kT}{h^2}\right)^{\frac{3}{2}} e^{-\frac{\chi}{kT}} E \qquad > -13.6\left(\frac{1}{\infty^2} - \frac{1}{n_{low}^2}\right) eV \tag{15}$$
$$n = 1 \rightarrow \infty \implies E > 13.6\,eV \tag{16}$$
$$n = 2 \rightarrow \infty \implies E > 3.4\,eV \tag{17}$$

<u>**Lecture 4**</u>

4.1 Binary Star Systems

- slide 8, binary system
- look at the semi-major axes of the orbits of the two stars around the centre of mass of the system
    - a_1 and a_2 for m_1 and m_2

$$P^2 = \frac{4\pi^2 a^3}{G(m_1 + m_2)} \tag{18}$$
$$a = a_1 + a_2 \tag{19}$$

- Smaller semi-major axis means larger mass
- similar to see-saw

$$m_1 a_1 = m_2 a_2 \implies \frac{m_1}{m_2} = \frac{a_2}{a_1} \tag{20}$$

- ratio of the semi-major axes gives ratio of masses
- actually measure α, angle of separation:
    - for d, distance from us

$$\alpha_n = \frac{a_n}{d} \implies \frac{m_1}{m_2} = \frac{\alpha_2}{\alpha_1} \tag{21}$$

4.2 Visual Binary Systems

Normal Example

- $d = 10\,pc$; $P = 200$ days
- $\alpha_1 = 0.02"$; $\alpha_2 = 0.08"$

$$a_1 = \alpha_1 d = 0.2\,Au\ ;\ a_2 = a_2 = \alpha_2 d = 0.8\,Au \tag{22}$$

$$a = a_1 + a_2 = 1\,Au \tag{23}$$

$$m_1 + m_2 = \frac{4\pi^2 a^3}{GP^2} = 3.4M_\odot = M_{tot} \tag{24}$$

$$\frac{m_1}{m_2} = \frac{\alpha_2}{\alpha_1} = \frac{a_2}{a_1} = 4.0 = M_{rot} \tag{25}$$

$$m_1 = \left[\frac{M_{rot}}{1+M_{rot}}\right] M_{tot} = 2.72M_\odot \tag{26}$$

$$m_2 = \left[\frac{1}{1+M_{rot}}\right] M_{tot} = 0.68M_\odot \tag{27}$$

Inclination Example

- For angled systems that aren't flat against our observations:

$$\hat{\alpha}_n = \alpha_n \cos i \tag{28}$$

$$m_1 + m_2 = \frac{4\pi^2}{G}\left(\frac{d}{\cos i}\right)\frac{\hat{\alpha}^3}{P^2} \tag{29}$$

$$\hat{\alpha} = \hat{\alpha}_1 + \hat{\alpha}_2 \tag{30}$$

- Has no effect on mass ratios observed - cos cancels
- Above equation means the actual masses will be affected by the inclination

4.3 Spectroscopic Binaries

- Correcting for inclination:

$$v_{nr}^{max} = v_n \sin i \tag{31}$$

- Assume $e << 1$

$$v_n = \frac{2\pi a_n}{P}\frac{m_1}{m_2} \qquad = \frac{v_2}{v_1} \tag{32}$$

- Same sort of stuff as visual binaries, but sin instead of cos basically

Special Case: Eclipsing Spectroscopic Binaries

- $i \approx 90°$
- don't need any corrections etc

<u>Lecture 5</u>

$$P = \underbrace{\frac{\rho k T}{\mu m_H}}_{\text{ideal gas law}} + \frac{1}{3}aT^4 \tag{33}$$

- Hydrostatic Equilibrium:

➥ Pressure force = Gravitational force

$$P\,on\,dA = [P(r+dr) - P(r)]dA \tag{34}$$
$$= dP\,dA \tag{35}$$
$$Gravitational = g\,\underbrace{\underbrace{dA\,dr}_{volume}\,\rho}_{mass},\;\; g = \frac{GM_r}{r^2} \tag{36}$$
$$dP\,dA = -g\rho\,dA\,dr \tag{37}$$
$$\frac{dP}{dr} = -\frac{GM_r\rho}{r^2} \tag{38}$$
$$\frac{dM_r}{dr} = 4\pi r^2\rho \tag{39}$$
$$M_r = \frac{4}{3}\pi r^2\rho \tag{40}$$
$$\frac{dP}{dr} = -G\frac{4}{3}\pi r\rho^2 \tag{41}$$
$$\int_{P_s}^{P_c} dP = -\frac{4}{3}\pi G\rho^2\int_R^0 r\,dr \tag{42}$$
$$P_c = \frac{2}{3}\pi G\rho^2 r^2,\;\; P_s = 0\,at\,r = R \tag{43}$$
$$= \frac{2}{3}\pi G r^2\left[\frac{3}{4}\frac{M}{\pi r^3}\right]^2 \tag{44}$$
$$= \frac{3}{8\pi}\frac{GM^2}{R^4} \tag{45}$$

➤ Example for our sun:

$$M = 2\times 10^{30}kg\;;\;\; R \approx 7\times 10^8 m \tag{46}$$
$$P_c \approx 10^{14}N\,m^{-2} \tag{47}$$
$$P_{c,\,true} \approx 2\times 10^{16}N\,m^{-2} \tag{48}$$

➤ out as assumed uniform density

Lecture 6

6.1 Virial Theorem

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2} \times V = \frac{4}{3}\pi r^3$$

$$V\frac{dP}{dr} = -\frac{GM\rho}{r^2}\frac{4}{3}\pi r^3$$

$$- \text{plug in } \frac{dm}{dr} = 4\pi r^2 \rho$$

$$V\frac{dP}{dr} = \frac{1}{3}\frac{GM}{r}\frac{dm}{dr}$$

$$\int_0^{P(R)} V\,dP = -\frac{1}{3}\underbrace{\int_0^M \frac{GM}{r}dm}_{\text{Total GPE}=U}$$

$$LHS: \int U\,dV = UV - \int V\,dU$$

$$\int_0^{P(R)} V\,dP = \underbrace{[PV]_0^{R_0}}_{=0} - \int_0^{V(R)} P\,dV = -\frac{1}{3}U$$

$$-3\int_0^{V(R)} P\,dV = U,\ dV = \frac{dm}{\rho} \implies$$

$$-3\int_0^M \frac{P}{\rho}dm = U \quad \text{- generalised form of Virial Theorem}$$

$$\text{Ideal Gas: } P = nkT = \frac{\rho kT}{\mu m_H}$$

$$\text{Average KE: } = \frac{3}{2}kT$$

$$\text{KE per kilo: } = \frac{3}{2}\frac{kT}{\mu m_H}$$

$$E_{KE} = \frac{3}{2}\frac{kT}{\mu m_H} = \frac{3}{2}\frac{P}{\rho}$$

$$-3\int_0^M \frac{P}{\rho}dm = U,\ \frac{P}{\rho} = \frac{2}{3}E_{KE}$$

$$\underbrace{\int_0^M E_{KE}\,dm}_{\text{Total KE, assume ideal gas}} = -\frac{1}{2}U$$

$$\implies K = -\frac{1}{2}U$$

6.1.1 Energy from Gravitational Collapse

$$
\begin{aligned}
dU_{g,i} &= -\frac{GM_r dm_i}{r} \text{ - GPE of point mass} \\
&\text{Consider shells of material} \\
dm &= 4\pi r^2 \rho dr \\
dU_g &= -\frac{GM_r 4\pi r^2 \rho}{r} dr \text{ - GPE of a shell} \\
U_g &= -4\pi G \int_0^R M_r \rho_r dr \\
M_r &= \frac{4}{3}\pi r^3 \bar{\rho} \text{ - avg density isn't too bad here} \\
U_g &= -\frac{16}{3}\pi^2 G \bar{\rho}^2 \int_0^R r^4 dr \\
&= -\frac{16}{15}\pi^2 G \bar{\rho}^2 R^5 \\
&\text{Convert back to mass} \\
U &= -\frac{9}{15}\frac{GM^2}{R} \text{ - GPE of the star} \\
K &= -\frac{1}{2}U \\
\implies E &= \frac{3}{10}\frac{GM^2}{R} \\
E &\approx \frac{3}{10}GM^2\Big[\frac{1}{R} - \frac{1}{R_{initial}}\Big] \\
&= \frac{3}{10}\frac{GM^2}{R} \iff R << R_{initial}
\end{aligned}
$$

6.2 Lecture 6

6.2.1 Binding Energies of Fusion

$$
\begin{aligned}
E_b(Z,N) &= \Delta mc^2 = [Zm_p + Nm_n - m(Z,N)]c^2 \\
E_b(4,0) &= [4m_p - m_{He,4}]c^2 = 26.731\, MeV \\
\frac{4m_p}{m_{He,4}} &= 1.007 \implies e = 0.7\% \\
E_\odot &= (0.1 \times M_\odot) \times 0.007 \times c^2 \\
&= 1.3 \times 10^{44} J \\
t &\approx \frac{E_\odot}{L_\odot} = 10^{10} yr
\end{aligned}
$$

6.2.2 Coulomb Barrier

- looking at probability that two particles are close enough for nuclear force to be important
- see figure on page 7 of slides
- using classical physics, we get

$$E = \frac{1}{2}mv^2 = \frac{3}{2}kT = \frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{r}$$

$$T = \frac{1}{6\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{rk} = \underbrace{1.1 \times 10^{10} K}_{r=10^{-15}m\ ;\ Z_1=Z_2=1}$$

- too high for our Sun
- use deBroglie wavelength and consider quantum effects

$$\begin{aligned}
\lambda &= \frac{h}{p},\ p = mv\ [m = \mu_m] \\
E &= \frac{1}{2}mv^2\ ;\ v^2 = \frac{p^2}{m^2} \\
E &= \frac{p^2}{2m} \\
p^2 &= \left(\frac{h}{\lambda}\right)^2 \\
E &= \frac{(\frac{h}{\lambda})^2}{2m} = \frac{h^2}{\lambda^2}\frac{1}{2m} \\
&= \frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{\lambda} = \frac{h^2}{\lambda^2}\frac{1}{2m} \\
\frac{1}{\lambda} &= \frac{2}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2 m}{h^2} \\
\text{replace } \frac{1}{r} &\text{ with } \frac{1}{\lambda} \\
T &= \frac{1}{12\pi^2\epsilon_0^2}\frac{Z_1^2 Z_2^2 e^4 m}{kh^2} = 9.8 \times 10^6 K
\end{aligned}$$

- this happens due to quantum tunneling

6.2.3 Probability of Nuclear Reactions

- see graph on page 13 of slides
- nuclear reaction probability is the product of Maxwell-Boltzmann and Tunneling Probability

6.3 Lecture 7

6.3.1 Nuclear Conservation Rules

1. electric charge must be conserved
2. nucleon umber must be conserved
   - $p, n = +1$
3. lepton number must be conserved
   - $e^{\mp} = \pm 1$
   - $\nu_e^{\mp} = \pm 1$

$${}^{A}_{Z}X$$

- A - atomic number for element X (nucleon number)
- Z - number of protons (electric charge)

6.3.2 Proton-Proton Chains

$$\begin{aligned}
&{}^1_1H + {}^1_1H \to {}^2_1H + e^+ + \nu_e \\
&{}^2_1H + {}^1_1H \to {}^3_2He + \gamma \\
&{}^3_2He + {}^3_2He \to {}^4_2He + {}^1_1H + {}^1_1H \\
\implies &4{}^1_1H \to {}^4_2He + \underbrace{2e^+ + 2\nu_e + 2\gamma}_{26.7\,MeV}
\end{aligned}$$

6.3.3 CNO Cycle

$$\begin{aligned}
&{}^{12}_6C + {}^1_1H \to {}^{13}_7N + \gamma \\
&{}^{13}_7N \to \underbrace{{}^{13}_6C + e^+ + \nu_e}_{\beta\text{ decay}} \\
&{}^{13}_6C + {}^1_1H \to {}^{14}_7N + \gamma \\
&{}^{14}_7N + {}^1_1H \to {}^{15}_8O + \gamma \\
&{}^{15}_8O \to \underbrace{{}^{15}_7N + e^+\nu_e}_{\beta\text{ decay}} \\
&{}^{15}_7N + {}^1_1H \to {}^{12}_6C + {}^4_2He \\
\text{Total: } &4{}^1_1H \to {}^4_2He + \underbrace{2e^+ + 2\nu_e + 3\gamma}_{E=26.7\,MeV}
\end{aligned}$$

6.4 Lecture 8

6.4.1 Energy produced in Stars

$$\begin{aligned}
dL &= \epsilon\, dm \quad [W] \\
\epsilon_{i,X} &= \epsilon_0 X_i X_X \rho^\alpha T^\beta \quad [W\, kg^{-1}] \\
dm &= 4\pi r^2 \rho\, dr \\
\implies \frac{dL}{dr} &= 4\pi r^2 \rho \epsilon
\end{aligned}$$

Slide 5 diagram

- Solid line just to do with fusion then no fusion
- Dashed line has that shape as volume increase so dL/dr does but then temperature starts falling so fusion decreases

6.4.2 Energy Seen on Earth

- Electrons lose energy travelling through sun

$$\frac{\lambda_{surface}}{\lambda_{core}} \approx 3 \times 10^6$$

6.4.3 Mean Free Paths

- vt - distance travelled

- n - particles per unit volume
- nvt - particle per unit area
- $n\sigma vt$ - number of interactions

$$\begin{aligned} l &= \frac{vt}{n\sigma vt} \\ &= \frac{1}{n\sigma} \end{aligned}$$

- This is the mean distance before a collision

$$\begin{aligned} d &= \sum_i l_i \\ d^2 &= d \cdot d \\ &= \sum_j \sum_i l_i \cdot l_j \end{aligned}$$

- When $i \neq j$, $l_i \cdot l_j = 0$

$$\begin{aligned} d^2 &= Nl^2 \\ \implies N &= \left(\frac{d}{l}\right)^2 \end{aligned}$$

- Use ideal gas law to help in questions of this

$$\begin{aligned} t_{total} &= t_{travel} + Nt_{scatter} \\ &= \frac{Nl}{c} + N \times 10^8 \\ &= 5700\, yrs + \cdots = 10^6\, yrs \end{aligned}$$

6.4.4 Radiation

$$\begin{aligned} P &= \frac{1}{3}aT^4 \\ \frac{dP}{P}dr &= \frac{dP}{dT}\frac{dT}{dr} \\ \frac{dP}{dr} &= \frac{4}{3}aT^3\frac{dT}{dr} \\ \frac{dP}{dr} &= -\frac{\kappa\rho}{c}F_{rad} \\ \kappa rho &= n\sigma \\ \frac{dT}{dr} &= -\frac{3}{4ac}\frac{\kappa\rho F_{rad}}{T^3} \\ L &= 4\pi r^2 F_{rad} \\ \frac{dT}{dr} &= -\frac{3}{16\pi ac}\frac{\kappa\rho L_r}{T^3 r^2} \end{aligned}$$

6.5 Lecture 9

6.5.1 Opacity

$$
\begin{aligned}
dI_\lambda &= -\kappa_\lambda \rho I_\lambda ds \\
\int_{I_{\lambda,0}}^{I_{\lambda,f}} \frac{dI_\lambda}{I_\lambda} &= -\int \kappa_\lambda \rho ds \\
\implies I_{\lambda,f} &= I_{\lambda,0} e^{-\int_0^s \kappa_\lambda \rho ds} \\
I_{\lambda,f} &= I_{\lambda,0} \quad \underbrace{e^{-\kappa_\lambda \rho s}}_{\text{optical depth, } \tau} \\
&= I_{\lambda,0} e^{-\tau},\ \tau = \kappa_\lambda \rho s
\end{aligned}
$$

- $\tau < 1$ - optically thin
- $\tau > 1$ - optically thick

Different sources of Opacity

- Two classes of opacity:
    1. Absorption - photon energy lost of KE of gas or degraded
    2. Scattering - photon reemitted at different direction, sometimes degraded

1. Bound-Bound transitions
    - typical temperature roughly $\leq 10^5$K
    - most effective for neutral gas
    - scattering and absorption
2. Bound-free transitions
    - typical temperature of $10^4 \to 10^6$K
    - partially ionised gas
    - absorption
3. Free-free emission
    - typical temperature of $10^4 \to 10^6$K
    - partially ionised gas
    - absorption
4. Electron scattering
    - dominant at roughly $\geq 10^6$K
    - fully ionised gas
    - scattering

6.6 Lecture 10

6.6.1 Schwarzchild Criterion for Convection

- slide 4 - 9

$$
\gamma = \frac{C_p}{C_V} = \frac{s+2}{s}
$$

- s is degrees of freedom

$$
\begin{aligned}
P &= k_a \rho^{\gamma} \\
\frac{dP}{P} &= \frac{\gamma d\rho}{\rho} \\
\gamma &= \frac{\rho}{P}\frac{dP}{d\rho}
\end{aligned}
$$

Surrounding gas

$$
\begin{aligned}
P &= nkT = \frac{\rho k T}{\mu m_H} \\
\frac{dP}{P} &= \frac{d\rho}{\rho} + \frac{dT}{T} \\
\frac{d\rho}{\rho} &= \frac{dP}{P} - \frac{dT}{T} \\
\frac{dP}{d\rho}_{sur} &> \frac{dP}{d\rho}_{adiab} \Bigg[\times \frac{\rho}{P} \\
\frac{\rho}{P}\frac{dP}{d\rho}_{sur} &> \frac{\rho}{P}\frac{dP}{d\rho}_{adiab} \\
\frac{\rho}{P}\frac{dP}{d\rho}_{sur} &> \gamma_{ad} \\
\frac{P}{dP}\left(\frac{dP}{P} - \frac{dT}{T}\right)_{sur} &< \frac{1}{\gamma_{adiab}} \\
\frac{P}{dP}\frac{dP}{P} - \frac{P}{dP}\frac{dT}{T} &< \frac{1}{\gamma_{adiab}} \\
1 - \left(\frac{P}{dP}\frac{dT}{T}\right)_{sur} &< \frac{1}{\gamma_{adiab}} \\
\frac{T}{P}\left(\frac{dP}{dT}\right)_{sur} &< \frac{\gamma_{adiab}}{\gamma_{adiab} - 1} \\
\left|\frac{dT}{dr}\right|_{sur} &> \left(\frac{\gamma_{adiab} - 1}{\gamma_{adiab}}\right)\frac{T}{P}\left|\frac{dP}{dr}\right|_{sur}
\end{aligned}
$$

Convection in the Sun For the sun:

$$
\begin{aligned}
-\frac{3}{16\pi ac}\frac{k\rho L_r}{T^3 r^2} &> \left(\frac{\gamma - 1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr} \\
\frac{dP}{dr} &= -\frac{GM_r\rho}{r^2} \\
\frac{L_r}{M_r} &> \frac{16\pi acG}{\kappa\rho}\frac{aT^4}{3}\frac{\gamma - 1}{\gamma} \\
&> \frac{16\pi acG}{\kappa\rho}P_{rad}\frac{\gamma - 1}{\gamma} \\
&> 1.9 \times 10^{-3}\, W\, kg^{-1}
\end{aligned}
$$

Mixing length

$$l = \alpha Hp$$

$$\frac{dP}{dr} = -\frac{GM_r\rho}{r^2} \implies \frac{1}{Hp} = -\frac{1}{P}\frac{dP}{dr}$$

$$Hp = \frac{Pr^2}{GM_r\rho}$$

$$l = \frac{\alpha Pr^2}{GM_r\rho}$$

6.7 Lecture 12

6.7.1 Cepheid Variables

$$\log\left(\frac{L}{L_\odot}\right) = 1.15\log_{10}\Pi^d + 2.47$$

$$\Pi^d = 10\,\text{days} \implies L = 4200\,L_\odot$$

$$\text{observed} < f > = 10^{-15} W\,m^{-2}$$

$$L = 4\pi d^2 < f >$$

$$d = \sqrt{\frac{L}{4\pi < f >}}$$

6.7.2 Stellar Pulsation

$$V_s = \sqrt{\frac{\gamma P}{\rho}},\ \gamma = \frac{C_p}{C_V}$$

$$\Pi = 2\int_0^R \frac{dr}{V_s}$$

$$\frac{dP}{dr} = -\frac{GM_r\rho}{r^2}$$

$$\text{const } p \implies \mu = \frac{4}{3}\pi r^3\rho$$

$$\frac{dP}{dr} = -\frac{4}{3}G\pi r\rho^2$$

$$dP = -\frac{4}{3}G\pi\rho^2\int_0^R r\,dr$$

$$P(r) = \frac{4}{3}G\pi\rho^2\left[\frac{R^2}{2} - \frac{r^2}{2}\right]$$

$$\begin{aligned}\Pi &= 2\int_0^R \frac{dr}{V_s} \\ &= 2\int_0^R \frac{dr}{\sqrt{\frac{2}{3}\gamma G\rho(R^2 - r^2)}} \\ &= 2\sqrt{\frac{3}{2\gamma\pi G\rho}}\left[\sin^{-1}\left(\frac{r}{R}\right)\right]_0^R \\ &= \sqrt{\frac{3\pi}{2G\rho\gamma}}\end{aligned}$$

6.8 Lecture 13

6.8.1 Jeans Mass

➤ For the gravitational collapse of a gas cloud:

$$
\begin{aligned}
GE = U &= -\frac{3}{5}\frac{GM^2}{R} \\
KE = K &= \frac{3}{2}NkT \\
&= \frac{3}{2}\frac{M_c}{\mu m_H}kT \\
2K &< |U| \\
2\left(\frac{3}{2}\frac{M_c kT}{\mu m_H}\right) &< \frac{3}{5}\frac{GM_c^2}{R_c} \\
R_c &= \left(\frac{3}{4}\frac{M_c}{\pi\rho_0}\right)^{\frac{1}{3}} \\
2\left(\frac{3}{2}\frac{M_c kT}{\mu m_H}\right) &< \frac{3}{5}GM_c^2\left(\frac{4}{3}\frac{\pi\rho_0}{M_c}\right)^{\frac{1}{3}} \\
\frac{5M_c kT}{\mu m H G} &< M_c^2\left(\frac{4}{3}\frac{\pi\rho_0}{M_c}\right)^{\frac{1}{3}} \\
M_c &< M_J \\
M_J &\approx \left(\frac{5kT}{G\mu m_H}\right)^{\frac{3}{2}}\left(\frac{3}{4\pi\rho_0}\right)^{\frac{1}{2}}
\end{aligned}
$$

6.8.2 Free-fall gravitational collapse

1. $M_c > M_J$
   - ➤ free fall collapse
   - ➤ optically thin
   - ➤ pressure increase
   - ➤ temperature constant
2. Fragmentation
   - ➤ optically thin
   - ➤ individual regions exceed local M_J
3. M_J minimised: Protostar
   - ➤ optically thick
   - ➤ pressure increase
   - ➤ temperature increase
   - ➤ Slow contraction (Kelvin-Helmholtz timescale)

6.9 Lecture 14

6.9.1 Stellar Evolution

1. Increase in μ (mean molecular mass) with time:

$$P = nkT = \frac{\rho k T}{\mu m_H}$$

As μ increases, ρ and T also increase for the pressure to remain constant.

Recall:

$$\epsilon_{i,X} = \epsilon_0 X_i X_X \rho^\alpha T^\beta, \alpha \approx 1$$

For proton-proton chain, $\beta \approx 4$
For CNO, $\beta \approx 17$

Luminosity increases with time.

6.9.2 Lifetime of Nuclear Fusion

$$\begin{aligned}
t &= \frac{E_{tot}}{L} = \frac{X\zeta Mc^2}{L} \\
\zeta_{pp} &= \frac{4m_p - m_{He}}{m_{He}} \approx 0.007 \\
t_\odot &= 10^{10} \text{ yrs} \\
L_{ms} &= L_\odot \left(\frac{M_\odot}{M}\right)^\alpha \\
t_{ms} &= \frac{X\zeta Mc^2}{L_\odot} \left(\frac{M_\odot}{M}\right)^\alpha \\
&= 10^{10} \frac{M}{M_\odot} \left(\frac{M_\odot}{M}\right)^\alpha \\
\therefore t_{ms} &= 10^{10} \left(\frac{M_\odot}{M}\right)^{\alpha - 1}
\end{aligned}$$

6.10 Lecture 15

6.10.1 Eddington Limit

$$\begin{aligned}
L_{Edd} &= \frac{4\pi c G M}{\kappa}, M = 100 M_\odot, \kappa = \kappa_{es} = 0.04\, kg\, m^{-2} \\
&= 3 \times 10^6 L_\odot
\end{aligned}$$

6.10.2 Photodisintegration

$$\begin{aligned}
\lambda_{max} &= \frac{2.9 \times 10^{-3}}{T},\ E = \frac{hc}{\lambda} \\
T_c &\geq 3 \times 10^9 K \implies E \geq 1\, MeV
\end{aligned}$$

Last Days of Fusion

- Shell fusion
- Silicon to Iron in Core
- P_{core} = high

Endothermic Release

- Iron breaking down into Helium and Helium breaking down in protons and neutrons
- still shell fusion ongoing
- P_{core} = rapidly decreasing

Electron capture

- very high density
- shell fusion
- $p + e^- \implies n + \nu_e$
- P_{core} = rapidly decreasing
- neutrino burst

Rapid core collapse

- shell fusion
- $P_{core} \approx 0$

Core rebound

- shell fusion
- $\rho > 8 \times 10^{18}\, kg\, m^{-3}$
- the strong force repels collapse and rebounds outwards

Supernova

- previous step drives supernova
- strong force drives high energy pushing
- generates a shock wave - more photodisintegration
- electron capture repeats and another neutrino burst
- nuclear synthesis of heavier elements, including beyond iron (endothermic)

6.11 Lecture 16

6.11.1 Electron Degeneracy Pressure

$$\Delta x \Delta p_x \approx \hbar$$
$$p_m in \approx \Delta p_x \approx \frac{\hbar}{\Delta x}$$
$$P \approx \frac{1}{2} n_e p v$$
$$n_e = \frac{\#e}{vol} = \frac{Z}{A} \frac{\rho}{m_H}$$
$$p_x = \Delta p_x = \frac{\hbar}{\Delta x}$$
$$\Delta x = n_e^{-1/3} \implies p_x = \hbar n_e^{1/3}$$
$$p^2 = p_x^2 + p_y^2 + p_z^2 = 3p_x^2$$
$$\implies p = \sqrt{3} p_x = \sqrt{3} \hbar n_e^{1/3}$$
$$p = mv = m_e v$$
$$\implies v = \frac{p}{m_e} = \frac{\sqrt{3}}{m_e} \hbar n_e^{1/3}$$
$$P = \frac{1}{3} n_e p v$$
$$p = \sqrt{3} \hbar \left[\left(\frac{Z}{A} \right) \frac{\rho}{m_H} \right]^{1/3}$$
$$v = \frac{\sqrt{3}}{m_e} \hbar \left[\left(\frac{Z}{A} \right) \frac{\rho}{m_H} \right]^{1/3}$$
$$\therefore P = \frac{\hbar^2}{m_e} \left[\left(\frac{Z}{A} \right) \frac{\rho}{m_H} \right]^{5/3}$$

6.11.2 White Dwarf Cooling

$$t_{cool} = \frac{E_{WD}}{L_{WD}} = \left(\frac{3kT_{c,WD}}{2} \right) \left(\frac{M_{WD}}{A m_H} \right) \left(\frac{1}{L_{WD}} \right)$$

6.12 Lecture 17

6.12.1 Rotation Period of Pulsars

$$\text{Centripetal Acceleration} = \text{Gravitational Acceleration}$$
$$\omega_{max}^2 R = \frac{GM}{R}$$
$$M = \frac{4}{3} \pi R^3 \rho$$
$$\omega_{max}^2 R = G \frac{4}{3} \pi R \rho$$
$$\omega = 2\pi f = \frac{2\pi}{P}$$
$$\frac{4\pi^2}{P^2} R = \frac{4}{3} G \pi R \rho$$
$$P_{min} = \left(\frac{3\pi}{G\rho} \right)^{1/2}$$

6.12.2 Stellar Core Rotation

Conservation of angular momentum:

$$\begin{aligned}
I_i\omega_i &= I_f\omega_f,\ I = CMR^2 \\
CMR_i^2\omega_i &= CMR_f^2\omega_f,\ \omega = \frac{2\pi}{P} \\
\frac{2\pi}{P_f} &= \frac{2\pi}{P_i}\left(\frac{R_i}{R_f}\right)^2 \\
P_f &= P_i\left(\frac{R_f}{R_i}\right)^2
\end{aligned}$$