Durham University

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Foundations of Physics 1

Waves and Optics

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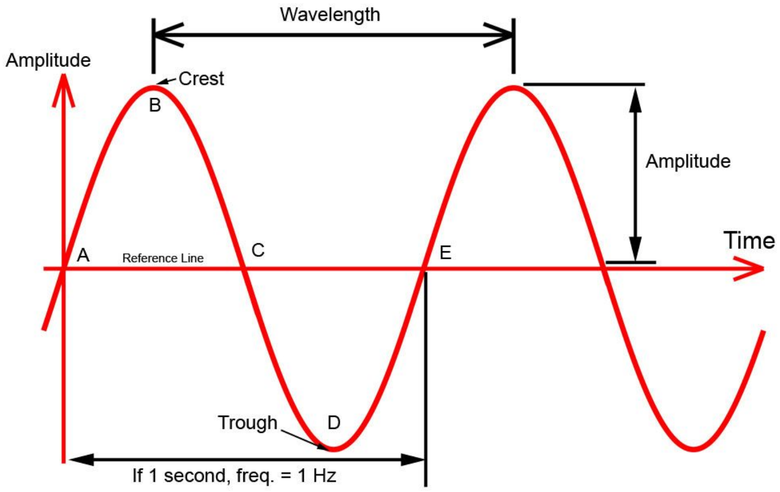
Lecture 1 Waves and Optics

1.1 Mechanical Waves

Mechanical Waves propagate through/on a system. Typically, the medium oscillates in a particular fashion:

- Transverse waves: Oscillations are perpendicular to the direction of motion
- Longitudinal waves: Oscillations are parallel to the direction of motion

1.2 Harmonic Waves



Point A shows the driver that generates the wave: each wave undergoes Simple Harmonic Motion. Wavelength – λ; Frequency – f; Angular frequency – $\omega = 2\pi f$; Amplitude – A; Period – $T = 1\frac{\lambda}{s} = \frac{1}{f} = \frac{2\pi}{\omega}$; Velocity – $v = \frac{\lambda}{T} = \lambda f$.

Basic mathematical form of a wave is $y = f(x \mp vt)$ – minus for moving right, plus for moving left. Can model any wave as $y = f(x,\, t)$.

For a SHM driver:

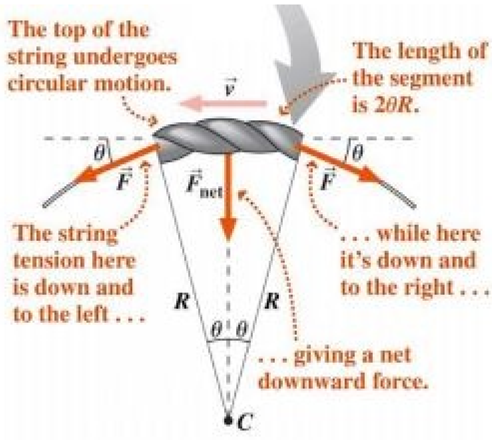
$$\delta y = A\cos(\omega t + \phi) \tag{1.1}$$

This tells us for the wire that $y(0,\, t) = A\cos(\omega t) \because \phi = 0$. The wave travels to the right, so y is a function of $(x - vt)$:

$$y = f(x - vt) \tag{1.2}$$
$$y(0,\, t) = f(-vt) = A\cos\omega t \tag{1.3}$$
$$y(x,\, t) = A\cos\left(\frac{\omega}{v}(x - vt)\right) = A\cos\left(\frac{\omega}{v}x - \omega t\right) \tag{1.4}$$
$$y(x,\, t) = A\cos\left(2\pi\left(\frac{x}{\lambda} - ft\right)\right) \tag{1.5}$$

This shows periodicity in x & t. Define the wave number, $k = \frac{2\pi}{\lambda}$ – this is the analogue of ω for x.

$$y(x,\, t) = A\cos(kx - \omega t) \text{ [For a harmonic wave moving to the right]} \tag{1.6}$$



1.3 The Wave Equation

Guess work from the previous equation gives:

$$\dot{y}(x,\,t) = \pm A\omega \sin{(kx \mp \omega t)} \qquad y'(x,\,t) = \pm Ak \sin{(kx \mp \omega t)} \tag{1.7}$$
$$\ddot{y}(x,\,t) = -A\omega^2 \cos{(kx \mp \omega t)} = -A\omega^2 y \qquad y''(x,\,t) = -Ak^2 \sin{(kx \mp \omega t)} = -Ak^2 y \tag{1.8}$$

Since $\frac{\delta y}{\delta(x,t)}$ changes sign for left- & right-moving waves, cannot use the wave equation

$$\text{But } \ddot{y} - \frac{\omega^2}{k^2}y'' = 0 \quad [\frac{\omega^2}{k^2} = v^2] \tag{1.9}$$

From this, the maths for a wire under a tension, F, can be derived: Resolve forces on the left- and right-hand sides:

$$RHS \qquad LHS \tag{1.10}$$
$$x_0 + \delta x \qquad x_0 \tag{1.11}$$
$$F_x = F\cos\theta_2 \qquad F_x = -F\cos\theta_1 \tag{1.12}$$
$$F_y = F\sin\theta_2 \qquad F_y = -F\sin\theta_1 \tag{1.13}$$

Use the small angle approximation: $\cos\theta \approx 1$, $\sin\theta \approx \theta$:

$$F_{Tot\,x} = F(\cos\theta_2 - \cos\theta_1) \approx 0 \tag{1.14}$$
$$F_{Tot\,y} = F(\sin\theta_2 - \sin\theta_1) \approx F(\theta_2 - \theta_1) \tag{1.15}$$
$$\approx F(\tan\theta_2 - \tan\theta_1) \tag{1.16}$$

But $\tan\theta$ is the gradient of the wire at each end:

$$F_{Tot\,y} = F\Big(\frac{\delta y}{\delta x}\Big|_{x_0+\delta x} - \frac{\delta y}{\delta x}\Big|_{x_0}\Big) \tag{1.17}$$

But recall when differentiating:

$$\frac{df}{dx} = \lim_{h\to 0}\frac{f(x+h) - f(h)}{h} \tag{1.18}$$

Here, $h = \delta x$ and $f = \frac{\delta y}{\delta x}$.

$$\frac{\delta y}{\delta x}\Big|_{x_0+\delta x} - \frac{\delta y}{\delta x}\Big|_{x_0} \approx \delta x\frac{\delta^2 y}{\delta x^2} \tag{1.19}$$
$$F_{Tot\,y} = F\delta x\frac{\delta^2 y}{\delta x^2} \tag{1.20}$$

Meanwhile, $F_{Tot\,y} = ma \to \mu\delta x\frac{\delta^2 y}{\delta t^2}$, where μ is the mass per unit length

$$F\frac{\delta^2 y}{\delta x^2}\delta x = \mu\delta x\frac{\delta^2 y}{\delta t^2} \tag{1.21}$$

The wave equation, therefore, is

$$\frac{\delta^2 y}{\delta t^2} = \frac{F}{\mu}\frac{\delta^2 y}{\delta x^2} \tag{1.22}$$

where:

$$v^2 = \frac{F}{\mu} \tag{1.23}$$

1.4 Energy In Wave Motion

As a wave moves, it transmits energy. At constant amplitude, energy is transmitted at a constant rate. Calculating for a harmonic wave: $y(x,\,t) = A\cos{(kx - \omega t)}$ & $E_{tot} = KE + PE$. Look at little piece of wire, $\delta x \to$ vibrates up, then falls back down:

$$\delta E_{KE} = \frac{1}{2}\delta m v^2 = \frac{1}{2}\mu\delta x\dot{y}^2 \tag{1.24}$$

Evaluate total energy when $PE = 0$, i.e. the wire is at the equilibrium position $\implies$ $\dot{y}$ is maximised so when $y = 0$, $\cos{(kx - \omega t)} = 0$ $\therefore$ $kx - \omega t = \frac{\pi}{2} + n\pi \;\forall\; n \in \mathbb{R}$:

$$\dot{y}(x,\,t) = A\omega\sin{(kx - \omega t)} \tag{1.25}$$

$$\dot{y} = \pm A\omega \tag{1.26}$$

$$\delta E_{KE} = \frac{1}{2}\mu\delta x A^2\omega^2 \tag{1.27}$$

Wave moves to the right at a constant velocity, so $\Delta x = v\,\Delta t$ gives distance through which energy propagates in time Δt. So energy is given by: $\delta E = \frac{1}{2}\mu v A^2\omega^2\,\delta t$. The rate at which energy is transmitted, or the transmission power P, is:

$$P = \frac{dE}{dt} = \frac{1}{2}\mu A^2\omega^2 v \tag{1.28}$$

$$P = \frac{1}{2}\sqrt{\mu F}A^2\omega^2 \tag{1.29}$$

1.5 Wave Intensity

For waves propagating through the unconfined media, intensity is a more useful concept than power. Intensity is defined as the rate of energy transmission per unit area, $I = \frac{P}{A}$. For a spherical wavefront, $I = \frac{P}{4\pi r^2}$. The total energy output rate is constant but spread over a larger area. This underlies the "standard candle" principal for distance in cosmology. Inverse square law for intensity:

$$4\pi r_1^2 I_1 = 4\pi r_2^2 I_2 \tag{1.30}$$

1.6 Waves At Boundaries

Waves reflect at boundaries. How they reflect depends on the nature of the boundary. Fixed boundaries return opposite amplitude wave – $y(x,\,t) = A\cos{(kx - \omega t)} + B\cos{(kx + \omega t)}$. Fixed at a boundary means $y(0,\,t) = 0 = A\cos{(\omega t)} + B\cos{(-\omega t)} \implies A = -B$. Therefore left-moving is inverted relative to right-moving wave.

Free boundaries returns same amplitude – consider a string attached to a pole by an unfixed ring. The ring moves up and down with the wave. Wire is always horizontal at transition: $\frac{\delta y(0,\,t)}{\delta x} = y'(0,\,t) = 0$

$$y'(0,\,t) = [-Ak\sin{(kx - \omega t)} - Bk\sin{(kx + \omega t)}]_{x=0} \tag{1.31}$$

$$y'(0,\,t) = Ak\sin{(\omega t)} - Bk\sin{(\omega t)} = 0 \tag{1.32}$$

$$A - B = 0 \implies A = B \tag{1.33}$$

1.7 Superposition of Waves

The wave equation is linear so:

$$\frac{\delta^2}{\delta t^2}(y_1 + y_2) = \ddot{y}_1 + \ddot{y}_2 \tag{1.34}$$
$$= v^2 y_1'' + v^2 y_2'' \tag{1.35}$$
$$= v^2 \frac{\delta^2}{\delta x^2}(y_1 + y_2) \tag{1.36}$$

If y_1 and y_2 both satisfy the wave equation, then so does $y_1 + y_2$. This implies the *principal of Superposition.*

1.8 Standing Waves On A String

Standing waves occurs when left- & right-moving waves with the same frequency & amplitude interfere

$$y = A[\sin(kx - \omega t) + \sin(kx + \omega t)] = 2A\sin(kx)\cos(\omega t) \tag{1.37}$$
$$y(0,\, t) = 0; \;\; y\left(\frac{n\pi}{k},\, t\right) = 0 \tag{1.38}$$

For length $\frac{n\pi}{k}$ of wire, $y(L,\, t) = 0$. y is also zero when $kx = n\pi$, $\{n \in \mathbb{N}\}$ – these points are called nodes. Maximal displacement is at an anti-node – $kx = \left(n + \frac{1}{2}\right)\pi$. Waves reflecting off fixed ends of string generate normal modes – they start & finish at boundaries of L. End points are nodes, so $L = \frac{n\lambda\pi}{2}$:

$$f_n = \frac{v}{\lambda n} = \frac{nv}{2L} \tag{1.39}$$
$$f_1 = \frac{1}{2L}\sqrt{\frac{F}{\mu}} \text{ – fundamental frequency} \tag{1.40}$$

1.9 Sound & Hearing

Sound is a sinusoidal longitudinal wave, usually travelling through air: $y(x,\, t) = A\cos(kx - \omega t)$: Use this to find the pressure wave and velocity:

$$y'(x,\, t) = -Ak\sin(kx - \omega t) \tag{1.41}$$
$$\dot{y}(x,\, t) = A\omega\sin(kx - \omega t) \tag{1.42}$$
$$\dot{y} = -\frac{\omega}{k}y' = -vy' \tag{1.43}$$

ΔP is given by ΔV and the bulk modulus:

$$\delta P = -B\frac{\delta V}{V} \tag{1.44}$$
$$V = S\delta x \tag{1.45}$$
$$\delta V = S(y(x_0 + \Delta x,\, t) - y(x_0,\, t)) = S\Delta x \frac{\delta y}{\delta x}\Big|_{x_0} \tag{1.46}$$
$$\delta P = -B\frac{\delta y}{\delta x} = Bk\sin(kx - \omega t) \tag{1.47}$$
$$P_{max} = BkA \tag{1.48}$$

For speed, relate δP to velocity differential:

$$\delta P = \frac{\delta F}{A} = \frac{1}{S}\frac{\delta p}{\delta t} \qquad \delta p = \delta m\dot{y} \tag{1.49}$$
$$\delta p = \rho \times S\Delta x \times \dot{y} \qquad -By' = \frac{\rho\Delta x\dot{y}}{\delta t} = -v^2\rho y' \tag{1.50}$$

$$v^2 = \frac{B}{\rho} \qquad\qquad v = \sqrt{\frac{B}{\rho}} \tag{1.51}$$

Recall that for ideal gases, $pV = nRT$ and the isothermal bulk modulus, $B = p$:

$$v^2 = \frac{nRT}{M} \implies v = \sqrt{\frac{nRT}{M}} \tag{1.52}$$

In a solid, use Young's Modulus:

$$v = \sqrt{\frac{Y}{\rho}} \tag{1.53}$$

1.10 Sound Intensity

Rate at which waves transport energy per unit area per unit time

$$\delta E = F\,\delta x = \delta P \times \delta S \times \delta x \tag{1.54}$$

$$\text{Intensity, } I = \frac{\delta E}{\delta S\,\delta t} = \frac{\delta P\,\delta S\,\delta x}{\delta S\,\delta t} \tag{1.55}$$

$$I = -By'\dot{y}BkA\sin(kx-\omega t) \times \omega A\sin(kx-\omega t) \tag{1.56}$$

$$I = Bk\omega A^2\sin^2(kx-\omega t) \tag{1.57}$$

Looking for $\langle I\rangle$:

$$\langle \sin^2 \rangle = \frac{1}{2} \tag{1.58}$$

$$I = \frac{1}{2}Bk\omega A^2 \tag{1.59}$$

$$k = \frac{\omega}{v} = \omega\sqrt{\frac{\rho}{B}} \tag{1.60}$$

$$I = \frac{1}{2}\sqrt{\rho B}\omega^2 A^2 \tag{1.61}$$

1.11 The Decibel Scale

Humans can detect sounds in a range of around 12 orders of magnitude so it easier to use a logarithmic scale:

$$\beta = 10\log\frac{I}{I_0} \tag{1.62}$$

$I_0 = 10^{-12} W\ m^{-2}\ \therefore\ 0 \to 120dB$ is hearing range.

1.12 Standing Waves and Normal Nodes

Sound Waves in pipes reflect from ends and interfere to form standing waves. Nodes & anti-nodes:

- node – no air motion/maximum pressure variation
- anti-node – maximal air motion/no pressure variation

There is a node at the end of a closed pipe and an anti-node at the end of an open pipe. 2 open ends: anti-node at either end with a node in the middle; $\lambda = 2L$. 1 open end, 1 closed end: node at closed end, anti-node at open end; $\lambda = 4L$. Odd harmonics in 1 open, 1 closed (harmonics giving using fundamental frequency equation).

1.13 Sound Phenomena

Sound waves interfere via principle of superposition. Phase and frequency are both important. For 2 waves of same amplitude and frequency, they can be:

- 'in-phase' for constructive interference
- 'out-of-phase' for destructive interference

$$y_1 = A\cos(kx - \omega t) \tag{1.63}$$
$$y_2 = A\cos(kx - \omega t + \phi) \tag{1.64}$$
$$y_1 + y_2 = A\Big[\cos\Big(kx - \omega t + \frac{\phi}{2} - \frac{\phi}{2}\Big) + \cos\Big(kx - \omega t + \frac{\phi}{2} + \frac{\phi}{2}\Big)\Big] \tag{1.65}$$
$$y_1 + y_2 = 2A\Big[\cos\Big(kx - \omega t + \frac{\phi}{2}\Big)\cos\Big(\frac{\phi}{2}\Big)\Big] - \text{amplitude } 2A\cos\Big(\frac{\phi}{2}\Big) \tag{1.66}$$

The most common reason for phase difference is path length.

1.14 Beats

When 2 waves are very nearly equal frequencies, beats occur. Sitting at $x = 0$:

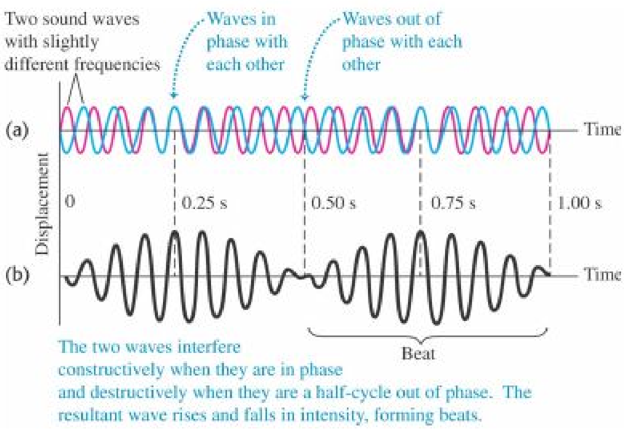
$$y_1 = A\cos(2\pi f_1 t) \tag{1.67}$$
$$y_2 = A\cos(2\pi f_2 t) \tag{1.68}$$
$$f_2 = f_1 + \delta f \tag{1.69}$$

Think of effect of δf as producing time-depended phase, $\phi \approx \delta f t$:

$$y_1 + y_2 = A[\cos(2\pi(f_1 + \delta f)t) + \cos(2\pi f_1 t)] \tag{1.70}$$
$$= 2A\cos(\pi \delta f t)\cos\Big(2\pi\Big(f_1 + \frac{\delta f}{2}\Big)t\Big) \tag{1.71}$$

This is nearly the same pitch as the original, the amplitude varies slowly over time. Consider the period of cos:

$$T = \frac{2\pi}{\pi \delta f} = \frac{2}{\delta f} \tag{1.72}$$
$$T >> \frac{1}{f_1} \tag{1.73}$$
$$f_{beat} = f_2 - f_1 = \delta f \tag{1.74}$$



1.15 The Doppler Effect

A shift in pitch due to motion. Key observation is that sound is transmitted through a medium that has a rest frame. Shift occurs whether it is listener source that is moving, though the effect is different in each case.

For a moving source:
f_s is fixed. Intuition – waves bunch up in front of the source and spread out behind:

$$\text{Forward:} \quad \lambda_f = \frac{c_f}{f_s} = \frac{c - v_s}{f_s} \qquad f_L = \frac{c}{\lambda_L} = \frac{c}{\lambda_f} = \frac{c}{c - v_s} f_s > f_s \tag{1.75}$$

$$\text{Back:} \quad \lambda_b = \frac{c + v_s}{f_s} \qquad f_b = \frac{c}{\lambda_b} = \frac{c}{c + v_s} f_s > f_s \tag{1.76}$$

For a moving listener:
λ_s is fixed. Intuition – listener is either catching up or receding from the wave, increasing/decreasing sound speed:

$$\text{Forward: } f_L = \frac{c_L}{\lambda_s} = \frac{c + v_L}{c} f_s \tag{1.77}$$

$$\text{Back: } f_L = \frac{c - v_L}{c} f_s \tag{1.78}$$

In general:

$$f_L = \frac{c \pm v_L}{c \pm v_s} f_s \tag{1.79}$$

If medium moves, $c \to c \pm v_{medium}$.

Examples:
Doppler flow meter – high frequency sound emitted towards body, reflected back to receiver allowing for example imaging of blood flow.

1.16 The Nature and Propagation of Light

Light is an electromagnetic wave – oscillations between electric and magnetic components.

- Light as a wave: satisfies the wave equation, has interference and diffraction
- Light as a particle: multiple of finite energy, $E = hf$

Wave Equation:

$$\frac{\delta^2}{\delta t^2} \underline{E} = c^2 \underline{\nabla}^2 \underline{E}; \qquad \frac{\delta^2}{\delta t^2} B = c^2 \underline{\nabla}^2 \underline{B} \tag{1.80}$$

This can be derived from a gauge potential:

$$\underline{E} = \underline{\nabla} \phi + \frac{d\underline{A}}{dt}; \qquad \underline{B} = \underline{\nabla} \times \underline{A} \tag{1.81}$$

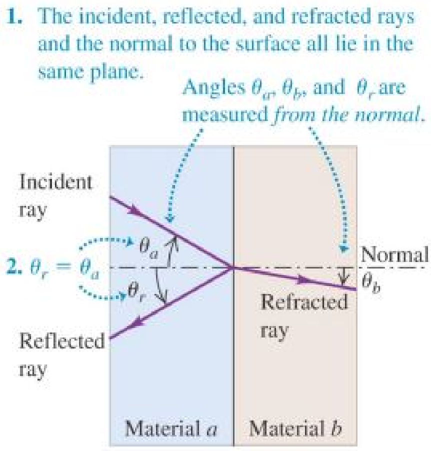
Visible light is a small part of the electromagnetic spectrum, with all wavelengths travelling at the same speed through a vacuum – $c \approx 3 \times 10^8 \; m \; s^{-1}$. Light slows down in a different medium – $c = \frac{1}{\sqrt{\epsilon \mu}}$. c is not frame-dependent, c is fundamental.

1.17 Reflection and Refraction

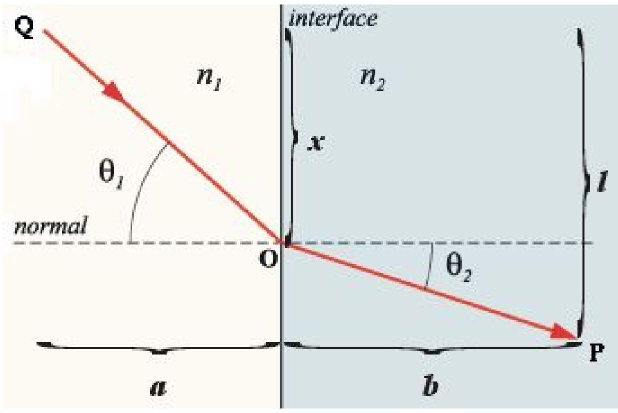
Light slows down in a medium, encoded in the refractive index:

$$n = \frac{c}{v_m} \geq 1 \tag{1.82}$$

When light hits interface between two media, some reflects, some doesn't.



Incident, refracted, and reflected rays are all in same plane. $\theta_r = \theta_i$ for reflection. For refraction, Snell's Law describes the relationship. This is easily derived from Fermat's principal of Least Action: Light takes "least-time path" between two points:



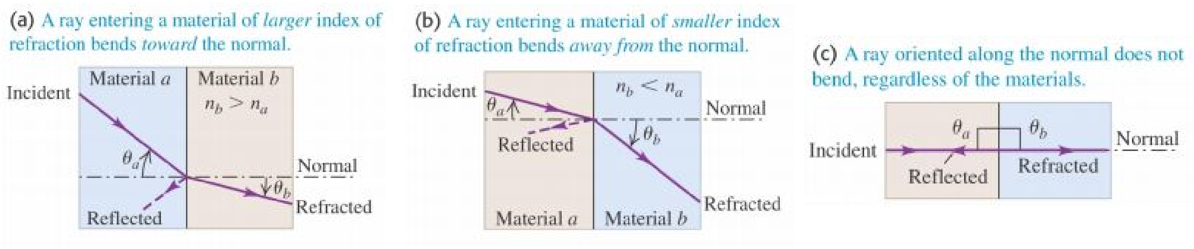
$$t_{AB} = \frac{\sqrt{a_x^2 + a_y^2}}{v_a} + \frac{\sqrt{b_x^2 + b_y^2}}{v_b} \tag{1.83}$$

$$\frac{dt_{AB}}{db_x} = 0 \tag{1.84}$$

$$\frac{dt_{AB}}{db_x} = \frac{b_x}{v_a\sqrt{a_x^2 + a_y^2}}\frac{da_x}{db_x} + \frac{b_x}{v_b\sqrt{b_x^2 + b_y^2}} \tag{1.85}$$

$$\frac{dt_{AB}}{db_x} = \frac{-\sin\theta_a}{v_a} + \frac{\sin\theta_b}{v_b} \tag{1.86}$$

$$n_a \sin\theta_a = n_b \sin\theta_b - \text{Snell's Law} \tag{1.87}$$



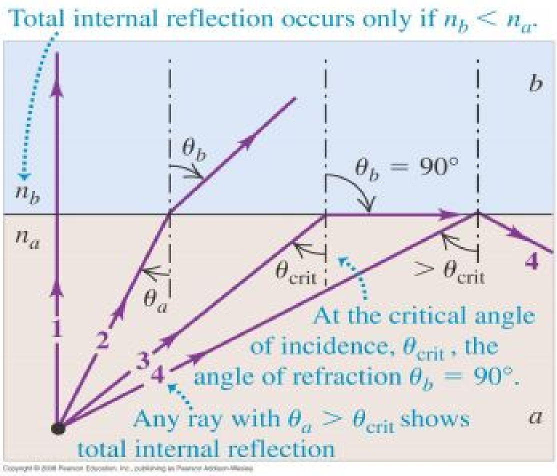
Refraction also depends on wavelength:

$$v = \lambda f = \frac{c}{n};\ f \text{ is unchanged} \tag{1.88}$$

$$\frac{\lambda}{\lambda_0} = \frac{v/f}{c/f} = \frac{1}{n} \tag{1.89}$$

1.18 Total Internal Reflection

Suppose $n_a > n_b$ (e.g. glass to air) i.e. $v_a < v_b$, then $\sin\theta_b = \frac{v_b}{v_a}\sin\theta_a > \sin\theta_a$. When $\sin\theta_a = \frac{v_a}{v_b}$, $\theta_b = \frac{\pi}{2}$ i.e. all light is reflected past that point. For $\theta > \theta_c$ $\left(\sin\theta_c = \frac{v_a}{v_b}\right)$, there is no solution for θ_b, so light is totally internally reflected.



Example: air to glass –

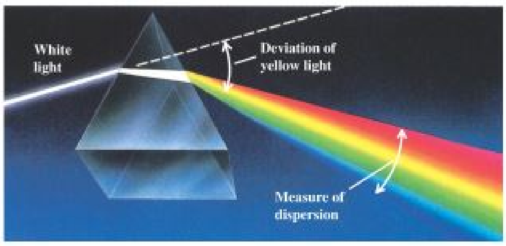
$$\sin\theta_c = \frac{1}{1.52} \approx 0.658 \tag{1.90}$$

$$\theta_c \approx 0.229\pi \text{ (Close to } \frac{\pi}{4}) \tag{1.91}$$

Light in glass is totally internally reflected for $\theta > 0.23\pi$, $\sin\theta_c = \frac{n_b}{n_a}$.

1.19 Dispersion

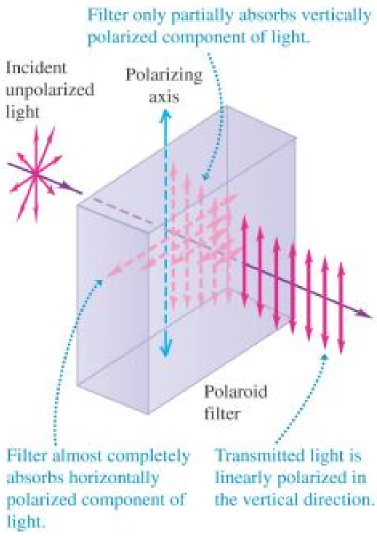
Speed of light in a vacuum is independent of wavelength, but refractive index has variation with wavelength – this gives dispersion. Usually n decreases as λ increases. This gives prism rainbow effect:



1.20 Polarisation

Transverse waves can be polarised, two normals to direction of motion. A polarising filter selects a single direction. In the electromagnetic spectrum, use $\underline{E}$ to define polarisation:

$$\text{y-polarised} \begin{cases} \underline{E} = \hat{j}E_{max}\cos(kx - \omega t) \\ \underline{B} = \hat{k}B_{max}\cos(kx - \omega t) \end{cases} \tag{1.92}$$



An ideal polariser would select precisely one polarised component and blocks light perpendicular to this axis. For 2 polarisers at an angle: $\underline{E} = E\cos\phi\underline{e}_1 + E\sin\phi\underline{e}_2$. $\underline{e}_1$ is transmitted, $\underline{e}_2$ is blocked. Transmitted light is most intense when polarisation axes are aligned, blocked when perpendicular. Intensity reduced by a factor of $\cos^2\phi$: $I = I_0\cos^2\phi$ – Malus' Law.

1.21 Polarisation By Reflection

For most angles, if $\underline{\mathrm{E}}$ is perpendicular to plane of incidence, light is more strongly reflected and partially polarised. For $\underline{\mathrm{E}}$ in plane of incidence, light more strongly refracted. There is a special angle at which no reflection occurs:

$$n_a \sin\theta_a = n_b \sin\theta_b \tag{1.93}$$

$$\theta_a + \theta_b = \frac{\pi}{2} \tag{1.94}$$

$$\tan\theta_a = \frac{n_b}{n_a} \text{ – Brewster Angle} \tag{1.95}$$

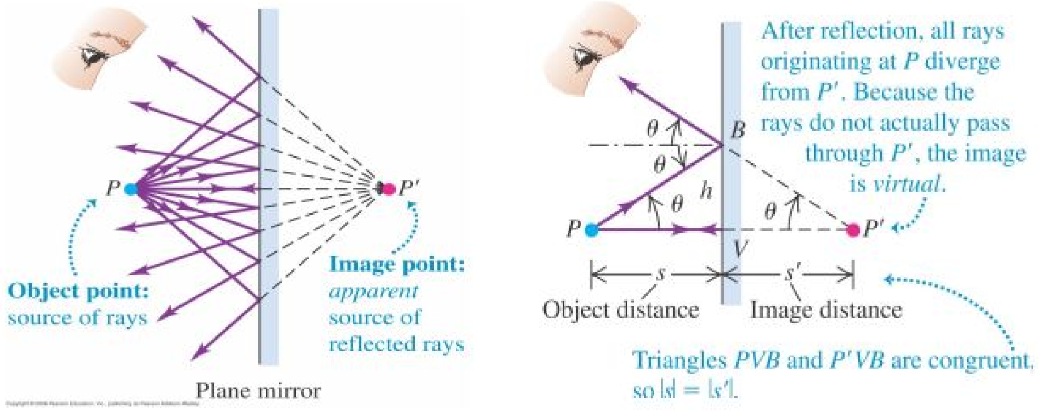
Light polarised completely perpendicular to the plane of incidence is completely reflected.

1.22 Huygen's principal

Empirical description that captures the essence of wave propagation and works with maths of boundary conditions (reflection/slits/refraction)
"Every point on a primary propagating wavefront serves as a source of spherical secondary wavelets that advance with speed and frequency identical to that of the primary wavefront. The 'next' primary wavefront is the envelope of these secondary wavefronts"

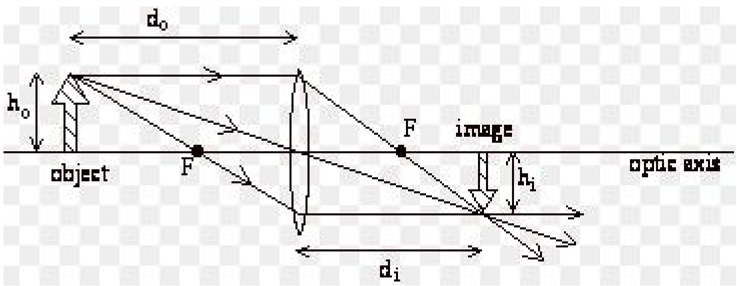
1.23 Ray Tracing

Ray represents a point on wavefront of light. We derive geometric relations for angles at which rays reflect or refract. Simple example: plane mirror. Light rays from P are uniformly reflected and appear to originate from P'. Here we trace rays back to P', the rays do not actually pass through P'. It is a *virtual image*. To find the location of P', note PVB and P'VB are the same (though inverted) triangles $\rightarrow |PV| = |P'V|$ or $|s| = |s'|$:



1.24 Sign Rules

When $s > 0$, the object is on the incoming side of the surface (i.e. a real object); when $s' > 0$, the object is on the outgoing side of the surface (i.e. virtual object), $s' < 0$ otherwise. For a curved surface with a radius of curvature, R: $R > 0$ when centre of curvature is on the outgoing side. Lateral magnification: $m > 0$ if image is upright, $m = \frac{y'}{y}$.



1.25 Spherical Mirror

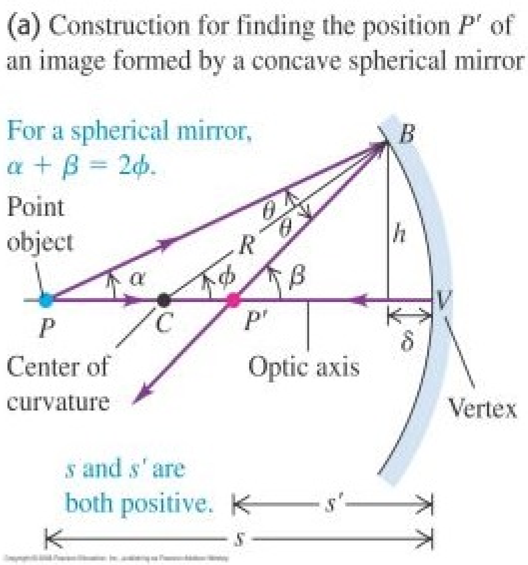


Fig. 7.3 A concave spherical mirror forms a real image of a point object P on the mirror's optic axis.

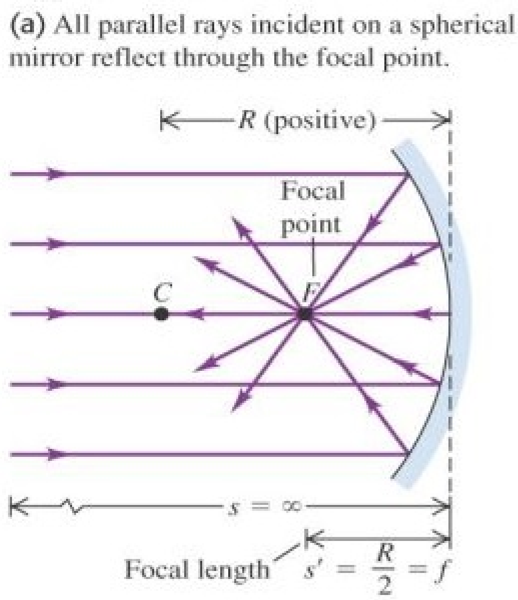


Fig. 7.4 The focal point and focal length of a concave mirror.

C – centre of curvature; R – radius of curvature; $\overrightarrow{VC}$ – principal axis of rotation (mirror is rotationally symmetric around $\overrightarrow{VC}$). Rays parallel to (and close to) VC all reflect through a single point F, the focal point or focus. $|VF|$ is the focal length. An approximation: rays further out focus closer to mirror (spherical aberration).

1.26 Sine Rule and Small Angle Approximation

$$\frac{|BC|}{\sin\alpha} = \frac{|PC|}{\sin\theta} \tag{1.96}$$

$$\frac{R}{\alpha} = \frac{s-R}{\theta} \tag{1.97}$$

$$s = \frac{R}{\alpha}(\alpha+\theta) \tag{1.98}$$

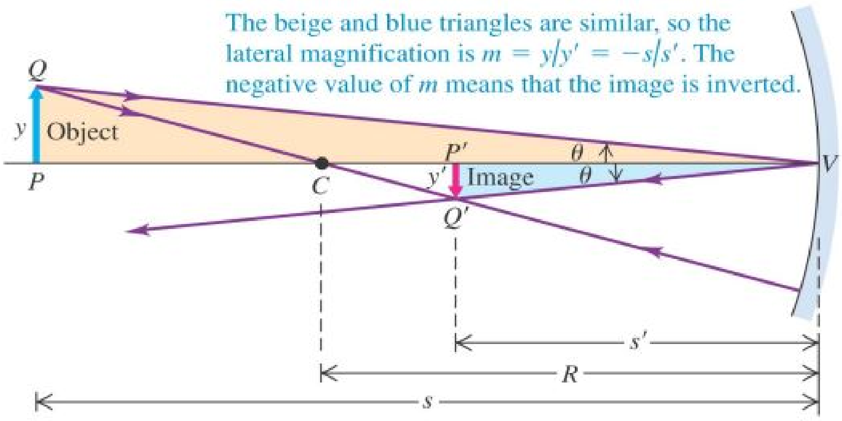
$$\frac{|BC|}{\sin(\pi-2\theta-\alpha)} = \frac{|P'C|}{\sin\theta} \tag{1.99}$$

$$\frac{R}{\sin(2\theta+\alpha)} = \frac{R}{2\theta+\alpha} = \frac{R-s'}{\theta} \tag{1.100}$$

$$s' = \frac{R(\alpha+\theta)}{\alpha+2\theta} \tag{1.101}$$

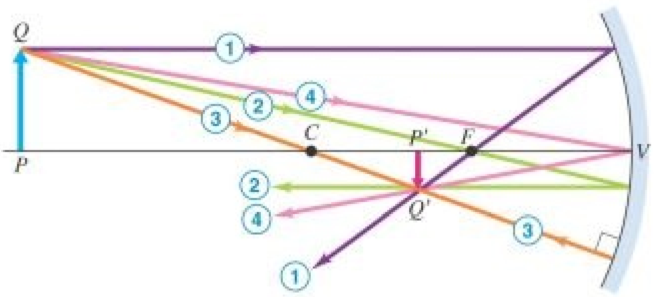
$$\frac{1}{s} = \frac{1}{s'} = \frac{\alpha}{R(\theta+\alpha)} + \frac{\alpha+2\theta}{R(\theta+\alpha)} = \frac{2}{R} = \frac{1}{f} \tag{1.102}$$

PQV and P'Q'V: $-\frac{y'}{s'} = \frac{y}{s}$. Similar triangles so $m = \frac{y'}{y} = -\frac{s'}{s}$.



1.27 Graphical Method

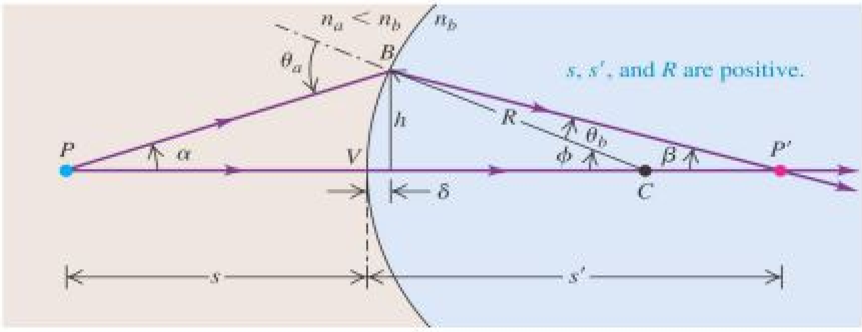
(a) Principal rays for concave mirror



Ray parallel to axis reflects through focal point ($\frac{R}{2}$); Ray through focal point reflects parallel to axis; Ray through C reflects back on itself; Ray through V reflects symmetrically about axis.

1.28 Refraction At A Spherical Surface

Images formed by refraction are our common perception of a lens. As before, study a spherical interface:



$$\beta = \theta_a - \theta_b - \alpha \tag{1.103}$$

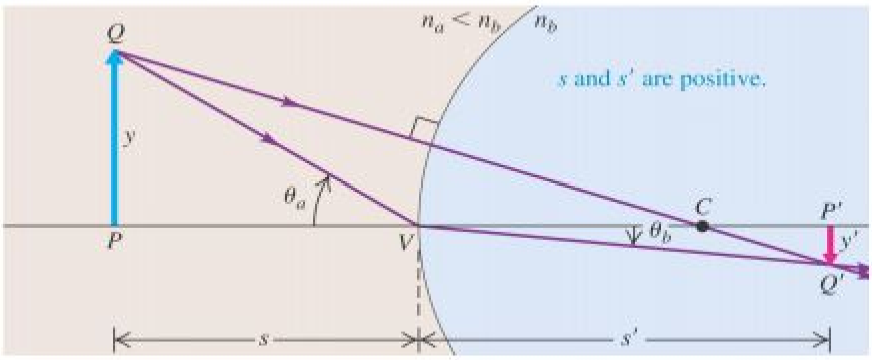
$$\text{From PCB: } \frac{R}{\alpha} = \frac{R}{\sin\alpha} = \frac{s+R}{\sin(\pi - \theta_a)} = \frac{s+R}{\theta_a} \implies s = (\theta_a - \alpha)\frac{R}{\alpha} \tag{1.104}$$

$$\text{From P'CB: } \frac{s'-R}{\sin\theta_b} = \frac{R}{\sin\beta} = \frac{R}{\sin(\theta_a - \theta_b - \alpha)} \implies \frac{R}{\theta_a - \theta_b - \alpha} \implies s' = \frac{R(\theta_a - \alpha)}{\theta_a - \theta_b - \alpha}\frac{1}{s'} = \frac{\theta_a - (n_a\theta_a/n_b) - \alpha}{(\theta_a - \alpha)R} \tag{1.105}$$

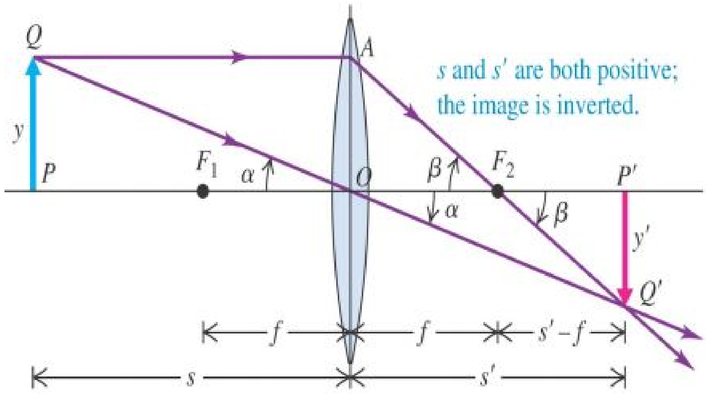
$$\frac{n_a}{s} + \frac{n_b}{s'} = \frac{n_b - n_a}{R} \tag{1.106}$$

$$y = s\sin\theta_a \ \& \ y' = s'\sin\theta_b \tag{1.107}$$

$$m = -\frac{y'}{y} = \frac{s'}{s}\frac{\sin\theta_b}{\sin\theta_a} = \frac{s'}{s}\frac{n_a}{n_b} \tag{1.108}$$



1.29 Optical Instruments



Lens consists of two refracting surfaces. Approximately a thin lens by neglecting detail of path inside lens. Ray parallel to f is focused at F_2, ray from F refracted parallel to axis. Ray through O is unaffected. Real (inverted) image: PQO & P'Q'O similar, $\frac{y}{s} = -\frac{y'}{s'}$ or $\frac{y'}{y} = \frac{s'}{s}$. OA$F_2$ and P'Q'F_2 similar:

$$\frac{y}{f} = -\frac{y'}{s'-f} \text{ or } \frac{y'}{y} = \frac{f-s'}{f} \tag{1.109}$$

$$\text{Together: } -\frac{s'}{s} = \frac{f-s'}{f} \tag{1.110}$$

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} \tag{1.111}$$

Also applies if lens is diverging. Lenses thicker in the middles are converging.

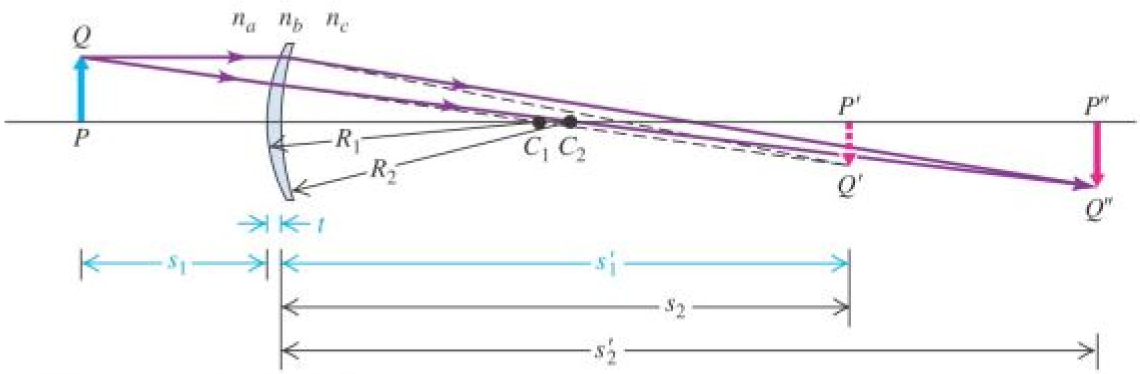
1.30 The Lensmaker's Equation

Relation between focal length of lens and radius of curvature and refractive index:

$$\frac{n_a}{s_1} + \frac{n_b}{s_1'} = \frac{n_b - n_a}{R_1} \tag{1.112}$$

$$\frac{n_b}{s_2} + \frac{n_c}{s_2'} = \frac{n_c - n_b}{R_2} \tag{1.113}$$

Know f is related to Object and Image distances, s & s'. Object ray first refracts through surface. The image (virtual) is object for the second refractive surface: s_1 = s; s_1' = -s_2 ∴ s_2' = s'.



$$\frac{n_a}{s_1} + \frac{n_c}{s_2^1} = \frac{n_a}{s} + \frac{n_a}{s_1} = n_a\left(\frac{1}{s} + \frac{1}{s'}\right) = LHS \tag{1.114}$$

$$\frac{n_b - n_a}{R_1} + \frac{n_c - n_b}{R_2} = n_b\left(\frac{1}{R_1} - \frac{1}{R_2}\right) - n_a\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = RHS \tag{1.115}$$

$$\frac{1}{f} = n\left(\frac{1}{R_1} - \frac{1}{R_2}\right) - \left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{1.116}$$

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{1.117}$$

Examples of lenses: Camera – Camera consists of a lens, shutter, receptor, & aperture control. Intensity of light on film is proportional to area viewed by camera. Aperture controls area $\propto D^2$. $I \propto \frac{D^2}{f^2}$ – f number = $\frac{f}{D}$.

1.31 Interference

Interference occurs when waves overlap, as with mechanical waves, we use principal of superposition. If two or more waves overlap, we add wave functions:

$$Q(\underline{x}, t) = Q_1(\underline{x}, t) + Q_2(\underline{x}, t) \tag{1.118}$$

Apply to light: electromagnetic waves

$$\text{Rep: } \underline{E} = \underline{E}_0 \cos(\underline{k} \cdot \underline{x} - \omega t) \tag{1.119}$$

$$\underline{E}_0 - \text{polarisation and amplitude; } A = |E_0| \tag{1.120}$$

$$\underline{k} - \text{wave-vector direction of travel} \tag{1.121}$$

$$\underline{k} \cdot \underline{E}_0 = 0 \tag{1.122}$$

Consider monochromatic coherent light:

$$\underline{E}_i \cos(\underline{k}_i \cdot \underline{x} - \omega t) \ \ \{i \in \mathbb{N}\};\ k - \text{possibly different, } \omega - \text{the same } \forall\, i. \tag{1.123}$$

Consider two sources of monochromatic light with the same amplitude:

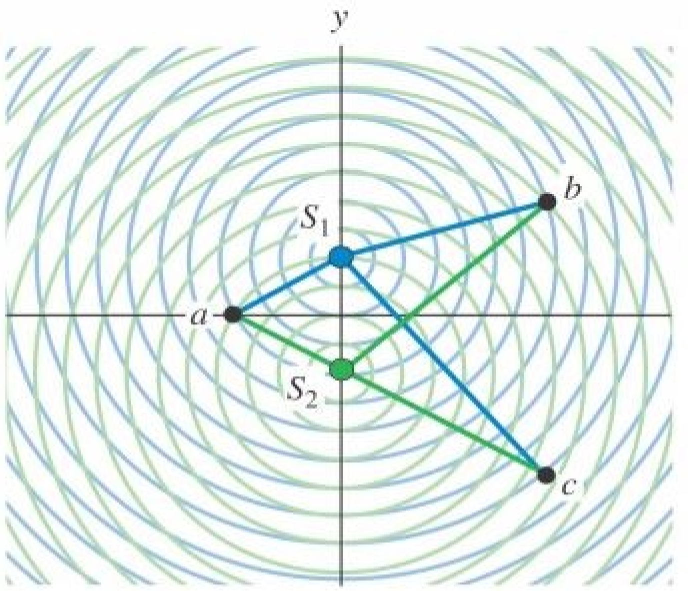


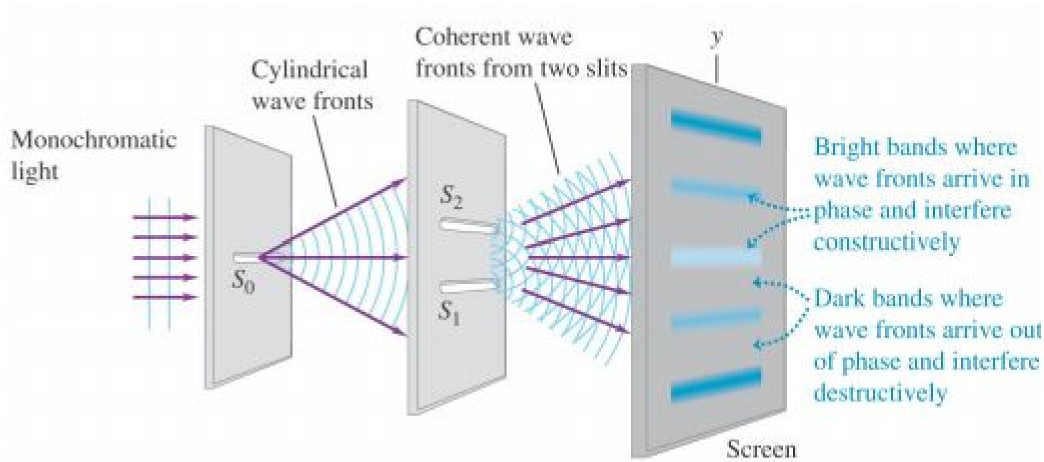
Fig. 9.1 A snapshot of sinusoidal waves spreading out from two coherent sources S_1 and S_2. Constructive interference occurs at point *a* (equidistant from the two sources) and at point *b*. Destructive interference occurs at point *c*.

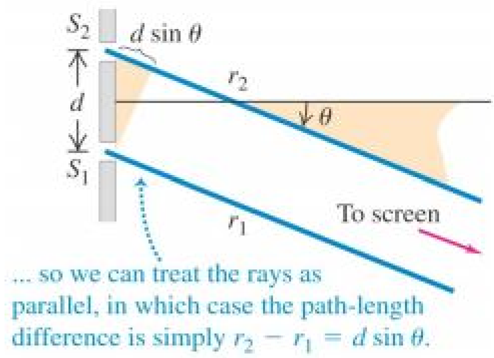
$$\text{Constructive: } |\overrightarrow{S_1P}| = |\overrightarrow{S_2P}| - n\lambda, \quad r_2 - r_1 = n\lambda,\ \{n \in \mathbb{Z}\} \tag{1.124}$$

$$\text{Destructive: } r_2 - r_1 = \left(n + \frac{1}{2}\right)\lambda \tag{1.125}$$

Lines of constructive interference are antinodal lines; lines of destructive interference are nodal lines.

1.32 Young's Slits





$R << d \therefore \theta$ is small. Extra distance for first ray gives phase difference $= d\sin\theta \approx d\theta$. For constructive interference: $n\lambda = d\theta$ $(d\sin\theta)$. Distance between fringes for the n$^{\text{th}}$ fringe:

$$y_n = R\tan\theta_n \approx R\theta_n \tag{1.126}$$

$$= R\frac{n\lambda}{d} \tag{1.127}$$

$$n\lambda = \frac{y_n d}{R} \tag{1.128}$$

Intensity is proportional to A^2: For a single electromagnetic wave –

$$I = \frac{1}{2}cE^2\epsilon_0 \text{ – E is amplitude} \tag{1.129}$$

How to add out-of-phase & differing amplitude? (still monochromatic)

$$E_1\cos(\omega t) + E_2\cos(\omega t + \phi) \tag{1.130}$$

There are two different methods that can be used:

1. Algebra

$$E\cos(\omega t + \alpha) = E_1\cos(\omega t) + E_2\cos(\omega t + \phi) \tag{1.131}$$

$$= E_1\cos(\omega t + \alpha - \alpha) + E_2\cos(\omega t + \alpha + \phi - \alpha) \tag{1.132}$$

$$\begin{aligned} &= E_1\cos(\omega t + \alpha)\cos\alpha + E_1\sin(\omega t + \alpha)\sin\alpha \\ &\quad + E_2\cos(\omega t + \alpha)\cos(\phi - \alpha) - E_2\sin(\omega t + \alpha)\sin(\phi - \alpha) \end{aligned} \tag{1.133}$$

$$\begin{aligned} &= [E_1\cos\alpha + E_2\cos(\phi - \alpha)]\cos(\omega t + \alpha) \\ &\quad + [E_1\sin\alpha - E_2\sin(\phi - \alpha)]\sin(\omega t + \alpha) \end{aligned} \tag{1.134}$$

Choose α to set 2^{nd} term to zero:

$$E_1 \sin\alpha = E_2 \sin(\phi - \alpha) = E_2 \sin(\sin\phi\cos\alpha - \cos\phi\sin\alpha) \tag{1.135}$$

$$\tan\alpha = \frac{E_2\sin\phi}{E_1 + E_2\cos\phi}, \quad \text{Check: } E_1 = E_2 \tag{1.136}$$

$$\tan\alpha = \frac{\sin\phi}{1+\cos\phi} = \frac{2\sin\phi/2\cos\phi/2}{2\cos^2\phi/2} = \tan\frac{\phi}{2} \tag{1.137}$$

$$\alpha = \frac{\phi}{2} \tag{1.138}$$

2. Phasors
   Represent the wave by a rotating vector. To add waves, add vectors.

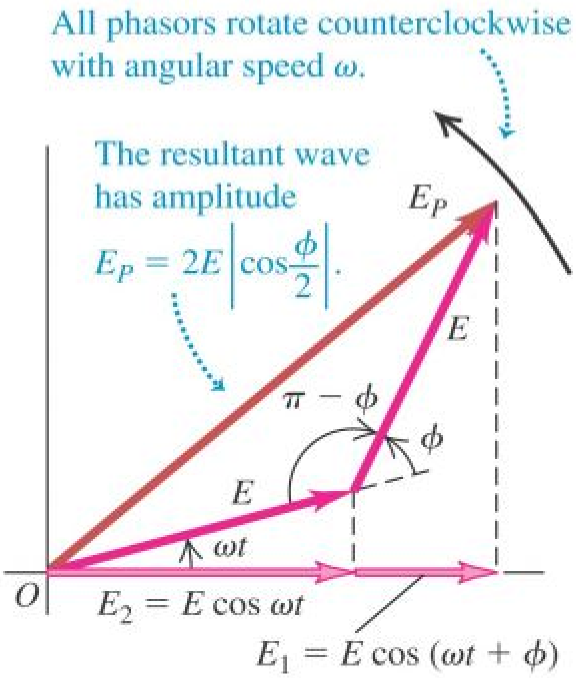


Fig. 9.5 Phasor diagram for the superposition at a point P of two waves of equal amplitude E with a phase difference ϕ.

From vector addition:

$$E_1\cos(\omega t) + E_2\cos(\omega t + \phi) \tag{1.139}$$

Find α at $t = 0$:

$$E_y = E_2\sin\phi = E\sin\alpha \tag{1.140}$$

$$E_x = E_1 + E_2\cos\phi \tag{1.141}$$

$$E^2 = (E_1 + E_2\cos\phi)^2 + E_2^2\sin^2\phi \tag{1.142}$$

$$= E_1^2 + E_2^2 + 2E_1E_2\cos\phi \tag{1.143}$$

$$E\sin\alpha = E_2\sin\phi \tag{1.144}$$

$$\tan\alpha = \frac{E_2\sin\phi}{E\cos\alpha} = \frac{E_2\sin\phi}{E_1 + E_2\cos\phi} \tag{1.145}$$

1.33 Intensity In Interference

Intensity of a electromagnetic wave:

$$I = \frac{1}{2}\epsilon_0 c E^2 = \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}E^2 \tag{1.146}$$

For 2 waves of the same amplitude, $E_{tot} = 2E\cos\left(\frac{\phi}{2}\right)$:

$$I = \left(2E\cos\left(\frac{\phi}{2}\right)\right)^2 \frac{\epsilon_0 c}{2} = 4I_0\cos^2\left(\frac{\phi}{2}\right) \tag{1.147}$$

Max intensity for $\phi = 0$. Average over all phase angles:

$$\langle I\rangle = \frac{I_0}{2\pi}\int_0^{2\pi}\cos^2\left(\frac{\phi}{2}\right)d\phi \implies \frac{4I_0}{2\pi}\int_0^{2\pi}\frac{1+\cos\phi}{2}d\phi = 2I_0 \tag{1.148}$$

Intensity in interference is redistributed. Central peak has intensity $4I_0$ here. Can relate phase to path length:

$$\phi = \frac{2\pi}{\lambda}(r_2 - r_1) \tag{1.149}$$

For Young's Slits:

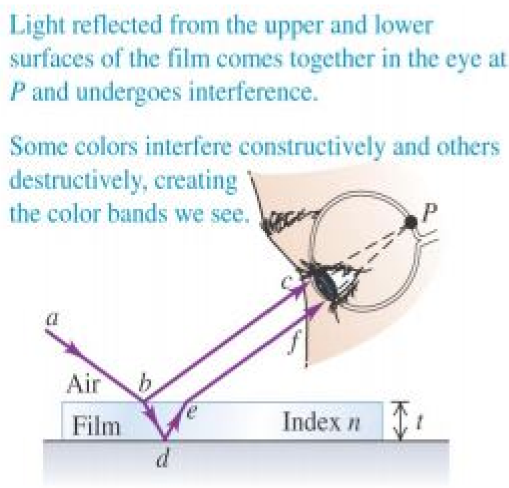
$$r_2 - r_1 = d\sin\theta = \frac{dy}{R} \tag{1.150}$$

$$\phi = kd\sin\theta = \frac{2\pi d}{\lambda}\sin\theta \tag{1.151}$$

$$I = I_0\cos^2\left(\frac{1}{2}kd\sin\theta\right) \tag{1.152}$$

$$I = 4I_0\cos^2\left(\frac{kdy}{2R}\right) \tag{1.153}$$

1.34 Interference In Thin Films



Light reflected from both interfaces has a phase difference giving interference. Interference depends on λ, so see different colours at different angles.

1.35 Phase & Reflection

When light hits a surface, typically some is reflected and some is refracted. The amount reflected depends on the angle of incidence, refractive indices, and polarisations. For nearly normal incidence,

$E_r = \frac{n_a - n_b}{n_a + n_b} E_i$. Although E is typically positive, keeping this as $n_a - n_b$ tells us about phase shift. Analogous to reflected and transmitted mechanical waves:

$$[I\cos(k_1 x - \omega t) + R\cos(k_1 x - \omega t)] + T\cos(k_2 x - \omega t) \tag{1.154}$$

$$v_{1/2} = \frac{\omega}{k_{1/2}} \tag{1.155}$$

$$\text{At } x = 0: \; y = I\cos(\omega t) + R\cos(\omega t) = T\cos(\omega t) \tag{1.156}$$

$$I + R = T \tag{1.157}$$

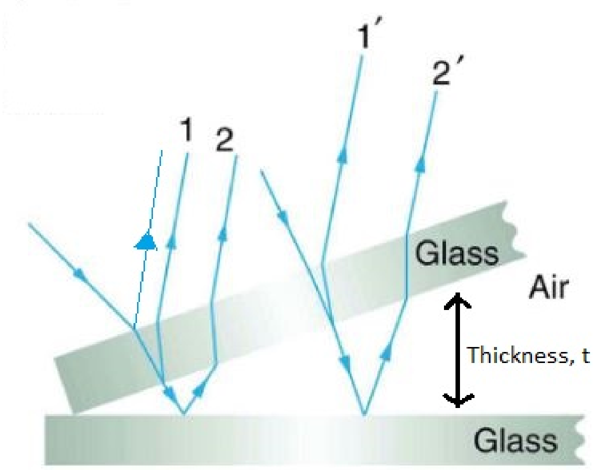
$$y' = -Ik_1 \sin(-\omega t) - Rk_1 \sin \omega t = -Tk_2 \sin(-\omega t) \tag{1.158}$$

$$k_2 T = k_1(I - R) \tag{1.159}$$

$$T = I + R = \frac{k_1}{k_2}(I - R) \tag{1.160}$$

$$R = \frac{k_1 - k_2}{k_1 + k_2} I \tag{1.161}$$

$n_a < n_b$ – phase shift of π in reflected light. If neither/both reflected waves have a phase shift, then the path difference gives the phase shift.



Otherwise:

$$\phi = \frac{2\pi}{\lambda}(r_2 - r_1) + \pi 2t \qquad = n\lambda \text{ – constructive, no phase shift} \tag{1.162}$$

$$2t = \left(n + \frac{1}{2}\right) \text{ – destructive, phase shift} \tag{1.163}$$

1.36 Michelson Interferometer

Exploits interference effects by splitting monochromatic light, sending on different paths, then recombing and looking for phases shifts.

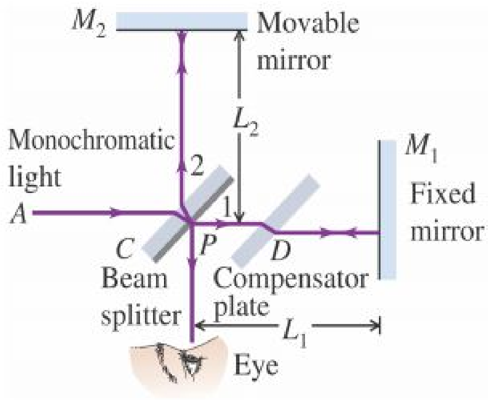


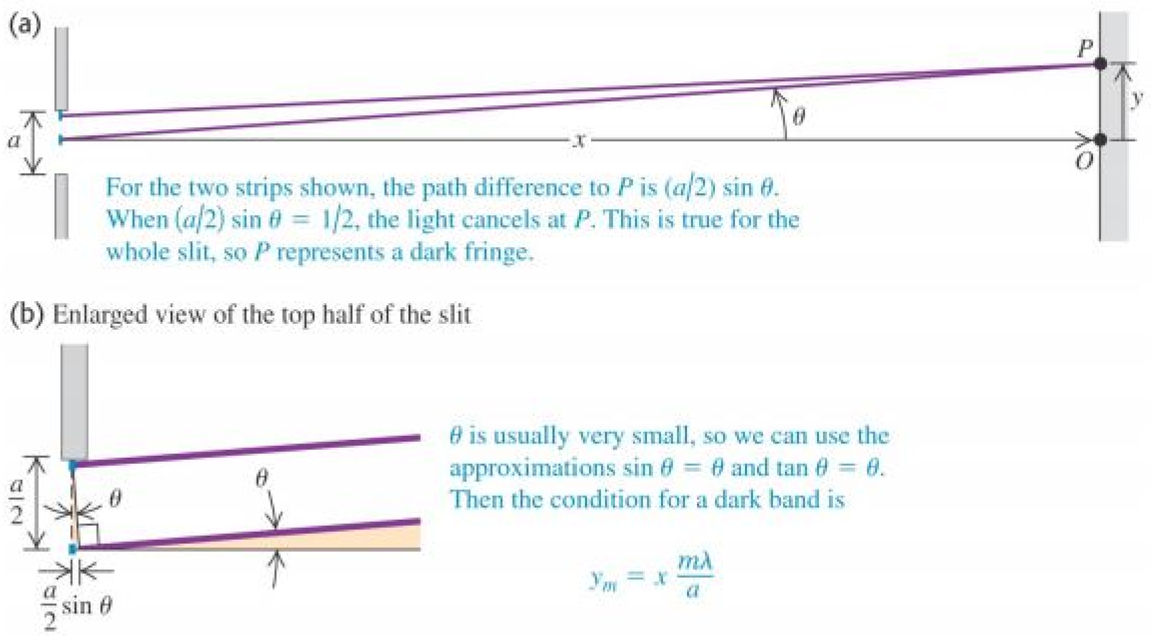
Fig. 10.4. A schematic Michelson interferometer. The observer sees an interference pattern that results from the difference in paths lengths from rays 1 and 2.

Compensator plate ensures both light paths travel through same amount of glass. In essence, same set-up as LIGO.

1.37 Diffraction

Wave nature of light means that boundaries do not cast precisely sharp shadows. Light "bends" around corners, and a close look reveals patterns of interference.
Full problem includes solving wave equation subject to boundary conditions (meaning wave from slits). Near-region diffraction is Fresnal. Far-field is Fraunhofer.

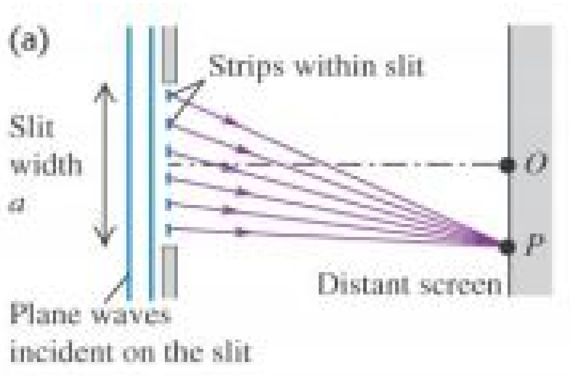
1.38 Fraunhofer Diffraction



Path difference: $r_2 - r_1$

$$r_2 - r_1 \approx \frac{a}{2}\sin\theta \tag{1.164}$$

Destructive interference if $\frac{a}{2}\sin\theta = n\frac{\lambda}{2}$. Expect dark fringes around a central bright band. Break slit into 4:



Path difference of $\frac{a}{4}\sin\theta$ between successive rays. Destructive at $\frac{a}{4}\sin\theta = \frac{\lambda}{2}$. Expect dark fringes at $a\sin\theta = n\lambda$. To find actual result, combine wavefronts from all points between $-\frac{a}{2}$ and $\frac{a}{2}$:

$$r_2 - r_1 = y\sin\theta \tag{1.165}$$

Add up combination from $y \in [-\frac{a}{2}, \frac{a}{2}]$. Integrate phase difference across slits: ($ky\sin\theta$ is the phase at y)

$$\begin{aligned}
\frac{1}{a}\int_{-a/2}^{a/2}\cos\left(ky\sin\theta\right)dy &= \frac{1}{a}\int_{-a/2}^{a/2}\cos\left(ky\sin\theta\right)dy && (1.166)\\
&= \frac{1}{a}\left[\frac{\sin\left(ky\sin\theta\right)}{k\sin\theta}\right]_{-a/2}^{a/2} && (1.167)\\
&= \frac{1}{ak\sin\theta}\left[\sin\left(\frac{ka}{2}\sin\theta\right) - \sin\left(-\frac{ka}{2}\sin\theta\right)\right] && (1.168)\\
&= \frac{2}{ak\sin\theta}\sin\left(\frac{ka}{2}\sin\theta\right) && (1.169)\\
&= \frac{\sin\left(\frac{ka}{2}\sin\theta\right)}{\frac{ka}{2}\sin\theta} && (1.170)\\
&= \frac{\sin x}{x} && (1.171)\\
x &= \frac{ka}{2}\sin\theta && (1.172)
\end{aligned}$$

Diffraction depends on:

- $k = \frac{2\pi}{\lambda}$, i.e. wavelengths
- a, the width of the slit
- $\sin\theta$, where we are on the screen

This gives our amplitude, so intensity patter is:

$$I = I_0\left(\frac{\sin\left(\frac{ka}{2}\sin\theta\right)}{\frac{ka}{2}\sin\theta}\right)^2 \tag{1.173}$$

At $\theta = 0$ (centre of the screen), $\frac{\sin x}{x} \to 1$. $\implies$ Intensity is maximum, $I = I_0$. 1^{st} Dark Fringe at (multiply RHS by n for other fringes):

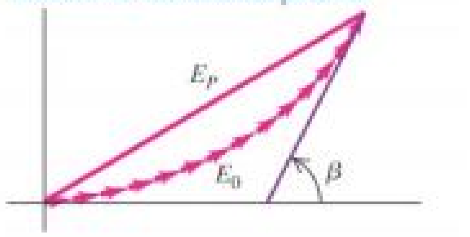
$$\begin{aligned}
\frac{ka}{2}\sin\theta &= \pi && (1.174)\\
\frac{a\pi}{\lambda}\sin\theta &= \pi && (1.175)\\
a\sin\theta &= \lambda && (1.176)
\end{aligned}$$

1.39 Phasor Derivation

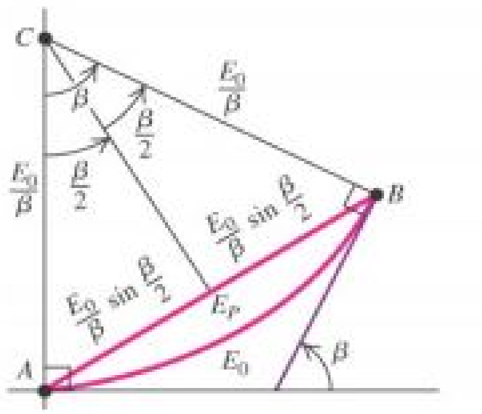
Split the slit into N source-lets and add the wave vectors for each

- Amplitude for each source – $\frac{A}{N}$
- Phase – $\frac{2\pi}{\lambda}\frac{a}{N}\sin\theta = \frac{ka}{N}\sin\theta$

(c) Phasor diagram at a point slightly off th center of the pattern: β = total phase differ between the first and last phasors.



(d) As in (c), but in the limit that the slit is subdivided into infinitely many strips



As $N \to \infty$, the phasors successively approximate an arc. Total phase is always $\beta = ka\sin\theta$. Total amplitude, $A_T = \frac{2A}{\beta}\sin\left(\frac{\beta}{2}\right)$.

1.40 Diffraction and Interference

Young's slits with finite width: For 2 slits we found that

$$E = 2E_0\cos\left(\frac{kd}{2}\sin\theta\right) \tag{1.177}$$
$$I = 2I_0\cos^2\left(\frac{kd}{2}\sin\theta\right) \tag{1.178}$$

and for diffraction:

$$E = E_0\frac{\sin\left(\frac{ka}{2}\sin\theta\right)}{\frac{ka}{2}\sin\theta} \tag{1.179}$$

The principal of superpositions means we superpose these effects:

$$E = E_0\frac{\sin\left(\frac{ka}{2}\sin\theta\right)}{\frac{ka}{2}\sin\theta}\left[2\cos\left(\frac{kd}{2}\sin\theta\right)\right],\ d = 4a \tag{1.180}$$
$$E = E_0\frac{\sin\left(\frac{ka}{2}\sin\theta\right)}{\frac{ka}{2}\sin\theta}\left[2\cos\left(2ka\sin\theta\right)\right] \tag{1.181}$$

The 4$^{\text{th}}$ maximum of interference is deleted by a minimum of the diffraction envelope.

1.41 Circular Apertures and Resolving Power

Any shape aperture forms a diffraction pattern. A circular aperture is common and forms an "Airy Disk".

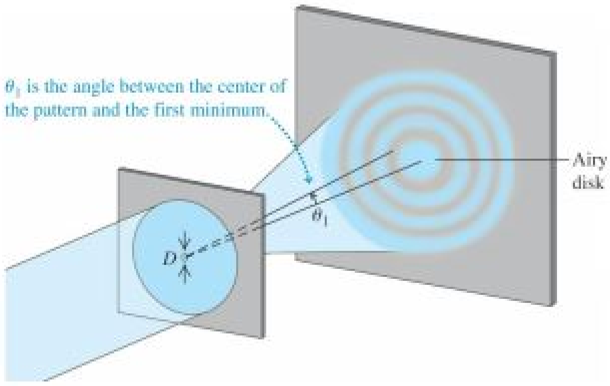


Fig. 12.5 The diffraction pattern formed by a single circular aperture of diameter D. The pattern consists of a central bright spot and alternating dark and bright circular rings.

For cylindrical or spherical wavefronts, we replace sin/cos by Bessel functions:

$$I = I_0 \left(2 \frac{J_1\left(\frac{kD}{2} \sin\theta\right)}{\frac{kD}{2}\sin\theta} \right)^2 \tag{1.182}$$

J_1 is a Bessel function; D is the diameter of the aperture

$$\text{First zero at } \frac{kD}{2}\sin\theta \approx 1.22\pi \tag{1.183}$$

$$\theta \approx 1.22\frac{\lambda}{D} \tag{1.184}$$

Sets limits on optical resolution. Need big D or smaller λ.

1.42 Diffraction with Multiple Slits

What happens with many slits? Look at $N = 4$.

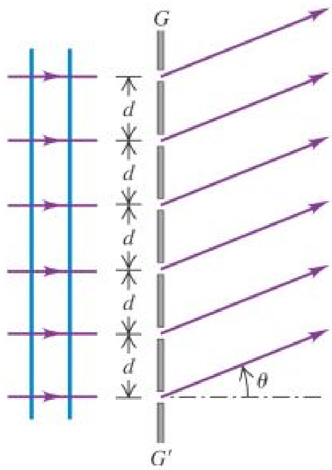


Fig. 12.2 A portion of a transmission diffraction grating. The separation between the centres of adjacent slits is *d*.

Path difference is $d\sin\theta$. If $d\sin\theta = n\lambda$, set constructive interference from all 4 sources: Let $\phi = kd\sin\theta$ – phase shift for adjacent sources. r_0 – distance of source to screen; E_0 – amplitude of one source. Overall amplitude:

$$E_{tot} = E_0[\cos(kr_0) + \cos(kr_0 + \phi) + \cos(kr_0 + 2\phi) + \cos(kr_0 + 3\phi)] \tag{1.185}$$

$$E_{tot} = E_0\left[2\cos\left(kr_0 + \tfrac{\phi}{2}\right)\cos\left(\tfrac{\phi}{2}\right) + 2\cos\left(kr_0 + \tfrac{5\phi}{2}\right)\cos\left(\tfrac{\phi}{2}\right)\right] \tag{1.186}$$

$$E_{tot} = 2E_0\cos\left(\tfrac{\phi}{2}\right)\left[2cos\left(kr_0 + \tfrac{3\phi}{2}\right)\cos(\phi)\right] \tag{1.187}$$

$$E_{tot} = 4E_0\cos\left(\tfrac{\phi}{2}\right)\cos(\phi)\cos\left(kr_0 + \tfrac{3\phi}{2}\right) \tag{1.188}$$

The 4 slits combine to give total amplitude:

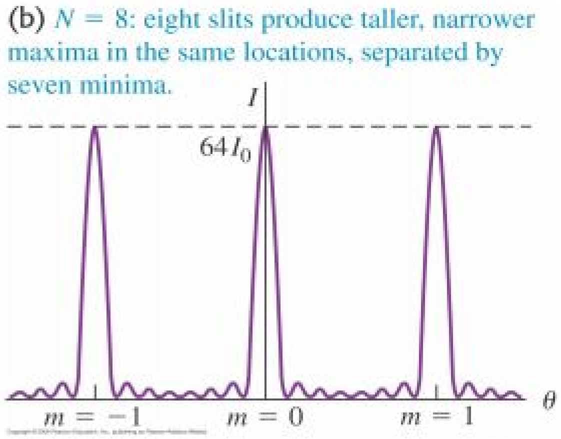
$$E_{tot} = 4E_0\cos\left(\tfrac{\phi}{2}\right)\cos(\phi) \tag{1.189}$$

$$\phi = 2\pi n \tag{1.190}$$

$$E_{tot} = 4E_0 \tag{1.191}$$

All 4 sources constructively interfere, but have $E_{tot} = 0$ for $\phi = \frac{n\pi}{2}$. Intensity $\propto E_{tot}^2$. Evenly spaced dark fringes at:

$$kd\sin\theta = \frac{2\pi}{n},\ n \in \mathbb{N} \tag{1.192}$$



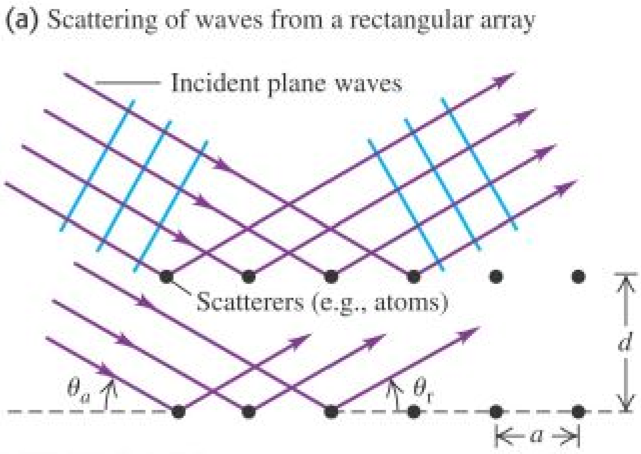
Added amplitudes out of phases – can use phasors. Easy to find dark fringes for $N = 4$ by adding vectors of phase differences. Deduce local maximums from this. For N slits, get principal maxima of intensity:

$$I = N^2 I_0 \text{ – Very Sharp Peak} \tag{1.193}$$

For dark fringe, $\phi = \frac{2\pi}{N}, \frac{4\pi}{N}... \rightarrow \frac{2(N-1)\pi}{N}$. So we have (N - 1) dark fringes between principal maxima. Local maxima are much lower. Also can get a diffraction envelope superposed if slit width is 'significant'.

1.43 Diffraction Grating and X-Ray Diffraction

An array of a large number of parallel slits with same spacing, d, and width, a, is a diffraction grating. Principal maxima occur at $d\sin\theta = n\lambda$ produce sharp maxima.



Consider X-rays incident on a 2D array of atoms.

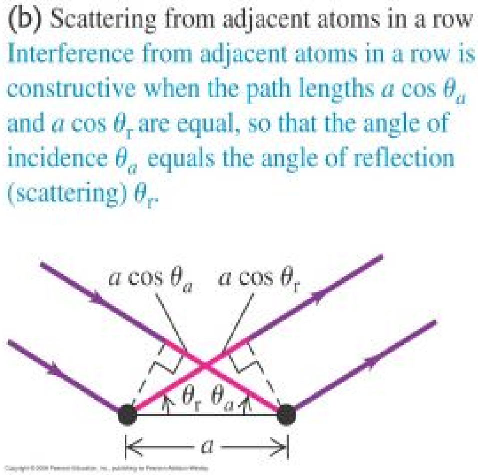
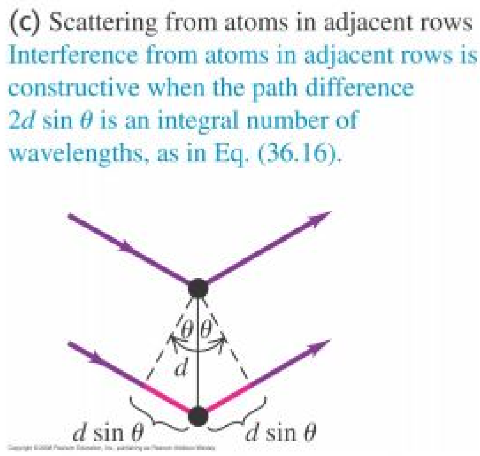


Fig. 12.3 A two-dimensional model of scattering from a rectangular array of point sources. Note that the angles in (b) are measured from the surface of the array rather than from the normal.



Total path length the same for $\theta_r = \theta_a$. Looked at scatter for adjacent rows also.