Particle Theory

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Contents

Part I

Relativistic Quantum Mechanics

Lecture 1 Recapitulation of important ingredients

1.1 Natural Units

$$\hbar = c = 1 \tag{1.1}$$

$$\text{energy} = \frac{1}{\text{length}} \tag{1.2}$$

$$\hbar c = 200\,\text{MeV}\cdot\text{fm} \tag{1.3}$$

$$\text{energy} = \text{momentum}\ (c = 3\times 10^8\ \text{ms}^{-1} = 3\times 10^{23}\ \text{fm s}^{-1}) \tag{1.4}$$

1.2 Four Vectors

$$p^{\mu=\{0,\ldots,3\}} = (E, \underline{p}) \tag{1.5}$$

$$p^\mu p_\mu = E^2 - \underline{p}^2 = E^2 - p_i p_i \tag{1.6}$$

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g_{\mu\nu} \tag{1.7}$$

$$g^\mu_\nu = I \tag{1.8}$$

$$\frac{\partial}{\partial x^\mu} = \partial_\mu \tag{1.9}$$

$$\frac{\partial}{\partial x_\mu} = \partial^\mu \tag{1.10}$$

$$\frac{\partial}{\partial x^\mu}(x\cdot p) = p_\mu \tag{1.11}$$

$$\frac{\partial}{\partial x_\mu}(x\cdot p) = p^\mu \tag{1.12}$$

Kronecker delta:

$$\delta_{ij} = \delta^{ij} = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases} \tag{1.13}$$

Levi-Civita tensor:

$$\epsilon_{ijk} = \epsilon^{ijk} = \begin{cases} 1 & \{ijk\}\ \text{cyclical perm of 123} \\ -1 & \{ijk\}\ \text{anti-cyclical perm} \\ 0 & \text{otherwise} \end{cases} \tag{1.14}$$

Anti-symmetric tensor:

$$\epsilon_{\mu\nu\rho\sigma} = \epsilon^{\mu\nu\rho\sigma} = \begin{cases} 1 & \{\mu\nu\rho\sigma\}\ \text{cyclical perm of 0123} \\ -1 & \{\mu\nu\rho\sigma\}\ \text{anti-cyclical perm} \\ 0 & \text{otherwise} \end{cases} \tag{1.15}$$

1.3 Lorentz Transformation

Boosts and rotations:

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \tag{1.16}$$

Lorentz boost along z-axis:

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \cosh\nu & 0 & 0 & -\sinh\nu \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\nu & 0 & 0 & \cosh\nu \end{pmatrix} \tag{1.17}$$

where rapidity

$$\cosh\nu = \gamma - \frac{1}{\sqrt{1-\nu^2}} \tag{1.18}$$

1.4 Lagrange formalism

$$L\left(q(t), \dot{q}(t)\right) = T - V \tag{1.19}$$

$$S(t, t_0) = \int_{t_0}^{t} L\ dt' \tag{1.20}$$

Minimise action, S, leads to Euler-Lagrange equations of motion:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \tag{1.21}$$

Introduce Hamilton function:

$$H(p,q) = \dot{q}\frac{\partial L}{\partial \dot{q}} - L = T + V,\ \dot{q} \to p = \frac{\partial L}{\partial \dot{q}} \tag{1.22}$$

$$\dot{q} = \frac{\partial H}{\partial p},\ \dot{p} = -\frac{\partial H}{\partial q} \tag{1.23}$$

1.5 Harmonic Oscillator, 1st quantisation

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 \tag{1.24}$$

$$[\hat{x}, \hat{p}] = i = \hat{x}\hat{p} - \hat{p}\hat{x} \tag{1.25}$$

Introducing the annihilation and creation operators:

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\sqrt{\omega}\hat{x} + \frac{i}{\sqrt{\omega}}\hat{p}\right) \tag{1.26}$$

$$\hat{a}^{\dagger} = \frac{1}{\sqrt{2}}\left(\sqrt{\omega}\hat{x} - \frac{i}{\sqrt{\omega}}\hat{p}\right) \tag{1.27}$$

$\hat{x}$,$\hat{p}$,$\hat{H}$ are Hermitian, so

$$[\hat{a}, \hat{a}^{\dagger}] = 1 \tag{1.28}$$

$$[\hat{a}, \hat{a}] = [\hat{a}^{\dagger}, \hat{a}^{\dagger}] = 0 \tag{1.29}$$

$$\hat{H} = \omega\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right) = \omega\left(\hat{N} + \frac{1}{2}\right) \tag{1.30}$$

$$\hat{N}|n\rangle = n|n\rangle \tag{1.31}$$
$$[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger \tag{1.32}$$
$$[\hat{N}, \hat{a}] = -\hat{a} \tag{1.33}$$
$$\hat{H}|E\rangle = E|E\rangle \tag{1.34}$$
$$\hat{H}\left(\hat{a}|E\rangle\right) = \left(\hat{a}\hat{H} + \hat{H}\hat{a} - \hat{a}\hat{H}\right)|E\rangle \tag{1.35}$$
$$= aE|E\rangle + \omega[\hat{N}, \hat{a}]|E\rangle \tag{1.36}$$
$$= (E - \omega)\hat{a}|E\rangle \tag{1.37}$$

Eigenvalues of Hermitian operators are real numbers, therefore the eigenvalues of their squares cannot be negative $\implies$ there must be a lowest state $|0\rangle$ (the ground state) such that

$$\hat{a}|0\rangle = 0 \implies E_0 = \frac{\omega}{2} \tag{1.38}$$

Lecture 2

2.1 Lagrange Formalism for Point Particles

Put point particles along one dimension at even intervals, $i = 1, 2, \ldots$. Can write a Lagrangian for the system which is the Lagrangian for all the points and their relative veolicties, $\mathcal{L}(q_i, \dot{q}_i, t)$.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \tag{2.1}$$

Now instead of these discrete labels, call the one dimension a continous "label" x. Now you have a chain where each "x" is a point on a continous chain, $\mathcal{L}\left(q(x), \dot{q}(x), t\right)$. Note that the time dependence is implicit in each variable of the Lagrangian - this reduces the discrete case to a function of one variable, however more complicated in the continous case.
This $q(x)$ is not a particle like the q_i, but a field.

2.2 Fields

Scalar fields are real- or complex-valued functions of space time.

$$\phi(\underline{x}, t) \in \mathbb{R}, \mathbb{C} \tag{2.2}$$

Now defining fields, the Lagrange function is now an integral of the Lagrange density, and then can consider the action principle.

$$L\left[\phi, \partial_\mu \phi\right] = \int \mathcal{L}\left(\phi(\underline{x}, t), \partial_\mu \phi(\underline{x}, t), t\right)\, d^3x \tag{2.3}$$

$$S(t, t_0) = \int_{t_0}^{t} L\, dt = \int_{t_0}^{t} \mathcal{L}\, d^4x \tag{2.4}$$

The dimension of the action must be zero as it gets exponentiated, and must result in zero mass.

$$\dim[S] = 0 \qquad\qquad \dim[dx] = -1 \tag{2.5}$$

These imply that the Lagrangian has mass dimension of 4.

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2}\phi^2 \tag{2.6}$$

By construction, the following holds:

$$\dim[\partial_\mu] = 1 \qquad\qquad \dim[\phi] = 1 \tag{2.7}$$

Now, following on using similar logic to Hamilton's principle and canonical conjugates for (2.6),

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \leftrightarrow \dot{\phi} \tag{2.8}$$

$$\mathcal{H} = \dot{\phi}\pi - \mathcal{L} = \frac{\pi^2}{2} + \frac{(\nabla\phi)^2}{2} + m^2\phi^2 \tag{2.9}$$

Now for the Equations of Motion:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{2.10}$$

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0 \tag{2.11}$$

$$\Box \phi + m^2 \phi = 0 \tag{2.12}$$

Example: Side Remark
For a free non-relativistic particle, we have

$$E = \frac{p^2}{2m} \tag{2.13}$$

The Schrodinger equation is essentially this. But then if we go relativistic (with a Fourier transform),

$$E^2 = \underline{p}^2 + m^2 \tag{2.14}$$
$$\partial_t^2 - \underline{\nabla}^2 - m^2 = \partial_\mu \partial^\mu - m^2 \tag{2.15}$$

We write our solution as

$$\phi(\underline{x}, t) = \sum_{\underline{p}} \left[A(\underline{p}) \cos(p \cdot x) + B(\underline{p}) \sin(p \cdot x) \right] \tag{2.16}$$
$$-p^2 + m^2 = 0 \tag{2.17}$$
$$p_0 = \omega = \sqrt{\underline{p}^2 + m^2} \tag{2.18}$$

Note that Eq (2.16) only works for the condition of Eq (2.18).
Instead of summing over momenta, we want to integrate over momenta, so try

$$\phi(\underline{x}, t) = \int \frac{d^4p}{(2\pi)^4} (2\pi) \delta(p^2 - m^2) \Theta(p_0) = \int \frac{d^3p}{2(2\pi)^3 p_0} \tag{2.19}$$

2.3 Making the field complex

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \tag{2.20}$$

From this, you get two sets of E.o.M, one for ϕ and one for ϕ^*.

$$\phi^* : \ 0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = (\Box + m^2)\phi^* \tag{2.21}$$
$$\phi : \ 0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} - \frac{\partial \mathcal{L}}{\partial \phi^*} = (\Box + m^2)\phi \tag{2.22}$$

Solutions as before:

$$\phi = A e^{ipx} \tag{2.23}$$
$$\phi^* = A^* e^{-ipx} \tag{2.24}$$

Now consider:

$$\phi \to \phi' = \phi e^{i\nu} \qquad \phi^* \to \phi'^* = e^{-i\nu}\phi^* \qquad \mathcal{L} = \mathcal{L}' \tag{2.25}$$

Now we demand the action is unchanged.

$$\delta S = 0 = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) + \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \phi \leftrightarrow \phi^* \right] \tag{2.26}$$
$$\delta\phi = \phi' - \phi = (e^{i\nu} - 1)\phi \implies \partial_\mu(\delta\phi) = \delta(\partial_\mu \phi) \tag{2.27}$$
$$\delta S = \int d^4x \left\{ \delta\phi \left[\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} + \frac{\partial \mathcal{L}}{\partial \phi} + \phi \leftrightarrow \phi^* + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \phi \right) \right] \right\} \tag{2.28}$$

Lecture 3

3.1 Conserved Current

The Lagrangian is invariant under transformations of the form

$$\phi \to \phi' = e^{i\theta}\phi \tag{3.1}$$

$$\delta\phi = \phi' - \phi = (e^{i\theta} - 1)\phi \tag{3.2}$$

$$\delta(\partial_\mu\phi) = \partial_\mu\phi' - \partial_\mu\phi = (e^{i\theta} - 1)\partial_\mu\phi \tag{3.3}$$

The next step from this is to figure out the change to the action. If the Lagrangian is invariant, the action should be - but this condition is a bit too tough. So the condition is that the action is invariant as it will be what changes the theory. Therefore, we demand that $\delta S = 0$.

$$\delta S = \delta\left(\int \mathcal{L}\ d^4x\right) \tag{3.4}$$

$$= \int \left\{\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) + (\phi \leftrightarrow \phi^*)\right\}\ d^4x \tag{3.5}$$

$$= \int \left\{\frac{\partial\mathcal{L}}{\partial\phi}(i\theta\phi) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(i\theta\partial_\mu\phi) + (\phi \leftrightarrow \phi^*, i \to -i)\right\}\ d^4x \tag{3.6}$$

Term two above:

$$\frac{\partial\mathcal{L}}{\partial(\partial(\partial_\mu\phi)}(i\theta\partial_\mu\phi) = \theta\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - i\theta\left(\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\phi \tag{3.7}$$

$$\implies \frac{\partial\mathcal{L}}{\partial\phi}(i\theta\phi) - i\theta\left(\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\phi = 0 \tag{3.8}$$

$$\delta S = \int\left\{i\theta\left[\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\phi\right) - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\phi^*\right)\right]\right\}\ d^4x = 0 \tag{3.9}$$

$$\implies \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\phi - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\phi^*\right] = 0 \tag{3.10}$$

$$j^\mu = (\partial^\mu\phi^*)\phi - (\partial^\mu\phi)\phi^* \tag{3.11}$$

This is the conserved current.

$$\partial_\mu j^r = 0 \qquad\qquad \partial_t\rho - \underline{\nabla}\underline{j} = 0 \tag{3.12}$$

We can then define conserved charge:

$$Q = j^0 = (\partial_t\phi^*)\phi - (\partial_t\phi)\phi^* \tag{3.13}$$

3.2 Quantising the field

1. We start with the Lagrangian. From this, we produce the conjugate momentum:

$$\mathcal{L}(\phi, \partial_\mu\phi) \to \pi = \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} = \dot{\phi} = \partial_t\phi \tag{3.14}$$

2. Go from Lagrangian to Hamiltonian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 \tag{3.15}$$

$$\mathcal{L} \to \mathcal{H} = \dot{\phi}\pi - \mathcal{L} \tag{3.16}$$

$$= \frac{1}{2}\pi^2 + \frac{1}{2}(\underline{\nabla}\phi)^2 + \frac{m^2}{2}\phi^2 \tag{3.17}$$

3. Go from a Hamiltonian to an operator Hamiltonian: $\mathcal{H} \to \hat{\mathcal{H}}$. We turn all classical fields into field operators, add hats. Lives in Fock space.
4. We demand equal time commutator.

$$[\hat{\phi}(\underline{x}, t), \hat{\pi}(\underline{y}, t)] = i\delta^3(\underline{x} - \underline{y}) \tag{3.18}$$

$$[\hat{\phi}(\underline{x}, t), \hat{\phi}(\underline{y}, t)] = [\hat{\pi}(\underline{x}, t), \hat{\pi}(\underline{y}, t)] = 0 \tag{3.19}$$

5. Define creation and annihilation operators, which will "inherit" commutator relations.

$$\phi(x) = \sum_{\underline{p}} \left[\hat{a}(p)e^{-ip\cdot x} + \hat{a}^\dagger(p)e^{ip\cdot x}\right] \quad \to \int \frac{d^4p}{(2\pi)^4} \left[\hat{a}(p)e^{-ipx} + \hat{(}p)e^{ipx}\right](2\pi)\delta(p^2 - m^2)\Theta(p_0) \tag{3.20}$$

The δ in the last equation is to show it must satisfy the energy-momentum equation, $E^2 - p^2 = m^2$.

$$\hat{\phi}(x) = \int_{p_0=\sqrt{\underline{p}^2+m^2}} \frac{d^3p}{(2\pi)^3 2p_0} \left[\hat{a}(p)e^{-ipx} + \hat{a}^\dagger(p)e^{ipx}\right] \tag{3.21}$$

$$\hat{\pi}(x) = ip_0 \int \frac{d^3p}{(2\pi)^3 2p_0} \left[-\hat{a}(p)e^{-ipx} + \hat{a}^\dagger(p)e^{ipx}\right] \tag{3.22}$$

$$ip_0\hat{\phi} + \hat{\pi} = 2ip_0 \int \frac{d^3p}{(2\pi)^3 2p_0} \hat{a}^\dagger(p)e^{ipx} \tag{3.23}$$

$$\hat{a}(p) = \int e^{ipx}\left(ip_0\hat{\phi} + \hat{\pi}\right) d^3x \tag{3.24}$$

$$\hat{a}^\dagger(p) = \int e^{-ipx}\left(ip_0\hat{\phi} - \hat{\pi}\right) d^3x \tag{3.25}$$

Now consider the ladder operators' commutations:

$$[\hat{a}(p), \hat{a}^\dagger(q)] = \delta^3(p - q) \tag{3.26}$$

$$[\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0 \tag{3.27}$$

Finally, we can write the Hamiltonian operator in terms of the ladder operators.

$$\hat{\mathcal{H}} = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2k_0} k_0 \left[\hat{a}^\dagger(k)\hat{a}(k) + \hat{a}(k)\hat{a}^\dagger(k)\right] \tag{3.28}$$

One interpretation of a Quantum Field Theory is as a continuous sum of harmonic oscillator Hamiltonians, one for each frequency vector $\underline{k}$.

6. Spectrum of states. We will start by defining a ground state, or a vacuum, $|0\rangle$, $\langle 0|0\rangle$. We can annihilate/create the vacuum with the ladder operators:

$$\hat{a}(\underline{k})|0\rangle = 0 \tag{3.29}$$

$$\hat{a}^\dagger(\underline{k})|0\rangle = |\underline{k}_1\rangle \tag{3.30}$$

$$\langle 0|\hat{\mathcal{H}}|0\rangle = \int d^3x \to \infty \tag{3.31}$$

But the vacuum is an eigenstate of a Hamiltonian with infinite energy. This is one of the many divergences in QFT. Now we want to use normal ordering to get rid of the infinities. We demand that $\langle 0| : \hat{\mathcal{H}} : |0\rangle$ is finite - this is *normal ordering.*

Lecture 4

We have to use creation and annihilation operators that propagate forwards and backwards in time for the plane wave to work. What propagates backwards in time though? Essentially anti-particles - recall the negative energy solutions from last year in Dirac theory. Particles propagating backwards in time are equivalent to anti-particles propagating forwards - this is where the backwards pointing arrows for Feynman diagrams come from.

4.1 Green's functions for QM

$$G(\underline{x},t;\underline{x}',t') : \ \psi(\underline{x},t) = \int d^3x \ G(\underline{x},t;\underline{x}',t)\psi(\underline{x}',t') \tag{4.1}$$

This makes use of the superposition in QM. In principle, you can use this with any theory. Note: the wavefunction used above is $\psi(\underline{x},t) \equiv \langle \underline{x}|\psi(t)\rangle$. THis solves the time-dependent Schrodinger equation,

$$(i\partial_t - \hat{H})|\psi\rangle = 0 \tag{4.2}$$

$$(i\partial_t - H)_x G(\underline{x},t;\underline{x}',t') = \delta^3(\underline{x}-\underline{x}')\delta(t-t') \tag{4.3}$$

$$G \to 0 \iff t < t' \tag{4.4}$$

This is called the retarded Green's function.

$$G(\underline{x},t;\underline{x}',t') = \langle \underline{x},t|\hat{U}(t,t')|\underline{x}',t'\rangle\Theta(t-t') \tag{4.5}$$

$\hat{U}$ is the time evolution operator. This solves everything really, but the issue is it doesn't give practical solutions.

$$\hat{U}(t-t') = \exp\left[-i\int_{t'}^{t} \hat{H}(\tau)\ d\tau\right] \to_{\cancel{\propto} t} \exp\left[-i\hat{H}(t-t')\right] \tag{4.6}$$

Now we take a free Hamiltonian:

$$\hat{H}_0 = \frac{\hat{p}^2}{2m} \tag{4.7}$$

This then gives the free Green's function, G_0.

$$\left(i\partial_t - \frac{\underline{p}^2}{2m}\right)_x G_0(\underline{x},t;\underline{x}',t') = \delta^3(\underline{x}-\underline{x}')\delta(t-t') \tag{4.8}$$

Can either go from p space to x space, or can transform time derivatives to energy and solve it in energe space.

$$\left(\omega - \frac{\underline{p}^2}{2m}\right) G_0(\underline{p},\omega) = 1 \tag{4.9}$$

$$G_0(\underline{p},\omega) = \frac{1}{\omega - \frac{\underline{p}^2}{2m}} \tag{4.10}$$

$$G_0(\underline{x},t;\underline{x}',t') = \int \frac{e^{-i\omega(t-t')-i\underline{p}(\underline{x}-\underline{x}')}}{w - \frac{\underline{p}^2}{2m}}\ d^3p\ d\omega \tag{4.11}$$

Notice that we have developed a problem here in the form of a pole in the denominator. This is solved using usual Cauchy tactics:

$$G_0^{(R)}(\underline{x},t;\underline{x}',t') = \int \frac{e^{-i\omega(t-t')-i\underline{p}(\underline{x}-\underline{x}')}}{w - \frac{\underline{p}^2}{2m} - i\epsilon^+}\ d^3p\ d\omega \tag{4.12}$$

This now transforms back to the retarded function, as the ϵ term gives back $\Theta(t-t')$.

$$G_0^{(R)} = \int \exp\left[-\frac{ip^2(t-t')}{2m}\right] \Theta(t-t') \exp\left[-ip(\underline{x}-\underline{x}')\right]\ d^3p \tag{4.13}$$

We can make this look very Gaussian for a simpler solve, quadratic in x and divided by t - a quadratic extension. Therefore, it is finite.
Now what happens adding a term to the Hamiltonian: $\hat{H} = \hat{H}_0 + V(\underline{x})$? Pretty much the only example of a term that is solvable under this is a harmonic oscillator. It becomes increasingly more complicated going down this path for real systems.
To solve this new Hamiltonian, must do something slightly different. Let us say we know the solution for the free Green's function:

$$\mathcal{F}\left[(i\partial_t - H_0)G_0\right] = 1 \tag{4.14}$$

$$\tilde{G}_0 = \frac{1}{i\partial_t - \tilde{H}_0} \tag{4.15}$$

$$(i\partial_t - H_0 - V)G = \delta^3(\underline{x}-\underline{x}')\delta(t-t') \tag{4.16}$$

$$\mathcal{F}\left[(i\partial_t - H_0 - V)G\right] = 1 \tag{4.17}$$

$$(\tilde{G}_0^{-1} - \tilde{V})\tilde{G} = 1 \tag{4.18}$$

From this, you can realise that you can write a formal solution for this as an expansion in how the particle interacts with the potential - Born potential from TP3.

$$G(\underline{x}_N, t_N; \underline{x}_0, t_0) = G_0(t_N, t_0) + G_0(t_N, t_1)V(t_1)G_0(t_1, t_0) + G_0(t_N, t_2)V(t_2)G_0(t_2, t_1)V(t_1)G_0(t_1, t_0) + \ldots \tag{4.19}$$

The second term here is the first Born approximation, but can then can be continued onward, on the condition of $t_N \geq t_{N_1} \geq \ldots t_2 \geq t_1 \geq t_0$. The Born expansion is just the first term in a process known as perturbative expansion, or the Dyson series. This is just a Taylor expansion in V.

Lecture 5

5.1 Causality

$$i\Delta(x-y) = [\hat{\phi}(x), \hat{\phi}(y)] \tag{5.1}$$

For equal times, $x_0 = y_0$, $i\Delta(x-y) = 0$ unless $\underline{x} = \underline{y}$. This is the equal time commutator, i.e. it is zero unless the two times are zero. The properties of $(x-y)$ depend on $(x-y)^2$, as $(x-y) = \sqrt{(x-y)^2}$ to make it meaningful and remove four-vector indices. For everything that x and y have space-like distance $(x-y)^2 \leq 0$, this must be zero; for light-like distances, it is something like the delta function.

$$\begin{aligned} i\Delta(x-y) &= \int \frac{d^3k\, d^3k'}{(2\pi)^3 2\omega_k (2\pi)^3 \omega_{k'}} \left\{ \left[\hat{a}(k), \hat{a}^\dagger(k')\right] e^{-ikx+ik'y} + \left[\hat{a}^\dagger(k), \hat{a}(k')\right] e^{ikx-ik'y} \right\} & (5.2)\\ &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)} \right] & (5.3) \end{aligned}$$

- manifestly Lorentz invariant (only scalar products of four-vectors)
- micro-causality: vanishes for space-like distances because $\Delta(x-y)$ vanishes for $t_x = t_y$ (equal time commutator) - this is true only because sum is over positive and negative energy waves, ignoring the negative energy solutions, Δ does no longer vanish for space-like distances, and signals could be transmitted with superluminal speed.
- $$\partial_t \Delta(x-y)|_{x_0=y_0} = -\delta^3(x-y) \tag{5.4}$$
- $\Delta(x-y)$ solves the Klein-Gordon equation:
  $$(\partial_\mu \partial^\mu + m^2)\Delta(x) = 0 \tag{5.5}$$
- Vacuum Expectation Value:
  $$\Delta_+(x-y) = \langle 0|\Delta(x-y)|0\rangle = \cdots \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-ik(x-y)} \tag{5.6}$$

  If $x \to y$, this will explode to infinity, i.e. all of space-time. To ignore this, define another operator

  $$\begin{aligned} \Delta_-(x-y) &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{ik(x-y)} & (5.7)\\ \langle 0|\Delta|0\rangle &= \Delta_+ - \Delta_- & (5.8) \end{aligned}$$

5.2 Green's function of Klein-Gordon field

$$\begin{aligned} (\Box + m^2)G(x,x') &= i\delta^4(x-x') & (5.9)\\ (-p^2 + m^2)G(p) &= i & (5.10)\\ G(p) &= \frac{-i}{p^2 - m^2 - i\epsilon^+} & (5.11) \end{aligned}$$

This is classical field theory. Must connect this with quantum operators and fields:

$$\begin{aligned} G(x,x') &= \langle 0|T[\hat{\phi}(x)\hat{\phi}(x')]|0\rangle = \Delta_F & (5.12)\\ T[\hat{\phi}(x)\hat{\phi}(x')] &= \Theta(t-t')\hat{\phi}(x)\hat{\phi}(x') + \Theta(t'-t)\hat{\phi}(x')\hat{\phi}(x) & (5.13) \end{aligned}$$

This is the time-ordered product. The first term describes forwards time, the second backwards time. Now we can look at $i\epsilon^+$ and convince ourselves that this is the equivalent of the Θ functions above:

$$\Theta(t) = -\int \frac{d\omega}{2\pi} \frac{ie^{-i\omega t}}{\omega - i\epsilon^+} = \Delta_F \tag{5.14}$$

The $-i\epsilon^+$ is the causal structure that leads into the time ordering of Eq (5.12). We call this then Δ_F, the Feynman propagator.

5.3 Interacting Theory

We want to apply the above theory to reality, to see it working on actual physics and prove it is not just a consistent mathematical framework, but also a valid physical one. We start with the ϕ^4 theory Lagrangian.

$$\mathcal{L} = \mathcal{L}_{free} + \mathcal{L}_{int} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{5.15}$$

This Lagrangian is said to not be realistic, as it is not one found in nature, but it is useful to test quantum field theory as it is useful for certain things and also awfully similar to the Higgs field after breaking electroweak symmetry, exlcuding the ϕ^3 term.
We want to calculate some scattering amplitude where some initial state becomes a final state, $M_{i\to f}$, or transition amplitude.

$$M_{i\to f} = \langle f|i\rangle \tag{5.16}$$

We pretend we are dealing with particles that are isolated from all other interactions coming from an infinite distance. This is not the case as there is constant "bubbles" of particles being created and annihilated instantaneously throughout space-time. We use perturbation theory to expand around the bubbles and subtract the eventual infinities that then plague us. The most useful quantity for now to calculate is the S-matrix:

$$M_{i\to f} = \langle f|\hat{S}|i\rangle = S_{fi} \tag{5.17}$$

Lecture 6

6.1 Pictures

- Schrodinger - $\hat{O}^s$ operators do no depend on time, but wavefunctions do, $\psi^s(t)$
- Heisenberg - $\hat{O}^H(t)$ operators do depend on time, but wavefunction does not, ψ^H. We can switch between Schrodinger and Heisenberg using the time-evolution operator:

$$\hat{O}^H(t) = \exp[i\hat{H}(t-t_0)]\hat{O}^s \exp[-i\hat{H}(t-t_0)] \tag{6.1}$$
$$\tag{6.2}$$

- Interaction - $\hat{H}$ is independent of time, and we define it as $\hat{H} = \hat{H}_0 + \hat{H}_{int}$, wehre $\hat{H}_0$ is the free particle Hamiltonian and $\hat{H}_{int}$ describes interactions,

$$\hat{O}^I(t) = \exp[i\hat{H}_0(t-t_0)]\hat{O}^s \exp[-i\hat{H}_0(t-t_0)] \tag{6.3}$$

We can connect the field operators of the Heisenberg and Interaction pictures too:

$$\hat{O}^H = \underbrace{\exp[i\hat{H}(t-t_0)]\exp[-i\hat{H}_0(t-t_0)]}_{\hat{U}^\dagger(t,t_0)}\hat{O}^I\underbrace{\exp[-\hat{H}_0(t-t_0)]\exp[-i\hat{H}(t-t_0)]}_{\hat{U}(t,t_0)} \tag{6.4}$$

$$i\partial_t\hat{U}(t,t_0) = \ldots\hat{H}^I_{int}\cdot\hat{U}(t,t_0) \tag{6.5}$$
$$\hat{U}(t,t_0) = 1 \implies \tag{6.6}$$
$$\hat{U}(t,t_0) = \sum_{n=0}^{\infty}(-i)^n\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\cdots\int_{t_0}^{t_{n-1}}dt_n\ \hat{H}^I_{int}(t_1)\hat{H}^I_{int}(t_2)\ldots\hat{H}^I_{int}(t_n) \tag{6.7}$$
$$= \exp\left[-iT\int_{t_0}^{t}\hat{H}^I_{int}(\tau)\,d\tau\right] \tag{6.8}$$

Note that $t_1 > t_2 > t_3 > \cdots > t_n$, and we put the T in the exponential to say that we have time-ordered to remove an incorrect factor $n!$. We can then notice that this is exactly the $\hat{S}$ matrix:

$$\hat{S} = T\exp\left[-i\int_{t_0}^{t}\hat{H}^I_{int}(\tau)\,d\tau\right] \tag{6.9}$$
$$= 1-\underbrace{i\int\hat{H}_{int}}_{\text{Born approx}}+(-i)^2\int\int\hat{H}_{int}+\ldots \tag{6.10}$$

6.2 Scattering in QFT

Look at $2\to 2$ scattering: $p_1+p_2\to q_1+q_2$. We set our incoming particles way back in the past so we can approximate with plane waves, and similarly in the future for outgoing particles - aymptotic states.

$${}_{t\to+\infty}\langle q_1,q_2;\mathrm{OUT}|p_1,p_2;\mathrm{IN}\rangle_{t\to-\infty} = \langle q_1,q_2|\hat{a}^\dagger_{in}(p_1)|p_2\rangle \tag{6.11}$$
$$= \langle q_1,q_2|\hat{a}^\dagger_{out}(p_1)|p_2\rangle + \langle q_1,q_2|\hat{a}^\dagger_{in}(p_1)-\hat{a}^\dagger_{out}(p_1)|p_2\rangle \tag{6.12}$$

The first term in this is zero unless $q_1,q_2=p_1$, which would lead to a "disconnected diagram", or no "real scattering". So this is irrelevant for $\hat{S}$ if $f\neq i$. We need to rewrite the creation and annihlation operators:

$$\hat{a}(k) = \int e^{ikx}\left[k_0\hat{\phi}(x,t)+i\hat{\pi}(x,t)\right]d^3x,\ \phi=\partial_t\hat{\phi},\ k_0=-i\partial_t e^{ikx} \tag{6.13}$$
$$= i\int\left[e^{ikx}(-i\overleftarrow{\partial_t})\hat{\phi}+e^{ikx}\partial_t(i\phi)\right]d^3x \tag{6.14}$$

$$= i \int i e^{ikx} (\breve{\partial}_t) \hat{\phi} \, d^3x \tag{6.15}$$

$$\hat{a}^\dagger(k) = i \int e^{-ikx} \breve{\partial}_t \hat{\phi} \tag{6.16}$$

$$\int f(x,t) \, d^3x = \int dt \, \frac{\partial}{\partial t} \int f(x,t) \, d^3x = \int d^4x \, \frac{\partial}{\partial t} f(x,t) \tag{6.17}$$

$$\langle q_1, q_2 | p_1, p_2 \rangle = i \int d^4x \, \left\{ e^{-ip_1 x} (\Box_x + m^2) \langle q_1, q_2 | \hat{\phi}(x,t) | p_2 \rangle \right\} \tag{6.18}$$

We can do this again to pull a particle out the "out" state:

$$\langle q_1, q_2 | p_1, p_2 \rangle = i \int d^4x \, e^{-ip_1 x + iq_1 y_1} (\Box_{x_1} + m^2)(\Box_{y_1} + m^2) \langle q_2 | T(\hat{\phi}(y_1)\hat{\phi}(x_1)) | p_2 \rangle \tag{6.19}$$

$$T\hat{\phi}(y)\hat{\phi}(x) = \Theta(y_0 - x_0)\hat{\phi}(y)\hat{\phi}(x) + \Theta(x_0 - y_0)\hat{\phi}(x)\hat{\phi}(y) \tag{6.20}$$

LSZ-theorem: external particle $\rightarrow \phi$ in time-ordered particle + plane wave part + diff. kernel.

Lecture 7

7.1 Wick's Theorem

To evaluate the perturbative series, resort to Wick's theorem. It tells us what happens with a time ordered product of two fields.

$$T[\hat{\phi}(x)\hat{\phi}(y)] = \underbrace{\Delta_F(x-y)}_{\text{Feynman propagator}} + \underbrace{:\hat{\phi}(x)\hat{\phi}(y):}_{\text{Normal ordered product}} \tag{7.1}$$

$$\langle 0|T[\hat{\phi}(x)\hat{\phi}(y)]|0\rangle = \langle 0|\Delta_F(x-y)|0\rangle + \langle 0| : \hat{\phi}(x)\hat{\phi}(y) : |0\rangle = \Delta_F(x-y) \tag{7.2}$$

$$\langle 0| : \hat{\phi}(x_1)\hat{\phi}(x_2)\dots\hat{\phi}(x_n) : |0\rangle = 0\forall n \tag{7.3}$$

$$\langle 0|\Delta_F(x-y)|0\rangle = \Delta_F(x-y) \tag{7.4}$$

$$\langle 0|T[\hat{\phi}_1\hat{\phi}_2\hat{\phi}_3\hat{\phi}_4]|0\rangle = \Delta_F(x_1-x_2)\Delta_F(x_3-x_4) + \Delta_F(x_1-x_3)\Delta_F(x_2-x_4) + \Delta_F(x_1-x_4)\Delta_F(x_2-x_3) \tag{7.5}$$

Back to scattering from before:

$$\begin{aligned}\langle q_1, q_2|p_1, p_2\rangle \propto \int \dots (\Box_{x_1} + m^2)(\Box_{x_2} + m^2)(\Box_{y_1} + m^2)(\Box_{y_2} + m^2) \\ \times \langle 0|T[\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(y_1)\hat{\phi}(y_2)]|0\rangle\end{aligned} \tag{7.6}$$

$$\langle q_1, q_2|T[i\lambda\phi^4(z)]|p_1, p_2\rangle \propto \dots \langle 0|T[\hat{\phi}(x_1)\hat{\phi}(x_2)i\lambda\phi^4(z)\hat{\phi}(y_1)\hat{\phi}(y_2)]|0\rangle \tag{7.7}$$

7.2 Feynman Rules

1. Draw all topologically distinct Feynman diagrams with n external legs, associating each leg with an external momentum.
2. For each **internal propagator** with momentum p add a term
$$\frac{d^4k}{(2\pi)^4}\frac{i}{(p^2-m^2-i\epsilon^+)} \tag{7.8}$$
3. For each **vertex** add a term
$$\frac{-i\lambda}{4!}(2\pi)^4\delta^4\left(\sum_i k_i\right) \tag{7.9}$$
where the sum goes over all momenta entering the vertices
4. Make sure you understand symmetry factor compensating the 4! in the vertex

7.3 The Dirac Equation

Dirac realised that the real relativistic equation was $p^2 = m^2$ which gives a quadratic in ∂_t, but this gave positive and negative energies, $E = \pm\sqrt{\underline{p}^2 + m^2}$. As well, the probabilistic interpretation no longer works as zero-components explode or go negative. Solution to both: abandon concept of a single-particle theory, construct a quantum field theory, and introduce anti-particles as negative energy solutions propagating backward in time.

$$i\partial_t\psi(\underline{x}, t) = -i\underline{\alpha}\cdot\underline{\nabla}\psi(\underline{x}, t) + \beta m\psi(\underline{x}, t) \tag{7.10}$$

We want to square this ansatz and get the Klein-Gordon equation back.
First requirement is that α and β are matrices. To get KG back when squaring, we require the following relations:

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij} \qquad\qquad \beta^2 = \alpha^2 = 1 \tag{7.11}$$

$$\{\alpha_i, \beta\} = 0 \qquad\qquad \mathrm{Tr}[\alpha_i] = \mathrm{Tr}[\beta] = 0 \tag{7.12}$$

$$\tag{7.13}$$

α_i conditions can be fulfilled by the Pauli matrices. α_i, β must be of even dimension, and then we see that because of $\{\alpha_i, \beta\} = 0$, the matrices must be of dim $= 4$ or $m = 0$.

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \qquad\qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7.14}$$

These matrices aren't very easy to work with though, so we can put them into a single four-vector:

$$\gamma^\mu_{\eta\xi} = (\beta, \beta\underline{\alpha})_{\eta\xi} \tag{7.15}$$

$$\{\gamma^\mu, \gamma^\nu\}_{\eta\xi} = 2g^{\mu\nu}1_{\eta\xi} \tag{7.16}$$

From this, we can re-write the Dirac equation using $\gamma^\mu\partial_\mu = \not{\partial}$ as

$$(i\not{p} - m)_{\eta\xi}\psi_\xi = 0 \tag{7.17}$$

How do we make a Lagrangian out of this? First we need a Hermitian conjugate of the equation, starting with the ansatz in Eq (7.10).

$$-i\partial_t\psi^\dagger = i\underline{\nabla}\psi^\dagger \cdot \underline{\alpha}^\dagger + m\psi^\dagger\beta^\dagger \tag{7.18}$$

$$\tag{7.19}$$

Then, multiplying from the right with $\beta^\dagger$ and using the matrix relations above, *put left arrow on slash*

$$-i\psi^\dagger\partial_\mu\gamma^\mu = m\psi^\dagger \tag{7.20}$$

$$\bar{\psi}(i\not{\partial} + m) = 0, \ \bar{\psi} = \psi^\dagger\gamma_0 \tag{7.21}$$

Now that we have both of these, we can write down our Lagrangian:

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi \tag{7.22}$$

put left right arrow above pslash.

Lecture 8

8.1 Quantising the Dirac equation

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi \tag{8.1}$$

$$\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = 0 \tag{8.2}$$

Perform a plane wave expansion:

$$\psi(x) = \int d^3x \left[e^{-ipx}u(p) + e^{ipx}v(p)\right] \tag{8.3}$$

u and v are spinor solutions of positive and negative energy respectively, or particle and anti-particle solutions.

$$(\not{p} - m)_{\alpha\beta}u_\beta(p) = 0 \qquad (\not{p} + m)v_\beta(p) = 0 \qquad p^\mu = (E, \underline{0}) = (m, \underline{0}) \tag{8.4}$$

So the only matrix that survives this condition is γ^0, or β.

$$(\beta - 1)u(m) = 0 \qquad (\beta + 1)v(m) = 0 \tag{8.5}$$

So this implies that we have two solutions for u and two for v:

$$u^{(1)}(m) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad u^{(2)}(m) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad v^{(1)}(m) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad v^{(2)}(m) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{8.6}$$

$u^{(1/2)}$ describes particles with spin up/down and $v^{(1/2)}$ describes anti-particles with spin up/down. To boost these spinors:

$$(\not{p} + m)(\not{p} - m)u = (p^2 - m^2)u \tag{8.7}$$

$$u(p) = \eta(\not{p} + m)u((m, \underline{0})) \tag{8.8}$$

Here, η is some renormalisation factor.

$$\not{p} = p^\mu\gamma_\mu = E\gamma^0 - p_x\gamma^1 - p_y\gamma^2 - p_z\gamma^3 \tag{8.9}$$

$$u(p) \to \bar{u}(p) = u^\dagger(p)\gamma^0 \tag{8.10}$$

We want to fix η so that a sum over the two u spinors that we have, we get a δ_{ij} and this is the same as negative sum over the two v spinors.

$$\bar{u}_i(p)u_j(p) = \delta_{ij} = -\bar{v}_i(p)v_j(p) \tag{8.11}$$

If you are not consistent with how you use your convention, then you will burn in hell.
Reverse the order of above and obtain the completeness relations:

$$\sum_{i=1}^{2} u_\alpha^{(i)}(p)\bar{u}_\beta^{(i)}(p) = \left(\frac{\not{p} + m}{2m}\right)_{\alpha\beta} \tag{8.12}$$

$$\sum_{i=1}^{2} v_\alpha^{(i)}(p)\bar{v}_\beta^{(i)}(p) = \left(\frac{\not{p} - m}{2m}\right)_{\alpha\beta} \tag{8.13}$$

Quantisation:

1. Construct Lagrangian
2. Make π:
$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \bar{\psi} i \gamma^0 = i\psi^\dagger \tag{8.14}$$
3. Construct Hamiltonian:
$$\mathcal{H} = \pi \dot{\psi} + \bar{\pi} \dot{\bar{\psi}} - \mathcal{L} \tag{8.15}$$
4. Make everything operators
5. Demand equal-time (anti-)commutators for bosons(fermions) - anti-commutator = 0 means no two particles in same state.
$$\left\{ \psi_\alpha(\underline{x}, t), \psi_\beta^\dagger(\underline{y}, t) \right\} = \delta_{\alpha\beta} \delta^3(x - y) \tag{8.16}$$
All others are zero.
6. Expand in waves with creation and annihilation operators
$$\psi_\alpha(\underline{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{m}{E} \sum_{i=1}^{2} \left[e^{-ipx} u_\alpha^{(i)}(p) \hat{b}_i(p) + e^{ipx} v_\alpha^{(i)}(p) \hat{d}_i^\dagger(p) \right] \tag{8.17}$$
$$\psi_\beta^\dagger(\underline{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{m}{E} \sum_{i=1}^{2} \left[e^{-ipx} \bar{u}_\beta^{(i)}(p) \hat{b}_i^\dagger(p) + e^{ipx} \bar{v}_\beta^{(i)}(p) \hat{d}_i(p) \right] \gamma^0 \tag{8.18}$$
7. Use completeness relations for $u(p)$ and $v(p)$:
$$\left\{ \hat{b}_\alpha(p), \hat{b}_\alpha^\dagger(q) \right\} = (2\pi)^3 \frac{E}{m} \delta^3(\underline{p} - \underline{q}) \delta_{\alpha\beta} \tag{8.19}$$
$$\left\{ \hat{d}_\alpha(p), \hat{d}_\alpha^\dagger(q) \right\} = (2\pi)^3 \frac{E}{m} \delta^3(\underline{p} - \underline{q}) \delta_{\alpha\beta} \tag{8.20}$$

Lecture 9

9.1 Dirac Propagator

$$\psi(x) = \sum_{i=1}^{2} \int d^3p \left[e^{-ipx} u^{(i)}(p) \hat{b}^{(i)} + e^{ipx} v^{(i)}(p) \hat{d}^{\dagger(i)} \right] \tag{9.1}$$

Want to have two different objects at positions x and y and have some amplitude between them, $\langle \psi(y) | \psi(x) \rangle$.

$$iS_F(y-x) = \langle 0 | \hat{\psi}(y) \hat{\psi}^\dagger(x) | 0 \rangle \Theta(y_0 - x_0) - \langle 0 | \hat{\psi}^\dagger(y) \hat{\psi}(x) | 0 \rangle \tag{9.2}$$

We subtract these two terms instead of add them due to Fermi algebra, as we are dealing with fermions now, not bosons, so we have a particle going one way with the first term and an anti-particle going the other with the latter term. So now, dropping the "hats" from the operators for convenience,

$$iS_F(x,y)_{\beta\alpha} = \langle 0 | T[\psi_\beta(y) \bar{\psi}_\alpha(x)] | 0 \rangle \tag{9.3}$$

$$= \int \frac{d^3p}{(2\pi)^2 2E_p} \left[e^{-ip(y-x)} \Theta(y_0 - x_0)(\not{p} + m) - e^{ip(y-x)}(\not{p} - m)\Theta(x_0 - y_0) \right]_{\beta\alpha} \tag{9.4}$$

$$= i \int \frac{d^4p}{(2\pi)^4} e^{-ip(y-x)} \left[\frac{\not{p} + m}{p^2 - m^2 + i\epsilon} \right]_{\beta\alpha} \tag{9.5}$$

Where in between the last two lines, we have used Cauchy's Integral formula.
In Feynman diagrams, we rarely deal with propagators in position space, so again, we will Fourier transform into momentum space, which is simply if you consider Eq (9.5) is already in the form of a Fourier transform.

$$iS_F(p) = \frac{i\not{p} + m}{p^2 - m^2 + i\epsilon^+} \tag{9.6}$$

9.2 Quantising Electrodynamics

We will start from scratch with electrodynamics, beginning with proposing two vector fields. We want these to be related somehow, so there is no point having two vectors, we will have a vector field $\underline{\mathbf{E}}$, and an axial vector field, $\underline{\mathbf{B}}$. We will also set the restraint that we can only use first derivates for equations of motion.

$$\underline{\nabla} \cdot \underline{\mathbf{E}} = 4\pi\rho, \text{ - scalar} \tag{9.7}$$

$$\underline{\nabla} \cdot \underline{\mathbf{B}} = 0, \text{ - pseudo-scalar} \tag{9.8}$$

$$\partial_t \underline{\mathbf{E}} - \underline{\nabla} \times \underline{\mathbf{B}} = -4\pi \underline{j}, \text{ - vectors} \tag{9.9}$$

$$\partial_t \underline{\mathbf{B}} - \underline{\nabla} \times \underline{\mathbf{E}} = 0, \text{ - axial-vectors} \tag{9.10}$$

The conditions we have used are that we have no magnetic monopoles, and $\underline{\mathbf{E}}$ and $\underline{\mathbf{B}}$ are light.

Now let's reintroduce the vector potential, $A^\mu = (\phi, \underline{A})$

$$E = -\underline{\nabla} A^0 - \partial_t \underline{A} \qquad \underline{\mathbf{B}} = \underline{\nabla} \times \underline{A} \tag{9.11}$$

So what is the free Lagrangian from this?

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \cancel{j^\mu A_\mu} \tag{9.12}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{9.13}$$

The cancelled term in the Lagrangian is what we would add for a term describing the source, but we have no source, so we set to 0. Using $F^{\mu\nu}$, we can re-write Maxwell's equations:

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{9.14}$$

$$\partial_\mu \tilde{F}^{\mu\nu} = 0,\ \tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \tag{9.15}$$

Eq (9.14) is in fact the Euler-Lagrange equation,

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0 \tag{9.16}$$

9.3 Gauge Invariance

Problems:

- $$\pi^0 = \frac{\partial \mathcal{L}}{\partial A^0} = 0 \implies [\hat{A}^0, \hat{\pi}^0] \propto \delta \tag{9.17}$$

- Gauge invariance:

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda \tag{9.18}$$

$$\implies F'_{\mu\nu} = F_{\mu\nu} \tag{9.19}$$

Our current standing of A^μ leaves us with four degrees of freedom, which is too many. We need to reduce the degrees of freedom by two if we want to develop a viable theory. A^μ is gauge dependent, whereas $F^{\mu\nu}$ is gauge independent, so $\underline{\mathbf{E}}, \underline{\mathbf{B}}$ are independent too.

There are some gauges we can choose to help us solve this:

- Coulomb gauge, $\underline{\nabla} \cdot \underline{A} = 0$
- Lorentz gauge, $\partial_\mu A^\mu = 0$
- Temporal gauge, $A_0 = 0$
- Axial gauge, $A_z = 0$

Any physical quantity we want to calculate must be gauge-independent.

Lecture 10

10.1 Quantisation in Coulomb Gauge

We start by recognising that A^0 is not dynamical: $\partial_t A^0 = 0$. We give a condition to fix the gauge:

$$\underline{\nabla} \cdot \underline{A} = 0. \tag{10.1}$$

This condition follows the logic of fixing the longitudinal component of $\underline{A}$ as a constant, which can just be set to zero since it has no bearing on the equations of motion. We then have two degrees of freedom left in the two transverse components of $\underline{A}$. For $\underline{A} \parallel \underline{z}$,

$$A_z = 0 \qquad\qquad A_{x,y} \neq 0 \tag{10.2}$$

$$\epsilon^\mu_{(1)} = (0,1,0,0) \qquad\qquad \epsilon^\mu_{(2)} = (0,0,1,0) \tag{10.3}$$

We then have a generalisation for any massless spin-1 field:

$$\epsilon_\mu \epsilon^\mu = 0 \qquad\qquad \epsilon^{(i)}_\mu \epsilon^{(j)\mu} = \delta_{ij} \tag{10.4}$$

We use the quantisation process discussed previously:

- Consider the Lagrangian of the gauge field,

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}. \tag{10.5}$$

  The objects we want to quantise are the fields, A^ν. Using the conjugate momenta,

$$\pi^\nu = \frac{\partial \mathcal{L}}{\partial \dot{A}^\nu} = \begin{cases} \nu = 0, & \pi^0 = 0 \\ \nu = i, & \pi^i = -E^i = F^{0i} \end{cases} \tag{10.6}$$

$$\tag{10.7}$$

- Construct the Hamiltonian:

$$\mathcal{H} = \pi^\nu \dot{A}^\nu - \mathcal{L} = \frac{1}{2}(\underline{\mathbf{E}}^2 + \underline{\mathbf{B}}^2) + \underline{\mathbf{E}}\underline{\nabla} A^0 \tag{10.8}$$

- Demand commutator relations:

$$[A^0(\underline{x}, t), \pi^0(\underline{y}, t)] = i\delta^3(\underline{x} - \underline{y}) \tag{10.9}$$

$$[A^i(\underline{x}, t), \pi^j(\underline{y}, t)] = i\delta_{ij}\delta^3(\underline{x} - \underline{y}) \tag{10.10}$$

  We want Eq (10.9) to be vanishing, so the fact it is not is problematic. Note that putting a nabla in front of A^0 will give you something of the form of Gauss' law, so this isn't bad physics; it's bad formalism. Our issue here is that our notation is not dynamical; A^0 is not an operator, just an inconsequential number. However, we cannot use this constraint on the commutators as one side will vanish while the other will not. Therefore, we cannot implement Gauss' law as an operator equation.
  What about trying the constraint on "physical states", $\underline{\nabla} \cdot \hat{\underline{\mathbf{E}}}|\psi\rangle = 0$? But this doesn't work either as it would violate our commutation relations again. The only way to get this to work is $\langle\psi|\underline{\nabla} \cdot \hat{\underline{\mathbf{E}}}|\psi\rangle = 0$ - **this means that Maxwell's laws are only realised as an average over physical states.** We will return to this solution when we address the Loretnz gauge.
  This makes sense because we don't care about the unphysical ones as long as they go into the corner and kill themselves.

➤ Our solution to the commutator problem is by replacing the δ-function:

$$[A^i, \pi^j] = i\delta_{ij}\delta^3(\underline{x} - \underline{y}) \tag{10.11}$$

$$\delta_{ij} \to \delta^{tr}_{ij}(\underline{x} - \underline{y}) = \int \frac{d^3k}{(2\pi)^3} e^{i\underline{k}(\underline{x}-\underline{y})} \left(\delta_{ij} - \frac{k_i k_j}{\underline{k}^2} \right) \tag{10.12}$$

$$\partial_i \delta^{tr}_{ij} = \partial_j \delta_{ij} = 0 \tag{10.13}$$

$$[A^i, \pi^j] = -\delta^{tr}_{ij}(\underline{x} - \underline{y}) = [A^i, -E^j] \tag{10.14}$$

Now, in this new form, $\underline{\nabla} \cdot \underline{\mathbf{E}}$ commutes with everything and we can implement it as an operator, where everything can be set to zero. So $\underline{\nabla} \cdot \underline{\hat{\mathbf{E}}} = 0$ is perfectly legitimate. As a by-product of this, we also have

$$[\underline{\nabla} \cdot \underline{A}_i, \underline{\hat{\mathbf{E}}}_j] = 0. \tag{10.15}$$

This clearly gives us the Coulomb gauge automatically, as $\underline{\nabla} \cdot \underline{A} = 0$.
Using δ^{tr} implies that A_i and E_j do commute at space-like distances - is this a problem?

- ➥ A is gauge-dependent and unphysical - it can not be measured anyway, and therefore it cannot harm causality
- ➥ From $[A_i, E_j]$ commutators for physical fields $[B_i, E_j]$ can be deduced - they vanish for space-like distance and infact are idential irrespective of choice of δ-function.

➤ Expanding fields in plane waves:

$$\underline{A}(\underline{x}, t) = \int \frac{d^3k}{(2\pi)^3(2k_0)} \sum_{\lambda=1}^{2} \left[\underline{\epsilon}^{(\lambda)}(k)\hat{a}(k,\lambda)e^{-ikx} + \underline{\epsilon}^{(\lambda)*}(k)\hat{a}^\dagger(k,\lambda)e^{ikx} \right] \tag{10.16}$$

$$[\hat{a}(k,\lambda), \hat{a}^\dagger(q,\kappa)] = (2\pi)^3 2k_0 \delta_{\lambda\kappa}\delta^3(\underline{k} - \underline{q}) \tag{10.17}$$

All of the other commutators vanish.

➤ **Summary:** Pain! Pain! Pain! *Cheat!* Phew!

Lecture 11

11.1 Quantisation in Lorentz Gauge

The Coulomb gauge is not manifestly Lorentz invariant - we introduced δ^{tr}_{ij}, so to achieve this we demand Lorentz-invariant form of equal-time commutators.

$$[\hat{A}^\mu(\underline{x},t), \hat{\pi}^\nu(\underline{y},t)] = ig^{\mu\nu}\delta^3(\underline{x}-\underline{y}) \tag{11.1}$$

This implies that all four components of A^μ must exist, but to achieve this, we must modify the Lagrangian.

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{\alpha}{2}(\partial_\mu A^\mu)^2 \tag{11.2}$$

A reminder that Coulomb gauge is $\underline{\nabla A} = 0$, but the Lorentz gauge is $\partial_\mu A^\mu = 0$. α here is a gauge parameter - it is unphysical, so physics must be independent of the choice of α, i.e. it is a Lagrange multiplier.

Now we can write our conjugate momenta

$$\pi^\mu = \frac{\partial\mathcal{L}}{\partial\dot{A}_\mu} = F^{\mu 0} - \alpha g^{\mu 0}(\partial_\mu A^\mu) = \begin{cases}\pi^0 = -\alpha(\partial_t A^0)\\ \pi^i = -E^i\end{cases} \tag{11.3}$$

What about Maxwell's equations?

$$\partial_\mu\partial^\mu A^\nu - (1-\alpha)\partial^\nu(\partial_\mu A^\mu) = 0 \tag{11.4}$$

We will choose $\alpha = 1$ to use from now on - this is the Feynman gauge.

Now, we want to find our polarisation vectors, ϵ^μ_λ. Naively because we have four degrees of freedom in A^μ, we could pick four independent polarisation vectors going down the indices, but this would create problems later, as some of these are unphysical. We demand that

$$\sum_\lambda \epsilon^\mu_\lambda \epsilon^{*\nu}_\lambda = -g^{\mu\nu}. \tag{11.5}$$

Now we can expand the fields, noting that we include the unphysical polarisations.

$$\hat{A}_\mu(x) = \int \frac{d^3k}{(2\pi)^3 2k_0}\sum_{\lambda=0}^{3}\left[\epsilon^{(\lambda)}_\mu(k)\hat{a}(k,\lambda)e^{-ikx} + \epsilon^{*(\lambda)}_\mu(k)\hat{a}^\dagger(k,\lambda)e^{ikx}\right] \tag{11.6}$$

The only non-vanishing commutator of creation and annihilation operators is

$$[\hat{a}(k,\lambda), \hat{a}^\dagger(q,\kappa)] = -(2\pi)^3(2k_0)g^{\lambda\kappa}\delta^3(k-q) \tag{11.7}$$

Now the normal-ordered Hamiltonian, i.e. creation to the left of annihilation,

$$:\mathcal{H}: = \int \frac{d^3k}{(2\pi)^3 2k_0}k_0\left[\sum_{i=1}^{3}\hat{a}^\dagger(k,\lambda)\hat{a}(k,\lambda) - \hat{a}^\dagger(k,0)\hat{a}(k,0)\right] \tag{11.8}$$

So before we had quanta that we counted into the Hamiltonian to each give a $+\omega$ of energy as they were particles. Now we see quanta that give us $-\omega$ like anti-particles even though we wrote them down as particles. So something has gone wrong. *It is important to reiterate however that* $\hat{a}^\dagger(k,\lambda)|0\rangle = 0$.

What if we create a state with a scalar photon ($\lambda = 0$), $|1_S\rangle$?

$$|1_S\rangle = \int \frac{d^3k}{(2\pi)^3 2k_0)} f(k)\hat{a}^\dagger(k,0)|0\rangle \tag{11.9}$$

$$\langle 1_S|1_S\rangle = \int \frac{d^3k}{(2\pi)^3 2k_0} \int \frac{d^3k'}{(2\pi)^3 2k_0} f(k) f^*(k') \langle 0|\hat{a}(k',0)\hat{a}^\dagger(k,0)|0\rangle \tag{11.10}$$

$$= -\langle 0|0\rangle \int \frac{d^3k}{(2\pi)^3 2k_0} |f(k)| < 0 \tag{11.11}$$

So we realise this is an unphysical state - we cannot have a real scalar photon, and should never have a negative norm. Is this a sign of failed quantisation? No, as we must still impose the constraint $\partial \cdot \hat{A} = 0$. So we implement gauge constraint by demanding that

$$\langle \psi|\partial \cdot \hat{A}|\psi\rangle = 0. \tag{11.12}$$

We can then decompose $\hat{A}$ into positive and negative energy modes:

$$0 = \langle \psi|(\partial \cdot \hat{A}^{(+)} + \partial \cdot \hat{A}^{(-)})|\psi\rangle \tag{11.13}$$

$$= \langle \psi| \left(\partial \cdot \hat{A}^{(+)}|\psi\rangle\right) + \left(\partial \cdot \hat{A}^{(+)}|\psi\rangle\right)^\dagger |\psi\rangle \tag{11.14}$$

So it is enough to demand that $\partial \cdot \hat{A}^{(+)}|\psi\rangle = 0$. We can use this constraint on the field expansion of $\hat{A}_\mu$,

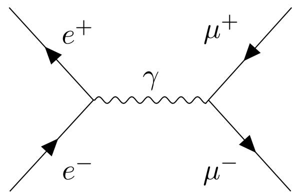
$$\partial \cdot \hat{A}^{(+)} = -i \int \frac{d^3k}{(2\pi)^3 2k_0} \sum_{\lambda=0}^{3} \hat{a}(k,\lambda) e^{-ikx} (k \cdot \epsilon^{(\lambda)}(k)) \tag{11.15}$$

If we assume $k \parallel \epsilon_z$, then $k \cdot \epsilon^{(1,2)}(k)$ is trivially fulfilled (transverse polarisations!) and we are left with demanding that

$$[\hat{a}(k,3) - \hat{a}(k,0)]|0\rangle = 0 \tag{11.16}$$

So the two unphysical polarisations compensate each other.

Lecture 12



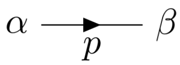
$$\mathcal{L}_{QED} = \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\alpha}{2}(\partial_\mu A^\mu)^2 - eA^\mu j^\mu \tag{12.1}$$

The first term of this is the free spinor field, the second is the free electromagnetic field, the third is a gauge-fixing term with $\alpha = 1$, and the fourth is the interaction term, with a current $j^\mu = \bar{\psi}\gamma^\mu\psi$.
If you're a physicist, we don't do social interactions, we just emulate them in whatever programming we run.

12.1 Feynman Rules

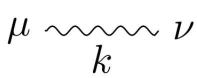
What are the "building blocks"?

- Terms with two fields give the propagators $\implies$ the Green's functions of the equations of motions
- Particles:



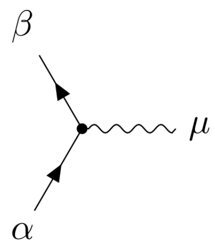
$$= \frac{i(\not{p} + m)_{\beta\alpha}}{p^2 - m^2 + i\epsilon^+}$$

- Propagators:



$$= \frac{-g^{\mu\nu}}{k^2 + i\epsilon^+}$$

- Vertices (interaction):

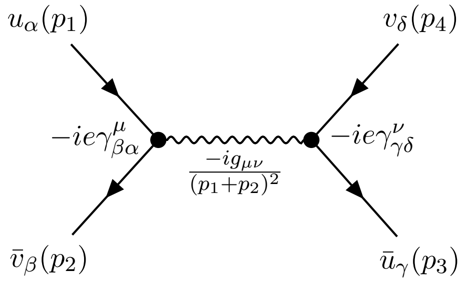


$$= -ie\gamma^\mu_{\beta\alpha}$$

Now the rules:

1. Draw all distinct Feynman diagrams, including labels for the momenta of the incoming and outgoing particles:
   - Incoming/outgoing (anti-)fermions with momentum p are represented with spinors: $u(p)/\bar{u}(p)$, $(\bar{v}(p)/v(p))$
   - Incoming/outgoing photons with momentum k are represented with polarisation vectors $\epsilon^\mu(k)$, $/\epsilon^{*\mu}(k)$
   - Propagators related to the external particles are "amputated"
2. Internal lines give rise to the corresponding propagators
3. Interactions are represented by vertices and corresponding expression. We impose four momentum conservation at each vertex, thereby fixing the momenta of internal lines.

4. i (imaginary unit) times the amplitude related to a single diagram is given by the product of all building blocks, the single-diagram amplitudes are summed to yield the overal amplitude.



$$i\mathcal{M} = \left[\bar{u}_\alpha(p_3)(-ie\gamma^\mu_{\alpha\beta})v_\beta(p_4)\right] \frac{-ig_{\mu\nu}}{(p_3+p_4)^2 + i\epsilon^+} \left[\bar{v}_\gamma(p_2)(-ie\gamma^\nu_{\gamma\delta}u_\delta(p_1)\right] \tag{12.2}$$

We now want to square this, so we sum over outgoing polarisations, average over incomings.

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = -\frac{e^4}{4}\sum_{spins}\left[\bar{u}_\alpha(p_3)\gamma^\mu_{\alpha\beta}v_\beta(p_4)\bar{v}_{\beta'}(p_4)\gamma^{\mu'}_{\beta'\alpha'}u_{\alpha'}(p_3)\right] \frac{g_{\mu\nu}g_{\mu'\nu'}}{(p_3+p_4)^4} \left[\bar{v}_\gamma(p_2)\gamma^\nu_{\gamma\delta}u_\delta(p_1)\bar{u}_{\delta'}(p_1)\gamma^{\nu'}_{\delta'\gamma'}v_{\gamma'}(p_2)\right] \tag{12.3}$$

$$\sum_{spin} u_\delta(p)\bar{u}_{\delta'}(p) = (\not{p} - m)_{\delta\delta'},\ \sum_{spin} v_\beta\bar{v}_{\beta'} = (\not{p} + m)_{\beta\beta'} \tag{12.4}$$

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{e^4}{4}\mathrm{Tr}\left[(\not{p}_3 - m_3)\gamma^\mu(\not{p}_4 + m_4)\gamma^\mu\right] \frac{g_{\mu\nu}g_{\mu'\nu'}}{(p_3+p_4)^4}\mathrm{Tr}\left[(\not{p}_2 + m_2)\gamma^\nu(\not{p}_1 - m_1)\gamma^{\nu'}\right] \tag{12.5}$$

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu},\ \mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma}) \tag{12.6}$$

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \mathrm{Tr}[pp\gamma^\rho\gamma^\mu\gamma^\delta\gamma^{\mu'}] - \mathrm{Tr}[m_3m_4\gamma^\mu\gamma^{\mu'}] \tag{12.7}$$

$$= 4p_{4\delta}p_{3\rho}\left(\gamma^{\rho\mu}g^{\delta\mu'} - g^{\rho\delta}g^{\mu\mu'} + g^{\rho\mu'}g^{\mu\delta}\right) - 4m_3m_4g^{\mu\mu'} \tag{12.8}$$

$$= \left[4(p_4^\mu p_3^{\mu'} + p_4^{\mu'}p_3^\mu - g^{\mu\mu'}p_3p_4) - 4m_\mu^2 g^{\mu\mu'}\right] \times \left[4(p_2^\nu p_1^{\nu'} + p_2^{\nu'}p_1^\nu - g^{\nu\nu'}p_1p_2) - 4m_e^2 g^{\nu\nu'}\right],\ m_i = 0 \tag{12.9}$$

$$= \frac{16e^4}{4(p_3+p_4)^4}\left[2(p_2p_4)(p_1p_3) + 2(p_1p_4)(p_2p_3) - 4(p_1p_2)(p_3p_4) + 4(p_1p_2)(p_3p_4)\right] \tag{12.10}$$

$$= \frac{8e^4}{(p_3+p_4)^2}\left[(p_2p_4)(p_1p_3) + (p_1p_4)(p_2p_3)\right] \tag{12.11}$$

Part II

Gauge Theories

Lecture 1

1.1 Plan

- Lectures 1-2: Introduction and Motivation, Intro to Group theory $\rightarrow$ Lie groups (continoues symmetries).
  - Why group theory?
    Gauge theories are quantum field theories with an emphasis on symmetry, as gauge theories have gauge symmetries. Mathematically, symmetries are described by group theory.
- Lecture 3: Different types of symmetries - global and local (gauge) symmetries.
- Lecture 4: From these gauge symmetries, we will construct Abelian, and non-Abelian gauge field theories.
- By the end of the course, we will learn about the Higgs mechanism and Spontaneous Symmetry Breaking, and ultimately reach the full Standard Model of Particle Physics.

The Standard Model is a gauge field theory of SU(3)×SU(2)×U(1) - this is the gauge group of the Standard Model.

- SU(3) is the gauge theory of strong interactions (QCD).
- SU(2)×U(1) is the gauge theory of the unified electroweak interactions.

1.2 Introduction to Group Theory

Groups are needed in order to describe and define symmetry transformations. So what is a group?

There are four properties that define a group:

1. **Closure** under group multiplication.

$$g_1 \cdot g_2 = g_3 \in G, \ \forall g_1, g_2 \in G \tag{1.1}$$

   So group multiplication is confined to within the bounds of the group.
2. **Associativity** of group multiplication.

$$g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3, \ \forall g_1, g_2, g_3 \in G \tag{1.2}$$

3. **Identity element**.

$$\exists\, e \in G \,:\, e \cdot g = g \cdot e = g, \ \forall g \in G \tag{1.3}$$

4. **Inverse element**.

$$\exists\, g^{-1} \in G \,:\, g^{-1} \cdot g = g \cdot g^{-1} = e, \ \forall g \in G \tag{1.4}$$

Now some notes:

- The group is called **Abelian** iff

$$g_1 \cdot g_2 = g_2 \cdot g_1, \ g_1, g_2 \in G \tag{1.5}$$

  This is equivalent to being called commutative, from $[g_1, g_2] = 0$.
- Then it holds that we have **non-Abelian** groups, where

$$g_1 \cdot g_2 \neq g_2 \cdot g_1 \tag{1.6}$$

  This is non-commutative, from $[g_1, g_2] \neq 0$.

Matrix multiplication (of square matrices) is an example of a non-Abelian group multiplication.

1.2.1 Important Examples

- SU(N) is a group of unitary $N \times N$ matrices, with det = 1. For SU(N), N is for $N \times N$ matrices, and the U says it is unitary. Unitary is defined by

$$SU(N) \ni U \, : \, U^\dagger \cdot U = U \cdot U^\dagger = \mathbb{I}_{N\times N}, \; U^\dagger = (U^*)^T \tag{1.7}$$

where $U^\dagger$ is called Hermitian conjugation. The S then defines the group as "Special", which means $\det(U) = 1$.
Let's check these group properties.
1. Matrix multiplication:

$$U_1 \in SU(N) : \; U_1^\dagger U_1 = \mathbb{I} \tag{1.8}$$

$$U_2 \in SU(N) : \; U_2^\dagger U_2 = \mathbb{I} \tag{1.9}$$

$$(U_1 U_2)^\dagger U_1 U_2 = U_2^\dagger \underbrace{U_1^\dagger U_1}_{\mathbb{I}} U_2 = U_2^\dagger U_2 = \mathbb{I} \tag{1.10}$$

$$\det(U_1 U_2) = \det(U_1) \cdot \det(U_2) = 1 \tag{1.11}$$

So we have **closure.**
2. Associativity is satisfied by the definition of matrix multiplication.
3. Unit matrix:

$$e = \mathbb{I}_{N\times N} \tag{1.12}$$

4. The inverse matrix element:

$$U^{-1} = U^\dagger \tag{1.13}$$

So we have a group that holds all the properties, a very important group at that.
Consider some general $U \in SU(N)$. How many real independent parameters (real degrees of freedom) does U have? An $N \times N$ complex matrix will have $2N^2$ real degrees of freedom. Now if we require unitarity, $U^\dagger = U$, there are N^2 constraints on the degrees of freedom, so now we are left with only N^2 degrees of freedom by this requirement. Now if we impose that $\det U = 1$, which is a single condition, we are left with $N^2 - 1$ real degrees of freedom.

- U(1) is a group of unitary 1×1 matrices, so $U^\dagger U = 1$.

$$\forall U \in U(1) \, : \, U = e^{i\alpha}, \; \alpha \in \mathbb{R} \tag{1.14}$$

Note we cannot require that the $\det U = 1$, otherwise we collapse down to a single value of this group, where $\alpha = 0$. We do not really need to check the group properties of U(1) as they are completely trivial.

- SO(N) is a group of $N \times N$ real-valued matrices which are orthogonal:

$$\forall O \in SO(N) \, : \, O^T \cdot O = O \cdot O^T = \mathbb{I} \tag{1.15}$$

So N is for $N \times N$, O is for orthogonal, and S again for det = 1.
SO(N) matrices are *proper* (we do not do parity transformations of $x \to -x$) rotations in the $\mathbb{R}^N$ (N-dimensional real vector space). It is again trivial to find the four group properties fully satisfied for SO(N), so these are groups again.

$$SO(2) \ni O = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{1.16}$$

We can see that for SO(2), we have one real parameter in $\theta \in \mathbb{R}, 0 \le \theta \le 2\pi$.

Lecture 2

Continuing last time, SO(2) is isomorphic to U(1):

$$SO(2):\ O = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad\qquad U(1):\ U = e^{i\theta} = \cos\theta + i\sin\theta \tag{2.1}$$

So both these groups depend only on the value of θ, and knowing any matrix in one of these groups allows us to construct the corresponding matrix in the other.

2.1 Group Theory Continued

2.1.1 Direct Products of Groups

Consider a group $G \ni \{g_1, g_2, \dots\}$, and another group $H \ni \{h_1, h_2, \dots\}$. We can define a product group from these, $G \times H$, which is also a group. The definition of this direct product is by construction. If we consider a matrix element of matrix $g \in G$, g_{ij}, and similarly for in H we consider the matrix element $h_{\alpha\beta}, h \in H$. Now we construct an object

$$g_{ij} \cdot h_{\alpha\beta} \equiv (gh)_{i\alpha;j\beta} \tag{2.2}$$

$$G \times H = \{g_{ij} \cdot h_{\alpha\beta}\} \tag{2.3}$$

Let us consider the example of $U(1) \times SU(2)$, where this direct product is equal to $U(2)$, which is a untary group, but the determinant is not $= 1$, but $\det = e^{i2\alpha}$, where α was the parameter of $U(1)$.

An important example to keep in mind is the Gauge Field Theory of the Standard Model: $SU(3) \times SU(2) \times U(1)$.

2.1.2 Simple vs Non-simple Groups

A simple group is defined as a group that cannot be written as a direct product of smaller groups, i.e. cannot be decomposed. A group U(N) is not simple, as

$$U(N) \approx U(1) \times SU(N), \tag{2.4}$$

where SU(N) and U(1) are both trivially simple groups.

2.1.3 Representations of Groups

We can describe group in two equivalent ways:

- A group is some formal mathematical structure - it is some set of elements which satisfies the four definitions of the group and some precise descriptions of what we mean by that.
- **The Fundamental Represenation of the Group** - we can derive a group via an explicit matrix representation, e.g. SU(N) are $N \times N$ complex matrices such that $U^\dagger U = 1 = UU^\dagger$ and $\det U = 1$.

Each group can have many different representations; the fundamental representation is what we used for its definitions. A group SU(N) in the fundamental representation is given by the $N \times N$ matrices. These matrices act on some N-dimensional complex vector space described by a N-vector.

$$\begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix} \cdot \begin{pmatrix} \bullet \\ \bullet \\ \bullet \end{pmatrix} \tag{2.5}$$

An N-vector, x_i, is transforming in the fundamental representation of SU(N).

We can also construct a tensor representation,

$$x_i = \begin{pmatrix} x_1 \\ x_2 \\ x_N \end{pmatrix} ; y_j = \begin{pmatrix} y_1 \\ y_2 \\ y_N \end{pmatrix} \in \mathbb{C}^N : \ x_i y_j = \text{rank-2 tensor, in SU(N)} \tag{2.6}$$

$$\sum_k \underbrace{U_{ik}}_{\text{SU(N) matrix}} \underbrace{x_k}_{\text{vector}} = \underbrace{x_i'}_{\substack{\text{transformed} \\ \text{vector}}} \quad \text{fund. rep.} \tag{2.7}$$

$$\sum_{j'} \sum_{i'} U_{ii'} U_{jj'} (x_{i'} y_{j'}) = (xy)_{ij} \tag{2.8}$$

These rank-two tensor representations can be decomposed into a singlet $\oplus$ traceless symmetric tensor $\oplus$ anti-symmetric tensor representations. These representations which cannot be reduced any further are called **irreducible representations.** So the fundamental representation of any group is irreducible, while rank-two tensor representation is reducible, as said above.

A group element in some general representation can be written as some matrix which can be brought into some block diagonal form, where off-diagonal elements are all zero, where each minimal block along the diagonal is an irreducible representation (irrep) of the group.

Lecture 3

3.1 Lie Groups

For a Lie group, G, with an element of this group, $a(\alpha^1, \ldots, \alpha^k)$, a depends continuously on parameters $\alpha^1, \ldots, \alpha^k$. Elements of Lie groups can be represented by

$$a = e^{-i\sum_{a=1}^{k} T^a\alpha^a}. \tag{3.1}$$

Here, the αs are our free parameters, and T^a are the generators of the Lie group, i.e. these are given matrices.

$$\alpha^i = 0\ \forall\, i \leq k,\ a = e^0 = \mathbb{I} \tag{3.2}$$

We can consider $\alpha^1, \ldots, \alpha^k \ll 1$ (infinitesimal),

$$a = e^{-i\sum_{a=1}^{k} T^a\alpha^a} = \mathbb{I} - i\sum_a T^a\alpha^a + \mathcal{O}(\alpha^2) \tag{3.3}$$

$$T^a = i\left.\frac{\partial a}{\partial \alpha^a}\right|_{\alpha^a=0} \tag{3.4}$$

For example, if $G = SU(2) \ni U_{2\times 2}$ (in the fundamental representation):

$$T^b_{2\times 2} = i\frac{\partial U_{2\times 2}}{\partial \alpha^b} \tag{3.5}$$

Now back to the general case of a Lie group (*from now on, the sum over repeated indices is assumed*),

$$G \ni a = \exp\left(-iT^a\alpha_a\right) \tag{3.6}$$

$$T^a = i\left.\frac{\partial a}{\partial \alpha^a}\right|_{\alpha^a=0} \tag{3.7}$$

This will find our generators for the Lie group, but these generators will not commute:

$$[T^a, T^b] = if^{abc}T^c \tag{3.8}$$

This is not an elephant, but another generator with some prefactor, where the f^{abc} is the structure constant of the Lie group. Following from the definition of the commutator relation, f^{abc} is completely anti-symmetric around its three indices.

Any given Lie group is defined by this relation in Eq (3.8). Essentially, the explicit form of the structure constants is what defines any given Lie group as distinct. From this relation, we can find the set of all generators, $\{T^a\}_{a=1}^k$, which will allow us to write down all elements of our Lie group, $a \in G$ through Eq (3.1).

3.1.1 Notations

Sometimes in the notes, we may use

$$a = e^{i\theta^a X_a},\ \theta^a = -\alpha^a,\ X^a = T^a \tag{3.9}$$

$$c^{abc} = f^{abc} \tag{3.10}$$

3.2 Some Simple Lie Groups

3.2.1 U(1) Group

The simplest example we'll have is $\mathrm{U}(1) \ni a$: here, we have $T = 1$, and the number of generators is also 1.

$$a = e^{-i\alpha\cdot 1},\ T = 1 \tag{3.11}$$

$$[T, T] = [1, 1] = 0 \implies f^{abc} = 0 \tag{3.12}$$

For all Abelian Lie groups, the commutators are zero (by definition).

3.2.2 SU(2) Group

$\mathrm{SU}(2) \ni U$:

$$U = e^{-i\alpha^a T_a} \tag{3.13}$$

So we want to know:

- How many T^as are there, i.e. k? And what are the generators of SU(2) in the fundamental representation?
  We will start by choosing the fundamental representation, where we have 2×2 complex matrices with $U^\dagger U = \mathbb{I}$ and $\det U = 1$.
  $$U = \exp\left(-i\sum_{a=1}^{3}\alpha^a \frac{\sigma^a}{2}\right) \tag{3.14}$$
  So we have three real parameters in α^a and the generators, $\frac{\sigma^a}{2}$ are the Pauli matrices (over 2). Is this right? Well Pauli matrices are Hermitian, and they provide us with a complete basis of 2×2 Hermitian matrices (-1, as it excludes the unit matrix). Let's consider the Hermitian conjugate of U to check:
  $$U^\dagger = \left[\exp\left(-i\alpha^a \frac{\sigma_a}{2}\right)\right]^\dagger = \exp\left(+i\alpha^a \frac{\sigma_a}{2}\right) = U^{-1} \tag{3.15}$$
  So we have unitarity, and we can easily check the other group properties if needed.
  We have learned that the generators of SU(2) (in the fundamental representation) are
  $$T^a = \frac{\sigma^a}{2},\ a = \{1, 2, 3\} \tag{3.16}$$
  This agrees with the argument of free parameters from last lecture where SU(N) has $N^2 - 1$ free parameters, which for SU(2) requires three free parameters, which we have in our three generators. But why is it $\frac{\sigma}{2}$ and not σ? The $\frac{1}{2}$ factor is due to normalisation, and depends on how we choose normalisation. In the fundamental representation, we choose
  $$\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}, \tag{3.17}$$
  and for $T^a = \frac{\sigma^a}{2}$, we fulfill this requirement.
- What is the Lie algebra of SU(2), i.e. f^{abc}?
  $$\left[\frac{\sigma^a}{2}, \frac{\sigma^b}{2}\right] = i\epsilon^{abc}\frac{\sigma^c}{2} \tag{3.18}$$
  We can check this directly using the defining properties of Pauli matrices, or just working it out by hand. So our structure constants of SU(2) are ϵ^{abc}, and now we have our full Lie algebra for SU(2):
  $$[T^a, T^b] = i\epsilon^{abc}T^c \tag{3.19}$$
- What about choosing in another representation than the fundamental one?
  We can choose any representation of SU(2), and we may get different descriptions of generators, but the Lie group is always defined by Eq (3.8), and for SU(2), $f^{abc} = \epsilon^{abc}$ in any representation, but the simplest one will always be the fundamental one.

Lecture 4

4.1 Symmetries: An Introduction

Recall: The Fundamental representation of SU(N) has $T^a_{N\times N}$ acting on $\phi = \begin{pmatrix} x_1 \\ n_N \end{pmatrix} \in \mathbb{C}$. The conjugate to the fundamental representation takes $\phi^\dagger$ instead.

In the adjoint representation, $(T^a)_{bc} = -if^{bca}$, a running from 1 to the number of generators, which is $N^2 - 1$ for SU(N).

Symmetries:
Let $\phi(x)$ be a field. Its Lagrangian is

$$\mathcal{L} = \frac{1}{\lambda}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\phi^4 \tag{4.1}$$

for a simply-interacting real scalar field, $\phi(x) \in \mathbb{R}$. The kinetic term is quadratic (bilinear) in the field, the interaction term is of a higher order. The kinetic terms yield propagators, here being $\frac{i}{p^2-m^2}$. The interaction terms yield vertices in the Feynman diagrams.

For a complex field,

$$\mathcal{L} = (\partial_\mu\phi^\dagger)(\partial^\mu\phi) - m^2\phi^\dagger\phi - \frac{\lambda}{4!}(\phi^\dagger\phi)^2 \tag{4.2}$$

For a general field, the action is

$$S = \int \mathcal{L}[\phi]\ d^4x \tag{4.3}$$

A transform of the fields which leaves the action invariant is called a symmetry. Additionally, a quantum theory must also leave the vacuum invariant.

Symmetries can be discrete, e.g. $\phi(x) \to -\phi(x)$, or continuous, e.g. $\phi(x) \to e^{-i\alpha(x)}\phi(x), \alpha \in \mathbb{R}$. The Lagrangian of a real scalar field is invariant under $\phi \to \phi(x)$, so the action is also invariant. Thus $\phi \to -\phi$ is a symmetry of this theory; for the complex scalar Lagrangian, $\phi \to e^{i\alpha}\phi$ is a symmetry. A continous symmetry can be local, depending on x_N, or global, being independent of x_N, i.e. if α in $e^{i\alpha}\phi$ is $\alpha(x)$ then it is local, and global if just α.
Continuous transforms are described by Lie groups. Global continuous symmetries provide conserved quantities - this is the basis of Noether's theorem which states that for every generator of a global continuous system, there exists a conserved current $j^\nu(x)$ such that $\partial^\nu j = 0$. This is a Noether current. For U(1), $\phi \to e^{-i\alpha}\phi$, which gives us the conserved electric charge,

$$Q = \int j^\nu(x)\ d^3x,\ \frac{dQ}{dt} = 0 \tag{4.4}$$

Noether's theorem does not necessarily hold for local symmetries. These are gauge symmetries. We consider a local U(1) transform, $\phi \to e^{-i\alpha(x)}\phi$, with the Dirac Lagrangian for free Dirac fermions,

$$\mathcal{L} = \bar{\psi}(i\gamma^\nu\partial_\nu - m)\psi \tag{4.5}$$

$$\mathcal{L} \to \mathcal{L} + \bar{\psi}i\gamma^\nu\partial_\nu(-i\alpha(x))\psi \text{ - not invariant} \tag{4.6}$$

To attain invariance, we add a gauge field A, that transforms as $A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x)$, e being the gauge coupling constant. We then replace the standard derivative with a covariant derivative of the form,

$$D_\mu = \partial_\mu + ieA_\mu \tag{4.7}$$

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi \tag{4.8}$$

We now have a gauge invariant Lagrangian above which describes not only the propagatino of $\bar{\psi}$ and ψ, but it also includes the interaction between A_μ and ψ. To give A_μ, the photon propagation, we include a term $-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ in the Lagrangian, where,

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{4.9}$$

which is plainly gauge invariant in the U(1) case. Thus,

$$\mathcal{L}_{QED} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{4.10}$$

Lecture 5

5.1 QED

The Lagrangian of QED:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{5.1}$$

QED is a U(1) gauge invariance theory, with Dirac fermion fields $\psi, \bar{\psi}$, and the gauge field A_μ, which is a vector field. The Dirac fields describe $e^\pm$, and the gauge field describes the photons, γ.

If we consider just the gauge field part of QED:

$$\mathcal{L}[A_\mu] = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{5.2}$$

We can then begin to write down equations of motion from this, first considering the action.

$$\mathcal{S} = \int \mathcal{L}[A_\mu]\, d^4x \tag{5.3}$$

We take the extremum of the action, where $\frac{\delta \mathcal{S}[A]}{\delta A_\nu} = 0$, to find the Euler-Lagrange equations.

$$\partial_\mu \frac{\partial \mathcal{L}[A]}{\partial(\partial_\mu A_\nu)} = \frac{\partial \mathcal{L}[A]}{\partial A_\nu} = 0 \tag{5.4}$$

$$\partial_\mu F^{\mu\nu} = 0 \tag{5.5}$$

However, if we look at the full Lagrangian again and fully express $D_\mu = \partial_\mu + ieA_\mu$, then Eq (5.5) is no longer equal to zero when fermions are present.

$$\partial_\mu F^{\mu\nu} = e\bar{\psi}\gamma^\nu\psi \equiv j^\nu \tag{5.6}$$

These are two of the Maxwell equations (out of four) for QED. The other two Maxwell equations are trivial in QED, following from the Bianchi identity,

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0. \tag{5.7}$$

This identity is automatically satisfied for $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. So all four classical Maxwell equations can be written in QED as

$$\frac{\partial S}{\partial A_\nu} = 0 \implies \partial_\mu F^{\mu\nu} = j^\nu \tag{5.8}$$

$$\frac{\partial S}{\partial \bar{\psi}} = 0 \implies (i\gamma^\mu D_\mu - m)\psi = 0 \tag{5.9}$$

Eq (5.9) is the Dirac equation with the field A_μ in the D_μ term.

5.1.1 How many degrees of freedom does the photon field have?

Naively, it looks like it has four degrees of freedom, as we have $A_\nu, \nu = 0, 1, 2, 3$. In reality, there are only two physical degrees of freedom of the photon. Why is that? A_0 field decouples $\rightarrow$ it is not a dynamical field as it does not have a kinetic term, i.e. $\frac{1}{2}(\partial_t A_0(x))^2$ is absent in the Lagrangian, but this term is required for any fieldto be kinetic as it would describe velocity.
We can always fix the gauge freedom by setting $A_0 \equiv 0$. We can further set $\partial_i A_i = 0$ - this is the Coulomb gauge. So we have two constraints from fixing the gauge, so two *unphysical* degrees of freedom

are removed, leaving us with $4-2=2$ physical degrees of freedom (assuming unbroken gauge invariance). These 2 degrees of freedom of the photon are its 2 transverse polarisations.

A_μ describes spin-1 fields (or particles), and we have the two gauges of $A_0 = 0$ and $\partial_i A_i = 0 \implies p_i A_i = 0$, where p_i is the three-momenta. If we then choose momentum to be wholly along the z-coordinate, so $\underline{p} = (0, 0, p)$, then $\underline{A} = (A_1, A_2, 0)$, and then we see that we have two transverse polarisations along x and y, while the momentum transfer is all in z - this is the simplest choice for example but any physical orientation of $\underline{p}$ and $\underline{A}$ will lead to the same two transverse polarisations.

5.1.2 What is allowed in QED?

Again, we write down the QED Lagrangian:

$$\begin{aligned}\mathcal{L}_{QED} =& -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \text{ - propagator of the photon field } A_\mu\text{, quadratic in it}\\ &+ \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \text{ - free propagation of fermions}\\ &- e\bar{\psi}\gamma^\mu A_\mu \psi \text{ - 3 point vertex describing interactions}\end{aligned} \tag{5.10}$$

$\mathcal{L}_{QED}$ is uniquely constructed from the requirement of gauge invariance. Can we add other gauge invariant interactions? Consider

$$\mathcal{L} = F_{\mu\nu}F^{\nu\alpha}F_\alpha{}^\mu \tag{5.11}$$

For mass dimension, [ ]: we have

$$[A_\mu] = 1 \qquad [\phi] = 1 \qquad [\mathcal{L}] = 4 \tag{5.12}$$
$$[\psi] = \frac{3}{2} \qquad [\bar{\psi}] = \frac{3}{2} \qquad [\mathcal{S}] = 0 \tag{5.13}$$

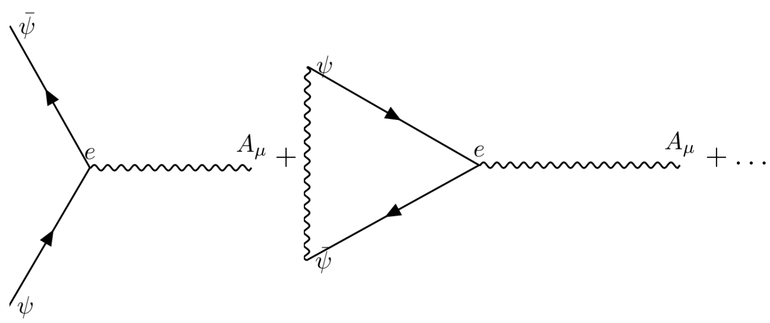
So we can see the mass dimension of the above $\mathcal{L}$ will be 6. We could add it to the QED Lagrangian with some pre-factor to get it to work?

$$\mathcal{L} = \mathcal{L}_{QED} + \frac{1}{M^2}F_{\mu\nu}F^{\nu\alpha}F_\alpha{}^\mu \tag{5.14}$$

Any terms of $\mathcal{L}$ that have a coefficient of negative mass dimension in front are not UV-renormalisable, and any operators is $\mathcal{L}$ of mass dimension greater than 4 are not renormalisable.

5.1.3 What is UV Renormalisation?

Any Quantum Field Theory which constains quantum corrections such as a tree level interaction, then one loop correction and up to higher loops, e.g.



(5.15)

All loop-level corrections contain ∞ in the UV, so we need to have a prescription to remove this divergence.

UV Renormalisation is the prescription to remove UV divergences.

So now $\mathcal{L}_{QED}$ is completed and is shown that we cannot remove or add anything from/to it.

Lecture 6

6.1 Non-Abelian Gauge Invariance

Start from a free Lagrangian for Dirac fermions

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \tag{6.1}$$

$\psi(x)$ transform in the fundamental representation of a Lie group, G. Assume that G is a SU(N) group, so $\psi(x)$ is a column vector with N rows $\bar{\psi}(x)$ will transform in the anti-fundamental representation, or conjugate to fundamental, as a row vector of N columns. This is invariant under global transformations.

$$\psi(x) \to U\psi(x),\ \bar{\psi}(x) \to \bar{\psi}(x)\, U^\dagger,\ U \in SU(N) \tag{6.2}$$

Here, U is not dependent on x, i.e. a global symmetry. The Lagrangian will be invariant. Upgrade this construction to $U(x)$, i.e. a gauge (local) symmetry.

$$\psi(x) \to U(x)\,\psi(x),\ \bar{\psi}(x) \to \bar{\psi}(x)\, U^\dagger(x),\ U(x) = e^{-i\alpha^a(x)T^a} \in SU(N) \tag{6.3}$$

Here, T^a are the generators of SU(N) in the fundamental representation. But here we will not see an invariant Lagrangian, so we must add something to the Lagrangian:

$$A_\mu(x) \to U(x)(A_\mu(x) + i\partial_\mu)U^\dagger(x) \tag{6.4}$$

In the fundamental representation, $U(x)$ is an $N \times N$ matrix, and A_μ must also be one for the equations to make sense - but this is still a single gauge field. The gauge field in matrix notation is then,

$$A_\mu(x) = gT^a A^a_\mu(x). \tag{6.5}$$

A^a_μ is not a gauge field but a component gauge field, $a = 0, \ldots, N^2 - 1$; g is the gauge coupling constant, i.e. a non-Abelian generalisation of e.

We can now form some trace identities (in the fundamental representation):

$$\mathrm{Tr}[T^a T^b] = \frac{1}{2}\delta^{ab} \qquad \mathrm{Tr}[A_\mu T^a] = \frac{g}{2}\sum_b A^b_\mu \delta^{ab} = \frac{g}{2}A^a_\mu \qquad A^a_\mu = \frac{2}{g}\mathrm{Tr}[A_\mu T^a] \tag{6.6}$$

Now we can set out requirements to make our original $\mathcal{L}$ gauge invariant:

- By transforming the derivative:

$$\partial_\mu \to D_\mu = \partial_\mu - iA_\mu \tag{6.7}$$

  This is the **Covariant Derivative.**
- We must form a kinetic term for A_μs so that these are dynamical gauge fields

$$\mathcal{L}_{kin} = -\frac{1}{2g^2}\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}] \tag{6.8}$$

  $F_{\mu\nu}$ is the field strength:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \tag{6.9}$$

  This is almost the field strength that we had for QED, but we must add the final commutator term in order to make it transform as we are now in non-Abelian algebra. $F_{\mu\nu}$ is the field strength in matrix notation, and can be written in component notation as well:

$$F_{\mu\nu}(x) = gT^a F^a_{\mu\nu}(x) \qquad F^a_{\mu\nu} = \frac{2}{g}\mathrm{Tr}[T^a F_{\mu\nu}] \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu \tag{6.10}$$

And our Lagrangian becomes

$$\mathcal{L} = -\frac{1}{2g^2}\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}] + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi. \tag{6.11}$$

We still need to check if this is gauge invariant or not. How do $D_\mu\psi$ and $F_{\mu\nu}$ transform under gauge transforms?

$$F_{\mu\nu} \to U\, F_{\mu\nu}\, U^\dagger \qquad\qquad D_\mu\psi \to U\, D_\mu\psi \tag{6.12}$$

So we see that $F_{\mu\nu}$ transforms in the adjoint representation, and $D_\mu\psi$ transforms as ψ. How do we check these?

➤ How to check for $F_{\mu\nu}$?

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu],\ A_{\mu,\nu} \to U(x)(A_{\mu,\nu} + i\partial_{\mu,\nu})U^\dagger(x) \tag{6.13}$$
$$\to U\, F_{\mu\nu}\, U^\dagger \tag{6.14}$$

➤ How to check for $D_\mu\psi$?

$$D_\mu\psi = (\partial_\mu + iA_\mu)\psi,\ \psi \to U(x)\psi(x),\ A_\mu \to U(x)(A_\mu + i\partial_\mu)U^\dagger(x) \tag{6.15}$$
$$\to U(x)D_\mu\psi \tag{6.16}$$

We notice that that gauge transformation for $A_\mu \to (A_\mu(x) + \partial_\mu)$ is really $A_\mu \to iD_\mu$.

So under gauge transformation, our Lagrangian transforms as:

$$\mathcal{L} \to -\frac{1}{2g^2}\mathrm{Tr}[UF_{\mu\nu}U^\dagger UF^{\mu\nu}U^\dagger] + \bar{\psi}U^\dagger(i\gamma^\mu UD_\mu\psi - mU\psi) \to \mathcal{L} \tag{6.17}$$

Hence, the Lagrangian is completely gauge invariant.

This is a universal prescription. For any Lagrangian with any matter fields (scalars, fermions, whatever else) that is invariant under global symmetry, performing the transformations

$$\partial_\mu \to D_\mu \qquad\qquad \oplus\mathcal{L}_{kin} = -\frac{1}{2g^2}\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}] \tag{6.18}$$

will induce a gauge-invariant Lagrangian. Thus, we have constructed a non-Abelian gauge theory.

Where does the kinetic pre-factor come from?

$$\mathcal{L}_{kin} = -\frac{1}{2g^2}\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}] = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{6.19}$$
$$F_{\mu\nu} = gT^aF^a_{\mu\nu},\ \mathrm{Tr}[T^aT^b] = \frac{1}{2}\delta^{ab} \tag{6.20}$$

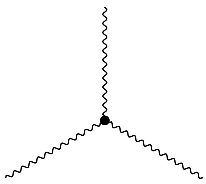
This gauge theory is known as Yang-Mills theory, which is just meaning a non-Abelian theory. This theory is no longer free. It automatically contains interactions through the inclusion of A_μ which forms three-point vertices (interactions): between the fermions and gauge bosons, three gauge bosons; and four-point vertices between four gauge bosons. This is not something we saw in QED, but nonetheless required for the Lagrangian to be physical at all.

Lecture 7

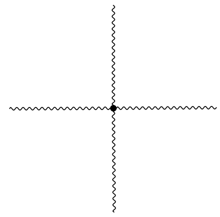
We want to find a Lagrangian with terms with powers > 2 of gauge fields, i.e. interactions of A_μs.

$$\mathcal{L} = \cdots - g f^{abc} \partial_\mu A_\nu^a A^{\mu b} A^{\nu c} - \frac{g^2}{4} f^{abc} f^{ade} A_\mu^b A_\nu^c A^{\mu d} A^{\nu e} + \text{ nothing else} \tag{7.1}$$

From the first term, we can see that there interactions involving three-point vertices of gauge fields of the form $g f^{abc} p^\mu$:



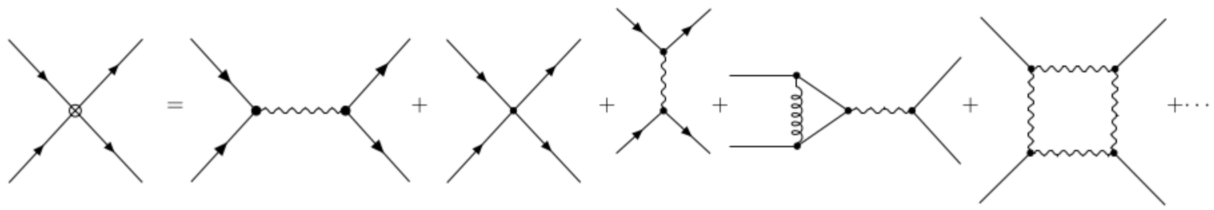
From the second, we find four-point vertices of the form $g f^{abc} f^{ade}$:



Interactions are automatically included and there are no higher levels of gauge field interactions. Both the terms above had to appear in the Lagrangian as they come from $\frac{1}{2g^2}\text{Tr}[F_{\mu\nu}F^{\mu\nu}]$. The relative coefficients between the three- and four-point vertices, and also between vertices with fermions, are fixed and uniquely determined by gauge invariance of the total Lagrangian. In a non-Abelian gauge theory, we cannot arbitrarily change the coupling constant g, e.g. if we coupled a matter field ϕ_1 to A_μ with a coupling constant g, and try to couple some other matter field ϕ_2 to A_μ with a coupling constant κg, then $\kappa = 1$. If $\kappa \neq 1$, then the f^{abc}s will be altered by this, which cannot be done as these define the group and will be constant. However, in an Abelian theory, we can couple different matter fields to gauge fields with κg, where κ is arbitrary.

7.1 Running Coupling Constants

In a classical gauge theory, the coupling constant is a constant. In a quantum gauge theory, it is no longer constant; instead, it depends on the energy scale at which we make observations. For a scattering experiment, at energy scale, $E = \sqrt{s}$, $s = (p_1 + p_2)^2$.



At tree level, processes are $\propto g^2$; at loop level, $\propto g^{2(1+\text{no. of loops})}$. When we extract the value of g^2 from any such measurement, we will get $g^2(E)$. We usually consider $g^2(p)$, where p is a momentum or energy value characteristic for the experiment, e.g. can be $p = E_{COM}$, p is either $\sqrt{s}$ or total transverse momentum (total momentum transfer). Consider the coupling constant defined

$$\alpha(p) \equiv \frac{g^2(p)}{4\pi}. \tag{7.2}$$

In QED, we have $\alpha_{QED} = \frac{e^2}{4\pi}$. In QCD, we have a SU(3) gauge theory with N_f flavour of quarks (fermions), with a coupling constant (computed to 1 loop order),

$$\alpha_s(p) = \frac{g_{SU(3)}^2(p)}{4\pi} = \frac{2\pi}{b_0 \log \frac{p}{\Lambda_{QCD}}} \tag{7.3}$$

Here, $\Lambda_{QCD} \approx 300\, MeV$, $b_0 = 11 - \frac{2}{3}N_f$.

Lecture 8

For non-Abelian gauge theory, all matter fields couple to A_μ^a with the same gauge coupling g:

$$D_\mu = \partial_\mu - igA_\mu^a T^a. \tag{8.1}$$

For Abelian, this is not the case:

$$D_\mu = \partial_\mu + ieYA_\mu, \tag{8.2}$$

where we define $Y \equiv$ hypercharge - an arbitrary factor that can be different between matter fields.

In a non-Abelian theory, we can compute the coupling constant $\alpha(p)$, defined in Eq (7.2), as

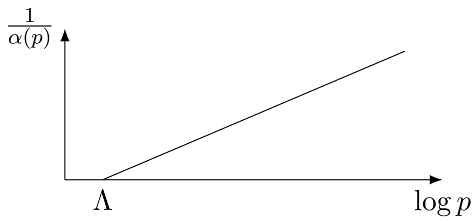
$$\alpha(p) = \frac{2\pi}{b_0 \log \frac{p}{\Lambda}} + \text{ higher-order corrections.} \tag{8.3}$$

$$\frac{2\pi}{\alpha(p)} = b_0 \log\left(\frac{p}{\Lambda}\right). \tag{8.4}$$

In an SU(N) gauge theory with N_f flavours of Dirac fermions,

$$b_0 = \frac{11}{3}N - \frac{2}{3}N_f. \tag{8.5}$$

Strong interactions in the Standard Model are described by QCD, the non-Abelian SU(3) gauge theory with $N_f = 6$ quarks. In QCD, $b_0 = 7$, and what is important to note about that is that it is positively valued. b_0 is called the first coefficient of the β-function - the sign of the β-function and the sign of its first coefficient determines how the coupling constant runs.



We can see at the scale of Λ, the coupling constant α becomes infinitely strong. More carefully:

- Assume that our 1-loop approximation to $\alpha(p)$ is correct
- $p \to \Lambda \implies \frac{1}{\alpha} \to 0 \implies \alpha \to \infty$, so we have an infinite strength interaction, which results in the confinement of quarks and gluons.

This is non actually a full proof as the assumption is incorrect, but it does result in the real picture of confinement.
Similarly, $p \to \infty \implies \frac{1}{\alpha(p)} \to \infty \implies \alpha(p) \to 0$, which results in *asymptotic freedom*, i.e. at high energies, QCD interactions become irrelevant. The theory of QCD at high energies becomes free, or non-interacting.

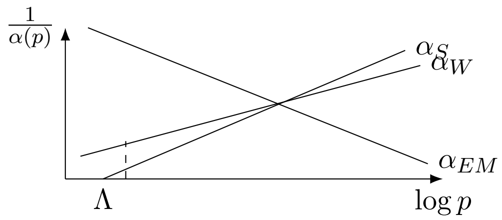
Lecture 9

9.1 The Standard Model

The Standard Model of particle physics is an SU(3)$_c$⊗SU(2)$_L$⊗U(1)$_Y$ gauge theory. SU(3) is the theory of strong interaction (QCD); and SU(2)⊗U(1) is the GWS electroweak theory, where SU(2) describes what will become weak interactions, and U(1) describes what will become electromagnetic interactions (after spontaneous symmetry breaking).

- $b_0 > 0$ for non-Abelian gauge theories (if $N_f <$ some critical number)
- $b_0 < 0$ for Abelian theories like U(1)

If we define three coupling constants: for QCD, α_S; for weak, α_W; and for QED, α_{EM}.



- At the electroweak scale $\approx m_{W,Z,H} \approx 100$ GeV, the weak force fails as its mediators are no longer present.
- EM will hit zero in the graph at what is known as Λ_{Landau} and freaks out from there.
- EM will fail like weak at the electron mass scale.
- All three coupling constants almost converge on a single point in the middle, but not quite. If they did, it would be indicative of a Grand Unified Theory (GUT) scale, i.e. SU(5)$_{GUT}$ → SU(3)⊗SU(2)⊗U(1). A single gauge theory of SU(5) would split into the three known groups at lower energy scales, below the GUT scale. SU(5) is currently ruled out, but maybe SO(10)?
- α_W is small for all regions where the weak force manifests, so it can always be studied in perturbation theory.
- α_{EM} is fine in the IR regime (low energy) but then freaks out at higher energies, and cannot be studied perturbatively.

9.2 QCD

- QCD has gauge group SU(3).
- What are the matter fields of QCD? Dirac fermions known as quarks, which transform in the fundamental representation of SU(3). Quarks are in triplet states of colour, $q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$, where q_i indicates the colour (1,2,3). There are six flavours of quarks as well, so q_i^f where f is the flavour running $1 \to 6$ and i is the colour running $1 \to 3$. (Of course, each quark is also a Dirac spinor of 4 components.)
- In the infrared regime (low energy), $\alpha_S \to \infty$ which implies colour confinement.
- We cannot find free quarks or gluons in nature (in the IR range).
- What we can observe instead are colourless (SU(3) singlets) composite particles, i.e. mesons and baryons.
- Mesons are quark-antiquark pairs: $\bar{q}_i^{f_1} q_i^{f_2}$, e.g. π-meson:

$$\pi^0 = \frac{\bar{u}u + \bar{d}d}{\sqrt{2}}, \qquad \pi^- = \bar{u}d, \qquad \pi^+ = \bar{d}u. \tag{9.1}$$

These are the lightest mesons using only the first generation of quarks. $Q(u) = \frac{2}{3}$, and $Q(d) = -\frac{1}{3}$.

- Baryons are three quark states each with different colour:

$$\sum_{ijk}^{3} \epsilon^{ijk} q_i^{f_1} q_j^{f_2} q_k^{f_3}. \tag{9.2}$$

For example, a proton is (uud) and neutron (udd).
- The gauge fields of QCD are the $A_\mu^{a=1\to 8}$, which are the massless gluons we know so we have an unbroken (exact) SU(3) gauge theory.

9.3 Electroweak theory

- For SU(2), we have the gauge fields $A_\mu^{a=1\to 3}$.
- What are the matter fields?
- There is a single scalar field of SU(2):

$$H = \begin{pmatrix} h^+ \\ h^0 \end{pmatrix}, \tag{9.3}$$

which is the Higgs field. H is a doublet of SU(2), as it transforms in the fundamental representation.
- If we compute a vacuum expectation value of the Higgs field,

$$\langle 0|H|0\rangle \equiv \langle H\rangle = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}, \tag{9.4}$$

where $v \neq 0$. In fact, $v \approx 246$GeV. More on this later.
- All other matter fields are fermions transforming under the fundamental representation of SU(2), so they are doublets. We have both quarks and leptons. Leptons are defined as fermions which do not transform under $\mathrm{SU}(3)_{QCD}$.
- The six quarks we have from 3 SU(2) doublets:

$$\begin{pmatrix} u \\ d \end{pmatrix}_L, \begin{pmatrix} c \\ s \end{pmatrix}_L, \begin{pmatrix} t \\ b \end{pmatrix}_L. \tag{9.5}$$

- Only left-handed fermions interact with SU(2):

$$\psi = \psi_L + \psi_R, \qquad \psi_L = \frac{(1-\gamma_5)}{2}\psi, \qquad \psi_R = \frac{(1+\gamma_5)}{2}\psi. \tag{9.6}$$

So the above quark doublets are all left-handed, the right-handed components are all SU(2) singlets (non-interacting), i.e. u_R, d_R etc.
- Leptons also form three doublets:

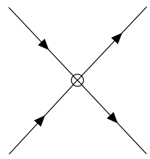
$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L. \tag{9.7}$$

Each lepton also has its right-handed singlet, such as e_R, μ_R, in principal. However no right-handed neutrinos have been observed so we say there are none.
- So we have three families, or generations, of both quarks and leptons.

Lecture 10

So far we discussed how the Standard Model is a gauge field theory with symmetry SU(3)$_{QCD}\otimes$ SU(2)$_W\otimes$U(1)$_Y$.

- We define the gauge fields of the electroweak sector as $W^{\pm}_{\mu}, Z^0_{\mu}, A_{\mu}$, so altogether four (3 from SU(2) and 1 from U(1)).
- The photon is strictly massless because the gauge symmetry of U(1)$_{QED}$ is unbroken.
- There is a way to break a gauge symmetry spontaneously to induce masses for the other electroweak gauge fields.
- SU(2)$_W\otimes$U(1)$_Y$ $\rightarrow$U(1)$_{QED}$ from spontaneous symmetry breaking.
- In the limit where the centre-of-mass energy $E_{com} \ll M^2_{Z,W^{\pm}}$, then the propagator can be reduced to $-\frac{g_w^2}{M^2_{Z,W^{\pm}}}$, and the interaction can be reduced to a point-like interaction between four fermions.



10.1 Spontaneous Symmetry Breaking

We can have:

- discrete symmetries, e.g. $\phi \rightarrow -\phi$
- continuous symmetries
  - global, e.g. $\phi(x) \rightarrow e^{-i\alpha}\phi(x)$
  - local (gauge), e.g. $\phi(x) \rightarrow e^{-i\alpha(x)}\phi(x)$

These symmetries can be one of three cases:

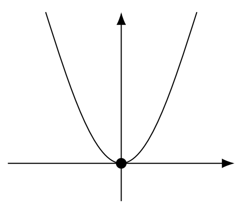
- exact, not broken
- explicitly broken, adding clear terms in the Lagrangian to break the symmetry
- spontaneously broken

Spontaneous symmetry breaking preserves the invariance of the Lagrangian and Action under the field transformation, but the vacuum state of the Hilbert space is not invariant.

- Take a real scalar field $\phi(x)$, and write its Lagrangian

$$\mathcal{L} = \underbrace{\frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi}_{K} \underbrace{-\frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\phi^4}_{-V(\phi)} \tag{10.1}$$

- Consider the discrete field transformation $\phi(x) \rightarrow -\phi(x)$: the Lagrangian is invariant.

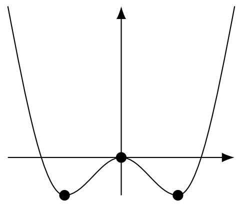


- There is a global minimum at $\phi = 0$; the ground state of the theory (or the vacuum state) is at $\phi = 0$.
- $\langle 0|\hat{\phi}(x)|0\rangle = \langle\phi\rangle = 0$: this the vacuum expectation value (VEV).
- In $\phi \rightarrow -\phi$, the symmetry is exact and is not spontaneously broken because $\langle\phi\rangle = 0$ ($0 \rightarrow -0$).
- What if we consider $m^2 \rightarrow -m^2$?

$$\mathcal{L} = \frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi + \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\phi^4 \tag{10.2}$$

$$V(\phi) = -\frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4 \tag{10.3}$$

So we now have not a minimum but a local maximum at $\phi = 0$, and two degenerate local minima at $\phi = \pm v$, $v = \frac{m}{\sqrt{\lambda}}$.



- $\langle \phi \rangle = \pm v$ - the universe has to spontaneously choose a vacuum, e.g. $\langle \phi \rangle = +v$ yields multiparticle states again using the creation operator $\hat{a}^\dagger$ and we have the Hilbert space, but the vacuum is not invariant under the transformation: $\langle \phi \rangle = v \to -v \neq v$. Thus, a spontaneously broken discrete symmetry of $\phi \to -\phi$.
- There are no interesting implications from this, so we need to look at continous symmetries to find some physical meaning.
- We are going to consider spontaneous breaking of a global continuous symmetry. Now taking a complex scalar field $\phi(x) \in \mathbb{C}$, and noting the Lagrangian as

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + m^2 |\phi|^2 - \frac{\lambda}{4} |\phi|^4. \tag{10.4}$$

- Again we consider $m^2 \to -m^2$, and look at the potential:

$$V(\phi) = -m^2 |\phi|^2 + \frac{\lambda}{4} |\phi|^4. \tag{10.5}$$

  We can equivalently rewrite it in the form

$$V(\phi) = \frac{\lambda}{4} \left(|\phi|^2 - v^2 \right)^2. \tag{10.6}$$

  $V(\phi)$ has the same form as it did before, but now rotated around z-axis as scalar field is complex, arriving at the Mexican hat potential. *find image of Mexican hat potential* There is now a whole circle of minima at $|\phi| = v$, so $\langle \phi \rangle = v e^{i\xi}$. The universe has to choose a single vacuum state spontaneously.
- Let's say we choose $\langle \phi \rangle = v$, i.e. $\xi = 0$. (It doesn't matter which we choose, but it's easiest to choose $\xi = 0$ for mathematical convenience.)
- So $\phi \to e^{-i\alpha} \phi$ keeps the Lagrangian invariant, but it does not keep the VEV invariant, so we do indeed have a spontaneously broken global continuous symmetry.

Lecture 11

11.1 Spontaneous Breaking of a Continuous Global Symmetry

For a scalar field $\phi(x) \in \mathbb{C}$, we consider a global U(1) symmetry, i.e. $\phi(x) \to e^{-i\alpha}\phi(x), \alpha \in \mathbb{R}$. We must look at the Lagrangian for this symmetry:

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - \lambda \left(|\phi|^2 - \frac{v^2}{2} \right)^2 . \tag{11.1}$$

Note we have made a slight change of normalisations from the previous lecture, but the constants don't matter as long as we are consistent throughout any example. Following the Mexican hat potential for $V(\phi)$, we have a circle of local minima at $|\phi| = \frac{v}{\sqrt{2}}$. Our U(1) symmetry is spontaneously broken like before, by the choice of a particular minima as the vacuum: $\langle \phi \rangle = \frac{v}{\sqrt{2}}$. Again, the Lagrangian will be invariant, but the vacuum state is not.
So now, we need to give a particle interpretation to the field. We build up the Hilbert space above the vacuum as small fluctuations from the vacuum to build a multi-particle state. To this end, we shift the scalar field $\phi(x)$ by the VEV:

$$\phi(x) - \frac{v}{\sqrt{2}} = \phi(x) - \langle \phi \rangle = \chi(x), \tag{11.2}$$

and it is fluctuations in this field χ which give rise to particle creation operators $\hat{a}^\dagger$, owing to the zero VEV of χ, $\langle \chi \rangle = 0$. We write χ as

$$\chi(x) = \frac{1}{\sqrt{2}} \left(\phi_1(x) + i\phi_2(x) \right), \tag{11.3}$$

where ϕ_i are both **real** scalar fields. So the original scalar field ϕ would be

$$\phi(x) = \frac{1}{\sqrt{2}} \left(v + \phi_1(x) + i\phi_2(x) \right). \tag{11.4}$$

We must check that this change of field variables is still allowed in the Lagrangian. The Lagrangian will become

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)(\partial^\mu \phi_1) + \frac{1}{2}(\partial_\mu \phi_2)(\partial^\mu \phi_2) - \frac{\lambda}{4} \left((v + \phi_1(x))^2 + \phi_2(x)^2 - v^2 \right)^2 . \tag{11.5}$$

We can see from direct comparison that this is the same equation as Eq (11.1) just with a variable change. So it is still invariant. If we consider the potential term of this,

$$\begin{aligned} -V(\phi) = & -\frac{\lambda}{4}(\phi_1^2 + \phi_2^2)^2 && \rightarrow \text{ four-point interactions} \\ & -\lambda v^2 \phi_1^2 && \rightarrow \text{ mass term} \\ & -\lambda v \phi_1 (\phi_1^2 + \phi_2^2) && \rightarrow \text{ three-point interactions.} \end{aligned} \tag{11.6}$$

Let us consider that mass term further:

$$-\frac{1}{2} m_1^2 \phi_2^2 \implies m_1^2 = 2\lambda v^2 \implies m_1 = \sqrt{2\lambda} v. \tag{11.7}$$

ϕ_2 is massless; it is called the **Goldstone boson**. It's massless because of a spontaneously broken global U(1) symmetry.

11.2 The Goldstone Theorem

If a continuous global symmetry G is broken spontaneously, there will be n massless Goldstone bosons, where n is the number of generators of G.

Let's consider QCD with $N_f = 2$ massless flavours of quarks, i.e. $m_{u,d} \to 0$ and $m_{t,b,c,s} \neq 0$. There is then a global symmetry of $\mathrm{SU}(2)_L$. Under this symmetry, the massless quarks will transform as

$$\begin{pmatrix} u_L \\ d_L \end{pmatrix} \to e^{-i\alpha^a T^{a\in\{1,2,3\}}} \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \qquad \begin{pmatrix} u_R \\ d_R \end{pmatrix} \to e^{+i\alpha^a T^{a\in\{1,2,3\}}} \begin{pmatrix} u_R \\ d_R \end{pmatrix}. \tag{11.8}$$

It is a phenomenological fact that $\mathrm{SU}(2)_L$ is spontaneously broken, which implies $N_f^2 - 1 = 3$ massless bosons. They are the three pions, π^0, π^+, π^-:

$$\pi^0 = \frac{\bar{u}u + \bar{d}d}{\sqrt{2}}, \qquad \pi^+ = u\bar{d}, \qquad \pi^- = d\bar{u}. \tag{11.9}$$

The mass of the pions $m_\pi \approx 140$ MeV $\ll 1$ GeV $= m_{proton}$, so it is an approximation, which can hold in certain schemes. So the pions can be **pseudo-Goldstone** bosons, which are not exactly massless since in reality $m_{u,d} \neq 0$.

11.3 Spontaneous Breaking of a Continuous Local (Gauge) Symmetry

We will consider the simplest case of the Higgs phenomenon: the Abelian Higgs model. For a scalar field $\phi(x) \in \mathbb{C}$, we consider a U(1) gauge symmetry, i.e. $\phi(x) \to e^{-i\alpha(x)}\phi(x)$. We must look at the Lagrangian for this symmetry:

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - \lambda \left(|\phi|^2 - \frac{v^2}{2} \right)^2. \tag{11.10}$$

This is fine, but we need to add some stuff on:

$$\begin{aligned} A_\mu(x) &\to A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x), && (11.11)\\ D_\mu &= \partial_\mu + ieA_\mu(x), && (11.12)\\ \mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - V(\phi). && (11.13) \end{aligned}$$

This form of the Lagrangian will be invariant under our gague transformations defined above. Again, the potential $V(\phi)$ will give us a Mexican hat potential with local minima at $|\langle\phi\rangle| = \frac{v}{\sqrt{2}}$. We choose the vacuum $\langle\phi\rangle = \frac{v}{\sqrt{2}}$ such that it breaks the gauge U(1) symmetry spontaneously.

Lecture 12

Last time, we did the Abelian Higgs model. A U(1) gauge theory with transforms

$$\phi(x) \to e^{-i\alpha(x)}\phi(x), \qquad A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x). \tag{12.1}$$

We have an invariant Lagrangian under U(1) symmetry:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D_\mu \phi|^2 - \lambda\left(|\phi|^2 - \frac{v^2}{2}\right)^2, \tag{12.2}$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{12.3}$$

$$D_\mu \equiv \partial_\mu + ieA_\mu. \tag{12.4}$$

We break the U(1) gauge symmetry spontaneously to get a Mexican hat potential with VEV $\langle\phi\rangle = \frac{v}{\sqrt{2}}$. The symmetry goes $U(1) \to \emptyset$.
Now we need to analyse the particle spectrum. We start by writing down our scalar field as

$$\phi(x) = \frac{1}{\sqrt{2}}\left(v + \rho(x)\right)e^{i\xi(x)}, \tag{12.5}$$

where $\phi(x)$ is a complex scalar field and we have no loss of generality as $\rho(x)$ and $\xi(x)$ are real fields which are our real field degrees of freedom which we needed as before. For a vev to be $\langle\phi\rangle = \frac{v}{\sqrt{2}}$, the two real fields ρ and ξ have vevs $= 0$.
Before, we worked with Cartesian coordinate degrees of freedom in ϕ_1 and ϕ_2, but now we are describing it in polar coordinates ρ and ξ. Now, we have a gauge theory so to do any analytical calculations (e.g. perturbation theory etc), we need to fix the gauge by removing the unphysical gauge degrees of freedom. So how do we fix the gauge? We will choose the unitary gauge,

$$\alpha(x) = \xi(x). \tag{12.6}$$

In the unitary gauge, the complex scalar field is going to become

$$\phi(x) = \frac{1}{\sqrt{2}}(v + \rho(x)), \tag{12.7}$$

where we have got rid of one real field degree of freedom. *Fixing the gauge is something which must be done in any gauge theory, not just when considering spontaneous symmetry breaking.* What is our Lagrangian now?

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu - ieA_\mu)(v+\rho(x)) \cdot (\partial_\mu + ieA_\mu)(v+\rho(x)) - \frac{\lambda}{4}\left((v+\rho(x))^2 - v^2\right)^2. \tag{12.8}$$

We want to know if the gauge fields A_μ and the real field $\rho(x)$ are massive, and what their masses are, i.e. m_A, m_ρ. For this, we need a term in the Lagrangian which looks like

$$\mathcal{L} \ni \frac{m_A^2}{2}A_\mu A^\mu - \frac{m_\rho^2}{2}\rho^2 + \dots \tag{12.9}$$

From the second and third terms in Eq (12.8) respectively, we get

$$\frac{1}{2}e^2 v^2 A_\mu A^\mu = \frac{1}{2}m_A A_\mu A^\mu \implies m_A = ev, \tag{12.10}$$

$$\lambda v^2 \rho(x)^2 = \frac{1}{2}m_\rho^2 \rho^2 \implies m_\rho = \sqrt{2\lambda}v. \tag{12.11}$$

So we see that the gauge field is massive, but we see it is proportional to the vev of the symmetry breaking, so it would become massless if we did not have this symmetry breaking. The $\rho(x)$ term is a real scalar field (i.e. the Higgs boson field) which is also massive through symmetry breaking. The Higgs field mass is also proportional to λ, which is the self-coupling constant of the Higgs field. In this case, we do not find a massless Goldstone boson. The unrealised Goldstone boson field is $\xi(x)$, which is "eaten" by the unitary gauge fixing. But what if we fixed the gauge in a different way? The calculation may be more complicated, but any gauge fixing would ultimately result in the removal of one real field degree of freedom which would be "eaten" in some way.

Let's count the degrees of freedom before and after spontaneous symmetry breaking:

- Before
  - Massless $A_\mu \implies$ 2 degrees of freedom: the 2 transverse polarisations (spin projections).
  - $\phi(x) \in \mathbb{C} \implies$ 2 degrees of freedom
  - $2 + 2 = 4$
- After spontaneous symmetry breaking:
  - A_μ with $m_A \neq 0 \implies$ 3 degrees of freedom in 2 transverse polarisations and 1 longitudinal.
  - However, $\phi(x)$ only has 1 degree of freedom in $\rho(x)$ (for the unitary gauge in polar coords)
  - $3 + 1 = 4$

We thus have studied the U(1) realisation of the Higgs mechanism. Key points:

- For every broken generator of the gauge group, the corresponding gauge boson becomes massive.
- There are no Goldstone bosons left in the spectrum. They now give rise to the longitudinal polarisations of massive gauge bosons.
- There is remaining massive scalar boson(s) $\rightarrow$ Higgs field(s).

12.1 The Final Section

For the Standard Model, we have the symmetry $\mathrm{SU}(3)_c \otimes \mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y$. We need to look through the matter fields of the theory.

Particle	$\mathbf{SU(3)}_c$	$\mathbf{SU(2)}_L$	$\mathbf{U(1)}_Y$
q_L	3	2	$+\frac{1}{6}$
$\bar{u}_R$	$\bar{3}$	1	$-\frac{2}{3}$
$\bar{d}_R$	$\bar{3}$	1	$+\frac{1}{3}$
L_L	1	2	$-\frac{1}{2}$
$\bar{\nu}_R$	1	1	0
$\bar{e}_R$	1	1	+1
H	1	2	$+\frac{1}{2}$

So we have left-handed quark and lepton doublets, q_L and L_L, and right-handed quark and lepton singlets. The Higgs doublet is defined as

$$H = \begin{pmatrix} h^+ \\ h^0 \end{pmatrix}, \qquad \langle H \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \tag{12.12}$$

where its vev is $v = 246\,\mathrm{GeV}$. The symmetry broken by the real Higgs mechanism in the SM is $\mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y \rightarrow \mathrm{U}(1)_{QED}$.

The full Covariant Derivative of the Standard Model is defined as

$$D_\mu = \partial_\mu - ig_1 \frac{Y}{2} B_\mu - ig_2 \frac{\sigma^i}{2} W^i_\mu - ig_3 \frac{\lambda^a}{2} G^a_\mu, \tag{12.13}$$

with $g_{1,2,3}$ the gauge couplings of $\mathrm{U}(1)_Y$, $\mathrm{SU}(2)_L$, and $\mathrm{SU}(3)_c$, where the gauge fields and generators of each group are in the term of their coupling. Now we write down the full Lagrangian of the Standard Model:

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + i\bar{\Psi} \not{D} \Psi + (D_\mu \Phi)^\dagger (D^\mu \Phi) - V(\Phi) - Y_{ij} \bar{\Psi}_i \Phi \Psi_j + h.c. \tag{12.14}$$

Part III

Particle Physics Phenomenology

Lecture 1 A Brief History...

The modern outlook of particle physics is based on these elementary particles: *draw table of all particles* Back in the 1940s, we did not have the same scope. We knew about protons, neutrons, and electrons. Then we discovered pions and muons coming in from the atmosphere, using cloud chambers and their difference of decay rate to distinguish them. So we added the muon to our elementary particles. Pions hinted towards the existence of quarks, made of u and d quarks.

1.1 ...of QCD

- Not long after, we discovered the Kaon as well. We saw something decay into two pions which had to be heavier. Kaons contain the s quark, so lead eventually to the higher generations of quarks.
- Over time, more particles were slowly discovered, e.g. myriads of mesons like π, K, ρ, η, etc.
- Gell-Mann realised that all these particles we were finding could be made up of more elementary particles called quarks, with different combinations and numbers yielding the different particles we knew at this time. There was no evidence at this time that this would be the case, it was just a useful thought experiment.
- In the late 60s, the Stanford accelerator used deep inelastic scattering to decompose the proton and resolve its constituents, i.e. the parton model leading to confirmation of quarks.
- Gell-Mann and others fledge out their theory of quarks into a full gauge theory into what we know today as SU(3) QCD. This was ultimately confirmed when the J/Ψ ($c\bar{c}$) was discovered by two separate accelerators, so now the quark model for the first two generations was found and made sense of the current catalogue of composite particles.
- Shortly after, we found experimental confirmation of the gluon, making sense of quarks as a gauge theory.
- We then discovered the Υ ($b\bar{b}$) meson in the mid 70s, which hinted at a third generation of quarks, but the top quark was to remain elusive until $\tilde{9}5$.

1.2 ...of GSW Theory

- In the mid 50s, we found interactions between protons and neutrinos to form neutrons and leptons, both for first and second generation.
- We required the same number of generations of quarks and leptons, and slowly we found the third generation of leptons by 2000 with the discovery of ν_τ.
- The interactions with neutrinos studied hinted to some other interaction besides electromagnetism and QCD, with its strength described in the Fermi constant. These interactions all seemed pointlike to us as the particle mediating them was so much heavier than the others.
- Glashow et al formed this into a gauge theory to attempt to describe this, finding the $W^{\pm}, Z$ bosons, as well as combining this with the electromagntic gauge theory to form the electroweak of SU(2)×U(1).
- In the 1980s at CERN, electrons and positrons were collided to produce the $W^{\pm}, Z$ bosons and measured their masses as $\tilde{8}0$ and 90 GeV respectively, values which were predicted back in the 60s by Weinberg and Salam.

1.3 ...of the Higgs theory

- The big issue we had was that all our theories worked on gauge invariance which would be broken by mass terms to form the masses we knew these particles had.
- Many people postulated what we now know as the Higgs mechanism at roughly the same time, in the 60s.
- Very skeptical for many years about this theory, although it was seen as the simplest way to get it done. Then in 2012, CERN found what we believe to be the Higgs boson, completing the current picture of particle physics, encompassing all forces, interactions, and particles predicted by the Standard Model.

1.4 Some Notes on Notation and Terminology

- Pions, Kaons, and any other particles made of one quark and an anti-quark are known as **Mesons**.
- Neutrons, protons, and other three-quark particles are known as **Baryons**.
- Overall, any particle made of quarks is called a **Hadron**.
- Leptons never really form bound states until electrons are bound by atoms, so there is not much terminology for them.

Next time, we will discuss particle colliders and their two parameters, COM energy and Luminosity. Collider physics is governed by the rate of events,

$$\frac{dN_{ev}}{dt} = L\sigma, \tag{1.1}$$

where L is luminosity and σ is the cross-section.

Lecture 2 The LHC

Collisions between two particles are the basis for experimental particle physics. Particles have four -momenta $p = (E_p, \underline{p})$, where the total four-momenta going in will be $p_T = (E_1 + E_2, \underline{p}_1 + \underline{p}_2)$. We can transform between coordinate systems of our four-momenta as

$$\frac{1}{\sqrt{1-v^2}}\begin{pmatrix} 1 & -\underline{v} \\ -\underline{v} & 1 \end{pmatrix}\begin{pmatrix} E_1 \\ \underline{p}_1 \end{pmatrix} = \frac{1}{\sqrt{1-v^2}}\begin{pmatrix} E_1 - \underline{v}\underline{p}_1 \\ -\underline{v}E_1 - p_1 \end{pmatrix} \tag{2.1}$$

$$= \Lambda_\mu{}^\nu p_\nu \tag{2.2}$$

We can choose the simplest frame for this, i.e. the COM frame:

$$(p_1 + p_2)_{cm} = \begin{pmatrix} E_1^{cm} + E_2^{cm} \\ 0 \end{pmatrix} \tag{2.3}$$

$$\frac{1}{\sqrt{1-v^2}}\begin{pmatrix} 1 & -\underline{v}^{cm} \\ -\underline{v}^{cm} & 1 \end{pmatrix}\begin{pmatrix} E_1 + E_2 \\ \underline{p}_1 + \underline{p}_2 \end{pmatrix} = \frac{1}{\sqrt{1-v^2}}\begin{pmatrix} E_1 + E_2 - \underline{v}^{cm}(\underline{p}_1 + \underline{p}_2) \\ -\underline{v}^{cm}(E_1 + E_2) - \underline{p}_1 - \underline{p}_2 \end{pmatrix} \tag{2.4}$$

$$s = (E_1^{cm} + E_2^{cm})^2 = (p_1 + p_2)^\mu (p_1 + p_2)_\mu = (E_1 + E_2)^2 - (\underline{p}_1 + \underline{p}_2)^2 \tag{2.5}$$

The LHC currently has a COM energy of 13 TeV, i.e. during proton-proton collisions, each proton as $E_p = 6.5\,\text{TeV}$, with three-momenta equal in magnitude with opposite signs. Consider proton at rest ($p_1 = (m_p, 0)$) colliding with electron ($p_2 = (E_2, \underline{p}_2)$):

$$s = (p_1 + p_2)^2 = (m_p + E_2)^2 = m_p^2 + 2E_2 m_p + \underbrace{E_2^2 - p_2^2}_{m_e^2} \tag{2.6}$$

$$E_{cm} = \sqrt{s} = 100\,\text{GeV} \tag{2.7}$$

So we switched from fixed targets to two moving targets as it massively increases COM energy available, although we will see that not all this energy is the energy available for particle production. We consider the cross-section concept for proton collisions, where

$$\frac{dN_{ev}}{dt} = 2vn_2N_1\sigma = L \times \sigma, \tag{2.8}$$

so the number of events occuring is dependent on the cross-section of the beams. We have defined L as the instantaneous *luminosity*, which is like flux in astronomy etc. So the number of events is dependent on the cross-section of beam collisions and the how often particles are included within the cross-section (in the Luminosity). We can describe each of the particles in these collisions using a Gaussian profile density of form

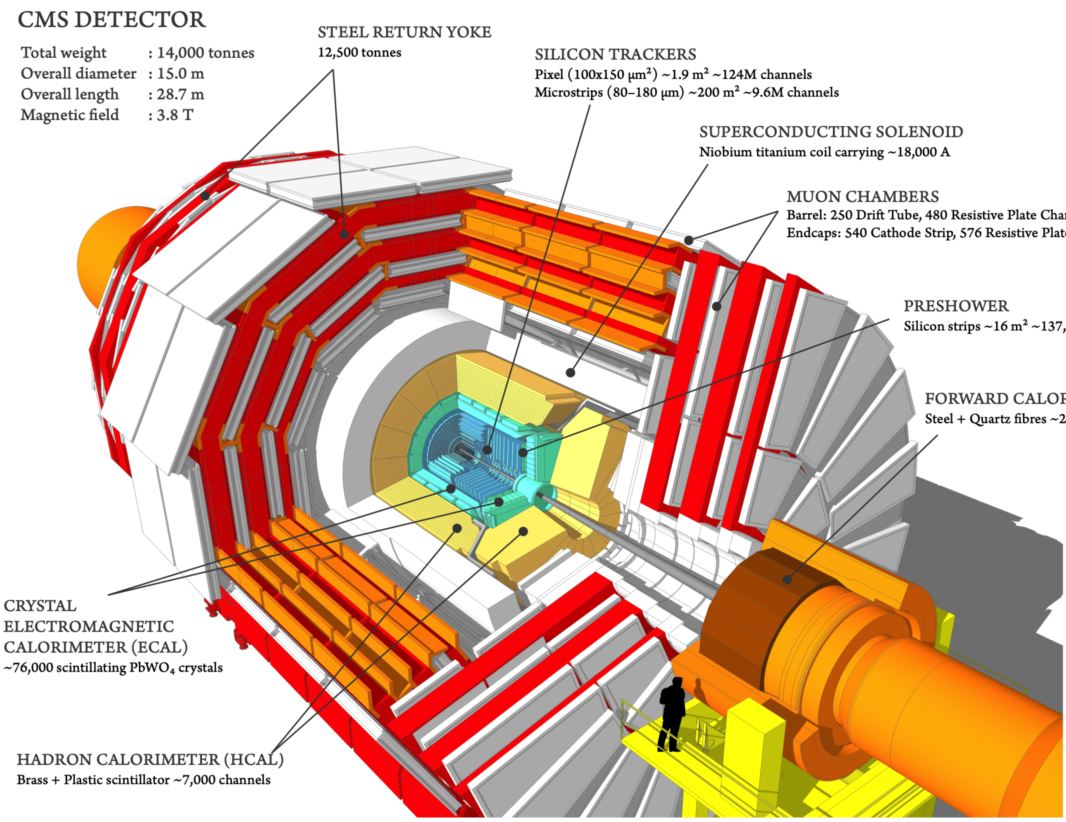
$$\rho = \exp\left(-\frac{x^2}{2\,dx^2} - \frac{y^2}{2\,dy^2} - \frac{z^2}{2\,dz^2}\right). \tag{2.9}$$

We can then write the Luminosity as

$$L = \frac{fN_1N_2N_b}{4\pi\,dx\,dy}. \tag{2.10}$$

We do not have a continuous beam of particles in these colliders, but a small collection of beams, where there will be N_b travelling in each direction. The Luminosity of the LHC is $L_{int} = 140\,\text{fb}^{-1}$, where fb is defined as femtobarn ($1\,\text{fb} = 10^{-15}\,\text{b} = 10^{-39}\,\text{cm}^2$). If we consider the cross-section of proton collisions for Higgs production, $\sigma_{pp\to h} = 4 \times 10^4\,\text{fb}$, we can calculate the number of Higgs produced at the LHC as $N_h = 5.6 \times 10^6$.

2.1 The CMS detector at CERN



- CMS (Compact Muon Solenoid) is a 14000 ton experiment, $15 \times 15\text{m}^2$.
- We have a "tracker" made out of silicon which tracks the particles, as charged particles' paths are bent moving through it due to the magnetic field generated by the surrounding superconductor.
- The particles will then collide into a "electromagnetic calorimeter" which allows us to measure their energy if they are electrons.
- There is then a "hadron calorimeter" which will collide with hadrons, i.e. pions, and measure their energy.
- Finally there is a muon chamber, which of course detects muons.
- Neutrinos will not be detected by any of these chambers, but we can infer if one has been produced through the starting energy/masses and the measured outputs of electrons, hadrons, and muons. Other low-interacting particles could be present in this as well, but so far all scattering events observed have been consistent with the missing particles being neutrinos.

We write length in units of $\frac{1}{\text{GeV}}$, which makes sense, when you multiply by $\hbar c$ and propagate through, it is then in units of approximately 2×10^{-16}m. This length scale will explain why we do not observe quarks on their own - they hadronise in a shorter time than it takes for us to observe them. Top quarks can however be observed on their own as their lifetime is shorted than the hadronisation time due to their significant mass, $m_t = 172.9\,\text{GeV}$.

Lecture 3 Path Integrals and Feynman Rules

Consider a quantum system with our conjugate operators $\hat{Q}$ and $\hat{P}$. These satisfy the familiar defintions:

$$[\hat{Q},\hat{P}] = i\hbar \qquad \hat{Q}|Q\rangle = Q|Q\rangle \tag{3.1}$$

$$\langle Q|P|\rangle = e^{iQP/\hbar} \qquad \hat{P}|P\rangle = P|P\rangle. \tag{3.2}$$

We can now consider the time evolution of the system using the time-dependent Schrodinger equation,

$$i\hbar\frac{\partial}{\partial t}|Q(t)\rangle = \mathcal{H}, \qquad \mathcal{H} = \frac{\hat{P}^2}{2} + V(\hat{Q}). \tag{3.3}$$

We consider a non-relativistic object moving in one dimension with unit mass $M = 1$. The amplitude is therefore

$$\langle Q_F|e^{-i\mathcal{H}T/\hbar}|Q_I\rangle. \tag{3.4}$$

We break the time T into $N+1$ intervals, such that $\delta t = T/(N+1)$, and evaluate the operators in terms of eigenvalues using several identities.

$$\langle Q_F|e^{-i\mathcal{H}T/\hbar}|Q_I\rangle = \langle Q_F|e^{-i\mathcal{H}\delta t/\hbar}\dots e^{-i\mathcal{H}\delta/\hbar}|Q_I\rangle \tag{3.5}$$

$$= \int \langle Q_F|e^{-i\mathcal{H}\delta t/\hbar}|Q_{N-1}\rangle\dots\langle Q_2|e^{-i\mathcal{H}\delta t/\hbar}|Q_1\rangle\langle Q_1|e^{-i\mathcal{H}\delta t/\hbar}|Q_I\rangle\prod_i dQ_i. \tag{3.6}$$

We can break down each product of this:

$$\langle Q_{j+1}|e^{-i\mathcal{H}\delta t/\hbar}|Q_j\rangle = \int \langle Q_{j+1}|e^{-i\mathcal{H}\delta t/\hbar}|P\rangle\langle P|Q_j\rangle\frac{dP}{2\pi} \tag{3.7}$$

$$= \int \langle Q_{j+1}|P\rangle e^{-i\frac{\delta t}{\hbar}(\frac{P^2}{2}+V(i\hbar\frac{\partial}{\partial P}))}\langle P|Q_j\rangle\frac{dP}{2\pi} \tag{3.8}$$

$$= \int e^{i\frac{Q_{j+1}P}{\hbar}}e^{-i\frac{\delta t}{\hbar}(\frac{P^2}{2}+V(i\hbar\frac{\partial}{\partial P}))}e^{-i\frac{Q_jP}{\hbar}}\frac{dP}{2\pi}. \tag{3.9}$$

The argument of the exponential is then:

$$-\frac{i\delta t}{\hbar}\left(\frac{P^2}{2} - \frac{P}{\delta t}(Q_{j+1}-Q_j) + V(Q_j)\right) = -\frac{i\delta t}{\hbar}\left(\frac{1}{2}\left(P - \frac{Q_{j+1}-Q_j}{\delta t}\right)^2 - \frac{(Q_{j+1}-Q_j)^2}{2\delta t^2} + V(Q_j)\right). \tag{3.10}$$

We can see that the integral in P is Gaussian which in general yields

$$\int_{-\infty}^{\infty}\exp\left(-\frac{(z-b)^2}{2a^2}\right)dz = \sqrt{2\pi a^2}, \tag{3.11}$$

so therefore the integral in P is

$$\int\langle Q_{j+1}|e^{-i\frac{\mathcal{H}\delta t}{\hbar}}|P\rangle\langle P|Q_j\rangle\frac{dP}{2\pi} = \sqrt{\frac{\hbar}{i2\pi\delta t}}\exp\left[\frac{i\delta t}{\hbar}\left(\frac{(Q_{j+1}-Q_j)^2}{2\delta t^2} - V(Q_j)\right)\right]. \tag{3.12}$$

The amplitude now reads

$$\begin{aligned}\langle Q_F|e^{-\frac{i}{\hbar}\mathcal{H}T}|Q_I\rangle = \left(\frac{-i\hbar}{2\pi\delta t}\right)^{\frac{N+1}{2}}\int\prod_j\left\{\exp\left[\frac{i\delta t}{\hbar}\left(\frac{(Q_{j+1}-Q_j)^2}{2\delta t^2} - V(Q_j)\right)\right]dQ_j\right\}\\ \times\exp\left[\frac{i\delta t}{\hbar}\left(\frac{(Q_1-Q_I)^2}{2\delta t^2} - V(Q_I)\right)\right].\end{aligned} \tag{3.13}$$

with $Q_{N+1} = Q_F$. We can then write the infintesimal limit of $\delta t \to 0$ to get

$$\langle Q_F|Q_I\rangle = \int_{Q_I}^{Q_F} \mathcal{D}Q e^{-\frac{i}{\hbar}S[Q]}, \text{ where} \tag{3.14}$$

$$S[Q] = \int_0^T \mathcal{L}(Q)\, dt = \int_0^T \left(\frac{\dot{Q}^2}{2} - V(Q)\right) dt, \tag{3.15}$$

$$\mathcal{D}Q = \lim_{\delta t\to 0} \left(\frac{-i\hbar}{2\pi\delta t}\right)^{\frac{N+1}{2}} \prod_i dQ_i. \tag{3.16}$$

We call $S[Q]$ the *action*, from which we can arrive at the Euler-Lagrange equations where $\frac{\delta S[Q]}{\delta Q} = 0$. So we sum over all paths and it is the action that determines the final results, the evolutions of the sytem. From here already one can see the importance of the action and Lagrangian. Finding the fundamental action that describes the world is key to predict and understand the possible outcomes. This makes the case for theorists to fixate with Lagrangians: there is the hope that the search for new phenomena will yield a complex description of Nature as specified by the Action.

Now we can connect to particle physics by

$$Q \to \phi(t, \underline{x}), \qquad\qquad P \to \partial_t\phi(t, \underline{x}) = \Pi, \tag{3.17}$$

where we can now define our Lagrangian as

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - m^2\phi^2. \tag{3.18}$$

The commutation relation can be defined as

$$[\hat{\phi}(\underline{x}), \hat{\Pi}(\underline{y})] = i\hbar\delta^3(\underline{x} - \underline{y}). \tag{3.19}$$

We have a Hamiltonian and Action from these, reading

$$\mathcal{H} = \frac{\partial_t\hat{\phi}^2}{2} + \frac{1}{2}\left(\nabla\hat{\phi}\right)^2 + V(\hat{\phi}), \tag{3.20}$$

$$S[\phi] = \int \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right) dt\, dt^3\underline{x}. \tag{3.21}$$

From here, we can build up our rules as previously, but we would only get a number of single-particles states. For this to be a description of nature, we need multi-particle states as well. To this end, we introduce the partition function:

$$Z[0] = \langle 0|e^{-\frac{i}{\hbar}\mathcal{H}T}|0\rangle_{T\to\infty} = \int \mathcal{D}e^{iS[\phi]} \tag{3.22}$$

$$Z[J] = \int \mathcal{D}\phi \exp\left[iS[\phi] + i\int \phi(x)J(x)\, d^4x\right]. \tag{3.23}$$

Lecture 4 Path Integrals and Feynman Rules, Contd.

From the scalar action of Eq. (3.21) and the two-point correlator,

$$\frac{\delta^2}{i\delta J(x) i\delta I(y)} \frac{Z[J]}{Z[0]}\bigg|_{j=0} = \frac{1}{Z[0]}\int \mathcal{D}\phi(x)\phi(y)e^{iS[\phi]}, \tag{4.1}$$

where functional derivatives are defined by

$$\frac{\delta J(y)}{\delta J(x)} = \delta^4(x-y). \tag{4.2}$$

Now we can rewrite the action as

$$\int \left(\frac{1}{2}\phi(-\Box - m^2)\phi + \phi J\right) d^4x = \in \left(\frac{1}{2}(\phi + \Delta J)(-\Box - m^2)(\phi + \Delta J) - \frac{1}{2}J\Delta J\right) d^4x \tag{4.3}$$

$$= \int \left(\frac{1}{2}\tilde{\phi}(-\Box - m^2)\tilde{\phi} - \frac{1}{2}J\Delta J\right) d^4x, \tag{4.4}$$

using integration by parts with $\Box = \partial^\mu \partial_\mu$. Here, $\Delta = (-\Box - m^2)^{-1}$, and $\tilde{\phi} = \phi + \Delta J$. This result back in the path integral gives

$$\frac{1}{Z[0]}\int \mathcal{D}\tilde{\phi}e^{i\S[\tilde{\phi}]}\frac{\delta^2}{i^2\delta J^2}e^{-\frac{i}{2}\int J\Delta J}\bigg|_{J=0} = \left(i\Delta + (\Delta J)^2\right)e^{\frac{i}{2}\int J\Delta J}\bigg|_{J=0} = i\Delta, \tag{4.5}$$

where we have identified the path integral in $\tilde{\phi}$ as $Z[0]$. Specifically,

$$(-\Box - m^2)(\Delta J)(x) = (-\Box - m^2)\int \Delta(x,y)J(y)\, d^4y, \tag{4.6}$$

$$\Delta(x,y) = \int \frac{e^{iq(x-y)}}{q^2 - m^2}\frac{d^4q}{(2\pi)^4}. \tag{4.7}$$

So all we had to do to compute the integral is to invert the operator of the quadratic action in ϕ. We obtain the propagation of a field from x to y in the absence of interactions, but now we want to include these. We can study the scattering matrix S, computed from the path integral for N particles as

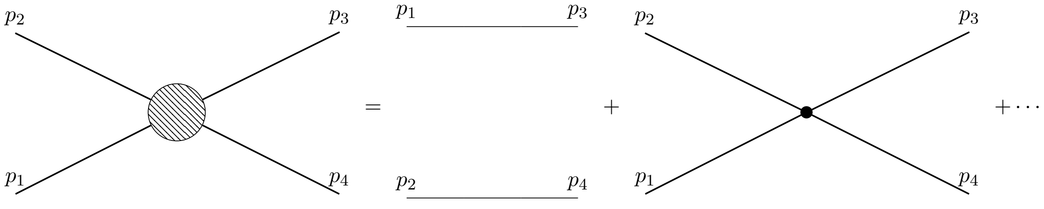
$$S = \frac{1}{Z[0]}\left((i\Delta)^{-1}\frac{\delta}{i\delta J}\right)^N Z[J]\bigg|_{J=0} = \frac{1}{Z[0]}\left((i\Delta)^{-1}\frac{\delta}{i\delta J}\right)^N \int \mathcal{D}\phi e^{iS[\phi]+\int J\phi}\bigg|_{J=0} \tag{4.8}$$

$$= \frac{1}{Z[0]}\int \mathcal{D}\tilde{\phi}e^{iS_0[\tilde{\phi}]}\left((i\Delta)^{-1}\frac{\delta}{i\delta J}\right)^N \exp\left[-\frac{i}{2}\int J\Delta J + iS_{int}[\tilde{\phi} - \Delta J]\right]\bigg|_{J=0} \tag{4.9}$$

$$= \frac{1}{Z[0]}\int \mathcal{D}\tilde{\phi}e^{iS_0[\tilde{[}\phi]-\frac{i}{2}\int J\Delta J}\left((i\Delta)^{-1}\frac{\delta}{i\delta J}\right)^N e^{iS_{int}[\tilde{\phi}-\Delta J]} + \text{ disconnected} \tag{4.10}$$

$$= \frac{1}{Z[0]}\int \mathcal{D}\phi e^{iS_0[\phi]}\left(\frac{\delta}{\delta\phi(p)}\right)^N e^{iS_{int}[\phi]} + \text{ disconnected.} \tag{4.11}$$

This is the **Lehmann Symanzyk Zimmerman** reduction formula and there is a lot to unpack. The disconnected terms are best understand with a diagrammatic approach:



Here we can see the disconnected refers to processes in which some particles travel freely and corresponds to letting $\frac{\delta}{\delta J}$ act on the free piece $\exp\left(-i\int J\Delta J/2\right)$. To get acquainted with this expression, consider the S matrix for the following interaction:

$$S_{int}[\phi] = -\int \frac{\lambda}{4!}\phi(x)^4\, d^4x = -\frac{\lambda}{4!}\int d^4x \prod_{i=1}^{4}\int e^{ip_ix}\phi(p_i)\, d^4p_i \tag{4.12}$$

$$= -\frac{\lambda}{4!}\prod_{i=1}^{4}\left[\int \phi(p_i)\, d^4p_i\right](2\pi)^4\delta^4\left(\sum p_i\right). \tag{4.13}$$

We consider 2 to 2 particle scattering with incoming momenta k_1, k_2 and outgoing momenta k_3, k_4. Now taking the derivative, the out-states have flipped momenta:

$$e^{iS_0[\phi]}\frac{\delta^2}{\delta\phi(k_1)\delta\phi(k_2)}\frac{\delta^2}{\delta\phi(-k_4)\delta\phi(-k_3)}e^{iS_{int}[\phi]} = e^{iS[\phi]}(-i\lambda)(2\pi)^2\delta(k_1+k_2-k_3-k_4)+\mathcal{O}(\lambda^2), \tag{4.14}$$

i.e. the path integral on e^{iS} cancels out with the factor $Z[0]$ in the denominator and we have therefore obtained the first order in λ matrix element S. The invariant matrix element is then defined as

$$S = \Pi - i(2\pi)^4\delta^4(p_I - p_F)\mathcal{M}, \tag{4.15}$$

where $p_{I,F}$ sum over the initial and final momenta. We then find that $-i\mathcal{M} = -i\lambda$.

Now consider the case of a proton with field $P(x)$ scattering off an electron with field $e(x)$ via the electromagnetic interaction:

$$D_\mu P(x) = (\partial_\mu + iQA_\mu)P(x) \tag{4.16}$$

$$S_{int} = \int \left(Q\bar{e}(x)\gamma_\mu e(x)A^\mu(x) - Q\bar{P}(x)\gamma_\mu P(x)A^\mu(x)\right)\, d^4x. \tag{4.17}$$

The S matrix is then computed from

$$\begin{aligned} & e^{iS_0[e,P,A_\mu]}\frac{\delta}{\delta\bar{e}(p_2)}\frac{\delta}{\delta e(p_1)}\frac{\delta}{\delta\bar{P}(k_2)}\frac{\delta}{\delta P(k_1)}e^{iS_{int}[e,P,A_\mu]} \\ =& e^{iS[e,P,A_\mu]}\int e^{i(p_1-p_2)x}[iQ\gamma_\mu]A^\mu(x)\, d^4x\int e^{i(k_1-k_2)y}[-iQ\gamma_\nu]A^\nu(y)\, d^4y + \mathcal{O}(Q^4) \\ =& e^{iS[e,P,A_\mu]-iS_0[A_\mu]}\int d^4x\int e^{i(p_1-p_2)x}(iQ\gamma_\mu)\left[A^\mu(x)A^\nu(y)e^{iS_0[A_\mu]}\right]e^{i(k_1-k_2)y}(-iQ\gamma_\nu)\, d^4y. \end{aligned} \tag{4.18}$$

We can perform the path integral in A_μ perturbatively and yield precisely the propagator

$$\begin{aligned} & Q^2\int d^4x\int d^4y e^{i(p_1-p_2)x}\gamma_\mu\int\frac{d^4q}{(2\pi)^4}\frac{-ie^{iq(x-y)}g^{\mu\nu}}{q^2}e^{i(k_1-k_2)y}\gamma_\nu \\ =& (2\pi)^4\int\delta^4(p_1-p_2-q)\, d^4q\ \gamma_\mu\frac{-iQ^2g^{\mu\nu}}{q^2}(2\pi)^4\delta^4(k_1-k_2+q)\gamma_\nu \\ =& (2\pi)^4\delta^4(p_1-p_2-k_2+k_1)\gamma_\mu\frac{-iQ^2g^{\mu\nu}}{(p_1-p_2)^2}\gamma_\nu, \end{aligned} \tag{4.19}$$

which has an overall momentum conservation Dirac delta and ends up being simpler than the derivation machinery might have suggested. To connect the above with the S matrix, we still must contract this with the spinors $u(\underline{p},s), \bar{u}(\underline{p},s)$ which are the connection between field and particle states. In these lectures, we will only use these up to spin 1.

$$\langle 0|\phi(x)|\underline{p}\rangle = e^{-ipx} \qquad \langle\underline{p}|\phi(x)|0\rangle = e^{ipx} \tag{4.20}$$

$$\langle 0|\psi(x)|\psi,\underline{p},s\rangle = u(\underline{p},s)e^{-ipx} \qquad \langle\psi,\underline{p},s|\bar{\psi}(x)|0\rangle = \bar{u}(\underline{p},s)e^{ipx} \tag{4.21}$$

$$\langle 0|\bar{\psi}(x)|\text{anti-}\psi,\underline{p},s\rangle = \bar{v}(\underline{p},s)e^{-ipx} \qquad \langle\text{anti-}\psi,\underline{p},s|\psi(x)|0\rangle = v(\underline{p},s)e^{ipx} \tag{4.22}$$

$$\langle 0|A_\mu(x)|\underline{p},\lambda\rangle = \epsilon_\mu(\underline{p},\lambda)e^{-ipx} \qquad \langle \underline{p},\lambda|A^\mu(x)|0\rangle = \epsilon^*_\mu(\underline{p},\lambda)e^{ipx} \tag{4.23}$$

$$\langle 0|W^+_\mu(x)|W^+,\underline{p},\lambda\rangle\epsilon_\mu(\underline{p},\lambda)e^{-ipx} \qquad \langle W^+,\underline{p},\lambda|W^-_\mu(x)|0\rangle = \epsilon^*_\mu(\underline{p},\lambda)e^{ipx} \tag{4.24}$$

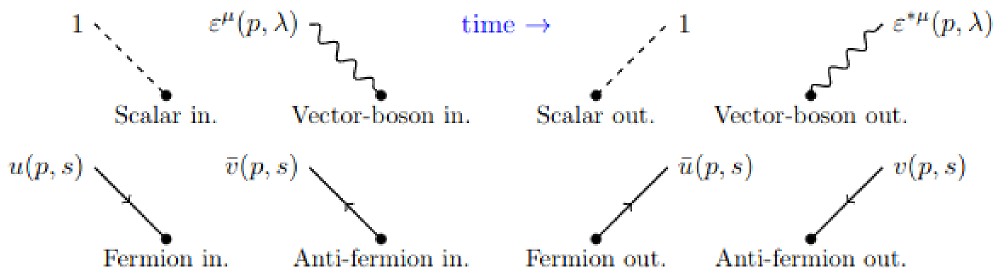
$$\langle 0|W^-_\mu(x)|W^-,\underline{p},\lambda\rangle\epsilon_\mu(\underline{p},\lambda)e^{-ipx} \qquad \langle W^-,\underline{p},\lambda|W^+_\mu(x)|0\rangle = \epsilon^*_\mu(\underline{p},\lambda)e^{ipx} \tag{4.25}$$

The simplicity of these spinors hints at a shortcut to the result which is simply a collection of rules which we can gather to skip all of the maths we have done above. These are the **Feynman rules:**

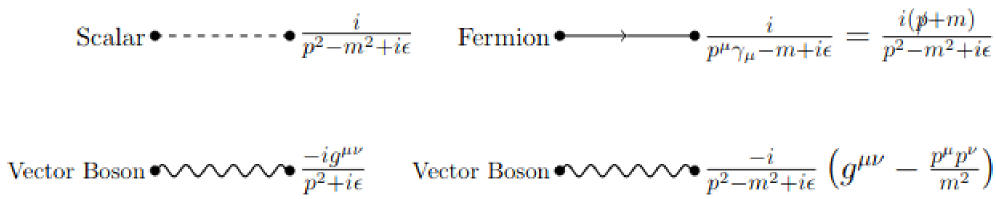
- **Interaction vertices** - to derive the Feynman rule for a given vertex, take the derivative of the interaction term in the Lagrangian with respect to fields until you obtain a constant and put an i into it. We have seen a couple of examples of this:

$$-\frac{\lambda}{4!}\phi^4 \to -i\lambda, \qquad \bar{e}A_\mu\gamma^\mu e \to i\gamma_\mu. \tag{4.26}$$

  The vertex is represented diagrammatically by each of the fields being a line joining in a point.
- For a initial/final state particle $\hat{a}^\dagger_{\underline{p},s}|0\rangle \equiv |\underline{p},s\rangle$ with momentum $\underline{p}$ and spin s, one must supplement the derivative with respect to the field with the field-state connection, meaning a factor.



- For internal lines which connect two vertices, we put in the propagator $i\Delta$. These are:



- For a given process, draw all possible diagrams (to a given order in your perturbative expansion) matching the external states and translate it into an amplitude $-i\mathcal{M}$ by summing over them and writing their contributions with the above prescriptions of factors for internal and external lines, and vertices. Impose momentum conservation on each vertex to fix the momenta of propagators as much as possible.

Lecture 5 Standard Model Overview

5.1 Path Integrals Conclusion

It can be shown that all diagrams at first order in our perturbative expansion have the momenta of propagaotrs fixed in terms of the momenta of external states. The next order does not and there's internal momenta which we have to integrate over. These are the rules, but one only really learns how to use them with examples. Finally, we can take the invariant matrix element $-i\mathcal{M}$ and find the cross-section. For two particles collding and producing n particles, we have

$$\sigma = \frac{1}{|\underline{v}_a - \underline{v}_b| 2E_{\underline{p}_a} 2E_{\underline{p}_b}} \int \left(\prod_{i=1}^{n} \frac{d^3\underline{p}_i}{2E_{\underline{p}_i}(2\pi)^3} \right) (2\pi)^4 \delta^4 \left(p_a + p_b 0 \sum_{i=1}^{n} p_i \right) |\mathcal{M}|^2, \tag{5.1}$$

where $\underline{v} = \underline{p}/E_{\underline{p}}$. The terms inside the inregral except for $\mathcal{M}$ constitute the **Lorentz Invariant Phase Space**, sometimes this will be called d**LIPS**, whereas the factors out front are related to our normalisation of states $\langle \underline{p}' | \underline{p} \rangle$. On the other hand, a decay rate in the particle's rest frame is

$$\Gamma = \frac{1}{2M_a} \int \prod_{i} \frac{d^3\underline{p}_i}{2E_{\underline{p}_i}(2\pi)^3} (2\pi)^4 \delta^4 \left(p_a - \sum_{i=1}^{n} p_i \right) |\mathcal{M}|^2, \tag{5.2}$$

with p_a the four-momenta of the decaying particle. So at last our trip from action to observables is done.

A number of useful relations for the square of the matrix elements when we sum over spins are:

$$\sum_s u(\underline{p}, s)\bar{u}(\underline{p}, s) = \not{p} + m, \qquad \sum_\lambda \epsilon_\mu(\underline{p}, \lambda)\epsilon^*_\nu(\underline{p}, \lambda) = -g_{\mu\nu}\,(m = 0), \tag{5.3}$$

$$\sum_s v(\underline{p}, s)\bar{v}(\underline{p}, s) = \not{p} - m, \qquad \sum_\lambda \epsilon_\mu(\underline{p}, \lambda)\epsilon^*_\nu(\underline{p}, \lambda) = \frac{p_\mu p_\nu}{m^2} - g_{\mu\nu}, \tag{5.4}$$

and since $(\gamma^0)^\dagger = \gamma^0$, $(\gamma^\mu)^\dagger\gamma^0 = \gamma^0\gamma^\mu$, we have, for example,

$$(\bar{u}\gamma^\mu v)^* = v^\dagger(\gamma^\mu)^\dagger(\gamma^0)^\dagger u = v^\dagger\gamma^0\gamma^\mu u = \bar{v}\gamma^\mu u. \tag{5.5}$$

5.2 Gauge Groups

The Standard Model is formed under the principles of gauge theory. The group of the Standard Model is $\mathrm{SU}(3)_c \otimes \mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y$, representing colour triplets, weak isospin doublets, and hypercharge respectively. $\mathrm{SU}(3)_c$ is the gauge group of colour (QCD), and the rest of the SM group is the electroweak theory which can be approximately split into the weak isospin and hypercharge, although not quite, and will result in electromagnetism after introducing the Higgs mechanism later. Let's overview how each gauge group is set up in terms of its bosons and generators:

	Color	**Weak Isospin**	**Hypercharge**
Group:	$\mathrm{SU}(3)_c$	$\mathrm{SU}(2)_L$	$\mathrm{U}(1)_Y$
Bosons:	${G_\mu}^a, a = 1 \to 8$	${W_\mu}^I, I = 1 \to 3$	B_μ
Generators:	$\frac{g_s}{2}T_a$	$\frac{g}{2}\sigma_I$	$Q_Y g' \mathbb{I}$

Here g_s, g, g' are the couplings of colour, weak, and hypercharge respectively. The matrices T_a and σ_I can be taken to be the Gell-Mann and Pauli matrices respectively, with the normalisations $\mathrm{Tr}(T_a T_b) = 2\delta_{ab}$

and $\mathrm{Tr}(\sigma_I \sigma_J) = 2\delta_{IJ}$. The field strengths for the gauge bosons transform in the adjoint representation and are defined as:

$$G_{\mu\nu} = \partial_\mu G_\nu{}^a T_a - \partial_\nu G_\mu{}^a T_a + \frac{ig_s}{2}\left[G_\mu{}^a T_a, G_\nu{}^b T_b\right], \tag{5.6}$$

$$W_{\mu\nu} = \partial_\mu W_\nu{}^I \sigma_I - \partial_\nu W_\mu{}^I \sigma_I + \frac{ig}{2}\left[W_\mu{}^I \sigma_I, W_\nu{}^J \sigma_J\right], \tag{5.7}$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu. \tag{5.8}$$

Gauge bosons couple to matter through the covariant derivative. We can simplify this and select certain terms for different interactions, so we only use terms appropriate, (e.g. we would not use the colour term for leptons), but in its full form for the SM, it is defined as

$$D_\mu = \partial_\mu + i\frac{g_s}{2} G_\mu{}^a T_a + i\frac{g}{2} W_\mu{}^I \sigma_I + ig' Q_Y. \tag{5.9}$$

For example, we would need this full derivative when dealing with left-handed quarks.

Lecture 6 Standard Model Overview Contd.

6.1 Matter

Matter can be defined by fields that are charged under the SM gauge groups, i.e. fermions and the Higgs doublet. First, we remind ourselves of the definitions for chirality:

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \qquad P_L \equiv \frac{1-\gamma_5}{2}, \qquad P_R \equiv \frac{1+\gamma_5}{2}, \tag{6.1}$$

where $P_{L,R}$ are the left/right-handed projectors. These projections are useful because it commutes with Lorentz transformations, i.e.

$$[\gamma_5, [\gamma_\mu, \gamma_\nu]] = 0. \tag{6.2}$$

So we can define the left- and right-handed components of the fermion fields by projecting using the above, which will remain invariant after a Lorentz transformation. The fermion fields (and Higgs) we then have are

Gauge Group	q_L	u_R	d_R	l_L	e_R	H
$SU(3)_c$	3	3	3	-	-	-
$SU(2)_L$	2	-	-	2	-	2
$U(1)_Y$	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	$\frac{1}{2}$

6.2 Lagrangian

As we've emphasised so far, the spacetime integral of the Lagrangian dictates the evolution of the system and possible outcomes; therefore it is the central construction from which we can derive observables. The construction of the Lagrangian of the Standard Model follows two rules: *Lorentz and gauge invariance.* These symmetries imply conserved currents which form the basis of our predictions. Using the field strength definitions,

$$G_{\mu\nu} = \partial_\mu G_\nu{}^a T_a - \partial_\nu G^a{}_\mu T_a + \frac{ig_s}{2}\left[G_\mu{}^a T_a, G^b{}_\mu T_b\right], \tag{6.3}$$

$$W_{\mu\nu} = \partial_\mu W_\nu{}^I \sigma_I - \partial_\nu W^I{}_\mu \sigma_I + \frac{ig}{2}\left[W_\nu{}^I \sigma_I, W^J{}_\nu \sigma_J\right], \tag{6.4}$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \tag{6.5}$$

we start by expressing the kinetic Lagrangian expressing interactions of matter and gauge bosons:

$$\begin{aligned}\mathcal{L}_{gauge} = &-\frac{1}{8}\mathrm{Tr}(G_{\mu\nu}G^{\mu\nu}) - \frac{1}{8}\mathrm{Tr}(W_{\mu\nu}) - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \\ &+ \sum_{\psi_L} i\bar{\psi}\gamma^\mu D_\mu \psi_L + \sum_{\psi_R} i\bar{\psi}\gamma^\mu D_\mu \psi_R + D^\mu H^\dagger D_\mu H.\end{aligned} \tag{6.6}$$

However, this Lagrangian does not include mass terms, which we will come onto later with the Higgs mechanism. We have a first guess of the Higgs potential to help introduce masses as the Mexican hat potential,

$$V(H) = -m_H^2 H^\dagger H + \lambda(H^\dagger H)^2. \tag{6.7}$$

From this, we can eventually arrive at the Yukawa interaction which will introduce fermion masses:

$$\mathcal{L}_{Yuk} = -\sum_{\text{gauge inv.}}\left(Y\bar{\psi}_L H\psi_R + Y\bar{\psi}_L\tilde{H}\psi_R\right) + h.c. \tag{6.8}$$

$$= -Y_u \bar{q}_L \tilde{H} u_R - Y_d \bar{q}_L H d_R - Y_e \bar{l}_L H e_R + h.c., \tag{6.9}$$

$$\tilde{H} = i\sigma_2 H^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} H^*, \tag{6.10}$$

where we have introduced $\tilde{H}$ to protect the conservation of hypercharge where H would not. We can now put everything together for the final SM Lagrangian:

$$\begin{aligned} \mathcal{L}_{SM} = & -\frac{1}{8}\mathrm{Tr}(G_{\mu\nu}G^{\mu\nu}) - \frac{1}{8}\mathrm{Tr}(W_{\mu\nu}W^{\mu\nu}) - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \\ & + \sum_{\psi_L} i\bar{\psi}\gamma^\mu D_\mu \psi_L + \sum_{\psi_R} i\bar{\psi}\gamma^\mu D_\mu \psi_R + D^\mu H^\dagger D_\mu H \\ & - Y_u \bar{q}_L \tilde{H} u_R - Y_d \bar{q}_L H d_R - Y_e \bar{l}_L H e_R + h.c. \\ & + m_H^2 H^\dagger H - \lambda (H^\dagger H)^2. \end{aligned} \tag{6.11}$$

This is the final result for a single generation of fermion. Eventually, we would have to add another index onto matter fields to sum over the three currently-known generations of particles, i.e. $u_R^i = (u_R^1, u_R^2, U_R^3)$. This will cause us to move from Yukawa coupling numbers Y to Yukawa 3×3 matrices, which introduces other phenomena later.

6.3 Conservation Laws

We have several conservation laws in the Standard Model. Through conserved currents, we say the charges are conserved over interactions. In addition, **Baryon number** is conserved, defined as

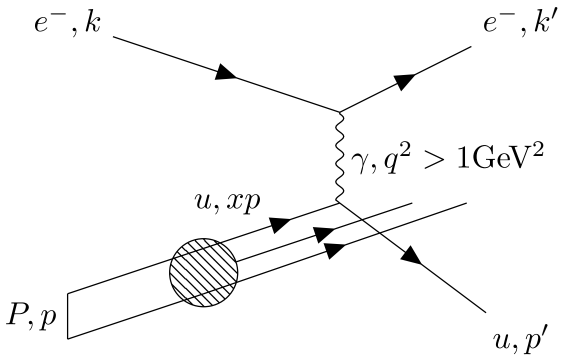
$$Q_B(q_L, u_R, d_R) = \frac{1}{3}(q_L, u_R, d_R). \tag{6.12}$$

Similarly, each generation of lepton has its own conservation, i.e. **Electron number**, **Muon number** are conserved, including their neutrinos in the count.

Lecture 7 Deep Inelastic Scattering and Partons

Quarks and gluons, known together as partons, are no observed as final states in experiments, yet are claimed to be components of mesons and baryons. What evidence is there then to this? We look to deep inelastic scattering and the parton model for an explanation.

Consider shooting electrons at protons at high CoM energy, $s \gg m_P^2 \approx 1\,\text{GeV}^2$. At these energies, electrons can probe the internal structure of the proton and 'catch' partons behaving as free particles inside the proton. This parton will receive a large momentum transfer and break away from the proton such that, after further hadronisation to conserve colour, the proton has 'broken' into various other hadrons.



To see how if this agrees with experimental data, let's consider the scattering at a partonic level, specifically choosing a u quark and the scattering $e + u \to e + u$. Let's assume the u quark carries a fraction x of the total momentum of the proton, then the partonic process is

$$-i\hat{\mathcal{M}} = ie\bar{u}_e(k')\gamma_\mu u_e(k)\frac{-ig^{\mu\nu}}{q^2}\left(-i\frac{2e}{3}\right)\bar{u}_u(p')\gamma_\nu u_u(xp), \tag{7.1}$$

with $q = k - k' = p' - xp$ and we use the 'hat' to denote partonic quantities. Recall the formula for the cross-section, which in this case we can simplify a bit since we have *relativistic particles* (we take $q^2 \gg m_i^2$),

$$d\hat{\sigma} = \frac{1}{2}\frac{1}{2|\underline{k}|2x|\underline{p}|}\frac{d^3\underline{p}'\,d^3\underline{k}'}{2|\underline{k}'|(2\pi)^3 2|\underline{p}'|(2\pi)^3}|\hat{\mathcal{M}}|^2(2\pi)^4\delta^4(xp + k - p' - k'). \tag{7.2}$$

If one works on the phase space for the final parton,

$$\begin{aligned}(2\pi)^4\delta^4(xp + q - p')\frac{d^3\underline{p}'}{(2\pi)^3 2|\underline{p}'|} &= \delta(x|\underline{p}| + q^0 0|x\underline{p} + \underline{q}|)\frac{2\pi}{2|x\underline{p} + \underline{q}|} \\ &= \frac{\pi}{p\cdot p'}\delta\left(x + \frac{q^2}{2p\cdot q}\right) = 2\pi\frac{p\cdot q}{p\cdot p'}\delta(2p\cdot qx + q^2),\end{aligned} \tag{7.3}$$

where $q = k - k'$. We can rewrite the above for the lepton phase space by changing the variables from $|\underline{k}|, \cos\theta$ is the CoM frame to $q^2, p\cdot q$ as

$$\frac{d^3\underline{k}'}{(2\pi)^3 2E_{\underline{k}'}} = \frac{d(q^2)\,d(p\cdot q)}{4(2\pi)^2 p\cdot k}, \tag{7.4}$$

where we also integrated over the angle $\phi \in [0, 2\pi)$ since the amplitude does not depend on it. For the matrix element, since we do not know the spin of the particles involved, we average over incoming and sum over outgoing as

$$\frac{1}{2^2}\sum_{s_e,s_u}\sum_{s_e',s_u'}\hat{\mathcal{M}}\hat{\mathcal{M}}^\dagger = \frac{1}{4}\sum_{s_e,s_u}\sum_{s_e',s_u'}\left|\bar{u}_e(k',s_e')\gamma_\mu u_e(k,s_e)\frac{e^2}{q^2}\frac{2}{3}\bar{u}_u(p',s_u')\gamma^\mu u_u(xp,s_u)\right|^2 \tag{7.5}$$

$$= \left(\frac{2e^2}{3q^2}\right)^2 4\mathrm{Tr}(\gamma_\mu x\not{p}\gamma_\nu\not{p}')\mathrm{Tr}(\gamma^\mu\not{k}\gamma^\nu\not{k}') \tag{7.6}$$

$$= \left(\frac{2e^2}{3q^2}\right)^2 8\left((xp\cdot k)(p'\cdot k') + (xp\cdot k')(p'\cdot k)\right) \tag{7.7}$$

$$= \left(\frac{2e^2}{3q^2}\right)^2 8\left((xp\cdot k)^2 + (xp\cdot k')^2\right). \tag{7.8}$$

We then put all this together for the cross-section and find

$$d\hat{\sigma} = \left(\frac{2e^2}{3q^2}\right)^2 \frac{1}{2(2xp^0)(2k^0)} 8\left((xp\cdot k')^2\right) \frac{d(q^2)\,d(p\cdot q)}{4(2\pi)^2 p\cdot k}\frac{\pi}{p\cdot q}\delta\left(x + \frac{q^2}{2p\cdot q}\right) \tag{7.9}$$

$$= x\left(\frac{2e^2}{3q^2}\right)^2 \frac{(p\cdot k)^2 + (p\cdot(k-q))^2}{8\pi p\cdot q(p\cdot k)^2}\delta\left(x + \frac{q^2}{2p\cdot q}\right) d(q^2)\,d(p\cdot q). \tag{7.10}$$

Now comes the part that we cannot compute: what is the probability of the photon bumping into parton with fraction of the momentum x? This is a magnitude which we cannot estimate in perturbation theory. Instead, we name it the **parton distribution function** $f_u(x)$ and sum over it:

$$\begin{aligned} d\sigma = d\hat{\sigma} f_u(x)\,dx &= \left(\frac{2e^2}{3q^2}\right)^2 \frac{p\cdot k}{4\pi}\left(1 + \frac{(p\cdot(k-q))^2}{(p\cdot k)^2}\right) d(p\cdot q) x f_u(x)\,dx \\ &= \frac{e^2}{8\pi}\frac{s}{q^4}\left(\frac{2e}{3}\right)^2 \left(1 + (1-y)^2\right)\, dy\, x f_u(x)\,dx, \end{aligned} \tag{7.11}$$

where we used the Dirac delta to set

$$x = -\frac{q^2}{(2p\cdot q)}, \qquad\qquad s = (p+k)^2 \approx 2p\cdot k, \tag{7.12}$$

and found appropriate to change variable from $p\cdot q$ to variable $y = \frac{p\cdot q}{p\cdot k}$. Finally we know it's not only the u quark in the proton but also the d, so we add it up too:

$$d\sigma_{eP\to eX} = \frac{e^2}{8\pi}\frac{s}{q^4}\left(1 + (1-y)^2\right)\, dy\, \left[\left(\frac{2e}{3}\right)^2 f_u(x) + \left(\frac{-e}{3}\right)^2 f_d(x)\right] dx, \tag{7.13}$$

where by eX in the final state, we sum over all possible products of the collision, termed *inclusive.* We do not know a priori $f_{u,d}(x)$, but this can still be a predictive results which can be tested against data. The general cross-section without assuming anything about the components of the proton, only using electromagnetic gauge invariance, reads

$$d\sigma_{eP\to eX} = \frac{e^4}{4\pi}\frac{s}{q^4}\left(xy^2F_1(x,y) + (1-y)F_2(x,y)\right)\, dx\, dy, \tag{7.14}$$

where $F_{1,2}$ arbitrary functions. Even if we start from arbitrary parton distribution functions, we cannot obtain arbitrary $F_{1,2}$, since one f only depends on x. Expressed in terms of $F_{1,2}$, the conditions that follow from our quark description read

$$F_2(x,y) = 2xF_1(x) = \sum_i Q_i^2 x f_i(x), \tag{7.15}$$

with Q_i the charge of the parton in units of e ($\frac{2}{3}, -\frac{1}{3}$ for u, d). This relation is known as the Callan-Gross equation. The fact that the functions $F_{1,2}$ depend only on x is knon as Bjorken scaling and a way to test it is to extract $F_{1,2}$ from experiment and plot them for fixed x and varying y; if they do not change, the Callan-Gross relation holds and the quark model prediction is right.

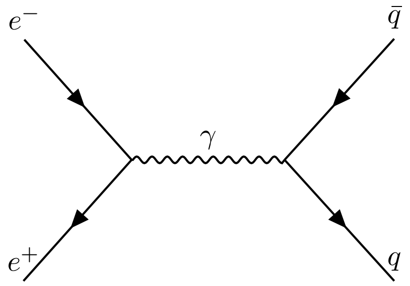
Lecture 8 PDFs and Hadronic vs Partonic

Studying deep inelastic scattering showed that the cross-section of the process $eP \to eX$ (where X represents anything that can be produced) can be expressed as an integral over the partonic process times a function $f(x)$ of the fraction of momentum x, i.e.

$$\int d\sigma_{eP\to eX} = \int \sum_i f_i(x)\, d\sigma_{ei\to ei}\, dx. \tag{8.1}$$

These functions f are called Parton Distribution Functions and their extraction from experiment (we cannot compute them) provides a window into the proton's inner structure. We

Lecture 9 Quantum Chromodynamics



$$-i\mathcal{M} = \bar{\nu}_e(ie\gamma_\nu)u_e \frac{-ig^{\mu\nu}}{s}\bar{u}_q(-ieQ_q)\gamma_\nu v_q \tag{9.1}$$

Here, $s = (p_{e^-} + p_{e^+})^2$. We then have the differential cross-section:

$$d\sigma = \frac{1}{4}\sum_{spin}|\mathcal{M}|(2\pi)^2\delta^2(p_{e^+} + p_{e^-} - p_q - p_{\bar{q}})\frac{d^3p_q\, d^3p_{\bar{q}}}{(2\pi)^6 2E_{p_q}E_{p_{\bar{q}}}} \tag{9.2}$$

The extra terms here are to integrate over **Lorentz Invariant Phase Space.** We want to reduce this, however, such that we get rid of the δ functions for convenience.

$$d\textbf{LIPS} = \frac{1}{(2\pi)^2}\delta(\sqrt{s} - E_{p_q} - E_{p_{\bar{q}}})\delta^3(-p_q - p_{\bar{q}})\frac{d^3p_q\, d^3p_{\bar{q}}}{2E_{p_q}2E_{p_{\bar{q}}}} \tag{9.3}$$

$$= \frac{1}{(2\pi)^2}\delta(\sqrt{s} - 2E_{p_q})\frac{d^3p_q}{4E_{p_q}^2} \tag{9.4}$$

$$= \frac{1}{(2\pi)^2}\delta(\sqrt{s} - 2E_{p_q})\frac{d\Omega\, p_q^2\, dp_q}{4E_{p_q}^2} \tag{9.5}$$

A useful trick for simplifying these δ functions is

$$\delta(f(x)) = \frac{1}{|f'(x)|}\delta(x - x_0). \tag{9.6}$$

So now we can use this on the modulus of the momenta:

$$\sqrt{s} - 2E_{p_q} = 0 \tag{9.7}$$

$$E_{p_q}^2 = m^2 + p_q^2 = \frac{s}{4} \tag{9.8}$$

$$|p_q|^2 = \frac{s - 4m^2}{4} \tag{9.9}$$

Back to our d**LIPS**:

$$d\textbf{LIPS} = \frac{1}{(2\pi)^2}\frac{1}{|2p_q/E_{p_q}|}\frac{d\Omega\, p_q^2\, dp_q}{4E_{p_q}^2}\delta(p_q - \sqrt{\frac{s - 4m^2}{4}}) \tag{9.10}$$

$$= \frac{d\Omega}{(2\pi)^2}\frac{|p_q|}{8E_{p_q}} = \frac{d\Omega}{(2\pi)^2}\frac{\sqrt{s - 4m^2}}{8\sqrt{s}} \tag{9.11}$$

Lecture 10 Electroweak Interactions

Spontaneous symmetry breaking induces SU(2)$_L$⊗U(1)$_Y$ →U(1)$_{EM}$. We have the Higgs potential for this as

$$V(H) = -m_H^2 H^\dagger H + \lambda(H^\dagger H)^2, \tag{10.1}$$

$$\langle 0|H|0\rangle = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}, \tag{10.2}$$

producing a Mexican hat potential with a VEV shown above. This Higgs mechanism allows us to generate gauge boson masses for the $W^\pm, Z$ bosons. For the vacuum value, we have

$$D_\mu\langle H\rangle = i \begin{pmatrix} \frac{g'}{2}B_\mu + \frac{g}{2}W_\mu^3 & \frac{g}{2}\left(W_\mu^1 - iW_\mu^2\right) \\ \frac{g}{2}\left(W_\mu^1 + iW_\mu^2\right) & \frac{g'}{2}B_\mu - \frac{g}{2}W_\mu^3 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} = \frac{iv}{\sqrt{2}} \begin{pmatrix} \frac{g}{2}(W_\mu^1 - iW_\mu^2) \\ \frac{g'}{2}B_\mu - \frac{g}{2}W_\mu^2 \end{pmatrix}. \tag{10.3}$$

The kinetic term for the vec of the Higgs is the modulus of this, i.e.

$$D_\mu\langle H\rangle^\dagger D^\mu\langle H\rangle = \frac{v^2}{2}\left(\frac{g^2}{4}|W_\mu^1 - iW_\mu^2|^2 + \left(\frac{g'}{2}B_\mu - \frac{g}{2}W_\mu^3\right)^2\right). \tag{10.4}$$

The gauge fields W^i, B do not themselves have mass terms, but from the above, we can introduce linear combinations of them which will:

$$W_\mu^\pm = \frac{1}{\sqrt{2}}\left(W_\mu^1 \mp iW_\mu^2\right) \qquad \frac{g}{2}W_\mu^3 - \frac{g'}{2}B_\mu = \frac{\sqrt{g^2+g'^2}}{2}Z_\mu, \tag{10.5}$$

so we now have the $W^\pm, Z$ fields corresponding to the real observable bosons, where the massless photon field A_μ comes from the relation

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_w & -\sin\theta_w \\ \sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} W_\mu^3 \\ B_\mu^3 \end{pmatrix}, \quad \cos\theta_w = \frac{g}{\sqrt{g^2+g'^2}}. \tag{10.6}$$

We can feed these fields back into Eq. (10.4) as

$$D_\mu\langle H\rangle^\dagger D^\mu\langle H\rangle = \frac{v^2}{2}\left(\frac{g^2}{2}W_\mu^+W^{-\mu} + \frac{g^2}{4\cos^2\theta_w}Z_\mu Z^\mu\right), \tag{10.7}$$

which produce mass terms in the Lagrangian appearing as

$$\mathcal{L}_M = M_W^2 W_\mu^+ W^{-\mu} + \frac{1}{2}M_Z^2 Z_\mu Z^\mu. \tag{10.8}$$

We can read the masses out from this, yielding

$$M_W = \frac{gv}{2} = 80\,\text{GeV}, \qquad M_Z = \frac{gv}{2\cos\theta_w} = 91\,\text{GeV}. \tag{10.9}$$

From all this, we can then write our lengthy descriptions of D_μ to see the full self-interactions of these bosons where we will see three- and four-point vertices emerges. It is then convenient to address the coupling of electroweak bosons to fermions. The electromagnetic coupling is the U(1) symmetry left over from SSB, and we define its coupling strength as

$$e = g\sin\theta_w, \tag{10.10}$$

where the only gauge invariant combination of the weak isospin and hypercharge charges we had previously is now the electric charge, given by

$$Q_{em}^L = \frac{\sigma_3}{2} + Q_Y^L, \quad Q_{em}^R = Q_Y^R \tag{10.11}$$

$$Q_{em} = \left(\frac{\sigma_3}{2} + Q_Y^L\right)P_L + Q_Y^R P_R, \tag{10.12}$$

where L, R denotes left- and right-handed fermions. It is worth noting that electromagnetism is not chiral, i.e. $Q_{em}^L = Q_{em}^R$ and there is no γ_5 term in the coupling. Chirality does however remain in the couplings of the $W^\pm, Z$ bosons, where $W^\pm$ bosons still only couple to left-handed fields.

Lecture 11 Electroweak Boson Properties

$W^\pm, Z$ bosons theorised in the 60s then discovered in the 80s. We perform electron-positron collisions to produce the Z boson, with a matrix element for the reaction $e^+e^- \to Z \to \mu^+\mu^-$,

$$-i\mathcal{M} = -\frac{ig}{\cos\theta_w}\bar{v}_e\gamma^\rho\left(-\frac{P_L}{2}+\sin^2\theta_w\right)u_e\frac{ig_{\rho\nu}+i\frac{p^Z_\rho p^Z_\nu}{M_Z^2}}{s-m_Z^2+i\epsilon}\left(-\frac{ig}{\cos\theta_w}\right)\bar{u}_{\mu^-}\gamma^\nu\left(-\frac{P_L}{2}+\sin^2\theta_w\right)v_{\mu^+}, \tag{11.1}$$

with $s=(p_{e^+}+p_{e^-})^2$. Initially, this was run with $s = 90\,\text{GeV}$, where the Z propagator seems to blow up. When the particle is produced on resonance, i.e. $s = M_Z^2$, we have to reconsider what's going on. In this regime, the Z boson is no longer a virtual short-distance mediator, but must be consider a possible final state itself, if the particle is stable. If the Z boson is an unstable particle, with a short lifetime, we need to modify the particle propagator to say how it evolves.

If we modify the propagator's denominator as $M_Z \to M_Z - \frac{i}{2}\Gamma_Z$ with Γ_Z the total decay width of the Z boson. This extram imaginary part in the action for the particle gives a time evolution, and hence an exponential decay for its probability. The resulting propagator and cross-section scales as

$$d\sigma \propto \frac{1}{|s-M_Z^2+i\Gamma_Z M_Z-\Gamma_Z^2/4|^2} \approx \frac{1}{(s-M_Z^2)^2+\Gamma_Z^2M_Z^2}. \tag{11.2}$$

This is called a Breit-Wigner distribution and looks like a peak at $s=M_Z^2$, which is sharper for smaller Γ_Z/M_Z. We call a small (large) Γ_Z/M_Z a narrow (broad) resonance. In the case of a well-defined peak, the Narrow Width Approximation applies and we can compute the cross-section as the exchange of an on-shell Z which implies production and decay are factorised which reads

$$\sigma_{N.W.A} = \frac{12\pi s}{M_Z^2}\frac{\Gamma(Z\to e^+e^-)\Gamma(Z\to\mu^+\mu^-)}{(s-M_Z^2)^2+M_Z^2\Gamma_Z^2}. \tag{11.3}$$

We can use this to break down the calculation into smaller pieces and look for different decays and reconstruct couplings of the Z boson. Let's focus on one decay in particular: $e^+e^- \to Z \to \bar{\nu}\nu$. The interaction term can be derived from the kinetic term, reading

$$\mathcal{L}_{int} = i(\bar{l}_L\gamma^\mu D_\mu l_L) = i\bar{\nu}\frac{igZ^\mu}{\cos\theta_w}\gamma_\mu\left(\frac{(\sigma_3)_{11}}{2}P_L-\sin^2\theta_w Q^\nu_{em}\right)\nu+\cdots, \tag{11.4}$$

$$= -\frac{gZ_\mu}{2\cos\theta_w}\bar{\nu}\gamma^\nu P_L\nu+\cdots, \tag{11.5}$$

$$-i\mathcal{M}_{Z\to\bar{\nu}\nu} = \epsilon^\mu(p_Z,\lambda)\bar{u}_\nu(p_\nu,s_\nu)\frac{-ig}{2\cos\theta_w}\gamma_\mu P_L v_\nu(p_{\bar{\nu}},s_{\bar{\nu}}). \tag{11.6}$$

For computing the rate, we average over the initial states for the polarisations $\lambda=\pm1,0$, and we sum over all possible final states, i.e.

$$\frac{1}{3}\sum_\lambda\sum_{s_\nu s_{\bar{\nu}}}\mathcal{M}\mathcal{M}^* = \frac{1}{3}\frac{g^2}{4\cos^2\theta_w}\sum_\lambda\sum_{s_\nu s_{\bar{\nu}}}\epsilon_\mu\epsilon^*_\rho\bar{u}\gamma^\mu P_L v v^\dagger P_L^\dagger(\gamma^\rho)^\dagger(\gamma^0)^\dagger u, \tag{11.7}$$

$$= \frac{1}{3}\frac{g^2}{4\cos^2\theta_w}\sum_\lambda\sum_{s_\nu s_{\bar{\nu}}}\epsilon_\mu\epsilon^*_\rho\bar{u}\gamma^\mu v\bar{v}\gamma^\rho P_L u \tag{11.8}$$

$$= \frac{1}{3}\frac{g^2}{4\cos^2\theta_w}\left(\frac{p_Z^\mu p_Z^\rho}{M_Z^2}-\eta^{\mu\rho}\right)\text{Tr}\left(\gamma_\mu P_L \not{p}_{\bar{\nu}}\gamma_\rho P_L \not{p}_\nu\right). \tag{11.9}$$

Now using the fact that we can bring one P_L to the other and $P_L^2 = P_L$, with the relation

$$\text{Tr}(\gamma_\mu\gamma_\alpha\gamma_\rho\gamma_\beta P_R) = 2\left(\eta^{\mu\alpha}\eta^{\rho\beta}+\eta^{\mu\beta}\eta^{\rho\beta}-\eta^{\rho\mu}\eta^{\alpha\beta}+i\epsilon^{\mu\alpha\rho\beta}\right), \tag{11.10}$$

we can find that

$$
\begin{aligned}
&\frac{1}{3}\frac{g^2}{4\cos^2\theta_w}\left(\frac{p_Z^\mu p_Z^\rho}{M_Z^2}-\eta^{\mu\rho}\right)\mathrm{Tr}(\gamma_\mu P_L \not{p}_{\bar{\nu}}\gamma_\rho P_L \not{p}_\nu)\\
=&\frac{g^2}{6\cos^2\theta_w}\left(\frac{p_Z^\mu p_Z^\rho}{M_Z^2}-\eta^{\mu\rho}\right)\Big((p_{\bar{\nu}})_\mu(p_\nu)_\rho+(p_{\bar{\nu}})_\rho(p_\nu)_\mu-\eta^{\rho\mu}p_{\bar{\nu}}\cdot p_\nu+i\epsilon^{\mu\alpha\rho\beta}(p_{\bar{\nu}})_\alpha(p_\nu)_\beta\Big) \qquad (11.11)\\
=&\frac{g^2}{6\cos^2\theta_w}\left(2\frac{p_Z\cdot p_\nu p_Z\cdot p_{\bar{\nu}}}{M_Z^2}+p_\nu\cdot p_{\bar{\nu}}\right),
\end{aligned}
$$

where given that $\epsilon^{\mu\nu\rho\sigma}$ is fully antisymmetric, the contraction with the averaged $\epsilon\epsilon^*$ cancels. Now working in phase space for the CoM frame, we get

$$
\begin{aligned}
\frac{d^3\underline{p}_\nu\, d^3\underline{p}_{\bar{\nu}}}{2|\underline{p}_\nu|2|\underline{p}_{\bar{\nu}}|(2\pi)^6}(2\pi)^4\delta(p_Z-p_\nu-p_{\bar{\nu}}) &= \frac{d^3\underline{p}_\nu}{4|\underline{p}_\nu|^2(2\pi)^2}\delta(M_Z-|p_\nu|-|p_\nu|) && (11.12)\\
&=\frac{\sin\theta\, d\theta\, d\phi}{4(2\pi)^2}\delta(M_Z-2|\underline{p}_\nu|)\, d|\underline{p}_\nu| && (11.13)\\
&=\frac{\sin\theta, d\theta\, d\phi}{8(2\pi)^2}. && (11.14)
\end{aligned}
$$

The product of the momenta p_i in the matrix element squared then read

$$
p_\nu\cdot p_{\bar{\nu}}=|\underline{p}_\nu||\underline{p}_{\bar{\nu}}|-\underline{p}_\nu\cdot\underline{p}_{\bar{\nu}}=2|\underline{p}_\nu|^2=\frac{M_Z^2}{2}, \qquad p_\nu\cdot p_Z=M_Z|\underline{p}_\nu|=\frac{M_Z^2}{2}, \tag{11.15}
$$

where we have used four-momentum conservation and in that the neutrinoes share the Z mass. Finally, overall for the decay width, we get

$$
\begin{aligned}
\Gamma_{Z\to\nu\bar{\nu}} &= \frac{1}{2M_Z}\int\frac{d^3\underline{p}_\nu\, d^3\underline{p}_{\bar{\nu}}}{2|\underline{p}_\nu|2|\underline{p}_{\bar{\nu}}|(2\pi)^6}(2\pi)^4\delta(p_Z-p_\nu-p_{\bar{\nu}})\frac{1}{3}\sum_\lambda\sum_{s_\nu s_{\bar{\nu}}}\mathcal{M}\mathcal{M}^* && (11.16)\\
&=\frac{1}{2M_Z}\frac{1}{8\pi}\frac{g^2}{6\cos^2\theta_w}M_Z^2=\frac{g^2M_Z}{96\pi\cos^2\theta_w}. && (11.17)
\end{aligned}
$$

How many neutrino flavours are there though? If we say there are N_ν, then

$$
\Gamma_{Z\to\sum_i\bar{\nu}_i\nu_i}=\frac{g^2N_\nu M_Z}{96\pi\cos^2\theta_w}. \tag{11.18}
$$

We can compare this to experiment to find N_ν. This isn't the easiest thing in experiment to pick up as neutrinos escape detection, but we can extract this from the total width of the Z boson, and subtracting all known decays. What is leftover, is the decay to neutrinos. Comparing this to Eq. (11.18), we find that

$$
N_\nu = 2.9840\pm 0.0082, \tag{11.19}
$$

which is a pretty clear indication that there are 3 neutrino flavours.

The W boson cannot be produced like the Z that is 'in the s channel', but we can produce $\pm$ pairs via, e.g., $e^+e^-\to\gamma/Z\to W^+W^-$. The condition for this is, however, stricter than for the Z boson. We are now producing two W bosons where before we only had to produce the one Z boson, so we must have a CoM energy $s>2M_W$. It is also worth nothing that we can study the chiral structure of the $W^\pm, Z$ couplings through the angular dependence of their decays. Another prediction from the SM is the mass ratio of the $W^\pm, Z$ bosons. These masses are directly related through the weak-mixing $\cos\theta_w$. Experimentally, we can find this ratio to be

$$
\frac{M_W^2}{\cos^2\theta_w M_Z^2}=1.0010\pm 0.0050, \tag{11.20}
$$

where the SM predicts this to be eaxctly one. We have another confirmation of the validity of the SM.