Atoms, Lasers, and Qubits

Dr Weatheril and Prof Adams

Michaelmas Term 2019 - Epiphany Term 2020

Contents

Part I

LASER PHYSICS

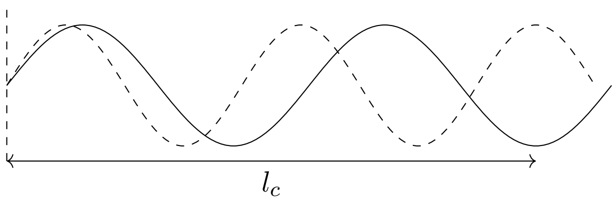
Lecture 1 Introduction

- **Note:** The course will be more reading based than math based. Read the references on each summary sheet.
- Need light oscillation, not just amplification.
- 1 in 10^{18} atomic clock accuracy.
- LD: laser diode
- Non-linear crystals allow different wavelengths
- Laser transitions based on E group of materials
- Learn the unit conversions
- Magneto-optical trap to cool atoms
- Optical frequency comb → accurate measurement of wavelength of light
- Sodium atoms in upper atmosphere which we fluoresce for AO
- **LEARN!** Q on paper always - contents of a laser
  1. More in excited state than ground
  2. Pump gets energy in
  3. Mirrors to make light bounce back and forward

1.1 Introduction to lasers

Lasers - coherence → 2 types - longitudinal and transverse

- Lasers are highly coherent, both transversely and longitudinally. Longitudinal and temporal coherence is related to linewidth, and will be discussed. Coherence length l_c and coherence time τ_c are the distance and time over which a coherent wave maintains a specified degree of coherence, i.e. when its phase is predictable.



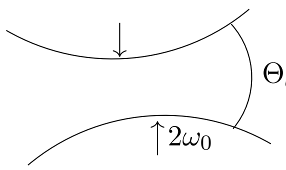
Coherence length and time:

$$l_c = \frac{2\pi c}{\delta\omega}, \quad \tau_c = \frac{2\pi}{\delta\omega} \tag{1.1}$$

- Can't have an infinitely narrow spectrum. Monochromaticity - laser has a spectral linewidth $\delta\omega$, this is much smaller than the actual carrier/centre frequency. $\delta\omega \ll \omega_0$ for a laser where ω_0 is the centre frequency. From mHz to GHz in range.
- Highly directional beam → energy contained in one region.

Directionality:

- Lasers have highly directional beams that diverge due to diffraction
- Beam will be larger
- Waist of beam, $2\omega_0$.



- All in a very low frequency range → all energy oscillating in small aarea in small frequency range - useful applications.

Brightness:

- Lasers are spectrally bright

- Definition of brightness - amount of power in particular area (solid angle) of the beam:

$$B_\omega = \frac{P}{A\Delta\Omega\Delta\omega}, \tag{1.2}$$

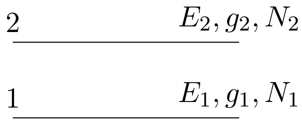
where A is the area, $\Delta\Omega$ is the solid angle, and $\Delta\omega$ is the linewidth.

Electromagnetic Field Modes - not examinable:

- 1st Chapter of 'Laser Physics' book
- Planck's radiation law
- Each unique solution of field is EM mode
- L^3 factored out when divided by volume
- Modes exist with or without energy

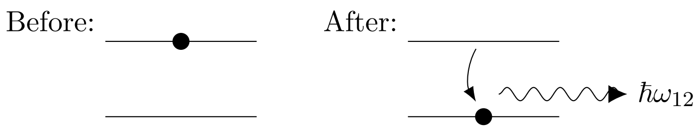
Lecture 2 Einstein's rate equations

A laser requires amplification due to stimulated emission of radiation.

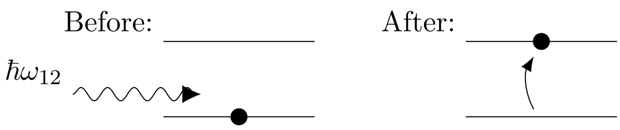


$$E_2 - E_1 = \hbar\omega_{12} \tag{2.1}$$

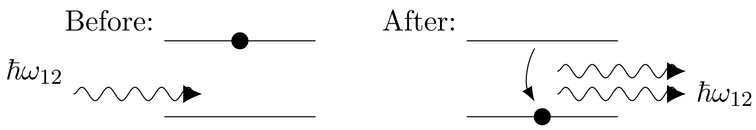
1. Spontaneous emission: Atom in some excited state, until some time later where it spontaneously decays into a lower state, with a photon emitted with energy shown in Eq (2.1). Rates: A_{21} per N_i atoms, or $A_{21}N_2$ per m^3.



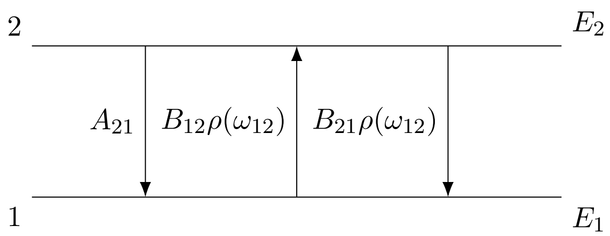
2. Absorption: Excite into excited state using energy of photon. Rates: $B_{12}\rho(\omega_{12})$, $B_{12}\rho(\omega_{12})N_1$.



3. Stimulated emission: At the initial time we have an atom in an excited state which we then apply a radiated field (photon) to. Later, the atom will decay into a lower state and there will be two outgoing photons - they have been emitted into the same mode. Rates: $B_{21}\rho(\omega_{12})$, $B_{21}\rho(\omega_{12})N_2$. Note:
   - Here, $\rho(\omega_{12})$ is the spectral energy density - the energy density, per unit angular frequency range at ω, with units of $J\,m^{-3}\,s$.
   - Generally, the light being used here is broad-band.



2.1 Apply the 3 processess



Conservation of atom number:

$$\frac{dN_2}{dt} = -\frac{dN_1}{dt} \tag{2.2}$$

$$N_1 + N_2 = N = \text{const} \tag{2.3}$$

$$N_1 B_{12}\rho(\omega_{12}) = N_2 A_{21} + N_2 B_{21}\rho(\omega_{12}) \tag{2.4}$$

Rearrange for the spectral energy density:

$$\rho(\omega_{12}) = \frac{N_2 A_{21}}{N_1 B_{12} - N_2 B_{21}} = \frac{\frac{A_{21}}{B_{21}}}{\frac{N_1}{N_2}\frac{B_{12}}{B_{21}} - 1} \tag{2.5}$$

Substitute using Boltzmann Law:

$$\frac{N_2}{N_1} = \frac{g_2}{g_1} \exp\left(-\frac{\hbar\omega_{12}}{k_B T}\right) \tag{2.6}$$

$$\implies \rho(\omega_{12}) = \frac{\frac{A_{21}}{B_{21}}}{\frac{g_1}{g_2}\frac{B_{12}}{B_{21}} \exp\left(\frac{\hbar\omega_{12}}{k_B T}\right) - 1} \tag{2.7}$$

Now look at Planck's Law:

$$\rho(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{\exp\left(\frac{\hbar\omega}{k_B T}\right) - 1} \tag{2.8}$$

Einstein realised there must be an extra condition to switch between these two forms. This reveals:

$$g_1 B_{12} = g_2 B_{21} \tag{2.9}$$

$$A_{21} = \frac{\hbar\omega_{12}^3}{\pi^2 c^3} B_{21} \tag{2.10}$$

Notes:

- Effectively only 1 coefficient as if we know A, we know B.
- A_{21} is the radiative decay rate,

$$A_{21} = \frac{1}{\tau_2} \tag{2.11}$$

- B has units $m^3\, J^{-1}\, s^{-2}$.
- A and B are constants for a particular atom.
- From the ω^3 term, we can see that an infrared transition may decay very fast, but a microwave transition may decay very slow.
- Ratio of $A/B \propto \omega^3$ - lasers at high frequency harder to achieve.
- The principle of detailed balance states that **in equilibrium, the total number of particles entering a quantum state by a particular rate per unit time is the same as the number leaving by the same rate.**

2.2 Steady State Solution

For simplicity, we will assume that $g_1 = g_2$ such that the B coefficients are the same.

$$N_1 B_{12} \rho(\omega_{12}) = N_2 A_{21} + N_2 B_{12} \rho(\omega_{12}) \tag{2.12}$$

$$N_1 = N - N_2 \tag{2.13}$$

Now we can rearrange to eliminate N_1.

$$\frac{N_2}{N} = \frac{B_{12}\rho(\omega_{12})}{A_{21} + 2B_{12}\rho(\omega_{12})} \tag{2.14}$$

Now consider the form of the above as the spectral energy density tends to infinity. What we see is that $N_2 \to \frac{N}{2}$. This tells us that for at least a two-level atom we cannot get population inversion, which is required for lasers, i.e. steady state inversion impossible.

2.3 Number of photons per mode

$\bar{n}$ can be thought of as the mean number of photons per mode, and $g(\omega)\, d\omega$ as the mode density.

$$\rho(\omega)\, d\omega = \bar{n} \times g(\omega)\, d\omega \times \hbar\omega \tag{2.15}$$

Standard result:

$$g(\omega)\, d\omega = \frac{\omega^2}{\pi^2 c^3}\, d\omega \tag{2.16}$$

$$\implies \bar{n} = \frac{\rho(\omega)}{\hbar\omega g(\omega)} = \frac{\pi^2 c^3}{\hbar\omega^3}\rho(\omega) \tag{2.17}$$

$$\bar{n} = \frac{B_{21}\rho(\omega)}{A_{21}} = \frac{\text{rate of stimulated emission}}{\text{rate of spontaneous emission}} \tag{2.18}$$

It follows that:

- $\bar{n} > 1$ - stimulated emission dominates $\implies$ LASERS
- $\bar{n} < 1$ - spontaneous emission dominates $\implies$ classical light source

For a black body:

$$\bar{n} = \frac{1}{\exp\left(\frac{\hbar\omega_{12}}{k_B T}\right) - 1} = \frac{B_{12}\rho(\omega)}{A_{21}} \tag{2.19}$$

These rates are equal when

$$\frac{\hbar\omega_{12}}{k_B T} = \ln(2) \tag{2.20}$$

For $\lambda = 500\, nm$, $T = 41400\, K$. So for most black bodies, stimulated emission is negligible.

Lecture 3 Linewidths and Lineshapes

2 types of broadening:

- Homogeneous - all atoms in sample affected the same
- Inhomogeneous - atoms in sample affected differently

3.1 Homogeneous Broadening

Every atom/molecule exhibits this type of broadening to varying magnitudes.

Radiative (natural) broadening:

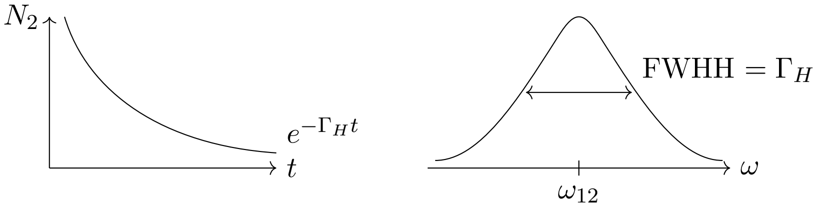
- Consider some excited state population, N_2. $e^{-\Gamma_H t}$ Leads to some spread in maximum.

$$FWHM = \Gamma_H \tag{3.1}$$

This broadening follows from the Heisenberg uncertainty principle: Finite lifetime $\implies$ spread in energy, $\Delta E \Delta t \approx \hbar/2$, then a spread in energy $\implies$ a spread in frequency,

$$\Delta E = \hbar \Delta\omega \implies \Delta\omega = \frac{1}{\tau_2} = A_{21} = \Gamma_H \tag{3.2}$$

This is only for a 2 level atom.

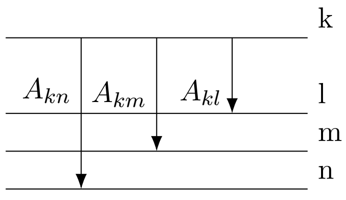


- Define the normalised lineshape function:

$$L_H(\omega) = \frac{\Gamma_H/2\pi}{(\omega-\omega_{12})^2 + \frac{\Gamma_H^2}{4}} \tag{3.3}$$

$$\int_0^\infty L_H(\omega)\, d\omega = 1 \tag{3.4}$$

- For multiple decay paths:

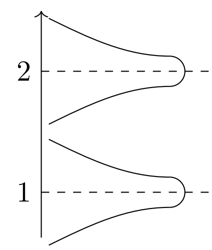


For level k

$$\frac{dN_k}{dt} = -N_k A_{kl} - N_k A_{km} - N_k A_{kn} \tag{3.5}$$

$$\tau_k = \frac{1}{\sum A_{ki}} = \frac{1}{\Gamma_H} \tag{3.6}$$

- When both levels decay:



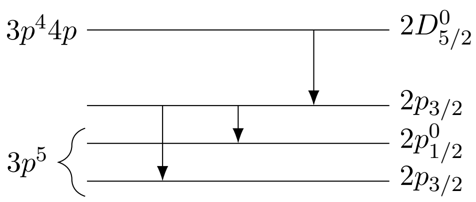
$$\Gamma_2 = \sum A_{2i} = \frac{1}{\tau_2} \tag{3.7}$$

$$\Gamma_1 = \sum A_{1i} = \frac{1}{\tau_1} \tag{3.8}$$

$$\Gamma_{21} = \Gamma_2 + \Gamma_1 \tag{3.9}$$

Γ_{21} is the emission linewidth.

Example: Argon ion laser



$\lambda = 488\,nm$, $A = 7.8 \times 10^7\ s^{-1}$, $1:\ 73,1\ nm;\ A = 4.5 \times 10^8\ s^{-1}$, $2:\ 72.3\ nm;\ A = 23 \times 10^8\ s^{-1}$.
Emission linewidth:

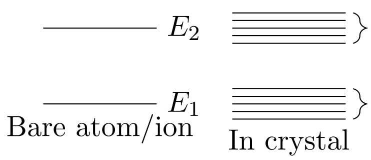
$$\sum A = 2.8 \times 10^9 = (2\pi)450\ MHz \tag{3.10}$$

Collisional/pressure broadening:

- Collisions between atoms - de-excite the atoms to reduce excited state lifetime which broadens transition
- Depends on pressure - important for gas lasers

Phonon broadening:

- In a crystal, two distinct groups of energy levels packed together, quantised vibrational modes $\implies$ phonons.
- Occurs in solid state lasers, temp-dependent
- Dominant broadening at room temperature



3.2 Inhomogeneous Broadening

Doppler Broadening:

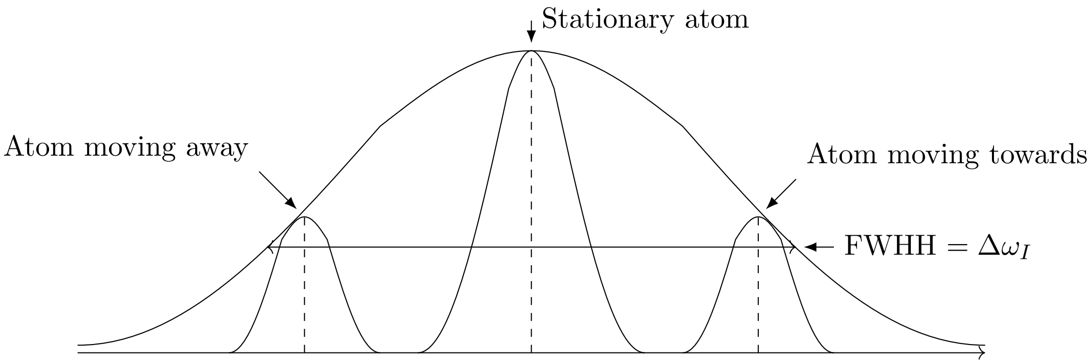
- Arises due to motion of atoms - when a moving atom emits, there is a Doppler shift dependent on the component of velocity along the direction of the emitted photon.

$$\omega = \omega_{12}\left(1 \pm \frac{v_z}{c}\right) \tag{3.11}$$

  $\pm$ for blue/red shift - blue as velocity towards observer, red away.
- Broadening arises due to Maxwell-Boltzmann distribution of velocities,

$$P(v_z)\,dv_z = \left(\frac{M}{2\pi k_B T}\right)^{1/2} \exp\left(-\frac{M v_z^2}{2k_B T}\right)\ dv_z \tag{3.12}$$

  Use correspondence between v_z and ω to get



$$P(\omega)\,d\omega = \frac{c}{\omega_{12}}\left(\frac{M}{2\pi k_B T}\right)^{1/2} \exp\left[-\frac{Mc^2}{2k_B T}\frac{(\omega-\omega_{12})^2}{\omega_{12}^2}\right] \tag{3.13}$$

➤ For Doppler broadening,

$$\Delta\omega_I = \frac{2\omega_{12}}{c}\left(\frac{2k_B T}{M}\ln 2\right)^{1/2} \tag{3.14}$$

$$= 7.16 \times 10^{-7}\omega_{12}\left(\frac{T}{M_A}\right)^{1/2} \tag{3.15}$$

$\Delta\omega_I$ is known as the Doppler width, M_A is using the mass in atomic units instead of kg as used so far for M.

Example: Argon ion laser 2

$M_A = 40$, $\lambda = 488\,nm$, Discharge temperature $\approx 1200°C$.

$$\Delta\omega_I = (2\pi)2.7\,GHz \tag{3.16}$$

This is many times larger than the natural broadening.

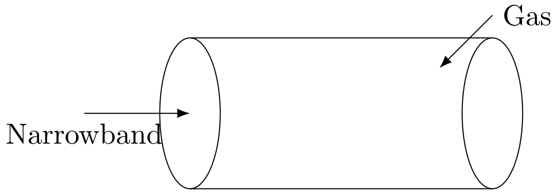
Amorphous crystal broadening:

➤ Occurs in glass materials

➤ Inhomogeneities are Gaussian (normal), the emission is also Gaussian

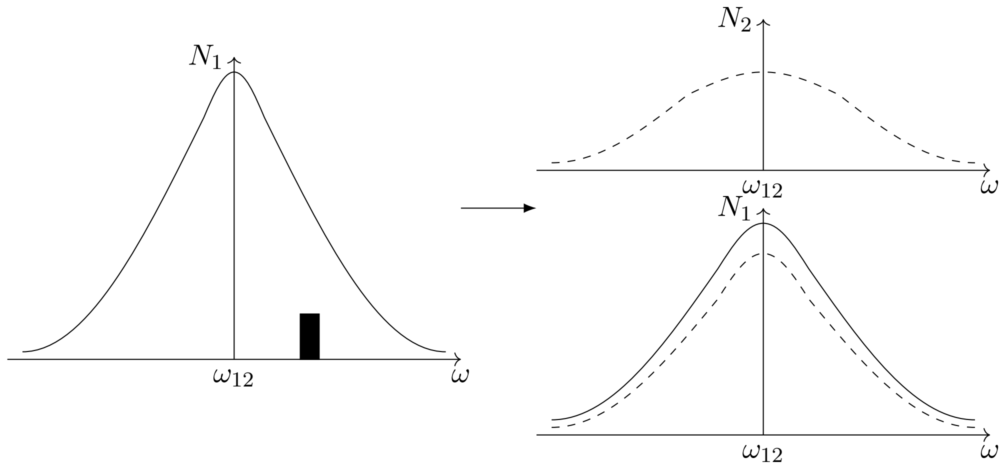
Lecture 4 Amplification by Stimulated Emission

Example:



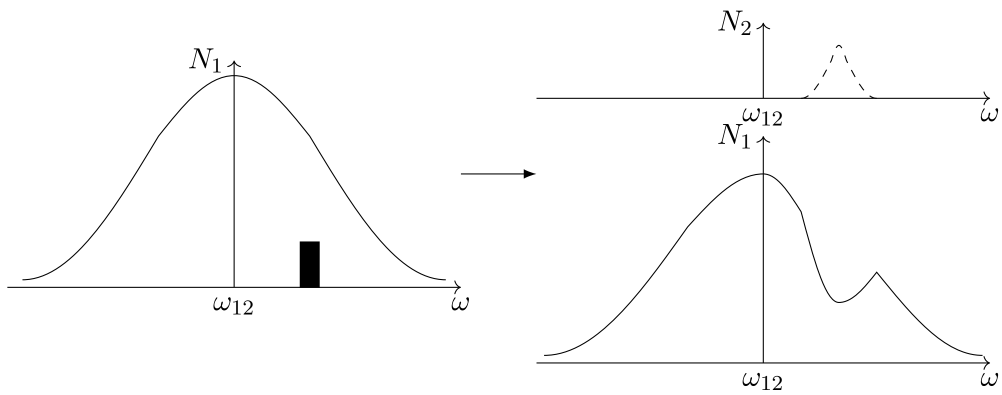
If the light beam is resonant with the atoms in the gas, we would expect some interaction. But how is the medium excited? It depends on the broadening.

1. Homogeneous broadening (Lorentzian):



   Turn on radiation within natural linewidth. All of these distributions will be the same - all atoms are affected equally, and the population of N_1 is reduced slightly as it moves into N_2.

2. Inhomogeneous broadening (Gaussian):



   Distributions are now very different - atoms are affected differently, and only a subset of the atoms interact. For example, with Doppler broadening, only particles with a specific velocity would be excited strongly by the light beam. The width of the N2 Gaussian is determined by natural broadening.

Reminder:

$$A_{21}N_2 \equiv \text{rate of spontaneous emission per } m^3 \tag{4.1}$$
$$B_{12}\rho(\omega_{12})N_1 \equiv \text{rate of absorption per } m^3 \tag{4.2}$$
$$B_{21}\rho(\omega_{12})N_2 \equiv \text{rate of stimulated emission per } m^3 \tag{4.3}$$

4.1 Homogeneous broadening

All atoms are affected the same $\implies$ simply define a new constant:

$$a_{21}(\omega) = A_{21}L_H(\omega) \qquad b_{12}(\omega) = B_{12}L_H(\omega) \qquad b_{21}(\omega) = B_{21}L_H(\omega) \tag{4.4}$$

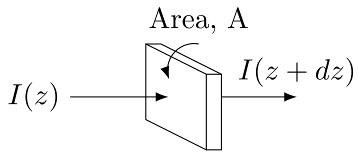
Then if we wanted to know the rate of emission now:

$$a_{21}(\omega)\, d\omega = \text{rate of emission in range } \omega \to \omega + d\omega \text{ per atom} \tag{4.5}$$

We have to multiply by the $d\omega$ because $L_H(\omega)$ has units $\frac{1}{\omega}$. Integrating recovers the result,

$$\int L_H(\omega)\, d\omega = 1 \tag{4.6}$$

Inhomogeneous broadening is more difficult. We want to do this, but we can use the results from homogeneous broadening in certain circumstances, i.e. for narrowband light and $\Gamma_H \ll \Delta\omega_I$.



Consider a weak narrowband beam of light, frequency ω and bandwidth $d\omega$.
Assume:

- No spontaneous emission
- Homogeneous broadening
- Steady state

Find change in power of beam.

$$(I(z + dz) - I(z))\, A = [N_2B_{21}L_H(\omega)\rho(\omega_{12})\, d\omega - N_1B_{12}L_H(\omega)\rho(\omega_{12})\, d\omega] \times \hbar\omega \times A\, dz \tag{4.7}$$

The term inside the square brackets above is the net rate of photons added to the field (m^{-3}).
For a beam, the intensity is described as

$$I(z) = c\rho(\omega)\, d\omega \tag{4.8}$$

Substitute this into Eq (4.7) and rearrange, also using the expressions for A and B coefficients found in lecture 2:

$$\frac{1}{I}\frac{dI}{dz} = \frac{\hbar\omega}{c}L_H(\omega)\,[N_2B_{21} - N_1B_{12}] \tag{4.9}$$
$$= \frac{\pi^2c^2}{\omega_{12}^2}A_{21}L_H(\omega)\left[N_2 - \frac{g_2}{g_1}N_1\right] \tag{4.10}$$

This uses the assumption that we are close to resonance.
The prefactor can be defined as the optical gain cross section,

$$\sigma(\omega) = \frac{\pi^2c^2}{\omega_{12}^2}A_{21}L_H(\omega). \tag{4.11}$$

Solve equation:

$$I(z) = I(0)\exp\left[\sigma(\omega)\times\left(N_2 - \frac{g_2}{g_1}N_1\right)z\right] \tag{4.12}$$

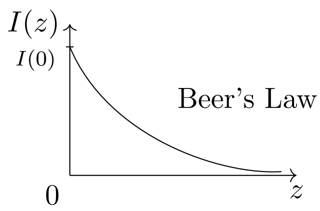
There are two cases:

1. Absorption:

$$N_2 < \frac{g_2}{g_1}N_1 \tag{4.13}$$

$$\implies I(z) = I(0)e^{-\kappa(\omega)z},\ \kappa(\omega) = \sigma(\omega)\left(\frac{g_2}{g_1}N_1 - N_2\right) \tag{4.14}$$

κ is the absorption coefficient.

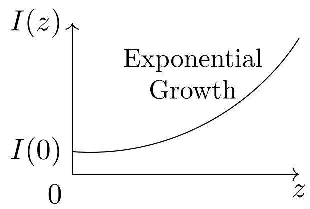


2. Amplification.

$$N_2 > \frac{g_2}{g_1}N_1 \tag{4.15}$$

$$I(z) = I(0)e^{g(\omega)z},\ g(\omega) = \sigma(\omega)\left(N_2 - \frac{g_2}{g_1}N_1\right) \tag{4.16}$$

$g(\omega)$ is the gain coefficient, $e^{g(\omega)z}$ is the gain.



The condition for gain is

$$N_2 > \frac{g_2}{g_1}N_1. \tag{4.17}$$

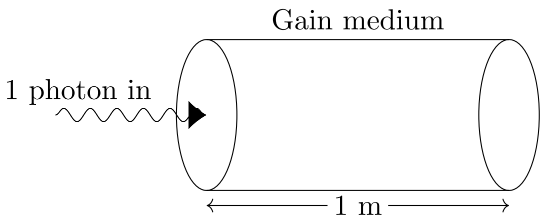
This is known as Population Inversion. Frequency dependence of the gain:

$$g(\omega) = \sigma(\omega)\times N^*,\ N^* = N_2 - \frac{g_2}{g_1}N_1 \tag{4.18}$$

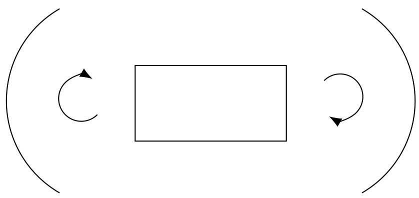
$$= \frac{\pi^2 c^2}{\omega_{12}^2}A_{21}L_H(\omega)\times\left[N_2 - \frac{g_2}{g_1}N_1\right] \tag{4.19}$$

We call N^* the population inversion density. **Note:** Frequency dependence is in the optical gain cross section via $L(\omega)$. $\sigma(\omega)$ is determined by the atom, we cannot control it. Broadening spreads gain over a range of frequencies.

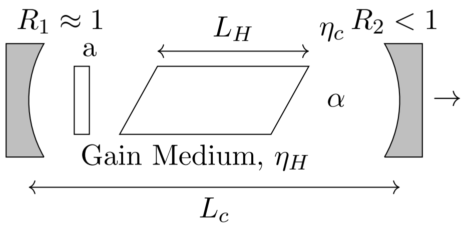
Lecture 5 The Laser Oscillator: Cavity Basics and Threshold



Imagine we have a sample which is our laser gain sample, and we will send one photon in. For most lasers, the peak gain, $g(\omega_{12}) \approx 0.01\ cm^{-1}$. So one photon in to our 1m long sample, means $e^1 = 2.7$ photons out. We must add a cavity to recirculate the light, and now using mirrors either side of the gain medium, we will pass through the gain medium 40 times, $e^{40} \approx 10^{17}$ photons.



5.1 General Cavity Design



- R_1 is mirror 1 and must have a reflectivity as close to 1 as possible
- R_2 is the output coupler and must have a lower reflectivity so that the laser can eventually escape through this
- L_c is the mirror separation, the length of the cavity
- α is the distributed loss over the length of the gain medium
- a is the intracavity element - it can make pulsed lasers or change the frequency, or induce a fixed loss
- The gain cell is angled, cut at Brewster's angle to minimise loss.
- The cavity imposes spectrum and characteristics to the laser.
- Distributed loss throughout cavity of α per meter.

5.2 Longitudinal Cavity Modes

The field in the cavity forms standing waves, i.e.

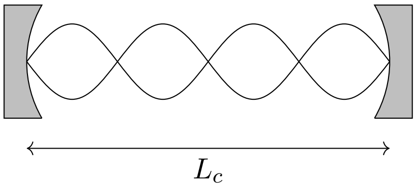
$$n\frac{\lambda_n}{2} = L_c \tag{5.1}$$

n is called the mode order. Mode separation,

$$\Delta\nu_{fsr} = \nu_{n+1} - \nu_n = \frac{c}{2L_c} \tag{5.2}$$

This is the Free Spectral Range.

Example: Mode order of 10cm cavity at $\lambda = 500\,nm$



$$n = \frac{2L_c}{\lambda} = 4 \times 10^5 \tag{5.3}$$
$$\Delta\nu_{fsr} = \frac{c}{2L_c} = 1.5\;Ghz \tag{5.4}$$

5.3 Cavity Losses

Trace intensity around the cavity:

$$I_0 \implies I_0 R_1 (1-a) R_2 (1-a) e^{-2\alpha L_c} \tag{5.5}$$

- Pass through the cavity twice, so $(1-a)$ twice.
- $e^{-2\alpha L_c}$ is the distributed loss over length, $2L_c$.

$$I_0 \implies I_0 R_1 R_2 (1-a)^2 e^{-2\alpha L_c} \tag{5.6}$$

Express as round trip loss,

$$I_0 \implies I_0 e^{-\delta_c} \tag{5.7}$$
$$\delta_c = \ln\left(\frac{1}{R_1 R_2 (1-a)^2}\right) + 2\alpha L_c \tag{5.8}$$
$$\frac{I_0 - I_0 e^{-\delta_c}}{I_0} = 1 - e^{-\delta_c} \approx 1 - (1-\delta_c) = \delta_c \tag{5.9}$$

This is the Fractional Round Trip Loss.

5.4 Cavity Lifetime

The lifetime of a photon in a cavity is a useful concept. Define cavity lifetime, τ_c.

$$I(t) = I_0 e^{-t/\tau_c} \tag{5.10}$$
$$= I_0 e^{-N\delta_c} = I_0 e^{-t\delta_c/T_{RT}} \tag{5.11}$$
$$\tau_c = \frac{2L_c}{c\delta_c} = \frac{T_{RT}}{\delta_c} \tag{5.12}$$

T_{RT} is called the Round Trip Time. Cavity has a quality factor,

$$Q = \frac{\omega}{\Delta\omega_c} = \omega\tau_c \tag{5.13}$$

So we can then think of cavities as having a linewidth, $\Delta\omega_c = \frac{1}{\tau_c}$.

5.5 Threshold Condition

$$\text{Round Trip Gain} \times \text{Round Trip Loss} = 1 \tag{5.14}$$

- Gain x loss > 1, intensity grows $\implies$ laser oscillation
- Gain x loss < 1, intensity decays $\implies$ no laser :(

Substitute for gain and loss:

$$e^{2g(\omega_{12}L_m} \times e^{-\delta_c} = 1 \tag{5.15}$$
$$g_{th}(\omega_{12}) = \frac{\delta_c}{2L_m} \tag{5.16}$$

Example: Helium Neon Laser

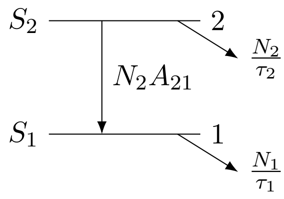
- Gain cell length $= L_m = L_c = 50\,cm$.
- Two mirrors with $R_1 = 0.998$ and $R_2 = 0.98$.
- Distributed losses are $\alpha = 0.02\,m^{-1}$.
- $A_{21} = 3.4 \times 10^6\,s^{-1},\ \lambda = 632.8\,nm$.
- Doppler width is dominant as it is a gas (could be pressure broadening, but assume not for this) - $\Delta\omega_I = 2\pi \cdot 1500\,MHz$.

So what is the population inversion density, N^*, required for laser oscillation?

- Considering round trip losses
- Fractional round trip loss
- Atomic properties for expression of gain
- Only thing left to then find is N^*

Lecture 6 The Laser Oscillator: Oscillation and Gain Saturation

6.1 Gain Saturation



Develop rate equations for N_1 and N_2.

$$\frac{dN_2}{dt} = S_2 - \frac{N_2}{\tau_2} \tag{6.1}$$

$$\frac{dN_1}{dt} = S_1 + N_2 A_{21} - \frac{N_1}{\tau_1} \tag{6.2}$$

In the steady state,

$$\frac{dN_2}{dt} = \frac{dN_1}{dt} = 0 \tag{6.3}$$

$$N_2 = S_2 \tau_2 \tag{6.4}$$

$$N_1 = S_1 \tau_1 + A_{21} \tau_1 N_2 \tag{6.5}$$

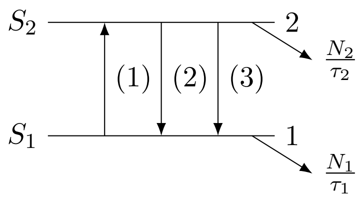
$$= S_1 \tau_1 + A_{21} \tau_1 S_2 \tau_2 \tag{6.6}$$

Population inversion density,

$$N_0^* = N_2 - \frac{g_2}{g_1} N_1 \tag{6.7}$$

$$= S_2 \tau_2 \left(1 - \frac{g_2}{g_1} A_{21} \tau_1\right) - \frac{g_2}{g_1} S_1 \tau_1 \tag{6.8}$$

'0' denotes small signal coefficient (in the absence of field).



$$(1) \quad \frac{g_2}{g_1} N_1 \sigma(\omega) \frac{I}{\hbar\omega} \tag{6.9}$$

$$(2) \quad N_2 \sigma(\omega) \frac{I}{\hbar\omega} \tag{6.10}$$

$$(3) \quad N_2 A_{21} \tag{6.11}$$

We have assumed $L(\omega) < \Gamma_{21}$.

$$\frac{dN_2}{dt} = S_2 - \frac{N_2}{\tau_2} - N^* \sigma(\omega) \frac{I}{\hbar\omega} \tag{6.12}$$

$$\frac{dN_1}{dt} = S_1 - \frac{N_1}{\tau_1} + N^* \sigma(\omega) \frac{I}{\hbar\omega} + A_{21} N_2 \tag{6.13}$$

$$N_2 = S_2 \tau_2 - N^* \left(\frac{\sigma I}{\hbar\omega}\right) \tau_2 \tag{6.14}$$

$$N_1 = S_1 \tau_1 + N^* \left(\frac{\sigma I}{\hbar\omega}\right) \tau_1 + A_{21} \tau_1 N_2 \tag{6.15}$$

$$N^* = \frac{S_2\tau_2\left(1 - \frac{g_2}{g_1}A_{21}\tau_1\right) - \frac{g_2}{g_1}S_1\tau_1}{1 + \left(\frac{\sigma I}{\hbar\omega}\right)\left[\tau_2 + \frac{g_2}{g_1}\left(1 - A_{21}\tau_2\right)\tau_1\right]} \tag{6.16}$$

Numerator is N_0^*:

$$N^* = \frac{N_0^*}{1 + \frac{I}{I_s}} \tag{6.17}$$

We can now define Saturation Intensity:

$$I_s(\omega) = \frac{\hbar\omega}{\sigma(\omega)}\frac{1}{\tau_2 + \frac{g_2}{g_1}\tau_1(1 - A_{21}\tau_2)} \tag{6.18}$$

Gain is saturated too, since

$$g(\omega) = \sigma(\omega)N^* = \frac{g_0(\omega)}{1 + \frac{I}{I_s(\omega)}} \tag{6.19}$$

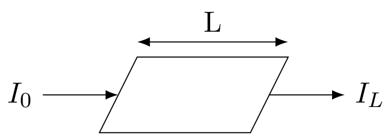
Comments:

- At $I = I_s(\omega) \to g(\omega) = \frac{g_0(\omega)}{2}$.
- $I_s(\omega)$ is frequency dependent through $\sigma(\omega)$, so will happen quicker at line centre.
- For an efficient laser, expect large population inversion $\implies \tau_2 \gg \tau_1$.
- All decays from state 2 into state 1 $\implies A_{21} \approx \frac{1}{\tau_2}$.

$$I_s(\omega) = \frac{\hbar\omega}{\sigma(\omega)\tau_2} \tag{6.20}$$

This is usually an excellent approximation.

Example: Linear Amplifier



$$\frac{dI}{dz} = g(\omega)I = \frac{g_0(\omega)I}{1 + \frac{I}{I_s(\omega)}} \tag{6.21}$$

We just integrate this, after length L:

$$\int_{I(0)}^{I(L)} \frac{1 + \frac{I}{I_s(\omega)}}{I}\, dI = \int_0^L g_0(\omega)\, dz \tag{6.22}$$

$$\ln\left(\frac{I_L}{I_0}\right) + \frac{I_L - I_0}{I_s(\omega)} = g_0(\omega)L \tag{6.23}$$

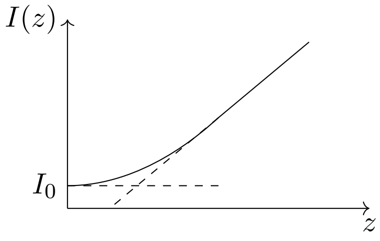
Two limiting cases:

1. $I/I_s(\omega) \ll 1 \implies$ exponential growth,

$$I_L = I_0 e^{g_0(\omega)L} \tag{6.24}$$

2. $I/I_s(\omega) \gg 1 \implies$ linear growth,

$$I_L = I_0 + g_0(\omega)I_s(\omega)L \tag{6.25}$$


I(z)
I0
z

Lecture 7   Multimode lasing and output power

Last time:

$$\frac{dN_2}{dt} = S_2 - \frac{N_2}{\tau_2} - \frac{N^*\sigma(\omega)I}{\hbar\omega} \tag{7.1}$$

$$\frac{dN_1}{dt} = S_1 - \frac{N_1}{\tau_1} + \frac{N^*\sigma(\omega)I}{\hbar\omega} + N_2A_{21} \tag{7.2}$$

We need a third equation to describe the system. Develop equation governing the the intensity build up. Consider photon density, n_ϕ.

$$\frac{dn_\phi}{dt} = \left(N^*\sigma(\omega)\frac{I}{\hbar\omega}\right)\left(\frac{L_m}{L_c}\right) - \frac{n_\phi}{\tau_c} \tag{7.3}$$

Intensity of beam,

$$I = \text{energy density} \times \text{velocity} \tag{7.4}$$

$$= n_\phi\hbar\omega \times c \tag{7.5}$$

$$n_\phi = \frac{I}{\hbar\omega c} \tag{7.6}$$

$$dn_\phi = \frac{I}{\hbar\omega c}dI \tag{7.7}$$

$$\frac{dI}{dt} = N^*\sigma(\omega)cI\left(\frac{L_m}{L_c}\right) - \frac{I}{\tau_c} \tag{7.8}$$

We know that

$$\tau_c = \frac{2L_c}{c\delta_c} \qquad g_{th} = \frac{\delta_c}{2L_m}. \tag{7.9}$$

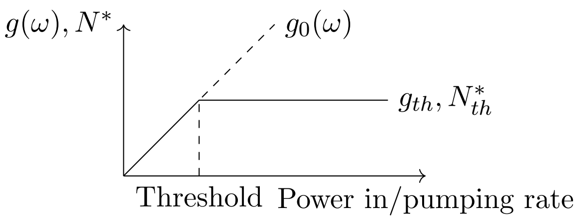
We then find the following useful for the final substitution:

$$\frac{1}{\tau_c} = \frac{c\delta_c}{2L_c} = c\left(\frac{L_m}{L_c}\right)g_{th} \tag{7.10}$$

$$g(\omega) = N^*\sigma(\omega) \tag{7.11}$$

$$\frac{dI}{dt} = c\left(\frac{L_m}{L_c}\right)\left[g(\omega) - g_{th}\right]I(t) \tag{7.12}$$

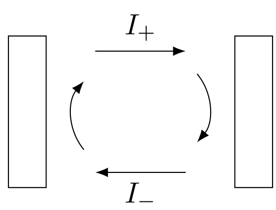
In the steady state, this should be zero which is found either if $I(t)$ is zero which is meaningless solution, or if $g_{ss}(\omega) = g_{th}$. This highlights the principle of gain saturation. We can take the expression found for gain last time and apply the above relation.



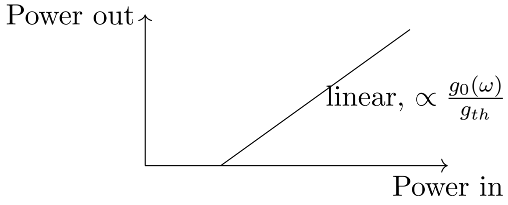
$$g_{ss}(\omega) = \sigma(\omega)N^* = \frac{g_0(\omega)}{1 + I_{ss}/I_s(\omega)} = g_{th} \tag{7.13}$$

$$I_{ss} = I_s(\omega)\left[\frac{g_0(\omega)}{g_{th}} - 1\right] \tag{7.14}$$

Consider simple cavity,



Mirror transmission, $T = (I_+ - R)$, $I_{out} = T \times I_+$. Assume low output coupling-uniform field approximation, i.e. $I_+ \approx I_-$.



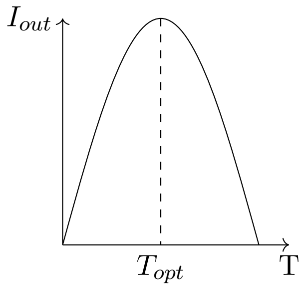
$$I_{ss} = I_+ + I_- \approx 2I_+ \tag{7.15}$$

$$I_{out} = T \times \frac{I_{ss}}{2} = T\frac{I_s(\omega)}{2}\left[\frac{g_0(\omega)}{g_{th}} - 1\right] \tag{7.16}$$

To get output power, we need to know mode area, A_{mode}.

$$P_{out} = A_{mode}I_{out} = \frac{1}{2}I_s(\omega)A_{mode}T\left[\frac{g_0(\omega)}{g_{th}} - 1\right] \tag{7.17}$$

7.1 Optimum Output Coupling



$$I_{out} = T\frac{I_s(\omega)}{2}\left[\frac{g_0(\omega)}{g_{th}} - 1\right] \tag{7.18}$$

$$g_{th} = \frac{\delta_c}{2L_m},\ \delta_c = T + \delta_{loss} \tag{7.19}$$

$$I_{out} = T\frac{I_s(\omega)}{2}\left[\frac{2g_0(\omega)L}{T + \delta_{loss}} - 1\right] \tag{7.20}$$

Differentiate expression for I_{out} with respect to T and set to zero:

$$\frac{dI_{out}}{dT} = \frac{I_s(\omega)}{2}\left[\frac{2g_0(\omega)L}{T+\delta_{loss}} - 1\right] - \frac{TI_s(\omega)}{2}\frac{2g_0(\omega)L}{(T+\delta_{loss})^2} \tag{7.21}$$

$$2g_0(\omega)L(T+\delta_{loss}) - (T+\delta_{loss})^2 = 2g_0(\omega)LT \tag{7.22}$$

$$(T+\delta_{loss})^2 = 2g_0(\omega)L\delta_{loss} \tag{7.23}$$

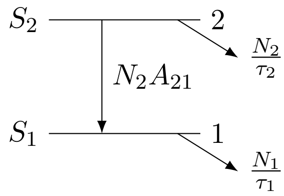
$$T_{opt} = \sqrt{2g_0(\omega)L\delta_{loss}} - \delta_{loss} \tag{7.24}$$

$$I_{opt} = \left[1 - \left(\frac{\delta_{loss}}{2g_0(\omega)L}\right)^{1/2}\right]^2 g_0(\omega)LI_s(\omega) \tag{7.25}$$

$$I_{max} = g_0(\omega)LI_s(\omega) \tag{7.26}$$

Lecture 8 Requirements for Population Inversion

8.1 Condition for Steady State Population Inversion



Develop rate equations for N_1 and N_2.

$$\frac{dN_2}{dt} = S_2 - \frac{N_2}{\tau_2} \tag{8.1}$$

$$\frac{dN_1}{dt} = S_1 + N_2 A_{21} - \frac{N_1}{\tau_1} \tag{8.2}$$

In steady state,

$$N_2 = \tau_2 S_2 \tag{8.3}$$

$$N_1 = \tau_1 S_1 + \tau_2 S_2 A_{21} \tau_1 \tag{8.4}$$

For population inversion, $N^* > 0$, or $\frac{N_2}{g_2} > \frac{N_1}{g_1}$.

$$\frac{\tau_2 S_2}{g_2} > \frac{\tau_1 S_1}{g_1} + \frac{\tau_2 S_2 A_{21} \tau_1}{g_1} \tag{8.5}$$

$$1 < \frac{g_1}{g_2} \frac{S_2 \tau_2}{S_1 \tau_1} \left[1 - \frac{g_2}{g_1} A_{21} \tau_1 \right] \tag{8.6}$$

We want:

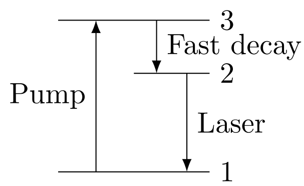
- $S_2 > S_1$ - selective pumping
- $\tau_2 > \tau_1$ - favourable lifetime ratio
- $g_1 > g_2$ - favourable degeneracy ratio
- The term in the square brackets only depends on the atomic properties, and we require

$$A_{21} < \frac{g_1}{g_2} \frac{1}{\tau_1} \tag{8.7}$$

  to be able to get laser oscillation. This is the **minimum** requirement for steady state inversion - alone, it may not be a sufficient condition, but it is necessary. The lower laser level must decay faster than it is being filled by spontaneous emission.

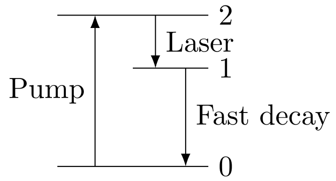
8.2 Three and Four Level Lasers

8.2.1 Traditional Solid State Three-Level Laser



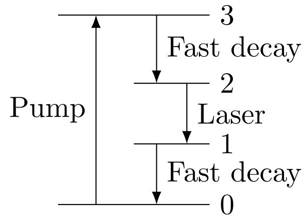
This is known as the traditional solid state three-level laser, e.g. Ruby.

8.2.2 Gas lasers



This is common for gas lasers, e.g. Ar+.

8.2.3 Solid state and dye lasers



This is most solid state lasers and dye lasers, e.g. Nd:YAG.

8.2.4 General comments

- We only show three/four levels here but each one of these levels may be a tight collection of thousands of levels which may be close enough to be treated as a single big level.
- For three-level schemes, one transition must be non-radiative to conserve parity.
  - For three-level solid state lasers, there is fast phonon decay from 3 to 2.
  - For gas lasers, both downward transitions are radiative but pumping is achieved by collisions.
- Traditional sold state laser will have a high pumping threshold because we need to transfer $\frac{N}{2}$ out of state 1 to get inversion, so these are not very efficient.
- Gas lasers have a low *quantum efficiency*.

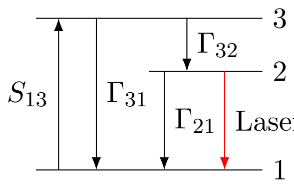
$$Q.E. = \frac{E_2 - E_1}{E_2 - E_0} = \frac{\text{laser photon energy}}{\text{pump `photon' energy}} = \frac{\hbar\omega_{12}}{\hbar\omega_{20}} \tag{8.8}$$

  because spontaneous emission rate $A_{21} \propto \omega_{12}^3$, hence to achieve $\tau_2 > \tau_1$ requires $\omega_{10} > \omega_{12}$.
- Four-level lasers are best, *cause bigger is better, baybeeeee.*

$$\text{Gain} = e^{g(\omega)L} = e^{\sigma(\omega)N^* L_m} \tag{8.9}$$

$\sigma(\omega)$ and L_m are fixed (or practically so), so for us to maximise our gain, we have to maximise N^* through pumping.

8.3 Three-Level Traditional Solid State Lasers



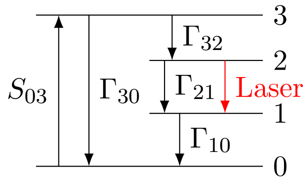
To study the steady state small signal population inversion, we ignore the effects of the laser field on the populations. The rate equations are:

$$\frac{dN_3}{dt} = S_{13}N_1 - (\Gamma_{31} + \Gamma_{32})N_3 \tag{8.10}$$

$$\frac{dN_2}{dt} = \Gamma_{32}N_3 - \Gamma_{21}N_2 \tag{8.11}$$

$$\frac{dN_1}{dt} = -S_{13}N_1 + \Gamma_{31}N_3 + \Gamma_{21}N_2 \tag{8.12}$$

Lecture 9 Population Inversion in Four-level lasers



Γ_{32} is a fast, non-radiative decay; Γ_{10} likewise.

$$\frac{dN_3}{dt} = N_0 S_{03} - (\Gamma_{30} + \Gamma_{32}) N_3 \tag{9.1}$$

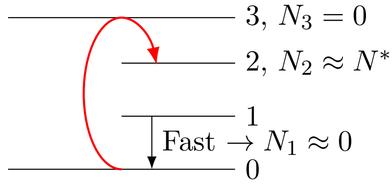
$$\frac{dN_2}{dt} = N_3 \Gamma_{32} - N_2 \Gamma_{21} \tag{9.2}$$

$$\frac{dN_1}{dt} = N_2 \Gamma_{21} - N_1 \Gamma_{10} \tag{9.3}$$

$$\frac{dN_0}{dt} = -N_0 S_{03} + N_3 \Gamma_{30} + N_1 \Gamma_{10} \tag{9.4}$$

In the ideal case, $\Gamma_{32} \gg \Gamma_{30}$, and $N_3 \approx 0$, i.e. decay is faster than pumping.

9.1 Idealised Four-level laser



The idealised four-level laser equations:

$$N_0 \approx N \qquad N_2 \approx \frac{S_{03}}{\Gamma_{21}} N_0 \qquad N_3 = 0 \tag{9.5}$$

9.2 Comparison of three- and four-level laser schemes

For a three-level scheme, we found that,

$$S_{13}^{th} = \frac{N + N_{th}^*}{N - N_{th}^*} \Gamma_{21} \approx \Gamma_{21}. \tag{9.6}$$

For four-level schemes, we found that

$$S_{03}^{th} = \frac{N_{th}^*}{N - N_{th}} \Gamma_{21} \approx \frac{N_{th}^*}{N} \Gamma_{21} \tag{9.7}$$

If we assume equal Γ_{21}s,

$$\frac{\text{four-level}}{\text{three-level}} = \frac{S_{03}^{th}}{S_{13}^{th}} = \frac{N_{th}^*}{N + N_{th}^*} \approx \frac{N_{th}^*}{N} \ll 1 \tag{9.8}$$

This reiterates that four-level lasers are much more efficient than three-level ones. Comparing power per unit volume absorbed at threshold:

$$\left(\frac{P}{V}\right)_{th,3} = \hbar\omega_{13} \Gamma_{21} \frac{N}{2} \qquad \left(\frac{P}{V}\right)_{th,4} = \hbar\omega_{30} N^* \Gamma_{21} \tag{9.9}$$

Example: Ruby vs Nd:YAG
Ruby doped at 1% by weight.

$$N_0 = 3.3 \times 10^{26}\, m^{-3} \qquad \Gamma_{21} = 3.3 \times 10^2\, s^{-1} \tag{9.10}$$

Assume pumping at $\lambda = 505\, nm$. Laser rod, diameter $6\, mm$, length $10\, cm$.

$$P_{th} = \left(\frac{hc}{\lambda}\right) \frac{N_0}{2} \Gamma_{21} \times V \approx 5.4\, kW \tag{9.11}$$

Now for the ND:YAG,

$$\Gamma_{21} = 1.8 \times 10^3\, s^{-1} \qquad \sigma = 9 \times 10^{-19}\, cm^2 \tag{9.12}$$

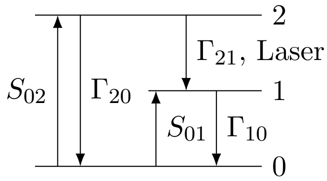
Assume cavity loss of $\delta_c = 0.05$, with diameter $4\, mm$, and length $5\, cm$. Pumped at $808\, nm$.

$$N^*_{th} = \frac{\delta_c}{2\sigma L_m} = 5.6 \times 10^{21}\, m^{-3} \tag{9.13}$$

$$P_{th} = \hbar\omega_{30} N^*_{th} \Gamma_{21} V \approx 1.66\, W \tag{9.14}$$

So Ruby has a massive energy cost compared to Nd:YAG, Nd:YAG much preferable.

Lecture 10 Population Inversion in Three-level Gas Lasers



Develop rate equations in absence of field:

$$\frac{dN_2}{dt} = S_{02}N_0 - (\Gamma_{20} + \Gamma_{21})N_2 \tag{10.1}$$

$$\frac{dN_1}{dt} = S_{01}N_0 + \Gamma_{21}N_2 - \Gamma_{10}N_1 \tag{10.2}$$

$$\frac{dN_0}{dt} = -(S_{02} + S_{01})N_0 + \Gamma_{20}N_2 + \Gamma_{10}N_1 \tag{10.3}$$

Using the steady state definition, $\frac{dN_i}{dt} = 0$:

$$N_2 = \frac{S_{02}}{\Gamma_{20} + \Gamma_{21}} N_0 \tag{10.4}$$

$$S_{01}N_0 + \Gamma_{21}\left[\frac{S_{02}}{\Gamma_{20} + \Gamma_{21}} N_0\right] - \Gamma_{10}N_1 = 0 \tag{10.5}$$

$$N_1 = \frac{S_{01}(\Gamma_{20} + \Gamma_{21}) + \Gamma_{21}S_{02}}{\Gamma_{10}(\Gamma_{20} + \Gamma_{21})} N_0 \tag{10.6}$$

For population inversion, we need

$$\frac{N_2}{N_1} > \frac{g_2}{g_1} \tag{10.7}$$

$$N^* = N_2 - \frac{g_2}{g_1} N_1 > 0 \tag{10.8}$$

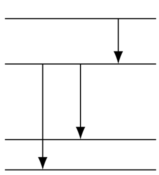
$$\frac{N_2}{N_1} = \frac{S_{02}\Gamma_{10}}{S_{01}(\Gamma_{20} + \Gamma_{21}) + \Gamma_{21}S_{02}} > \frac{g_2}{g_1} \tag{10.9}$$

We have to find another pumping technique as we cannot use light. Both down transitions are optical (radiative), so S_{02} cannot be optical due to parity conservation, and Γ_{20} is negligible.

$$\frac{N_2}{N_1} = \frac{S_{02}\Gamma_{10}}{(S_{01} + S_{02})\Gamma_{21}} > \frac{g_2}{g_1} \tag{10.10}$$

$$\frac{S_{02}}{S_{01}} > \left(\frac{g_1\Gamma_{10}}{g_2\Gamma_{21}} - 1\right)^{-1} \tag{10.11}$$

Example: Argon ion laser



$\lambda = 488.0nm$ with $A = 7.8 \times 10^7 s^{-1}$, decays): $73.1nm$, $A = 4.5 \times 10^8 s^{-1}$; $72.3nm$, $A = 23 \times 10^8 s^{-1}$. We have a degeneracy of $g = 2J + 1$, so $g_1 = 4, g_2 = 6$. $\Gamma_{21} = 7.8 \times 10^7\, s^{-1}$, $\Gamma_{10} = 27.5 \times 10^8\, s^{-1}$.
From this, we get a ratio of 0.044. For this particular system, we can pump into the lower level 22.5 times as quickly as into the upper one and still maintain population inversion due to fast natural decay rates. **Striking result!**
This can happen because $A \propto \omega^3$ so massively different for the higher frequencies of the lower level

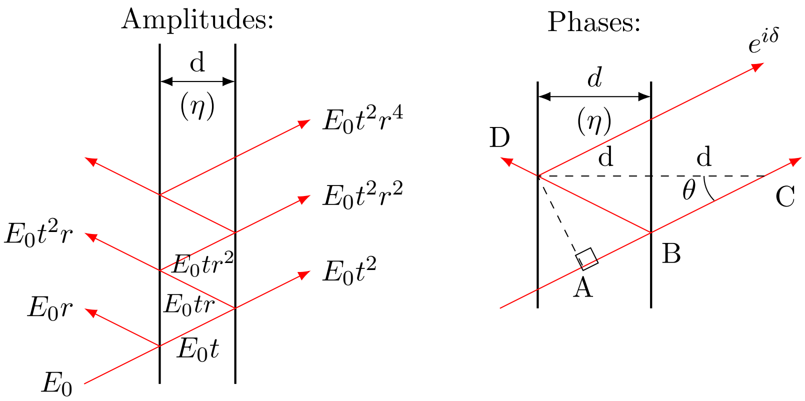
> decay. The downside is low quantum efficiency.

Gas lasers cannot be optically pumped, so they use collisions, or **particle pumping.**

Lecture 11 The Fabry-Perot Etalon and Laser Cavity Modes

11.1 The Fabry-Perot Elaton

Consider two identical semi-reflecting parallel mirrors:



t and r are the amplitude transmission and reflection coefficients respectively.

11.1.1 Path different between succesive rays

$$\begin{aligned} P.D. &= AB + BD = AB + BC = AC \quad (11.1)\\ &= 2d\cos\theta \quad (11.2)\\ \delta &= 2kd\cos\theta = \frac{4\pi d}{\lambda}\cos\theta \quad (11.3)\end{aligned}$$

11.1.2 Transmitted amplitude

$$\begin{aligned} E_T &= E_0t^2\left(1 + r^2e^{i\delta} + r^4e^{2i\delta} + r^6e^{3i\delta} + \dots\right) \text{ - Geometric Progression} \quad (11.4)\\ &= \sum_0^\infty r^{2n}e^{in\delta} = \frac{1}{1-r^2e^{i\delta}} \quad (11.5)\\ &= \frac{E_0T}{1-Re^{i\delta}},\ T = |t|^2,\ R = |r|^2 \quad (11.6)\end{aligned}$$

11.1.3 Transmitted Intensity

$$I_T = |E_T|^2 = \frac{I_0T^2}{|1-Re^{i\delta}|^2} \quad (11.7)$$

One can easily show that

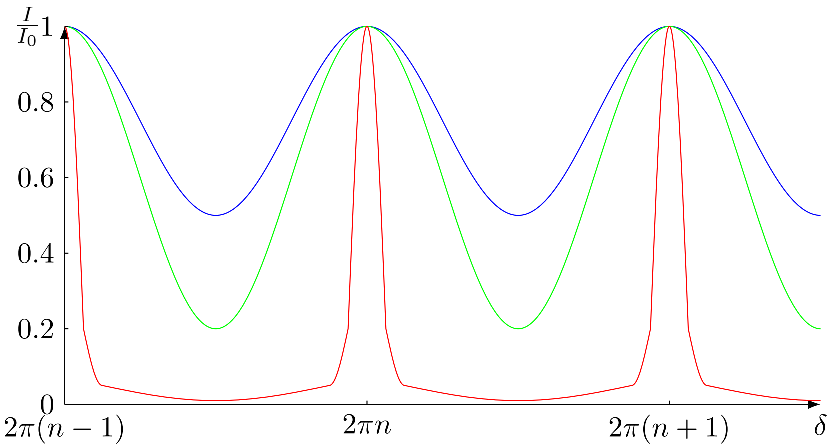
$$|1-Re^{i\delta}|^2 = (1-R)^2\left[1 + \frac{4R}{(1-R)^2}\sin^2\left(\frac{\delta}{2}\right)\right] \quad (11.8)$$

Hence, we have

$$\frac{I_T}{I_0} = \frac{T^2}{(1-R)^2}\frac{1}{1+\frac{4R}{(1-R)^2}\sin^2\frac{\delta}{2}} = \frac{T^2}{(1-R)^2}\frac{1}{1+\frac{4F^2}{\pi^2}\sin^2\frac{\delta}{2}} \tag{11.9}$$

Where we can define **Finesse:**

$$F = \frac{\pi\sqrt{R}}{1-R} \tag{11.10}$$



11.1.4 Points to note

- For mirrors, $T+R+A=1$, absorption $A\approx 0$, so $T=1-R$.

$$\left.\frac{I_T}{I_0}\right|_{max} = 1 \tag{11.11}$$

So all light trnasmitted at cavity resonance.

- Phase difference:

$$\delta = \frac{4\pi d}{\lambda}\cos\theta \tag{11.12}$$

Dependent on angle, separation, and wavelength.

- For bright fringes, $\sin\frac{\delta}{2}=0$ so $\delta=2n\pi$, or $2d\cos\theta = n\lambda$.
- Free spectral range, $\Delta\nu_{fsr}$. At maxima,

$$\lambda = \frac{2d\cos\theta}{n} \implies \nu = \frac{c}{\lambda} = \frac{nc}{\cos\theta} \tag{11.13}$$

$$\Delta\nu_{fsr} = \nu_{n+1}-\nu_n = \frac{c}{2d\cos\theta} \tag{11.14}$$

Then, at normal incidence,

$$\Delta\nu_{fsr} = \frac{c}{2d} \tag{11.15}$$

- Fringe width. What is the full width at half maximum (FWHM) of fringes? Assume $\frac{I_T}{I_0}=\frac{1}{2}$ when $\delta = 2\pi n \pm \delta_{1/2}$.

$$\frac{1}{1+\frac{4F^2}{\pi^2}\sin^2(\frac{\delta_{1/2}}{2})} = \frac{1}{2} \tag{11.16}$$

$$\sin^2\left(\frac{\delta_{1/2}}{2}\right) = \frac{\pi^2}{4F^2} \tag{11.17}$$

$$\delta_{1/2} = 2\sin^{-1}\left(\frac{\pi}{2F}\right) \tag{11.18}$$

$$\delta_{1/2} \approx \frac{\pi}{F},\ F >> 1 \tag{11.19}$$

$$\text{FWHM} = 2\delta_{1/2} = \frac{2\pi}{F} \tag{11.20}$$

Separation of adjacent peaks (in radians) $= 2\pi$. Hence, finesse,

$$F = \frac{\Delta\nu_{fsr}}{\Delta\nu_{FWHM}}. \tag{11.21}$$

So high finesse $\rightarrow$ sharp fringes, and vice versa. Before,

$$F = \frac{\pi\sqrt{R}}{1-R} \implies \Delta\nu_{FWHM} = \frac{c}{2d}\left(\frac{1-R}{\pi\sqrt{R}}\right). \tag{11.22}$$

Example:
30cm long etalon with mirrors, $R = 0.99$.

$$\Delta\nu_{fsr} = \frac{c}{2d} = 500\,MHz \tag{11.23}$$

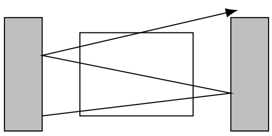
$$F = \frac{\pi\sqrt{R}}{1-R} \approx 313 \tag{11.24}$$

$$\Delta\nu_{FWHM} = \frac{\Delta\nu_{fsr}}{F} = 1.6\,MHz \tag{11.25}$$

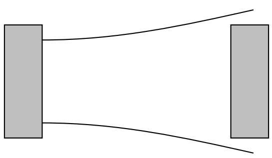
In visible range, say $\lambda = 550\,nm,\ \nu = 545\,THz$. We can use this etalon to reolve to 1 part in 10^8.

11.2 Laser Cavities

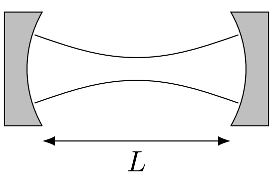
The Fabry-Perot etalon is not a good laser cavity. We need to place a gain medium between mirrors.

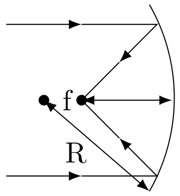


This cavity however is difficult to align. More fundamentally, diffraction will cause losses.



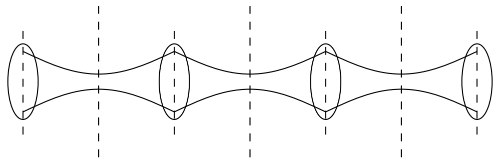
Diffraction leads to large losses. The solution is to use curved mirrors: Each curved mirror acts as a lens,





$$f = \frac{R}{2}. \tag{11.26}$$

where here R is the radius of curvature. Cavity acts as a sequence of lenses.



Related by Fourier transform, so essentially we want a function that is its own Fourier transform, we will see Gauss-Hermite.

Lecture 12 Gaussian Beams and Cavity Stability

The electric field distribution needs to reproduce after a round trip, so it needs to be its own Fourier transform, i.e. Gause-Hermite modes.

$$U = \frac{U_0}{q} e^{i(kz-\omega t)} e^{ikr^2/2q} H_l\left(\frac{\sqrt{2}x}{w}\right) H_m\left(\frac{\sqrt{2}y}{w}\right) e^{-i(l+m)\alpha} \tag{12.1}$$

$$q = z - z_0 - iZ_R,\ r^2 = x^2 + y^2,\ \tan\alpha = \frac{z}{Z_R} \tag{12.2}$$

Where Z_R is the Rayleigh range, and z_0 is the location of beam waist. Gaussian beam (TEM_{00}):

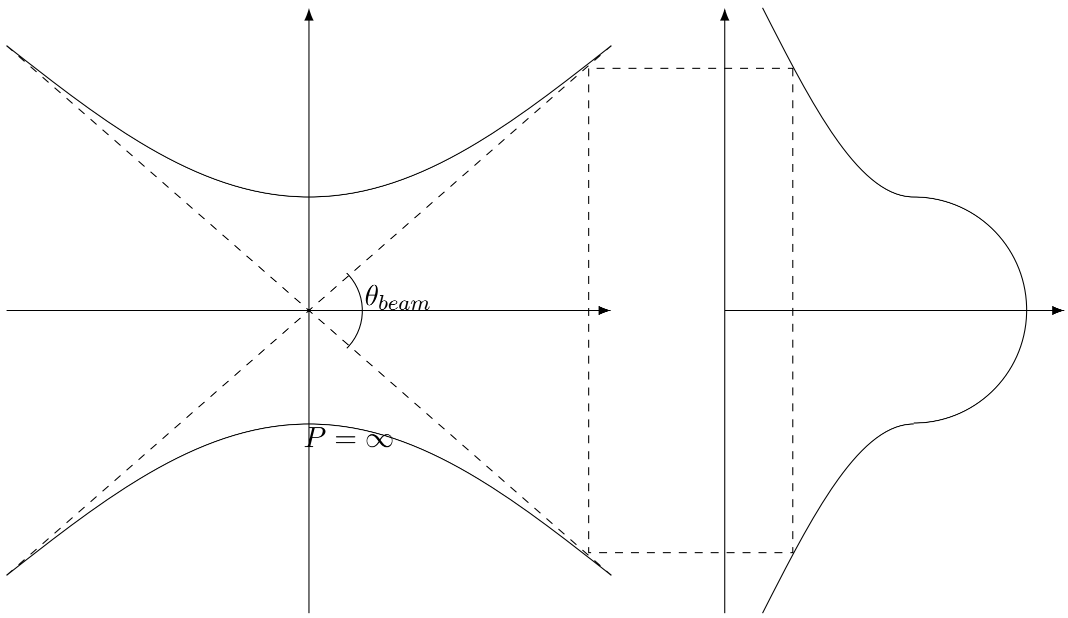
$$U = U_0 e^{i(kz-\omega t)} \frac{e^{ikr^2/2q}}{q} \tag{12.3}$$

12.1 Propagation of Gaussian beams

$$w(z) = w_0\left(1 + \left(\frac{z}{z_R}\right)^2\right)^{1/2} \implies \frac{w(z)^2}{w_0^2} - \frac{z^2}{z_R^2} = 1 \tag{12.4}$$

This is the equation of hyperbola with asymptotes at

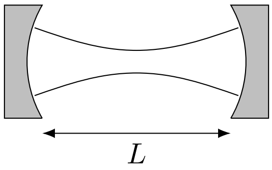
$$w(z) = \pm\frac{w_0 z}{z_R} \tag{12.5}$$



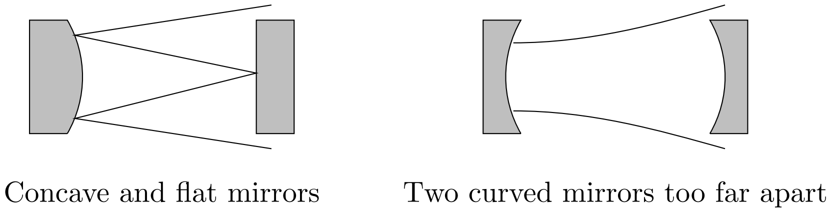
$$\theta_{beam} = \frac{2w_0}{z_R} = \frac{2\lambda}{\pi w_0} = 1.27\frac{\lambda}{w_0} \tag{12.6}$$

So a Gaussian beam is diffraction limited, as we get a very similar result here as you would for a circular aperture.

$$I = \frac{2P}{\pi w(z)^2} e^{-2r^2/w(z)^2} \tag{12.7}$$



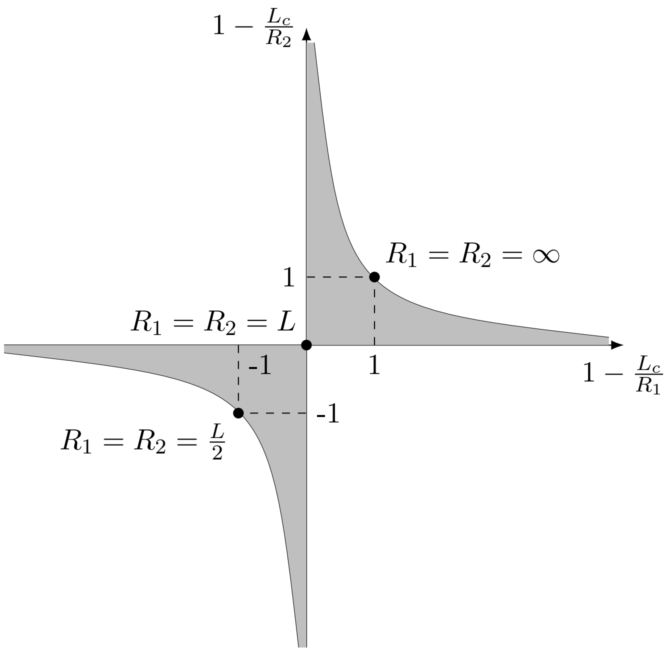
What values of L_c, R_1, and R_2 lead to a stable solution? Some things won't work.



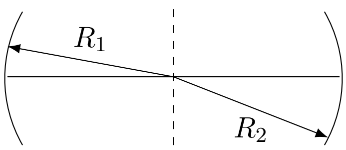
The stability condition is

$$0 < \left(1 - \frac{L_c}{R_1}\right)\left(1 - \frac{L_c}{R_2}\right) < 1. \tag{12.8}$$

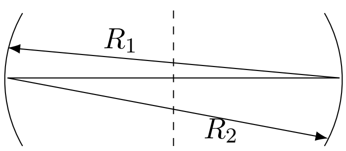
This can often be represented graphically:



1. Symmetric concentric:



2. Symmetric confocal:

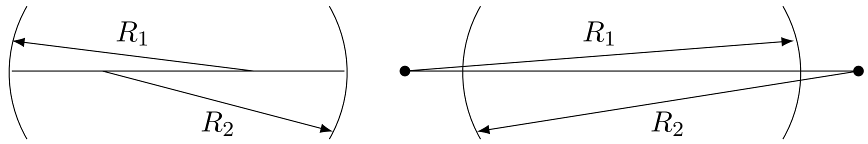


- ➤ The difference between concentric and confocal is where the radius of curvature begins - at the centre, or at each end.

3. Plane parallel:

We want to design cavity to be in the shaded regions of the graph, and this requires satisfying two conditions in the inequality:

- ➤ $L_c < R_1 + R_2$
- ➤ The centres of curvature must **both** be inside or outside the cavity.



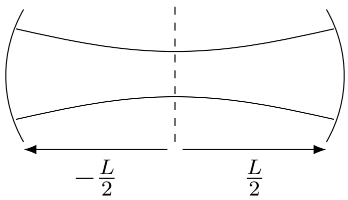
Example:
A laser has mirrors with $R_1 = 1.0m$, and $R_2 = 0.35m$. What ranges of L_c are stable?

From the second requirement, we know there are two ranges.

- ➤ Both inside $\rightarrow 1.0 < L_c < 1.35$
- ➤ Both outside $\rightarrow 0 < L_c < 0.35$

12.2 Finding the mode shape for a cavity

Simple case, **Symmetric:**
The waist must be at the centre, so $R_1 = R_2 = R$.



At each mirror, the radius of curvature of the beam is the same as the mirror. Recall

$$R(z) = \frac{z^2 + z_R^2}{z} \tag{12.9}$$

$$R\left(\frac{L_c}{2}\right) = R = \frac{\frac{L_c^2}{4} + z_R^2}{\frac{L_c}{2}} \tag{12.10}$$

$$z_R^2 = \frac{L_c}{4}(2R - L_c) \equiv \left(\frac{\pi w_0^2}{\lambda}\right)^2 \tag{12.11}$$

$$w_0^2 = \frac{\lambda}{2\pi}\left[L_c(2R - L_c)\right]^{1/2} \tag{12.12}$$

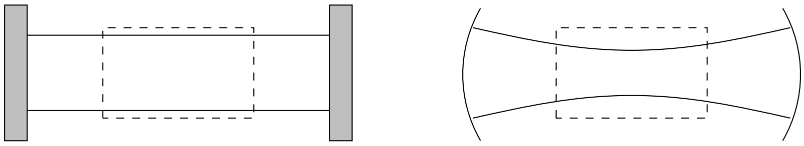
We know the beam waist is inside a symmetric cavity from R and L_c.

12.3 Mode Volume

How do we choose what cavity parameters to use? Try to match mode volume to Gain medium.

$$V \leq \pi w_0^2 L_c \tag{12.13}$$

This assume the beam is not spreading, i.e. $Z_R \gg L_c$.



Lecture 13 Cavity Effects: Single Frequency Operation

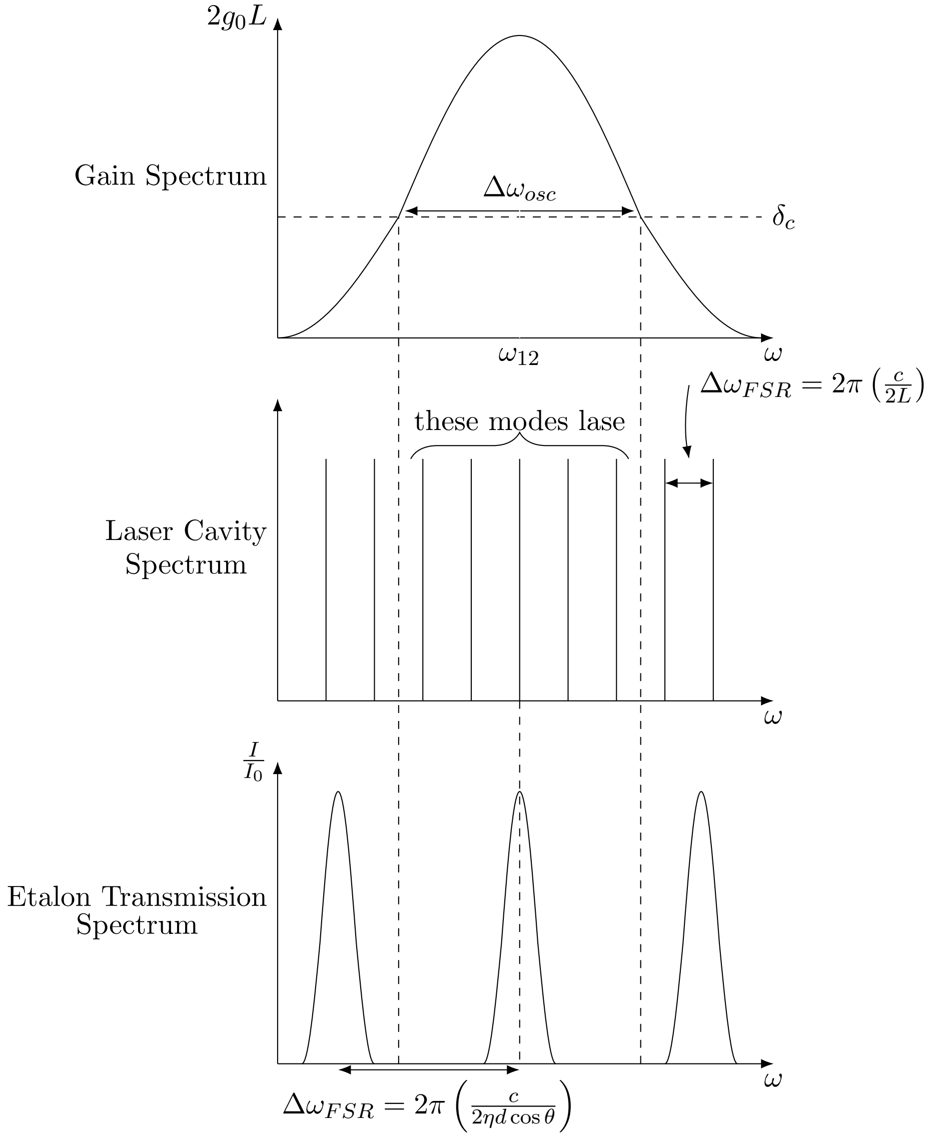
Spectral and spatial hole burning occur for homegeneous and inhomogeneous systems $\rightarrow$ multimode lasing:

- ➤ Homogeneous - **spatial** hole burning
- ➤ Inhomogeneous - **spectral** hole burning

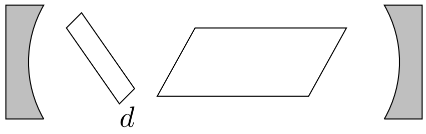
This multimode (longitudinal) behaviour can be eliminated but to do so in each case is different.

13.1 Inhomogeneous - spectral hole burning

In a Doppler broadened gain medium, it is likely the laser will oscillate simultaneously on several longitudinal modes if the cavity mode spacing is less than the range of frequencies over which there is net gain. $\Delta\omega_{osc}$ = range over which the laser can lase on; $\Delta\omega_{FSR}$ = the mode separation.



A common solution to multimode lasing is to introduce an **intracavity etalon** (Fabry-Perot Etalon), whose thickness is such that only one longitudinal cavity mode lases.

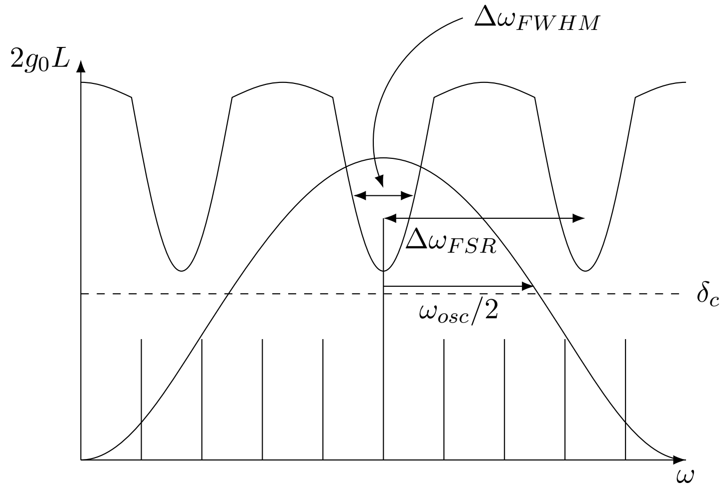


As you decrease d, the free spectral range decreases.
Etalon is usually solid and mounted at a small angle. This may allow for turning of the etalon transmission, since

$$\Delta\omega_{FSR} = 2\pi\left(\frac{c}{2\eta d\cos\theta}\right) \tag{13.1}$$

This can be a frequency control (moves transition peak around).

There are now two cavities to consider: cavity length, L; and etalon length, d. It is instructive to think of the etalon as altering the round trip lenses; the etalon loss is high when light is reflected from the etalon, but then drops on its resonance. So we end up with a single mode to lase on.



The etalon must satisfy two conditions:

1. Etalon free spectral range must be greater than half the cavity spectral range → to prevent modes in wings from lasing.

$$\Delta\omega_{FSR} = 2\pi\left(\frac{c}{2\eta d\cos\theta}\right) > \frac{\Delta\omega_{osc}}{2} \tag{13.2}$$

2. To select only one mode, less than the mode FSR:

$$\Delta\omega_{FWHM} < 2\pi\left(\frac{c}{2L_c}\right) \tag{13.3}$$

Since

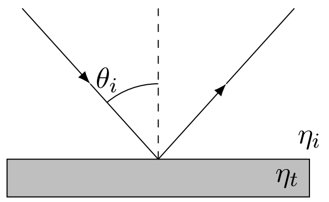
$$\Delta\omega_{FSR} = \frac{2\pi c}{2\eta d\cos\theta}, \qquad F = \frac{\Delta\omega_{FSR}}{\Delta_{FWHM}}, \tag{13.4}$$

the above conditions imply conditions on d and R → etalon thickness and reflectivity.

13.2 Homogeneous - Spatial Hole Burning

For a homogeneously broadened gain medium, we may get **multimode operation** due to spatial hole burning. We have a standing wave in the cavity of $\cos^2$ nature which depletes gain in some regions, but leaves unused gain in other regions. This is bad because the laser generally finds a way to use this gain by running on additional modes. Solution: eliminate the standing wave to remove spatial hole burning. We can do this by using a **ring cavity** (a square or ring), in which the laser mode only propagates in one direction using an optical diode.

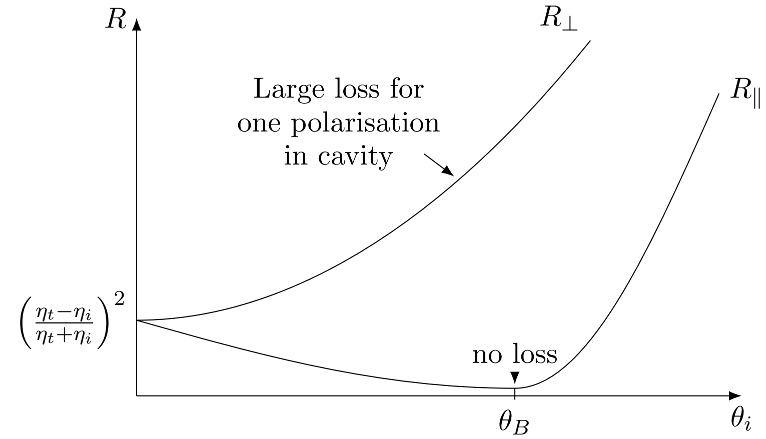
1. Mirror (M1) transmits the pump light (but reflects laser light) and allows pump beam into cavity. The beam is focused to a similar waist to the cavity mode within the crystal. Pump is Ar^+ or doubled Nd:YAG.
2. Crystal



Cut such that it is inserted into the cavity as Brewster's angle. Typically $\approx 2cm$ long and $5mm$ diameter. Brewster's angle means we have zero reflection at the air-crystal boundary for one polarisation. Boundary conditions $\implies$ Fresnal coefficients.

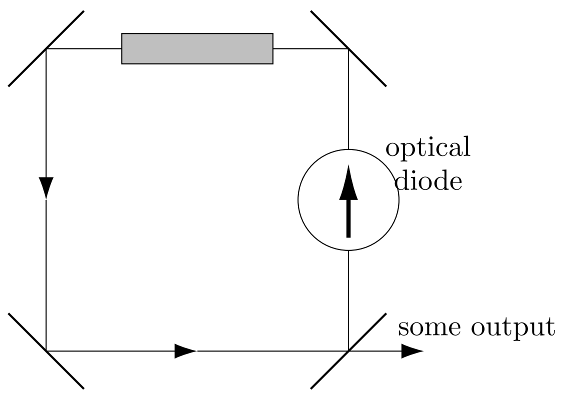
$$\tan\theta_B = \frac{\eta_t}{\eta_i} \tag{13.5}$$

For Ti:Sapphire: $\eta = 1.76 \implies \theta_B \approx 60°$. Note: for one population in the cavity, there is no loss at the crystal; for the other, there is a very large reflection loss.



3. Optical Diode: Consists of 2 elements that rotate the polarisation of the light. One is the standard waveplate, $\frac{\lambda}{2}$. Other is a Faraday Rotator; this is a device based on the Faraday effect - for some materials placed in a magnetic field, the light polarisation is rotated. The direction of rotation depends on the direction of light propagation with respect to the direction of the $\underline{\mathbf{B}}$-field.
   There is no time reversal symmetry $\rightarrow$ it imposes directionality on the cavity such that if it is the wrong way, then there is complete loss. So the laser is forced to lase in a single direction; elimates spatial hole burning and makes it easier to achieve single frequency oscillation.

A simple ring cavity might look something like:

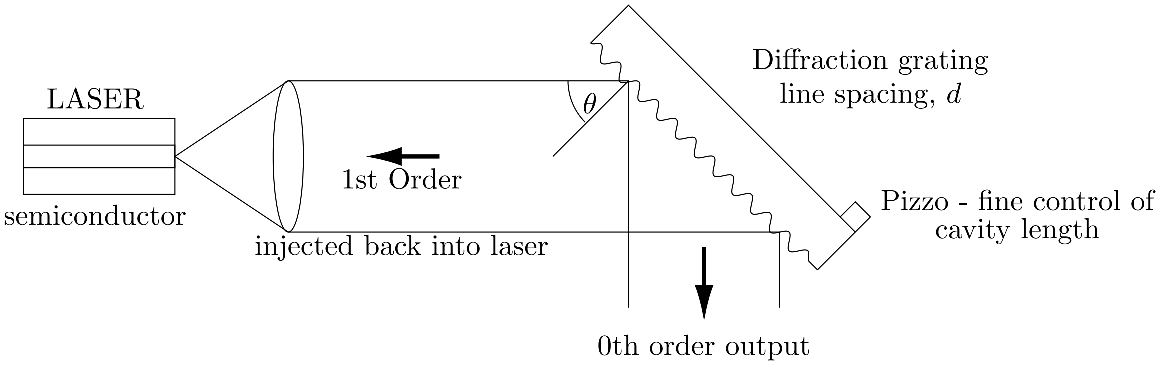


Instead of two mirrors, we now have multiple.

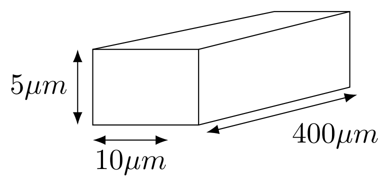
How much is single frequency output, single frequency?

13.3 External Cavity Diode Laser

Another example of using the elements of a cavity to achieve tunable single frequency.



Semiconductor emits light at recombination across pn junction; lens to focus diffracted light.
A typical semiconductor looks like:



Usually high refractive index, $\eta = 3.6$.

$$\Delta\nu_{FSR} = \frac{c}{2\eta d} \approx 100\,GHz \tag{13.6}$$

Relectivity of uncoated ends:

$$R = \left(\frac{\eta - 1}{\eta + 1}\right)^2 \approx 0.32 \tag{13.7}$$

I.e. expect 32% reflection $\to$ just from interface of semiconductor and air.

We define Finesse:

$$F = \frac{\pi\sqrt{R}}{1-R} = 2.6 \text{ (low)} \tag{13.8}$$

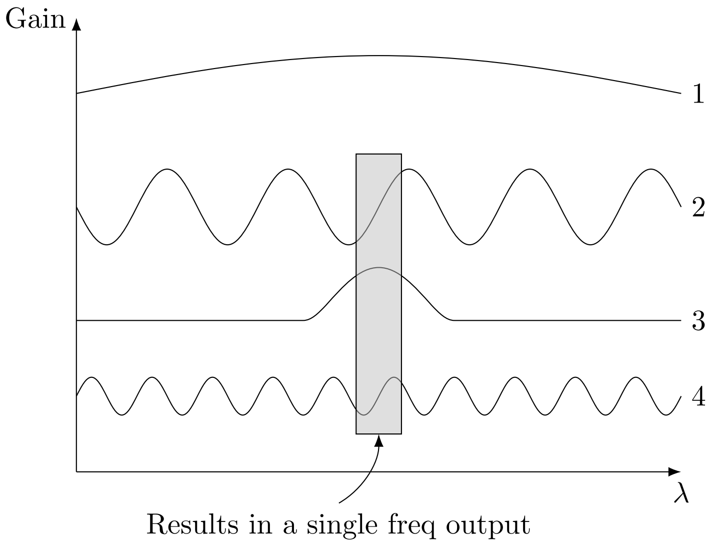
$$\Delta\nu_{FWHM} = \frac{\Delta\nu_{FSR}}{F} \approx 40\,GHz \tag{13.9}$$

Diffraction grating does two things:

- Selects λ via angle: $2d\sin\theta = n\lambda$. For 1800 lines mm^{-1}, $\lambda = 780nm \implies \theta = 45°$. Selects a range of frequencies by resolving power:

$$\frac{\nu}{\Delta\nu} = N = \text{total no. of lines illuminated} \tag{13.10}$$

- Forms an extended cavity
  1. Broad gain of laser from semiconductor itself.
  2. Internal cavity modes from semiconductor $\implies$ expect low finesse oscillation from low F cavity.
  3. Diffraction grating feedback - from a secondary cavity as some light is reflected straight back to initial cavity, which results in a longer cavity and extra spectra structure.
  4. External Cavity modes.

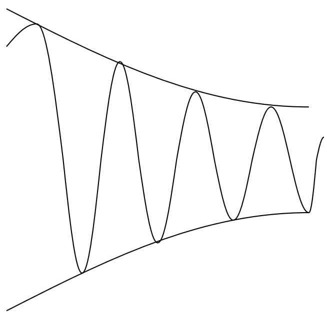


We want to tune all these such that lasing occurs where gain is large enough relative to loss. We can have a very narrow frequency output from these systems but it cannot be infinitely small $\implies$ there is a limit on size range.

From Lecture 5, we saw **cavity linewidth,**

$$\Delta\omega_c = \frac{1}{\tau_c} \tag{13.11}$$

It depends on the cavity lifetime - if we had a cavity with some light in it, the light would escape and the field in the cavity would decay away exponentially with some time constant, τ_c.



A Fourier transform gives $\Delta\omega$ spread in frequency. Here we now have an oscillating field that doesn't die away. The τ_c is infinitely long lifetime and the gain exactly balances the loss and there is no decay of the field in the cavity. We get a perfect single frequency out of the laser! So what have we missed..? $\implies$ **Spontaneous Emission.**

Spontaneous Emission can occur instead of an atom being stimulated by the laser field; stimulated emission adds coherently to the cavity field whereas spontaneous emission does not. It has no phase relationship to the cavity field and adds incoherently. It also has an inherently Lorentzian distribution of frequencies. Note: spontaneous emission has little effect on amplitude/intensity of laser output - on a plot of Power vs time, the power would appear very stable and constant, with only small oscillations about the central value.

$$P_{em} = \frac{\hbar\omega_{12}^3}{\pi^2 c^3}(1+\bar{n})B_{21},\ \bar{n} \geq 10^6 \tag{13.12}$$

Where $(1+\bar{n})$ represents the spontaneous and stimulated effects respectively, combined; and $\bar{n}$ is the number of photons per mode. Remember:

$$A_{21} = \frac{\hbar\omega_{12}^3}{\pi^2 c^3}B_{21} \tag{13.13}$$

However, spontaneous emission causes phase of the electric field to wander randomly - noticeable as the variation in phase leads to some linewidth.

$$\omega \propto \frac{d\phi}{dt} \tag{13.14}$$

A full quantum optics treatment leads to the **Schawlow-Townes formula:**

$$\Delta\omega_{laser} = \frac{\hbar\omega}{2P_{out}}\Delta\omega_c^2 \tag{13.15}$$

This is the 'Quantum limit to laser linewidth' - can't get any better than this.

Example: HeNe Laser
$1mW$ - low output laser, $\lambda = 632.8nm$. Cavity: $L_c = 1m,\ \delta_c = 1\%$.

$$\Delta\omega_c = \frac{c\delta_c}{2L_c} = 2\pi \times 0.24\, MHz \text{ - Cavity Linewidth} \tag{13.16}$$

$$\Delta\omega_{laser} = \frac{\hbar\omega}{2P_{out}}\Delta\omega_c^2 = 2\pi \times 0.006\, MHz \text{ - Laser linewidth} \tag{13.17}$$

The laser linewidth is the frequency spread of output.

Such low linewidths are very difficult to achieve as they require phenomenably stable cavities.

13.4 Practical Limit

Recall modes:

$$\omega_n = 2\pi\left(\frac{c}{2L_c}\right)n \tag{13.18}$$

$$\frac{\Delta\omega}{\omega} = -\frac{\Delta L_c}{L_c} \tag{13.19}$$

So what limits the laser linewidth? Consider the HeNe example - to reach the fundamental limit:

$$\Delta\omega_{laser} = 2\pi \times 0.006\, MHz \implies \frac{\Delta\omega_{laser}}{\omega} \approx 10^{-19} \implies \Delta L \approx 10^{-19} M \tag{13.20}$$

This is a tiny fraction of the size of the atom (better than LIGO!). If we stabilise the cavity to 0.1% of the optical wavelength:

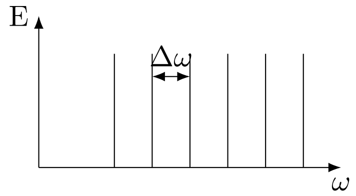
$$\frac{\Delta L}{L} \approx 6 \times 10^{-10} \implies \Delta\omega_{laser} \approx 2\pi \times 0.3\, MHz \tag{13.21}$$

Lecture 14 Pulsed Lasers

We will consider two types of pulsed operation:

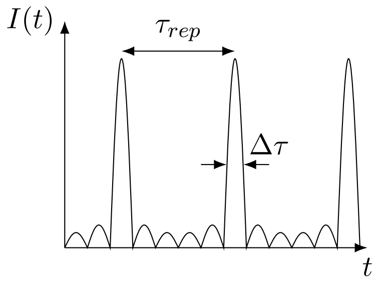
1. Mode Locking
2. Q-Switching

14.1 Mode Locking



Modes - each with its own frequency and phase. If the phases are 'locked', the intensity output of the laser is

$$I(t) = \frac{I_0 \sin^2\left(\frac{1}{2}N\Delta\omega t\right)}{\sin^2\left(\frac{1}{2}\Delta\omega t\right)} \tag{14.1}$$



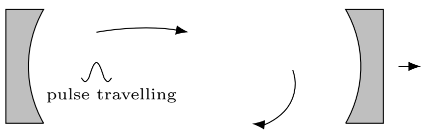
There are $N-2$ intermediate peaks, where N is the number of modes lasing. What is the time separation between these peaks, τ_{rep}?

14.1.1 Repetition Rate

When numerator and denominator are simultaneously zero, i.e. when $\frac{1}{2}\Delta\omega t = m\pi$. Hence,

$$\tau_{rep} = \frac{2\pi}{\Delta\omega} = \frac{2L}{c} \tag{14.2}$$

This is just the cavity round trip time - every time the beam in the cavity hits the output coupler, some beam gets emitted again.



14.1.2 Pulse Duration

Time separation between peak pulse and first minima, $\Delta\tau$. This will occur when numerator is zero, i.e.

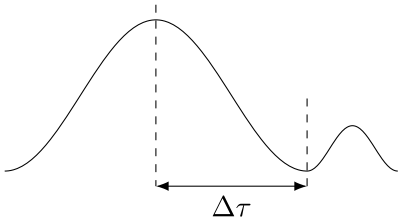
$$\sin\frac{1}{2}N\Delta\omega t = 0 \tag{14.3}$$

$$\frac{1}{2}N\Delta\omega\Delta\tau = \pi \tag{14.4}$$

$$\Delta\tau = \frac{2\pi}{N\Delta\omega} = \frac{\tau_{rep}}{N} \tag{14.5}$$

$$\Delta\tau = \frac{2\pi}{\Delta\omega_{osc}} \approx \frac{1}{\text{Gain Bandwidth}} \tag{14.6}$$

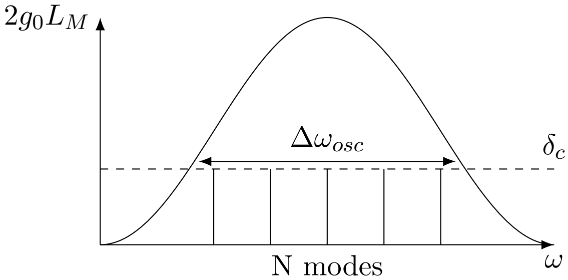
The final approximation is accurate when assuming very large N.



14.1.3 Maximum Intensity

$$I(t) = N^2 I_0 \tag{14.7}$$

This can be huge. N can be > 1000. You can deduce $\Delta\omega_{osc}$ from gain and loss:



14.1.4 Examples

- HeNe laser:
  $L = 0.5m$, $\lambda = 632.8nm$, Doppler width - $\Delta\nu_D = 1.5GHz$.

$$\tau_{rep} = \frac{2L}{c} \approx 3ns \qquad \Delta\tau = \frac{1}{\Delta\nu_D} \approx 0.7ns \tag{14.8}$$

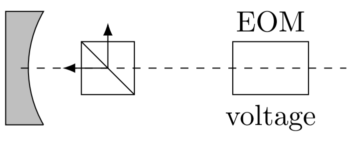
  There are four modes a lasing, 3 nanoseconds, 2 lectures left, and a partidge in a pear treeeeeee.
- Ti:Sapphire:
  $L = 2m$, Gain from $700nm - 1000nm \rightarrow 120THz$.

$$\tau_{rep} = \frac{2L}{c} \approx 12ns \qquad \Delta\tau = \frac{1}{\Delta\nu} \approx 10fs \tag{14.9}$$

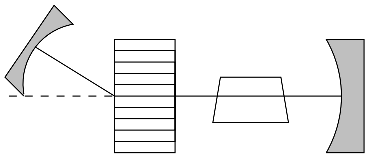
  This is roughly equivalent to 1.6 million modes lasing.

Mode locking can be achieved using active or passive methods.

- Active Mode Locking
  - Electro-Optic Modulator: used with a polarisor, applying a voltage can let light through at repetition rate, by rotating polarisation.

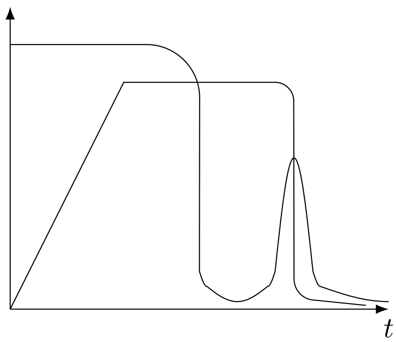


- ➥ Acousto-Optic Modulator: sound wave in crystal using Bragg diffraction.



14.2 Q Switching

Reminder - cavity linewidth $\Delta\omega_c = \frac{1}{\tau_c}$, and a Quality factor, $Q = \frac{\omega}{\Delta\omega_c} = \omega\tau_c$. The cavity Q factor is switched dynamically to control the build up of gain.



- ➤ Initially cavity has high loss (low Q)
- ➤ Pumping leads to build up of population in upper level (requires long τ_c to stop decay)
- ➤ When population inversion = gain reaches steady state, switch to low loss-high Q
- ➤ Suddenly there is a cavity and radiation builds up.
  - ➥ Exponential growth leads to development of a single giant pulse which extracts a large fraction of the energy from the gain medium (depletes N^*)

Historically achieved using rotating mirrors, while modern lasers use the AOM or EOM methods - pulses limited to $\approx 100ps$.

Lecture 15 Nonlinear Optics

We have net non-linear optical effects as **saturation**, **frequency doubling**, and **Keir lensing**; here we consider the physics of optical non-linearities, and how non-linearities are used in laser physics and fundamental research.
Nonlinear optics - *"the modification of the optical properties of a medium by the presence of light."* Typically, only laser light is intense enough to do this.

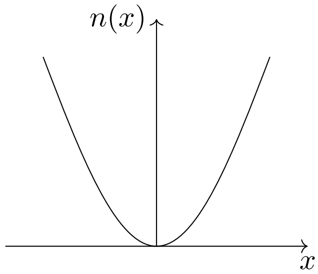
Consider the **dipole moment per unit volumen**, or **polarisation**:

$$P(t) = \epsilon_0 \chi E(t) \tag{15.1}$$

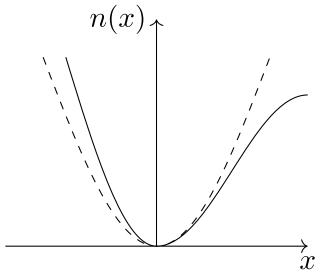
Where our variables are: ϵ_0 - permittivity of free space; $E(t)$ - electric field; χ - electric susceptibility, which is related to refractive index η, by

$$\eta = \sqrt{1+\chi}. \tag{15.2}$$

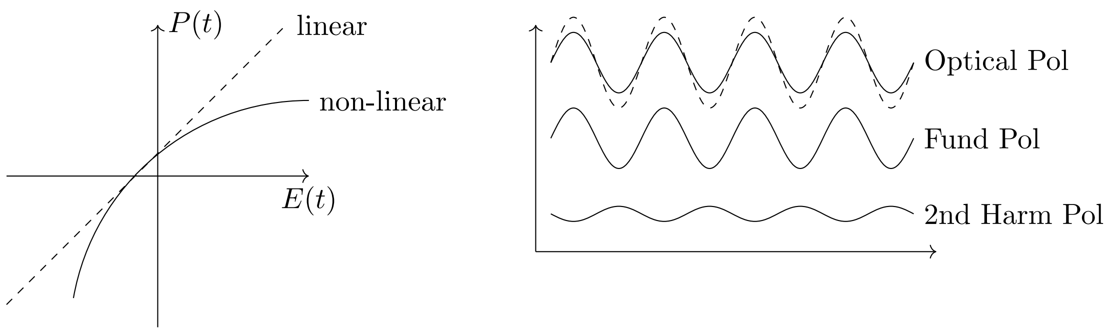
- Electron moving in a parabolic (harmonic) potential:
  Charge radiated with the same frequency as the driving field.



- If the potential is **anharmonic**, the picture changes:
  This is what the potential might look like for a non centro-symmetric material.



For low amplitude driving, the potential looks harmonic - high intensity lasers are needed to observe non-linear effects.



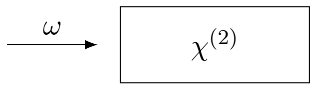
Nonlinear optics result in frequencies which pick up other harmonics, not just the fundamental frequency. For an anharmonic potential, express the polarisation as a power series in field strength:

$$P(t) = \epsilon_0 \left[\chi^{(1)}E(t) + \chi^{(2)}E^2(t) + \chi^{(3)}E^3(t) + \dots\right] \tag{15.3}$$

$$P(t) = P^{(1)}(t) + P^{(2)}(t) + P^{(3)}(t) + \dots \tag{15.4}$$

The (i) numbers are just notation differentiating $\chi^{(i)}$ as the different order susceptibilities. We take these as scalar quantities for simplicity ($P^{(1)}$ etc). In general, χ are frequency dependent but we neglect that here. Also assume the response to driving is instantaneous.

15.1 Second Order Effects



Consider the time dependence of the electric field.

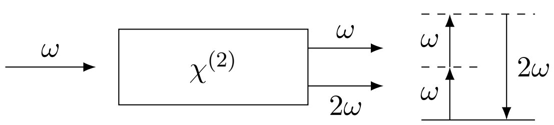
$$E(t) = E_0 e^{-i\omega t} + E_0^* e^{i\omega t} \tag{15.5}$$

Second Order polarisation:

$$\begin{aligned} P^{(2)}(t) &= \epsilon_0 \chi^{(2)} E(t)^2 && (15.6)\\ &= \epsilon_0 \chi^{(2)} \left[E_0^* E_0 + E_0 E_0^* + E_0^2 e^{-i2\omega t} + E_0^{*2} e^{-2\omega t}\right] && (15.7)\\ &= 2\epsilon_0 \chi^{(2)} E_0 E_0^* + \epsilon_0 \chi^{(2)} E_0^2 e^{-i2\omega t} + \epsilon_0 \chi^{(2)} E_0^{*2} e^{i2\omega t} && (15.8) \end{aligned}$$

The first term describes the time-dependent DC field referred to as **optical rectification**; the latter terms which contain frequencies at twice the original frequency.

15.1.1 Second Harmonic Generation



- Not all light converted but can be highly efficient.
- For every two photons with angular momentum ω, we get one out with 2ω.
- Note: spin is not conserved here (but not of concern).
- Initial path is shining a laser onto a crystal.
- This is used for **frequency doubling**, e.g. Nd:YAG at $1064nm \rightarrow 532mm$.

15.2 Third Order Polarisation

$$P^{(3)}(t) = \epsilon_0 \chi^{(3)} E(t)^3 \tag{15.9}$$

Consider applied field,

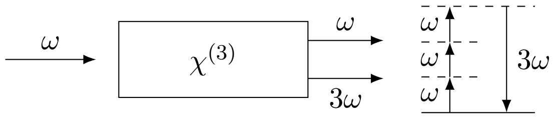
$$E(t) = E_0 \cos(\omega t) \tag{15.10}$$

Using the trig identity:

$$\begin{aligned} \cos^3(\omega t) &= \frac{1}{4}\cos(3\omega t) + \frac{3}{4}\cos(\omega t) && (15.11)\\ \implies P^{(3)}(t) &= \frac{1}{4}\epsilon_0 \chi^{(3)} E_0^3 \cos(3\omega t) + \frac{3}{4}\epsilon_0 \chi^{(3)} E_0^3 \cos(\omega t) && (15.12)\\ &= \chi^{(3)} \epsilon_0 E_0^3 \left[\frac{1}{4}\cos(3\omega t) + \frac{3}{4}\cos(\omega t)\right] && (15.13) \end{aligned}$$

Similar to the second order case, we can achieve Third Harmonic Generation.

15.2.1 Third Harmonic Generation



Sum and difference generation can also be achieved; typically we will have $\chi^{(2)} \approx 10^{-11}[mV^{-1}]$, $\chi^{(3)} \approx 10^{-22}[m^2V^{-2}]$.

15.3 Intensity Dependent Refractive Index

Consider a component of the field at frequency ω:

$$P = \epsilon_0 \left(\chi^{(1)} + \frac{3}{4}\chi^{(3)}E_0^2 \right) E_0 \cos(\omega t) \tag{15.14}$$

This looks like a linear susceptibility with an additional non-linear term:

$$\chi = \chi_{linear} + \chi_{non-linear} = \chi^{(1)} + \frac{3}{4}\chi^{(3)}E_0^2 \tag{15.15}$$

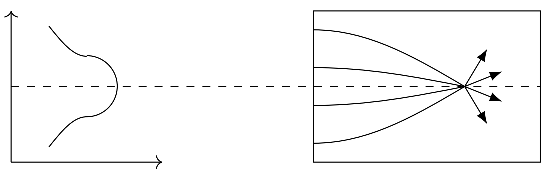
and since the refractive index

$$\begin{aligned} \eta &= \sqrt{1+\chi} && (15.16)\\ &= \sqrt{1+\chi_L+\chi_{NL}},\ \eta_0 = \sqrt{1+\chi_L} && (15.17)\\ &= \eta_0\sqrt{1+\frac{\chi_{NL}}{2\eta_0^2}} && (15.18) \end{aligned}$$

Using a Taylor expansion, since $\chi_{NL} \ll \eta_0^2$:

$$\eta = \eta_0 + \frac{3}{8}\frac{\chi^{(3)}}{\eta_0}E_0^2 = \eta_0 + \eta_2 I \tag{15.19}$$

η_2 is the **Second Order Non-linear Refractive Index.** It is very small for most materials, e.g. $10^{-20}m^2/W$ for glass. This is often called the optical (or AC) Kerr effect, and leads to **self-focusing**:



- ➤ Care is required as it can lead to damage due to 'runaway' effect.
- ➤ Used to create ultrafast pulses in mode-locked lasers.
- ➤ The intensity profile of the beam creates equivalent to graded refractive index (GRIN) lens.
- ➤ Also interesting physics, e.g. optical solitons.

Part II

Quantum Information and Computing

Lecture 1

1.1 What is a quantum computer?

It's like a classical computer, but we replace 'bits' (0s and 1s) with *qubits.* But what is a qubit? A qubit is a 2-level quantum system, with the quantum levels referred to $|0\rangle$ and $|1\rangle$.

Examples of 2-level systems:

- Spin-$\frac{1}{2}$ particle: 2 states are spin 'up' and spin 'down'.
- Photon: 2 polarisations, e.g. vertical and horizontal or left-circular and right-circular.
- Atoms, ions, molecules with many energy levels and we can select 2 as our qubit states.
- 'Artifical atoms' in solid state, e.g. quantum dots in semiconductors or LC resonator in a superconductor.

List five physical implementations of qubits and their problems:

- Two energy levels in an atom trapped by an optical tweezer - difficult to localise.
- Two energy levels in an ion trapped using electrodes - it's only in one dimension (scaling problem).
- Two energy levels of an impurity ion (spin) in a semi-conductor (e.g. phosphorous in Si) - interacts with surroundings, i.e. Silicon.
- Two energy levels of an LC circuit in a superconductor - very bulky and needs $10mK$ cryostat.
- Two polarisation modes of a photon - photons don't interact.

Lecture 2

2.1 DiVincenzo Criteria

The DiVincenzo criteria are often used to frame discussions about the advantages and disadvantages of different quantum computing platforms. The five criteria are:

1. Initialisation (**state preparation**) - typically means the ability to prepare identical qubits through cooling and address each qubit independently (localisation).
2. A universal set of quantum **gates** - single- and two-qubit gates at minimum.
3. Measurement (**read out**) of $|0\rangle$ or $|1\rangle$.
4. Low **decoherence** - qubits isolated from environment (external world).
5. **Scalability** - the ability to scale up to say 100 or 1000 or more qubits.

2.2 Why Quantum Computing?

1. Moore's law - as transistor size is reduced, we approach atomic dimensions.
2. Energy efficiency - replace dissipative classical gates with reversible quantum gates.
3. Quantum 'advantage' - quantum computers can store more information and compute (certain problems) much faster.

A classical bit has states 0 and 1: N bits have $2N$ states; a qubit has states $0\rangle$ and $|0\rangle$: N qubits have 2^N states, i.e. exponential scaling. To see why, we need the **Qubit State Vector**.

$$|\psi\rangle = a|0\rangle + b|1\rangle, \tag{2.1}$$

where a and b are complex coefficients that may be time-dependent which obey the normalisation criterion.

Now we want a 2 qubit state vector, where our two qubits are A and B, as a normalised product state:

$$\begin{aligned} |\Psi\rangle_{AB} &= (a|0\rangle_A + b|1\rangle_A) \otimes (c|0\rangle_B + d|1\rangle_B), && (2.2) \\ &= ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle. && (2.3) \end{aligned}$$

In addition to product, we can have **entangled states** that are not factorisable into products.

What about 3 qubits, A, B, C?

$$|\Psi\rangle_{ABC} = c_{000}|000\rangle + c_{001}|001\rangle + c_{010}|010\rangle + c_{011}|011\rangle + c_{100}|100\rangle + c_{101}|101\rangle + c_{110}|110\rangle + c_{111}|111\rangle, \tag{2.4}$$

which is $2^3 = 8$ states. For 4 qubits, we would have $2^4 = 16$ states; for N qubits, 2^N states. For 40 qubits, $2^{40} \approx 10^{12}$; for 100 qubits, $2^{100} \approx 10^{30}$.

Lecture 3 Two-level quantum mechanics

Can think of the state vector similar to spin:

$$|\psi\rangle = a|0\rangle + b|1\rangle = a|\uparrow\rangle + b|\downarrow\rangle = \begin{pmatrix} a \\ b \end{pmatrix}. \tag{3.1}$$

How does the time-dependence appear? $0\rangle$ and $|1\rangle$ are solutions of a Schrodinger equation,

$$i\hbar\frac{\partial}{\partial t}|\alpha\rangle = H_0|\alpha\rangle, \tag{3.2}$$

with energies E_0 and E_1:

$$i\hbar\frac{\partial}{\partial t}|0\rangle = E_0|0\rangle,\ i\hbar\frac{\partial}{\partial t}|1\rangle = E_1|1\rangle, \tag{3.3}$$

$$|\psi(t=0)\rangle = a(0)|0\rangle \implies a(t) = a(0)e^{-iE_0t/\hbar}, \tag{3.4}$$

$$|\psi\rangle = ae^{-iE_0t/\hbar}|0\rangle + be^{-iE_1t/\hbar}|1\rangle, \tag{3.5}$$

$$= e^{-E_0t/\hbar}\left(a|0\rangle + be^{-i\omega_0t}|1\rangle\right). \tag{3.6}$$

ω_0 is the angular resonant frequency of the qubit, $\omega_0 = (E_1 - E_0)/\hbar$. We define

$$G \equiv e^{-iE_0t/\hbar} \text{ - global phase,} \qquad R \equiv e^{-i\omega_0t} \text{ - relative phase.} \tag{3.7}$$

$$|a|^2 + |b|^2 = 1 \tag{3.8}$$

$$|\psi\rangle = a|0\rangle + be^{i\phi}|1\rangle, \tag{3.9}$$

So only have two free parameters in the ratio $\frac{a}{b}$ and the phase ϕ.

$$|\psi\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} a \\ be^{i\phi} \end{pmatrix} \tag{3.10}$$

$\tan\frac{\theta}{2} = \frac{b}{a}$ and θ and ϕ are angles (from z down and x round respectively as spherical coordinates. So we can talk about our state vector as a Bloch vector, with all possible states of the qubit are points on the Bloch sphere.

3.1 Single-qubit gates

All single qubits are rotations on the Bloch sphere. These rotations are described by unitary operator, e.g. U,

$$|\psi_f\rangle = U|\psi_i\rangle, \tag{3.11}$$

where U is a 2×2 matrix. We can write U as a sum of Pauli spin matrices (and the identity matrix),

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \hat{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.12}$$

Example: Density operator

The density operator is defined as:

$$\hat{\rho} = |\psi\rangle\langle\psi|. \tag{3.13}$$

We will write this in the form of the Pauli matrices and identity, then as a single matrix:

$$\hat{\rho} = \frac{1}{2}\left(\hat{\sigma}_0 + u\hat{\sigma}_x + v\hat{\sigma}_y + w\hat{\sigma}_z\right) \tag{3.14}$$

$$= \frac{1}{2}\begin{pmatrix} 1+w & u-iv \\ u+iv & 1-w \end{pmatrix}. \tag{3.15}$$

u, v, w are the expectation values of $\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$ for the state $|\psi\rangle$.

Example: Short Question #7

Let's take a particular state,

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\phi}|1\rangle\right). \tag{3.16}$$

What is the value of θ?

$$\cos\frac{\theta}{2} = \sin\frac{\theta}{2} = \frac{1}{\sqrt{2}} \implies \frac{\theta}{2} = \frac{\pi}{4} \implies \theta = \frac{\pi}{2}. \tag{3.17}$$

Bloch vector is in equatorial plane. Now calculate expectation values of $\hat{\sigma}_i$:

$$\langle\psi|\hat{\sigma}_x|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & e^{-i\phi} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{i\phi} \end{pmatrix} \tag{3.18}$$

$$= \frac{1}{2}\left(e^{i\phi} + e^{-i\phi}\right) = \cos\phi. \tag{3.19}$$

$\phi = 0 \implies u = +1, \phi = \pi \implies u = -1.$

$$\langle\psi|\hat{\sigma}_y|\psi\rangle = \frac{1}{2}\begin{pmatrix} 1 & e^{-i\phi} \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\phi} \end{pmatrix} \tag{3.20}$$

$$= \frac{1}{2i}\left(e^{i\phi} + e^{-i\phi}\right) = \sin\phi. \tag{3.21}$$

$$\langle\psi|\hat{\sigma}_z|\psi\rangle = \frac{1}{2}\begin{pmatrix} 1 & e^{-i\phi} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\phi} \end{pmatrix} \tag{3.22}$$

$$= \frac{1}{2}(1-1) = 0. \tag{3.23}$$

So the Bloch vector is in equatorial plain as thought.

Lecture 4

- Any qubit can be represented by a point on the Bloch sphere.
- We have two parameters: the polar angle, θ; and the azimuthal angle, ϕ.
- The qubit state vector is defined as

$$|\psi\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}. \tag{4.1}$$

In Cartesian coordinates, the qubit state is given by the Bloch vector $\underline{b} = (u, v, w)$, with $\sqrt{u^2+v^2+w^2} = 1$. This Cartesian description allows for

$$\sqrt{u^2+v^2+w^2} < 1, \tag{4.2}$$

i.e. point inside the sphere. In the real world, we have decoherence where the effect is to shorten the Bloch vector. We have two types of states:

- Pure states: $\sqrt{u^2+v^2+w^2} = 1$, on the Bloch sphere, represented by the wavefunction $|\psi\rangle$.
- Mixed states: $\sqrt{u^2+v^2+w^2} < 1$, inside the Bloch sphere.

We cannot describe mixed states by a wavefunction, so we introduce the density matrix.

4.1 Density matrix for a pure state of a qubit

We define the density matrix as:

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2} \end{pmatrix} \tag{4.3}$$

$$= \begin{pmatrix} \cos^2\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+\cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & 1-\cos\theta \end{pmatrix} \tag{4.4}$$

$$= \frac{1}{2}\begin{pmatrix} 1+\cos\theta & \sin\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\phi + i\sin\theta\sin\phi & 1-\cos\theta \end{pmatrix} \tag{4.5}$$

$$\rho = \frac{1}{2}\begin{pmatrix} 1+w & u-iv \\ u+iv & 1-w \end{pmatrix} = \frac{1}{2}\left(\hat{\sigma}_0 + u\hat{\sigma}_x + v\hat{\sigma}_y + w\hat{\sigma}_z\right). \tag{4.6}$$

What is the expectation value of ρ for our wavefunction? If we have a normalised wavefunction, $\langle\psi|\rho|\psi\rangle = \langle\psi|\psi\rangle\langle\psi|\psi\rangle = 1$.

$$\langle\psi|\rho|\psi\rangle = \frac{1}{2}\left(\langle\psi|\sigma_0|\psi\rangle + u\langle\psi|\sigma_x|\psi\rangle + \cdots\right) \tag{4.7}$$

$$= \frac{1}{2}\left(1+u^2+v^2+w^2\right) = 1 \tag{4.8}$$

For mixed states, $\sqrt{u^2+v^2+w^2} < 1$, so this will no longer be the case. We will explore that properly later.

4.2 Qubit Rotations

All unitary operations (that conserve the norm $\langle\psi|\psi\rangle$) can be described as rotations of the Bloch vector. A general rotation by an angle Θ about a unit vector $\underline{n} = (n_x, n_y, n_z)$. We define the Rotation operator,

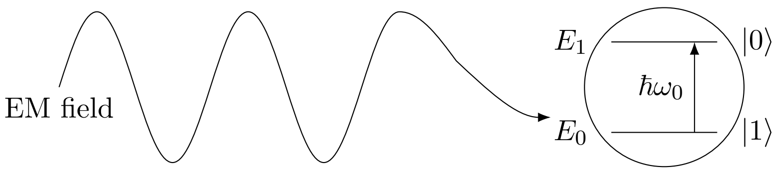
$$\hat{R} = \exp\left(-i\frac{1}{2}\underline{\hat{\sigma}}\cdot\underline{n}\Theta\right) \tag{4.9}$$

$$= \hat{\sigma}_0\cos\frac{\Theta}{2} - i\underline{\hat{\sigma}}\cdot\underline{n}\sin\frac{\Theta}{2} \tag{4.10}$$

$$= \begin{pmatrix} \cos\frac{\Theta}{2} - in_z\sin\frac{\Theta}{2} & (-in_x - ny)\sin\frac{\Theta}{2} \\ (-in_x + n_y)\sin\frac{\Theta}{2} & \cos\frac{\Theta}{2} + in_z\sin\frac{\Theta}{2} \end{pmatrix} \tag{4.11}$$

Lecture 5

How do we implement rotation on the Bloch sphere? We need to drive the qubit with a resonant (or near-resonant) EM field.



The qubit is like an oscillator we can drive. EM field:

$$\mathcal{E} = \mathcal{E}_0 \cos(\phi_L - \omega t), \tag{5.1}$$

where ϕ_L is the laser phase. We could include any phase offset. We assume the qubit is at $\underline{r} = 0$, and $\phi = 0$ at $t = 0$, so $\phi_L = \underline{k} \cdot \underline{r}$. This then depends on the relative position of the qubit and field. Note that since there is always some δr uncertainty in position of the qubit, this gives rise to a range of phase ϕ_L which can influence the qubit behaviour, e.g. phonons are solid vibrating all the time $\rightarrow$ phase is changing. Hence inside the sphere for the qubit and not on the sphere. We can try to reduce this by lowering temperatures. Other notation:

$$\Delta = \omega - \omega_0, \tag{5.2}$$

where ω_0 is the qubit frequency, ω the field frequency, and Δ is defined as the qubit-field detuning. If the qubit is subject to an external field, the coefficients a and b may become functions of time, and we should write the qubit state vector in the form

$$|\psi(t)\rangle = a(t)|0\rangle e^{-iE_0 t/\hbar} + b(t)|1\rangle e^{-iE_1 t/\hbar}. \tag{5.3}$$

The time dependence of the coefficients a and b depends on the interaction between the qubit and the EM field. We can use the Schrodinger equation:

$$i\hbar \frac{\partial}{\partial t}|\psi\rangle = (\mathcal{H}_0 + \mathcal{H}')|\psi\rangle. \tag{5.4}$$

Previously we just had $\mathcal{H}_0$ which describes the qubit, but now we add $\mathcal{H}'$ which is the perturbation due to the EM field. Typically, this interaction is of the form

$$\mathcal{H}' = -\hat{\underline{d}} \cdot \underline{\mathcal{E}}, \qquad = -d\mathcal{E}_0 \cos(\phi_L - \omega t), \tag{5.5}$$

where $\underline{d} = -e\hat{\underline{r}}$ is the electric dipole operator. and we can also define the Rabi frequency in Hz as

$$\Omega = -\frac{d\mathcal{E}_0}{\hbar}, \tag{5.6}$$

$$\mathcal{H}' = \hbar\Omega \cos(\phi_L - \omega t). \tag{5.7}$$

The Rabi frequency is a measure of the coupling between the field and the qubit. If the field is put on and left on, then the state will oscillate between $|0\rangle$ and $|1\rangle$ at the Rabi frequency. Subbing the perturbation, we get two equations for a and b:

$$i\dot{a}(t) = \frac{1}{2}\Omega e^{-i\phi_L} e^{i\Delta \cdot t} b, \qquad i\dot{b}(t) = \frac{1}{2}\Omega e^{i\phi_L} e^{-i\Delta \cdot t} a. \tag{5.8}$$

These equations have an explicit time dependence. To deal with this, we can go into a rotating frame using a substitution:

$$\tilde{a} = a e^{-it\cdot\Delta/2}, \qquad \tilde{b} = b e^{it\cdot\Delta/2}. \tag{5.9}$$

This is so that we spin at the same frequency as the qubit and since we are interested in the relative phase of the field and qubit, this makes sense. The Schrodinger equation for the interaction between a qubit and an oscillatory EM field is

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \mathcal{H}_{int}|\psi\rangle, \tag{5.10}$$

where for the vector for of the state vector, the **interaction Hamiltonian** is a 2×2 matrix:

$$\mathcal{H}_{int} = \frac{\hbar}{2}\begin{pmatrix} \Delta & \Omega e^{-i\phi_L} \\ \Omega e^{i\phi_L} & -\Delta \end{pmatrix}, \tag{5.11}$$

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} \Delta & \Omega e^{-i\phi_L} \\ \Omega e^{i\phi_L} & -\Delta \end{pmatrix}\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix}. \tag{5.12}$$

We can also write $\mathcal{H}_{int}$ in terms of the spin matrices. Note that we have the detuning on the diagonals (like $\hat{\sigma}_z$) and to get the exponential, we need cos and sin terms:

$$\mathcal{H}_{int} = \frac{\hbar}{2}\left[\Delta\hat{\sigma}_z + \Omega\left(\cos\phi_L\hat{\sigma}_x + \sin\phi_L\hat{\sigma}_y\right)\right], \tag{5.13}$$

where $\hat{\sigma}_y$ contains the imaginary values necessary for the exponents. Consider zero detuning, i.e. $\Delta = 0, \omega_0 = \omega$, and $\phi_L = 0$:

$$i\dot{a} = \frac{1}{2}\Omega b, \qquad i\dot{b} = \frac{1}{2}\Omega a. \tag{5.14}$$

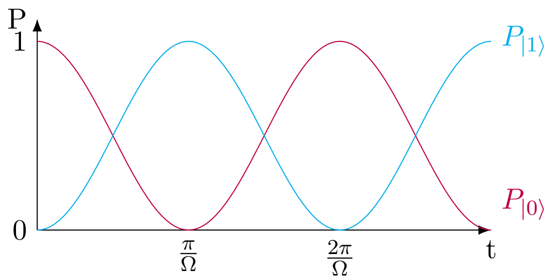
Taking time derivatives again:

$$i\ddot{a} = \frac{1}{2}\Omega\dot{b} = -\frac{i}{4}\Omega a \implies \ddot{a} = -\frac{1}{4}\Omega^2 a. \tag{5.15}$$

For $a(0) = 1, b(0) = 0$, $a = \cos\left(\frac{\Omega t}{2}\right)$. Getting probabilities:

$$P_{|0\rangle} = |a|^2 = \cos^2\left(\frac{\Omega t}{2}\right) = \frac{1}{2}(1 + \cos\Omega t), \tag{5.16}$$

$$P_{|1\rangle} = |b|^2 = \sin^2\left(\frac{\Omega t}{2}\right) = \frac{1}{2}(1 - \cos\Omega t), \tag{5.17}$$



$\theta = \frac{\Omega t}{2}$ is the rotation angle. The effect of the resonant EM field is to drive the qubit from $|0\rangle \rightarrow |1\rangle$ and back at Ω frequency.

The qubit follows a line of longitude. ϕ_L determines the direction in the xy plane about which we rotate, i.e. can change which longitude line we travel around. Derive rotation matrix $\hat{R}$ from the interaction Hamiltonian:

$$|\psi(t)\rangle = e^{-i\mathcal{H}_{int}t/\hbar}|\psi(0)\rangle = \hat{R}|\psi(0)\rangle. \tag{5.18}$$

To find the rotation matrix, we rewrite the interaction in the form:

$$\hat{R} = e^{-i\mathcal{H}_{int}t/\hbar} = e^{-(i/2)\hat{\sigma}\cdot\underline{\hat{n}}\Theta}, \quad \Theta = t\sqrt{\Omega^2 + \Delta^2}. \tag{5.19}$$

We had defined $\mathcal{H}_{int}$ as

$$\mathcal{H}_{int} = \frac{\hbar}{2}\left[\Delta\hat{\sigma}_z + \Omega\left(\cos\phi_L\hat{\sigma_x} + \sin\phi_L\hat{\sigma}_y\right)\right], \tag{5.20}$$

so using this in $\hat{R}$ above, we can determine $\underline{\hat{n}}$ to be

$$\underline{\hat{n}} = \frac{1}{\Theta}\begin{pmatrix}\Omega\cos\phi_L & \Omega\sin\phi_L & \Delta\end{pmatrix}. \tag{5.21}$$

Lecture 6

From normalisation we define $\Theta = t\sqrt{\Omega^2+\Delta^2}$, with the quantity in brackets known as the *effective Rabi frequency.* Substituting the unit vector $\underline{\hat{n}}$ above into the rotation matrix:

$$\hat{R} = \begin{pmatrix} \cos\frac{\theta}{2} - in_z\sin\frac{\theta}{2} & (-in_x - n_y)\sin\frac{\theta}{2} \\ (-in_x + n + y)\sin\frac{\theta}{2} & \cos\frac{\theta}{2} + in_z\sin\frac{\theta}{2} \end{pmatrix} \tag{6.1}$$

$$= \begin{pmatrix} \cos\frac{\theta}{2} - i\frac{\Delta}{\Theta}\sin\frac{\theta}{2} & -i\frac{\Omega}{\Theta}e^{i\phi_L}\sin\frac{\theta}{2} \\ -i\frac{\Omega}{\Theta}e^{-i\phi_L}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} + i\frac{\Delta}{\Theta}\sin\frac{\theta}{2} \end{pmatrix}. \tag{6.2}$$

This is known as the **Rabi solution.** We consider the first of 3 special cases:
Case 1: $\Delta = 0$, i.e. on resonance. The Rabi solution reduces to

$$\hat{R} = \begin{pmatrix} \cos\frac{\Omega t}{2} & -ie^{i\phi_L}\sin\frac{\Omega t}{2} \\ -ie^{-i\phi_L}\sin\frac{\Omega t}{2} & \cos\frac{\Omega t}{2} \end{pmatrix}, \tag{6.3}$$

which corresponds to a rotation about a vector in the equatorial plane in the Bloch sphere. For $\Delta = 0$, we often use notation $\hat{R}(\theta, \phi_L)$ where $\theta = \frac{\Omega t}{2}$ is the rotation angle and ϕ_L determines the direction in the xy plane about which we rotate. For an atom initially in state $|0\rangle$, i.e.

$$|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle, \tag{6.4}$$

and an interaction of duration t, the Rabi solution gives

$$|\psi(t)\rangle = \hat{R}|\psi(0)\rangle = \begin{pmatrix} \cos\frac{\Omega t}{2} \\ -ie^{-i\phi_L}\sin\frac{\Omega t}{2} \end{pmatrix} = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}, \tag{6.5}$$

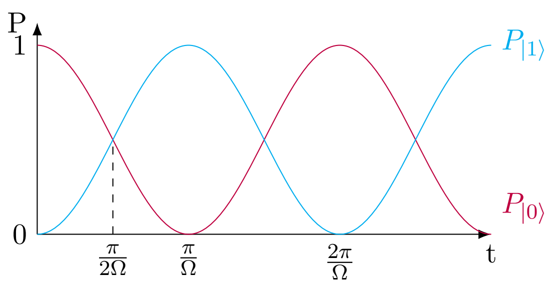
$$a(t) = \cos\frac{\Omega t}{2}, \qquad b(t) = -ie^{-i\phi_L}\sin\frac{\Omega t}{2}, \tag{6.6}$$

and we get the populations in states $|0\rangle, |1\rangle$ from

$$P_{|0\rangle} = |a(t)|^2 = \cos^2\frac{\Omega t}{2}, \tag{6.7}$$

$$P_{|1\rangle} = |b(t)|^2 = \sin^2\frac{\Omega t}{2}, \tag{6.8}$$

which are the **Rabi oscillations.** The population oscillates between the two states at the Rabi frequency, Ω. The Rabi frequency is proportional to the field amplitude, i.e. the square root of the field intensity.



Another interesting case is when there is no interaction ($\Omega = 0$) but the field and the qubit have a different frequency $\Delta \neq 0$, i.e. *free evolution rotations.* In this case, the rotation matrix is

$$\hat{R} = \begin{pmatrix} \exp\left(-i\frac{\Delta t}{2}\right) & 0 \\ 0 & \exp\left(i\frac{\Delta t}{2}\right) \end{pmatrix}, \tag{6.9}$$

which corresponds to a rotation about the z-axis in the Bloch sphere. We use this matrix to describe the free evolution in a *Ramsey interferometer.*

6.1 Single Qubit Rotations

The Rabi solution describes how a near-resonance EM field may be used to drive the qubit from any initial state to any final state. Refer to this operation as a **single-qubit rotation** or **single-qubit gate.** Three useful interactions come from the times you pulse, e.g. as above the $\frac{\pi}{2}$-, π-, and 2π-pulse. For all except the **Hadamard gate**, we use resonant driving $\Delta = 0$.

- $\frac{\pi}{2}$-pulse (particularly useful):
  For a $\frac{\pi}{2}$ pulse, we choose the EM field intensity and the pulse duration such that $\Omega t = \frac{\pi}{2}$. This corresponds to a rotation of 90° on the Bloch sphere which could take us from pole to equator, i.e. a qubit initially in $|0\rangle$ or $|1\rangle$ is excited $\rightarrow$ an equal superposition on the equator (or start on the equator and rotate to a pole). Substituting $\Omega t = \frac{\pi}{2}$ in the Rabi solution, we obtain

$$|\psi(t_{\pi/2})\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -ie^{i\phi_L} \\ -ie^{-i\phi_L} & 1 \end{pmatrix} |\psi(0)\rangle. \tag{6.10}$$

  To simplify the matrix, we can choose a particular laser phase, e.g. $\phi_L = \frac{\pi}{2}$, so:

$$\hat{R}(\theta, \phi_L) = \hat{R}\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \tag{6.11}$$

  and acting on our initial state $|\psi(0)\rangle = |0\rangle$,

$$|\psi(t_{\pi/2})\rangle = \hat{R}\left(\frac{\pi}{2}, \frac{\pi}{2}\right) |\psi(0)\rangle \tag{6.12}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{6.13}$$

$$= \frac{1}{\sqrt{2}} \left(|0\rangle - |1\rangle\right), \tag{6.14}$$

  which is a superposition between $|0\rangle$ and $|1\rangle$. This corresponds to a rotation around the y axis, ending up in the $-x$ direction, i.e. the expectation value $\langle\hat{\sigma}\rangle = -1$. If we apply an arbitrary phase ϕ_L to $|1\rangle$, we find

$$|1\rangle \rightarrow \frac{1}{\sqrt{2}} \left(-ie^{i\phi_L}|0\rangle + |1\rangle\right). \tag{6.15}$$

  For $\phi_L = 0$, we would rotate around the x axis and end up along y. So the laser phase sets the vector in the xy plane that we rotate around. If we had started with a superposition, e.g. in Eq (6.14), it carries on to the $|1\rangle$ state if we apply the same $\hat{R}\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$:

$$\hat{R}\left(\frac{\pi}{2}, \frac{\pi}{2}\right) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{6.16}$$

  So 2 $\frac{\pi}{2}$ pulses are equivalent to a π-pulse and takes us from $|0\rangle \rightarrow |1\rangle$. We shouldn't be concerned about the '-1' as when we take the modulus this global phase disappears.
- π-pulse: For $\Omega t_\pi = \pi$,

$$|\psi(t_\pi)\rangle = \begin{pmatrix} 0 & -ie^{i\phi_L} \\ -ie^{-i\phi_L} & 0 \end{pmatrix} |\psi(0)\rangle, \tag{6.17}$$

$$|0\rangle \rightarrow -ie^{-i\phi_L}|1\rangle, \qquad |1\rangle \rightarrow -ie^{i\phi_L}|0\rangle. \tag{6.18}$$

  For $\phi_L = 0$ and $\phi_L = \frac{\pi}{2}$ respectively, the rotation matrix is

$$\hat{R}(\pi, 0) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}; \qquad \hat{R}\left(\pi, \frac{\pi}{2}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{6.19}$$

  Except for the sign change, this is equivalent to a '*bit-flip*' or *NOT gate*.

➤ 2π-pulse: For $\Omega t_{2\pi} = 2\pi$, we have

$$\hat{R}\left(2\pi, \phi_L\right) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\hat{\sigma}_0, \tag{6.20}$$

so it effectively just multiplies the entire wavefunction by -1, which is a global phase factor (and property of spin-$\frac{1}{2}$ particles) that can be useful in gate operations. In the case of two qubits, if one of them picks up a -1, then it does become significant as they are entangled and we can now detect this.

6.2 Hadamard Gate

The application of two successive $\frac{\pi}{2}$-pulses inverts the state. However, it is also useful to have a pulse that takes us into a superposition and back to the same state. This is a **Hadamard gate** or **Hadamard transform**. The Hadamard operator is

$$\mathcal{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{6.21}$$

and the application of two of these leave the state unchanged, i.e. $\mathcal{H}^2 = \hat{\sigma}_0$ (as long as nothing happens in-between applying them). There are different ways to implement a Hadamard. One os to set $\Delta = \Omega$, in which case the Rabi solution becomes

$$\hat{R} = \begin{pmatrix} \cos \frac{\Omega}{\sqrt{2}} t - \frac{i}{\sqrt{2}} \sin \frac{\Omega}{\sqrt{2}} t & -\frac{i}{\sqrt{2}} e^{i\phi_L} \sin \frac{\Omega}{\sqrt{2}} t \\ -\frac{i}{\sqrt{2}} e^{-i\phi_L} \sin \frac{\Omega}{\sqrt{2}} t & \cos \frac{\Omega}{\sqrt{2}} t + \frac{i}{\sqrt{2}} \sin \frac{\Omega}{\sqrt{2}} t \end{pmatrix}, \tag{6.22}$$

and if we set $\frac{\Omega}{\sqrt{2}} t = \frac{\pi}{2}$ and $\phi_L = 0$, we obtain

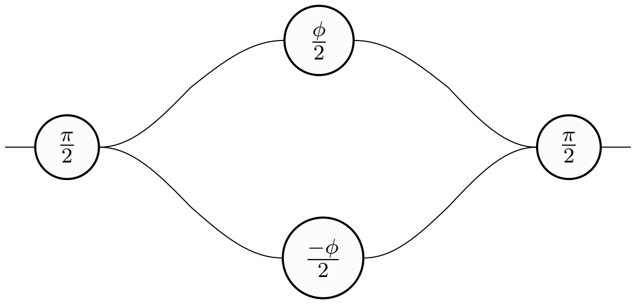
$$\hat{R} = -\frac{i}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{6.23}$$

so $\hat{R}$ is the Hadamard up to some global factor which can be ignored. The Hadamard is a rotation around some 45°; applying twice brings us back to the start.

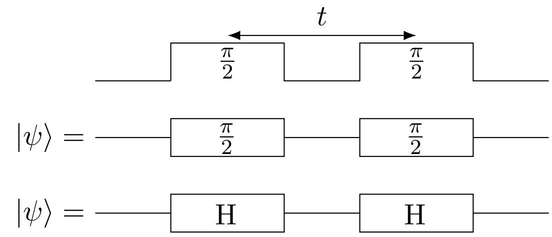
Lecture 7

7.1 Ramsey Interferometry

We have two $\frac{\pi}{2}$ pulses separated by time t (or Hadamards):



We can also schematically represent paths in Hilbert space:



There is a read-out of relative phase. This interval t is often called the *free evolution time,* but this could include an interaction. During this free evolution, the qubit Bloch vector rotates around the z-axis. We can describe this precession using a rotation matrix:

$$\hat{R}_z(\phi) = \begin{pmatrix} e^{i-\phi/2} & 0 \\ 0 & e^{-\phi/2} \end{pmatrix} = \begin{pmatrix} e^{-i\Delta t/2} & 0 \\ 0 & e^{i\Delta t/2} \end{pmatrix}, \tag{7.1}$$

so we can say that the Bloch vector precesses around the z-axis at a rate given by the difference between the field and qubit evolution, $\Delta = \omega - \omega_0$. The total rotation angle is $\phi = \Delta t$.

The complete Ramsey sequence is given by two $\frac{\pi}{2}$ pulses separated by free evolution. The $\frac{\pi}{2}$ pulse matrix is obtained from the Rabi solution for $\Delta = 0$.

$$\hat{R} = \begin{pmatrix} \cos\frac{\theta}{2} & -ie^{i\phi_L}\sin\frac{\theta}{2} \\ -ie^{-i\phi_L}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}, \tag{7.2}$$

where $\theta = \Omega t$. We choose $\phi_L = \frac{\pi}{2}$:

$$\hat{R}\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \hat{R}_y\left(\frac{\pi}{2}\right). \tag{7.3}$$

Our complete sequence is given by $\hat{R}\left(\frac{\pi}{2}\right)\hat{R}\left(\phi\right)\hat{R}_y\left(\frac{\pi}{2}\right)$, where we write our sequence of rotations in reverse order.

$$\hat{R}_y\left(\frac{\pi}{2}\right)\hat{R}(\phi)\hat{R}\left(\frac{\pi}{2}\right) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{7.4}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^{-i\phi/2} & e^{-i\phi_2} \\ -e^{-\phi/2} & e^{i\phi/2} \end{pmatrix} \tag{7.5}$$

$$= \begin{pmatrix} -i\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \\ -\cos\frac{\phi}{2} & i\sin\frac{\phi}{2} \end{pmatrix}. \tag{7.6}$$

Initial Condition (at the North Pole):

$$|\psi\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{7.7}$$

$$\hat{R}\left(\frac{\pi}{2}\right)\hat{R}_z(\phi)\hat{R}_y\left(\frac{\pi}{2}\right)|\psi\rangle = \begin{pmatrix} -\sin\frac{\phi}{2} \\ -\cos\frac{\phi}{2} \end{pmatrix} \tag{7.8}$$

Our output state has:

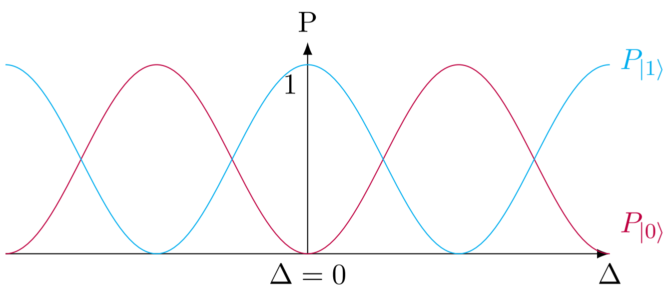
$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -\sin\frac{\phi}{2} \\ -\cos\frac{\phi}{2} \end{pmatrix} \tag{7.9}$$

The probability to be in state $|0\rangle$ at the output is then

$$P_{|0\rangle} = |a|^2 = \frac{1}{2}(1-\cos\phi) \tag{7.10}$$

$$P_{|1\rangle} = |b|^2 = \frac{1}{2}(1+\cos\phi) \tag{7.11}$$

For $\phi = \Delta t$,



At $\Delta = 0$, we see $P_{|0\rangle} \to 0$, because the second $\frac{\pi}{2}$ pulse completes the excitation to state $|1\rangle$. The fringes we see here are known as the Ramsey fringes.

7.2 Two qubits

We like to name our qubits, so we may have:

- A and B
- C (control) and T (target)

Now we need to write our two-qubit state vector to describe our two-qubit system:

$$|\psi\rangle_{AB} = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle. \tag{7.12}$$

Here, $|ij\rangle = |0\rangle \otimes |1\rangle$, i.e. it is an abbreviation for the tensor product. The state vector can also be represented as a column vector:

$$|\psi\rangle_{AB} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, \tag{7.13}$$

where it has four components, a, b, c, d. All two-qubit operators, therefore, are 4×4 matrices.

How do we construct two-qubit operators? As an example, we apply the interaction Hamiltonian $\mathcal{H}_{int}$ to both qubits A and B:

$$\mathcal{H}_2 = \mathcal{H}_{int} \otimes \hat{\sigma}_0 + \hat{\sigma}_0 \otimes \mathcal{H}_{int}, \tag{7.14}$$

where the first term takes our previous 2×2 interaction Hamiltonian and places it in the new 4×4 matrix so it interacts with A and does nothing to B, and the second term such that it interacts with B and not A. For $\Delta = 0$:

$$\mathcal{H}_2 = \frac{\hbar}{2}\begin{pmatrix} 0 & \Omega \\ \Omega & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{\hbar}{2}\begin{pmatrix} 0 & \Omega \\ \Omega & 0 \end{pmatrix} \tag{7.15}$$

$$= \frac{\hbar}{2}\begin{pmatrix} 0 & 0 & \Omega & 0 \\ 0 & 0 & 0 & \Omega \\ \Omega & 0 & 0 & 0 \\ 0 & \Omega & 0 & 0 \end{pmatrix} + \frac{\hbar}{2}\begin{pmatrix} 0 & \Omega & 0 & 0 \\ \Omega & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega \\ 0 & 0 & 0 & \Omega \end{pmatrix} \tag{7.16}$$

$$= \frac{\hbar}{2}\begin{pmatrix} 0 & \Omega & \Omega & 0 \\ \Omega & 0 & 0 & \Omega \\ \Omega & 0 & 0 & \Omega \\ 0 & \Omega & \Omega & 0 \end{pmatrix}. \tag{7.17}$$

It follows that the operators acting on three-qubit systems correspond to 8×8 matrices, and in general, operators acting on N-qubit systems correspond to $2^N \times 2^N$ matrices.

Lecture 8

8.1 Entanglement

Entanglement is what gives quantum computing its 'advantage'. It is essential in all forms of two-qubit gates, such as CNOT ('Control-NOT'). An entangled state cannot be separated into a product of individual qubit state vectors.

8.1.1 Examples of entangled states

The complete set of "maximally entangled" two-qubit states are the so-called *Bell states:*

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle \pm |11\rangle\right) \tag{8.1}$$

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle \pm |10\rangle\right) \tag{8.2}$$

We then can tell something about B if we know A and vice versa: for (8.1), if we measure A in 0, then we know B is 0 too regardless of where B is (non-local effect). We cannot transmit information though, so the limit of transferring is still less than the speed of light; we may know something about B if we observe A, but we cannot tell B that.

8.1.2 Measuring a part of a two-qubit state

This can be particularly interesting for entangled states. We introduce the idea of *projection operators.* If we have a state $|\psi\rangle_{AB}$, what is the probability that A is in the state $|\phi_A\rangle$ regardless of B or that both A and B are in the state $|\phi_{A,B}\rangle$?

We use a projection operator $\hat{P}$ that projects $|\psi_{AB}\rangle$ onto the state we want to see.

$$\text{Probability} = \langle\hat{P}\rangle = \mathrm{Tr}(\hat{\rho}_{AB}\hat{P}), \text{ where } \hat{\rho} = |\psi\rangle_{AB\,AB}\langle\psi| \tag{8.3}$$

is the density matrix and $\hat{P}$ is the projection operator. The trace Tr is the sum of diagonal elements.

Example:
Consider the entangled Bell state

$$|\Phi^{+}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right). \tag{8.4}$$

What is the probability of observing qubit A in state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$? We need to find a projection operator that

1. projects A on $|+\rangle$,
2. and leaves B unchanged.

This operator will read

$$\hat{P} = |+\rangle\langle+| \otimes \hat{\sigma}_0. \tag{8.5}$$

For a single qubit,

$$\hat{P} = \langle\psi|+\rangle\langle+|\psi\rangle = |\langle+|\psi\rangle|^2, \tag{8.6}$$

i.e. the probability overlap between $|\psi\rangle$ and $|+\rangle$. For two qubits, the operator reads

$$\hat{P} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \tag{8.7}$$

$$= \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{8.8}$$

$$= \frac{1}{2}\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\end{pmatrix}. \tag{8.9}$$

Now we define the density matrix as

$$\hat{\rho} = |\Phi^+\rangle\langle\Phi^+| \tag{8.10}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\\ 0\\ 1\end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0 & 1\end{pmatrix} \tag{8.11}$$

$$= \frac{1}{2}\begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 1\end{pmatrix}. \tag{8.12}$$

The product of these is then

$$\hat{\rho}\hat{P} = \frac{1}{2}\begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 1\end{pmatrix} \frac{1}{2}\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\end{pmatrix} \tag{8.13}$$

$$= \frac{1}{4}\begin{pmatrix}1 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 1 & 1 & 1\end{pmatrix} \tag{8.14}$$

$$P_{A+} = \mathrm{Tr}(\hat{\rho}\hat{P}) = \frac{1}{2}, \tag{8.15}$$

i.e. the probability that A is in state $|+\rangle$ is $\frac{1}{2}$. The probability that B is in state $|+\rangle$ is also $\frac{1}{2}$.

What is the probability that both A and B are in state $|+\rangle$? Classically, we would say that $P_{AB} = P_A P_B = \frac{1}{4}$; this is wrong. A and B now have a correlation between them in the quantum system, so we use the projection operators to find the probability that both A and B are in $|+\rangle$. The projection operator reads

$$\hat{P} = |+\rangle\langle+| \otimes |+\rangle\langle+| \tag{8.16}$$

$$= \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \otimes \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \tag{8.17}$$

$$= \frac{1}{4}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\end{pmatrix}. \tag{8.18}$$

Our product reads

$$\hat{\rho}\hat{P} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \tag{8.19}$$

$$= \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}. \tag{8.20}$$

$$\mathrm{Tr}(\hat{\rho}\hat{P}) = \frac{1}{2}, \tag{8.21}$$

i.e. the probability that both are in state $|+\rangle$ is $\frac{1}{2}$. Therefore, $P_{AB} \neq P_A P_B$, because A and B are **correlated** - whenever A is $|+\rangle$, then B is in $|+\rangle$.

Lecture 9

Last time, we looked at entanglement in the Bell states - which are maximally entangled:

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle \pm |11\rangle\right). \tag{9.1}$$

We calculated the probability P_A that A is in the state $|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$, which we found was $P_A = \frac{1}{2}$. The probability P_B that B is in the state $|+\rangle$ was also $P_B = \frac{1}{2}$. Then, we asked what the joint probability that both A and B are in the state $|+\rangle$. Classically, we would say that $P_{AB} = P_A P_B = \frac{1}{4}$, but here this is not the case as these states are correlated, i.e. whenever A is in the state $|+\rangle$, then B must also be in the state $|+\rangle$.
Entanglement implies correlation.

To see this correlation, we can rewrite the Bell state by defining the $|+\rangle, |-\rangle$ basis as

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \qquad |-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right), \tag{9.2}$$

$$|0\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right) \qquad |1\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right). \tag{9.3}$$

Using these definitions, we simply rearrange the Bell state:

$$\begin{aligned}
|\Phi^{+}\rangle &= \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) && (9.4)\\
&= \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)\otimes\frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right) + \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right)\otimes\frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right)\right] && (9.5)\\
&= \frac{1}{\sqrt{2}}\left[\frac{1}{2}\left(|++\rangle + |-+\rangle + |+-\rangle + |--\rangle + |++\rangle - |-+\rangle - |+-\rangle + |--\rangle\right)\right] && (9.6)\\
&= \frac{1}{\sqrt{2}}\left(|++\rangle + |--\rangle\right). && (9.7)
\end{aligned}$$

This is an example of the rotational invariance of the Bell state. So the Bell state appears the same in this basis as before - we have simply 'rotated' the state vector around the Bloch sphere to align along the x-axis. In fact, the Bell state will appear the same regardless of the basis/rotation. The Bell state represents quantum entanglement and correlations which is invariant under rotations, *providing that we choose to measure A and B in the same basis.*

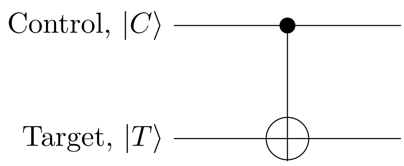
9.1 Qubit Gates

We have two basic types of gates:

- The NOT gate - a single qubit gate
- The Controlled NOT gate - two qubit gate that works on correlation for its output

All computations can be built up from these two basic gates.

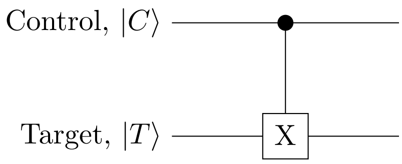
Let's consider the CNOT gate. This is a universal gate, i.e. having just CNOT gates + single qubit rotations is sufficient to build a universal quantum computer. We have:



What does it do? If C is in state $|1\rangle$, then NOT on T, otherwise we do nothing. This can be written as a matrix:

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \approx \begin{Bmatrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{Bmatrix}, \tag{9.8}$$

where the final term is in roughly what each line represents in the 4×4 matrix. For the first two lines, where A is in $|0\rangle$, we do nothing to B, hence the identity form; the latter two lines, where A is in $|1\rangle$, we perform NOT on B - this resembles the Pauli $\hat{\sigma}_x$ matrix. This is also called the Controlled-X (CX) gate, where X represents that Pauli $\hat{\sigma}_x$ matrix.

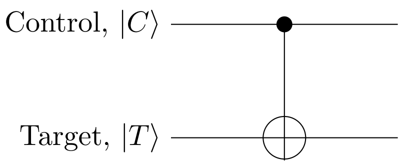


CNOT clearly involves entanglement then. We cannot separate CNOT into a product of single-qubit matrices, e.g.

$$\text{CNOT} \neq \hat{\sigma}_0 \otimes \hat{\sigma}_x. \tag{9.9}$$

So CNOT is an entangling operator.

Example: CNOT creates entanglement
Our Control and Target inputs are:



$$|C\rangle = |+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \tag{9.10}$$

$$|T\rangle = |0\rangle. \tag{9.11}$$

The full input state is then

$$|C\rangle \otimes |T\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes |0\rangle \tag{9.12}$$

$$|CT\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |10\rangle\right). \tag{9.13}$$

This is a **product state** - there is no entanglement between these yet. Now we pass this product state through the CNOT operator to give

$$\text{CNOT}|CT\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right), \tag{9.14}$$

where we not have the Bell state which is maximally entangled by definition. Therefore, we see *CNOT creates entanglement.*

Let's work through this again in matrix form.

$$|CT\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{9.15}$$

$$\mathrm{CNOT}|CT\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{9.16}$$

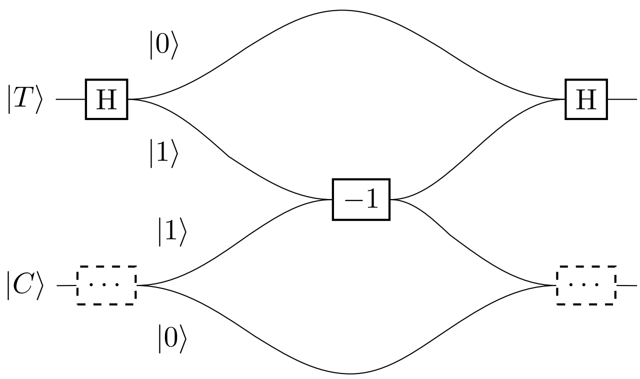
$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \approx \begin{Bmatrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{Bmatrix} \tag{9.17}$$

$$= \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) = |\Phi^+\rangle. \tag{9.18}$$

We get the same result as before.

9.1.1 How to realise CNOT?

We consider a conditional Ramsey interferometer:



If the Control is in state $|1\rangle$, then it applies a -1 factor to the state vector and this -1 factor flips the output of the Ramsey interferometer, i.e. perform the NOT.

9.2 Implementing Quantum Computing in real physical systems

We have a choice of our five possiblities:

1. Atoms
2. Ions
3. Semiconductors
4. Superconductors
5. Photons

We are going to use atoms from here, but the question is how to satisfy the 5 diVincenzo criteria:

1. Initialisation: prepare qubits
2. Gates: single-qubit rotations and two-qubit gates (CNOT)
3. Read-out.
4. Low decoherence.
5. Scalability.

Lecture 10 Rydberg Atom Quantum Computer

1. Preparation: laser cooling (Nobel Prize 1997), and trapping with optical tweezers (Nobel Prize 2018) are used to assemble the atoms into our qubit states, all into state $|0\rangle$ for example to begin with.
2. Gates: qubit gates created using Rydberg states and dipole-dipole interactions.

Strike Impacts:

- Thursday Week 16 and possibly workshop Week 16
- Tuesday Week 17 and possibly workshop Week 17
- All lectures Week 18 and 19

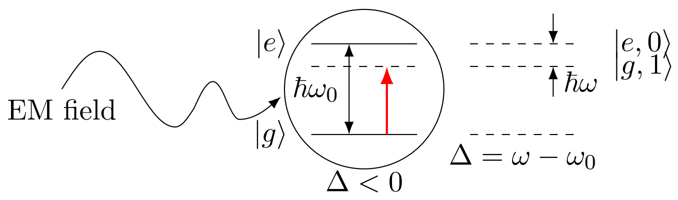
These are the possible impacts of striking, if Charles strikes. We need to believe in the many worlds interpretation of quantum mechanics and only by coming to the lectures and collapsing the wavefunction will we find out if we are in a universe where he is striking or not. The question of what universe Charles is in has been one considered by some of the greatest thinkers of modern times; the results are inconclusive.

10.1 Optical Tweezing

We use one focussed laser beam to move an atom and place it exactly where we want it. Light exerts a force on atoms and they are forced either to the ends of the beam or the focus of it. Two types of light force:

- Optical dipole force - cycles of absorption (stimulated emission and trapping)
- Spontaneous scattering force - cycles of absorption (spontaneous emissing and cooling)

For our optical dipole potential, we recall the two state model, with $\hbar\omega_0$ between $|g\rangle$ and $|e\rangle$, leading to the detuning frequency Δ.



We can form our interaction operator as we have done previously when looking at the Rabi model, yielding

$$\mathcal{H}_{int} = \frac{\hbar}{2}\begin{pmatrix} \Delta & \Omega \\ \Omega & -\Delta \end{pmatrix}. \tag{10.1}$$

We do not consider between $|g\rangle$ and $|e\rangle$, but $|g, 1\rangle$ (the ground state plus a photon) and $|e, 0\rangle$ (the excited state with no photon). These states are only slightly separated by $\hbar(\omega_0 - \omega)$, i.e. $-\hbar\Delta$, from which we can see we have chosen negative detuning where $\Delta < 0$. This gives our bare states when $\Omega = 0$, and then we see that increasing Ω *dresses* the states, increasing the upper state and decreasing the lower state.

$$\mathcal{H}_{int} = \frac{\hbar}{2}\begin{pmatrix} \Delta & \Omega \\ \Omega & -\Delta \end{pmatrix}. \tag{10.2}$$

The energies of the states in the laser field are given by the eigenvalues of $\mathcal{H}_{int}$:

$$\left(\mathcal{H}_{int} - \frac{\hbar}{2}\lambda\hat{\sigma}_0\right)\begin{pmatrix} a \\ b \end{pmatrix} = 0 \tag{10.3}$$

$$\frac{\hbar}{2}\begin{vmatrix} \Delta - \lambda & \Omega \\ \Omega & -\Delta - \lambda \end{vmatrix} = 0 \tag{10.4}$$

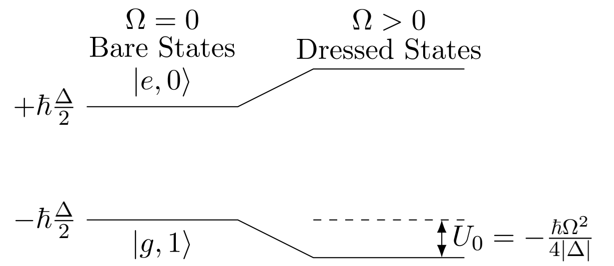
$$-\Delta^2 + \lambda^2 - \Omega^2 = 0 \tag{10.5}$$

$$\lambda^2 = \Delta^2 + \Omega^2 \tag{10.6}$$

$$\lambda = \pm\sqrt{\Delta^2 + \Omega^2} \tag{10.7}$$

$$E_\pm = \pm\frac{\hbar}{2}\sqrt{\Delta^2 + \Omega^2}, \tag{10.8}$$

where $E_\pm$ are our eigenenergies. We choose $\Delta < 0$ (red detuning), and $|\Delta| \gg \omega$ (far detuned to avoid spontaneous emission). We then get



$$E_\pm = \pm\frac{\hbar}{2}|\Delta|\sqrt{1 + \frac{\Omega^2}{|\Delta|^2}} \tag{10.9}$$

$$= \pm\frac{\hbar}{2}|\Delta|\left(1 + \frac{1}{2}\frac{\Omega^2}{|\Delta|^2}\right) \tag{10.10}$$

$$= \pm\frac{\hbar}{2}|\Delta| \pm \frac{\hbar\Omega^2}{4|\Delta|}. \tag{10.11}$$

The shift of the lower state,

$$U_0 = -\frac{\hbar\Omega^2}{4|\Delta|}, \tag{10.12}$$

is known as the **Light Shift** (or ac-Stark shift). This tells use the depth of the optical tweezer indent(handwriting?). We want to rewrite this in terms of laser intensity. Recall that

$$\Omega = -\frac{d\mathcal{E}}{\hbar}, \qquad I = \frac{1}{2}c\epsilon_0\mathcal{E}^2. \tag{10.13}$$

The dipole moment d is related to the spontaneous emission rate of the excited state $|e\rangle$, Γ, by

$$\Gamma = \frac{d^2}{3\pi\epsilon_0(\lambda_0/2\pi)^3}, \tag{10.14}$$

where λ_0 is the wavelength of the transition $|g\rangle \to |e\rangle$, $\omega_0 = \frac{2\pi c}{\lambda_0}$. Using these,

$$\Omega = \Gamma\sqrt{\frac{I}{2I_s}}, \text{ where } I_s = \frac{\pi}{3}\frac{hc\Gamma}{\lambda_0^3} \tag{10.15}$$

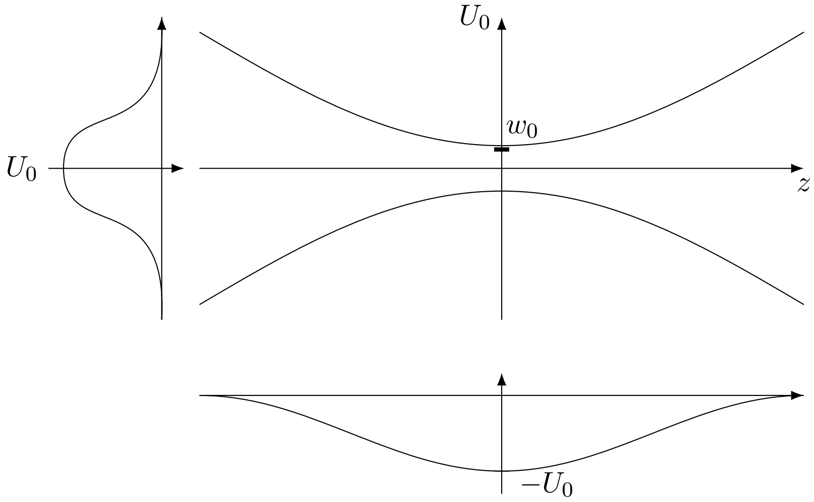
is the saturation intensity. Our optical tweezer trap depth is now

$$U_0 = -\frac{\hbar\Omega^2}{4|\Delta|} = -\frac{\hbar\Gamma^2}{4|\Delta|}\frac{I}{2I_s}. \tag{10.16}$$

The trap depth U_0 is proportional to the laser intensity and inversely proportional to detuning, $|\Delta|$. We set the limit of large detuning to avoid spontaneous emission, but this impacts the trap depth. To get this back, however, we use a large laser intensity.
A laser beam is described by

$$I(x, y, z) = I_0\frac{w_0^2}{w^2}e^{-2(x^2+y^2)/w^2}, \qquad w = w_0\left(1 + \frac{z^2}{z_R^2}\right)^{1/2}, \qquad z_R = \frac{\pi w_0^2}{\lambda}. \tag{10.17}$$

We can therefore plot what the potential U_0 looks like as $U_0(x, y, z)$:



The atom is now trapped at the focus in all 3 dimensions, giving us an optical tweezer. Consider

$$\int_{-\infty}^{\infty} I(z=0)\,dx\,dy = P \implies I_0 = \frac{2P}{\pi w_0^2}, \tag{10.18}$$

where P is the laser power. This tells us that we want to have as powerful a laser as possible, with a smaller beam waist size. However, we cannot focus the beam waist arbitrarily small because diffraction and other effects will take hold. Typically, $w_{0,min} \approx \lambda$.

Lecture 11

Example: Workshop 2, Question 2: Ramsey Interferometry and Decoherence

The Hadamard is how we read out the 'phase' of the state vector. Why is that? We take a general state (as in part c of question),

$$|\psi\rangle = \begin{pmatrix} ae^{-i\Delta t/2} \\ be^{i\Delta t/2} \end{pmatrix}. \tag{11.1}$$

Then we must take the density matrix for this state - where it is a pure state, so it is completely coherent and has a well-defined state. To describe decoherence, which is due to interaction with the environment, we use the density matrix description because it allows for mixed states, i.e. states without perfectly defined phases.

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} |a|^2 & ab^* e^{-i\Delta t} \\ ba^* e^{i\Delta t} & |b|^2 \end{pmatrix}, \tag{11.2}$$

where the diagonal terms represent the probability to be in each state, and the off-diagonal terms are called **Coherences** which contain all the information about phase.
These may contain all the *wave* information, but not necessarily all the *quantum* information. If a state decoheres completely, i.e. coherence decay to zero:

$$\rho_{dec} = \begin{pmatrix} |a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix} \tag{11.3}$$

$$= |a|^2|0\rangle\langle 0| + b^2|1\rangle\langle 1|, \tag{11.4}$$

so we still have a superposition, but no phase information. This is called a **mixed state.**

For the read-out process, we go back to earlier on in the question now. We define a general density matrix as

$$\rho = \frac{1}{2}\begin{pmatrix} 1+w & u-iv \\ u+iv & 1-w \end{pmatrix}. \tag{11.5}$$

We now use the Hadamard to modify the density matrix, yielding

$$\rho' = \hat{H}\rho\hat{H}^\dagger = \frac{1}{2}\begin{pmatrix} 1+u & w+iv \\ w-iv & 1-u \end{pmatrix}. \tag{11.6}$$

The probability to be in $|0\rangle$ is then

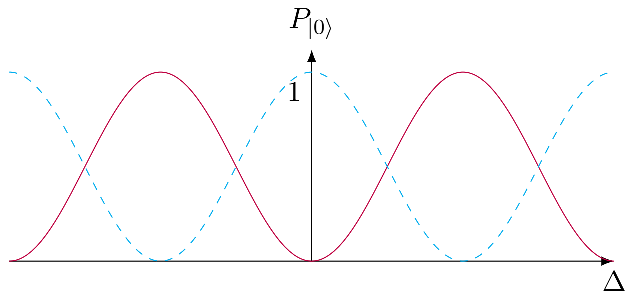
$$P_{|0\rangle} = \frac{1}{2}(1+u), \tag{11.7}$$

where we can now read out the real part of coherence. The Hadamard is used to measure the coherence of the qubit, by projecting the real part of the coherence u from the off-diagonals to the diagonals.

Now let's jump down to part f of the question because Charles says so. We transform $u \to ue^{-it/T_2}$, where T_2 is some time constant representing how the coherence decays. The probability will be modified by this, showing a decaying and oscillating amplitude:

$$P_{|0\rangle} = \frac{1}{2}(1+ue^{-t/T_2}), \quad u = \cos(\Delta t) \tag{11.8}$$

$$= \frac{1}{2}(1+\cos(\Delta t)e^{-t/T_2}). \tag{11.9}$$



These are the Ramsey fringes for $t \ll T_2$. Pure states are good because we have no decoherence, so quantum computing will work. Over time, the oscillations can get smaller, begin to vibrate around $\frac{1}{2}$, so the Ramsey fringe visibility is reduced, leading to decoherence. Working at low temperatures, i.e. mK, reduces likelihood of decoherence.

11.1 Optical Tweezing Ctd.

We used a laser to trap an atom qubit in its beam, with

$$U_0 = -\frac{\hbar\Omega}{4|\Delta|}, \qquad \Omega = \Gamma\sqrt{\frac{I}{2I_s}}. \tag{11.10}$$

The trap depth is proportional to the laser intensity (and power):

$$I = I_0 \frac{w_0^2}{w^2} e^{-2(x^2+y^2)/w_0^2}. \tag{11.11}$$

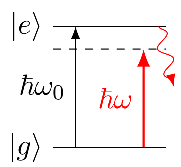
doodle The trap depth forms a harmonic potential, $\frac{1}{2}m\omega_z z^2$. This means we want to cool the atom to its Harmonic Oscillator ground state. *doodle* This is the minimum size of the atomic wave packet. We then send in our laser beam with an electric field,

$$\mathcal{E} = \mathcal{E}_0 \cos(\phi_L - \omega t), \quad \phi_L = \underline{k} \cdot \underline{r}. \tag{11.12}$$

There is now an error in the phase,

$$\Delta\phi_L = k\Delta x, \tag{11.13}$$

if the field propagates in the x-direction (switch out x for y or z depending on propagation direction). This error is a source of decoherence, so we need to try and minimise it. However, the scaling is very bad, on the order of needing 10^4 more laser power to decrease Δx by a factor of 10.



$$\Delta = \omega - \omega_0 < 0, \tag{11.14}$$
$$|\Delta| \gg \Omega. \tag{11.15}$$

We need $|\Delta|$ large to avoid the decay of our excited state $|e\rangle$. Calculating the probability that qubit scatters a photon, we have

- spontaneous decay rate of $|e\rangle$ is Γ
- the photon scattering rate is $R = \Gamma P_{|e\rangle}$
- $P_{|e\rangle}$ is the probability to be in state $|e\rangle$; for a two-level model, this is $|b|^2$

This model, however, did not include the possiblity of decay, so we need to add decay to our equations of motion of the Bloch vector. From Workshop 1,

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} 0 & -\Delta & 0 \\ \Delta & 0 & -\Omega \\ 0 & \Omega & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}. \tag{11.16}$$

Where the RHS is Torque×Bloch vector. We now have

$$\dot{u} = -\frac{u}{T_2} - \Delta v, \tag{11.17}$$

$$\dot{v} = \Delta u - \frac{v}{T_2} - \Omega w, \tag{11.18}$$

$$\dot{w} = \Omega v - \frac{1}{T_1}(w-1). \tag{11.19}$$

T_2 is the decay of coherence; T_1 is the decay of population. These are the **Optical Bloch Equations.**

Lecture 12 Spontaneous Photon Scattering

In our optical tweezer model described above, not all photons going into the qubit will hit the same way, so sometimes we will have scattering of photons off the photon. Last time we discussed spontaneous emission from the excited state, Γ, where the rate of spontaneous scattering can then be described as

$$R = \Gamma P_{|e\rangle}. \tag{12.1}$$

12.1 Optical Bloch Equations

If spontaneous emission is the only decoherence/decay mechanism, then

$$T_1 = \frac{1}{\Gamma}, \qquad T_2 = \frac{2}{\Gamma}, \tag{12.2}$$

$$\frac{1}{T_1} = \Gamma = \frac{1}{\tau}, \qquad \frac{1}{T_2} = \frac{\Gamma}{2}, \tag{12.3}$$

where τ is the lifetime of the excited state. We can then substitute these into the optical Bloch equations defined last lecture, yielding

$$\dot{u} = -\frac{\Gamma}{2}u - \Delta v, \tag{12.4}$$

$$\dot{v} = \Delta u - \frac{\Gamma}{2}v - \Omega w, \tag{12.5}$$

$$\dot{w} = \Omega v - \Gamma(w-1). \tag{12.6}$$

From here, we solve for the steady state, $\dot{u} = \dot{v} = \dot{w} = 0$. Following lots of algebra, we find

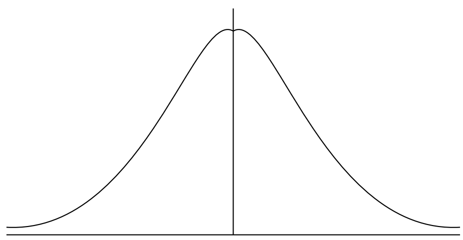
$$P_{|e\rangle} = \frac{1}{2}(1 - w_{ss}) = \frac{\Omega^2/4}{\Delta^2 + \Gamma^2/4 + \Omega^2/2}. \tag{12.7}$$

This is one of our Very Important Equations. We want to plot this for three cases:

1. Small Ω: $\Omega \ll |\Delta|$ and Γ. (NB: not the case for optical tweezer!)

$$P_{|e\rangle} \approx \frac{\Omega^2/4}{\Delta^2 + \Gamma^2/4}. \tag{12.8}$$

This corresponds to weak driving, where it will result in low intensity and low probability. This equation will look familiar as a Lorentzian, or a Cauchy distribution.



Our Full Width Half Maximum is then

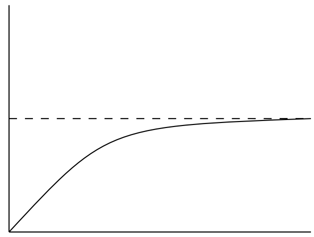
$$\text{FWHM} = \Gamma = \frac{1}{T_1} = \frac{1}{\tau}, \tag{12.9}$$

which is the natural width of $|g\rangle \to |e\rangle$ transition.

2. $\Delta = 0$, i.e. on resonance. Again, not like our tweezer where we are very far from resonance. Now, we have a probability of

$$R_{|e\rangle} = \frac{\Omega^2/4}{\Gamma^2/4 + \Omega^2/2}. \tag{12.10}$$

We can then plot the probability, now against Ω^2:



$$\frac{\Omega^2}{2} = \frac{\Gamma^2}{4} \implies P_{|e\rangle} = 0.25 \tag{12.11}$$

This is linear for $\Omega < \Gamma$. Then

$$\Omega^2 = \Gamma^2 \frac{I}{2I_S}, \qquad P_{|e\rangle} = 0.25 \text{ for } I = I_S, \tag{12.12}$$

where I_S is the saturation intensity.

3. $|\Delta| \gg \Gamma, \Omega$, i.e. the case for optical tweezing. $\Omega \gg \Gamma$ also. We have strong driving and strong detuning.

$$P_{|e\rangle} = \frac{\Omega^2/4}{\Delta^2} = \frac{\Omega^2}{4\Delta^2}. \tag{12.13}$$

Therefore, the photon scattering rate in an optical tweezer is

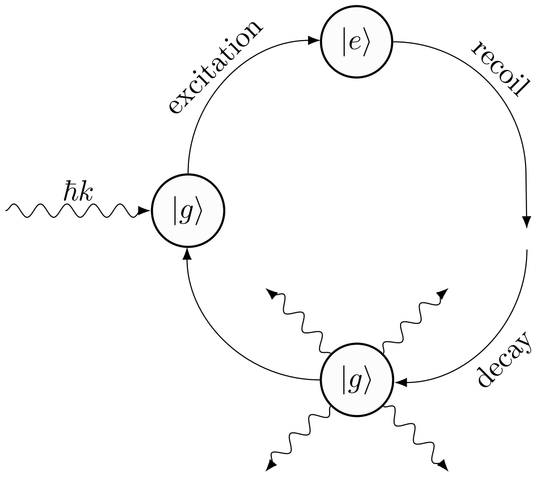
$$R = \Gamma P_{|e\rangle} = \Gamma \frac{\Omega^2}{4\Delta^2}. \tag{12.14}$$

Now to the trap depth,

$$U_0 = -\frac{\hbar\Omega^2}{4|\Delta|} \implies \frac{|U_0|}{R} = \frac{\hbar\Omega^2/4|\Delta|}{\Gamma\Omega^2/4\Delta^2} = \hbar\left(\frac{\Delta}{\Gamma}\right). \tag{12.15}$$

So we want to use $\Delta \gg \Gamma$ to minimise scattering for a given trap depth, i.e. *far detuning.*

We can use spontaneous scattering for cooling! Laser cooling works through a spontaneous scattering force. Each scattering event cause a recoil on the qubit or atom. *doodle*



Cycles of excitation and decay transfer, on average, $\hbar \underline{k}$ of momentum to the atom per cycle. This exerts a force

$$\underline{F} = \hbar \underline{k} R = \hbar \underline{k} \Gamma P_{|e\rangle} \tag{12.16}$$

$$= \hbar \underline{k} \Gamma \frac{\Omega^2/4}{\Delta^2 + \Gamma^2/4 + \Omega^2/2}. \tag{12.17}$$

This force is called the **Spontaneous Scattering Force.** The properties of this force will then follow the previous discussions on the behaviour of the probability in different regimes.

How will we use this force for cooling? Use 2 counter-propagating lasers, both red detuned. *doodle* The force on the atom is then the sum of the force from laser 1 and laser 2:

$$F = F_+ + F_-. \tag{12.18}$$

If the atom is moving, however, there is a Doppler shift.

- Laser 1: it's moving away, so detuning, $\Delta \to \Delta - kv_{atom}$.
- Laser 2: it's moving towards, so detuning, $\Delta \to \Delta + kv_{atom}$.

$$F_\pm = \pm \hbar \underline{k} \Gamma \frac{\Omega^2/4}{\Gamma^2/4 + \Omega^2/2 + (\Delta \mp kv_{atom})^2}. \tag{12.19}$$

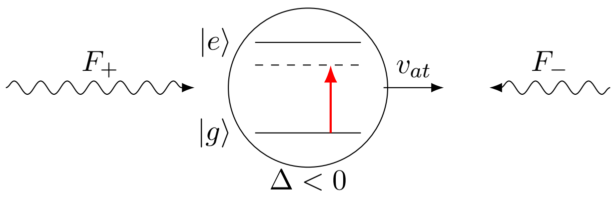
Lecture 13 Laser Cooling

We have cycles of absorption and spontaneous emission which give rise to the spontaneous force,

$$F = \hbar \underline{k} R, \text{ where } R = \Gamma P_{|e\rangle} = \frac{\Gamma \Omega^2/4}{\Omega^2/2 + \Gamma^2/4 + \Delta^2}. \tag{13.1}$$

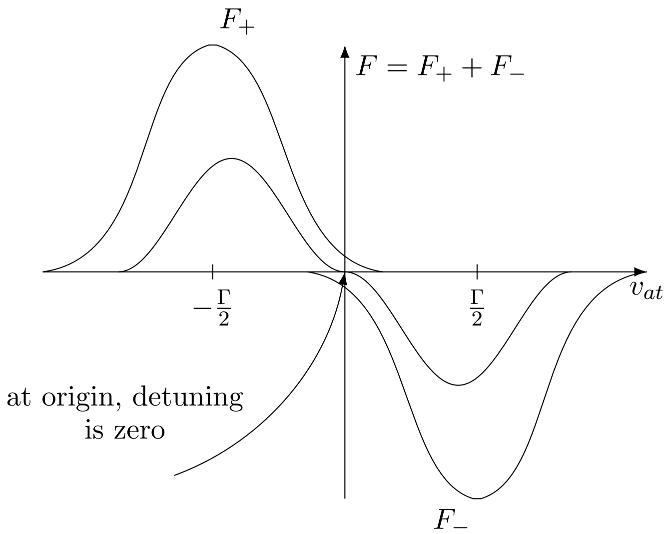
Note that $R_{max} = \frac{\Gamma}{2}$, for $\Omega \gg |\Delta|, \Gamma$.

Laser cooling, counter-propagating red detuned ($\Delta < 0$) laser beams.



$$F_\pm = \pm \frac{\Gamma}{2} \frac{I/I_r}{1 + 4(\Delta \mp kv)^2/\Gamma^2}, \text{ where } \Omega^2 = \Gamma^2 \frac{I}{2I_s}. \tag{13.2}$$

If $I < I_s$:



$$\text{for} |v_{at}| < \frac{\Gamma}{k}, \quad \underline{F} = -\alpha \underline{v}_{at} \tag{13.3}$$

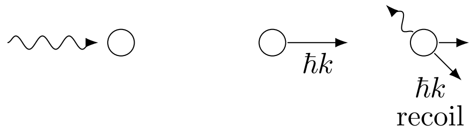
Defining the atomic momentum as p, we can use $F = \frac{dp}{dt} = -\alpha \frac{p}{m}$ to find the rate of change of kinetic energy:

$$\left(\frac{dE}{dt}\right) = \frac{d}{dt}\left(\frac{p^2}{2m}\right) = \frac{p}{m}\frac{dp}{dt} = -\alpha \left(\frac{p}{m}\right)^2, \tag{13.4}$$

where the negative means the rate is decreasing. We now have laser cooling!

13.1 How cold can we cool?

There is also a heating due to random recoils due to spontaneous emission.



- Heating Rate:

$$\left(\frac{dE}{dt}\right)_{heat} = \frac{d}{dt}\left(\frac{p^2}{2m}\right). \tag{13.5}$$

- Equilibrium temp (minimum temperature occurs at $\Delta = \frac{\Gamma}{2}$:

$$\left(\frac{dE}{dt}\right)_{cool} = -\left(\frac{dE}{dt}\right)_{heat}, \tag{13.6}$$

$$k_B T_{dopp} = m v_{at}^2 = \frac{\hbar\Gamma}{2}. \tag{13.7}$$

  This is the Doppler Cooling Limit. Typically, $T_{dopp} \approx 1 - 100\,\mu$K - the coldest thing in the Universe.

We can use laser cooling to load cold atoms into optical tweezers. As with before, we have cycles of absorption and spontaneous emission, but this is only found in simple atoms as we need a **Closed Transition.**

Let's focus on a particular atom. e.g. Cs (single valence electron).

- Ground state $|g\rangle$, 6s $(n = 6, l = 0)$
- Excited state $|e\rangle$, 6p $(n = 6, l = 1)$

Laser cooling on $|g\rangle \rightarrow |e\rangle$ transition.

Cs atom has nuclear spin $I = \frac{7}{2}$ and electron spin $s = \pm\frac{1}{2}$. The total 'spin' (atomic angular momentum) is $F = I + S = 3, 4$. For each F, we ahve $2F + 1$ magnetic sub-levels labelled m_F. So the ground state has 16-fold degeneracy. We select 2 of these 16 as our qubit states $|0\rangle$ and $|1\rangle$:

$$|F = 3, m_f = 3\rangle = |0\rangle,$$
$$|F = 4, m_F = 4\rangle = |1\rangle.$$

As both are ground states, they do not decay (not spontaneous emission), i.e. low decoherence (**DiVincenzo #4).**

Lecture 14

Lecture 15 Rydberg Atoms

Let's look at the DiVincenzo criteria for alkalia-atom QC!

1. Initialisation
   (a) Optical tweezer traps ✓
   (b) Laser cooling ✓
   (c) Optical pumping to initialise qubit state ($|0\rangle$ or $|1\rangle$) ✓
2. Gates
   (a) Single qubit gates ✓
   (b) Stimulated Raman transition ✓
   (c) Two-qubit gates ✘
3. Read-out
   (a) Fluorescence on $|g\rangle \to |e\rangle$ transition ✓ *doodle*
4. Low decoherence
   (a) $|0\rangle$ and $|1\rangle$ are both 'ground' state, i.e. hyperfine states with the $|g\rangle$ manifold ✓
5. Scaling
   (a) Optical tweezer array ✓

The only thing we have left to do is the two-qubit gates, i.e. we need interacting qubits.

15.1 Two-Qubit Gates

Atoms in states $|g\rangle$ and $|e\rangle$ do not interact unless they are very close ($\ll 100\,\mathrm{nm}$ apart), but we will create our atom qubit array with spacings of order $5 - 10\,\mu\mathrm{m}$. We need some way of introducing interactions at these scales.

To create interaction, we excite to **Highly-Excited Rydberg States,** i.e. states with principal quantum number $n \gtrapprox 50$.

15.2 Rydberg Scalings

If we consider the Bohr model, and take a state with principal quantum number n, then

$$mv_n r_n = n\hbar. \tag{15.1}$$

Equating the centripetal force with the Coulomb force, we find

$$r_n = n^2 a_0. \tag{15.2}$$

This tells us that the size of the electronic wavefunction is $\propto n^2$; for $n \approx 100$, the atom is $> 1\,\mu\mathrm{m}$ across. The dipole D is also $\propto n^2$. The dipole-dipole $\propto D^2 \propto n^4$, so using high n we can induce strong interactions.

Using the Bohr model and virial theorem, the energy is $\frac{1}{2}$ the potential energy:

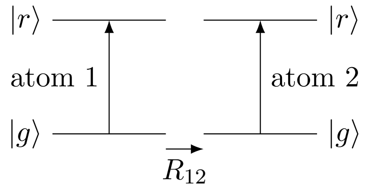
$$E_n = -\frac{e^2}{2(4\pi\epsilon_0)r_n} = -\frac{R_H}{n^2}, \tag{15.3}$$

where R_H is the Rydberg constant for Hydrogen. For Cs,

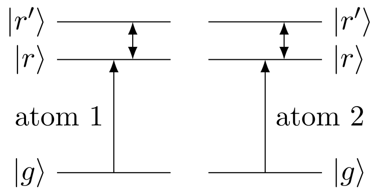
$$E_n = -\frac{R_{Cs}}{n^*}, \quad n^* = n - \delta, \tag{15.4}$$

where n^* is the reduced principal quantum number offset from n by δ.

15.3 Rydberg Interactions



finish doodle Exciting two atoms into states $|r\rangle$ allows them to strongly via dipole-dipole interactions (DDI). To calculate the DDI, we need to consider couplings to other Rydberg states, e.g. $|r'\rangle$.



The DDI mixes some of $|r'\rangle$ into the state $|r\rangle$ which gives rise to the Van Der Waals energy shift. This can be written as a matrix

$$\hat{\mathcal{H}} = \hbar \begin{pmatrix} 0 & 0 & 0 & V \\ 0 & \omega_{rr'} & V & 0 \\ 0 & V & \omega_{rr'} & 0 \\ V & 0 & 0 & 2\omega_{rr'} \end{pmatrix}, \tag{15.5}$$

where $\hbar\omega_{rr'}$ is the energy splitting between r and r', and

$$V = \frac{D^2/\hbar}{4\pi\epsilon_0 R_{12}^3}. \tag{15.6}$$

The basis for $\hat{\mathcal{H}}_{DDI}$ is $|rr\rangle, |rr'\rangle, |r'r\rangle, |r'r'\rangle$. Consider the sub-bases $|rr\rangle, |r'r'\rangle$:

$$\hat{\mathcal{H}} = \hbar \begin{pmatrix} 0 & V \\ V & 2\omega_{rr'} \end{pmatrix}. \tag{15.7}$$

This should be familiar from previous examples. This is analagous to the light shift $\frac{\Omega^2}{4\Delta}$. *doodle* This is the Van Der Waals Interaction.

$$V \propto \frac{1}{R_{12}^3}, \quad V_{vdW} \propto \frac{1}{R_{12}^6} \tag{15.8}$$

$$V \propto D^2, \quad D \propto n^2 \tag{15.9}$$

$$V \propto n^4 \tag{15.10}$$

$$E_n = -\frac{R_{Cs}}{n^2}, \quad E_{n+1} - N \propto \frac{1}{n^3},\ n \gg 1, \tag{15.11}$$

$$V_{vdW} \propto \frac{V^2}{\omega_{rr'}} \propto \frac{n^8}{n^{-3}} \propto n^{11}. \tag{15.12}$$

So van der Waals has n^{11} scaling! Tunability is strength of interactions.

Lecture 16 Rydberg Blockade and Entanglement

doodle Excitation linewidth $|g\rangle \to |r\rangle$ for $\Omega \gg \Gamma$ is of order Ω. If $V_{vdW} > \Omega$ (linewidth), the excitation is blocked. This is called **Rydberg Blockade.**
This occurs at distance R_{12}, where

$$V_{vdW} = \frac{C_6}{R_{12}^6} = \Omega \tag{16.1}$$

$$R_{12} = \left(\frac{C_6}{\Omega}\right)^{1/6} = R_b, \tag{16.2}$$

where R_b is the blockade radius, $R_b \approx 10\mu$m which is relatively large! *doodle* We cannot excite the second Rydberg atom within this sphere.

Now we consider starting with both atoms in the ground state, and drive the $|g\rangle \to |r\rangle$ transition on resonance. *doodle* We have two two-level systems, giving a four state basis: $|gg\rangle, |gr\rangle, |rg\rangle, |rr\rangle$. Recall with one atom and $\Delta = 0$, our interaction Hamiltonian was given as

$$\hat{\mathcal{H}}_{int} = \frac{\hbar}{2}\begin{pmatrix} 0 & \Omega \\ \Omega & 0 \end{pmatrix}. \tag{16.3}$$

Now for two atoms, our interaction Hamiltonian reads

$$\mathcal{H} = \mathcal{H}_{int} \otimes \sigma_0 + \sigma_0 \otimes \mathcal{H}_{int} \tag{16.4}$$

$$= \begin{pmatrix} 0 & \Omega \\ \Omega & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & \Omega \\ \Omega & 0 \end{pmatrix} \tag{16.5}$$

$$= \begin{pmatrix} 0 & 0 & \Omega & 0 \\ 0 & 0 & 0 & \Omega \\ \Omega & 0 & 0 & 0 \\ 0 & \Omega & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \Omega & 0 & 0 \\ \Omega & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega \\ 0 & 0 & \Omega & 0 \end{pmatrix} \tag{16.6}$$

$$= \begin{pmatrix} 0 & \Omega & \Omega & 0 \\ \Omega & 0 & 0 & \Omega \\ \Omega & 0 & 0 & \Omega \\ 0 & \Omega & \Omega & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & V_{vdW} \end{pmatrix} \tag{16.7}$$

$$\mathcal{H}_{|g\rangle\to|r\rangle} = \frac{\hbar}{2}\begin{pmatrix} 0 & \Omega & \Omega & 0 \\ \Omega & 0 & 0 & \Omega \\ \Omega & 0 & 0 & \Omega \\ 0 & \Omega & \Omega & 2V_{vdW} \end{pmatrix} \tag{16.8}$$

This is valid for $R_{12} < R_b, V_{vdW} \gg \Omega$ ($\frac{\hbar}{2}$ should be throughout it, but he missed it out and I'm lazy). The $|rr\rangle$ state is completely decoupled by the vdW shift, so we can neglect this and turn it into a 3 state basis as

$$\mathcal{H}_{|g\rangle\to|r\rangle} = \frac{\hbar}{2}\begin{pmatrix} 0 & \Omega & \Omega \\ \Omega & 0 & 0 \\ \Omega & 0 & 0 \end{pmatrix}. \tag{16.9}$$

We have the eigenvectors,

$$|gg\rangle, \quad \frac{1}{\sqrt{2}}\left(|gr\rangle + |rg\rangle\right), \quad \frac{1}{\sqrt{2}}\left(|gr\rangle - |rg\rangle\right).$$

If we then rewrite the interaction in the basis of these states, we get

$$\mathcal{H} = \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{2}\Omega & 0 \\ \sqrt{2}\Omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{16.10}$$

where we can now see that $|gg\rangle$ couples to $\frac{1}{\sqrt{2}}(|gr\rangle + |rg\rangle)$ with Rabi frequency $\sqrt{2}\Omega$, but it does not couple to $\frac{1}{\sqrt{2}}(|gr\rangle - |rg\rangle)$ at all. *doodle* The effect of the Rydberg blockade is that we excite an entangled Bell state where only one atom is in the Rydberg state but we do not know (and cannot know) which one.