General Relativity

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Just intro stuff

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2.1 Intro to Tensor Notation

Consider the cartesian definition for $\underline{r}$:

$$\underline{r} = x\underline{i} + y\underline{j} + \underline{z}. \tag{2.1}$$

We have the basis vector $\{\underline{i}, \underline{j}, \underline{k}\}$ and coordinate values $\{x, y, z\}$. We can write this in a different form as

$$\underline{r} = x^1\underline{e}_1 + x^2\underline{e}_2 + x^3\underline{e}_3. \tag{2.2}$$

Note: $x^2 \neq x * x$. The 2 is an index, not a power. If we want to square something, we will write $(x^1)^2 = x^1x^1$. We can rewrite the above again as

$$\underline{r} = \sum_{i=1}^{3} x^i\underline{e}_i. \tag{2.3}$$

We can then simplify this further using the **Einstein summation convention**:

$$\underline{r} = x^i\underline{e}_i, \tag{2.4}$$

i.e. whenever there is a repeated index, we sum over them. Different letters will imply different things:

- ➤ Roman letters $i, j, \ldots$ - summing over 3D space
- ➤ Roman letters $a, b, c, \ldots$ - summing over ND space
- ➤ Roman letters $A, B, \ldots$ - summing over 2D space
- ➤ Greek letters $\alpha, \beta, \mu, \nu, \ldots$ - summing over 4D space-time $\{x^0, x^1, x^2, x^3\}$, starting from 0 as time is different slightly, so $\{ct, x^i\}$

2.2 Coordinate Transformation

You may be used to

$$x' = \gamma\left(x - \frac{vct}{c}\right), \tag{2.5}$$

where the extra c factor to make time space-like. This notation can get confusing so instead we use:

$$x^{\bar{1}} = \gamma\left(x^1 - \frac{v}{c}x^0\right), \tag{2.6}$$

where the 'bar' indicates new coordinate system.

For a minute vector difference between points P and Q $d\underline{r}$ in two coordinate systems, we can define $\underline{e}_a$:

$$\underline{r}(P) = \underline{e}_a x^a \qquad \underline{r}(P) = \underline{e}_{\bar{b}} x^{\bar{b}} \tag{2.7}$$

$$d\underline{r} = dx^a\underline{e}_a \tag{2.8}$$

$$\frac{\partial \underline{r}}{\partial x^a} = \underline{e}_a \qquad \frac{\partial \underline{r}}{\partial x^{\bar{b}}} = \underline{e}_{\bar{b}} \tag{2.9}$$

But what is the relationship between these two coordinate systems? Start with $x^{\bar{b}} = x^{\bar{b}}(x^a)$, and consider a general function

$$f = f(x^1, x^2, x^3) \tag{2.10}$$
$$\Delta f = \frac{\partial f}{\partial x^1}\Delta x' + \frac{\partial f}{\partial x^2}\Delta x^2 + \frac{\partial f}{\partial x^2}\Delta x^3 = \frac{\partial f}{\partial x^a}\Delta x^a \tag{2.11}$$

How do we get a small change in $x^{\bar{b}}$?

$$\Delta x^{\bar{b}} = \frac{\partial x^{\bar{b}}}{\partial x^a}\Delta x^a \tag{2.12}$$
$$dx^{\bar{b}} = \frac{\partial x^{\bar{b}}}{\partial x^a}dx^a \tag{2.13}$$
$$dx^{\bar{a}} = \frac{\partial x^{\bar{a}}}{\partial x^b}dx^b \tag{2.14}$$

Notice how we can simply just switch round the indices - **these are all dummy variables and as long as the index notation is consistent, it is completely arbitrary which letter is used,** i.e. the letters themselves mean nothing.

2.3 Tensors

Any quantity which transforms as

$$A^{\bar{b}} = \frac{\partial x^{\bar{b}}}{\partial x^a}A^a \tag{2.15}$$

is a Rank (1,0) or order 1 contravariant tensor. What about $\underline{e}_a$?

$$\underline{r} = x^a \underline{e}_a = x^{\bar{b}} \underline{e}_{\bar{b}} \tag{2.16}$$
$$\underline{e}_{\bar{b}} = \frac{\partial \underline{r}}{\partial x^{\bar{b}}} = \frac{\partial \underline{r}}{\partial x^a}\frac{\partial x^a}{\partial x^{\bar{B}}} = \frac{\partial x^a}{\partial x^{\bar{b}}}\underline{e}_a \tag{2.17}$$

So now we have reversed the position of the indices in Eq (2.15).

How do we define scalars?

$$\nabla \phi = \frac{\partial \phi}{\partial x^i}\underline{e}_i \tag{2.18}$$
$$\frac{\partial \phi}{\partial x^{\bar{j}}} = \frac{\partial x^i}{\partial x^{\bar{j}}}\frac{\partial \phi}{\partial x^i} \tag{2.19}$$

In general, we have

$$A_{\bar{j}} = \frac{\partial x^i}{\partial x^{\bar{j}}}A_i, \tag{2.20}$$

which we call a Rank (0,1) or order 1 covariant tensor.

Lecture 3

3.1 Higher order tensors

Consider

$$T^{ab} = A^a B^b, \tag{3.1}$$

$$T^{\bar{c}\bar{d}} = A^{\bar{c}} B^{\bar{d}} = \left(\frac{\partial x^{\bar{c}}}{\partial x^a} A^a\right)\left(\frac{\partial x^{\bar{d}}}{\partial x^b} B^b\right) = \frac{\partial x^{\bar{c}}}{\partial x^a}\frac{\partial x^{\bar{d}}}{\partial x^b} A^a B^b = \frac{\partial x^{\bar{c}}}{\partial x^a}\frac{\partial x^{\bar{d}}}{\partial x^b} T^{ab}. \tag{3.2}$$

This is the definition of a second order contravariant tensor.

3.2 Tensor Equations

We can write a basic tensor equation,

$$T^a = k(A^a + B^a), \tag{3.3}$$

and wonder how this would look in a transformed coordinate system?

$$T^{\bar{b}} = \frac{\partial x^{\bar{b}}}{\partial x^a} T^a = k\left(\frac{\partial x^{\bar{b}}}{\partial x^a} A^a + \frac{\partial x^{\bar{b}}}{\partial x^a} B^a\right) \tag{3.4}$$

$$= k(A^{\bar{b}} + B^{\bar{b}}). \tag{3.5}$$

So if a tensor equation is true, it is true in all coordinate systems.

3.3 The metric tensor

What is the metric? *The metric is a measure of space.* We define the metric tensor,

$$g_{ab} = \underline{e}_a \cdot \underline{e}_b = g_{ba}, \tag{3.6}$$

so it is symmetric. We can use this when calculating spacetime distances:

$$ds^2 = \underline{dr} \cdot \underline{dr} = (dx^a \underline{e}_a) \cdot \left(dx^b \underline{e}_b\right) \tag{3.7}$$

$$= (\underline{e}_a \cdot \underline{e}_b) dx^a\, dx^b = g_{ab} dx^a\, dx^b. \tag{3.8}$$

Is it a tensor?

$$g_{\bar{a}\bar{b}} = (\underline{e}_{\bar{a}} \cdot \underline{e}_{\bar{b}}) = \left(\frac{\partial x^c}{\partial x^{\bar{a}}} \underline{e}_c\right) \cdot \left(\frac{\partial x^d}{\partial x^{\bar{b}}} \underline{e}_d\right) \tag{3.9}$$

$$= \frac{\partial x^c}{\partial x^{\bar{a}}}\frac{\partial x^d}{\partial x^{\bar{b}}} (\underline{e}_c \cdot \underline{e}_d) = \frac{\partial x^c}{\partial x^{\bar{a}}}\frac{\partial x^d}{\partial x^{\bar{b}}} g_{cd}, \tag{3.10}$$

so it transforms as a tensor; a second order covariant tensor.

3.4 Kronecker Delta

We can write an arbitrary vector as

$$\underline{A} = A^a \underline{e}_a = A^{\bar{b}} \underline{e}_{\bar{b}} = \left(\frac{\partial x^{\bar{b}}}{\partial x^a} A^a\right)\left(\frac{\partial x^d}{\partial x^{\bar{b}}} \underline{e}_d\right) \tag{3.11}$$

$$= \left(\frac{\partial x^{\bar{b}}}{\partial x^a}\frac{\partial x^d}{\partial x^{\bar{b}}}\right) A^a \underline{e}_d = \left(\frac{\partial x^d}{\partial x^a}\right) A^a \underline{e}_d \tag{3.12}$$

$$= \delta_a{}^d A^a \underline{e}_d = A^d \underline{e}_d = A^a \underline{e}_a \tag{3.13}$$

Lecture 4

Lecture 5

Lecture 6

Asbolute Derivative:

$$\frac{D\lambda^a}{ds} = \frac{d\lambda^a}{ds} + \Gamma^a{}_{bc}\lambda^b \frac{dx^c}{ds} \tag{6.1}$$

Covariant Derivative:

$$\lambda^a{}_{;c} = \frac{\partial \lambda^a}{\partial x^c} + \Gamma^a{}_{bc}\lambda^b \tag{6.2}$$

Christoffel Symbols:

$$\Gamma^c{}_{ab}\underline{e}_c = \frac{\partial \underline{e}_a}{\partial x^b}, \quad \Gamma^c{}_{ab} = \Gamma^c{}_{ba} \tag{6.3}$$

Other stuff:

$$\frac{\partial g_{ab}}{\partial x^c} = \Gamma^d{}_{ac}g_{bd} + \Gamma^d{}_{bc}g_{ad} \tag{6.4}$$

$$\frac{\partial g_{bc}}{\partial x^a} = \Gamma^d{}_{ba}g_{cd} + \Gamma^d{}_{ca}g_{bd} \tag{6.5}$$

$$\frac{\partial g_{ca}}{\partial x^b} = \Gamma^d{}_{cd}g_{ad} + \Gamma^d{}_{ab}g_{cd} \tag{6.6}$$

$$2\Gamma^d{}_{ac}g_{bd} = \frac{\partial g_{ab}}{\partial x^c} + \frac{\partial g_{bc}}{\partial x^a} - \frac{\partial g_{ca}}{\partial x^b} \tag{6.7}$$

$$\Gamma^f{}_{ac} = \frac{1}{2}g^{fb}\left(\frac{\partial g_{bc}}{\partial x^a} - \frac{\partial g_{ca}}{\partial x^b} + \frac{\partial g_{ab}}{\partial x^c}\right) \tag{6.8}$$

$$= \frac{1}{2}g^{fb}\left(\partial_a g_{bc} - \partial_b g_{ca} + \partial_c g_{ab}\right) \tag{6.9}$$

We multiplied lefthandside of (6.7) by $\delta^f{}_d$.

Example: 2D flat space

$x^A = \{x, y\}$:

$$g_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{diag}(1, 1) \tag{6.10}$$

$$\Gamma^A{}_{BC} = 0 \tag{6.11}$$

So we don't have to deal with these in Cartesian coordinates. What about polar coordinates? $x^A = \{r, \theta\}$:

$$ds^2 = dr^2 + r^2\, d\theta^2 \tag{6.12}$$

$$g_{AB} = \mathrm{diag}(1, r^2) \tag{6.13}$$

$$\Gamma^A{}_{BC} \neq 0 \tag{6.14}$$

So we can still get non-zero Christoffel symbols even for flat space, but it is still "boring" really.

Let's consider something more interesting, i.e. curved. For 3D space, we have

$$ds^2 = dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \tag{6.15}$$

But we want to use just the surface of a sphere, so fixed $r = a$:

$$ds^2 = a^2\, d\theta^2 + a^2 \sin^2\theta\, d\phi^2 = g_{AB}dx^A\, dx^B \tag{6.16}$$

$$g_{AB} = \text{diag}(a^2, a^2 \sin^2 \theta) \tag{6.17}$$

We have g_{AB}, but we want g^{AB}. Recall

$$g^{AB} g_{BC} = \delta^A{}_C. \tag{6.18}$$

So we have a set of 4 simultaneous equations:

$$g^{A1} g_{1C} + g^{A2} g_{2C} = \delta^A{}_C. \tag{6.19}$$

For diagonal g_{AB} **ONLY**:

$$g^{AB} g_{BA} = g^{AA} g_{AA} = 1 \implies g^{AA} = \frac{1}{g_{AA}} \tag{6.20}$$

$$g^{AB} = \text{diag}\left(\frac{1}{a^2}, \frac{1}{a^2 \sin^2 \theta}\right) \tag{6.21}$$

So now we want to calculate

$$\Gamma^\theta{}_{\theta\theta} = \frac{1}{2} g^{\theta B} \left(\partial_\theta g_{B\theta} - \partial_B g_{\theta\theta} + \partial_\theta g_{\theta B}\right), \quad g^{\theta B} = 0, B \neq \theta \tag{6.22}$$

$$= \frac{1}{2}\frac{1}{a^2} \left(\partial_\theta g_{\theta\theta} - \partial_\theta g_{\theta\theta} + \partial_\theta g_{\theta\theta}\right) = 0 \tag{6.23}$$

$$\Gamma^\theta{}_{\phi\theta} = \Gamma^\theta{}_{\theta\phi} = \frac{1}{2} g^{\theta B} \left(\partial_\theta g_{B\phi} - \partial_B g_{\phi\theta} + \partial_\phi g_{\theta B}\right) \tag{6.24}$$

$$= \frac{1}{2} g^{\theta\theta} \left(\partial_\theta g_{\theta\phi} - \partial_\theta g_{\phi\theta} + \partial_\phi g_{\theta\theta}\right) = 0 \tag{6.25}$$

$$\Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta \tag{6.26}$$

$$\Gamma^\phi{}_{\theta\phi} = \cot\theta \tag{6.27}$$

The rest of the Christoffel symbols for this example are 0 (there are $2^3 = 8$ in total?).

6.1 Geodesic Equations

The velocity is a tensor,

$$\underline{v} = v^\alpha \underline{e}_\alpha = \frac{\partial x^\alpha}{\partial \tau} \underline{e}_\alpha \tag{6.28}$$

If there's no force, then there's no change in the velocity vector doesn't change, but its components might change. No force means the absolute derivative of the components:

$$\frac{D v^\alpha}{d\tau} = 0 \tag{6.29}$$

By an affine parameter, we mean a linear function of path length $u = A + Bs$, such as the proper time τ.

$$\frac{dv^\alpha}{d\tau} + \Gamma^\alpha{}_{\beta\gamma} v^\beta \frac{dx^\gamma}{d\tau} = 0 \tag{6.30}$$

$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha{}_{\beta\gamma} \frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0 \tag{6.31}$$

$$\ddot{x}^\alpha + \Gamma^\alpha{}_{\beta\gamma} \dot{x}^\beta \dot{x}^\gamma = 0 \tag{6.32}$$

Let's guess and make a solution for the sphere, $s = a\theta$, so we are just going around the circumference of the sphere at constant ϕ. For θ:

$$\frac{d^2\theta}{ds^2} + \Gamma^\theta{}_{BC} \frac{dx^B}{ds}\frac{dx^c}{ds} = 0 + \Gamma^\theta{}_{\phi\phi} \frac{d\phi}{ds}\frac{d\phi}{ds} = 0 \tag{6.33}$$

We get a big tick and a gold star! For ϕ:

$$\frac{d^2\phi}{ds^2} + \Gamma^\phi{}_{BC} \frac{dx^B}{ds}\frac{dx^C}{ds} = 0 + \Gamma^\phi{}_{\theta\phi} \frac{d\theta}{ds}\frac{d\phi}{ds} + \Gamma^\phi{}_{\phi\theta} \frac{d\phi}{ds}\frac{d\theta}{ds} = 0 \tag{6.34}$$

So it's a geodesic path! Yayyyyyy!

Lecture 7

Lecture 8

Last lecture:

- ➤ Euler-Lagrange equations
- ➤ 'easier' way to find ${\Gamma^a}_{bc}$
- ➤ how to find Geodesic paths

This lecture:

- ➤ more tensor derivatives

Recall absolute derivative again:

$$\frac{D\lambda^a}{ds} = \frac{d\lambda^a}{ds} + {\Gamma^a}_{bc}\lambda^b \frac{dx^c}{ds}. \tag{8.1}$$

The first term above is the total change, and then the second is to "subtract off the change due to the coordinate system". So for parallel transport, this means there is no physical change, i.e.

$$\frac{D\lambda^a}{ds} = 0. \tag{8.2}$$

The absolute derivative obeys normal rules for derivatives.

- ➤ Linear operator -

$$\frac{D}{ds}\left(\lambda^a + k\mu^a\right) = \frac{D\lambda^a}{ds} + k\frac{D\mu^a}{ds}. \tag{8.3}$$

- ➤ The (Leibniz) chain rule -

$$\frac{D}{ds}\left(\lambda^a\mu^b\right) = \mu^b\frac{D\lambda^a}{ds} + \lambda^a\frac{D\mu^b}{ds}. \tag{8.4}$$

What is the absolute derivative of a scalar, ϕ? ϕ does not depend on the coordinates as tensors do, so it would just be the normal derivative, i.e.

$$\frac{D\phi}{ds} = \frac{d\phi}{ds}. \tag{8.5}$$

We have defined the absolute derivative of a contravariant tensor, but now what about a covariant tensor μ_a? We can write a scalar as $\phi = \lambda^a\mu_a$, so we can write

$$\frac{D\phi}{ds} = \frac{D}{ds}(\lambda^a\mu_a) = \mu_a\frac{D\lambda^a}{ds} + \lambda^a\frac{D\mu_a}{ds}, \tag{8.6}$$

$$\frac{d\phi}{ds} = \mu_a\left(\frac{d\lambda^a}{ds} + {\Gamma^a}_{bc}\lambda^b\frac{dx^c}{ds}\right) + \lambda^a\frac{D\mu_a}{ds}, \tag{8.7}$$

$$\lambda^a\frac{d\mu_a}{ds} = \mu_a{\Gamma^a}_{bc}\lambda^b\frac{dx^c}{ds} + \lambda^a\frac{D\mu_a}{ds}, \tag{8.8}$$

$$= \mu_b{\Gamma^b}_{ac}\lambda^a\frac{dx^c}{ds} + \lambda^a\frac{D\mu_a}{ds}, \tag{8.9}$$

where λ^a is any tensor, so if this is true, it must be true for any λ^a. We can then 'cancel' λ^a through unity, as the remaining equation must also be true:

$$\frac{d\mu_a}{ds} = {\Gamma_{ac}}^b\mu_b\frac{dx^c}{ds} + \frac{D\mu_a}{ds}, \tag{8.10}$$

$$\frac{D\mu_a}{ds} = \frac{d\mu_a}{ds} - {\Gamma_{ac}}^b\mu_b\frac{dx^c}{ds}. \tag{8.11}$$

This is the absolute derivative of a convariant tensor.

What is the absolute derivative of a rank (1,1) tensor $\tau^a{}_b = \lambda^a \mu_b$?

$$\frac{D\tau^a{}_b}{ds} = \frac{D(\lambda^a \mu_b}{ds} = \mu_b \frac{D\lambda^a}{ds} + \lambda^a \frac{D\mu_b}{ds}, \tag{8.12}$$

$$= \mu_b \left[\frac{d\lambda^a}{ds} + \Gamma^a{}_{dc} \lambda^d \frac{dx^c}{ds} \right] + \lambda^a \left[\frac{d\mu_b}{ds} - \Gamma_{bc}{}^d \mu_d \frac{dx^c}{ds} \right], \tag{8.13}$$

$$\frac{D\tau^a{}_b}{ds} = \frac{d}{ds}(\lambda^a \mu_b) + \Gamma^a{}_{dc} \tau^d{}_b \frac{dx^c}{ds} - \Gamma_{bc}{}^d \tau_d{}^a \frac{dx^c}{ds}. \tag{8.14}$$

This is the absolute derivative for a rank (1,1) tensor.

What about the covariant derivative? It is defined as:

$$\lambda^a{}_{;c} = \frac{\partial \lambda^a}{\partial x^c} + \Gamma^a{}_{bc} \lambda^b \tag{8.15}$$

$$\frac{D\lambda^a}{ds} = \lambda^a{}_{;c} \frac{dx^c}{ds} \tag{8.16}$$

$$= \frac{\partial \lambda^a}{\partial x^c} \frac{dx^c}{ds} + \Gamma^a{}_{bc} \lambda^b \frac{dx^c}{ds} = \frac{d\lambda^a}{ds} + \dots. \tag{8.17}$$

All the rules still apply, so we can write out the covariant derivative for a scalar, a covariant, and a higher order tensor:

$$\phi_{;c} = \frac{\partial \phi}{\partial x^c}, \tag{8.18}$$

$$\mu_{a;c} = \frac{\partial \mu + a}{\partial x^c} - \Gamma_{ac}{}^b \mu_b, \tag{8.19}$$

$$\lambda_{ab;c} = \frac{\partial \lambda_{ab}}{\partial x^c} - \Gamma_{ac}{}^d \lambda_{db} - \Gamma_{bc}{}^d \lambda_{ad}. \tag{8.20}$$

We can also consider the metric:

$$g_{ab;c} = ? \tag{8.21}$$

$$\frac{\partial g_{ab}}{\partial x^c} = \frac{\partial}{\partial x^c} (\underline{e}_a \cdot \underline{e}_b), \tag{8.22}$$

$$= \frac{\partial \underline{e}_a}{\partial x^c} \cdot \underline{e}_b + \underline{e}_a \cdot \frac{\partial \underline{e}_b}{\partial x^c}, \tag{8.23}$$

$$= \left(\Gamma_{ac}{}^d \underline{e}_d \right) \cdot \underline{e}_b + \underline{e}_a \cdot \left(\Gamma_{bc}{}^d \underline{e}_d \right), \tag{8.24}$$

$$= \Gamma_{ac}{}^d g_{db} + \Gamma_{bc}{}^d g_{ad}, \tag{8.25}$$

$$g_{ab;c} = \frac{\partial g_{ab}}{\partial x^c} - \Gamma_{ac}{}^d g_{db} - \Gamma_{bc}{}^d g_{ad} = 0, \tag{8.26}$$

where we used the previous definitions of the covariant derivative to find this definition. Why is this important? The metric allows us to switch coordinate systems as

$$R_a = g_{ab} R^b. \tag{8.27}$$

Now suppose we want to find the covariant derivative:

$$R_{a;c} = \cancel{g_{ab;c} R^b} + g_{ab} R^b{}_{;c} \tag{8.28}$$

$$= g_{ab} R^b{}_{;c}. \tag{8.29}$$

Lecture 9

9.1 The Riemann Curvature Tensor

- how do we know if space is curved?
- second derivatives of the metric
- 'space-sing' around a loop
- convergence of geodesic paths

The Riemann curvature tensor tells us *how much the direction of a vector changes as it goes round a loop*, or the *tidal forces of gravity.*

$$\left(\lambda^a{}_{;b}\right)_{;c} = \frac{\partial \lambda^a{}_{;b}}{\partial x^c} + \Gamma^a{}_{ec}\lambda^e{}_{;b} - \Gamma_{bc}{}^f \lambda_{;f}{}^a \tag{9.1}$$
$$= \frac{\partial}{\partial x^c}\left(\frac{\partial \lambda^a}{\partial x^b} + \Gamma^a{}_{bd}\lambda^d\right) + \cdots \tag{9.2}$$
$$\lambda^a{}_{;b;c} = \frac{\partial^2 \lambda^a}{\partial x^c \partial x^b} + \Gamma^d{}_{bd}\frac{\partial \lambda^d}{\partial x^c} + \lambda^d \frac{\partial \Gamma^a{}_{bd}}{\partial x^c} + \Gamma^a{}_{ec}\lambda^a{}_{;b} - \Gamma_{bc}{}^f \lambda^a{}_{;f} \tag{9.3}$$

In flat space, the Christoffel symbols go to zero, so we can sway $;b$ and $;c$ indices, but **not in curved space.**

$$\lambda^a{}_{;c;b} = \frac{\partial^2 \lambda^a}{\partial x^b \partial x^c} + \Gamma^a{}_{cd}\frac{\partial \lambda^d}{\partial x^b} + \lambda^d \frac{\partial \Gamma^a{}_{cd}}{\partial x^b} + \Gamma^a{}_{eb}\lambda^e{}_{;c} - \Gamma_{cb}{}^f \lambda^a{}_{;f} \tag{9.4}$$
$$\lambda^a{}_{;c;b} - \lambda^a{}_{;b;c} = \left(\Gamma^a{}_{cd}\frac{\partial \lambda^d}{\partial x^b} - \Gamma^a{}_{bd}\frac{\partial \lambda^d}{\partial x^c}\right) + \lambda^d\left(\frac{\partial \Gamma_{cd}{}^a}{\partial x^b} - \frac{\partial \Gamma_{bd}{}^a}{\partial x^c}\right) + \left(\Gamma^a{}_{eb}\lambda^e{}_{;c} - \Gamma^a{}_{ec}\Gamma^e{}_{;b}\right) \tag{9.5}$$
$$= \Gamma^a{}_{cd}\left(\frac{\partial \lambda^a}{\partial x^b} - \lambda^d{}_{;b}\right) + \Gamma^a{}_{bd}\left(\lambda^d{}_{;c} - \frac{\partial \lambda^d}{\partial x^c}\right) + \lambda^d\left(\frac{\partial \Gamma^a{}_{cd}}{\partial x^b} - \frac{\partial \Gamma^a{}_{bd}}{\partial x^c}\right) \tag{9.6}$$
$$= \left(\Gamma^a{}_{be}\Gamma^e{}_{cd} - \Gamma^a{}_{ce}\Gamma^e{}_{bd} + \frac{\partial \Gamma^a{}_{cd}}{\partial x^b} - \frac{\partial \Gamma^a{}_{bd}}{\partial x^c}\right)\lambda^d = R^a{}_{dbc}\lambda^d \tag{9.7}$$

We have arrived at the Riemann curvature tensor. Consider: *doodle diagram*

$$\lambda^a(B^1) = \lambda^a(A) + \lambda^a{}_{;b}\delta x^b + \lambda^a{}_{;c}\delta y^c + \lambda^a{}_{;b;c}\delta x^b \delta y^c \tag{9.8}$$
$$\lambda^a(B^2) = \lambda^a(A) + \lambda^a{}_{;c}\delta y^c + \lambda^a{}_{;b}\delta x^b + \lambda^a{}_{;c;b}\delta y^c \delta x^b \tag{9.9}$$
$$\Delta\lambda^a = \lambda^a(B^2) - \lambda^a(B^1) = \left(\lambda^a{}_{;c;b} - \lambda^a{}_{;b;c}\right)\delta x^b \delta y^c \tag{9.10}$$
$$= R^a{}_{dbc}\lambda^a \cdot \text{ area of loop} \tag{9.11}$$

In flat space, any two lines have a separation that increases linearly with distance s, e.g. $\partial_s^2 = 0$. But in curved space, our two lines can converge or diverge as they travel from initial parallel conditions. Consider two lines $x^a(s)$ and $\tilde{x}^a(s)$ with a separation $\zeta^a = \tilde{x}^a - x^a$. We can write our geodesic equation for this as

$$\frac{d^2 x^a}{ds^2} + \Gamma^a{}_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds} = 0 \tag{9.12}$$
$$\frac{d^2 \tilde{x}^a}{ds^2} + \tilde{\Gamma}^a{}_{bc}\frac{d\tilde{x}^b}{ds}\frac{d\tilde{x}^c}{ds} = 0 \tag{9.13}$$
$$\ddot{\zeta}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{\zeta}^c + \Gamma^a{}_{bc}\dot{\zeta}^b\dot{x}^c + \frac{\partial \Gamma^a{}_{bc}}{\partial x^d}\zeta^d\dot{x}^b\dot{x}^c = 0 \tag{9.14}$$
$$\frac{D^2\zeta^a}{ds^2} = \frac{D}{ds}\left(\dot{\zeta}^a + \Gamma^a{}_{bc}\zeta^b\dot{x}^c\right) \tag{9.15}$$
$$= \ddot{\zeta}^a + \Gamma^a{}_{bc}\zeta^b\dot{x}^c + \frac{d}{ds}\left(\Gamma^a{}_{bc}\zeta^b\dot{x}^c\right) + \Gamma^a{}_{ef}\left(\Gamma^e{}_{bc}\zeta^b\dot{x}^c\right)\dot{x}^f \tag{9.16}$$

Lecture 10

Things are about to get a little more physics! - Richard Bower.
In this lecture:

- ➤ Geodesic convergence.
- ➤ Symmetry of the Riemann tensor,

$$R^a{}_{dbc} = \Gamma^a{}_{bc}\Gamma^e{}_{cd} - \Gamma^a{}_{ce}\Gamma^e{}_{bd} + \partial_b\Gamma^a{}_{cd} - \partial_c\Gamma^a{}_{bd}. \tag{10.1}$$

- ➤ We will look towards Einstein's equations by combining our knowledge of movement in curved space with gravity.
- ➤ We will do this through the stress-energy tensor, $T^{\mu\nu}$. (Not as simple as it could be, R has four indices, this has two, but we'll get to that.)
- ➤ This will lead towards to conservation laws in GR.

At the end of last lecture, we considered divergent lines with a spatially-dependent separation ζ:

$$\ddot{x}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{x}^c = 0 \tag{10.2}$$
$$\ddot{\tilde{x}}^a + \tilde{\Gamma}^a{}_{bc}\dot{x}^b\dot{x}^c = 0 \tag{10.3}$$
$$\ddot{\zeta} + \Gamma^a{}_{bc}\dot{x}^b\dot{\zeta}^c + \Gamma^a{}_{bc}\dot{\zeta}^b\dot{x}^c + \frac{\partial\Gamma^a{}_{bc}}{\partial x^d}\zeta^d\dot{x}^b\dot{x}^c = 0. \tag{10.4}$$

We want to find the Tensor equation from this!

$$\frac{D^2\zeta^a}{ds^2} = \ddot{\zeta}^a + \Gamma^a{}_{bc}\dot{\zeta}^b\dot{x}^c + \frac{d}{ds}\left(\Gamma^a{}_{bc}\zeta^b\dot{x}^c\right) + \Gamma^a{}_{ef}\left(\Gamma^a{}_{bc}\zeta^b\dot{x}^c\right)\dot{x}^f. \tag{10.5}$$
$$\frac{d}{ds}\left(\Gamma^a{}_{bc}\zeta^b\dot{x}^c\right) = \Gamma^a{}_{be}\zeta^b\ddot{x}^e + \Gamma^a{}_{bc}\dot{\zeta}^b\dot{x}^c + \zeta^b\dot{x}^c\left(\frac{\partial\Gamma^a{}_{bc}}{\partial x^d}\dot{x}^d\right). \tag{10.6}$$

So we expanded out the derivative to get rid of something we didn't want, and now combine for the final result (as well as using the definition of a geodesic path, $\ddot{x}^e = -\Gamma^e{}_{cd}\dot{x}^c\dot{x}^d$, in Eq. (10.5)):

$$\frac{D^s\zeta^a}{ds^2} + \left(\Gamma^a{}_{be}\Gamma^e{}_{cd} - \Gamma^a{}_{ce}\Gamma^e{}_{bd} + \partial_b\Gamma^a{}_{cd} - \partial_c\Gamma^a{}_{bd}\right)\zeta^b\dot{x}^c\dot{x}^d = 0. \tag{10.7}$$

This looks very confused as it is, but we can simplify by now defining the Riemann tensor:

$$\frac{D^2\zeta}{ds^2} + R^a{}_{dbc}\zeta^b\dot{x}^c\dot{x}^d = 0, \tag{10.8}$$
$$R^a{}_{dbc} = \Gamma^a{}_{bc}\Gamma^e{}_{cd} - \Gamma^a{}_{ce}\Gamma^e{}_{bd} + \partial_b\Gamma^a{}_{cd} - \partial_c\Gamma^a{}_{bd}. \tag{10.9}$$

From this definition, we can quickly infer the symmetry relation

$$R^a{}_{dcb} = -R^a{}_{dbc}. \tag{10.10}$$

10.1 Symmetries of the Riemann tensor

In 4D space-time, the Riemann tensor contains $4 \times 4 \times 4 \times 4 = 256$ numbers. This isn't very fun unless you are a computer, says Richard. However, using the symmetry of the Riemann tensor, there are only 20 indepedent elements, which can be a lot easier. How do we demonstrate this though? We can use mathematical tricks for tensors, whereby if we prove something in one coordinate system, it must be true in all coordinate systems.

10.1.1 Local Geodesic Coordinates

Local geodesic coordinates describe a flat space at a local point in curved space. We know, therefore, that we can write the metric at this point as

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{10.11}$$

which is known as the **Minkowski metric** for flat space. This is familiar from special relativity, where it represents the usual Minkowski product,

$$ds^2 = c^2\, d\tau^2 = c\, dt^2 - dx^2 - dy^2 - dz^2. \tag{10.12}$$

So the Christoffel symbol at this local point is

$$\Gamma^{\alpha}{}_{\beta\gamma} = 0, \tag{10.13}$$

where we have only chosen a point to set this locally to zero. However, the derivatives will not be zero, as if we move away from this local point, space will become curved again:

$$\frac{\partial \Gamma^{\alpha}{}_{\beta\gamma}}{\partial x^{\delta}} = 0. \tag{10.14}$$

Now this will allow us to simplify the Riemann tensor from its definition in Eq. (10.9), where it can now be written as

$$R^{a}{}_{dbc} = \partial_b \Gamma^{a}{}_{cd} - \partial_c \Gamma^{a}{}_{bd}. \tag{10.15}$$

What about the symmmetry relations of this simplified Riemann tensor?

$$R^{a}{}_{cdb} = \partial_d \Gamma^{a}{}_{bc} - \partial_b \Gamma^{a}{}_{dc}, \tag{10.16}$$
$$R^{a}{}_{bcd} = \partial_c \Gamma^{a}{}_{db} - \partial_d \Gamma^{a}{}_{cb}. \tag{10.17}$$

What about the sum of these three tensors? Due to the symmetry of the Christoffel symbols' lower indicies, we can see that there will be lots of cancellations in this sum, leading to

$$R^{a}{}_{dbc} + R^{a}{}_{cdb} + R^{a}{}_{bcd} = 0, \tag{10.18}$$

which is known as the cyclic relation of the Riemann tensor. The Riemann tensor can also be written in its covariant form:

$$R_{abcd} = g_{ae} R^{e}{}_{bcd}. \tag{10.19}$$

In its covariant form, we can also go through similar steps to above to prove other symmetry relations, including:

$$R_{abcd} = -R_{bacd}, \qquad R_{abcd} = -R_{abdc}, \qquad R_{abcd} = R_{cdab}. \tag{10.20}$$

From these relations, we see that repeated indices can lead to zero, such as

$$R_{aacd} = -R_{aacd} \implies R_{aacd} = 0, \tag{10.21}$$
$$R_{abcc} = -R_{abcc} \implies R_{abcc} = 0. \tag{10.22}$$

This allows us to neglect many of the 256 numbers as they will be zero, reducing the total of free elements of the Riemann tensor to 20.

10.2 How can we relate this to gravity?

Let's consider Eq. (10.4) for low speeds, i.e. non-relativistic.

$$\zeta^\alpha = \begin{pmatrix} 0, & 0, & y, & 0 \end{pmatrix}, \tag{10.23}$$

i.e. we have a simple separation in the y-direction between the two geodesic paths. We would then get something like

$$\frac{D^y}{ds^2} = -R^2{}_{020} y c^2, \tag{10.24}$$

where we only consider the indices relating to y and time. This should look somewhat familiar. What would we have written for this in Newtonian physics, if we have these two objects both moving towards a large mass M with a separation y and a distance from the massive object r?

$$\frac{d^2y}{dt^2} = -\frac{GM}{r^2} \cdot \frac{y}{r}. \tag{10.25}$$

Important note: the massive object is the Sun or a molecular cloud or something, according to Richard. **A black hole wouldn't be happy, you wouldn't be able to tell.**

Using $ds = c\,dt$:

$$c^2 \frac{d^2y}{ds^2} = -\frac{GM}{r^3} y, \tag{10.26}$$

$$-\frac{GM}{r^3} = -R^2{}_{020}. \tag{10.27}$$

So we now have an equivalence of curved space and Newtonian gravitational force. We can see that the Newtonian expression in Eq. (10.27) is the energy density of space, if mass and energy can be equivalent. This is what lead Einstein to his famous equations.

10.3 What is energy?

Energy is special relativity was defined in

$$E = \gamma m_0 c^2, \tag{10.28}$$

where we can now relate the mass and energy of objects. We also have a number density, n_0, for the number of particles "in a box" or in the universe or whatever you're considering. Recall from special relativity the four-velocity of a system:

$$u^\alpha = \frac{dx^\alpha}{dt} = \gamma \begin{pmatrix} c, & -\frac{dx^i}{dt} \end{pmatrix}. \tag{10.29}$$

There is also the four-momentum of the system:

$$p^\alpha = m_0 u^\alpha = \gamma m_0 \begin{pmatrix} c, & -\frac{dx^i}{dt} \end{pmatrix}. \tag{10.30}$$

The number per unit volume is γn_0. We also have a flux through the surface,

$$N^\rho = n_0 u^\rho. \tag{10.31}$$

Using all this, we can write down an expression for the energy-momentum density:

$$T^{\mu\nu} = N^\mu p^\nu = n_0 u^\mu m_0 u^\nu = \rho_0 u^\mu u^\nu. \tag{10.32}$$

In the rest frame, these two velocities will each be c, so the T^{00} term will get us back to Eq. (10.28) as $\rho_0 c^2$.