Durham University

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Advanced Theoretical Physics

Quantum Optics

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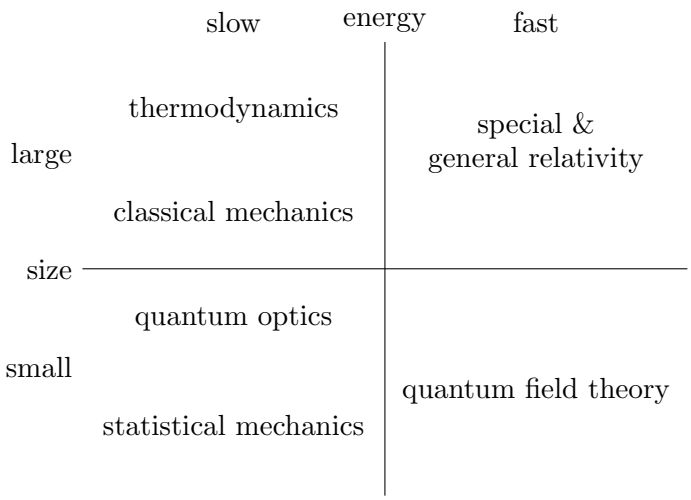


Ingredients:

- ➤ harmonic oscillators
- ➤ Gaussian integrals
- ➤ Hamiltonian mechanics (canonical variables q and p)
- ➤ maths of operators - adjoint, self-adjoint, Hermitian, commutation relations
- ➤ QM in both Schrodinger and Heisenberg pictures
- ➤ density matrices
- ➤ classical EM - Maxwell's equations in Coulomb gauge - especially plane waves and dipoles

Hanbury Brown and Tiss:

$$G(\tau) = I_A(t)I_B(t+\tau) \tag{1.1}$$

Lecture 2 Creation and Annihilation Operators

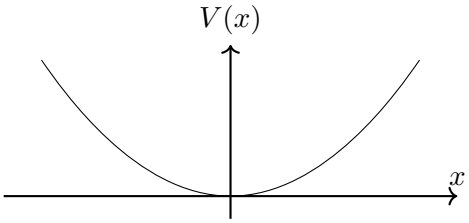
2.1 Learning Outcomes

To be able to state, explain and apply the operator formalism of the quantum harmonic oscillator, including:

- the Hamiltonian in terms of the creation and annihilation operators
- the number operator and number states, eigenstates of the Hamiltonian
- definition of the creation and annihilation operators, commutation relations, and the adjoint and self-adjoint operators
- mathematical properties of the number states, completeness
- systems of two or more independent oscillators

2.2 Quantum Harmonic Oscillator

$$F = ma = m\ddot{x} \tag{2.1}$$
$$= -kx \tag{2.2}$$
$$x(t) = x_0 \sin \omega t \tag{2.3}$$
$$p_x(t) = p_0 \cos \omega t \tag{2.4}$$
$$V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2 \tag{2.5}$$
$$\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi \tag{2.6}$$
$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \tag{2.7}$$



Start with writing the Hamiltonian, then turn everything into operators

$$H = \frac{p^2}{2m} + \underbrace{\frac{1}{2}m\omega^2 x^2}_{V(x)} \tag{2.8}$$
$$p \to \hat{p} = -i\hbar\frac{d}{dx},\ x \to \hat{x} \tag{2.9}$$
$$[\hat{x}, \hat{p}] = i\hbar \tag{2.10}$$
$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \tag{2.11}$$
$$\hat{a} = \frac{1}{\sqrt{2m\hbar\omega}}\left(m\omega\hat{x} + i\hat{p}\right) \tag{2.12}$$
$$\hat{a}^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}\left(m\omega\hat{x} - i\hat{p}\right) \tag{2.13}$$

$$\hat{x} = \left(\frac{\hbar}{2m\omega}\right)^{1/2} (\hat{a} + \hat{a}^\dagger) \tag{2.14}$$

$$\hat{p} = -i\left(\frac{m\hbar\omega}{2}\right)^{1/2} (\hat{a} - \hat{a}^\dagger) \tag{2.15}$$

$$[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1 \tag{2.16}$$

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}) \tag{2.17}$$

$$\hat{a}^\dagger\hat{a} = \hat{n},\ \hat{n}|n\rangle = n|n\rangle \tag{2.18}$$

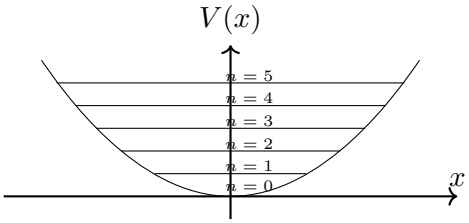
$$\hat{H}|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle = E_n|n\rangle \tag{2.19}$$

How do the annihilation and creation operators, $\hat{a}$ and $\hat{a}^\dagger$ interact with the number states, $|n\rangle$?

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \tag{2.20}$$

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \tag{2.21}$$

Together, the creation and annihilation operators are known as the *ladder operators.* Ladder operators move the system up or down the energy levels of the harmonic potential.



We now have a partly new mathematical representation. Notice that the potential still remains positive, it does not go negative. Therefore we must have:

$$\hat{a}|0\rangle = 0, \tag{2.22}$$

$$\hat{n} = \hat{a}^\dagger\hat{a}|0\rangle = 0. \tag{2.23}$$

$$\implies \hat{H}|0\rangle = E_0|0\rangle = \frac{1}{2}\hbar\omega|0\rangle \tag{2.24}$$

So the ground state is labelled '0' but does not have $E = 0$.
Now we introduce $\hat{O}^\dagger$ as the adjoint of $\hat{O}$ if

$$\langle\psi|\hat{O}|\phi\rangle = \langle\phi|\hat{O}^\dagger|\psi\rangle^* \ \forall\psi, \phi \tag{2.25}$$

A self-adjoint operator is equivalent to a Hermitian operator, i.e. $\hat{n}, \hat{H}$.
For adjoint operators:

$$(\hat{A} + \hat{B})^\dagger = \hat{A}^\dagger + \hat{B}^\dagger \tag{2.26}$$

$$(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger \tag{2.27}$$

$$(c\hat{A})^\dagger = c^*\hat{A}^\dagger \tag{2.28}$$

$$(\hat{A}^\dagger)^\dagger = \hat{A} \tag{2.29}$$

More on the number states:

- ➤ they are orthogonal

$$\langle n|n\rangle = 1 \tag{2.30}$$

$$\langle n|m\rangle = 0,\ n \neq m \tag{2.31}$$

$$\langle n|m\rangle = \delta_{n,m} \tag{2.32}$$

$$\tag{2.33}$$

➤ they form a basis (note: not mathematically a Hilbert space, but a Banah(?) space)

$$|\psi\rangle = \sum_n c_n |n\rangle \tag{2.34}$$

$$0 \leq n \leq \infty \tag{2.35}$$

2.3 Two Oscillators - independent

$$|\psi_0\rangle = \sum_n c_n |n\rangle_0 \tag{2.36}$$

$$|\psi_1\rangle = \sum_m c_m |m\rangle_1 \tag{2.37}$$

$$|\psi_{01}\rangle = \sum_{n,m} c_{n,m} |n\rangle_0 |m\rangle_1 \tag{2.38}$$

What we are doing is "tensoring" the Hilbert spaces: $\mathcal{H}_0 \otimes \mathcal{H}_1$:

$$|n\rangle_0 |m\rangle_1 \equiv |n\rangle_0 \otimes |m\rangle_1. \tag{2.39}$$

Now we have the operators, $\hat{a}_0, \hat{a}_0^\dagger, \hat{a}_1, \hat{a}_1^\dagger$:

$$\hat{a}_0 \otimes \mathbb{I}_1, \ \mathbb{I}_0 \otimes \hat{a}_1, \ldots \tag{2.40}$$

$$[\hat{a}_0, \hat{a}_1] = [\hat{a}_0, \hat{a}_1^\dagger] = 0 \tag{2.41}$$

$$\hat{H} = \hat{H}_0 \otimes \mathbb{I}_1 + \mathbb{I}_1 \otimes \hat{H}_1 \tag{2.42}$$

Note this is for non-interacting oscillators. For interacting,

$$\hat{H} = \hat{H}_0 \otimes \mathbb{I}_1 + \mathbb{I}_1 \otimes \hat{H}_1 + \mathcal{H}_{int}. \tag{2.43}$$

Lecture 3 Quantisation of the Electromagnetic Field

3.1 Learning Outcomes

To be familiar with the route to quantisation of the electromagnetic field, in particular to:

- ➤ Explain and state the description of the electromagnetic field in terms of modes and polarization
- ➤ Be familiar with the equivalence between a mode of the field and a quantum harmonic oscillator
- ➤ To explain the form of (but not derive) expressions for the Hamiltonian of the electromagnetic field, and the electric and magnetic fields in terms of the creation and annihilation operators
- ➤ To recognise and explain the concepts of the Schrodinger and Heisenberg representations, and to explain which is being applied
- ➤ To explain and apply the concepts of adjoint and self-adjoint operators and their matrix elements

3.2 Quantising the EM field

Consider an EM scalar potential, $\phi = 0$ (no free charges), and a vector potential, $\underline{A}$.

$$\underline{\mathbf{E}}(\underline{r},t) = \frac{\partial}{\partial t}\underline{A}, \quad \underline{\mathbf{B}}(\underline{r},t) = \nabla \times \underline{A}(\underline{r},t) \tag{3.1}$$

$$\underline{\nabla}\left[\underline{\nabla}\cdot\underline{A}\right] - \nabla^2\underline{A} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\underline{A} = 0 \tag{3.2}$$

Coulomb gauge, $\nabla \cdot \underline{A} = 0$.

$$\underline{A} = \sum_{\underline{k}} \left\{ \underline{A}_{\underline{k}} \exp\left[i(\underline{k}\cdot\underline{r} - \omega_k t)\right] + \underline{A}^*_{\underline{k}} \exp\left[-i(\underline{k}\cdot\underline{r} - \omega_k t)\right] \right\} \tag{3.3}$$

$$\omega_k = c|k|, \;\; \underline{k}\cdot\underline{A}_k = 0 \tag{3.4}$$

Polarisation vectors, $\underline{e}_{k1}, \underline{e}_{k2}$ - orthonormal vectors perpendicular to $\underline{k}$.

$$\underline{A}_k = A_{k1}\underline{e}_{k1} + A_{k2}\underline{e}_{k2} \tag{3.5}$$

$$\underline{A} = \sum_{\underline{k},s} A_{\underline{k},s}\underline{e}_{\underline{k},s} \exp\left\{i(\underline{k}\cdot\underline{r} - \omega_k t)\right\} + A^*_{\underline{k},s}\underline{e}_{\underline{k},s} \exp\left\{-i(\underline{k}\cdot\underline{r} - \omega_k t)\right\} \tag{3.6}$$

The labels of the modes are $\underline{k}, s$, $s \in 1, 2$. They gives us the: direction; wavelength, $\frac{2\pi}{|\underline{k}|}$; and polarisation, s. To quantise this classically:

$$H = \frac{1}{2}\epsilon_0 \int \left(\underline{\mathbf{E}}\cdot\underline{\mathbf{E}} + c^2\underline{\mathbf{B}}\cdot\underline{\mathbf{B}}\right) dV = 2\epsilon_0 V \sum_{\underline{k},s} \omega_k^2 A_{\underline{k},s} A^*_{\underline{k},s} \tag{3.7}$$

$$A_{\underline{k},s} = \frac{1}{2\omega_k\sqrt{\epsilon_0 V}}\left\{\omega_k q_{\underline{k},s} + i p_{\underline{k},s}\right\} \tag{3.8}$$

$$A^*_{\underline{k},s} = \frac{1}{2\omega_k\sqrt{\epsilon_0 V}}\left\{\omega_k q_{\underline{k},s} - i p_{\underline{k},s}\right\} \tag{3.9}$$

$q_{\underline{k},s}, p_{\underline{k},s}$ canonical coordinates (x, p).

$$H_{\underline{k},s} = \frac{1}{2}\left(p^2_{\underline{k},s} + \omega_k^2 q_{\underline{k},s}\right) \tag{3.10}$$

Harmonic oscillator $m = 1$, $x \leftrightarrow p$. To transfer this from classical to quantum, you simply convert everything to its operator form. For a single mode:

$$\hat{H}_{\underline{k},s} = \left(\hat{a}^\dagger_{\underline{k},s}\hat{a}_{\underline{k},s} + \frac{1}{2}\right)\hbar\omega_k, \qquad [\hat{a}_{\underline{k},s}, \hat{a}^\dagger_{\underline{k},s}] = 1, \qquad \hat{a}^\dagger_{\underline{k},s}\hat{a}_{\underline{k},s} = \hat{n}_{\underline{k},s}. \tag{3.11}$$

Now we have eigenstates, $|n\rangle_{\underline{k},s}$. Note: modes are not always equal to photons, but you can have photons spread over several modes.
Going back on the substitution:

$$\hat{A}_{\underline{k},s} = \sqrt{\frac{\hbar}{2\omega_k \epsilon_0 V}}\hat{a}_{\underline{k},s} \qquad \hat{A}^\dagger_{\underline{k},s} = \sqrt{\frac{\hbar}{2\omega_k \epsilon_0 V}}\hat{a}^\dagger_{\underline{k},s} \tag{3.12}$$

From these, we can find the quantised electric and magnetic field expressions. We will mostly be concerned with the electric field throughout this course as it has a much stronger interaction with matter than the magnetic.

$$\hat{\underline{\mathbf{E}}}_{\underline{k},s}(\underline{r},t) = i\left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right)^{1/2} \underline{e}_{\underline{k},s}\left[\hat{a}_{\underline{k},s}\exp\{i(\underline{k}\cdot\underline{r}-\omega_k t)\} - \hat{a}^\dagger_{\underline{k},s}\exp\{-i(\underline{k}\cdot\underline{r}-\omega_k t)\}\right] \tag{3.13}$$

3.3 Multimode Fields

$$\hat{H}_{\underline{k},s} = \sum_{\underline{k},s}\hbar\omega_k\left(\hat{a}^\dagger_{\underline{k},s}\hat{a}_{\underline{k},s} + \frac{1}{2}\right) \tag{3.14}$$

So the modes are independent of each other, but will interact through matter. We have a basis of

$$|n_1 n_2 n_3 \ldots\rangle \equiv |n_1\rangle_{\underline{k}1,s} \otimes |n_2\rangle_{\underline{k}2,s} \otimes \ldots \tag{3.15}$$

Now we can write the electric field operator:

$$\hat{\underline{\mathbf{E}}}(\underline{r},t) = \sum_{\underline{k},s}\hat{\underline{\mathbf{E}}}_{\underline{k},s}(\underline{r},t) \tag{3.16}$$

$$= \sum_{\underline{k},s} i\left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right)^{1/2} \underline{e}_{\underline{k},s}\left\{\hat{a}_{\underline{k},s}\exp[i(\underline{k}\cdot\underline{r}-\omega_k t)] + \hat{a}^\dagger_{\underline{k},s}\exp[-i(\underline{k}\cdot\underline{r}-\omega_k t)]\right\} \tag{3.17}$$

This is written in the Heisenberg representation. Now if we look at the expectation value, for one mode of the electric field

$${}_{\underline{k},s}\langle n|\hat{\underline{\mathbf{E}}}(\underline{r},t)|n'\rangle_{\underline{k},s} \tag{3.18}$$

This is time dependent as seen by the field operator and will oscillate in time through some means. As a reminder, consider an operator in the Heisenberg picture:

$$\hat{O}_H(t) = \hat{U}^\dagger(t,t_0)\hat{O}\hat{U}(t,t_0) \tag{3.19}$$

$$\hat{U}(t,t_0) = \exp\left[-i\frac{\hat{H}(t-t_0)}{\hbar}\right] \tag{3.20}$$

Lecture 4 Quantum States of the Electromagnetic Field.

4.1 Learning Outcomes

We will cover three important families of quantum states of light: number states, coherent states, and squeezed states. After studying this part, you should:

- ➤ recognise each of these types of states, and be able to provide a simple definition
- ➤ be familiar with the expansion of coherent states in terms of number states
- ➤ be able to calculate expectation values and variances of physical quantities (e.g. electric field, photon number, quadratures, etc) for number states and coherent states
- ➤ be familiar with and able to define the quadrature operators
- ➤ be able to sketch phase space diagrams of coherent states and squeezed states
- ➤ to provide and explain the definition of a quadrature squeezed state and its key properties
- ➤ to describe the uncertainty relation and the concept of minimum uncertainty states

4.2 Single Mode Fields

$$\hat{n}_{\underline{k},s} = \hat{a}^{\dagger}_{\underline{k},s}\hat{a}_{\underline{k},s} \tag{4.1}$$

$$\hat{H}_{\underline{k},s} = \left(\hat{n}_{\underline{k},s} + \frac{1}{2}\right)\hbar\omega_k \tag{4.2}$$

The eigenstates are the number states, a.k.a Fock states, $|n\rangle_{\underline{k},s}$.

$$\hat{n}_{\underline{k},s}|n\rangle_{\underline{k},s} = n|n\rangle_{\underline{k},s} \tag{4.3}$$

$$\hat{H}_{\underline{k},s}|n\rangle_{\underline{k},s} = \hbar\omega_k\left(n + \frac{1}{2}\right)|n\rangle_{\underline{k},s} \tag{4.4}$$

It follows that the vacuum state has energy as well - $|0\rangle_{\underline{k},s}$, with energy $\frac{1}{2}\hbar\omega_k$.

$$\hat{a}^{\dagger}_{\underline{k},s}|0\rangle_{\underline{k},s} = |1\rangle_{\underline{k},s} \tag{4.5}$$

$$\hat{a}_{\underline{k},s}|1\rangle_{\underline{k},s} = |0\rangle_{\underline{k},s} \tag{4.6}$$

4.3 Multimode fields

For multimode states, we just do the sum of all of these states.

$$\hat{H} = \sum_{\underline{k},s}\left(\hat{n}_{\underline{k},s} + \frac{1}{2}\right)\hbar\omega_k \tag{4.7}$$

$$|n_0 n_1 n_2 \dots n_k \dots\rangle = |n_0\rangle_0 \otimes |n_1\rangle \otimes \dots \otimes |n_k\rangle_k \otimes \dots \tag{4.8}$$

The modes cannot interact with each other, so are independent unless there is matter to interact with.

$$\left[\hat{a}^{\dagger}_{\underline{k},s}, \hat{a}_{\underline{k}',s'}\right] = \delta_{kk'}\delta_{ss'} \tag{4.9}$$

$$_{\underline{k},s}\langle n_k|n_{k'}\rangle_{\underline{k}',s} = 0,\ \forall k \neq k', s \neq s' \tag{4.10}$$

4.4 Electric Field of a Single Field Number State

What is the expectation value, $\langle\hat{\underline{\mathbf{E}}}_{\underline{k},s}(\underline{r},t)\rangle$?

$$_{\underline{k},s}\langle n|\hat{\underline{\mathbf{E}}}_{\underline{k},s}(\underline{r},t)|n\rangle_{\underline{k},s} = \underline{e}_{\underline{k},s}i\left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right)^{1/2}\left[_{\underline{k},s}\langle n|\hat{a}_{\underline{k},s}|n\rangle_{\underline{k},s}e^{i(\underline{k}\cdot\underline{r}-\omega_k t)} - _{\underline{k},s}\langle n|\hat{a}^{\dagger}_{\underline{k},s}|n\rangle_{\underline{k},s}e^{-i(\underline{k}\cdot\underline{r}-\omega_k t)}\right] \tag{4.11}$$

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \implies \langle n|n-1\rangle = 0 \tag{4.12}$$

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \implies \langle n|n+1\rangle = 0 \tag{4.13}$$

$$\implies \langle \hat{\underline{\mathbf{E}}}_{\underline{k},s}(\underline{r},t)\rangle = 0, \ \forall n \tag{4.14}$$

I.e. the mean electric field is zero for number states. Now consider the expectation value for the square of the electric field, $\langle \hat{\underline{\mathbf{E}}}_{\underline{k},s}(\underline{r},t)^2\rangle$.

$$\langle \hat{\underline{\mathbf{E}}}_{\underline{k},s}(\underline{r},t)^2\rangle = 2\left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right)\left(n+\frac{1}{2}\right) \tag{4.15}$$

Now we can work out the variance:

$$\langle(\Delta\underline{\mathbf{E}})^2\rangle = \langle\hat{\underline{\mathbf{E}}}^2\rangle - \langle\hat{\underline{\mathbf{E}}}\rangle^2 = 2\left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right)\left(n+\frac{1}{2}\right) \tag{4.16}$$

$$H = \frac{1}{2}\epsilon_0 \int \left(\underline{\mathbf{E}}^2 + \frac{1}{c^2}\underline{\mathbf{B}}^2\right) dV \tag{4.17}$$

From analogy to the classical Hamiltonian, we can see how the variance would be related to the energy. The vacuum state fluctuates in its energy around zero, and this has observable effects.

4.5 Electric Field in Multimode Fields

For a multimode field's vacuum,

$$\langle(\Delta\underline{\mathbf{E}})^2\rangle = \sum_{\underline{k},s}\left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right). \tag{4.18}$$

This term sums to infinity, which can be a problem, and leads to some effects:

- The Lamb shift in atoms' energy levels
- The Casimir effect causes a series of modes to form in between two plates that can be zero at the plates, while the modes outside the plates have no restrictions. The difference in the two areas of modes causes a force that pushes the plates together. The Casimir effect has a classical analogue when boats close to a harbor wall are pushed into the wall by the difference in waves from out in the open water and between the boat and the wall.

4.6 The Number States - A Summary

- Complete orthonormal basis
- Well defined photon number and energy
- Zero mean electric field - not at all classical
- E-field fluctuates, even for $|0\rangle$, and E-field fluctuations increase with n
- Most non-classical states you can get, and you can do experiments with them to show this using single or a few photons in the modes

Lecture 5 Towards the Classical Limit

5.1 States with Classical Limits

$$E_x(z,t) \propto \sin(kx - \omega t) \tag{5.1}$$
$$\langle n|\hat{a}|n\rangle = \langle n|\hat{a}^\dagger|n\rangle = 0 \tag{5.2}$$
$$|\psi\rangle = \sum_n c_n|n\rangle \tag{5.3}$$

We want to look for eigenstates of $\hat{a}$:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \alpha \in \mathbb{C} \tag{5.4}$$

We did this because right eigenstates of $\hat{a}^\dagger$ do not exist.

$$\hat{a}^\dagger|\beta\rangle \neq \beta|\beta\rangle, \forall\beta \in \mathbb{C} \tag{5.5}$$
$$\langle\alpha|\hat{a}^\dagger = \alpha^*\langle\alpha| \tag{5.6}$$

However, $\hat{a}^\dagger$ does form eigenstates with left states.

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n|n\rangle \tag{5.7}$$
$$\hat{a}|\alpha\rangle = \sum_{n=1}^{\infty} c_n\sqrt{n}|n-1\rangle = \alpha\sum_{n=0}^{\infty} c_n|n\rangle \tag{5.8}$$
$$= \sum_{n=0}^{\infty} c_{n+1}\sqrt{n+1}|n\rangle = \alpha\sum_{n=0}^{\infty} c_n|n\rangle \tag{5.9}$$
$$c_{n+1} = \frac{\alpha}{\sqrt{n+1}}c_n \tag{5.10}$$

Use the fact that the states are normalised to find c_0:

$$\langle\alpha|\alpha\rangle = 1 \to c_0 \tag{5.11}$$
$$= |c_0|^2 \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha^*)^m}{\sqrt{m!}}\frac{(\alpha)^n}{\sqrt{n!}}\langle m|n\rangle \tag{5.12}$$
$$|c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = |c_0|^2 \exp|\alpha|^2 = 1 \tag{5.13}$$

If we take c_0 to be real and positive,

$$|\alpha\rangle = \exp\left\{-\frac{|\alpha|^2}{2}\right\}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{5.14}$$

These are the eigenstates of $\hat{a}$ with eigenvalue $\alpha \in \mathbb{C}$, $\alpha = 0 \implies |0\rangle$.

$$\langle\alpha|\hat{a}|\alpha\rangle = \alpha \tag{5.15}$$
$$\langle\alpha|\hat{a}^\dagger|\alpha\rangle = \alpha^* \tag{5.16}$$

The states $|\alpha\rangle$ we found are right eigenstates of $\hat{a}$ and left eigenstates of $\hat{a}^\dagger$,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \tag{5.17}$$

$$\langle\alpha|\hat{a}^\dagger = \alpha^*\langle\alpha| \tag{5.18}$$

So what is the mean photon number of these states, $\langle\alpha|\hat{n}|\alpha\rangle$?

$$\langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = |\alpha|^2 \tag{5.19}$$

So α behaves like a mean amplitude. What we want to calculate now is the variance of photon number, i.e. $\langle(\Delta n)^2\rangle = \langle n^2\rangle - \langle n\rangle^2$.

$$\langle n\rangle^2 = |\alpha|^4 \tag{5.20}$$

$$\langle\alpha|\hat{n}^2|\alpha\rangle = \langle\alpha|\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}|\alpha\rangle \tag{5.21}$$

We wish to normal order Eq (5.21) by switching the middle two operators. Using commutators, we have:

$$[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1 \tag{5.22}$$

$$\implies (5.21) = \langle\alpha|\hat{a}^\dagger(\hat{a}^\dagger\hat{a} + 1)\hat{a}|\alpha\rangle \tag{5.23}$$

$$= \langle\alpha|\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}|\alpha\rangle + \langle\alpha|\hat{a}^\dagger\hat{a}|\rangle \tag{5.24}$$

$$= |\alpha|^4 + |\alpha|^2 \tag{5.25}$$

$$\langle(\Delta n)^2\rangle = |\alpha|^2 \tag{5.26}$$

$$\langle n\rangle = \langle(\Delta n)^2\rangle \tag{5.27}$$

So the mean is equal to the variance. This is what is described in the Poisson distribution.

$$\frac{\Delta n}{n} = \langle n\rangle^{-1/2} \tag{5.28}$$

So this gets smaller as n gets larger, i.e. more classical for large n.

5.2 The electric field of a more classical state

$$\langle\hat{\underline{\mathbf{E}}}_x(z,t)\rangle = \langle\alpha|\hat{\underline{\mathbf{E}}}_x(z,t)|\alpha\rangle \tag{5.29}$$

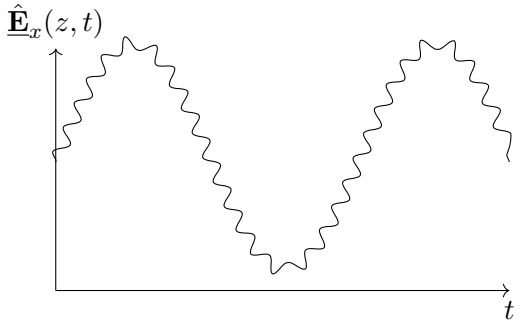
$$= \langle\alpha|\left\{\left(\frac{i\hbar\omega_k}{2\epsilon_0 V}\right)^{1/2}\underline{e}_{\underline{k},x}\left[\hat{a}\exp(i(kz-\omega_k t)) - \hat{a}^\dagger\exp(-i(kz0\omega_k t))\right]\right\}|\alpha\rangle \tag{5.30}$$

$$= 2|\alpha|\left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right)^{1/2}\sin\{\omega t - kz - \theta\}\,\underline{e}_{\underline{k},x},\ \alpha = |\alpha|e^{i\theta} \tag{5.31}$$

$$\langle\hat{\underline{\mathbf{E}}}_x^2(z,t)\rangle = \frac{\hbar\omega_k}{2\epsilon_0 V}\left[1 + 4|\alpha|^2\sin^2(\omega t - kz - \theta)\right] \tag{5.32}$$

$$\langle(\Delta\hat{\underline{\mathbf{E}}}_x(z,t))^2\rangle = \frac{\hbar\omega_k}{2\epsilon_0 V} \tag{5.33}$$

This variance has no α in it, so the $|\alpha\rangle$ states only have vacuum fluctuations. The variance of coherent states = minimum possible, and doesn't depend on α or n. This looks like classical EM field, with $|\alpha|$ the amplitude, and $\langle n\rangle = |\alpha|^2$, but with an uncertainty fluctuation around the path.



$$|\alpha\rangle = \exp\left\{-\frac{|\alpha|^2}{2}\right\}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{5.34}$$

$$|\beta\rangle = \exp\left\{-\frac{|\beta|^2}{2}\right\}\sum_{m=0}^{\infty}\frac{\beta^m}{\sqrt{m!}}|m\rangle \tag{5.35}$$

$$\langle\beta|\alpha\rangle = \exp\left\{-\frac{|\beta|^2+|\alpha|^2}{2}\right\}\sum_{m,n}\frac{(\beta^*)^m(\alpha)^n}{\sqrt{m!n!}}\langle m|n\rangle \tag{5.36}$$

$$= \exp\left\{-\frac{|\beta|^2+|\alpha|^2}{2}\right\}\sum_{n}\frac{(\beta^*\alpha)^n}{n!} \tag{5.37}$$

$$= \exp\left\{-\frac{|\beta|^2+|\alpha|^2}{2}\right\}\exp\{\beta^*\alpha\} \tag{5.38}$$

$$\langle\beta|\alpha\rangle^2 = \exp\left\{-|\alpha-\beta|^2\right\} \tag{5.39}$$

So two states will depend on how much they differ from one another, they form an over-complete basis.

$$\frac{1}{\pi}\int|\alpha\rangle\langle\alpha|\;d^2\alpha = \sum_n|n\rangle\langle n| = \mathbb{I} \tag{5.40}$$

$$|\phi\rangle = \frac{1}{\pi}\int|\alpha\rangle\langle\alpha|\phi\rangle\;d^2\alpha \tag{5.41}$$

Lecture 6 Harmonic Oscillators

$$\langle \hat{\underline{\mathbf{E}}}_x(z,t)\rangle = 2|\alpha\rangle \left(\frac{\hbar\omega_k}{3\epsilon_0 V}\right)^{1/2} \sin\{\omega t - kz - \theta\}\, \underline{e}_{\underline{k},\alpha} \tag{6.1}$$

$$\alpha = |\alpha| e^{i\theta} \tag{6.2}$$

Now back to thinking about the harmonic oscillator, using $\hat{p}, \hat{q}$.

$$\hat{X}_1 = \frac{1}{2}\left(\hat{a} + \hat{a}^\dagger\right) \to \hat{q}, \quad \hat{X}_2 = \frac{1}{2i}\left(\hat{a} - \hat{a}^\dagger\right) \to \hat{p}, \quad [\hat{X}_1, \hat{X}_2] = \frac{i}{2} \tag{6.3}$$

$$\langle \hat{X}_1\rangle_\alpha = \langle\alpha|\hat{X}_1|\alpha\rangle = \frac{1}{2}\left\{\langle\alpha|\hat{a}|\alpha\rangle + \langle\alpha|\hat{a}^\dagger|\alpha\rangle\right\} = \frac{1}{2}\{\alpha + \alpha^*\} = \mathrm{Re}(\alpha) \tag{6.4}$$

$$\langle \hat{X}_2\rangle_\alpha = \frac{1}{2i}\{\alpha - \alpha^*\} = \mathrm{Im}(\alpha) \tag{6.5}$$

$$\langle(\Delta\hat{X}_1)^2\rangle = \langle(\Delta\hat{X}_2)^2\rangle = \frac{1}{4} \tag{6.6}$$

Consider two Hermitian operators, $\hat{A}, \hat{B}$, then the modulus commutator is:

$$\Delta A \cdot \Delta B \geq \frac{1}{2}\left|\langle\psi|[\hat{A},\hat{B}]|\psi\rangle\right| \tag{6.7}$$

$$(\Delta A)^2 = \langle(\Delta A)^2\rangle = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2 \tag{6.8}$$

$$\langle(\Delta\hat{X}_1)^2\rangle\langle(\Delta\hat{X}_2)^2\rangle \geq \frac{1}{4}\left|[\hat{X}_1,\hat{X}_2]\right|^2 = \frac{1}{16} \tag{6.9}$$

For coherent states, Eq (6.12) is equal to $\frac{1}{16}$. So the coherent state is a minimum uncertainty state, where the remaining uncertainty is all in the vacuum fluctuations. We can represent the classical harmonic oscillator in phase space, but we want to do the same thing for a quantum harmonic oscillator, for a state $|\alpha\rangle$:

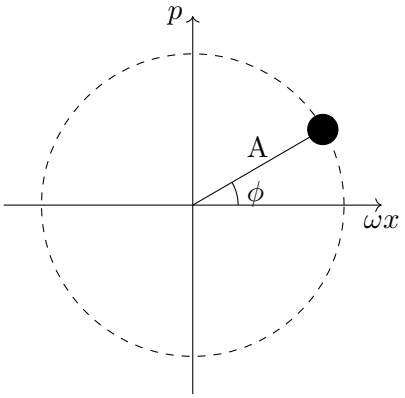


Figure 6.1: Classical HO

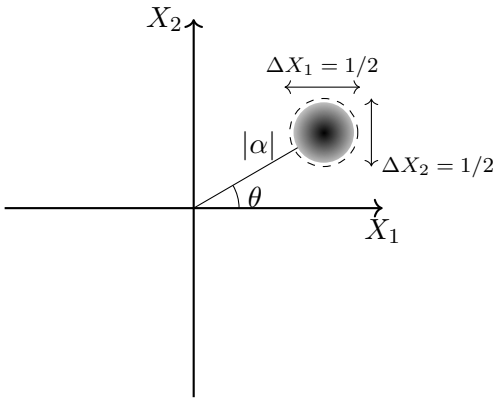


Figure 6.2: Quantum HO

How does the vacuum state look in this picture?

$$\langle(\Delta\hat{X}_1)^2\rangle\langle(\Delta\hat{X}_2)^2\rangle \geq \frac{1}{16} \tag{6.10}$$

So if you make one of these smaller than $\frac{1}{4}$, you can satisfy this equation by making the other larger - this forms squeezed states. You can also squeeze the vacuum state.

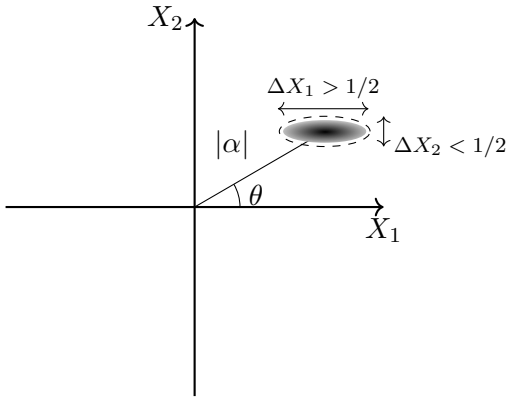


Figure 6.3: Squeezed quadrature state

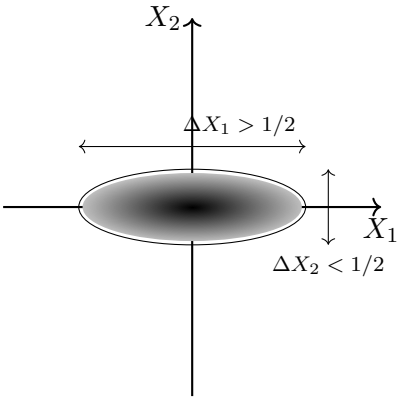


Figure 6.4: Squeezed vacuum state

You can squeeze the amplitude and make the phase more uncertain by showing the ellipse at an angle:

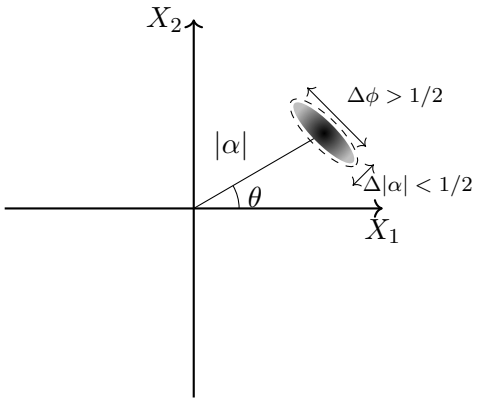


Figure 6.5: Squeezed phase state

To create these states, you need a new operator:

$$\hat{s}(\epsilon) = \exp\left[\frac{1}{2}\left(\epsilon^*\hat{a}^2 - \epsilon(\hat{a}^\dagger)^2\right)\right] \tag{6.11}$$

You can create this operator using:

- ➤ non-linear crystals
- ➤ atomic gases
- ➤ 4-wave mixing

$$\hat{H} = \underbrace{\hbar\omega\hat{a}^\dagger\hat{a}}_{\text{signal mode}} + \underbrace{\hbar\omega_p\hat{b}^\dagger\hat{b}}_{\text{pump mode}} + i\hbar\chi^{(2)}\left(\hat{a}^2\hat{b}^\dagger - (\hat{a}^\dagger)^2\hat{b}\right) \tag{6.12}$$

$\chi^{(2)}$ is known as the second order non-linear susceptibility. So we will treat the pump mode as almost classical,

$$\hat{b} \to \beta e^{i\omega_p t} \qquad \hat{b}^\dagger \to \beta^* e^{-i\omega_p t} \qquad \eta = \chi^{(2)}\beta \tag{6.13}$$

From this, we can write our Hamiltonian in the parametric approximation:

$$\hat{H}^{(PA)} = \hbar\omega\hat{a}^\dagger\hat{a} + i\hbar\left(\eta^*\hat{a}^2 e^{i\omega_p t} - \eta(\hat{a}^\dagger)^2 e^{-i\omega_p t}\right) \tag{6.14}$$

Then in the interaction picture,

$$\hat{H}_{int} = i\hbar\left(\eta^*\hat{a}^2 e^{i(\omega_p - 2\omega)t} - h.c.\right) = i\hbar\left(\eta^*\hat{a}^2 - \eta(\hat{a}^\dagger)^2\right),\ \omega_p = 2\omega \tag{6.15}$$

$$U = \exp\left(-\frac{i}{\hbar}\hat{H}_{int}\right) = \exp\left(\eta^*\hat{a}^2 - \eta(\hat{a}^\dagger)^2\right) \tag{6.16}$$

Lecture 7 Atom-Light Interactions

7.1 Learning Outcomes

We will develop a fully quantum mechanical model for the interaction of light with atoms. After studying this part, you should:

- Be familiar with the atom-light interaction Hamiltonian in the electric dipole approximation, and explain this approximation.
- Be familiar with the Pauli operators as quantum description of the two-level atom.
- Explain the physical basis of the Jaynes-Cummings model for atom-light interactions. Explain and use, but not derive, the resulting Hamiltonian.
- Be familiar with the concept of dressed state, and vacuum Rabi splitting in the context of the Jaynes Cummings Hamiltonian.
- Be able to perform simple calculations with combinations of atomic and field operators.
- Provide a simple explanation of a photodetector in the context of quantum optics.
- Be familiar with the operator expression for the photodetection probability, and understand the meaning of normal ordering in this context.

7.2 Atom-Light Interactions

$$\hat{H} = \hat{H}_{atom} + \hat{H}_{field} + \hat{H}_{int} \tag{7.1}$$

$$\hat{H}_{atom} = \frac{\hat{p}^2}{2m} + \hat{V}(r) \tag{7.2}$$

So for the EM field, classical version in something of $\hat{A}$. Interaction: force on e^- due to EM field,

$$\underline{F} - q\left[\underline{\mathbf{E}}(\underline{r}, t) + \underline{v} \times \underline{\mathbf{B}}(\underline{r}, t)\right] \tag{7.3}$$

where $\underline{F}$ is the Lorentz force. *We do not need to know the derivation for this, only to use it.* This is known as the minimal coupling Hamiltonian. Choose Coulomb gauge, $\phi = 0$, $\nabla \cdot A = 0$, but note that this is not Lorentz invariant. Now we apply the electric dipole approximation, where light $\gg$ atoms.

$$A_{\underline{k},s}(\underline{r}, t) = A_{\underline{k},s} \exp\left[i(\underline{k} \cdot \underline{r} 0\omega_k t)\right] + c.c. \tag{7.4}$$

The length scale of the light in this is given by the wavevector, $\approx |\underline{k}| = 2\pi/\lambda$. For visible light, $\lambda \approx 400 - 700\, nm$; for microwaves, $\lambda \approx 1\, cm$. But for atoms, $\approx 0.1\, nm$, so as far as atoms are concerned, photons are huge. If $\underline{r}$ is the size of the atom, $\underline{k} \cdot \underline{r} \ll 1$. So $A(\underline{r}, t) \approx A(t)$ over the size of the atom.

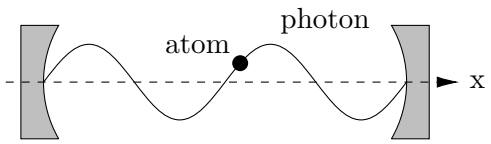
$$\hat{H}_{int} = e\hat{\underline{r}} \cdot \hat{\underline{\mathbf{E}}}(t) \tag{7.5}$$

The sign of e is positive, so e^- is $-e$. $e\hat{\underline{r}} = \hat{\underline{d}}$, the electric dipole operator.

$$\hat{H}_{int} = \hat{\underline{d}} \cdot \hat{\underline{\mathbf{E}}}(t) \tag{7.6}$$

So we want a single mode field, and the best way to think about this is place the light into a cavity.

$$\hat{\underline{\mathbf{E}}} = e_{\underline{k}} \left(\frac{\hbar\omega_k}{\epsilon_0 V}\right)^{1/2} (\hat{a} + \hat{a}^\dagger) \sin(kx) \tag{7.7}$$



Two level atom - the two that match EM field.

$$\hat{H} = \hat{H}_{atom} + \hat{H}_{field} + \hat{H}_{int} \tag{7.8}$$

$$\hat{H}_{field} = \hbar\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \tag{7.9}$$

The $\frac{1}{2}$ is the zero-point energy. A two-level atom is equivalent to a spin-$\frac{1}{2}$ particle.

$$\hat{\sigma}_+ = |e\rangle\langle g| \qquad \hat{\sigma}_- = |g\rangle\langle e| = \hat{\sigma}_+^\dagger \tag{7.10}$$

$$\hat{\sigma}_z = |e\rangle\langle e| - |g\rangle\langle g| \tag{7.11}$$

$$\hat{\sigma}_+ = \hat{\sigma}_x + i\hat{\sigma}_y \qquad \hat{\sigma}_- = \hat{\sigma}_x - \hat{\sigma}_y \tag{7.12}$$

Now we can write the Hamiltonian for the atom:

$$\hat{H}_{atom} = \frac{1}{2}(E_e - E_g)\hat{\sigma}_z = \frac{1}{2}\hbar\omega_0\hat{\sigma}_z \tag{7.13}$$

$$\hat{H}_{int} = -\hat{\underline{d}}\cdot\hat{\underline{\mathbf{E}}} = g\hat{d}(\hat{a} + \hat{a}^\dagger) \tag{7.14}$$

$$g = -\left(\frac{\hbar\omega}{\epsilon_0 V}\right)^{1/2}\sin(kx) \tag{7.15}$$

$$\langle g|\hat{\underline{d}}|g\rangle = \langle e|\hat{\underline{d}}|e\rangle = 0 \tag{7.16}$$

$$\hat{\underline{d}} = d|g\rangle\langle e| + d^*|e\rangle\langle g| \tag{7.17}$$

$$= d(\hat{\sigma}_+ + \hat{\sigma}_-) \tag{7.18}$$

$$\hat{H}_{int} = \hbar\left(\frac{dg}{\hbar}\right)(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a} + \hat{a}^\dagger) \tag{7.19}$$

$$\hat{H} = \frac{1}{2}\hbar\omega_0\hat{\sigma}_z + \hbar\omega_k\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \hbar\lambda(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a} + \hat{a}^\dagger) \tag{7.20}$$

Note that λ here is not a wavelength but just the constant from Eq (7.19).

7.3 Rotating Wave Approximation

So in the Heisenberg picture, we have

$$\hat{a}(t) = \hat{a}(0)e^{-i\omega t} \qquad \hat{a}^\dagger(t) = \hat{a}^\dagger(0)e^{+i\omega t} \tag{7.21}$$

$$\hat{\sigma}_+(t) = \hat{\sigma}_+(0)e^{+i\omega_0 t} \qquad \hat{\sigma}_-(t) = \hat{\sigma}_-(0)e^{-i\omega_0 t} \tag{7.22}$$

$\hat{H}_{int}$ has:

$$\hat{\sigma}_+\hat{a} \approx e^{i(\omega_0-\omega)t} \qquad \hat{\sigma}_-\hat{a}^\dagger \approx e^{i(w-\omega_0)t} \tag{7.23}$$

$$\hat{\sigma}_+\hat{a}^\dagger \approx e^{i(\omega+\omega_0)t} \qquad \hat{\sigma}_-\hat{a} \approx e^{-i(\omega+\omega_0)t} \tag{7.24}$$

Eqs (7.23) oscillate slowly and conserve energy, while Eqs (7.24) oscillate fast and do not conserve energy. So we want to drop the fast oscillating terms as these are not useful to our picture of the Hamiltonian, and this is the rotating wave approximation. From this, we arrive at the Jaynes-Cummings Hamiltonian:

$$\hat{H}_{JC} = \frac{1}{2}\hbar\omega_0\hat{\sigma}_z + \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \hbar\lambda(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger) \tag{7.25}$$

This describes:

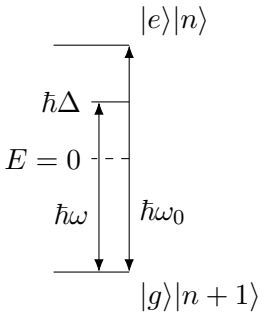
- ➤ energy of atom
- ➤ energy of photons/cavity mode (single mode)
- ➤ how energy can be exchanged between atom and cavity mode

$$\hat{\sigma}_+\hat{a}|g\rangle|n\rangle \implies |e\rangle|n-1\rangle \tag{7.26}$$

$$\hat{\sigma}_-\hat{a}^\dagger|e\rangle|n\rangle \implies |g\rangle|n+1\rangle \tag{7.27}$$

So these behave like raising and lowering operators as they shift the atom around the cavity.
So what assumptions did we use?

- ➤ electron can't go anywhere else outside the cavity
- ➤ number of photons is fixed, such that there were no other losses to outside the cavity
- ➤ only need $|g\rangle, |e\rangle$ for the two levels of the atom and $|n\rangle, |n+1\rangle$ for the states of the photons



We can then form two basis states:

$$|\psi_1, n\rangle \equiv |e\rangle|n\rangle = |i\rangle, \quad |\psi_2, n\rangle \equiv |g\rangle|n+1\rangle = |f\rangle \tag{7.28}$$

$$\hat{H}(n) = \hbar \begin{pmatrix} n\omega + \frac{1}{2}\omega_0 & \lambda\sqrt{n+1} \\ \lambda\sqrt{n+1} & (n+1)\omega - \frac{1}{2}\omega_0 \end{pmatrix} \tag{7.29}$$

Matrix elements obtained from $\langle i|\hat{H}(n)|i\rangle$, etc.

Lecture 8 Rotating Wave Approximation

8.1 Rotating Wave Approx Ctd

We will take $\Delta = 0$, and for an initial state $|i\rangle = |e\rangle|n\rangle$,

$$|\psi(t)\rangle = c_i(t)|i\rangle + c_f(t)|f\rangle \tag{8.1}$$

$$E_i = \frac{1}{2}\hbar\omega + n\hbar\omega,\ E_f = -\frac{1}{2}\hbar\omega + (n+1)\hbar\omega \tag{8.2}$$

So the energies are equal, and conserved. We can then solve the Schrodinger equation for Eq (8.1), and find relations for the co-factors:

$$\dot{c}_i = -i\lambda\sqrt{n+1}c_f \qquad \ddot{c}_i + \lambda^2(n+1)c_i = 0 \tag{8.3}$$

$$c_i(t) = \cos\left(\lambda t\sqrt{n+1}\right) \qquad c_f(t) = i\sin\left(\lambda t\sqrt{n+1}\right) \tag{8.4}$$

$$P_i(t) = |c_i(t)|^2 = \cos^2\left(\lambda t\sqrt{n+1}\right) \tag{8.5}$$

To figure this out, we can go through many different solutions, but the simplest in this problem is to use what is called the **atomic inversion**:

$$\omega(t) = \langle\psi(t)|\hat{\sigma}_z|\psi(t)\rangle \tag{8.6}$$

$$= P_i(t) - P_f(t) = \cos\left(2\lambda t\sqrt{n+1}\right) \tag{8.7}$$

Now we will define the **Rabi frequency**, as

$$\Omega(n) = 2\lambda\sqrt{n+1} \tag{8.8}$$

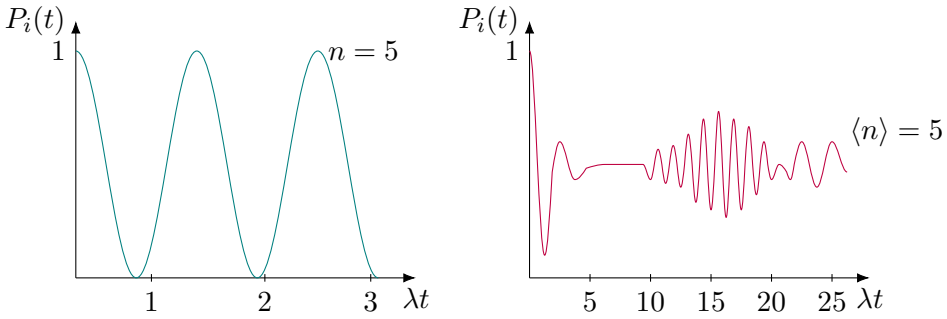
Note that there are even Rabi oscillations for $n = 0$. This may seem a strange possibility at first, but there are many experiments to prove this is in fact the case, where it occurs in our cavity model as the vacuum of the single mode focused on the atom. This can appear quite classical at first, but in the more general case as you increase the number of states involved, it will seem more quantum. We can write general descriptions for our ground and excited states:

$$|\psi_g(t)\rangle = -i\sin\sum_{n=0}^{\infty} c_n \sin\left(\frac{1}{2}\Omega(n)\right)|n+1\rangle \tag{8.9}$$

$$|\psi_e(t)\rangle = \sum_{n=0}^{\infty} c_n \cos\left(\frac{1}{2}\Omega(n)t\right)|n\rangle \tag{8.10}$$

$$\omega(t) = \sum_{n=0}^{\infty} |c_n|^2 \cos\left(\Omega(n)t\right),\ c_n = \frac{\alpha^n}{\sqrt{n!}}e^{-|\alpha|^2/2} \tag{8.11}$$

For a single number state, P_i oscillates between ground and excited in a classical sinusoidal appearance, but when we consider a coherent state with average $\langle n\rangle$, it drops down into being a superposition of being ground and excited, then after a while excites into partial oscillations again, and continues this interesting pattern.



Now, we want to analyse the Jaynes-Cummings Hamiltonian through the calculation of the matrix elements instead, so we will start with our same two pairs of states, this time with $\Delta \neq 0$.

$$|i\rangle = |e\rangle|n\rangle, \quad |f\rangle = |g\rangle|n+1\rangle \tag{8.12}$$

$$\hat{H}(n) = \hbar \begin{pmatrix} n\omega + \frac{1}{2}\omega_0 & \lambda\sqrt{n+1} \\ \lambda\sqrt{n+1} & (n+1)\omega - \frac{1}{2}\omega_0 \end{pmatrix} \tag{8.13}$$

This has eigenenergies with the Rabi frequency, which is the same as before if $\Delta = 0$.

$$E_\pm = \hbar \left\{ \left(n + \frac{1}{2}\right)\omega \pm \frac{1}{2}\Omega_n(\Delta) \right\} \tag{8.14}$$

$$\Omega_n(\Delta) = \left[\Delta^2 + 4\lambda^2(n+1)\right]^{1/2} \tag{8.15}$$

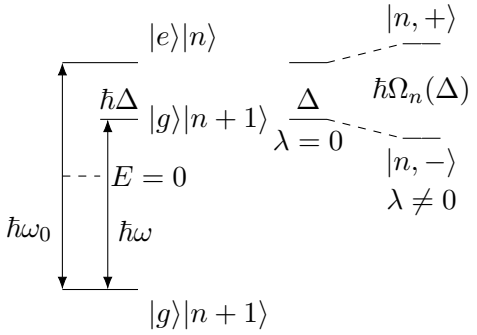
We then have eigenstates,

$$|n,+\rangle = \cos\left(\frac{1}{2}\Phi_n\right)|i\rangle + \sin\left(\frac{1}{2}\Phi_n\right)|f\rangle \qquad |n,-\rangle = -\sin\left(\frac{1}{2}\Phi_n\right)|i\rangle + \cos\left(\frac{1}{2}\Phi_n\right)|f\rangle \tag{8.16}$$

We have used a new angle here, Φ_n, where this is defined as

$$\Phi_n = \arctan\left(\frac{2\lambda\sqrt{n+1}}{\Delta}\right) = \arctan\left(\frac{\Omega_n(0)}{\Delta}\right). \tag{8.17}$$

We are happy with this definition even in the limit of $\Delta \to 0$, as tans have known angles even at ∞. The eigenstates above are known as *dressed states* of the atom, so when the atom is in a single-mode field like this, the field changes the eigenstates so they become eigenstates of the whole system. This looks like the energies of the eigenstates being changed by the application of the field. The original eigenstates we considered, $|e\rangle|n\rangle$ and $|g\rangle|n+1\rangle$, are called *bare states.* These bare states do not have the same energy, unless $\Delta = 0$.

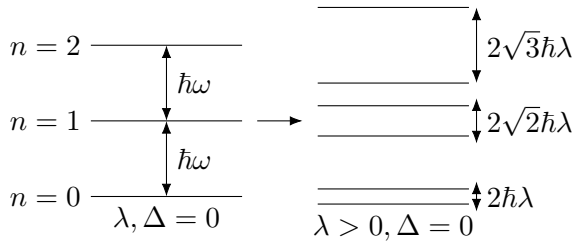


So if we define $\Delta = \omega_0 - \omega$, we can see that now Δ is the energy splitting between the states $|e\rangle|n\rangle$ and $|g\rangle|n+1\rangle$, but this is for $\lambda = 0$ and we can see the dependence on λ on these energy states through the Rabi frequency. It is then clear that this energy splitting Δ, will increase for the eigenenergies, $|n,-\rangle$ and $n,+\rangle$. This is known as a *Stark Shift*, since it is interacting through the E-field of light. Specifically, it is known as the AC or dynamic Stark shift, as the source of the electric field is not static, but dynamic.

So as $\Delta \to 0$, we are on resonance, and the bare states are degenerate, but the dressed states are still split.

$$\arctan\left(\frac{2\lambda\sqrt{n+1}}{\Delta \to 0}\right) \to \frac{\pi}{2} \qquad \cos\left(\frac{1}{2}\Phi_n\right) = \sin\left(\frac{1}{2}\Phi_n\right) \to \frac{1}{\sqrt{2}} \tag{8.18}$$

$$|n,+\rangle = \frac{1}{\sqrt{2}}\left(|e\rangle|n\rangle + |g\rangle|n+1\rangle\right) \qquad |n,-\rangle = \frac{1}{\sqrt{2}}\left(-|e\rangle|n\rangle + |g\rangle|n+1\rangle\right) \tag{8.19}$$



The phenomenon shown above is known as vacuum Rabi splitting, where we get anharmonic energy levels due to coupling as $\lambda \neq 0$. This is the underlying physics that allows us to engineer quantum technology by compressing waves down several orders of magnitude. This all begins as the light and atom when allowed to interact begin to behave as a composite system.

Lecture 9 Photodetection and Correlation Functions

9.1 Photodetection

We are interested in detecting low numbers of photons, so we will make use of the physics previously discussed, in terms of Hamiltonians

$$\hat{H} = \hat{H}_{field} + \hat{H}_{atom} + \hat{H}_{int} \tag{9.1}$$

$$\hat{H}_{field} = \sum_{\underline{k},s} \hbar\omega_k \left(\hat{a}^\dagger_{\underline{k},s} \hat{a}_{\underline{k},s} + \frac{1}{2} \right) \tag{9.2}$$

$$\hat{H}_{atom} = \hbar\omega_0 \hat{\sigma}_z, \ |k| = \frac{2\pi}{\lambda} \tag{9.3}$$

$$\tag{9.4}$$

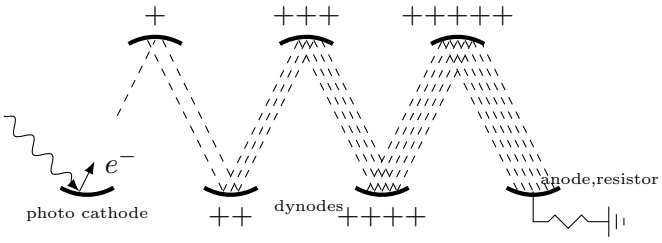
We then want to use the electric dipole approximation to build our interaction Hamiltonian,

$$\hat{H}_{int} = e\hat{\underline{r}} \cdot \hat{\underline{\mathbf{E}}} = -\hat{d} \cdot \hat{\underline{\mathbf{E}}}(\underline{R}, t) \tag{9.5}$$

We are aiming to ionise the atom, because then we have a free electron to detect as well. Once we have one electron freed, we can pull many more until we have multiple electrons to form a current, which we can detect. Of course to ionise the atom, we require the photon to add enough energy to free the electron.

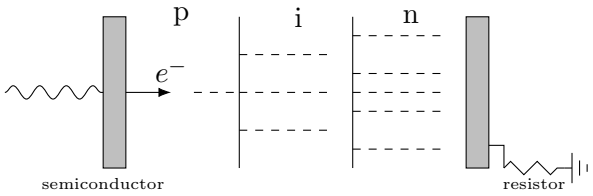
Two types of photodetectors:

- Photomultiplier tube (PMT):

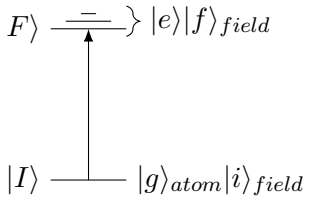


  It is in a tube so it has a high vacuum. This tube can result in a gain $\approx 10^8$.

- Avalanche Photo Detector (APD)



  We do not have as high a gain here, but using a semiconductor instead of a high vacuum is more practical and easier to make.



For a free electron,

$$|e\rangle = \frac{1}{\sqrt{V}} e^{i\underline{q}\cdot\underline{r}} \tag{9.6}$$

So for our interaction Hamiltonian then,

$$\hat{H}_{int} = -\hat{d}\cdot\hat{\underline{\mathbf{E}}}(\underline{r},t) \tag{9.7}$$

$$\hat{\underline{\mathbf{E}}}(\underline{r},t) = i\sum_{\underline{k},s}\left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right)^{1/2} \underline{e}_{\underline{k},s}\left[\hat{a}^\dagger_{\underline{k},s}(t)e^{i\underline{k}\cdot\underline{r}} - \hat{a}_{\underline{k},s}(t)e^{-i\underline{k}\cdot\underline{r}}\right] \tag{9.8}$$

This is clearly in the Heisenberg picture as we have time-dependent operators. Here, $\underline{e}$ is the polarisation vector.

$$\hat{a}_{\underline{k},s}(t) = \hat{a}_{\underline{k},s}e^{-i\omega_k t} \qquad\qquad \hat{a}^\dagger_{\underline{k},s}(t) = \hat{a}^\dagger_{\underline{k},s}e^{i\omega_k t} \tag{9.9}$$

We want to split the electric field into two parts for absorption and emission:

$$\hat{\underline{\mathbf{E}}}_+(\underline{r},t) \equiv i\sum_{\underline{k},s}\left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right)^{1/2} \underline{e}_{\underline{k},s}\hat{a}_{\underline{k},s}(t)e^{i\underline{k}\cdot\underline{r}} \tag{9.10}$$

$$\hat{\underline{\mathbf{E}}}_-(\underline{r},t) \equiv -i\sum_{\underline{k},s}\left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right)^{1/2} \underline{e}_{\underline{k},s}\hat{a}^\dagger_{\underline{k},s}(t)e^{-i\underline{k}\cdot\underline{r}} \tag{9.11}$$

These have been defined such that $\hat{\underline{\mathbf{E}}}_- \equiv \hat{\underline{\mathbf{E}}}_+^\dagger$ and vice versa; these operators have taken on the properties of the creation and annihilation operators. We only want to use $\hat{\underline{\mathbf{E}}}_+$ as this deals with absorption, which is what we're interested in.

$$\langle F|\hat{H}|I\rangle \approx -\underbrace{\langle e|\hat{d}|g\rangle}_{\text{atom}}\underbrace{\langle f|\hat{\underline{\mathbf{E}}}_+(\underline{r},t)|i\rangle}_{\text{field}} \tag{9.12}$$

The atom term above is equal to d, but the field term isn't so clear. We must sum over all possible final states to find this expectation value, using the **broadband detector assumption.**

$$P_f = |\langle F|\hat{H}|I\rangle|^2 \propto |\langle f|\hat{\underline{\mathbf{E}}}_+(\underline{r},t)|i\rangle|^2 \tag{9.13}$$

$$P_i \propto \sum_f |\langle f|\hat{\underline{\mathbf{E}}}_+(\underline{r},t)|i\rangle|^2 \tag{9.14}$$

$$\propto \sum_f \langle i|\hat{\underline{\mathbf{E}}}_-(\underline{r},t)|f\rangle\langle f|\hat{\underline{\mathbf{E}}}_+(\underline{r},t)|i\rangle \tag{9.15}$$

If this sum is over all possible f and doesn't depend on any of the other terms, then the $|f\rangle\langle f|$ will sum to the identity.

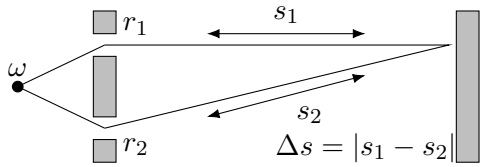
$$P_i \propto \langle i|\hat{\underline{\mathbf{E}}}_-(\underline{r},t)\hat{\underline{\mathbf{E}}}_+(\underline{r},t)|i\rangle \tag{9.16}$$

$$\hat{\underline{\mathbf{E}}}_+(\underline{r},t) \equiv i\sum_{\underline{k},s}\left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right)^{1/2} \underline{e}_{\underline{k},s}\hat{a}_{\underline{k},s}(t)e^{i\underline{k}\cdot\underline{r}} \tag{9.17}$$

Depending on what our state is, wrapping it around the electric field operators can produce some fairly non-trivial outcomes. Note that the ordering of these operators is important; if the operators were the other way around, they would be acting on the f bra/kets instead of the i bra/kets, and we would not have been able to sum and cancel the fs.
Reminding ourselves that we are trying to count photons, we can see that $\hat{\underline{\mathbf{E}}}_-\hat{\underline{\mathbf{E}}}_+ \propto \hat{a}^\dagger\hat{a} = \hat{n}$, the number operator. Therefore our current result makes sense as it will indeed count photons, for discrete states or coherent states (detecting average number of photons instead). We will see more on this when we discuss correlation functions in lecture 11, but we first need to understand beamsplitters in lecture 10.

9.2 Coherence/correlation functions

First, we will remind ourselves of the classical functions. An early example is Young's slits experiment:



We need

$$\Delta s \leq \frac{c}{\Delta\omega}, \qquad \Delta t = \frac{\Delta s}{c} = \frac{1}{\Delta\omega} \tag{9.18}$$

$\frac{c}{\Delta\omega}$ is the coherence length, and Δt is the coherence time. Now what about the field at the detector?

$$t_1 = t - \frac{s_1}{c},\ t_2 = t - \frac{s_2}{c} \tag{9.19}$$

$$\underline{\mathbf{E}}(\underline{r}, t) = k_1\underline{\mathbf{E}}(\underline{r}_1, t_1) + k_2\underline{\mathbf{E}}(\underline{r}_2, t_2) \tag{9.20}$$

$$I(\underline{r}) = \langle|\underline{\mathbf{E}}(\underline{r}, t)|^2\rangle_t \tag{9.21}$$

$$= |k_1|^2\langle|\underline{\mathbf{E}}(\underline{r}_1, t_1)|^2\rangle + |k_2|^2\langle|\underline{\mathbf{E}}(\underline{r}_2, t)|^2\rangle + 2\mathrm{Re}\left[k_1^* k_2\langle\underline{\mathbf{E}}^*(\underline{r}_1, t_1)\underline{\mathbf{E}}(\underline{r}_2, t_2)\rangle\right] \tag{9.22}$$

$$= I_1 + I_2 + 2\mathrm{Re}[\text{interference}] \tag{9.23}$$

So we introduce $\gamma^{(1)}(x_1, x_2)$. This is the first order normalised mutual coherence function.

$$\gamma^{(1)}(x_1, x_2) = \frac{\langle\underline{\mathbf{E}}^*(\underline{r}_1, t_1)\underline{\mathbf{E}}(\underline{r}_2, t_2)\rangle}{\left\{\langle|\underline{\mathbf{E}}(\underline{r}_1, t_1)|^2\rangle\langle|\underline{\mathbf{E}}(\underline{r}_2, t_2)|^2\rangle\right\}^{1/2}} \tag{9.24}$$

$$I(\underline{r}) = I_1 + I_2 + 2\sqrt{I_1 I_2}\mathrm{Re}\left[k_1^* k_2\gamma^{(1)}(x_1, x_2)\right] \tag{9.25}$$

Now we set $k_j = |k_j|e^{i\phi_j}$.

$$I(\underline{r}) = I_1 + I_2 + 2\sqrt{I_1 I_2}|\gamma^{(1)}(x_1, x_2)|\cos(\phi_1 - \phi_2) \tag{9.26}$$

$$\tag{9.27}$$

The cos term in this is the phase difference from path difference. The value of $\gamma^{(1)}$ sets the amount of coherence:

$$\gamma^{(1)}\begin{cases} = 1 & \text{full coherence} \\ > 0, < 1 & \text{partial coherence} \\ = 0 & \text{no coherence} \end{cases} \tag{9.28}$$

Lecture 10 Quantum Theory of Beamsplitters

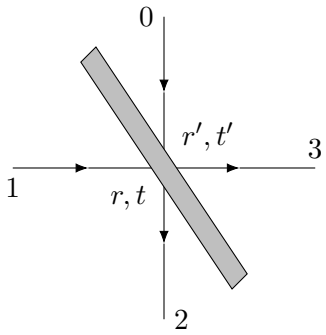
10.1 Learning Outcomes

We will develop the quantum mechanical theory of the beamsplitter. After studying this section, you should

- ➤ Be familiar with the classical description of a lossless beamsplitter.
- ➤ Be able to derive equations that relate the creation and annihilation operators for a lossless beamsplitter.
- ➤ Understand the meaning of coincidence in the context of the beamsplitter.
- ➤ Understand the meaning of unitary transforms in the context of the beamsplitter.
- ➤ Be familiar with the significance of a beamsplitter for single photon states, and the results when the two input modes each contain a single photon (Hong-Ou-Mandel effect).
- ➤ Be familiar with the operation of a beamsplitter with coherent states.
- ➤ Be able to calculate the joint output state of a beamsplitter for any number state or coherent state input, or combinations thereof.

10.2 The beam splitter

A general beamsplitter will look like the following:



r, t, r', t' are complex, so they introduce phases. We will only deal with **lossless beamsplitters:** $|r| = |r'|$, $|t| = |t'|$. Beamsplitters mix modes, so 2 and 3 are both linear combinations of 0 and 1. In a quantum beamsplitter, there will **always** be two inputs; if we do not put two in, then the second will be the vacuum, which must still be considered. For now, we will not consider polarisation dependence, but this will be addressed later on. Classically,

$$\begin{pmatrix} a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} t' & r \\ r' & t \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \tag{10.1}$$

For the quantum case,

$$\begin{pmatrix} \hat{a}_2 \\ \hat{a}_3 \end{pmatrix} = \begin{pmatrix} t' & r \\ r' & t \end{pmatrix} \begin{pmatrix} \hat{a}_0 \\ \hat{a}_1 \end{pmatrix} \tag{10.2}$$

and similarly for $\hat{a}^\dagger$ but with the complex conjugates of t, r etc. We shall fix the phase shift so reflected vs transmitted $\pm i$, $\exp\left(\pm i\frac{\pi}{2}\right)$. Using a 50:50 beam splitter, $|r| = |t| = \frac{1}{\sqrt{2}}$.

$$\begin{pmatrix} \hat{a}_2 \\ \hat{a}_3 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} \hat{a}_0^\dagger \\ \hat{a}_1^\dagger \end{pmatrix} \tag{10.3}$$

This is the balanced Hadamard operator used in Quantum information.
We want to calculate $\langle \hat{n}_2 \rangle$. So for the input, we have

$$|0\rangle_0|1\rangle_1 \equiv |0_0 1_1\rangle \tag{10.4}$$

$$\langle \hat{n}_2 \rangle = \langle 0_0 1_1 | \hat{a}_2^\dagger \hat{a}_2 | 0_0 1_1 \rangle \tag{10.5}$$

So how do we define the output operators?

$$\hat{a}_2 = \frac{1}{\sqrt{2}} \left(\hat{a}_0 + i\hat{a}_1 \right) \qquad \hat{a}_3 = \frac{1}{\sqrt{2}} \left(i\hat{a}_0 + \hat{a}_1 \right) \tag{10.6}$$

$$\hat{a}_2^\dagger = \frac{1}{\sqrt{2}} \left(\hat{a}_0^\dagger - i\hat{a}_1^\dagger \right) \qquad \hat{a}_3^\dagger = \frac{1}{\sqrt{2}} \left(-i\hat{a}_0^\dagger + \hat{a}_1^\dagger \right) \tag{10.7}$$

So we can input these into the expectation value equation:

$$\langle \hat{n}_2 \rangle = \langle 0_0 1_1 | \frac{1}{2} \left(\hat{a}_0^\dagger - i\hat{a}_1^\dagger \right) \left(\hat{a}_0 + i\hat{a}_1 \right) | 0_0 1_1 \rangle \tag{10.8}$$

$$= \langle 0_0 1_1 | \frac{1}{2} \left\{ \hat{a}_0^\dagger \hat{a}_0 - i\hat{a}_1^\dagger \hat{a}_0 + i\hat{a}_0^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_1 \right\} | 0_0 1_1 \rangle \tag{10.9}$$

$$= \langle 0_0 1_1 | \frac{1}{2} \left\{ 0 + 0 + 0 + 1 \right\} | 0_0 1_1 \rangle = \frac{1}{2} = \langle \hat{n}_3 \rangle \tag{10.10}$$

Naturally, this is exactly what we expected from a 50:50 beamsplitter - $\hat{n}_2$ and $\hat{n}_3$ have an even split. However, if we consider

$$\langle 0_0 1_1 | \hat{n}_2 \hat{n}_3 | 0_0 1_1 \rangle = 0 \neq \langle \hat{n}_2 \rangle \langle \hat{n}_3 \rangle \tag{10.11}$$

The photons will only ever come out at one port at a single time, so on average it is 50:50, but will alternate between the ports, so one port is always at 0 - **photon anti-bunching.** This is not what we would expect classically, where $\langle I_2 I_3 \rangle = \langle I_2 \rangle \langle I_3 \rangle$. So putting single photons through a beam splitter is something very non-classical.

We can approach this problem in another way:

$$|0\rangle_0|1\rangle_1 = \hat{a}_1^\dagger |0\rangle_0 |0\rangle_1 \tag{10.12}$$

So this time we start with two vacuum states and use the creation operator to add photons.

$$\hat{a}_2^\dagger = \frac{1}{\sqrt{2}} \left\{ \hat{a}_0^\dagger - i\hat{a}_1^\dagger \right\} \qquad \hat{a}_3^\dagger = \frac{1}{\sqrt{2}} \left\{ -i\hat{a}_0^\dagger + \hat{a}_1^\dagger \right\} \tag{10.13}$$

$$\hat{a}_1^\dagger = \frac{1}{\sqrt{2}} \left\{ i\hat{a}_2^\dagger + \hat{a}_3^\dagger \right\} \qquad \hat{a}_0^\dagger = \frac{1}{\sqrt{2}} \left\{ \hat{a}_2^\dagger + i\hat{a}_3^\dagger \right\} \tag{10.14}$$

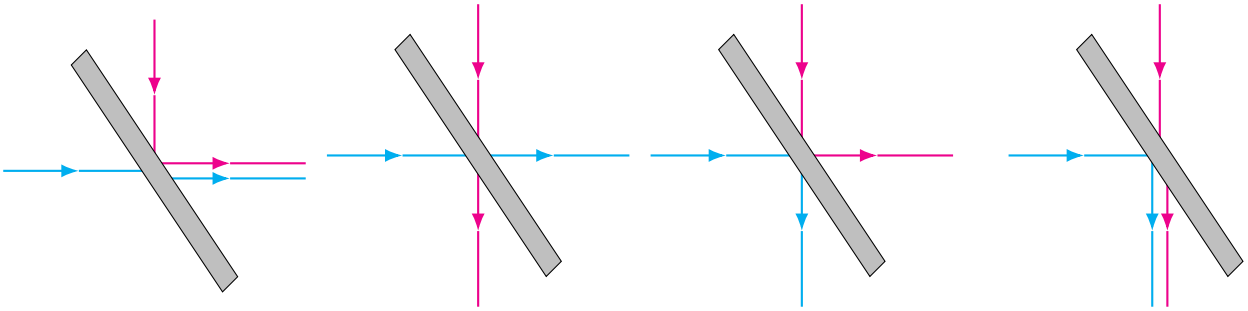
Note that vacuum in means vacuum out, i.e. $|0_0 0_1\rangle \rightarrow |0_2 0_3\rangle$. Now let's apply $\hat{a}_1^\dagger$, which means adding a photon to port 1.

$$|0_0 1_1\rangle \rightarrow \frac{1}{\sqrt{2}} \left\{ i\hat{a}_2^\dagger + \hat{a}_3^\dagger \right\} |0_2 0_3\rangle \tag{10.15}$$

$$= \frac{1}{\sqrt{2}} \left\{ i|1_2 0_3\rangle + |0_2 1_3\rangle \right\} \tag{10.16}$$

$$\langle \hat{n}_2 \rangle = \langle \hat{n}_3 \rangle = \frac{1}{2}, \ \langle \hat{n}_2 \hat{n}_3 \rangle = 0 \tag{10.17}$$

So this method reproduces the same results as before. Note that we cannot write Eq (10.16) as a product state, i.e. $|1_2 0_3\rangle \neq |1_2\rangle |0_3\rangle$. Instead, this is known as an entangled state.
Now, what happens if we put one photon into each input at once? We now have four equally likely possibilities:

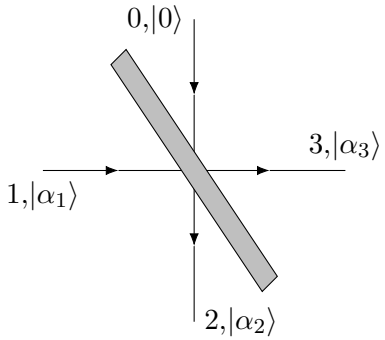


$$|1_0 1_1\rangle = \hat{a}_0^\dagger \hat{a}_1^\dagger |0_0 0_1\rangle \rightarrow \frac{1}{2}\left(\hat{a}_2^\dagger + i\hat{a}_3^\dagger\right)\left(i\hat{a}_2^\dagger + \hat{a}_3^\dagger\right)|0_2 0_3\rangle \tag{10.18}$$

$$\rightarrow \frac{1}{2}\left\{i\hat{a}_2^\dagger \hat{a}_2^\dagger - \hat{a}_3^\dagger \hat{a}_2^\dagger + \hat{a}_3^\dagger \hat{a}_2^\dagger + i\hat{a}_3^\dagger \hat{a}_3^\dagger\right\}|0_2 0_3\rangle \tag{10.19}$$

$$\rightarrow \frac{i}{2}\left\{\sqrt{2}|2_2 0_3\rangle + \sqrt{2}|0_2 2_3\rangle\right\} \tag{10.20}$$

10.3 Coherent States and Beamsplitters



$$|0\rangle_0 |\alpha_1\rangle_1 \rightarrow |\alpha_2\rangle_2 |\alpha_3\rangle_3 \tag{10.21}$$

So inputting one coherent state with a vacuum, yields coherent states in both outputs. For 50:50,

$$\alpha_2 = \frac{i\alpha_1}{\sqrt{2}} \qquad \alpha_3 = \frac{\alpha_1}{\sqrt{2}} \qquad BS = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \tag{10.22}$$

Now what if we have

$$|\alpha_0\rangle_0 |\alpha_1\rangle_1 \rightarrow |\alpha_2\rangle_2 |\alpha_3\rangle_3 \tag{10.23}$$

We want to have the same amplitude going into both 1 and 0, but with a phase difference, so $|\alpha_0| = |\alpha_1| = |\alpha|$, with $\alpha_1 = |\alpha|\epsilon^{i\theta}$ and $\alpha_0 = |\alpha|$. We will find that

$$P_2 \propto \sin^2 \frac{\theta}{2} \tag{10.24}$$

This can be compared to the coherent state results in Lecture 6.

Lecture 11 Interference and Coherence

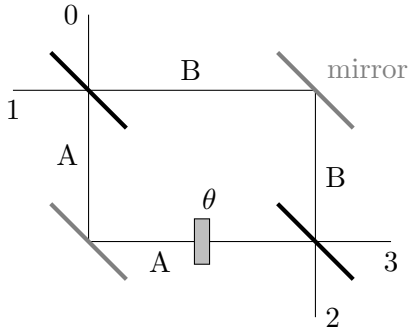
11.1 Learning Outcomes

We will discuss interference, and both the classical and quantum definitions of first and second order coherence. After studying this section, you should

- Be able to calculate the output state of an interferometer with number states or coherent states in the input modes.
- Understand and be able to apply the definition of first-order coherence, and be familiar with its link to the visibility of interference fringes.
- Be familiar with, and able to apply, the definition of second-order coherence.
- In each of the above cases, to be familiar with and able to apply the quantum definitions in terms of correlation functions involving electric field operators.
- To understand, and be able to apply the terms incoherent, partially coherent, and fully coherent in the context of first-order coherence.
- To understand and be able to apply the terms Poissonian, sub-Poissonian, bunching, and anti-bunching in the context of second-order coherence.
- To be familiar with the first- and second-order coherence properties of single-mode number states, coherent states, and thermal states.

11.2 Interferometers

The Mach-Zehnder interferometer is the simplest and cleanest interferometer:



Let's work through it. Going through the first beamsplitter, we get:

$$|0\rangle_0|1\rangle_1 \to \frac{1}{\sqrt{2}}\left(|0\rangle_A|1\rangle_B + i|1\rangle_A|0\rangle_B\right) \tag{11.1}$$

The phase from the mirrors is $e^{i\pi/2}$ on both, but we don't need to know it.
Now, we need to put in a phase shift, $e^{i\theta}$. We can only phase shift the second term, as we can't phase shift the vacuum - it has no well defined phase to shift.

$$|0\rangle_0|1\rangle_1 \to \frac{1}{\sqrt{2}}\left(|0\rangle_A|1\rangle_B + i|1\rangle_A|0\rangle_B\right) \tag{11.2}$$

$$\to \frac{1}{\sqrt{2}}\left(|0\rangle_A|1\rangle_B + ie^{i\theta}|1\rangle_A|0\rangle_B\right) \tag{11.3}$$

Then, we have the second beamsplitter, which transforms as:

$$|0\rangle_A|1\rangle_B \to \frac{1}{\sqrt{2}}\left(|1\rangle_2|0\rangle_3 + i|0\rangle_3|1\rangle_3\right) \tag{11.4}$$

$$|1\rangle_A|0\rangle_B \to \frac{1}{\sqrt{2}}\left(|0\rangle_2|1\rangle_3 + i|1\rangle_2|0\rangle_3\right) \tag{11.5}$$

So we can then put both beamsplitters together for the full transform:

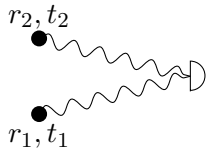
$$|0\rangle_0|\rangle_1 \to \frac{1}{2}\left(i(1+e^{i\theta})|0\rangle_2|1\rangle_3 + (1-e^{i\theta})|1\rangle_2|0\rangle_3\right) \tag{11.6}$$

$$P_2 = \frac{1}{4}(1-e^{i\theta})^2 = \frac{1}{2}(1-\cos\theta) \tag{11.7}$$

$$P_3 = \frac{1}{2}(1+\cos\theta) \tag{11.8}$$

Single photons → wavelike interference, averaging over many.

What we had before interferometers was double beam experiments:



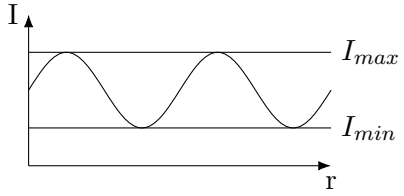
$$\underline{\mathbf{E}}(vr,t) = k_1\underline{\mathbf{E}}(\underline{r}_1,t_1) + k_2\underline{\mathbf{E}}(\underline{r}_2,t_2) \tag{11.9}$$

From this, we found the first-order normalised mutual coherence function:

$$\gamma^{(1)}(\underline{r}_1,t_1,\underline{r}_2,t_2) = \frac{\langle\underline{\mathbf{E}}^*(\underline{r}_1,t_1)\underline{\mathbf{E}}(\underline{r}_2,t_2)\rangle_t}{\{\langle|\underline{\mathbf{E}}(\underline{r}_1,t_1)|^2\rangle_t\langle|\underline{\mathbf{E}}(\underline{r}_2,t_2)|^2\rangle_t\}^{1/2}} \tag{11.10}$$

$$I(\underline{r}) = I_1 + I_2 + \sqrt{I_1 I_2}|\gamma^{(1)}(\underline{r}_1,t_1,\underline{r}_2,t_2)|\cos(\phi_1-\phi_2) \tag{11.11}$$

$$\gamma^{(1)}\begin{cases} = 1 & \text{fully coherent} \\ < 1 & \text{partially coherent} \\ = 0 & \text{fully incoherent} \end{cases} \tag{11.12}$$



Rayleigh fringe visibility is defined as:

$$\nu = \frac{I_{max}-I_{min}}{I_{max}+I_{min}} \tag{11.13}$$

$$\nu = \begin{cases} 1 & \text{fully coherent} \\ 0 & \text{fully incoherent} \end{cases} \tag{11.14}$$

Reminder that our first order coherence depends on $\langle\underline{\mathbf{E}}^*(\underline{r}_1,t_1)\underline{\mathbf{E}}(\underline{r}_2,t_2)\rangle_t$, but

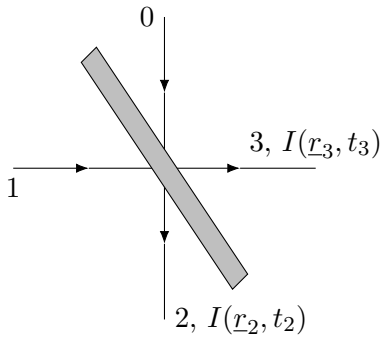
$$P_i \propto \langle i|\hat{\underline{\mathbf{E}}}_-(\underline{r},t)\hat{\underline{\mathbf{E}}}_+(\underline{r},t)|i\rangle \propto \hat{a}^\dagger\hat{a} = \hat{n} \tag{11.15}$$

This is the quantum version of the first order coherence. What about our **quantum** first-order normalised mutual coherence function?

$$g^{(1)}(\underline{r}_1,t_1,\underline{r}_2,t_2) = \frac{\langle\hat{\underline{\mathbf{E}}}_-(\underline{r}_1,t_1)\hat{\underline{\mathbf{E}}}_+(\underline{r}_2,t_2)\rangle}{\left\{\langle\hat{\underline{\mathbf{E}}}_-\hat{\underline{\mathbf{E}}}_+(\underline{r}_1,t_1)\rangle\langle\hat{\underline{\mathbf{E}}}_-\hat{\underline{\mathbf{E}}}_+(\underline{r}_2,t_2)\rangle\right\}^{1/2}} \tag{11.16}$$

We find that both number and coherent states can be fully coherent, i.e. $g^{(1)} = 1$. The first order coherent function tells you how coherent you are and how visible your fringes will be.

11.3 Second Order Coherence



Let's consider a classical second-order normalised mutual coherence function:

$$\gamma^{(2)}(\underline{r}_3, t_3, \underline{r}_2, t_2) = \frac{\langle I(\underline{r}_3, t_3) I(\underline{r}_2, t_2)\rangle_t}{\langle I(\underline{r}_3, t_3)\rangle \langle I(\underline{r}_2, t_2)\rangle_t} \tag{11.17}$$

This is second order as $I \propto \underline{\mathbf{E}}^* \underline{\mathbf{E}}$, so each term has to $\underline{\mathbf{E}}$-fields, instead of one as previously. Now let's look at the quantum version again:

$$g^{(2)}(\underline{r}_3, t_3, \underline{r}_2, t_2) = \frac{\langle \hat{\underline{\mathbf{E}}}_-(\underline{r}, t_3) \hat{\underline{\mathbf{E}}}_-(\underline{r}_2, t_2) \hat{\underline{\mathbf{E}}}_+(\underline{r}_2, t_2) \hat{\underline{\mathbf{E}}}_+(\underline{r}_3, t_3)\rangle}{\langle \hat{\underline{\mathbf{E}}}_- \hat{\underline{\mathbf{E}}}_+(\underline{r}_3, t_3)\rangle \langle (\hat{\underline{\mathbf{E}}}_- \hat{\underline{\mathbf{E}}}_+(\underline{r}_2, t_2)\rangle} \tag{11.18}$$

Lecture 12 Beam Splitters and Interferometry

For the Mach-Zehnder interferometer, what if $\theta = 0$? If for a single photon going through one beamsplitter, we have the Hadamard operator,

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \tag{12.1}$$

then, for two beamsplitters, as seen in the interferometer, without a phase shift, we would have the Hadamard operator squared,

$$H^2 = \frac{1}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = I \tag{12.2}$$

which makes the identity matrix and the one photon will simply come out of a single port again.

12.1 Single mode field (no beamsplitter)

The only difference here is that everything that commutes can be cancelled out, and we are left with the dagger operators:

$$g^{(2)}_{\text{single}} = \frac{\langle \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{a}^\dagger \hat{a} \rangle} \tag{12.3}$$

$$= \frac{\langle \hat{n}(\hat{n}-1) \rangle}{\langle \hat{n} \rangle^2} \tag{12.4}$$

Where we use the ladder operators' commutation relation to redefine.

1. Coherent state:

$$\langle \alpha | \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} | \alpha \rangle = |\alpha|^4 \tag{12.5}$$

$$\langle \alpha | \hat{n} | \alpha \rangle = |\alpha|^2 \tag{12.6}$$

$$g^{(2)} = 1 \tag{12.7}$$

2. Number states:

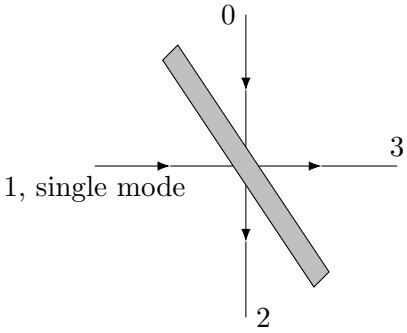
$$\langle 1 | \hat{n}(\hat{n}-1) | 1 \rangle = 0 \tag{12.8}$$

$$\frac{\langle n | \hat{n}(\hat{n}-1) | n \rangle}{\langle \hat{n} \rangle^2} = \frac{n^2 - n}{n^2} = 1 - \frac{1}{n} \tag{12.9}$$

For $g^{(2)}$, number states with more photons look more classical. For a $g^{(2)} < 1$, we call it sub-Poissonian, because for a coherent state with $g^{(2)} = 1$, we have a Poisson distribution.

12.2 Two modes

The Hanburg, Brown, and Twiss experiment (didn't quite manage this), actually Jeff Kimble and Herbert Waller.



$$|0_0 1_1\rangle \to \frac{1}{\sqrt{2}}\{i|1_2 0_3\rangle + |0_2 1_3\rangle\} \tag{12.10}$$

So single time, two modes.

$$g^{(2)}(t_2 = t_3) = \frac{\langle \hat{a}_3^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_3\rangle}{\langle \hat{a}_3^\dagger \hat{a}_3\rangle\langle \hat{a}_2^\dagger \hat{a}_2^\dagger\rangle} = \frac{\langle \hat{n}_3 \hat{n}_2\rangle}{\langle \hat{n}_3\rangle\langle \hat{n}_2\rangle} = 0 \tag{12.11}$$

But if we take it at different times, then the input is now $|0_0 1_1\rangle_{t_2}|0_0 1_1\rangle_{t_3}$, so now $\langle \hat{n}_2 \hat{n}_3\rangle = \frac{1}{4}$, and

$$g^{(2)}(t_2 - t_3) = 1 \tag{12.12}$$

Now if we put light into both inputs:

$$|1_0 1_1\rangle \to \frac{i}{\sqrt{2}}\{|0_2 2_3\rangle + |2_2 0_3\rangle\} \tag{12.13}$$

$$g^{(2)} = \frac{\langle \hat{n}_2 \hat{n}_3\rangle}{\langle \hat{n}_2\rangle\langle \hat{n}_3\rangle} = 0 \tag{12.14}$$

So this is definitely non-classical.

12.3 Coherent states in beamsplitter

$$|0_1 \alpha_1\rangle \to \frac{i}{\sqrt{2}}|\alpha_2, \alpha_3\rangle \tag{12.15}$$

$$\langle \hat{n}_2\rangle\hat{n}_3\rangle = \langle \hat{n}_2 \hat{n}_3\rangle \implies g^{(2)} = 1 \tag{12.16}$$

12.4 Summary

For first order, we have $\gamma^{(1)}$ for classical, and $g^{(1)}$ for quantum. We are looking at amplitudes:

$$\frac{\langle \hat{\underline{\mathbf{E}}}_+^{(1)} \hat{\underline{\mathbf{E}}}_-^{(2)}\rangle}{(\langle \hat{\underline{\mathbf{E}}}_+^{(1)} \hat{\underline{\mathbf{E}}}_-^{(1)}\rangle\langle \hat{\underline{\mathbf{E}}}_+^{(2)} \hat{\underline{\mathbf{E}}}_-^{(2)}\rangle)^{1/2}} \tag{12.17}$$

Amplitudes of interference fringes - fringe visibility.
How coherent the EM fields are:

$$0 \le \gamma^{(1)}, \text{ (incoherent)} \tag{12.18}$$

$$g^{(1)} \le 1, \text{ (coherent)} \tag{12.19}$$

For second order, we have $\gamma^{(2)}$ for classical, and $g^{(2)}$ for quantum.

$$1 \le \gamma^{(2)} < \infty \tag{12.20}$$

$$0 \le g^{(2)} < 1 \tag{12.21}$$

We looked at small number states, where

$$g^{(2)} = 1 - \frac{1}{n} \tag{12.22}$$

We also looked at coherent states, where we have:

$$g^{(2)} = 1, \text{ (Poissonian)} \tag{12.23}$$

$$g^{(2)} < 1, \text{ (sub-Poissonian)} \tag{12.24}$$

Lecture 13 Polarisation in Two-Mode States

13.1 Learning Outcomes

We will discuss polarisation in terms of two-mode states and projective measurements. After studying this section, you will:

- Be able to calculate the probability of detecting a photon after a linear polariser, for any two-mode input state (number, coherent) and any combination of polariser angle and initial polarisation.
- Understand the action of a linear polariser in terms of projective measurements; be familiar with the use of the projection operator, and be able to discuss the state before and after the measurement.

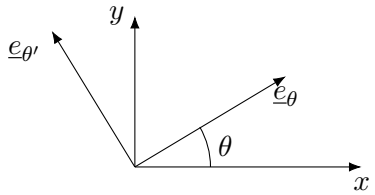
13.2 Polarisation

$$\hat{\underline{\mathbf{E}}}(\underline{r},t) = i\sum_{\underline{k},s}\omega_k \underline{e}_{\underline{k},s}\left[A_{\underline{k},s}\exp\left\{i(\underline{k}\cdot\underline{r}-\omega_k t)\right\}+c.c.\right] \tag{13.1}$$

We are now concerned with $\underline{e}_{\underline{k},s}$, the polarisation vector of the field. There are two polarisations to make up a complete basis, contained in the index s. Above is what we will see classically.

Consider two modes with same $\underline{k},\omega_k$, where $\underline{k}\propto\hat{z}$, so we have $\underline{e}_x,\underline{e}_y$.

$$\hat{\underline{\mathbf{E}}}(\underline{r},t) = i\omega_k\left[(A_x\underline{e}_x+A_y\underline{e}_y)\exp\left\{i(\underline{k}\cdot\underline{r}-\omega_k t)\right\}+c.c.\right] \tag{13.2}$$



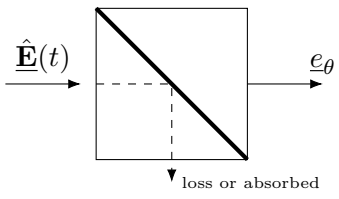
$$\underline{e}_\theta = \cos\theta\underline{e}_x+\sin\theta\underline{e}_y \tag{13.3}$$

$$\underline{e}_{\theta'} = -\sin\theta\underline{e}_x+\cos\theta\underline{e}_y \tag{13.4}$$

Hence, for the amplitudes, $A_x,A_y\to A_\theta,A_{\theta'}$:

$$\begin{pmatrix}A_\theta\\A_{\theta'}\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\-\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}A_x\\A_y\end{pmatrix} \tag{13.5}$$

Now we must have our polarising beamsplitters:



$$A_y = 0 \tag{13.6}$$

$$A_\theta = A_x\cos\theta \tag{13.7}$$

$$I_\theta \propto |A_x|^2\cos^2\theta \tag{13.8}$$

What about the quantum setting? We have our operator that we will split into the different Cartesian polarisations, $\hat{a}_{\underline{k},s}\to\hat{a}_x,\hat{a}_y$, and we have a rotation matrix, $\hat{a}_\theta,\hat{a}_{\theta'}$. Note that

$$[\hat{a}_\theta,\hat{a}_{\theta'}] = 0 \tag{13.9}$$

$$[\hat{a}_\theta,\hat{a}_\theta^\dagger] = 1 \tag{13.10}$$

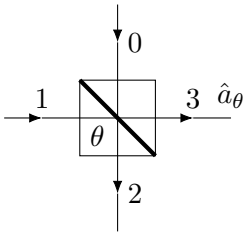
So these are just the creation and annihilation operators in a specific orientation.

Example: Number state with one photon polarised along θ
We have a two mode field (with $|0\rangle$ the single mode vacuum):

$$|1_\theta, 0_{\theta'}\rangle = \hat{a}^\dagger_\theta|0,0\rangle \tag{13.11}$$
$$= (\hat{a}^\dagger_x \cos\theta + \hat{a}^\dagger_y \sin\theta)|0,0\rangle \tag{13.12}$$
$$= \cos\theta|1_x, 0_y\rangle + \sin\theta|0_x, 1_y\rangle \tag{13.13}$$

We can create from the vacuum state in any way we want to, so we don't have to worry about what we are using for this, and whether we use a number or coherent state.
We can show that

$$\hat{n}|1_\theta, 0_{\theta'}\rangle = \sum_{\underline{k},s} \hat{n}_{\underline{k},s}|1_\theta, 0_{\theta'}\rangle \tag{13.14}$$
$$= |1_\theta, 0_{\theta'}\rangle \tag{13.15}$$
$$\langle\hat{n}\rangle = 1 \tag{13.16}$$



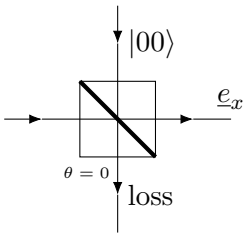
$$\text{Prob} \propto \langle\hat{n}\rangle = \langle\psi|\hat{a}^\dagger_\theta\hat{a}_\theta|\psi\rangle \tag{13.17}$$

So what if we have a state $|\psi\rangle$ of one photon, but we don't know polarisation. Depending on this polarisation, we will be likelier to get different measurements. In a single measurement, we record 0 or 1.

$$P_0 = |\langle 0,0|\psi\rangle|^2 \qquad\qquad P_1 = |\langle 1_{\theta P}|\psi\rangle|^2 \tag{13.18}$$

All we can do with one measurement is if we detect photon, i.e. 1. So then we know something: that is definitely not polarised in θ' direction, because then it definitely wouldn't have come through. To determine θ, we need to make many measurements, or many copies, of $|\psi\rangle$ - this need gives rise to the concept of ensembles, where the ratio of 0 to 1 lets you work out what θ is.

We can now try another experiment:



$$|\psi\rangle = |1_{\pi/4}, 0_{3\pi/4}\rangle \tag{13.19}$$

So we have one photon, polarised at 45 degrees. The probability of detecting one photon is then

$$P_1 = |\langle 1_x, 0_x|1_{\pi/4}, 0_{3\pi/4}\rangle|^2 = \frac{1}{2}, \ (\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}) \tag{13.20}$$

Now let's compare with a random mixture of two states, so half the time we have $|1_x, 0_y\rangle$, and half the time we have $|0_x, 1_y\rangle$. Again however, we get $P_1 = \frac{1}{2}$. How do we tell the difference between the two cases?

If we set the polariser to $\theta_p = \frac{\pi}{4}$, $|1_{\pi/4}, 0_{3\pi/4}\rangle \implies P_1 = 1$, but the fifty-fifty mixture still yields $P_1 = 1$.

What we have been writing here are the projection operators for these states:

$$|\psi\rangle = \sum_j c_j |j\rangle \tag{13.21}$$

where $\{j\}$ is a complete orthonormal basis.

$$P_j = |\langle j|\psi\rangle|^2 = |c_j|^2 \tag{13.22}$$
$$= \langle\psi|j\rangle\langle j|\psi\rangle \tag{13.23}$$
$$= \langle\psi|\hat{P}_j|\psi\rangle,\ \hat{P}_j = |j\rangle\langle j| \tag{13.24}$$
$$\hat{P}_j|\psi\rangle = |j\rangle\langle j|\psi\rangle = c_j|j\rangle \tag{13.25}$$

So this projection operator, $\hat{P}_j$, is projecting the state $|\psi\rangle$ onto $|j\rangle$ - however this is not normalised as it is. So we need as set of these so the normalisation sums to one.

$$P_x = |\langle 1_x 0_y|\psi\rangle|^2 \tag{13.26}$$
$$|\psi\rangle = \frac{1}{\sqrt{2}}\{|1_x 0_y\rangle + |0_x 1_y\rangle\} = |1_{\pi/4}, 0_{3\pi/4}\rangle \tag{13.27}$$
$$P_x = \frac{1}{2}, \hat{P}_x = |1_x 0_y\rangle\langle 1_x 0_y| \tag{13.28}$$
$$\hat{P}_x|\psi\rangle = \frac{1}{\sqrt{2}}|1_x 0_y\rangle \tag{13.29}$$

If we apply $\hat{P}_x$ but "fail", i.e. no photon, then we must have $|0_x 1_y\rangle$, so we have then define

$$\hat{P}_y = |0_x 1_y\rangle\langle 0_x 1_y| \tag{13.30}$$
$$\hat{P}_y|\psi\rangle = \frac{1}{\sqrt{2}}|0_x 1_y\rangle \tag{13.31}$$

Again, this is not normalised by itself.
So for measurements there will be at least two outcomes (if only one, then there is clearly no need to make the measurement!), and we need a complete set of projectors which sum to identity, $\left\{\hat{P}_x, \hat{P}_y\right\}$.

$$\hat{P}_x + \hat{P}_y = \mathbb{I}^2 = |1_x 0_y\rangle\langle 1_x 0_y| + |0_x 1_y\rangle\langle 0_x 1_y| \tag{13.32}$$
$$\sum_j \hat{P}_j = \mathbb{I} \tag{13.33}$$

So we have out complete set of projective measurements. *Other types of measurements are possible but these are beyond the current scope.*

Lecture 14 Density Matrices and Quantum States

14.1 Learning Outcomes

Density matrices, pure states, mixed states, and entangled states. To be familiar with the concepts of pure states, mixed states, and the density matrix. To apply them to entangled states and product states, and use the partial trace. After studying this section you will be able to:

- Explain and state the definitions of pure states, mixed states, and the density matrix.
- Perform calculations using the density operator and the trace (e.g. trace of the density operator, expectation values) for pure states and mixed states.
- Explain and state the definitions of entangled states and product states for bi-partite systems in a pure state.
- To be familiar with the definition and properties of the partial trace and its application to the joint density matrix.
- To be familiar with and apply the partial trace over a sub-system of a bipartite density matrix, and understand how the outcome differs for product states and entangled states.

14.2 The Density Matrix

Thus far, we have been working with pure states, $|\psi\rangle$, which represent everything about a particular quantum state.

$$|\psi\rangle = \sum_j c_j |j\rangle \tag{14.1}$$

So we can form a density matrix for a pure state, $\hat{\rho}$ - it is the projector of the state onto itself.

$$\hat{\rho} = |\psi\rangle\langle\psi| = \sum_j c_j |j\rangle \sum_m c_m^* \langle m| \tag{14.2}$$

$$= \sum_{j,m} c_j c_m^* |j\rangle\langle m| \tag{14.3}$$

Example: Horizontal and Vertical polarisation

$$|L\rangle = \frac{1}{\sqrt{2}} \{|1_H, 0_V\rangle + i|0_H, 1_V\rangle\} \tag{14.4}$$

$$\hat{\rho}_L = |L\rangle\langle L| = \frac{1}{2} \{|H\rangle\langle H| + i|V\rangle\langle H| - |H\rangle\langle V| + |V\rangle\langle V|\} \tag{14.5}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \tag{14.6}$$

Note: this is Hermitian, and the diagonal terms give likelihoods of being in H or V, while the off-diagonals form coherences, i.e. the relative phases between the different basis states.
For a pure state, the Schrodinger equation can be written as

$$\frac{d}{dt}\hat{\rho} = \frac{i}{\hbar}[\hat{\rho}, \hat{H}]. \tag{14.7}$$

This is known as the van-Neumann equation.

So let's go back to the example of source giving horizontal or vertical polarisations, but we don't know which one we were getting. We can write this in terms of a density matrix,

$$\hat{\rho}_{H/V} = \frac{1}{2}|H\rangle\langle H| + \frac{1}{2}|V\rangle\langle V| = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{14.8}$$

The difference here is there is no off-diagonal terms, so there is no coherences in this ensemble, i.e. a mixed state. Above, we just had the projector of one state $|L\rangle$, but now we have the sum of two projectors. So the coherences will depend on whether we have a mixed or pure state.

More generally, we can have

$$\hat{\rho} = \sum_j p_j |\psi_j\rangle\langle\psi_j|. \tag{14.9}$$

This will work for any (not necessarily orthogonal) set of states $|\psi_j\rangle$. In general, this decomposition is not unique, unless you know it is from a particular set.

Example: Pure or Mixed?

$$\hat{\rho} = \frac{1}{2}|L\rangle\langle L| + \frac{1}{2}|H\rangle\langle H| \tag{14.10}$$

$$= \frac{1}{2}\cdot\frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{14.11}$$

$$= \frac{1}{4}\begin{pmatrix} 3 & i \\ -i & 1 \end{pmatrix} \tag{14.12}$$

There is an easy way to test if this state is pure or mixed. We square $\hat{\rho}$, or $\hat{\rho}^\dagger\hat{\rho}$, but $\hat{\rho}$ is Hermitian, so it is just $\hat{\rho}^2$. For a pure state, $\hat{\rho}_L$,

$$\hat{\rho}_L = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \qquad \hat{\rho}_L^2 = \frac{1}{4}\begin{pmatrix} 2 & 2i & -2i & 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = \hat{\rho}_L \tag{14.13}$$

So we want $\hat{\rho} = \hat{\rho}^2$ for a pure state. Let's check for the state above:

$$\rho = \frac{1}{4}\begin{pmatrix} 3 & i \\ -i & 1 \end{pmatrix} \qquad \hat{\rho}^2 = \frac{1}{16}\begin{pmatrix} 4 & 4i \\ -4i & 2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & i & -i & 2 \end{pmatrix} \neq \hat{\rho} \tag{14.14}$$

So the above is not a pure state, but a mixed state.

It is very easy to see why this is the case for a pure state generally, as

$$\hat{\rho} = |\psi\rangle\langle\psi| \qquad \hat{\rho}^2 = |\psi\rangle\underbrace{\langle\psi|\psi\rangle}_{1}\langle\psi| = \hat{\rho} \tag{14.15}$$

If the diagonal elements of $\hat{\rho}$ describe the likelihoods of being in whichever state, then we have to look at the trace - the sum of the diagonal entries. When performing in terms of an operator,

$$\mathrm{Tr}[\hat{A}] = \sum_j \langle j|\hat{A}|j\rangle \tag{14.16}$$

for some orthonormal basis $\{|j\rangle\}$. This is invariant of basis, so any basis will give the same trace.

We can find the traces of a multiplication of operators too:

$$\mathrm{Tr}[\hat{\rho}\cdot\hat{P}_n] = \langle n|\hat{\rho}|n\rangle, \;\; \hat{P}_n = |n\rangle\langle n| \tag{14.17}$$

So the second operator will "wrap" itself around the first essentially.

What about the expectation value of $\hat{A}$ on $\hat{\rho}$?

$$\mathrm{Tr}[\hat{\rho}\cdot\hat{A}] = \langle\hat{A}\rangle_{\hat{\rho}} \tag{14.18}$$

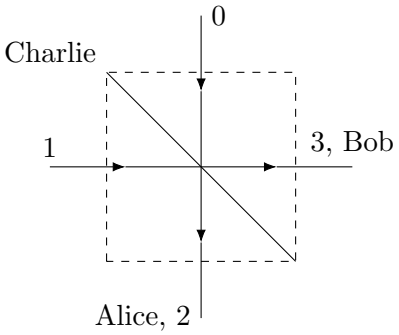
For a pure state, we can see this is true easily from $\langle\psi|\hat{A}|\psi\rangle$ and $\hat{\rho} = |\psi\rangle\langle\psi|$, and by linearity, it can also be proven for mixed states.

14.3 Useful Properties of Trace

- $\text{Tr}[\hat{A}\hat{B}] = \text{Tr}[\hat{B}\hat{A}]$, even if $[\hat{A}, \hat{B}] \neq 0$. This comes from cyclic permutation: $\text{Tr}[\hat{A}\hat{B}\hat{C}] = \text{Tr}[\hat{B}\hat{C}\hat{A}] = \text{Tr}[\hat{C}\hat{A}\hat{B}]$.
- $\text{Tr}[\hat{A} + \hat{B}] = \text{Tr}[\hat{A}] + \text{Tr}[\hat{B}]$.
- The condition for $\hat{\rho}$ to be a density operator comes from conservation of probability and is (for pure and mixed states): $\text{Tr}[\hat{\rho}] = 1$.
  Whereas $\text{Tr}[\hat{\rho}^2] = \text{Tr}[\hat{\rho}]$ **only for pure states.**

14.4 Entangled States

Entangled states are quantum correlation, where these are more correlated than it is possibly to get in classical physics.



Charlie has 0 and 1 and makes the states. So we will input $|+\rangle$ states, where

$$|+\rangle = \frac{1}{\sqrt{2}}|H + V\rangle, \tag{14.19}$$

with a polariser of $\frac{\pi}{4}$.

$$|+_0\rangle|+_1\rangle = \frac{1}{2}\{(H_0 + V_0)(H_1 + V_1)\} \tag{14.20}$$

$$= \frac{1}{2}\{|H_0H_1\rangle + |H_0V_1\rangle + |V_0H_1\rangle + |V_0V_1\rangle\} \tag{14.21}$$

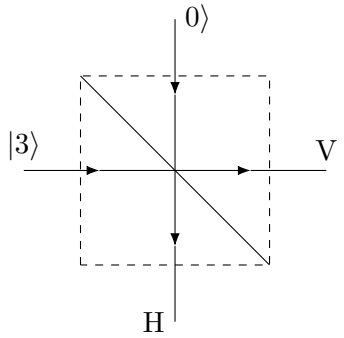
$$= \frac{1}{2}\{-|H_2H_3\rangle + i|H_3V_3\rangle + i|V_2H_2\rangle + |V_2V_3\rangle\} \tag{14.22}$$

$$= \frac{1}{2}\{|V_2V_3\rangle - |H_2H_3\rangle + i(|H_3V_3\rangle + |H_2V_2\rangle)\} \tag{14.23}$$

We will discard the imaginary term in what is called post selection and renormalise, by considering when both outputs get one photon, but we don't know which. The imaginary term describes when either state gets both photons so we ignore it.

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|V_2V_3 - H_2H_3\rangle) \tag{14.24}$$

What happens if we measure if it has horizontal or vertical polarised photons? We send into another beamsplitter and see where it comes out.



But how do we write the state $|3\rangle$ that we are putting into the polarising beamsplitter?

$$|\psi_{AB}\rangle\langle\psi_{AB}| = \hat{\rho}_{AB} \tag{14.25}$$

$$\mathrm{Tr}_B[\hat{\rho}_{AB} \equiv \mathrm{Tr}_B[\hat{\rho}_{AB}\mathbb{I}_B] \tag{14.26}$$

$$\hat{\rho}_{AB} = \frac{1}{2}\left(|V_2V_3\rangle\langle V_2V_3| - |V_2V_3\rangle\langle H_2H_3| - |H_2H_3\rangle\langle V_2V_3| + |H_2H_3\rangle\langle H_2H_3|\right) \tag{14.27}$$

$$\mathbb{I}_B = |H_3\rangle\langle H_3| + |V_3\rangle\langle V_3| \tag{14.28}$$

Note the projector is not normalised. We are going to measure then ignore the outcome.

$$\hat{\rho}_A = \mathrm{Tr}_B[\hat{\rho}_{AB}] = \frac{1}{2}\left(|H_2\rangle\langle H_2| + |V_2\rangle\langle V_2|\right) \tag{14.29}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{14.30}$$

So for either Alice or Bob it looks like what they're getting is a stream of photons which are randomly either horizontally or vertically polarised but there are no coherences here. This is one of the signatures of entanglement - *you start with a pure state, you trace out one of the parties, and the other one will see something mixed.*

Now let's see what happens when they do the measurements. So Charlie is sending a stream of photons like this, and Alice and Bob can see the following:

Alice	0	0	H	2	V	H	2	2	V	V
Bob	2	2	H	0	V	H	0	0	V	V

So we get a whole stream of different outcomes, but the key thing is is that whenever they get a single photon each, they will get the same outcomes for these photons and know what each gets. What makes this Quantum is that you can do this in any pair of bases and you will always get this level of correlation, regardless of the polarising angle of the beamsplitter (as long as it is a shared angle) and this will not work on classical states.

This example is the first half of how you do quantum key distribution for quantum cryptography - the other half is how to stop somebody intercepting the photons and cheating on you.