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Mathematical Methods in Physics

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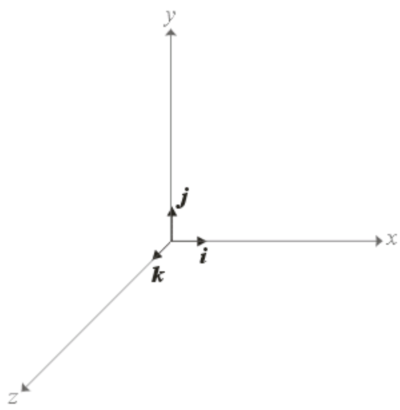
Part I

Lecture 1

1.1 Geometrical Applications of Vectors in R3

$$\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \iff \underline{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \iff \underline{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$



We consider a Cartesian system:

$\{\hat{i}, \hat{j}, \hat{k}\}$ = 'standard basis'

This set is an orthonormal set of vectors:

The vectors $\hat{i}, \hat{j}, \&\hat{k}$ are orthogonal and have a modulus of 1

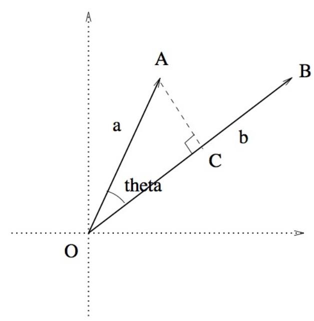
$$\hat{i} \perp \hat{j}; \quad \hat{i} \perp \hat{k}; \quad \hat{j} \perp \hat{k}$$

$$|\hat{i}| = |\hat{j}| = |\hat{k}| = 1$$

1.2 Scalar (or dot) Product

$$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta$$

$$\underline{a} \cdot \underline{b} = a_1b_1 + a_2b_2 + a_3b_3$$



$\underline{a}$ can be split into components $\perp$ and $\|$ to $\underline{b}$:

$$\underline{a} = \underline{a}_{\|} + \underline{a}_{\perp}$$

- $\underline{a}_{\|} \equiv \underline{OC}$ is the orthogonal projection of $\underline{a}$ on to the direction of $\underline{b}$
- Its modulus is $|\underline{a}| \cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}|}$

$$\underline{a}_{\|} = \left(\frac{\underline{a} \cdot \underline{b}}{|\underline{b}|}\right)\frac{\underline{b}}{|\underline{b}|} \implies \underline{a}_{\perp} = \underline{a} - \underline{a}_{\|}$$

The dot product is symmetrical so:

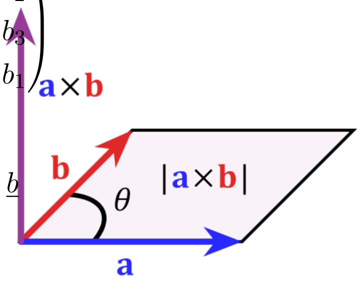
$$\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$$

$$|\underline{a}| = \sqrt{\underline{a} \cdot \underline{a}}$$

1.3 Vector (or cross) Product

$$\underline{a} \times \underline{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

$$|\underline{a} \times \underline{b}| = |\underline{a}||\underline{b}| \sin \theta$$



Notice that $|\underline{a} \times \underline{b}|$ is the area of the parallelogram with sides $\underline{a}$ and $\underline{b}$

The cross product is anti-symmetric:

$$\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$$

1.4 Scalar Triple Product

$$[\underline{a}, \underline{b}, \underline{c}] = \underline{a} \cdot (\underline{b} \times \underline{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The absolute value of the scalar triple product for three arbitrary vectors $\underline{a}$, $\underline{b}$, and $\underline{c}$ corresponds to the volume of the parallelepiped with sides $\underline{a}$, $\underline{b}$, and $\underline{c}$

$$|[\underline{a}, \underline{b}, \underline{c}]| = |\underline{a}||\underline{b} \times \underline{c}| \cos \phi = |\underline{a}||\underline{b}||\underline{c}| \sin \theta \cos \phi$$

It is unchanged under an even permutation of the vectors:

$$[\underline{a}, \underline{b}, \underline{c}] = [\underline{b}, \underline{c}, \underline{a}] = [\underline{c}, \underline{a}, \underline{b}]$$

It changes sign under an odd permutation:

$$[\underline{a}, \underline{b}, \underline{c}] = -[\underline{b}, \underline{a}, \underline{c}] = -[\underline{a}, \underline{c}, \underline{b}] = -[\underline{c}, \underline{b}, \underline{a}]$$

It vanishes if any two vectors are the same

1.5 Einstein Summation Convention for Subscripts

Any index that appears twice in a given term of an expression is understood to be summed over all the values that an index can take

The summed-over subscripts are called dummy subscripts and the others, free subscripts

$$\sum_{i=1}^{n} a_i b_i \equiv a_i b_i$$

$$a_{ij} b_{jk} = \sum_{j=1}^{3} a_{ij} b_{jk} = a_{i1} b_{1k} + a_{i2} b_{2k} + a_{i3} b_{3k}$$

$$a_{ij} b_{jk} c_k = \sum_{j=1}^{3} \sum_{k=1}^{3} a_{ij} b_{jk} c_k \quad \text{(Gives 9 terms)}$$

1.6 Kronecker Delta in R3

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} ; \quad i, j = 1, 2, 3$$

$$b_i\delta_{ij} = b_1\delta_{1j} + b_2\delta_{2j} + b_3\delta_{3j} \begin{cases} j=1 & \to \; b_1 \\ j=2 & \to \; b_2 \\ j=3 & \to \; b_3 \end{cases} \Bigg\} \implies b_j$$

$$b_i\delta_{ij} = b_j$$

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$$

$$\underline{a} \cdot \underline{b} = a_i b_i = \delta_{ij} a_i b_j$$

1.7 Levi-Civita Symbol in R3

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if i,j,k is an even permutation of 1,2,3} \\ & \epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1 \\ -1 & \text{if i,j,k is an odd permutation of 1,2,3} \\ & \epsilon_{213} = \epsilon_{132} = \epsilon_{321} = -1 \\ 0 & \text{otherwise} \end{cases}$$

1.7.1 Features

1. $\epsilon_{ijk} = \epsilon_{jki}$ (even permutation)
   It does not change sign
2. $\epsilon_{ijk} = -\epsilon_{jik}$ (odd permutation)
   It changes sign under the interchange of any pair of indices
3. $\epsilon_{ijj} = \epsilon_{iii} = 0$

1.7.2 Exercises

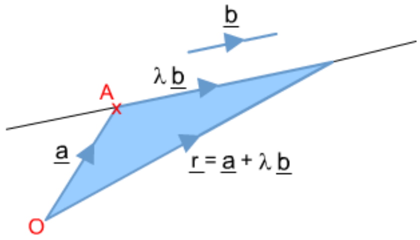
1. $(\underline{a} \times \underline{b})_i = \epsilon_{ijk} a_j b_k$
   $(\underline{a} \times \underline{b})_1 = \epsilon_{1jk} a_j b_k = \epsilon_{123} a_2 b_3 + \epsilon_{132} a_3 b_2 = a_2 b_3 - a_3 b_2$
2. $[\underline{a}, \underline{b}, \underline{c}] = \epsilon_{ijk} a_i b_j c_k = a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_1 b_3 c_2 - a_2 b_1 c_3 - a_3 b_2 c_1$

Lecture 2

2.1 Lines in R3

Consider a point A and a direction $\hat{b}$:

$$(\underline{r} - \underline{a}) = \hat{b}\lambda$$
$$\underline{r} = \underline{a} + \hat{b}\lambda$$



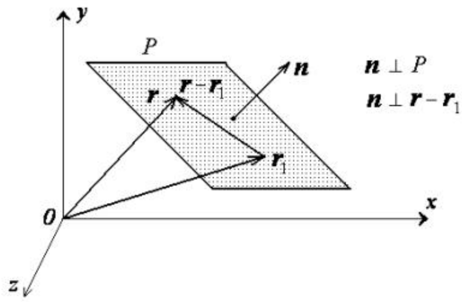
Note that $\underline{r} = \underline{r}(\lambda)$ (parametric form)

Note also that by taking the vector product with $\hat{b}$, we obtain another equation for the line:

$$(\underline{r} - \underline{a}) \times \hat{b} = 0$$

2.2 Equation of a plane in R3

A plane through a point A with position vector, $\underline{a}$, and perpendicular to uni vector, $\hat{n}$, is:



$$(\underline{r} - \underline{a}) \cdot \hat{n} = 0$$
$$\underline{r} \cdot \hat{n} = \underline{a} \cdot \hat{n} = d$$

This is the Cartesian form for the equation of a plane

Consider a plane with points A, B, and C with corresponding position vectors:

$$t_1(\underline{b} - \underline{a}) + t_2(\underline{c} - \underline{a} = \underline{r} - \underline{a})$$

This is the parametric equation for a plane

2.3 Linear Vector Spaces

Found in Chapter 8 of Riley, Hobson, and Bence

A vector space, V, is a set whose elements are called "vectors" and such that there are two operations defined on them:

- you can add vectors to each other
- you can multiply vectors by a scalar

Those operations must obey certain simple rules; these rules are called *axioms*
The Axioms for a Vector Space are:

1. The vector space is closed under addition and scalar multiplication
   - If $\underline{v}$ and $\underline{u} \in$ V, then $\underline{v} + \underline{u} \in$ V
   - If $\underline{v} \in$ V, then $\alpha\underline{v} \in$ V
2. Associativity
   - $(\underline{u} + \underline{v}) + \underline{w} = \underline{u} + (\underline{v} + \underline{w})$
   - $(\alpha\beta)\underline{v} = \alpha(\beta\underline{v})$
3. There exists a *zero element* or *neutral element*, $\underline{0}$
   - $\underline{0} + \underline{v} = \underline{v}$
4. There exists an *inverse element*, $-\underline{v}$

- $\underline{v} + (-\underline{v}) = \underline{0}$

5. Commutativity
   - $\underline{u} + \underline{v} = \underline{v} + \underline{u}$
6. Distributivity
   - $\alpha(\underline{v} + \underline{u}) = \alpha\underline{v} + \alpha\underline{u}$
   - $(\alpha + \beta)\underline{v} = \alpha\underline{v} + \beta\underline{v}$
7. Scalar multiplication by 1 leaves $\underline{v}$ unchanged
   - $1\underline{v} = \underline{v}$

Note that by scalar, we usually mean $\in \mathbb{R}$
In this case, we refer to V as a *real vector space*

It is also possible for scalars $\in \mathbb{C}$
in this case, we have *complex vector spaces*

2.3.1 Examples

1. $\mathbb{R}$
2. Generalisation to $\mathbb{R}^n$ for Euclidean vector spaces
3. Further generalised to $\mathbb{C}^n$
4. The set of all real functions, $f(x)$, with no restrictions on x and with the usual (calculus) addition and scalar multiplication
   - $(f_1 + f_2)(x) = f_1(x) + f_2(x)$
   - $(\alpha f)(x) = \alpha f(x)$
5. The matrices of size $(n \times m)$ with real elements and the usual (calculus) addition and scalar multiplication of matrices
6. The set of vectors in the 3D space for which $2X - 3Y + 11Z + 2 = 0$ **is not a vector space**
   - $\underline{0} \notin$ V
   - $2 \cdot 0 - 3 \cdot 0 + 11 \cdot 0 + 2 \neq 0$
7. $2X - 3Y + 11Z = 0$ is a vector space
8. Consider a second order, linear, homogeneous differential equation of the form:
   - $p(x)\frac{d^2f}{dx^2} + q(x)\frac{df}{dx} + r(x)f = 0$
   - p, q, and r are fixed functions

   The space of the solutions of such an equation forms a vector space under the usual addition and scalar multiplication

Lecture 3

For k vectors, $\{\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_k\}$ in V, the expression $\{\alpha_1\underline{v}_1, \alpha_2\underline{v}_2, \cdots, \alpha_k\underline{v}_k\}$ is called a linear combination.

The set of all linear combinations of $\{\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_k\}$ is called a span of $\{\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_k\}$:

$$\text{Span}(\underline{v}_1, \cdots, \underline{v}_n) = \left\{ \sum_{i=1}^{k} \alpha_i \underline{v}_i \ ; \ \alpha_i \in \mathbb{R}/\mathbb{C} \right\}$$

3.1 Examples

1. A span of a single vector is the set of all scalar multiples of this vector
   ➤ it is a line through 0 in the direction of the vector
2. The span of two vectors, provided they are not multiples of each other, can be seen as the plane through 0 containing these vectors

3.2 Definition

A set of vectors, $\{\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_k\} \in V$ is called linearly independent if

$$\sum_{i=1}^{k} \alpha_i \underline{v}_i = 0 \implies \alpha_i = 0 \ \forall \ i$$

Otherwise, the vectors are called linearly dependent
That is, these vectors are linearly dependent if

$$\sum_{i=1}^{k} \alpha_i \underline{v}_i = 0 \implies \alpha_i \neq 0 \text{ for one } i$$

3.2.1 Claim

The vectors $\{\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_k\}$ are linearly dependent $\iff$ $\underline{v}_i$ can be written as a linear combination of the other vectors

3.2.2 Examples

1.

$$\underline{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \ ; \ \underline{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \ ; \ \underline{v}_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$\alpha_1\underline{v}_1 + \alpha_2\underline{v}_2 + \alpha_3\underline{v}_3 = \begin{pmatrix} \alpha_3 \\ \alpha_1 + \alpha_2 + \alpha_3 \\ \alpha_1 + 2\alpha_2 - \alpha_3 \end{pmatrix} = 0$$

$$\alpha_3 = 0 \ ; \ \alpha_1 = -\alpha_2 \ ; \ \alpha_1 = \alpha_2 = 0$$

These vectors are linearly independent

2.

$$\underline{v}_1 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \ ; \ \underline{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \ ; \ \underline{v}_3 = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$$

$$\alpha_1\underline{v}_1 + \alpha_2\underline{v}_2 + \alpha_3\underline{v}_3 = \begin{pmatrix} -2\alpha_1 + \alpha_2 \\ \alpha_2 + 2\alpha_3 \\ \alpha_1 + \alpha_2 + 3\alpha_3 \end{pmatrix} = 0$$

$$\alpha_1 = 1 \ ; \ \alpha_2 \ ; \ \alpha_3 = -1$$

These vectors are linearly dependent

Notice that $\underline{v}_3 = \underline{v}_1 + 2\underline{v}_2$
Can calculate linear dependence using determinant:

$$det(\underline{v}_1, \underline{v}_2, \underline{v}_3) = 0 \implies \text{linearly dependent}$$

3. The set of polynomials of degree 2 or less with coefficients in $\mathbb{R}$ form a vector space
Consider these three polynomials:

$$\left\{ \underbrace{1 + x + x^2}_{\underline{v}_1} \ ; \ \underbrace{1 - x + 3x^2}_{\underline{v}_2} \ ; \ \underbrace{1 + 3x - x^2}_{\underline{v}_3} \right\}$$

$$\alpha_1\underline{v}_1 + \alpha_2\underline{v}_2 + \alpha_3\underline{v}_3 = \alpha_1(1 + x + x^2) + \alpha_1(1 - x + 3x^2) + \alpha_3(1 + 3x - x^2) = 0$$

$$x^2 \underbrace{(\alpha_1 + 3\alpha_2 - \alpha_3)}_{0} + x \underbrace{(\alpha_1 - \alpha_2 + 3\alpha_3)}_{0} + \underbrace{(\alpha_1 + \alpha_2 + \alpha_3)}_{0} = 0$$

$$\alpha_1 = -2\alpha_2 \ ; \ \alpha_3 = \alpha_2 \ ; \ \alpha_2 = n, \ n \in \mathbb{R}$$

$$\underline{v}_3 = 2\underline{v}_1 - \underline{v}_2$$

3.3 Definition

The minimal set of vectors that span a vector space is called a basis for that space.

A set of vectors $\{\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_k\} \in V$ is called a basis $\iff$

1. $\{\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_k\}$ are linearly independent
2. $V = \text{Span}(\{\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_k\})$

Then we have that:

- The number of vectors in a basis is the dimension of a vector space
- If $\{\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_k\}$ is a basis in V, any $\underline{v}_n \in V$ can be written as a unique linear combination of the vectors in the basis, $\underline{v}_n = \alpha_i\underline{v}_i$
  - Coefficients α_i are called the components of $\underline{v}$ wrt to the basis

3.3.1 Examples

1. Previous example 2(i). It is a basis in $\mathbb{R}^3$ (dim 3)
2. For the 2x3 matrices with entries $\mathbb{R}$, a basis is given by:

$$E_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ ; \ E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ ) & 0 & 0 \end{pmatrix} \text{etc}$$

$$E_{ij}, \ i = 1,2 \ ; \ j = 1,2,3$$

3. Polynomials of degree 2 or less with coefficients in $\mathbb{R}$
Basis is:

$$\left\{ 1 \ ; \ x \ ; \ x^2 \right\} \text{(dim 3)}$$

3.4 Inner (or scalar) product

Consider a vector space, V
The inner product of V is a scalar function denoted $\langle \underline{v} \mid \underline{w} \rangle$ that satisfies the following properties:

1. $\langle \underline{v} \mid \underline{w} \rangle = \langle \underline{w} \mid \underline{v} \rangle^*$
2. $\langle \underline{v} \mid \alpha\underline{w} + \beta\underline{u} \rangle = \alpha\langle \underline{v} \mid \underline{w} \rangle + \beta\langle \underline{v} \mid \underline{u} \rangle$
3. $\langle \underline{v} \mid \underline{v} \rangle > 0,\ \underline{v} \neq 0$

Notes:

- Two vectors are orthogonal if $\langle \underline{v} \mid \underline{w} \rangle = 0$
- Length of a vector (norm) is $|\underline{v}| = \sqrt{\langle \underline{v} \mid \underline{v} \rangle}$

3.4.1 Examples

1. In $\mathbb{R}$, the dot product:

$$\langle \underline{v} \mid \underline{w} \rangle = \underline{v}^\dagger \cdot \underline{w} = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

$$\implies v_1 w_1 + v_2 w_2 + v_3 w_3$$

2. In $\mathbb{C}$:

$$\langle \underline{v} \mid \underline{w} \rangle = \underline{v}^\dagger \cdot \underline{w} = (\underline{v}^T)^* \cdot \underline{w} = \begin{pmatrix} v_1^* & v_2^* & v_3^* \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

$$\implies v_1^* w_1 + v_2^* w_2 + v_3^* w_3 = v_i^* w_i \implies |\underline{v}| = \sqrt{\langle \underline{v} \mid \underline{v} \rangle} = \sqrt{\underline{v}^T \cdot \underline{v}}$$

E.g.

$$\underline{v} = \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix} \implies |\underline{v}| = \sqrt{\begin{pmatrix} 1 & 0 & -i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}} = \sqrt{1+1} = \sqrt{2}$$

Lecture 4

4.1 Matrices

From now on, we work in $\mathbb{R}$ or $\mathbb{C}$

4.1.1 Definition

An operator is an object that associates a vector to another vector

1. Matrices are good examples of operators:

$$A\bar{a} = \bar{b} \; ; \; A_{ij}a_j = b_i$$

   A_{ij} is an element of the matrix, A
2. Matrices are linear operators

$$\mathbf{A}(\bar{a} + \bar{b}) = \mathbf{A}\bar{a} + \mathbf{A}\bar{b}$$
$$\mathbf{A}(\alpha\bar{a}) = \alpha\mathbf{A}\bar{a}$$

4.2 Operators with Matrices

1. Matrix addition and matrix multiplication
   Note that matrix multiplication is not commutative: $\mathbf{AB} \neq \mathbf{BA}$
2. The transpose of a matrix:

$$(\mathbf{A^T})_{ij} = \mathbf{A}_{ji}$$
$$(\mathbf{ABCD})^{\mathbf{T}} = \mathbf{D^T C^T B^T A^T}$$

3. The complex conjugation of a matrix:

$$(\mathbf{A}^*)_{ij} = (\mathbf{A}_{ij})^*$$

4. The Hermitian conjugate of a matrix: (adjoint)

$$(\mathbf{A}^\dagger)_{ij} = (\mathbf{A}_{ij})^\dagger$$

5. The trace of a square matrix:
   It is the sum of diagonal elements

$$\mathrm{Tr}(\mathbf{A}) = \mathbf{A}_{ii}$$

   Notice that the trace is invariant under cyclic permutations, i.e.

$$\mathrm{Tr}(\mathbf{AB}) = \mathrm{Tr}(\mathbf{BA})$$
$$\mathrm{Tr}(\mathbf{ABC}) = \mathrm{Tr}(\mathbf{BCA}) = \mathrm{Tr}(\mathbf{CAB})$$

6. The inverse of a square matrix:
   The inverse of a matrix, $\mathbf{A}$, is a matrix denoted by $\mathbf{A}^{-1}$ such that $\mathbf{AA^{-1}} = \mathbf{A^{-1}A} = \mathbf{I}$

   Identity Matrix: $\mathbf{I}_{ij} = \delta_{ij}$

   Note that a matrix, $\mathbf{A}$, could not have an inverse. If $\nexists\ \mathbf{A}^{-1}$, then matrix $\mathbf{A}$ is said to be singular

4.2.1 Properties of the Inverse

1.

$$(\mathbf{ABCD})^{-1} = \mathbf{D}^{-1}\mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$$

2.

$$(\mathbf{A^T})^{-1} = (\mathbf{A^{-1}})^T$$

The most straight-forward way for calculating the inverse of a matrix is the *Gauss-Jordan method.*

This uses *Elementary Row Operations*:

- ➤ Multiply any row by a non-zero constant
- ➤ Interchange any two rows
- ➤ Add some multiple of one row to another

4.2.2 Example

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 2 & 2 & 7 \end{pmatrix}. \text{ Find the inverse of } \mathbf{A} \text{ if it exists.}$$

Use an augmented matrix of $\mathbf{A}|\,\mathbf{I}$:

$$\left.\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 1 & 3 & 5 & 0 & 1 & 0 \\ 2 & 2 & 7 & 0 & 0 & 1 \end{array}\right. \Longrightarrow \left.\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & -2 & -1 & -2 & 0 & 1 \end{array}\right. \Longrightarrow \left.\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -4 & 2 & 1 \end{array}\right. \Longrightarrow$$

$$\left.\begin{array}{ccc|ccc} 1 & 0 & 2 & 3 & -2 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -4 & 2 & 1 \end{array}\right. \Longrightarrow \left.\begin{array}{ccc|ccc} 1 & 0 & 0 & 11 & -6 & -2 \\ 0 & 1 & 0 & 3 & -1 & -1 \\ 0 & 0 & 1 & -4 & 2 & 1 \end{array}\right. \Longrightarrow$$

$$\mathbf{A}^{-1} = \begin{pmatrix} 11 & -6 & -2 \\ 3 & -1 & -1 \\ -4 & 2 & 1 \end{pmatrix}$$

4.3 Determinant of a square matrix

4.3.1 Definition

The minor $|\mathbf{A_{ij}}|$ associated with element $\mathbf{A_{ij}}$ of a $(n \times m)$ matrix, $\mathbf{A}$, is the determinant of the $(n-1) \times (n-1)$ matrix obtained by removing all elements in the ith row and the jth column

$$|\mathbf{A}| = \det(\mathbf{A}) = \mathbf{A_{11}}|\mathbf{A_{11}}| - \mathbf{A_{12}}|\mathbf{A_{12}}| + \mathbf{A_{13}}|\mathbf{A_{13}}| \cdots + (-1)^{m-1}\mathbf{A_{1m}}|\mathbf{A_{1m}}|$$

This is called the Laplace expansion along the first column
The Laplace expansion can be performed along any row or column

4.3.2 Example

1.

$$\mathbf{A} = \begin{pmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} \\ \mathbf{A_{21}} & \mathbf{A_{22}} \end{pmatrix}$$

$$|\mathbf{A}| = \mathbf{A_{11}}\mathbf{A_{22}} - \mathbf{A_{12}}\mathbf{A_{21}}$$

2.

$$\mathbf{A} = \begin{pmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} & \mathbf{A_{13}} \\ \mathbf{A_{21}} & \mathbf{A_{22}} & \mathbf{A_{23}} \\ \mathbf{A_{31}} & \mathbf{A_{32}} & \mathbf{A_{33}} \end{pmatrix}$$

$$|\mathbf{A}| = \mathbf{A_{11}} \begin{vmatrix} \mathbf{A_{22}} & \mathbf{A_{23}} \\ \mathbf{A_{32}} & \mathbf{A_{33}} \end{vmatrix} - \mathbf{A_{12}} \begin{vmatrix} \mathbf{A_{21}} & \mathbf{A_{23}} \\ \mathbf{A_{31}} & \mathbf{A_{33}} \end{vmatrix} + \mathbf{A_{13}} \begin{vmatrix} \mathbf{A_{21}} & \mathbf{A_{22}} \\ \mathbf{A_{31}} & \mathbf{A_{32}} \end{vmatrix}$$

$$|\mathbf{A}| = (\mathbf{A_{11}A_{22}A_{33}} + \mathbf{A_{12}A_{23}A_{31}} + \mathbf{A_{13}A_{21}A_{32}}) - (\mathbf{A_{11}A_{23}A_{32}} + \mathbf{A_{12}A_{21}A_{33}} + \mathbf{A_{13}A_{22}A_{31}})$$

$$|\mathbf{A}| = \mathbf{A_{1i}A_{2j}A_{3k}}\epsilon_{ijk}$$

4.3.3 Properties of the Determinant

1.

$$|\mathbf{ABCD}| = |\mathbf{A}||\mathbf{B}||\mathbf{C}||\mathbf{D}| \quad (|\mathbf{AB}| = |\mathbf{BA}|)$$

2.

$$|\mathbf{A}|^{\mathbf{T}} = |\mathbf{A}| \ ; \ |\mathbf{A}^*| = |\mathbf{A}|^* \ ; \ |\mathbf{A}^\dagger| = |\mathbf{A}|^*$$

$$|\mathbf{A^{-1}}| = |\mathbf{A}|^{\mathbf{-1}} \implies |\mathbf{A}| = 0 \to \nexists \ \mathbf{A^{-1}}$$

3. If two rows or columns are linearly dependent, then:

$$|\mathbf{A}| = 0$$

4. If $\mathbf{B}$ is obtained from $\mathbf{A}$ by interchanging two rows or columns then:

$$|\mathbf{B}| = -|\mathbf{A}|$$

5. If $\mathbf{B}$ is obtained from $\mathbf{A}$ by multiplying the elements of any row or column by α, then:

$$|\mathbf{B}| = \alpha|\mathbf{A}|$$

$$\mathbf{B} = \alpha\mathbf{A} \implies |\mathbf{B}| = \alpha^k|\mathbf{A}| \ ; \ \text{k is number of rows or columns}$$

Lecture 5

5.1 The Eigenvalue Problem

Consider an $(n \times n)$ matrix. We want to answer the following question:

Are there any vectors $\bar{x} = 0$ which are transformed by $\mathbf{A}$ into multiples of themselves,

$$\mathbf{A}\bar{x} = \lambda\bar{x}$$

If it exists, $\bar{x}$ is an eigenvector, and λ is its eigenvalue

$(\mathbf{A} - \lambda\mathbf{I})\bar{x} = 0$ represents a set of homogeneous linear equations
Such a set of equations will only have a non-trivial solution set if $|\mathbf{A} - \lambda\mathbf{I}| = 0$

5.1.1 Definition

$|\mathbf{A} - \lambda\mathbf{I}| = 0$ is called the characteristic equation, or polynomial of degree n

The eigenvalues, λ, are the n solutions of this equation

5.1.2 Example

Construct the characteristic equation, $|\mathbf{A} - \lambda\mathbf{I}| = 0$:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & 2 & 1 \\ 2 & 1-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\cdots \implies$$

$$\lambda = 1, 4, -1$$

For $\lambda = 1$, solve $(\mathbf{A} - \mathbf{I})\bar{x} = 0$

$$\begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \implies \begin{aligned} 2x_2 + x_3 &= 0 \\ 2x_1 + x_3 &= 0 \\ x_1 + x_2 + x_3 &= 0 \end{aligned}$$

$$\bar{x} = \begin{pmatrix} x_1 \\ x_1 \\ -2x_1 \end{pmatrix} \rightarrow e.g.\ \bar{x} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \implies \hat{x} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

Can repeat for other λs

Note that eigenvectors are generally linearly dependent
Eigenvectors are mutually orthogonal in special cases

If a $(n \times n)$ matrix, $\mathbf{A}$, has n distinct eigenvalues, then the set of corresponding eigenvectors represent a basis in the vector space on which the matrix acts

If the eigenvectors are not all distinct (i.e. degenerate), the basis may or may not exist

If $\mathbf{A}$ has zero eigenvalues, then $\mathbf{A}$ must be singular $\implies |\mathbf{A} - \lambda\mathbf{I}| = |\mathbf{A}| = 0$

5.1.3 Example

$$\mathbf{A} = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix} \implies |\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0 \implies \lambda = 5, -3, -3$$

Degenerate eigenvalue of -3

For $\lambda = 5$, solve $(\mathbf{A} - 5\mathbf{I}) = 0$:

$$\bar{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

For $\lambda_2 = \lambda_3 = -3$, solve $(\mathbf{A} + 3\mathbf{I}) = 0$:

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \implies \left.\begin{array}{r} x_1 + 2x_2 - 3x_3 = 0 \\ 2x_1 + 4x_2 - 6x_3 = 0 \\ -x_1 - 2x_2 + 3x_3 = 0 \end{array}\right\} \textit{same equation}$$

$$x_1 = -2x_2 + 3x_3 \implies \bar{x} = \begin{pmatrix} -2x_2 + 3x_3 \\ x_2 \\ x_3 \end{pmatrix}$$

This yields linearly independent eigenvectors

5.2 Special Matrices

1. Symmetric Matrix: $\mathbf{A} = \mathbf{A^T}$
2. Hermitian Matrix: $\mathbf{A} = \mathbf{A}^\dagger$
   Theorem: The eigenvalues of an Hermitian or Symmetric matrix are real
3. Antisymmetric Matrix: $\mathbf{A^T} = -\mathbf{A}$
4. Anti-Hermitian Matrix: $\mathbf{A}^\dagger = -\mathbf{A}$
   Theorem: The eigenvalues of an Antisymmetric or Anti-Hermitian matrix are purely imaginary or zero
5. Orthogonal Matrix: $\mathbf{A^T} = \mathbf{A^{-1}} \implies \mathbf{A^T A} = \mathbf{I}$
6. Unitary Matrix: $\mathbf{A}^\dagger = \mathbf{A^{-1}} \implies \mathbf{A}^\dagger\mathbf{A} = \mathbf{I}$
   Theorem: The eigenvalues of an Unitary or Orthogonal matrix have unit modulus, i.e $|\lambda|^2 = 1$

Lecture 6

6.1 Special Matrices Continued

6.1.1 Theorem - Eigenvalues of a Hermitian or Symmetric Matrix are real

$$\begin{array}{rcl}
\mathbf{A}\bar{x} = \lambda\bar{x} & \implies & \bar{x}\mathbf{A}^{\dagger} \ = \lambda^{*}\bar{x}^{\dagger} \\
\Downarrow & \implies & \bar{x}^{\dagger}\mathbf{A} \ = \lambda^{*}\bar{x}^{\dagger} \\
\bar{}^{\dagger}\mathbf{A}\bar{x} = \lambda\bar{x}^{\dagger}\bar{x} & \& & \bar{x}^{\dagger}\bar{x} = \lambda^{*}\bar{x}^{\dagger}\bar{x} \\
(\lambda^{*} - \lambda)\bar{x}^{\dagger}\bar{x} = 0 \quad \lambda^{*} = \lambda & & \lambda \in \mathbb{R}
\end{array}$$

6.1.2 Theorem - Eigenvectors of special matrices are linearly independent

In addition, they can be chosen such that they are mutually orthogonal

Definition Two matrices, $\mathbf{A}$ and $\mathbf{A'}$ are said to be similar if $\mathbf{A'} = \mathbf{S^{-1}AS}$ (similarity transformation)

$\mathbf{A}$ and $\mathbf{A'}$ represent the same linear operator in different bases
These bases are related by $\mathbf{S}$
$\mathbf{A}$ and $\mathbf{A'}$ share a few basis-independent properties:

1. $|\mathbf{A}| = |\mathbf{A'}|$
2. $\mathrm{Tr}(\mathbf{A}) = \mathrm{Tr}(\mathbf{A'})$
3. $\{\lambda s \, of \, \mathbf{A}\} = \{\lambda s \, of \, \mathbf{A'}\}$

6.2 Diagonalisation of a Matrix

If the new basis is chosen to be a set of eigenvectors of $\mathbf{A}$, then the matrix $\mathbf{A'} = \mathbf{D}$ is diagonal

$$\mathbf{D} = \mathbf{S^{-1}AS}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \; ; \; \mathbf{S} \begin{pmatrix} \vdots & \vdots & & \vdots \\ \bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_n \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \end{pmatrix}$$

$$
\begin{aligned}
\mathbf{AS} = &\quad \mathbf{A}\begin{pmatrix} \vdots & \vdots & & \vdots \\ \bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_n \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \end{pmatrix} \\
= &\begin{pmatrix} \vdots & \vdots & & \vdots \\ \mathbf{A}\bar{x}_1 & \mathbf{A}\bar{x}_2 & \cdots & \mathbf{A}\bar{x}_n \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \end{pmatrix} \\
= &\begin{pmatrix} \vdots & \vdots & & \vdots \\ \lambda_1\bar{x}_1 & \lambda_2\bar{x}_2 & \cdots & \lambda_n\bar{x}_n \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \end{pmatrix} \\
= &\begin{pmatrix} \vdots & \vdots & & \vdots \\ \bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_n \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \end{pmatrix}\mathbf{D}
\end{aligned}
$$

$$\mathbf{AS} = \mathbf{SD}$$

$$\mathbf{D} = \mathbf{S^{-1}AS}$$

1. $|\mathbf{A}| = |\mathbf{D}| = \prod_{i=1}^{n} \lambda_i = \lambda_1 \times \lambda_2 \times \cdots \times \lambda_n$
2. $\mathrm{Tr}(\mathbf{A}) = \mathrm{Tr}(\mathbf{D}) = \sum_{i=1}^{n} \lambda_i = \lambda_1 + \lambda_2 + \cdots + \lambda_n$

6.2.1 Example

$$
\mathbf{A} = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix} \implies \begin{matrix} \lambda_1 = 6 & \lambda_2 = 1 \\ \bar{x}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix} & \bar{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{matrix}
$$

$$
\mathbf{D} = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{S} = \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix} \quad \mathbf{S^{-1}} = \frac{1}{5}\begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}
$$

$$\mathrm{Tr}(\mathbf{A}) = 7 = \mathbf{D}$$

$$|\mathbf{A}| = 6 = |\mathbf{D}|$$

Consider special matrices
Since it is always possible to find a basis of eigenvectors, then these matrices are always diagonisable.

Since the eigenvectors can be chosen to be an orthonormal set then the matrix $\mathbf{S}$ becomes unitary, i.e. $\mathbf{D} = \mathbf{S^{-1}AS}$ becomes $\mathbf{D} = \mathbf{S}^{\dagger}\mathbf{AS}$ ($S^{-1} = S^{\dagger}$)

For an orthonormal set, $\{\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n\} \to \underline{\bar{x}_i \cdot \bar{x}_j = \delta_{ij}}$, $\quad \mathbf{S}^{\dagger}\mathbf{S} = \mathbf{I}$

$$
\mathbf{S}^\dagger\mathbf{S} = \begin{pmatrix} \cdots & \bar{x}_1^* & \cdots & \cdots \\ \cdots & \bar{x}_2^* & \cdots & \cdots \\ & \vdots & & \\ \cdots & \bar{x}_n^* & \cdots & \cdots \end{pmatrix} \begin{pmatrix} \vdots & \vdots & & \vdots \\ \bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_n \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \end{pmatrix}
$$

$$
= \begin{pmatrix} \bar{x}_1^\dagger\bar{x}_1 & \bar{x}_1^\dagger\bar{x}_2 & \cdots & \bar{x}_1^\dagger\bar{x}_n \\ \bar{x}_2^\dagger\bar{x}_1 & \cdots & & \vdots \\ \vdots & \ddots & & \vdots \\ \bar{x}_n^\dagger\bar{x}_1 & \cdots & \cdots & \bar{x}_n^\dagger\bar{x}_n \end{pmatrix}
$$

$$
= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \cdots & & \vdots \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & \cdots & 1 \end{pmatrix} = (\delta_{ij})
$$

6.2.2 Example

For a symmetric matrix:

$$
A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & 1 \end{pmatrix} \qquad \lambda_1 = 4 \qquad \lambda_{2/3} = -2
$$

$$
\bar{x}_1 = \begin{pmatrix} a \\ 0 \\ a \end{pmatrix} \qquad \bar{x}_{2/3} = \begin{pmatrix} b \\ c \\ -b \end{pmatrix}
$$

$$
\bar{x}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \bar{x}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \bar{x}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}
$$

$$
\mathbf{S} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & -1 & 0 \end{pmatrix} \qquad \mathbf{S}^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \end{pmatrix}
$$

$$
\mathbf{D} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & -1 & 0 \end{pmatrix} = \mathbf{S}^\dagger\mathbf{A}\mathbf{S}
$$

6.3 Application: Power of Matrices

$$
\mathbf{A}^n = \underbrace{\mathbf{A}\mathbf{A}\cdots\mathbf{A}}_{\text{n times}} \text{ if } \mathbf{D} = \mathbf{S^{-1}AS} \text{ then}
$$

$$
\begin{aligned}
\mathbf{A} = \mathbf{SDS^{-1}} \implies \mathbf{A}^n &= (\mathbf{SDS^{-1}})^n \\
&= \underbrace{(\mathbf{SDS^{-1}})(\mathbf{SDS^{-1}})\cdots(\mathbf{SDS^{-1}})}_{\text{n times}} \\
&= \mathbf{SD^nS^{-1}}
\end{aligned}
$$

$$\mathbf{D}^n = \begin{pmatrix} \lambda_1^n & & & \\ & \lambda_2^n & & \\ & & \ddots & \\ & & & \lambda_n^n \end{pmatrix}$$

Lecture 7

7.1 Fourier Series (FS)

FS are series of cos and sin - trig series

They are used to represent periodic functions
A function f(x) is called periodic if $f(x + l) = f(x)\ \forall\ x$
$l > 0$, called the period

The Fourier expansion of a periodic function, f(x), with period L is:

$$f(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty} \left[a_r \cos\left(\frac{2\pi rx}{L}\right) + b_r \sin\left(\frac{2\pi rx}{L}\right) \right]$$

$$a_r = \frac{2}{L} \int_{x_0}^{x_0+L} f(x) \cos\left(\frac{2\pi rx}{L}\right) dx,\ \ r = 0, 1, 2, \cdots$$

$$b_r = \frac{2}{L} \int_{x_0}^{x_0+L} f(x) \sin\left(\frac{2\pi rx}{L}\right) dx,\ \ r = 1, 2, 3, \cdots$$

x_0 is arbitrary since the integral is a function with period L

FS are infinite sums so need to be sure they converge

An FS converges if f(x) satisfies the Dirichlet Conditions

7.2 Dirichlet Conditions

In the interval L, the periodic function f(x)

1. is single-valued $\forall\ x_0 \le x \le x_0 + L$
2. has a finite number of finite discontinuities
3. has a finite number of extreme values, i.e. maxima and minima
   - ➤ This implies that we can represent non-continuous functions by FS

The set of all periodic functions on the interval L that can be represented by FS forms a Vector Space:

1. Operation: Standard addition and scalar multiplication
2. Basis:

$$1 \qquad \sin\left(\frac{2\pi rx}{L}\right) \qquad \cos\left(\frac{2\pi rx}{L}\right) \qquad r > 0$$

   - ➤ Infinite number of elements, i.e. infinite dimension vector space

3. A general element of the space can be written as a linear combination of the basis elements:

$$f(x) = \frac{a_0}{2} \cdot 1 + \sum_{r=1}^{\infty} \left[a_r \cos\left(\frac{2\pi rx}{L}\right) + b_r \sin\left(\frac{2\pi rx}{L}\right) \right]$$

4. Inner product: $\langle f \,|\, g \rangle$

$$\langle f \,|\, g \rangle = \frac{2}{L} \int_0^L f(x) g(x)\, dx$$

   The basis is orthogonal so:

$$\langle f \,|\, g \rangle = \frac{2}{L} \Bigg[$$

$$\int_0^L \sin\left(\frac{2\pi rx}{L}\right) \cos\left(\frac{2\pi px}{L}\right) dx = 0$$

$$+\int_0^L \sin\left(\frac{2\pi rx}{L}\right)\sin\left(\frac{2\pi px}{L}\right)dx = \begin{cases} 0 & r \neq p \\ \frac{L}{2} & r = p \end{cases}$$

$$\left. + \int_0^L \cos\left(\frac{2\pi rx}{L}\right)\cos\left(\frac{2\pi px}{L}\right)dx = \begin{cases} 0 & r \neq p \\ \frac{L}{2} & r = p > 0 \\ L & r = p = 0 \end{cases} \right]$$

- ➤ If f(x) is even, i.e. $f(-x) = f(x)$, then $b_r = 0 \ \forall\ r \in \mathbb{N}$
- ➤ If f(x) is odd, i.e. $f(-x) = -f(x)$, then $a_r = 0 \ \forall\ r \in \mathbb{N}$

7.2.1 Example

A function in the interval of $-\pi \leq x \leq \pi$:

$$f(x) = \begin{cases} -x & -\pi \leq x \leq 0 \\ x & 0 \leq x \leq \pi \end{cases}$$
$$L = 2\pi$$

f(x) is even so $b_r = 0$

$$\begin{aligned} a_r &= \frac{1}{\pi}\int_{-\pi}^{0}(-x)\cos(rx)\,dx + \frac{1}{\pi}\int_0^{\pi} x\cos(rx)\,dx \\ &= \frac{2}{\pi}\int_0^{\pi} x\cos(rx)\,dx \end{aligned}$$

Could choose the interval of $0 \leq x \leq 2\pi$ instead for the same function

$$\begin{aligned} f(x) &= \begin{cases} x & 0 \leq x \leq \pi \\ 2\pi - x & \pi \leq x \leq 2\pi \end{cases} \\ a_r &= \frac{1}{\pi}\int_0^{\pi} x\cos(rx)\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(2\pi - x)\cos(rx)\,dx \\ &= \begin{cases} a_r = \frac{2}{\pi}\frac{(-1)^r - 1}{r^2} & r > 0 \\ a_0 = \pi \end{cases} \end{aligned}$$

7.2.2 Example

$$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases} \quad L = 2\pi$$

f(x) is odd so $a_r = 0$

$$\begin{aligned} b_r &= \frac{1}{\pi}\int_0^{\pi}\sin(rx)\,dx - \frac{1}{\pi}\int_{-\pi}^{0}\sin(rx)\,dx \\ &= \frac{2}{\pi}\sin(rx)\,dx = \frac{2}{r\pi}(1 - (-1)^r),\ r > 0 \\ b_r &= \begin{cases} 0 & r \text{ even} \\ \frac{4}{r\pi} & r \text{ odd} \end{cases} \\ f(x) &= \frac{4}{\pi}\sum_{r=1}^{\infty}\frac{\sin((2r-1)x)}{2r-1} \\ &= \frac{4}{\pi}\left(\sin(x) + \frac{\sin(3x)}{3} + \cdots\right) \end{aligned}$$

Function at $x = n\pi$ is discontinuous - the value of the FS at these values is zero

The sum of the FS at a jump (discontinuity), x_0, is equal to the average of the two function values on either side, i.e.

$$\frac{1}{2}\left[f(x_0^-) + f(x_0^+)\right]$$

Lecture 8

8.1 FS Continued

Sometimes we have functions that are defined only on a finite interval. In order to calculate the FS for this, we need to extend the function by means of functions that are periodic.

8.1.1 Example

$$f(x) = x^2,\ 0 \leq x \leq 2$$

In order to calculate FS, we need to think of possible extensions:

1. Extend to $x^2, \quad -2 \leq x \leq 2$ - this is even and continuous
2. $$f(x) = \begin{cases} x^2 & 0 \leq x < 2 \\ -x^2 & -2 < x \leq 0 \end{cases}$$

   This is odd and non-continuous
3. Extend as $x^2,\ 0 < x < 2$ - this is non-continuous

All these extensions are fine in the sense that they are a fine representation of the function $f(x) = x^2,\ 0 \leq x \leq 2$

However, they have a different value at $x = 2$
In general, continuous extensions are preferable - Gibb's phenomenon at points of discontinuity

FS evaluated at specific points can be used to calculate series of constant terms
Consider Example 2 in Lecture 7:

$$\begin{aligned} f(x) &= \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases} \\ f(x) &= \frac{4}{\pi}\Big(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots \Big) \end{aligned}$$

Choose $f(x = \frac{\pi}{2}) = 1$:

$$\begin{aligned} f(x = \frac{\pi}{2}) &= \frac{4}{\pi}(\sin\frac{\pi}{2} + \cdots) \\ 1 &= \frac{4}{\pi}\Big(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\Big) \\ \frac{\pi}{4} &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} \end{aligned}$$

8.2 Calculus of FS

1. If we integrate a FS with respect to x, we get a power of r at the denominator of each coefficient. This results in a more rapid convergence, meaning a FS can always be integrated.
2. If we differentiate a FS with respect to x, we get a power of r at the numerator of each coefficient. This reduces the rate of convergence, so must be careful with differentiation.

8.2.1 Example

Consider $f(x) = x^2,\ 0 \leq x \leq 2$
In order to write FS, we need a periodic function

Let's choose the even one looked at previously ($b_r = 0$):

$$f(x) = x^2,\ 2 \leq x \leq 2,\ L = 4$$
$$f(x) = \frac{4}{3} + 16\sum_{r=1}^{\infty} \frac{(-1)^r}{(\pi r)^2} \cos\left(\frac{\pi r x}{2}\right) \equiv x^2,\ 0 \leq x \leq 2$$

This FS represents the function $f(x) = x^2,\ 0 \leq x \leq 2$

8.3 Integrating FS

$$\int \frac{4}{3}dx + 16\sum_{r=1}^{\infty} \frac{(-1)^r}{(\pi r)^2} \int \cos\left(\frac{\pi r x}{2}\right) dx = \int x^2 dx$$
$$\frac{4}{3}x + 32\sum_{r=1}^{\infty} \frac{(-1)^r}{(\pi r)^3} \sin\left(\frac{\pi r x}{2}\right) + C = \frac{1}{3}x^3$$

This is not a FS - C and $\frac{4}{3}x$ are not in terms of sin and cos

8.4 Differentiating FS

$$\frac{d}{dx}\left(\frac{4}{3}\right) + 16\sum_{r=1}^{\infty} \frac{(-1)^r}{(\pi r)^2} \frac{d}{dx}\left(\cos\left(\frac{\pi r x}{2}\right)\right) = \frac{d}{dx}\left(x^2\right)$$
$$-8\sum_{r=1}^{\infty} \frac{(-1)^r}{\pi r} \sin\left(\frac{\pi r x}{2}\right) = 2x$$
$$\implies -4\sum_{r=1}^{\infty} \frac{(-1)^r}{\pi r} \sin\left(\frac{\pi r x}{2}\right) = x$$

Use this result in the integrated expression to resolve issues:

$$-16\sum_{r=1}^{\infty} \frac{(-1)^r}{\pi r} \sin\left(\frac{\pi r x}{2}\right) + 96\sum_{r=1}^{\infty} \frac{(-1)^r}{(\pi r)^3} \sin\left(\frac{\pi r x}{2}\right) + C' = x^3$$

Almost have a FS for $g(x) = x^3,\ 0 \leq x \leq 2$

$$g(0) = 0 \implies C' = 0$$
$$-16\sum_{r=1}^{\infty} \frac{(-1)^r}{\pi r} \sin\left(\frac{\pi r x}{2}\right) + 96\sum_{r=1}^{\infty} \frac{(-1)^r}{(\pi r)^3} \sin\left(\frac{\pi r x}{2}\right) = x^3$$

This is the FS of $g(x) = x^3,\ 0 \leq x \leq 2$

8.5 Complex FS

The FS's can be written in complex form:

$$f(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty} \left(a_r \cos\left(\frac{2\pi r x}{L}\right) + b_r \sin\left(\frac{2\pi r x}{L}\right)\right)$$

$$= \frac{a_0}{2} + \sum_{r=1}^{\infty} \left(\frac{a_r}{2} \left(e^{i\frac{2\pi rx}{L}} + e^{-i\frac{2\pi rx}{L}} \right) + \frac{b_r}{2i} \left(e^{i\frac{2\pi rx}{L}} - e^{-i\frac{2\pi rx}{L}} \right) \right)$$

$$= \frac{a_0}{2} + \sum_{r=1}^{\infty} \left(e^{i\frac{2\pi rx}{L}} \left(\frac{a_r}{2} + \frac{b_r}{2i} \right) + e^{-i\frac{2\pi rx}{L}} \left(\frac{a_r}{2} - \frac{b_r}{2i} \right) \right)$$

$$\frac{a_r - b_r i}{2} \equiv c_r \ ; \ \frac{a_r + b_r i}{2} \equiv d_r$$

$$a_r = a_{-r} \ ; \ b_r = b_{-r}$$

$$d_r = c_{-r}$$

$$f(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty} \left(c_r e^{i\frac{2\pi rx}{L}} + c_{-r} e^{-i\frac{2\pi rx}{L}} \right)$$

$$= \sum_{-\infty}^{\infty} c_r e^{i\frac{2\pi rx}{L}}, \ c_0 = \frac{a_0}{2}$$

Lecture 9

9.1 Integral Transforms

A function $g(y)$ defined by the equation

$$g(y) = \int_{-\infty}^{\infty} k(x,y) f(x)\, dx = I[f(x)](y)$$

is called the integral transform of $f(x)$

1. The function $k(x,y)$ is called the kernel of the transform
2. I is linear, i.e.
$$I[c_1 f_1 + c_2 f_2] = c_1 I[f_1] + c_2 I[f_2]$$
3. If I is given, can introduce the inverse, I^-1, such that
$$I[f] = g \to I^{-1}[g] = f$$

There are several types of ITs. Consider the Fourier and the Laplace transforms

9.2 Fourier Transforms (FTs)

FT of f is defined by

$$\mathcal{F}[f(t)](\omega) = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \underbrace{e^{-i\omega t}}_{k(x,y)} dt$$

The integral exists if:

1. f has a finite number of finite discontinuities
2. $\int_{-\infty}^{\infty} |f(t)|\, dt$ is finite, i.e. it converges

If f is continuous, can define the inverse FT as

$$\mathcal{F}^{-1}[\hat{f}(\omega)](t) = f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega$$

There are different forms for the FTs:

1. $$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \ ; \ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega$$
2. $$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt \ ; \ f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-i\omega t} d\omega$$
3. $$\omega = 2\pi v \to \hat{f}(v) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi v t} dt \ ; \ f(t) = \int_{-\infty}^{\infty} \hat{f}(v) e^{i2\pi v t} dv$$

There are functions that are not periodic, therefore, cannot use FS for them. Can imagine that these functions are defined over an infinite interval.

Consider complex FS:

$$f(t) = \sum_{-\infty}^{\infty} c_n e^{i\frac{2\pi n t}{L}} = \sum_{-\infty}^{\infty} \left(\frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(t) e^{-i\frac{2\pi n t}{L}} dt \right) e^{i\frac{2\pi n t}{L}}$$

$$\frac{2\pi n}{L} = \omega_n \implies \Delta\omega = \omega_{n+1} - \omega_n = \frac{2(n+1)\pi}{L} - \frac{2\pi n}{L} = \frac{2\pi}{L}$$

$$f(t) = \sum_{-\infty}^{\infty} \left(\frac{\Delta\omega}{2\pi} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(t) e^{-it\omega_n} dt \right) e^{it\omega_n}$$

As $L \to \infty$:

- $$\left[-\frac{L}{2},\frac{L}{2}\right] \to (-\infty,\infty)$$
- $$\omega_n \to \omega$$
- $$\Delta\omega \to d\omega$$
- $$\sum_{-\infty}^{\infty} \to \int_{-\infty}^{\infty}$$

$$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} \underbrace{\left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} f(t)e^{-i\omega t}dt\right)}_{\hat{f}(\omega)} e^{i\omega t}$$

Fourier Series	Fourier Transforms
Periodic functions	Non-periodic functions
Finite period	Infinite period
Discrete spectrum	Continuous spectrum

9.2.1 Example

$$f(t) = \begin{cases} 0 & -\frac{L}{2} < t < -\frac{a}{2} \\ 1 & -\frac{a}{2} < t < \frac{a}{2} \\ 0 & \frac{a}{2} < t < \frac{L}{2} \end{cases}$$
$$f(t) = f(t+L)$$

FS is complex:

$$c_n = \frac{a}{L}\frac{\sin\left(\frac{n\pi a}{L}\right)}{\frac{n\pi a}{L}}$$
$$f(t) = \sum_{-\infty}^{\infty} \frac{a}{L}\frac{\sin\left(\frac{n\pi a}{L}\right)}{\frac{n\pi a}{L}} e^{i\frac{2\pi n t}{L}}$$

The c_n, called the spectral coefficient of the n^{th} harmonic, form a discrete complex spectrum

$$|c_n| \approx |\frac{\sin t_n}{t_n}|$$

As the period increases, the separation between the pulses increases as well. In the limit, $L \to \infty$, only a single pulse remains and the resulting function is:

$$f(t) = \begin{cases} 1 & -\frac{a}{2} < t < \frac{a}{2} \\ 0 & \text{otherwise} \end{cases}$$

This is non-periodic
The FT is:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \frac{1}{\sqrt{2\pi}}\int_{-\frac{a}{2}}^{\frac{a}{2}} e^{-i\omega t}dt$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-i\omega t}}{-i\omega} \right]_{-\frac{a}{2}}^{\frac{a}{2}} = \frac{2}{\sqrt{\pi}} \frac{1}{-2i\omega} \left(e^{-i\frac{\omega a}{2}} - e^{i\frac{\omega a}{2}} \right)$$

$$= \frac{a}{\sqrt{2\pi}} \frac{\sin\left(\frac{a\omega}{2}\right)}{\frac{a\omega}{2}}$$

Lecture 10

10.1 Fourier Transforms Continued

$$\mathcal{F}[f(t)](\omega) = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$
$$\mathcal{F}^{-1}[\hat{f}(\omega)](t) = f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega$$

10.1.1 Properties

- Scaling

$$\begin{aligned}
\mathcal{F}[f(at)](\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(at) e^{-i\omega t} dt, \ (at = t') \\
&= \frac{1}{|a|} \mathcal{F}[f(t')]\Big(\frac{\omega}{a}\Big) = \frac{1}{|a|\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t') e^{-i\omega \frac{t'}{a}} dt' \\
&= \frac{1}{|a|} \hat{f}\Big(\frac{\omega}{a}\Big)
\end{aligned}$$

- Transposition

$$\mathcal{F}[f(t+a)](\omega) = e^{i\omega a} \mathcal{F}[f(t)](\omega) = e^{i\omega a} \hat{f}(\omega)$$

- Exponential Multiplication

$$\mathcal{F}[e^{at} f(t)](\omega) = \mathcal{F}[f(t)](\omega + ia) = \hat{f}(\omega + ia)$$

10.1.2 Example

$$\begin{aligned}
\mathcal{F}\Big[f\big(\frac{t}{2}\big)\cos(\alpha t)\Big](\omega) &= \frac{1}{2}\mathcal{F}\Big[f\big(\frac{t}{2}\big)e^{i\alpha t}\Big](\omega) + \frac{1}{2}\mathcal{F}[f\big(\frac{t}{2}\big)e^{-i\alpha t}](\omega) \\
&= \frac{1}{2} \cdot 2\mathcal{F}[f(t)e^{2i\alpha t}](2\omega) + \frac{1}{2} \cdot 2\mathcal{F}[f(t)e^{-2i\alpha t}](2\omega) \\
&= \mathcal{F}[f(t)](2\omega - 2\alpha) + \mathcal{F}[f(t)](2\omega + 2\alpha)
\end{aligned}$$

10.2 Fourier Transform of a Derivative (differential rule)

$$\mathcal{F}[f'(t)](\omega) = i\omega \hat{f}(\omega)$$
$$\mathcal{F}[f''(t)](\omega) = -\omega^2 \hat{f}(\omega)$$
$$\mathcal{F}[f^{(n)}(t)](\omega) = (i\omega)^n \hat{f}(\omega)$$

10.3 Convolution

The convolution of two functions is defined as

$$h(y) = \int_{-\infty}^{\infty} f(x) g(y-x)\, dx = (f \star g)(y) = (g \star f)(y)$$

10.4 Convolution Theorem

The FT of the convolution $h(y)$ is the product of the FTs of f and g:

$$\begin{aligned}
\hat{h}(k) &= \sqrt{2\pi}\hat{f}(k)\hat{g}(k) \\
\hat{h}(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(y)e^{-iyk}dy = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x)g(y-x)\,dx\right)e^{-iyk}dy \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\left(\int_{-\infty}^{\infty} g(y-x)e^{-iyk}dy\right)dx,\ (y-x=z) \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\left(\int_{-\infty}^{\infty} g(z)e^{-i(z+x)k}dz\right)dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ixk}dx\left(\frac{1}{\sqrt{2\pi}}\sqrt{2\pi}\int_{-\infty}^{\infty} g(z)e^{-izk}dz\right) \\
&= \sqrt{2\pi}\hat{f}(k)\hat{g}(k)
\end{aligned}$$

10.4.1 Example

$$\frac{d^2f}{dt^2} + 2\frac{df}{dt} + f(t) = h(t)$$

Apply FT on both sides:

$$\begin{aligned}
\mathcal{F}\left[\frac{d^2f}{dt^2}\right](\omega) + 2\mathcal{F}\left[\frac{df}{dt}\right](\omega) + \mathcal{F}[f(t)](\omega) &= \mathcal{F}[h(t)](\omega) \\
-\omega^2\hat{f}(\omega) + 2i\omega\hat{f}(\omega) + \hat{f}(\omega) &= \hat{h}(\omega) \\
\hat{f}(\omega) &= \frac{\hat{h}(\omega)}{(1+2i\omega-\omega^2)} \\
&= \frac{\hat{h}(\omega)}{(i\omega+1)^2}
\end{aligned}$$

1.
$$f(t) = \mathcal{F}^{-1}\left[\frac{\hat{h}(\omega)}{(i\omega+1)^2}\right]$$

2.
$$\begin{aligned}
\hat{f}(\omega) &= \frac{\hat{h}(\omega)}{(i\omega+1)^2} = \sqrt{2\pi}\hat{h}(\omega)\hat{g}(\omega),\quad \left[\hat{g}(\omega) = \frac{1}{\sqrt{2\pi}}\frac{1}{(i\omega+1)^2}\right. \\
f(t) &= \int_{-\infty}^{\infty} g(t')h(t-t')\,dt'
\end{aligned}$$

10.5 Dirac delta-function

Consider a pulse,

$$\delta_n(x) = \begin{cases} n & -\frac{1}{2n} < x < \frac{1}{2n} \\ 0 & \text{otherwise} \end{cases}$$

If we take the duration of the pulse to decrease while at the same time retaining a unit area, then in the limit, we are lead to the notion of the Dirac δ-function:

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\delta_n(x)\,dx = \int_{-\infty}^{\infty}\delta(x)\,dx = 1$$

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\delta_n(x)f(x)\,dx=\int_{-\infty}^{\infty}\delta(x)f(x)\,dx=f(0)$$

δ-function is not a standard function. It is a generalised function distribution.
It is defined as the limit of a sequence of functions (not a unique sequence)

Its defining properties are:

1. $$\delta(x-a)=0,\ x\neq a$$

2. $$\int_{\alpha}^{\beta}f(x)\delta(x-a)\,dx=\begin{cases}f(a) & \alpha<a<\beta\\ 0 & \text{otherwise}\end{cases}$$

10.6 Example

1. $$\int_{-4}^{4}\delta(x-\pi)\cos(x)\,dx=\cos\pi=-1$$

2. $$\hat{\delta}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(t)e^{-i\omega t}dt=\frac{1}{\sqrt{2\pi}}$$

Lecture 11

11.1 Integral Representation of the delta-function

$$\delta_n(t-x) = \frac{\sin(n(t-x))}{\pi(t-x)} = \frac{1}{2\pi}\int_{-n}^{n} e^{i\omega(t-x)}d\omega$$

Because this is a δ-function sequence, we can write

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty}\delta(t-x)f(t)\,dt = \lim_{n\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\left(\int_{-n}^{n} e^{i\omega(t-x)}d\omega\right)dt \\ &= \int_{-\infty}^{\infty} f(t)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-x)}d\omega\right)dt \\ \implies \delta(t-x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-x)}d\omega \end{aligned}$$

11.1.1 Example

Inverse FT of a constant, $\frac{1}{\sqrt{2\pi}}$:

$$\mathcal{F}^{-1}\left[\frac{1}{\sqrt{2\pi}}\right](t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{i\omega t}d\omega = \delta(t)$$

11.1.2 Properties

1. $$\delta(x) = \delta(-x)\text{- even function}$$

2. $$\delta(g(x)) = \underbrace{\sum_a \frac{\delta(x-a)}{|g'(a)|}}_{g(a)=0,\ g'(a)\neq 0}$$

11.1.3 Example

Calculate

$$I = \int_{-\infty}^{\infty}\delta(x^2-b^2)f(x)\,dx$$

Consider

$$\delta(x^2-b^2)\ ;\ g(x) = x^2-b^2$$

$g(x)$ is a polynomial with two roots:

$$\begin{aligned} x &= \pm b \\ g'(x) &= 2x \\ g'(\pm b) &= \pm 2b \neq 0 \end{aligned}$$

So then:

$$\begin{aligned} \delta(x^2-b^2) &= \frac{\delta(x+b)}{|-2b|} + \frac{\delta(x-b)}{|2b|} \\ &= \frac{\delta(x-b)+\delta(x+b)}{2b} \end{aligned}$$

$$\begin{aligned}\implies I &= \frac{1}{2b}\int_{-\infty}^{\infty}\delta(x-b)f(x)\,dx + \frac{1}{2b}\int_{-\infty}^{\infty}\delta(x+b)f(x)\,dx \\ &= \frac{1}{2b}(f(b)+f(-b))\end{aligned}$$

11.2 Heaviside Step Function

This is also a distribution

$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

This is also used as $\Theta(x)$

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)H'(x)\,dx &= \Big[f(x)H(x)\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)H(x)\,dx \\ &= f(\infty) - \int_{0}^{\infty} f'(x)\,dx = f(\infty) - f(\infty) + f(0) \\ &= f(0) = \int_{-\infty}^{\infty} f(x)\delta(x)\,dx\end{aligned}$$

11.3 Laplace Transforms

Definition:

$$\mathcal{L}[f(t)](s) = \bar{f}(s) = \int_0^{\infty} f(t)e^{-st}dt$$

We take s to be real

11.3.1 Examples

1. Consider $f(t) = t$

$$\begin{aligned}\bar{f}(s) &= \int_0^{\infty} te^{-st}dt = \Big[\frac{te^{-st}}{-s}\Big]_0^{\infty} + \frac{1}{s}\int_0^{\infty} e^{-st}dt \\ &= \frac{1}{s}\Big[\frac{e^{-st}}{-s}\Big]_0^{\infty} \\ &= \frac{1}{s^2},\ s > 0\end{aligned}$$

2.

$$\begin{aligned} f(t) &= \cosh(kt) \\ \implies \bar{f}(s) &= \frac{s}{s^2-k^2},\ s > |k|\end{aligned}$$

3. Consider $H(t-a)$

$$\begin{aligned}\mathcal{L}[H(t-a)](s) &= \int_0^{\infty} H(t-a)e^{-st}dt = \int_a^{\infty} e^{-st}dt \\ &= \Big[\frac{e^{-st}}{-s}\Big]_a^{\infty} \\ &= \frac{e^{-sa}}{s},\ s > 0\end{aligned}$$

11.3.2 Properties

1. $$\mathcal{L}[H(t-a)f(t-a)](s) = e^{-sa}\mathcal{L}[f(t)](s) = e^{-as}\bar{f}(s)$$
2. $$\mathcal{L}[e^{at}f(t)](s) = \mathcal{L}[f(t)](s-a) = \bar{f}(s-a)$$
3. $$\mathcal{L}[f(at)](s) = \frac{1}{|a|}\bar{f}(s),\ a \neq 0$$
4. $$\mathcal{L}[t^n f(t)](s) = (-1)^n \frac{d^n \bar{f}(s)}{ds^n}, n \in \mathbb{N}$$
5. Laplace transform of a derivative:
$$\begin{aligned}
\mathcal{L}[f'(t)](s) &= \int_0^\infty f'(t)e^{-st}dt = \Big[f(t)e^{-st}\Big]_0^\infty + s\int_0^\infty f(t)e^{-st}dt \\
&= -f(0) + s\bar{f}(s),\ s > 0 \\
\mathcal{L}[f''(t)](s) &= s^2\bar{f}(s) - sf(0) - f'(0) \\
\mathcal{L}[f^{(n)}(t)](s) &= s^n\bar{f}(s) - s^{(n-1)}f(0) - s^{(n-2)}f^{(1)}(0) - \cdots - s^0 f^{(n-1)}(0)
\end{aligned}$$

11.3.3 Example

$$\begin{aligned}
\mathcal{L}[\sinh(kt)](s) &= \mathcal{L}\left[\frac{d}{dt}\Big(\frac{\cosh(kt)}{k}\Big)\right] = -\frac{1}{k} + \frac{s}{k}\frac{s}{s^2-k^2} \\
&= \frac{k}{s^2-k^2},\ s > |k| \\
\mathcal{L}[t\sinh(kt)](s) &= (-1)\frac{d}{ds}\Big(\frac{k}{s^2-k^2}\Big) \\
&= \frac{2ks}{s^2-k^2},\ s > |k|
\end{aligned}$$

11.4 Convolution Theorem

If the functions f and g have LTs $\bar{f}(s)$ and $\bar{g}(s)$, then

$$\begin{aligned}
\mathcal{L}[(f\star g)](s) = \mathcal{L}[(g\star s)] &= \mathcal{L}\Big[\int_0^t f(u)g(t-u)\,du\Big](s) \\
&= \bar{f}(s)\bar{g}(s) \\
\bar{f}(s)\bar{g}(s) &= \int_0^\infty f(u)e^{-su}du\int_0^\infty g(v)e^{-sv}dv \\
&= \int_0^\infty\int_0^\infty e^{-s(u+v)}f(u)f(v)\,du\,dv,\ (u+v=t) \\
&= \int_0^\infty du\int_0^\infty e^{-st}f(u)f(t-u)\,dt
\end{aligned}$$

Can choose for integration:

Vertical Strips	Horizontal Strips
$0 < u < \infty$	$0 < t < \infty$
$u < t < \infty$	$0 < u < t$

$$\begin{aligned}
\bar{f}(s)\bar{g}(s) &= \int_0^\infty e^{-st}\left(\int_0^t f(u)g(t-u)\,du\right)dt \\
&= \mathcal{L}\Big[\int_0^t f(u)g(t-u)\,du\Big](s)
\end{aligned}$$

Lecture 12

12.1 Inverse of a LT

$$\mathcal{L}^{-1}[\bar{f}(s)](t) = f(t)$$

The general method requires complex analysis, however, it is possible to perform the inverse of a LT by:

1. Inspection - use partial fraction decomposition and LT properties and table of transforms
2. Convolution Theorem

12.1.1 Table of Laplace Transforms

$f(t)$	$\bar{f}(s)$	$s_0 (s > s_0)$
c	$\frac{c}{s}$	0
e^{at}	$\frac{1}{s-a}$	a
ct^n	$\frac{cn!}{s^{n+1}}$	0
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	a

12.1.2 Examples

1. Partial Fractions decomposition:
   Find $f(t)$ if $\bar{f}(s) = \frac{s+3}{s(s+1)}$

$$\begin{aligned}
\bar{f}(s) &= \frac{s+3}{s(s+1)} = \frac{3}{s} - \frac{2}{s+1} = \bar{f}_1(s) + \bar{f}_2(s) \\
\mathcal{L}^{-1}[\bar{f}_1(s)](t) &= 3,\ s > 0 \\
\mathcal{L}^{-1}[\bar{f}_2(s)](t) &= -2e^{-t},\ s > -1 \\
\implies \mathcal{L}^{-1}[\bar{f}(s)](t) &= 3 - 2e^{-t},\ s > 0
\end{aligned}$$

2. Convolution Theorem Find $f(t)$ if $\bar{f}(s) = \frac{2}{s^2(s+1)^2}$

$$\begin{aligned}
\bar{f}(s) &= \frac{2}{s^2} \cdot \frac{1}{(s+1)^2} = \bar{f}_1(s) \cdot \bar{f}_2(s) \\
\mathcal{L}^{-1}[\bar{f}_1(s)](t) &= 2t,\ s > 0 \\
\mathcal{L}^{-1}[\bar{f}_2(s)](t) &= te^{t},\ s > 1 \\
\implies \mathcal{L}^{-1}[\bar{f}(s)](t) &= \int_0^t 2(t-u)ue^{-u}du \\
&= 2e^{-t}(t+2) + 2(t-2)
\end{aligned}$$

3. The LTs are used to solve ODEs:

$$\frac{df}{dt} + 2f = e^{-t},\ f(0) = 3$$

Apply LT across ODE

$$\begin{aligned}
&\implies \mathcal{L}\Big[\frac{df}{dt}\Big](s) + 2\mathcal{L}[f](s) = \mathcal{L}[e^{-t}](s) \\
&\implies -f(0) + s\bar{f}(s) + 2\bar{f}(s) = \frac{1}{s+1} \\
&\implies \bar{f}(s) = \frac{3s+4}{(s+2)(s+1)} = \frac{1}{s+1} + \frac{2}{s+2} \\
&\implies f(t) = e^{-t} + 2e^{-2t}
\end{aligned}$$

4. The Hamiltonian for a harmonic oscillator

$$H(p,x) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = E$$
$$-\frac{\hbar^2}{2m}\frac{d^2\Psi(x)}{dx^2} + \frac{1}{2}m\omega^2 x^2\Psi(x) = E\Psi(x)$$
$$\Psi_0(x) = e^{-\frac{m\omega}{2\hbar}x^2},\ E_0 - \frac{\hbar}{2}\omega$$

What is the wavefunction in momentum-space?
In order to see that, apply Fourier analysis, $x \to k,\ p = \hbar k$

$$g_0(p) = \hat{\Psi}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{m\omega}{2\hbar}x^2} e^{-ikx} dx$$
$$= e^{-\frac{p^2}{2\hbar m\omega}}$$
$$\Psi_0(x) = e^{-\frac{m\omega}{2\hbar}x^2}\ ,\ g_0(p) = e^{-\frac{p^2}{2\hbar m\omega}}$$

The width of a Gaussian, $e^{-\frac{x^2}{\sigma^2}}$, is $\sqrt{\frac{\sigma^2}{2}}$:

$$\Delta x = \sqrt{\frac{\hbar}{m\omega}}\ ;\ \Delta p = \sqrt{\hbar m\omega}$$

This is a Quantum Mechanical effect and shows the Uncertainty Principle:

$$\Delta x \Delta p = \hbar$$

Lecture 13

13.1 Vector Calculus/Vector fields

These are vectors whose components are functions of one or more variables.

$$\bar{a}(u) = a_x(u)\hat{i} + a_y(u)\hat{j} + a_z(u)\hat{k}$$

A vector function defines a vector field.

13.1.1 Derivative of a Vector Function

The derivative is obtained by differentiating each component.

$$\bar{a}'(u) = \frac{d\bar{a}}{du} = \frac{da_x}{du}\hat{i} + \frac{d\bar{a}_y}{du}\hat{j} + \frac{d\bar{a}_z}{du}\hat{k}$$

Note that in Cartesian coordinates, the vectors $\hat{i}, \hat{j}, \hat{k}$ are constant.

13.2 Rules of Differentiation

1. $$\frac{d}{du}(\phi\bar{a}) = \frac{d\phi}{du}\bar{a} + \phi\frac{d\bar{a}}{du}$$
2. $$\frac{d}{du}(\bar{a} \cdot \bar{b}) = \frac{d\bar{a}}{du}\bar{b} + \bar{a}\frac{d\bar{b}}{du}$$
3. $$\frac{d}{du}(\bar{a} \times \bar{b}) = \frac{d\bar{a}}{du} \times \bar{b} + \bar{a} \times \frac{d\bar{b}}{du}$$
4. $$\frac{d\bar{a}}{du}(\phi(u)) = \frac{d\bar{a}}{d\phi}\frac{d\phi}{du}$$

13.2.1 Differential of a Vector Function

$$d\bar{a} = \frac{\bar{a}}{du}du = \bar{a}'(u)du$$

If the vector function depends on more than one variables, then:

$$\bar{a}(u, v, \cdots) = a_x(v, u, \cdots)\hat{i} + a_y(u, v, \cdots)\hat{j} + a_z(u, v, \cdots)\hat{k}$$

Derivative:

$$\frac{\partial\bar{a}}{\partial u} = \frac{\partial\bar{a}_x}{\partial u}\hat{i}\cdots \; ; \; \frac{\partial\bar{a}}{\partial v} = \cdots$$

Differential:

$$d\bar{a} = \frac{\partial\bar{a}}{\partial u}du + \frac{\partial\bar{a}}{\partial v}dv + \cdots$$

13.2.2 Example

$$\bar{a} = \bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$\frac{\partial\bar{r}}{\partial x} = \hat{i} \; ; \; \frac{\partial\bar{r}}{\partial y}\cdots$$
$$d\bar{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$$

13.3 Curve and Vector Fields

A curve, C, can be represented by a vector function of the type:

$$\bar{r}(u) = x(u)\hat{\imath} + y(u)\hat{\jmath} + z(u)\hat{k}$$

This is called a parametric representation of the curve, C, and u is the parameter of the representation.

13.3.1 Examples

1.

$$y = -x, \quad -1 \leq x \leq 1$$
$$\implies \bar{r}(u) = u\hat{\imath} - u\hat{\jmath}, \quad -1 \leq u \leq 1$$

2.

$$x^2 + y^2 = 4$$
$$\bar{r}(u) = (3\cos(u))\hat{\imath} + (2\sin(u))\hat{\jmath}, \ 0 \leq u \leq 2\pi$$

3.

$$\frac{x^2}{4} + y^2 = 1, \ y \geq 0, \ z = 3$$
$$\bar{r}(u) = (2\cos(u))\hat{\imath} + (\sin(u))\hat{\jmath} + 3\hat{k}, \ 0 \leq u \leq 2\pi$$

13.3.2 Features of the Curve

- The derivative $\bar{r}'(u)$ is a vector tangent to the curve at each point.

$$\frac{d\bar{r}}{du} = \bar{r}'(u) = x'(u)\hat{\imath} + y'(u)\hat{\jmath}_z'(u)\hat{k}$$

- The arc length, S, measured along C from some fixed points satisfies:

$$\left(\frac{dS}{du}\right)^2 = \frac{d\bar{r}}{du} \cdot \frac{d\bar{r}}{du} \implies dS = \pm\sqrt{\left(\frac{d\bar{r}}{du} \cdot \frac{d\bar{r}}{du}\right)}\, du$$

  The sign fixes the direction of measuring, for increasing or decreasing u.

13.3.3 Examples

Consider a helix:

$$\bar{r}(u) = 3\cos u\hat{\imath} + 3\sin u\hat{\jmath} + 4u\hat{k}$$

We want to measure the arc length between $u = 0$ and $u = 4$:

$$\bar{r}'(u) = -3\sin u\hat{\imath} + 3\cos u\hat{\jmath} + 4\hat{k}$$
$$dS = \sqrt{(+16)}\, du = 5\, du$$
$$\implies S = \int_0^4 5\, du = 20$$

The parameterisation can be changed from $u \to v$ then:

$$\frac{d\bar{r}}{dv} = \frac{d\bar{r}}{du}\frac{du}{dv}$$

The tangent vector changes size but not direction.
If $v = S$, then

$$\frac{d\bar{r}}{dS} = \frac{d\bar{r}}{du}\frac{du}{dS} = \frac{\bar{t}}{|t|} = \hat{t}$$

dS is the line element of C

Since $\hat{t} = \frac{d\bar{r}}{dS}$ is a unit tangent vector, if we take the derivative of $\hat{t}^2 = \hat{t} \cdot \hat{t} = 1 \implies$

$$\hat{t} \cdot \hat{t}' = 0 \implies \hat{t} \perp \hat{t}'$$

That is

$$\frac{d^2\bar{r}}{dS^2} = \hat{t}'$$

defines a direction perpendicular to C at each point

$$\frac{d^2\bar{r}}{dS^2} = \frac{d\hat{t}}{dS} = \frac{\hat{n}}{\rho}$$

where $\hat{n}$ is called the principal normal and ρ is the radius of curvature.

Consider a helix

$$\begin{aligned}
\bar{r} &= 3\cos u\hat{i} + 3\sin u\hat{j} + 4u\hat{k} \\
\frac{d\bar{r}}{du} &= \bar{t} = -3\sin u\hat{i} + 3\cos u\hat{j} + 4\hat{k} \\
\implies \hat{t} &= \frac{d\bar{r}}{dS} = \frac{d\bar{r}}{du}\frac{du}{dS} = (-3\sin u\hat{i} + 3\cos u\hat{j} + 4\hat{k})\frac{1}{5} \\
\hat{t} &= -\frac{3}{5}\sin(\frac{S}{5})\hat{i} + \frac{3}{5}\cos(\frac{S}{5})\hat{j} + \frac{4}{5}\hat{k} \\
\frac{d\hat{t}}{dS} &= \frac{d\hat{t}}{du}\frac{du}{dS} = -\frac{3}{25}\cos(\frac{S}{5})\hat{i} - \frac{S}{25}\sin(\frac{5}{5})\hat{j} = \frac{\hat{n}}{\rho} \\
\frac{1}{\rho} &= \frac{3}{25} \implies \rho = \frac{25}{3}
\end{aligned}$$

Lecture 14

14.1 Scalar Functions and Fields

A scalar function defines a scalar field, e.g. $\phi(u,v), \phi(x,y,x), \phi(r), r = |\underline{r}| = \sqrt{x^2+y^2+z^2}$

14.1.1 Gradient of a Scalar Function in Cartesian Coordinates

This operation allows us to establish a relation between scalar and vector functions.
For a given scalar field, $\phi(x,y,z)$, the gradient of ϕ is:

$$\text{grad}\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{\imath} + \frac{\partial\phi}{\partial y}\hat{\jmath} + \frac{\partial\phi}{\partial z}\hat{k}$$

14.1.2 Properties of Del

1. ∇ =del, or nabla, operator: $\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$
   It is a vector differential operator.
2. $\nabla\phi$ is a vector function
3. $$\nabla(\phi+\psi) = \nabla\phi + \nabla\psi$$
4. $$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$$
5. $$\nabla(\psi(\phi)) = \psi'(\phi)\nabla\phi$$
6. Special cases: $r = |\underline{r}| = \sqrt{x^2+y^2+z^2}$
   $$\nabla r = \frac{\underline{r}}{r}\ ;\ \nabla\left(\frac{1}{r}\right) = -\frac{\underline{r}}{r^3}\ ;\ \nabla(\phi(r)) = \phi'(r)\nabla r$$

Example

$$\begin{aligned}
\nabla r &= \frac{\partial}{\partial x}r\hat{\imath} + \frac{\partial}{\partial y}r\hat{\jmath} + \frac{\partial}{\partial z}r\hat{k} \\
&= \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{\imath} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{\jmath} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{k} \\
&= \frac{1}{r}(x\hat{\imath} + y\hat{\jmath} + z\hat{k}) = \frac{\underline{r}}{r}
\end{aligned}$$

14.2 Surfaces and Vector Fields

A surface S can be represented by a vector function of the type

$$\underline{r}(u,v) = x(u,v)\hat{\imath} + y(u,v)\hat{\jmath} + z(u,v)\hat{k}$$

This is called a parametric representation of the surface S

14.2.1 Features

- $\frac{\partial \underline{r}}{\partial u}$ and $\frac{\partial \underline{r}}{\partial v}$ are tangent vectors to a curve C on S with v and u constant respectively.
  These vectors are linearly independent, and their cross-product defines a vector, $\underline{n}$, which is normal to S.
$$\underline{n} = \frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v} \neq 0$$
- The small changes du and dv produce a small parallelogram on S. We have:
$$dS = \left| \frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v} \right| du\, dv = |\underline{n}|\, du\, dv$$
  This is called the scalar area element.
- The vector area element is:
$$d\underline{S} = \left(\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v} \right) du\, dv = \underline{n}\, du\, dv$$
- The order of the parameters u and v specifies an orientation for S, which is equivalent to a choice of normal, $\underline{n}$.

A surface S is said to be orientable if the vector $\underline{n}$ is determined everywhere by a choice of sign.

A surface can be represented by the equation $\phi(\underline{r}) = c$
Consider any curve $\underline{r}(u)$ in S, i.e.

$$\begin{aligned}
\phi(\underline{r}(u)) &= c,\ \underline{r}(u) = x(u)\hat{i} + y(u)\hat{j} + z(u)\hat{k} \\
\frac{d\phi}{du} &= \nabla\phi \cdot \frac{d\underline{r}}{du} = 0 \\
d\phi &= \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz \\
&= \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k}) \\
&= \nabla\phi \cdot d\underline{r} = 0
\end{aligned}$$

Hence $\nabla\phi \perp$ to any vector tangent to the surface. $\nabla\phi$ is normal to the surface.

14.2.2 Example

$$\begin{aligned}
\phi(\underline{r}) &= x^2 + y^2 + z^2 = c \text{ - sphere with radius } \sqrt{c} \\
\nabla\phi &= 2x\hat{i} + 2y\hat{j} + 2z\hat{k} = 2\underline{r}
\end{aligned}$$

14.2.3 Definition

A surface can be defined to have a boundary, ∂S, consisting of a smooth closed curve. A surface is bounded if it can be contained within some solid sphere. A bounded surface with no boundary is closed.

Lecture 15

15.1 Divergence of a Vector Field in Cartesian Coordinates

Consider a vector field $\underline{a}(x, y, z)$, then the divergence is defined as

$$\nabla \cdot \underline{a} = \text{div}\underline{a} = \frac{\partial}{\partial x}\underline{a}_x + \frac{\partial}{\partial y}\underline{a}_y + \frac{\partial}{\partial z}\underline{a}_z$$

15.1.1 Properties

1. $$\nabla \cdot (\underline{a} + \underline{b}) = \nabla \cdot \underline{a} + \nabla \cdot \underline{b}$$

2. $$\nabla \cdot (\phi\underline{a}) = \nabla\phi \cdot \underline{a} + \phi\nabla\underline{a}$$

3. $$\nabla(\underline{a} \times \underline{b}) = \underline{b}(\nabla \times \underline{a}) - \underline{a}(\nabla \times \underline{b})$$

4. $$\begin{aligned} \underline{r} &= x\hat{i} + y\hat{j} + z\hat{k} \\ \nabla \cdot \underline{r} &= 3 \end{aligned}$$

15.2 Laplacian of a Scalar Field

$$\begin{aligned} \nabla(\nabla \cdot \phi) = \nabla^2\phi &= \frac{\partial^2}{\partial x^2}\phi + \frac{\partial^2}{\partial y^2}\phi + \frac{\partial^2}{\partial z^2}\phi \\ \nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \end{aligned}$$

It is a scalar differential operator

15.3 Curl of a Vector Field

$$\nabla \times \underline{a} = \text{curl}\underline{a} = \hat{i}\Big(\frac{\partial}{\partial y}\underline{a}_z - \frac{\partial}{\partial z}\underline{a}_y\Big) + \hat{j}\Big(\frac{\partial}{\partial z}\underline{a}_x - \frac{\partial}{\partial x}\underline{a}_z\Big) + \hat{k}\Big(\frac{\partial}{\partial x}\underline{a}_y - \frac{\partial}{\partial y}\underline{a}_x\Big)$$

15.3.1 Properties

1. $$\nabla \times (\underline{a} + \underline{b}) = \nabla \times \underline{a} + \nabla \times \underline{b}$$

2. $$\nabla \times (\phi\underline{a}) = \nabla\phi \times \underline{a} + \phi(\nabla \times \ )$$

3. $$\nabla \times (\underline{a} \times \underline{b}) = (\underline{b} \cdot \nabla)\underline{a} - (\nabla \cdot \underline{a})\underline{b} - (\underline{a} \cdot \nabla)\underline{b} + (\nabla \cdot \underline{b})\underline{a}$$

4. $$\nabla \times \underline{r} = 0$$

Important point to remember:

➤ Since ∇ is an operator, ordering is important, i.e.

$$\underbrace{\nabla \cdot \underline{a}}_{\text{scalar}} \neq \underbrace{\underline{a} \cdot \nabla}_{\text{scalar differential operator}}$$
$$\underbrace{\nabla \times \underline{a}}_{\text{vector}} \neq \underbrace{\underline{a} \times \nabla}_{\text{vector differential operator}}$$

15.4 Line Integrals

Consider a smooth curve, C, ion space or a plane defined by an equation of $\underline{r}(u)$ with end points $\underline{r}(\alpha) = \underline{A}, \underline{r}(\beta) = \underline{B}$. A direction along C must be specified, e.g. $\underline{A} \to \underline{B}$. We have an oriented curve.

The line integral of a vector field $\underline{a}(\underline{r})$ along C is

$$\int_C \underline{a}(\underline{r}) \cdot d\underline{r} = \int_\alpha^\beta \underline{a}(\underline{r}(u)) \cdot \frac{d\underline{r}}{du} du$$

15.4.1 Example

Consider

$$\underline{a}(\underline{r}) = xe^y \hat{\imath} + z^2 \hat{\jmath} + xy\hat{k}$$

Evaluate

$$\int_C \underline{a}(\underline{r}) \cdot d\underline{r}$$

Consider different paths that can be taken in $\mathbb{R}$

1.

$$\begin{aligned}
\underline{r}(u) &= u\hat{\imath} + u\hat{\jmath} + u\hat{k},\ 0 \leq u \leq 1 \\
\underline{a}(\underline{r}) &= ue^u \hat{\imath} + u^2 \hat{\jmath} + u^2 \hat{k} \\
\frac{d\underline{r}}{du} &= \underline{r}'(u) = \hat{\imath} + \hat{\jmath} + \hat{\jmath} \\
\underline{a}(\underline{r}) \cdot \frac{d\underline{r}}{du} &= ue^u + 2u^2 \\
\int_{C_1} \underline{a}(\underline{r}) \cdot d\underline{r} &= \int_0^1 (ue^u + 2u^2)\, du = \frac{5}{3}
\end{aligned}$$

2.

$$\begin{aligned}
\underline{r}(u) &= u\hat{\imath} + u^2\hat{\jmath} + u^3\hat{k},\ 0 \leq u \leq 1 \\
\underline{a}(\underline{r}) &= ue^{u^2} \hat{\imath} + u^6 \hat{\jmath} + u^3 \hat{k} \\
\underline{r}'(u) &= \hat{\imath} + 2u\hat{\jmath} + 3u^2\hat{k} \\
\underline{a}(\underline{r}) \cdot \frac{d\underline{r}}{du} &= ue^{u^2} + 2u^7 + 3u^5 \\
\int_{C_2} \underline{a}(\underline{r}) \cdot d\underline{r} &= \int_0^1 (ue^{u^2} + 2u^7 + 3u^5)\, du = \frac{e}{2} + \frac{1}{4}
\end{aligned}$$

15.4.2 Properties

1. Integral, $\int_C \underline{a}(\underline{r}) \cdot d\underline{r}$, in general depends not only on the end points $\underline{a}$ and $\underline{B}$ but also on the path C itself.
2. If C is a curve with an orientation, $\underline{A} \to \underline{B}$, then -C is a curve with orientation $\underline{B} \to \underline{A}$ and:

$$\int_{-C} \underline{a}(\underline{r}) \cdot d\underline{r} = \int_\beta^\alpha \underline{a}(\underline{r}) \cdot d\underline{r} = -\int_C \underline{a}(\underline{r}) \cdot d\underline{r}$$

3. If $C = C_1 + C_2 + C_3 + \cdots$, then

$$\int_C \underline{a}(\underline{r}) \cdot d\underline{r} = \int_{C_1} \underline{a}(\underline{r}) \cdot d\underline{r} + \int_{C_2} \underline{a}(\underline{r}) \cdot d\underline{r} + \cdots$$

   Each curve, C_i, needs to have:
   - ➤ a regular parameterisation,
   - ➤ the end points of successive segments need to coincide,
   - ➤ and there is a compatible choice of parameterisations

   Each parameterisation can be chosen independently.

   This allows for useful constructions that rely on segments with opposite orientations that cancel.

4. Other kinds of line integral are possible:

$$\int_C \phi \, d\underline{r} \ ; \ \int_C \underline{a} \times d\underline{r} \ ; \ \int \phi \, dS \ ; \ \int \underline{a} \, dS$$

   If $\phi = 1$, then $\int dS$ is the length of the curve

Lecture 16

16.1 Line Integrals Continued

$$\int_C \underline{a} \cdot d\underline{r} = \int \underline{a}(\underline{r}) \cdot \frac{d\underline{r}}{du} du, \ \underline{r} = \underline{r}(u)$$

16.1.1 Example

Evaluate $\int_C \phi dS$, where $\phi(\underline{r}) = (x-y)^2$

$$\begin{aligned}
\underline{r}(u) &= a\cos u\hat{\imath} + a\sin u\hat{\jmath}, \ 0 \leq u \leq \pi \\
\phi(\underline{r}(u)) &= (a\cos u - a\sin u)^2 = a^2(\cos u - \sin u)^2 \\
dS &= \sqrt{\frac{d\underline{r}}{du} \cdot \frac{d\underline{r}}{du}}\, du = \sqrt{(-a\sin u\hat{\imath} + a\cos u\hat{\jmath})(-a\sin u\hat{\imath} + a\cos u\hat{\jmath})}\, du \\
&= a\, du \\
\implies \int_C \phi dS &= \int_0^\pi a^3(\cos u - \sin u)\, du = \pi a^3
\end{aligned}$$

16.1.2 Simple Connection

A region D is simply connected if every closed path within D can be shrunk to a point without leaving the region, i.e. the region D does not have any holes.

16.1.3 Theorem

Consider a vector function $\underline{a}(\underline{r})$ and a path C in a region D which is simply connected. Then the following statements are equivalent:

1. The integral $I = \int_C \underline{a}(\underline{r}) \cdot d\underline{r}$ is independent of C for some given end points and orientation.
2. $\underline{a}(\underline{r}) = \nabla\phi$ for some scale field $\phi(\underline{r})$
3. $\nabla \times \underline{a}(\underline{r}) = 0$

The vector field $\underline{a}(\underline{r})$ is said to be conservative and ϕ is said to be its potential.

In addition,

$$\begin{aligned}
I &= \int_C \underline{a}(\underline{r}) \cdot d\underline{r} = \int_C \nabla\phi \cdot d\underline{r} \\
&= \int_C \nabla\phi \cdot \frac{d\underline{r}}{du} du = \int_\alpha^\beta \frac{d}{du}(\phi(\underline{r}(u)))\, du \\
&= [\phi(\underline{r}(u))]_\alpha^\beta = \underbrace{\phi(\underline{B}) - \phi(\underline{A})}_{\underline{r}(\alpha)=\underline{A} \ \underline{r}(\beta)=\underline{B}}
\end{aligned}$$

That is, the integral depends only on the end points, not the path C joining them.

When C is closed, $I = \oint_C \underline{a}(\underline{r}) \cdot d\underline{r} = 0$
For a conservative force, $\underline{r} = -\nabla V$, where $V(\underline{r})$ is the potential, we have

$$W = \int_C \underline{F} \cdot d\underline{r} = V(\underline{A}) - V(\underline{B})$$

The work done is equal to the loss in potential energy.

16.1.4 Example

Consider a vector field, $\underline{a}(\underline{r}) = (xy^2 + z)\hat{\imath} + (x^2y + 2)\hat{\jmath} + xhatk$. Show that it is conservative and find its potential, ϕ.

$$
\begin{aligned}
\nabla \times \underline{a}(\underline{r}) &= \hat{\imath}\Big(\frac{\partial}{\partial y}\underline{a}_z - \frac{\partial}{\partial z}\underline{a}_y\Big) + \hat{\jmath}\Big(\frac{\partial}{\partial z}\underline{a}_x - \frac{\partial}{\partial x}\underline{a}_z\Big) + \hat{k}\Big(\frac{\partial}{\partial x}\underline{a}_y - \frac{\partial}{\partial y}\underline{a}_x\Big) \\
&= (0-0)\hat{\imath} + (1-1)\hat{\jmath} + (2xy - 2xy)\hat{k} = 0 \\
\underline{a}(\underline{r}) &= \nabla\phi = \frac{\partial}{\partial x}\phi\hat{\imath} + \frac{\partial}{\partial y}\phi\hat{\jmath} + \frac{\partial}{\partial z}\phi\hat{k} \\
\implies \frac{\partial \phi}{\partial x} &= xy^2 + z \to \phi = \frac{x^2y^2}{2} + zx + f(y,z) \\
\implies \frac{\partial \phi}{\partial y} &= x^2y + 2 \to f(y,z) = 2y + h(z) \\
\implies \phi &= \frac{x^2y^2}{2} + 2x + 2y + h(z) \\
\implies \frac{\partial \phi}{\partial z} &= x \to h(z) = c \\
\implies \phi &= \frac{x^2y^2}{2} + 2x + 2y + c
\end{aligned}
$$

Evaluate $I = \int_C \underline{a}(\underline{r}) \cdot d\underline{r}$ along C

$$
\begin{aligned}
\underline{r}(u) &= \epsilon u\hat{\imath} + \frac{\epsilon}{u}\hat{\jmath} + h\hat{k} \\
\underline{A} &= (\epsilon, \epsilon, h)\ ;\ \underline{B} = (2\epsilon, \frac{\epsilon}{2}, h)
\end{aligned}
$$

$$
I = \int_C \underline{a}(\underline{r}) \cdot d\underline{r} = \phi(\underline{B}) - \phi(\underline{A}) = \frac{(2\epsilon)^2}{2}(\frac{\epsilon}{2})^2 + 2\epsilon h + \epsilon + c - (\frac{\epsilon^4}{2} + \epsilon h + 2\epsilon + c) = \epsilon(h-1)
$$

Evaluate the integral explicitly

$$
\begin{aligned}
\underline{a}(\underline{r}) &= (\epsilon u\frac{\epsilon^2}{u^2} + h)\hat{\imath} + (\epsilon^2u^2\frac{\epsilon}{u} + 2)\hat{\jmath} + \epsilon u\hat{k} \\
\underline{r}(u) &= \epsilon\hat{\imath} - \frac{\epsilon}{u^2}\hat{\jmath} \\
\underline{a}(\underline{r}(u)) \cdot \underline{r}'(u) &= (\frac{\epsilon^4}{u} + h\epsilon) - \frac{\epsilon^4}{u} - \frac{2\epsilon}{u^2} \\
I &= \int_{u=1}^{u=2} (h\epsilon - \frac{2\epsilon}{u^2})\, du \\
&= \epsilon(h-1)
\end{aligned}
$$

Lecture 17

17.1 Surface Integrals

Let S be a smooth surface defined by $\underline{r}(u,v)$ with S being the appropriate region in the parametric space Δ. The surface integral of a vector function $\underline{a}(\underline{r})$ over S with orientation given by the unit normal vector $\hat{n}$ is

$$\int_S = \underline{a}(\underline{r}) \cdot d\underline{S} = \int \underline{a}(\underline{r}) \cdot \hat{n} dS = \int_D \underline{a}(\underline{r}(u,v)) \cdot \Big(\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\Big)\, du\, dv$$

17.1.1 Example

Evaluate

$$I = \int_S \underline{a} \cdot d\underline{r}$$

where $\underline{a} = x\hat{i}$ and S is the surface of the hemisphere $x^2+y^2+z^2=a^2$, with $z \geq 0$.
A suitable parameterisation for the surface makes use of spherical polar coordinates:

$$\underline{r}(\theta,\phi) = a\sin\theta\cos\phi\hat{i} + a\sin\theta\sin\phi\hat{j} + a\cos\theta\hat{k}$$

$0 \leq \theta \leq \frac{\pi}{2}$ and $0 \leq \phi \leq 2\pi$
Need to compute $d\underline{S}$:

$$\begin{aligned}
\frac{\partial \underline{r}}{\partial \theta} &= a\cos\theta\cos\phi\hat{i} + a\cos\theta\sin\phi\hat{j} - a\sin\theta\hat{k}\\
\frac{\partial \underline{r}}{\partial \phi} &= -a\sin\theta\sin\phi\hat{i} + a\cos\theta\cos\phi\hat{j}\\
d\underline{S} = \Big(\frac{\partial \underline{r}}{\partial \theta} \times \frac{\partial \underline{r}}{\partial \phi}\Big)\, d\theta\, d\phi &= a^2(\sin^2\theta\cos\phi\hat{i} + \sin^2\theta\sin\phi\hat{j} + \sin\theta\cos\theta\hat{k})\, d\theta\, d\phi\\
|d\underline{S}| = dS &= a^2\sin\theta\ d\theta\, d\phi\\
\implies d\underline{S} &= a^2\sin\theta\, \underbrace{(\sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k})}_{\hat{n}}\ d\theta\, d\phi\\
&= a^2\sin\theta(\frac{\underline{r}}{a})\ d\theta\, d\phi\\
\underline{a} &= a\sin\theta\cos\phi\hat{i}\\
\underline{a}\cdot d\underline{S} &= a^3\sin^3\theta\cos^2\phi\\
I &= \int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} d\theta\ a^3\sin^3\theta\cos^2\phi\\
&= a^3\int_0^{2\pi} d\phi\ \cos^2\phi \int_0^{\frac{\pi}{2}} d\theta\ \sin\theta(1-\cos^{\theta}) = \frac{2\pi a^3}{3}
\end{aligned}$$

17.1.2 Observations

1. $\int \underline{a}\cdot d\underline{S}$ depends on the orientation of S. Changing the orientation implies that the sign of the unit vector $\hat{n}$ changes, which is equivalent to changing the order of u and v in the definition of S, which is equivalent to change the sign of the integral.
   If the surface is closed, the convention is that $d\underline{S}$ is pointing outwards - the volume is enclosed.
2. Other kinds of integrals:

$$\int_S \phi\, dS\ ;\ \int_S \underline{a}\times d\underline{S}\ ;\ \int \phi\, d\underline{S}\ ;\ \int \underline{a}\, dS$$

For the first integral, setting $\phi = 1$ makes the integral into the area of the surface. Considering Example 17.1.1,

$$\begin{aligned}\text{Area} &= \int_S dS = \int a^2 \sin\theta \; d\theta\phi \\ &= \int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} d\theta \; a^2 \sin\theta = 2\pi a^2\end{aligned}$$

3. In order to parameterise the surface, it is often useful to use alternative coordinates systems, e.g.
   (a) Spherical polars:

$$\begin{aligned} x &= r\sin\theta\cos\phi & 0 \leq &\phi \leq 2\pi \\ y &= r\sin\theta\sin\phi & 0 \leq &\theta \leq \pi \\ z &= r\cos\theta & r &\geq 0 \end{aligned}$$

   (b) Cylindrical polars:

$$\begin{aligned} x &= \rho\cos\phi & \rho &\geq 0 \\ y &= \rho\sin\phi & 0 \leq &\phi \leq 2\pi \\ z &= z & -\infty < &z < \infty \end{aligned}$$

17.2 Volume Integrals

Let V be a volume described by $\underline{r}(u, v, w)$ with V being the appropriate region in the parameter space D. The volume integral of a function ϕ is:

$$\int_V \phi \, dV = \int_V \phi(\underline{r}(u, v, w)) \left| \frac{\partial \underline{r}}{\partial u} \cdot \left(\frac{\partial \underline{r}}{\partial v} \times \frac{\underline{r}}{\partial w} \right) \right| du \, dv \, dw$$

17.2.1 Example

Consider the density of electric charge, $\rho(r) = \frac{\rho_0 z}{a}$ in a hemisphere of radius a with $z \leq 0$ and ρ_0 constant. What is the total charge, H?

$$\begin{aligned} Q &= \int_H \rho(\underline{r}) \, dV \\ \frac{\partial \underline{r}}{\partial r} &= \sin\theta\cos\phi\hat{i} + \sin\theta\cos\phi\hat{j} + \cos\theta\hat{k} \\ \frac{\partial \underline{r}}{\partial \theta} &= r\cos\theta\cos\phi\hat{i} + r\cos\theta\cos\phi\hat{j} - r\sin\theta\hat{k} \\ \frac{\partial \underline{r}}{\partial \phi} &= r\cos\theta\cos\phi\hat{i} + r\cos\theta\cos\phi\hat{j} - r\sin\theta\hat{k} \\ dV &= \left| \frac{\partial \underline{r}}{\partial r} \left(\frac{\partial \underline{r}}{\partial \theta} \times \frac{\partial \underline{r}}{\partial \phi} \right) \right| dr \, d\theta \, d\phi = r^2 \sin\theta \; dr \, d\theta \, d\phi \\ Q &= \int_0^a dr \int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} d\theta \; \frac{\rho_0}{a} (r\cos\theta) r^2 \sin\theta \\ &= \frac{\rho_0}{a} \int_0^a r^3 dr \int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} d\theta \; \frac{\sin 2\theta}{2} \\ &= \frac{\pi a^3 \rho_0}{4} \end{aligned}$$

Lecture 18

18.1 Divergence Theorem

This is also called Gauss' Theorem.

$$\int_V \nabla \cdot \underline{a}\, dV = \int_S \underline{a} \cdot d\underline{S}$$

18.1.1 Example

Take V to be the hemisphere $x^2 + y^2 + z^2 \geq a^2, z \geq 0, \underline{a} = (z + a)\hat{k}$, then $\partial V = S_{1,\,hemi} + S_{2,disc}$

On the left handside of the divergence theorem:

$$\nabla \cdot \underline{a} = 1 \implies \int_V \nabla \underline{a}\, dV = \int_V dV = \frac{2}{3}\pi a^3$$

On the right handside:

- For S_1, use spherical polar coordinates, that is:

$$\begin{aligned}
\underline{r}_1(\theta, \phi) &= a \sin\theta \cos\phi \hat{i} + a \sin\theta \sin\phi \hat{j} + a \cos\theta \hat{k} \\
d\underline{S}_1 &= a \sin\theta \underline{r}_1 \; d\theta\, d\phi \\
\underline{a} = a(\cos\theta + 1)\hat{k} &\implies \underline{a} \cdot d\underline{S}_1 = a^2(\cos\theta + 1) \sin\theta \cos\theta \; d\theta\, d\phi \\
\int_{S_1} a^3 \sin\theta \cos\theta (\cos\theta + 1) \; d\theta\, d\phi &= a^3 2\pi \int_0^{\frac{\pi}{2}} (\cos^2\theta \sin\theta - \sin\theta \cos\theta) \; d\theta = \frac{5}{3}\pi a^3
\end{aligned}$$

- For S_2, use spherical polar coordinates as well. That is:

$$\begin{aligned}
\underline{r}_2(r, \phi) &= r \cos\phi \hat{i} + r \sin\phi \hat{j} \\
d\underline{S}_2 &= \left(\frac{\partial \underline{r}_2}{\partial r} \times \frac{\partial \underline{r}_2}{\partial \phi}\right) dr\, d\phi = r \; dr\, d\phi\, \hat{k} \\
&= -r \; dr\, d\phi\, \hat{k} \\
\underline{a} \cdot d\underline{S}_2 &= -ar \; dr\, d\phi \\
\implies -\int_{S_2} ar \; dr\, d\phi &= -2\pi a \int_0^a r\, dr = -\pi a^3 \\
\implies \int_{S_1} + \int_{S_2} &= \frac{5}{3}\pi a^3 - \pi a^3 = \frac{2}{3}\pi a^3
\end{aligned}$$

18.2 Stokes' Theorem

$$\int_S (\nabla \times \underline{a}) \cdot d\underline{S} = \int_C \underline{a} \cdot d\underline{r}$$

- $\underline{a}$ is a vector function
- S is a bounded smooth surface with a boundary $\partial S = C$
- C and S have compatible orientations

Imagine you are walking on the surface (side with the normal $d\underline{S}$ pointing out). If you walk near the edge of the surface in the direction corresponding to the orientation C, then the surface must be to your left.

18.2.1 Example

Take $\underline{a} = xz\hat{j}$ and S to be the section of the cone $x^2 + y^2 = z^2, a \leq z \leq b, b > a > 0$, then $\partial S = C_b + C_a$

- On the left handside, using cylindrical polar coordinates:

$$\begin{aligned}
\underline{r}(\rho_1, \phi) &= \rho \cos\phi\hat{i} + \rho \sin\phi\hat{j} + \rho\hat{k},\ z = \rho \\
d\underline{S} &= \left(\frac{\partial \underline{r}}{\partial \rho} \times \frac{\partial \underline{r}}{\partial \phi}\right) d\rho\, d\phi = \rho(-\cos\phi\hat{i} - \sin\phi\hat{j} + \hat{k})\ d\rho\, d\phi \\
\nabla \times \underline{a} &= -x\hat{i} + z\hat{k} = \rho(-\cos\phi\hat{i} + \hat{k}) \\
\implies \int_S (\nabla \times \underline{a}) \cdot d\underline{S} &= \int_0^{2\pi} d\phi \int_a^b (\rho^2 \cos^2\phi + \rho^2)\ d\rho = \pi(b^3 - a^3)
\end{aligned}$$

- On the right handside, we have two circles, C_a and C_b, so use polar coordinates. For the orientation, look down the z axis.

$$\begin{aligned}
\underline{r}_b(\phi) &= (b\cos\phi\hat{i} + b\sin\phi\hat{j} + b\hat{k}) \\
\underline{r}_b'(\phi) &= -b\sin\phi\hat{i} + b\cos\phi\hat{j} \\
\underline{a} &= = b^2\cos\phi\hat{j} \\
\implies \int_{C_b} \underline{a} \cdot d\underline{r}_b &= b^3 \int_0^{2\pi} \cos^2\phi\ d\phi = b^3\pi
\end{aligned}$$

Notice that C_a has opposite orientation, then remember that $\int_{-C} = -\int_C$

$$\begin{aligned}
\int_{C_a} \underline{a} \cdot d\underline{r}_a &= -a^3 \int_0^{2\pi} \cos^2\phi\ d\phi = -a^3\pi \\
\implies \int_{C_b} + \int_{C_a} &= b^3\pi - a^3\pi = \pi(b^3 - a^3)
\end{aligned}$$

Lecture 19

19.1 Orthogonal curvilinear coordinates

Consider 3 coordinates u, v, w in $\mathbb{R}^3$, then we have the following position vector:

$$\underline{r}(u,v,w) = x(u,v,w)\hat{\imath} + y(u,v,w)\hat{\jmath} + z(u,v,w)\hat{k}$$

x, y, and z are differentiable and continuous.
The line element is:

$$d\underline{r} = \frac{\partial \underline{r}}{du}du + \frac{\partial \underline{r}}{\partial v}dv + \frac{\partial \underline{r}}{\partial w}dw$$

For a good parameterisation, the partial vectors are required to be linearly independent:

$$\frac{\partial \underline{r}}{\partial u} \cdot \left(\frac{\partial \underline{r}}{\partial v} \times \frac{\partial \underline{r}}{\partial w}\right) \neq 0$$

19.1.1 Definition

The coordinates u, v, and w are said to be orthogonal curvilinear coordinates if the partial vectors are orthogonal.

19.1.2 Properties

1. Set
$$\frac{\partial vr}{\partial u} = h_u \hat{e}_u \ ; \ \frac{\partial vr}{\partial v} = h_v \hat{e}_v \ ; \ \frac{\partial vr}{\partial w} = h_w \hat{e}_w,$$
then h_u, h_v, h_w are called scale factors and the unit vectors $\hat{e}_n$ form the orthonormal basis of the vector space $\mathbb{R}^3$
2. The line element is
$$d\underline{r} = h_u \hat{e}_u + h_v \hat{e}_v + h_w \hat{e}_w$$
The scale factors determine the changes in length along each orthogonal direction resulting from changes in u, v, and w.
3. The volume element is
$$dV = \left|\frac{\partial \underline{r}}{\partial u}\left(\frac{\partial \underline{r}}{\partial v} \times \frac{\partial \underline{r}}{\partial w}\right)\right| du\,dv\,dw = h_u h_v h_w \ du\,dv\,dw$$
4. Consider a surface, for instance, w constant. Then the surface is parameterised by u and v. The vector area element is
$$d\underline{S} = \left(\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\right) du\,dv = h_u h_v \hat{e}_w \ du\,dv$$

19.1.3 Examples

➤ Cartesian coordinates:

$$\begin{aligned} \underline{r}(x,y,z) &= x\hat{\imath} + y\hat{\jmath} + z\hat{k} \\ \frac{\partial \underline{r}}{\partial x} &= \hat{\imath} \ ; \ \frac{\partial \underline{r}}{\partial y} = \hat{\jmath} \ ; \ \frac{\partial \underline{r}}{\partial z} = \hat{k} \\ h_x = h_y = h_z &= 1 \ ; \ \hat{e}_x = \hat{\imath} \ ; \ \hat{e}_y = \hat{\jmath} \ ; \ \hat{e}_z = \hat{k} \\ dV &= dx\,dy\,dz \end{aligned}$$

➤ Cylindrical polar coordinates:

$$\begin{aligned}
\underline{r}(\rho,\phi,z) &= \rho\cos\phi\hat{\imath}+\rho\sin\phi\hat{\jmath}+z\hat{k} \\
\frac{\partial\underline{r}}{\partial\rho} &= \cos\phi\hat{\imath}+\sin\phi\hat{\jmath}\ ;\ \frac{\partial\underline{r}}{\partial\phi}=-\rho\sin\phi\hat{\imath}+\rho\cos\phi\hat{\jmath}\ ;\ \frac{\partial\underline{r}}{\partial z}=\hat{k} \\
h_\rho &= 1\ ;\ h_\phi=\rho\ ;\ h_z=1\ ;\ dV=\rho\,d\rho\,d\phi\,dz \\
\hat{e}_\rho &= \cos\phi\hat{\imath}+\sin\phi\hat{\jmath}\ ;\ \hat{e}_\phi=-\sin\phi\hat{\imath}+\cos\phi\hat{\jmath}\ ;\ \hat{e}_z=\hat{k}
\end{aligned}$$

19.2 Grad, div, and curl

Consider a scalar function $f(u,v,w)$, then:

$$df=\frac{\partial f}{\partial u}du+\frac{\partial f}{\partial v}dv+\frac{\partial f}{\partial w}dw=\nabla f\cdot d\underline{r} \tag{19.1}$$

We show this formula in Cartesian coordinates:

$$\nabla f\cdot d\underline{r}=\Big(\hat{\imath}\frac{\partial f}{\partial x}+\hat{\jmath}\frac{\partial f}{\partial y}+\hat{k}\frac{\partial f}{\partial z}\Big)\cdot\Big(\hat{\imath}dx+\hat{\jmath}dy+\hat{k}dz\Big)$$

In the case of general orthogonal curvilinear coordinates,

$$\nabla f\cdot d\underline{r}=(?)\cdot(h_u\hat{e}_u du+h_v\hat{e}_v dv+h_w\hat{e}_w dw)$$

The gradient of a set of orthogonal curvilinear coordinates is:

$$\nabla f=\Big(\frac{1}{h_u}\hat{e}_u\frac{\partial f}{\partial u}+\frac{1}{h_v}\hat{e}_v\frac{\partial f}{\partial v}+\frac{1}{h_w}\hat{e}_w\frac{\partial f}{\partial w}\Big)$$

19.2.1 Example

Consider $f=\rho\cos\phi$ in cylindrical polars:

$$\nabla f=\hat{e}_\rho\frac{\partial f}{\partial\rho}+\frac{\hat{e}_\phi}{\rho}\frac{\partial f}{\partial\phi}+\hat{e}_z\frac{\partial f}{\partial z}=\hat{e}_\rho\cos\phi-\hat{e}_\phi\sin\phi$$

The differential operator del, ∇, in orthogonal curvilinear coordinates is:

$$\nabla=\Big(\frac{1}{h_u}\hat{e}_u\frac{\partial}{\partial u}+\frac{1}{h_v}\hat{e}_v\frac{\partial}{\partial v}+\frac{1}{h_w}\hat{e}_w\frac{\partial}{\partial w}\Big)$$

19.2.2 Div

Consider a vector field $\underline{a}=a_u\hat{e}_u+a_v\hat{e}_v+a_w\hat{e}_w$, then:

$$\nabla\cdot\underline{a}=\frac{1}{h_uh_vh_w}\left[\frac{\partial}{\partial u}(h_vh_wa_u)+\frac{\partial}{\partial v}(h_uh_wa_v)+\frac{\partial}{\partial w}(h_uh_va_w)\right]$$

Consider $\nabla\cdot(a_u\hat{e}_u)$:

$$\hat{e}_u=h_u\nabla u \qquad \hat{e}_v=h_v\nabla v\hat{e}_w \qquad =h_w\nabla w$$

$$\begin{aligned}
\hat{e}_u&=\hat{e}_v\times\hat{e}_w=h_vh_w(\nabla v\times\nabla w)\\
\nabla(a_uh_vh_w(\nabla v\times\nabla w))&=\underbrace{\nabla(a_vh_vh_w)(\nabla v\times\nabla w)}_{A}+\underbrace{a_uh_vh_w(\nabla v\times\nabla w)}_{B=0}\\
A:\implies\ \nabla(a_vh_vh_w)\Big(\frac{\hat{e}_v}{h_v}\times\frac{\hat{e}_w}{h_w}\Big)&=\nabla(a_vh_vh_w)\cdot\frac{\hat{e}_u}{h_vh_w}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{h_u h_v h_w} \frac{\partial}{\partial u}(a_u h_v h_w) \\
B: \implies \nabla(\nabla v \times \nabla w) &= \nabla w \cancel{(\nabla \times \nabla v)} - \nabla v \cancel{(\nabla \times \nabla w)}
\end{aligned}
$$

The curl of a gradient function, $\operatorname{curl}\operatorname{grad}\phi$, is zero.

$$
\nabla \times \underline{a} = \frac{1}{h_u h_v h_w} \begin{vmatrix} h_u \hat{e}_u & h_v \hat{e}_v & h_w \hat{e}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u a_u & h_v a_v & h_w a_w \end{vmatrix}
$$

Part II

Lecture 1

1.1 Introduction

This part of the course will mainly deal with differential equations.

1.1.1 Classes of Differential Equation

- Ordinary Differential Equations, e.g.

$$\frac{dy}{dx} + f(x, y) = 0 \tag{1.1}$$

- Partial Differential Equations, e.g.

$$\frac{\partial g(x, y)}{\partial x} + \frac{\partial g(x, y)}{\partial y} = 0 \tag{1.2}$$

1.1.2 Order of ODEs

The order of an ODE is the value of the highest derivative present.

$$\frac{dy}{dx} + y^2 + \sqrt{xy} = 0 \tag{1st Order}$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + f(x, y) = 0 \tag{2nd Order}$$

1.1.3 Degree of ODEs

The degree of an ODE is the power of the highest order term after all the derivatives have been rationalised.

$$\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^3 + f(x, y) = 0 \tag{2nd Degree}$$

$$\left(\frac{d^2y}{dx}\right) + \left(\frac{dy}{dx}\right)^{3/2} + y = 0 \tag{4th Degree}$$

$$\frac{d^2y}{dx^2} + \sqrt{xy} = 0 \tag{2nd Degree}$$

1.1.4 Solution to ODEs

- The most general function that solves the equation.
- "General solution" contains integration constants that are not fixed by the equation.
- These constants can be fixed by boundary conditions which leads to a "particular solution".
- An nth order ODE has n integration constants.

1.2 1st Order, 1st Degree ODEs

$$\frac{dy}{dx} = f(x, y) \to A(x, y)\, dx + B(x, y)\, dy = 0 \tag{1.3}$$

1.2.1 Separable Equations

$$f(x,y) = f(x)g(y) \tag{1.4}$$
$$\frac{dy}{dx} = f(x)g(y) \tag{1.5}$$
$$\int \frac{dy}{g(y)} = \int f(x)\,dx \tag{1.6}$$

Example:

$$x^2 \frac{dy}{dx} = 1 + y \tag{1.7}$$
$$\frac{dy}{dx} = \frac{1+y}{x^2} = (1+y) \cdot \frac{1}{x^2} \tag{1.8}$$
$$\int \frac{dy}{1+y} = \int \frac{dx}{x^2} \tag{1.9}$$
$$\ln(1+y) = -\frac{1}{x} + c \tag{1.10}$$
$$1 + y = Ae^{-\frac{1}{x}} \tag{1.11}$$
$$y = Ae^{-\frac{1}{x}} - 1 \tag{1.12}$$

1.2.2 Exact Equations

$$A(x,y)\,dx + B(x,y)\,dy = 0 \tag{1.13}$$
$$\partial_y A(x,y) = \partial_x B(x,y) \tag{1.14}$$
$$U(x,y) \to dU = \partial_x U\,dx + \partial_y U\,dy \tag{1.15}$$
$$dU = 0 \to U = c \tag{1.16}$$
$$A(x,y) = \partial_x U \to U(x,y) = \int A(x,y)\,dx + F(y) \tag{1.17}$$
$$B(x,y) = \partial_y U \tag{1.18}$$
$$\partial_y U = \partial_y \left[\int A(x,y)\,dx \right] + F'(y) = B(x,y) \tag{1.19}$$

Example:

$$\frac{x}{2}\frac{dy}{dx} + x^2 + \frac{y}{2} = 0 \tag{1.20}$$
$$A = x^2 + \frac{y}{2}\ ;\ B = \frac{x}{2} \tag{1.21}$$
$$\partial_y A = \frac{1}{2}\ ;\ \partial_x B = \frac{1}{2} \implies \text{Exact} \tag{1.22}$$
$$x^2 + \frac{y}{2} = \frac{\partial U}{\partial x} \to U(x,y) = \frac{x^3}{3} + \frac{xy}{2} + F(y) \tag{1.23}$$
$$\partial_y U(x,y) = \frac{x}{2} + F'(y) = \frac{x}{2} \implies F'(y) = 0 \to F(y) = c \tag{1.24}$$
$$U(x,y) = \frac{x^3}{3} + \frac{xy}{2} = d \tag{1.25}$$

$$y = -\frac{2}{3}x^2 + \frac{2d}{x} \tag{1.26}$$

1.2.3 The Integrating Factor

$$A(x,y)\,dx + B(x,y)\,dy = 0 \tag{1.27}$$
$$\partial_y A \neq \partial_x B \to \mu(x,y)A(x,y)\,dx + \mu(x,y)B(x,y)\,dy = 0 \tag{1.28}$$
$$\partial_y[\mu A] = \partial_x[\mu B] \tag{1.29}$$

μ is called the integrating factor

- If $\mu = \mu(x)$:

$$\partial_y[\mu A] = \mu\partial_y A = \mu' B + \mu\partial_x B \tag{1.30}$$
$$\frac{d\mu}{\mu} = \frac{1}{B}(\partial_y A - \partial_x B) = f(x) \tag{1.31}$$

- If $\mu = \mu(y)$:

$$\frac{d\mu}{\mu} = \frac{1}{A}(\partial_x B - \partial_y A) = g(y) \tag{1.32}$$

- Special case: linear equations

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{1.33}$$
$$A = P(x)y - Q\ ;\ B = 1 \tag{1.34}$$
$$\frac{1}{B}(\partial_y A - \partial_x B) = \frac{1}{1}(P - 0) = P(x) \tag{1.35}$$
$$\frac{d\mu}{\mu} = P(x) \to \mu = e^{\int P(x)\,dx} \tag{1.36}$$

Example:

$$\frac{dy}{dx} + xy + x^2 = 0 \tag{1.37}$$
$$dy + \left(\frac{y}{x} + x^2\right)\,dx = 0 \tag{1.38}$$
$$\mu = e^{\int \frac{dx}{x}} = x \tag{1.39}$$
$$x\,dy + (y + x^3)\,dx = 0 \tag{1.40}$$
$$\partial_x B = 1\ ;\ \partial_y A = 1 \tag{1.41}$$

Lecture 2

2.1 Simplifying Equations by Change of Variables

2.1.1 Homogeneous Equations

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right), \rightarrow y = v \cdot x \tag{2.1}$$

$$y' = v'x + v = f(v) \implies \frac{dx}{x} = \frac{dv}{f(v) - v} \tag{2.2}$$

$$f(x,y) = \frac{A(x,y)}{B(x,y)} \rightarrow \begin{cases} A(\lambda x, \lambda y) & = \lambda^n A(x,y) \\ B(\lambda x, \lambda y) & = \lambda^n B(x,y) \end{cases} \tag{2.3}$$

Example:

$$xy\frac{dy}{dx} + 3x^2 - y^2 = 0 \tag{2.4}$$

$$\frac{dy}{dx} = \frac{y^2 - 3x^2}{xy} \rightarrow y = vx \tag{2.5}$$

$$xv' + v = \frac{v^2x^2 - 3x^2}{vx^2} + v = \frac{v^2 - 3}{v} + v = -\frac{3}{v} \tag{2.6}$$

$$\frac{dv}{dx}x = -\frac{3}{v} \rightarrow v\,dv = -3\frac{dx}{x} \tag{2.7}$$

$$\frac{v^2}{2} = -3\ln x + c \rightarrow v^2 = d - 6\ln x \tag{2.8}$$

$$v = \pm\sqrt{d - 6\ln x} \tag{2.9}$$

2.1.2 Isobaric Equations

- Give $x\,dx$ weight 1
- Give $y\,dy$ weight m
- If everywhere is the same power, again separable: $y = vx^m$

$$(\underbrace{1}_{0} + \underbrace{xy}_{1\,m})\underbrace{dy}_{m} + \underbrace{y^2}_{2m}\underbrace{dx}_{1} = 0 \tag{2.10}$$

$$m = 2m + 1 \rightarrow m = -1 \rightarrow y = \frac{v}{x} \tag{2.11}$$

$$\frac{dy}{dx} = \frac{v'}{x} - \frac{1}{x^2}v \tag{2.12}$$

$$(1+v)\left(\frac{v'}{x} - \frac{1}{x^2}v\right) + \frac{v^2}{x^2} = 0 \tag{2.13}$$

$$\frac{v'}{x}(1+v) = \frac{v}{x^2} \rightarrow dv\left(\frac{1}{v} + 1\right) = \frac{dx}{x} \tag{2.14}$$

$$\ln v + v = \ln x + c \rightarrow \ln y + \cancel{\ln x} + xy = \cancel{\ln x} + c \tag{2.15}$$

$$\ln y + xy = c \tag{2.16}$$

2.1.3 Bernoulli Equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \to v = y^{1-n} \tag{2.17}$$
$$v' = (1-n)y^{-n} = (1-n)y^{-n}\left[Q(x)y^n - P(x)y\right] \tag{2.18}$$
$$= (1-n)Q(x) - P(x)(1-n) \times y^{1-n} \therefore \text{Linear} \tag{2.19}$$

2.2 Linear Higher Order ODEs

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x) \tag{2.20}$$
$$f(x)\begin{cases} = 0 & \text{homogeneous} \\ \neq 0 & \text{inhomogeneous} \end{cases} \tag{2.21}$$

- General solution will have n integration constants
- There are n independent solutions
- To solve:
  1. Set $f(x) = 0$ to get the complementary equation
  2. Solve the complementary equation for n independent solutions
  3. Most general solution, $\{y_i\}$:

     $$y_c = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n \tag{2.22}$$

     You will have n linearly independent solutions
  4. $\{y_i\}$ linearly independent?

     $$\sum_{i=1}^{n} c_i y_i = 0 \iff c_i = 0 \ \forall i \in N \tag{2.23}$$

     How do you check? The Wronskian Technique:

     $$\sum c_i y_i = 0 \ ; \ \sum c_i y_i' = 0 \ ; \ \sum c_i y_i'' = 0 \tag{2.24}$$

     Can be written in matrix form to solve:

     $$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \\ v_1^{(n-1)} & \cdots & \cdots & y_n^{n-1} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \underline{0} \tag{2.25}$$
     $$\underline{W} \cdot \underline{C} = \underline{0} \tag{2.26}$$

     If $\underline{W}$ is invertible:

     $$\underline{c} = (W^{-1}) \cdot \underline{0} = 0 \tag{2.27}$$
     $$\det W = |W| \neq 0 \tag{2.28}$$

     This leads to linearly independent
  5. Solve full equation
  6. Find any solution of the full equation, the particular solution
  7. The most general solution is

     $$y + y_p + y_c \tag{2.29}$$

     if y_p and y_c are linearly independent

- $$\sum_{i=1}^{n} a_i y^(i) = 0,\ a_i \in \mathbb{R} \tag{2.30}$$

Try $y = Ae^{\lambda x}$:

$$y' = \lambda y \to y'' \lambda^2 y \cdots \tag{2.31}$$

$$\sum_{i=1}^{n} a_i \lambda^i y = 0 \to \sum_{i=1}^{n} a_i \lambda^i = 0 \tag{2.32}$$

This is the auxiliary equation.

- $\{\lambda_i\}_{i=1\cdots n}$ roots
- If all roots $\neq$: There are n solutions using equation above
- If some roots repeat: $\{\lambda_1, \lambda_1, \cdots\}$
  This is two-fold degenerate
- $e^{\lambda x},\ xe^{\lambda_n x} \to$ k-fold degree

$$\{e^{\lambda ix}, xe^{\lambda ix}, x^2 e^{\lambda ix}, \cdots, x^{k-1} e^{\lambda ix}\} \tag{2.33}$$

Lecture 3

3.1 Linear Higher Order ODEs with Constant Coefficients

$$\sum_{i=0}^{\mathbb{R}} a_i y^{(i)} = f(x) \tag{3.1}$$

1. Look at y_c:
$$\sum_{i=0}^{\mathbb{N}} a_i y^{(i)} = 0 \tag{3.2}$$

2. Try $y_c = Ae^{\lambda x}$ (auxiliary equation for lambda):
$$\sum_{i=0}^{\mathbb{N}} a_i \lambda^i = 0 \tag{3.3}$$

   (a) All roots are different, $\{\lambda_i\}_{i\in\mathbb{N}}, \lambda_i \neq \lambda_j$:
$$y_c = \sum c_i e^{\lambda_i x}, \ \{e^{\lambda_i x}\} \text{ are independent} \tag{3.4}$$

   (b) Some root is repeated:
$$\{\lambda_1, \lambda_2, \underbrace{\lambda_3, \lambda_3, \cdots, \lambda_3}_{\times K}, \lambda_4, \cdots\} \tag{3.5}$$
$$\{e^{\lambda_3 x}, xe^{\lambda_3 x}, x^2 e^{\lambda_3 x}, \cdots\} \tag{3.6}$$

Example:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0 \tag{3.7}$$

Try $y = Ae^{\lambda x}$:

$$\lambda^2 Ae^{\lambda x} - 5\lambda e^{\lambda x} + 6Ae^{\lambda x} = 0 \tag{3.8}$$
$$\lambda^2 - 5\lambda + 6 = 0 \tag{3.9}$$
$$\lambda = \frac{5 \pm \sqrt{25-24}}{2} = 3, 2 \tag{3.10}$$
$$(\lambda - 3)(\lambda - 2) = 0 \tag{3.11}$$
$$y_1 = e^{3x}, \ y_2 = e^{2x} \tag{3.12}$$

Now check for independence (Wronskian):

$$W = \left|\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\right| = \begin{vmatrix} e^{3x} & e^{2x} \\ 3e^{3x} & 2e^{2x} \end{vmatrix} \tag{3.13}$$
$$= 2e^{2x}e^{3x} - 3e^{3x}e^{2x} = -e^{5x} \neq 0 \tag{3.14}$$
$$y_c = c_1 e^{3x} + c_2 e^{2x} \tag{3.15}$$

Example:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0 \tag{3.16}$$

$$\lambda^2 - 2\lambda + 2 = 0 \tag{3.17}$$

$$\lambda = \frac{1 \pm \sqrt{1-2}}{1} = 1 \pm i \tag{3.18}$$

λ and λ^* are solutions

$$\begin{aligned} y_c &= c_1 e^{(1+i)x} + c_2 e^{(1-i)x} && (3.19)\\ e^{(1\pm i)x} &= e^x[\cos(x) \pm i\sin(x)] && (3.20)\\ y_c &= e^x[A\cos(x) + B\sin(x)] && (3.21)\\ A &= c_1 + c_2,\ B = i(c_1 - c_2) && (3.22) \end{aligned}$$

$y_c \in \mathbb{R}$ if boundary conditions are real
Use Wronskian to check independence again:

$$\begin{aligned} y_1 &= e^x\cos(x),\ y_2 = e^x\sin(x) && (3.23)\\ y_1' &= e^x\cos(x) - e^2\sin(x),\ y_2' = e^x\sin(x) + e^x\cos(x) && (3.24)\\ W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x\cos(x) & e^x\sin(x) \\ e^x\cos(x) - e^2\sin(x) & e^x\sin(x) + e^x\cos(x) \end{vmatrix} && (3.25)\\ &= e^{2x} \neq 0 && (3.26) \end{aligned}$$

N.B. - Three equivalent ways of writing y_c:

$$\begin{aligned} y_c &= e^x(A\cos(x) + B\sin(x)) && (3.27)\\ &= e^x\alpha\cos(x+\beta) = e^x\alpha[\cos(x)\cos(\beta) + \sin(x)\sin(\beta)],\ (A = \alpha\cos(\beta), B = \alpha\sin(\beta)) && (3.28)\\ &= e^x\alpha\sin(x+\beta) = e^x\alpha[\sin(x)\cos(\beta) + \cos(x)\sin(\beta)],\ (A = \alpha\sin(\beta), B = \alpha\cos(\beta)) && (3.29) \end{aligned}$$

Example:

$$\begin{aligned} &\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0 && (3.30)\\ &\lambda^2 - 4\lambda + 4 = 0 && (3.31)\\ &(\lambda - 2)^2 = 0 \implies \lambda = 2, \text{(2-fold soln)} && (3.32) \end{aligned}$$

$\{e^{2x}, xe^{2x}\}$ are solutions

$$\begin{aligned} y_1 &= e^{2x},\ y_2 = xe^{2x} && (3.33)\\ y_1' &= 2e^{2x}.\ y_2' = e^{2x}(1+2x) && (3.34)\\ W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x}(1+2x) \end{vmatrix} && (3.35)\\ &= e^{4x} = 0 && (3.36) \end{aligned}$$

3.2 Finding Particular Solutions

➤ Try simple functions similar to $f(x)$

Example:

$$\begin{aligned} &\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^x && (3.37)\\ &y = y_c + y_p = c_1e^{2x} + c_2e^{3x} + y_p && (3.38) \end{aligned}$$

Try $y_p = Ae^x$:

$$Ae^x - 5Ae^x + 6Ae^x = e^x \tag{3.39}$$

$$A(1-5+6) = 1 \implies A = \frac{1}{2} \tag{3.40}$$

$$y_p = \frac{1}{2}e^x \tag{3.41}$$

y_c, y_p must be linearly independent

- If $f(x)$ already features in y_c, try $xf(x)$

Lecture 4

Example:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y + 3x^2 - 2x + 12 = 0 \tag{4.1}$$
$$y_c \to \frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0 \tag{4.2}$$
$$\lambda^2 - 2\lambda - 3 = 0 \tag{4.3}$$
$$\lambda = 1 \pm \sqrt{1+3} = 3, -1 \tag{4.4}$$
$$y_c = c_1 e{-x} + c_2 e^{3x} \tag{4.5}$$

What about the particular solution?
Try:

$$y_p = \alpha x^2 \beta x + \gamma \tag{4.6}$$
$$y_p' = 2\alpha x + \beta \tag{4.7}$$
$$y_p'' = 2\alpha \tag{4.8}$$
$$\implies 2\alpha - 2[2\alpha x + \beta] - 3[\alpha x^2 + \beta x + \gamma] + 3x^2 - 2x + 12 = 0 \tag{4.9}$$

This solution must be valid $\forall\ x$

$$x^2[3 - 3\alpha] + x[-4\alpha - 3\beta - 2] + [2\alpha - 2\beta - 3\gamma + 12] = 0 \tag{4.10}$$
$$\alpha = 1 \implies \beta = -2 \implies \gamma = 6 \tag{4.11}$$
$$y_p = x^2 - 2x + 6 \tag{4.12}$$
$$y = c_1 e^{-x} + c_2 e^{3x} + x^2 - 2x + 6 \tag{4.13}$$

Now fix c_i by requiring $y(0) = y'(0) = 0$:

$$y(0) = c_1 + c_2 + 6 = 0 \tag{4.14}$$
$$y'(0) = -c_1 + 3c_2 - 2 = 0 \tag{4.15}$$
$$y(0) + y'(0) = 4c_2 + 4 = 0 \tag{4.16}$$
$$\implies c_2 = -1 \to c_1 = -5 \tag{4.17}$$
$$y = -5e^{-x} - e^{3x} + x^2 - 2x + 6 \tag{4.18}$$

4.1 Laplace Transforms

For a function, $f(x)$:

$$\mathcal{L}[f(x)] = \bar{f}(s) = \int_0^\infty f(x)e^{-sx}dx \tag{4.19}$$

The Laplace Transform is invertible so can go back and forth across the map.

Example:

$$\mathcal{L}[e^{ax}] = \frac{1}{s-a}\ ;\ \bar{f}(s) = \frac{1}{s-3} \to f(x) = e^{3x} \tag{4.20}$$
$$\mathcal{L}[x^n] = \frac{n!}{n^{1+x}} \tag{4.21}$$

$$\mathcal{L}[\cos(\alpha x)] = \frac{s}{\alpha^2 + s^2} \tag{4.22}$$

Example:

$$\bar{f}(s) = \frac{1}{(s+1)(s+3)} \rightarrow f(x)? \tag{4.23}$$

Use partial fractions to find $f(x)$:

$$\bar{f}(s) = \frac{A}{s+1} + \frac{B}{s+3} \tag{4.24}$$

$$\frac{(s+1)}{(s+1)(s+3)} = \left(\frac{A}{s+1} + \frac{B}{s+3}\right)(s+1) \tag{4.25}$$

$$\frac{1}{s+3} = A + B\frac{s+1}{s+3} \underbrace{\Longrightarrow}_{s=-1} \frac{1}{-1+3} = A \rightarrow A = \frac{1}{2} \tag{4.26}$$

$$\frac{s+3}{(s+1)(s+3)} = A\frac{s+3}{s+1} + B \underbrace{\Longrightarrow}_{s=-3} \frac{1}{-3+1} = B \rightarrow B = -\frac{1}{2} \tag{4.27}$$

$$\Longrightarrow \frac{1}{(s+1)(s+3)} = \frac{1}{2}\frac{1}{s+1} - \frac{1}{2}\frac{1}{s+3} \tag{4.28}$$

$$f(x) = \mathcal{L}^{-1}[\frac{1}{2}\frac{1}{s+1}] - \mathcal{L}^{-1}[\frac{1}{2}\frac{1}{s+3}] = \frac{1}{2}\mathcal{L}^{-1}[\frac{1}{s+1}] - \frac{1}{2}\mathcal{L}^{-1}[\frac{1}{s+3}] \tag{4.29}$$

$$= \frac{1}{2}e^{-x} - \frac{1}{2}e^{-3x} \tag{4.30}$$

4.1.1 Laplace Transform of a Derivative

See previous notes on this in Part I

$$\sum_{i=0}^{n} a_i y^{(i)} = f(x) \tag{4.31}$$

$$\mathcal{L}\left[\sum_{i=0}^{n} a_i y^{(i)}\right] = \sum_{i=0}^{n} \mathcal{L}\left[a_i y^{(i)}\right] = \sum_{i=0}^{n} a_i \mathcal{L}\left[y^{(i)}\right] = \mathcal{L}[f] \tag{4.32}$$

$$\left(\sum_{i=0}^{n} k_i s^i\right)\bar{f}(s) = g(s) \tag{4.33}$$

Example:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 9e^{-2x},\ y(0) = y'(0) = 0 \tag{4.34}$$

Perform Laplace transforms on this:

$$s^2\bar{y} - 5(s\bar{y}) + 4\bar{y} = \frac{9}{s+2} \tag{4.35}$$

$$\bar{y}[s^2 - 5s + 4] = \frac{9}{s+2} \Longrightarrow \bar{y} = \frac{9}{(s+2)(s^2-5s+4)} \tag{4.36}$$

$$\bar{y} = \frac{9}{(s+2)(s-4)(s-1)} = \frac{A}{s+2} + \frac{B}{s-4} + \frac{C}{s-1} \tag{4.37}$$

$$\Longrightarrow A = \frac{1}{2}\ ;\ B = \frac{1}{2}\ ;\ C = -1 \tag{4.38}$$

$$\bar{y} = \frac{1}{2}\frac{1}{s+2} + \frac{1}{2}\frac{1}{s-4} - \frac{1}{s-1} \tag{4.39}$$

$$y(x) = \frac{1}{2}e^{-2x} + \frac{1}{2}e^{4x} - e^{x} \tag{4.40}$$

Lecture 5

5.1 Techniques for Linear ODEs, with Generic Coefficients

➤ In general, there is no universal technique

5.1.1 Legendre (Euler) Linear ODEs

$$a_n(\alpha x+\beta)^n y^{(n)} + a_{n-1}(\alpha x+\beta)^{n-1} y^{(n-1)} + \cdots + a_1(\alpha x+\beta)y' + a_0 y = f(x) \tag{Legendre}$$

When $\alpha = 1, \beta = 0$, it becomes the Euler equation

Change of variables leads to constant coefficients:

$$\alpha x+\beta = e^t \tag{5.1}$$
$$t = \ln(\alpha x+\beta) \tag{5.2}$$
$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\left[\frac{\alpha}{\alpha x+\beta}\right] \tag{5.3}$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{\alpha}{\alpha x+\beta}\frac{dy}{dt}\right] = -\frac{\alpha^2}{(\alpha x+\beta)^2}\frac{dy}{dt} + \frac{\alpha^2}{(\alpha x+\beta)^2}\frac{d^2y}{dt^2} \tag{5.4}$$
$$\frac{d^3y}{dx^3} = \frac{\alpha^3}{(\alpha x+\beta)^3}\frac{d}{dt}\left[\frac{d}{dt}-1\right]\left[\frac{d}{dt}-2\right]y \tag{5.5}$$
$$\frac{d^ny}{dx^n} = \frac{\alpha^n}{(\alpha x+\beta)^n}\frac{d}{dt}\left[\frac{d}{dt}-1\right]\left[\frac{d}{dt}-2\right]\cdots\left[\frac{d}{dt}-(n-1)\right]y \tag{5.6}$$

After this change of variable:

$$\tilde{a}_n\frac{d^ny}{dt^n} + \cdots + y(t) = f(x) \tag{5.7}$$

Example:

$$(x+1)^2y'' + 4(x+1)y' + 2y = \ln(x+1) + \frac{3}{2} \tag{5.8}$$
$$t = \ln(x+1) \tag{5.9}$$
$$(x+1)^2\frac{1}{(x+1)^2}\frac{d}{dt}\left[\frac{d}{dt}-1\right]y + 4(x+1)\frac{1}{(x+1)}\frac{dy}{dt} + 2y = t + \frac{3}{2} \tag{5.10}$$
$$\ddot{y} - \dot{y} + 4\dot{y} + 2y = t + \frac{3}{2} \tag{5.11}$$
$$\ddot{y} + 3\dot{y} + 2y = t + \frac{3}{2} \tag{5.12}$$
$$\lambda^2 + 3\lambda + 2 = 0 \implies (\lambda+2)(\lambda+1) = 0 \tag{5.13}$$
$$y_c = c_1e^{-t} + c_2e^{-2t} \tag{5.14}$$
$$y_p = a + bt,\ y_p' = b,\ y_p'' = 0 \implies y_p = \frac{t}{2} \tag{5.15}$$
$$y(t) = c_1e^{-t} + c_2e^{-2t} + \frac{t}{2} \tag{5.16}$$
$$y(x) = c_1\frac{1}{1+x} + c_2\frac{1}{(1+x)^2} + \frac{\ln(1+x)}{2} \tag{5.17}$$

5.2 Variation of Parameters

Imagine you know the complementary equation, but can't find the particular solution:

$$\sum_{i=0}^{n} a_i(x) y^{(i)} = f(x) \tag{5.18}$$

$$y_c = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n \tag{5.19}$$

Use the trick: $c_i \to c_i(x)$

$$\tilde{y} = c_1(x) y_1 + \cdots + c_n(x) y_n \tag{5.20}$$

Clearly $\tilde{y}$ does not solve homogeneous problem
Can we choose $c_i(x)$ such that they solve the inhomogeneous problem?
$\{c_i(x)\} \to$ n functions
1 constant (solve DE) $\to (n-1)$ free conditions
Choose $c_i' = 0$

$$c_1' y_1 + c_2' y_2 + \cdots + c_n' y_n = 0 \tag{5.21}$$

$$c_1' y_1' + c_2' y_2' + \cdots + c_n' y_n' = 0 \tag{5.22}$$

$$c_1' y_1'' + c_2' y_2'' + \cdots + c_n' y_n'' = 0 \tag{5.23}$$

$$\vdots \; (n-1) \text{ constraints on } c_1 \tag{5.24}$$

$$c_1' y_1^{(n-2)} + c_2' y_2^{(n-2)} + \cdots + c_n' y_n^{(n-2)} = 0 \tag{5.25}$$

$$- - - - - - - - - - \tag{5.26}$$

$$\tilde{y}' = \left[\sum c_i y_i\right]' = \cancel{\sum c_i' y_i} + \sum c_i y_i' \tag{5.27}$$

$$\tilde{y}'' = \cancel{\sum c_i' y_i'} + \sum c_i y_i'' \tag{5.28}$$

$$\tilde{y}^{(n-2)} = \sum c_i y_i^{(n-2)} \tag{5.29}$$

$$\tilde{y}^{(n-1)} = \sum c_i' y^{(n-2)} + \sum c_i y^{(n-1)} \tag{5.30}$$

Plug in to differential equation:

$$a_n(x) \left[c_1' y_1^{(n-1)} + \cdots + c_n' y_n^{(n-1)}\right] = f(x) \tag{5.31}$$

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & \cdots & & \\ \vdots & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \\ \vdots \\ c_n' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \frac{f(x)}{a_n(x)} \end{pmatrix} \tag{5.32}$$

$$\begin{pmatrix} c_1' \\ \vdots \\ c_n' \end{pmatrix} = M_W^{-1} \cdot \begin{pmatrix} 0 \\ \vdots \\ \frac{f(x)}{a_n(x)} \end{pmatrix} \tag{5.33}$$

Found $c_i(x)$ that solve inhomogeneous problem

Example: 2nd Order
Second Order $\to 2 \times 2$ Wronskian

$$y_c = c_1 y_1 + c_2 y_2 \to c_i = c_i(x) \tag{5.34}$$

$$M_W = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{f(x)}{a_2(x)} \end{pmatrix} \tag{5.35}$$

$$M_W^{-1} = \frac{1}{W}\begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix}\begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{f(x)}{a_2} \end{pmatrix} \tag{5.36}$$

$$\frac{1}{W}\begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \frac{f(x)}{a_2} \end{pmatrix} = \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} \tag{5.37}$$

$$\begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \frac{1}{W}\begin{pmatrix} -y_2 \frac{f(x)}{a_2} \\ y_1 \frac{f(x)}{a_2} \end{pmatrix} \tag{5.38}$$

General solution:

$$y_p = c_1 y_1 + c_2 y_2 \tag{5.39}$$

$$y_p = -y_1 \int \left(\frac{y_2}{W} \frac{f(x)}{a_2} \right) dx + y_2 \int \left(\frac{y_1}{W} \frac{f(x)}{a_2} \right) dx \tag{5.40}$$

This is the Wronskian technique.

Lecture 6

6.1 Linear ODEs Continued

- Found $y_c = c_1y_1 + \cdots + c_ny_n$, c_i are constants
- inhomogeneous problem?
- try $c_i \to c_i(x)$
- $\tilde{y} = \sum \tilde{c}_i(x)y_i(x)$ - no longer a solution of the homogeneous problem
- $\{c_i(x)\}, i \in N$ "$tildey$ in 1" is a solution of the differential equation
- useful constraint:

$$c_1'y_1 + c_2'y_2 + \cdots + c_n'y_n = 0 \tag{6.1}$$
$$c_1'y_1' + c_2'y_2' + \cdots + c_n'y_n' = 0 \tag{6.2}$$
$$c_1'y_1'' + c_2'y_2'' + \cdots + c_n'y_n'' = 0 \tag{6.3}$$
$$\vdots\, (n-1) \text{ constraints on } c_1 \tag{6.4}$$
$$c_1'y_1^{(n-2)} + c_2'y_2^{(n-2)} + \cdots + c_n'y_n^{(n-2)} = 0 \tag{6.5}$$
$$-------------- \tag{6.6}$$
$$c_1'y^{(n-1)} + c_2'y_2^{(n-1)} + \cdots + c_n'y_n^{(n-1)} = 0 \tag{6.7}$$
$$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & \cdots & & \\ \vdots & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \\ \vdots \\ c_n' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \frac{f(x)}{a_n(x)} \end{pmatrix} \tag{6.8}$$

6.2 Second Order Equations

$$\begin{pmatrix} y_1 & y_2 \\ y_! & y_2' \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{f(x)}{a_2(x)} \end{pmatrix} \tag{6.9}$$
$$\begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \frac{1}{W} \begin{pmatrix} y_2' & -y_2 \\ y_1' & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{f(x)}{a_2(x)} \end{pmatrix} = \frac{1}{W} \begin{pmatrix} -y_2\frac{f(x)}{a} \\ y_1\frac{f(x)}{a} \end{pmatrix} \tag{6.10}$$
$$c_1' = -\frac{y_2 f}{Wa} \ ; \ c_2' = \frac{y_1 f}{Wa} \tag{6.11}$$

Example:
Solve with variation of parameters

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = e^{2x} \tag{6.12}$$
$$\lambda^2 - \lambda - 2 = 0 \implies (\lambda - 2)(\lambda + 1) = 0 \tag{6.13}$$
$$y_c = c_1e^{-x} + c_2e^{2x} \tag{6.14}$$
$$\tilde{y} = c_1(x)e^{-x} + c_2(x)e^{2x} \tag{6.15}$$
$$y_1 = e^{-x},\ y_2 = e^{2x},\ f = e^{2x},\ a = 1,\ W = 3e^x \tag{6.16}$$
$$c_1 = -\int \frac{e^{2z}e^{2z}}{e^{3z}}dz = -\int \frac{e^{3z}}{3}dz = -\frac{e^{3z}}{9} \tag{6.17}$$
$$c_2 = \int \frac{e^{-z}e^{2z}}{3e^z}dz = \int \frac{dz}{3} = \frac{x}{3} \tag{6.18}$$

$$y_p = -\frac{e^{3x}}{9}e^{-x} + \frac{x}{3}e^{2x} \tag{6.19}$$
$$= e^{2x}\left(\frac{x}{3} - \frac{1}{9}\right) \tag{6.20}$$

6.3 Green's Functions

- It is important to only be dealing with linear problems
-

$$\sum_{i=0}^{n} a_i(x)y^{(i)} = f(x) \tag{6.21}$$
$$\sum_{i=0}^{n}\left[a_i(x)\frac{d^n}{dx^n}\right]y(x) = f(x) \tag{6.22}$$

- Use Laplace transform as an operator on y

$$\mathcal{L}\cdot y = f \tag{6.23}$$

-

$$\hat{\mathcal{L}}G(x,z) = \delta(x-z) \tag{6.24}$$
$$y = \int G(x,z)f(z)\,dz \tag{6.25}$$
$$\hat{\mathcal{L}}y = \int\left[\hat{\mathcal{L}}G(x,z)\right]f(z)\,dz = \int\delta(x-z)f(z)\,dz = f(x) \tag{6.26}$$

- Boundary conditions:
  - Homogeneous conditions:

$$y(a) = 0,\ y'(a) = 0 \tag{6.27}$$

Always $y \to a\tilde{y}+$ polynomial

$$y(x) = G(x,z)f(z)\,dz \tag{6.28}$$
$$y(a) = \int G(a,z)f(z)\,dz = 0,\ [G(a,z) = 0] \tag{6.29}$$

Identical for $\partial_x G(x,z)|_{x=a} = 0$

- $G^{(n)}(x,z)$ contains δ
-

$$\int_{z-\epsilon}^{z+\epsilon}\sum_{i=0}^{n}a_i(x)G^{(i)}(x,z)\,dx = \int_{z-\epsilon}^{z+\epsilon}\delta(x-z)\,dx = 1,\ \epsilon\to 0 \tag{6.30}$$
$$\int_{z-\epsilon}^{z+\epsilon}a_n(x)G^{(n)}(x,z)\,dx = \int_{z-\epsilon}^{z+\epsilon}\frac{d}{dx}\left[a_nG^{(n-1)}\right]dx - \int_{z-\epsilon}^{z+\epsilon}\cancel{\left(\frac{d}{dx}a_n\right)}G^{(n-1)} \tag{6.31}$$
$$= a_nG^{(n-1)}\Big|_{z-\epsilon}^{z+\epsilon} = 1 \tag{6.32}$$

-

$$\hat{\mathcal{L}}\left[G(x,z)\right] = \delta(x-z) \tag{6.33}$$
$$a_nG(x,z)\Big|_{z-\epsilon}^{z+\epsilon} = 1 \tag{6.34}$$

G has same boundary conditions in x and in y

Example:

$$\frac{d^2y}{dx^2} = y = f(x) \tag{6.35}$$

$$\hat{\mathcal{L}} = \frac{d^2}{dx^2} + 1 \tag{6.36}$$

$$\hat{\mathcal{L}}\left[G(x,z)\right] = \delta(x-z) = 0 \iff x \neq z \tag{6.37}$$

$$\implies y_c = c_1 \sin(x) + c_2 \cos(x) \tag{6.38}$$

$$G(x,z) = \begin{cases} A_1(z)\sin(x) + A_2(z)\cos(x) & x > z \\ B_1(z)sin(x) + B_2(z)\cos(x) & x > z \end{cases} \tag{6.39}$$

$$\implies G(x,z) = \begin{cases} A_1(z)\sin(x) & x > z \\ B_2(z)\cos(x) & x > z \end{cases} \tag{6.40}$$

G is continuous in $x = z \implies A_1(x)\sin(z) = B_2(x)\cos(z)$
G' has unit disc in $x = z \implies -B_2\sin(x) - A_1\cos(x) = 1$

$$G(x,z) = \begin{cases} -\cos(z)\sin(x) & x > z \\ \sin(z)\cos(x) & x > z \end{cases} \tag{6.41}$$

$$y(x) = \int_0^{\frac{\pi}{2}} G(x,z)f(x)\,dz \tag{6.42}$$

Constant limit of linear sup?

Lecture 7

7.1 Linear Second Order Homogeneous Equations

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \tag{7.1}$$

$$\implies \frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \tag{7.2}$$

7.1.1 Series Solutions

For $x \approx x_0$:

$$y = \sum_{i=0}^{\infty} a_i(x - x_0)^i \tag{7.3}$$

Can we find a_i?
In general, $f(x)$ is not a Taylor expansion around x_0, e.g. $\frac{1}{x}$

Let's assume series solution expansion (y is smooth enough)
If y admits a series representation around $x = x_0 \implies p, q$ smooth in x_0
If p, q are smooth, then the series expansion exists

What if p, q not regular in x_0?
Consider $y = \sqrt{x}, x \approx 0$, regular in 0
$\implies y' = \frac{1}{\sqrt{x}}$, not regular in 0

One can define a generalisation of Taylor expansion - "Frobenius Expansion":

$$f = x^\sigma \sum_{i=0}^{\infty} a_i x^i,\ \sigma \in \mathbb{C},\ [a_0 \neq 0] \tag{7.4}$$

e.g., $\sqrt{x}\sin(x)$

If y is Frobenius expansion in $x = x_0 = 0$:

$$y = x^\sigma \sum_{i=0}^{\infty} a_i x^i,\ x \to 0,\ y \approx x^\sigma \tag{7.5}$$

$$y' \approx \sigma x^{\sigma-1} \tag{7.6}$$

$$y'' \approx x^{\sigma-2} \tag{7.7}$$

If y is well defined in 0 $\implies x^\sigma toc \implies xy', x^2y''$ are well defined

If $y = x^\sigma \sum a_i x^i$:

$$\overbrace{\frac{d^2y}{dx^2}}^{x^{\sigma-2}} + \overbrace{p(x)\frac{dy}{dx}}^{x^{\sigma-1}} + \overbrace{q(x)y}^{x^\sigma} = 0,\ y = x^\sigma \sum a_i x^i \tag{7.8}$$

➤ If p, q in $x = x_0 \implies$ Taylor series solution
➤ If p, q are singular in $x = x_0 \implies$
1. If $\lim_{x\to x_0}(x-x_0)p(x)$ is finite and
2. If $\lim_{x\to x_0}(x-x_0)^2 q(x)$ is finite

➤ Solution as a Frobenius series exists - x_0 "regular singular point"
➤ else x_0 "essential singular point" - irregular

Example:
Find all singular points and classify

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + ky = 0 \tag{7.9}$$
$$\frac{d^2y}{dx^2} - \frac{2x}{(1-x^2)}\frac{dy}{dx} + \frac{k}{(1-x^2)}y = 0 \implies x = \pm 1 \tag{7.10}$$

Singular point at $x = 1$:

$$p = \frac{2x}{(1-x)(1+x)} \qquad q = \frac{k}{(1-x)(1+x)} \tag{7.11}$$
$$\implies = \lim_{x\to 1} \cancel{(x-1)}\frac{2x}{\cancel{(1-x)}\cancel{(1+x)}} = \text{finite} \qquad = \lim_{x\to 1}(x-1)^2\frac{k}{(1-x)(1+x)} = 0 \tag{7.12}$$

This implies $x = \pm 1$ as a regular singular point

$x \to \infty$:
Consider $x \to \frac{1}{w}, w = 0$ is a singular point?

$$\frac{dy}{dx} = \frac{dy}{dw}\frac{dw}{dx} = -\frac{1}{x^2}\frac{dy}{dw} = -w^2\frac{dy}{dw} \tag{7.13}$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[-\frac{1}{x^2}\frac{dy}{dw}\right] = \frac{2}{x^3}\frac{dy}{dw} + \frac{1}{x^4}\frac{d^2y}{dw^2} = 2w^3\frac{dy}{dw} + w^3\frac{d^2y}{dw^2} \tag{7.14}$$
$$\left(1 - \frac{1}{w^2}\right)w^3\left[2\frac{dy}{dw} + w\frac{d^y}{dw^2}\right] + \frac{2}{w}w^2\frac{dy}{dw} + ky = 0 \tag{7.15}$$
$$w^2\left(w^2-1\right)\frac{d^2y}{dw^2} + 2w^3\frac{dy}{dw} + ky = 0 \tag{7.16}$$
$$\frac{d^y}{dw^2} + 2\frac{w}{w^2-1}\frac{dy}{dw} + \frac{k}{w^2(w^2-1)} = 0 \tag{7.17}$$
$$p(w) = \frac{2w}{w^2-1},\ w\cdot p \to^{w\to 0} 0 \tag{7.18}$$
$$q(w) = \frac{k}{w^2(w^2-1)},\ w^2 q \to^{w\to 0} -k \text{ (finite)} \tag{7.19}$$

Regular singular point

Example:

$$\frac{d^2y}{dx^2} + y = 0 \tag{7.20}$$

Series around $x = 0$

$$y = \sum_{i=0}^{\infty} a_i x^i \qquad \frac{dy}{dx} = \sum_{i=0}^{\infty} i a_i x^{i-1} \qquad \frac{d^2y}{dx^2} = \sum_{i=0}^{\infty} i(i-1)a_i x^{i-2} \tag{7.21}$$

$$\sum_{i=0}^{\infty}\left[i(i-1)a_x x^{i-2} + a_i x^i\right] = 0 \tag{7.22}$$

$$\sum_{i=2}^{\infty} i(i-1)a_i x^{i-2} = \sum_{i=0}^{\infty}(i+2)(i+1)a_{i+2}x^i \tag{7.23}$$

$$\sum_{i=0}^{\infty}\left[(i+1)(i+2)a_{i+2} + a_i\right] x^i = 0 \tag{7.24}$$

$$\implies a_{i+2} = -\frac{a_i}{(i+1)(i+2)} \tag{7.25}$$

A regular relation

- Odd/even terms are independent:
  - $a_0 = 0, a_1 = 1 \implies a_{2n} = 0$

$$\implies y = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sin(x) \tag{7.26}$$

  - $a_0 = 1, a_1 = 0 \implies a_{2n+1} = 0$

$$\implies y = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \cos(x) \tag{7.27}$$

Lecture 8

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(y)y = 0 \tag{8.1}$$

The structure of y depends on singularity structure of p, q
If p, q are regular in $x = x_0$, $y = \sum_{i=0}^{\infty} a_i x^i$

➤ Is it possible to find a polynomial solution?

$$y = \sum_{i=0}^{\infty} a_i x^i,\ \ a_i = 0 \forall i > N \tag{8.2}$$

Example:

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + ky = 0,\ \ x \approx 0 \tag{8.3}$$

$$(1-x^2)\sum_{i=0}^{\infty}\left(i(i-1)a_i x^{i-2}\right) - 2x\sum_{i=0}^{\infty} i a_i x^{i-1} + k\sum_{i=0}^{\infty} a_i x^i = 0 \tag{8.4}$$

$$\sum((i+2)(i+1)a_{i+2} - i(i-1)a_i - 2ia_i + ka_i)x^i = 0 \tag{8.5}$$

$$(i+2)(i+1)a_{i+2} - a_i i(i+1) + ka_i = 0 \tag{8.6}$$

$$a_{i+2} = a_i \frac{i(i+1) - k}{(i+1)(i+2)} \tag{8.7}$$

Can the series terminate?
This always happens if $k = \lambda(\lambda+1), \lambda \in \mathbb{N}$ - series terminates at $O(\lambda)$

8.1 Regular Singular Points

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(y)y = 0 \tag{8.8}$$

$$\text{If: } \lim_{x\to x_0} p(x) \text{ Or } \lim_{x\to x_0} q(x) \nexists \implies \text{ singular point} \tag{8.9}$$

$$\text{If: } \lim_{x\to x_0}(x-x_0)p(x) \text{ and } \lim_{x\to x_0}(x-x_0)^2 q(x) \exists \implies \text{ regular singular point} \tag{8.10}$$

From now on, $x_0 = 0$:

$$xp \equiv s\ ;\ x^2 q \equiv t,\ \text{regular in } 0 \tag{8.11}$$

$$\frac{d^2y}{dx^2} + \frac{s(x)}{x}\frac{dy}{dx} + \frac{t(x)}{x^2}y = 0 \tag{8.12}$$

$$y = s^{\sigma}\sum_{i=0}^{\infty} a_i x^i,\ \ a_0 \neq 0 \tag{8.13}$$

$$y = \sum_{i=0} a_i x^{i+\sigma} \tag{8.14}$$

$$\sum_{i=0}^{\infty}\left[(i+\sigma)(i+\sigma-1)a_i x^{i+\sigma-2} + \frac{s(x)}{x}(i+\sigma)a_i x^{i+\sigma-1} + \frac{t(x)}{x^2}a_i x^{i+\sigma}\right] = 0 \tag{8.15}$$

$$\sum_{i=0}^{\infty}\left[(i+\sigma)(i+\sigma-1) + s(x)(i+\sigma) + t(x)\right]a_i x^{i+\sigma-2} = 0 \tag{8.16}$$

$$\sum_{i=0}^{\infty}\left[(i+\sigma)(i+\sigma-1)+s(x)(i+\sigma)+t(x)\right]a_i x^i=0,\ \forall x \tag{8.17}$$

$$x=0 \implies [\sigma(\sigma-1)+s(0)\sigma+t(0)]a_0=0 \tag{8.18}$$

This is the indicial equation.

$$\sigma(\sigma-1)+s(0)\sigma+t(0)=0 \tag{8.19}$$

$$(\sigma-\sigma_1)(\sigma-\sigma_2)=0 \tag{8.20}$$

Solutions:

1. $\sigma_1=\sigma_2 \implies$ one solution
2. $\sigma_1\neq\sigma_2 \implies$:
   (a) the largest root leads to solution $[\sigma_1],\ (\sigma_1>\sigma_2)$
   (b) if $\sigma_1-\sigma_2\notin\mathbb{N}$, then σ_2 also leads to independent solution:

$$y_1\approx x^{\sigma_1},\ y_2\approx x^{\sigma_2},\ \frac{y_2}{y_1}\approx x^{[\sigma_2-\sigma_1]} \tag{8.21}$$

   (c) if $\sigma_1-\sigma_2\in\mathbb{N}$, σ_2 sometimes leads to independent solution, sometimes not

Example:

$$4x\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0,\ x\approx 0 \tag{8.22}$$

$$\frac{d^2y}{dx^2}+\frac{1}{2x}\frac{dy}{dx}+\frac{1}{4x}y=0 \tag{8.23}$$

$$\implies s=\frac{1}{2},\ t=\frac{x}{4} \tag{8.24}$$

$$y=x^{\sigma}\sum_{i=0}^{\infty}a_i x^i \tag{8.25}$$

$$\sum_{i=0}^{\infty}\left[(\sigma+1)(\sigma+i-1)+\frac{1}{2}(\sigma+i)+\frac{x}{4}\right]x^{\sigma+i-2}a_i=0 \tag{8.26}$$

$$\sigma(\sigma-1)+\frac{1}{2}\sigma=0=\sigma\left(\sigma-\frac{1}{2}\right)=\begin{cases}\sigma_1 & =\frac{1}{2}\\ \sigma_2 & =0\end{cases} \tag{8.27}$$

$$[(\sigma+i)(\sigma+i-1)+\frac{1}{2}(\sigma+i)]a_i+\frac{1}{4}a_{i-1}=0 \tag{8.28}$$

$$y(x,\sigma)\equiv x^{\sigma}\sum_{i=0}^{\infty}a_i(\sigma)x^i \tag{8.29}$$

If $\sigma=\sigma_1,\sigma_2,\ y(\sigma,x)$ solves one

$$\left[\frac{d^2}{dx^2}+\frac{1}{2x}\frac{d}{dx}+\frac{1}{4x}\right]x^{\sigma}\sum_{i=0}^{\infty}a_i(\sigma)x^i=\cdots \tag{8.30}$$

$$\implies \sum_{i=0}^{\infty}\left[(\sigma+i)\left(\sigma+i-\frac{1}{2}\right)\right]a_i x^{\sigma+i-2}+\sum_{i=1}\infty\frac{x^{\sigma+i-2}a_{i-1}}{4} \tag{8.31}$$

$$\implies \left[\sigma\left(\sigma-\frac{1}{2}\right)\right]a_0x^{\sigma-2}+\sum_{i=1}^{\infty}\cancel{\left[(\sigma+1)(\sigma+i-\frac{1}{2})a_i+\frac{a_{i-1}}{4}\right]}x^{\sigma+i-2} \tag{8.32}$$

$$\left[\frac{d^2}{dx^2}+\frac{1}{2x}\frac{d}{dx}+\frac{1}{4x}\right]y(x,\sigma)=\sigma\left(\sigma-\frac{1}{2}\right)a_0x^{\sigma-2}\propto\sigma\left(\sigma-\frac{1}{2}\right)x^{\sigma} \tag{8.33}$$

$$\mathcal{L}y(x,\sigma)=(\sigma-\sigma_1)(\sigma-\sigma_2)a_0x^{\sigma-2} \tag{8.34}$$

Solve Recursion for $\sigma-\{\sigma_1,\sigma_2\}$

$\sigma_1 = \frac{1}{2}$:

$$\left[(\frac{1}{2}+i)(\frac{1}{2}+i-1)+\frac{1}{2}(\frac{1}{2}+i)\right]a_i+\frac{1}{4}a_{i-1}=0 \tag{8.35}$$

$$2i(2i+1)a_i+a_{i-1}=0 \tag{8.36}$$

$$a_i=-\frac{a_{i-1}}{2i(2i+1)} \tag{8.37}$$

$$y=x^\sigma\sum_{i=0}^{\infty}a_ix^i=\sqrt{x}\left[1-\frac{x}{3!}+\frac{x^2}{5!}-\cdots\right] \tag{8.38}$$

$$=\sqrt{x}-\frac{(\sqrt{x})^3}{3!}+\frac{(\sqrt{x})^6}{5!}-\cdots \tag{8.39}$$

$$=\sin(\sqrt{x}) \tag{8.40}$$

Do the same for $\sigma=\sigma_2 \implies y=\cos(\sqrt{x})$

$$y=c_1\sin\sqrt{x}+c_2\cos\sqrt{x} \tag{8.41}$$

Lecture 9

Example: Singular Points

$$x(x-1)\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+y=0 \tag{9.1}$$
$$\frac{d^2y}{dx^2}+\frac{3}{x-1}\frac{dy}{dx}+\frac{1}{x(x-1)}y=0 \tag{9.2}$$
$$y=x^\sigma\sum_{i=0}^{\infty}a_ix^i,\ a_0\neq 0 \tag{9.3}$$
$$(x-1)\frac{d^2y}{dx^2}+3\frac{dy}{dx}+\frac{1}{x}y=0 \tag{9.4}$$
$$(x-1)\sum_{i=0}^{\infty}[a_i(\sigma+i-1)x^{\sigma+i-2}+3(\sigma+i)a_ix^{\sigma+i-1}+a_ix^{\sigma+i-1}]=0 \tag{9.5}$$
$$\sum_{i=0}^{\infty}[a_i(\sigma+i)(\sigma+i-1)x^{\sigma+i-1}-a_i(\sigma+i)(\sigma+i-1)x^{\sigma+i-2}+3(\sigma+i)a_ix^{\sigma+i-1}+x^{\sigma+i-1}a_i]=0 \tag{9.6}$$

So the indicial equation is

$$\sigma(\sigma-1)=0 \implies \sigma=0,1 \tag{9.7}$$
$$a_{i-1}[(\sigma+i)^2]-\cancel{[\sigma+i]}[\sigma+i-1]a_i=0 \tag{9.8}$$
$$\implies a_i=\frac{\sigma+i}{\sigma+i-1}a_{i-1} \tag{9.9}$$
$$\sigma_1=1 \implies a_i=\frac{1+i}{i}a_{i-1} \tag{9.10}$$
$$\implies y=x\sum_{i=0}^{\infty}(1+2x+3x^2+4x^3+\cdots)=x\frac{1}{(1-x)^2} \tag{9.11}$$
$$\sigma_2=0 \implies a_i=\frac{i}{i-1}a_{i-1} \implies \nexists \tag{9.12}$$

How can we find the second solution then?

1. Wronskian method:

$$W=\begin{vmatrix}y_1 & y_2\\ y_1' & y_2'\end{vmatrix}=y_1y_2'-y_2y_1' \tag{9.13}$$
$$W'=\cancel{y_1'y_2'}+y_1y_2''-\cancel{y_2'y_1'}-y_2y_1'' \tag{9.14}$$
$$=y_1(-py_2'-qy_2)-y_2(-py_1'-qy_2) \tag{9.15}$$
$$=-p(y_1y_2'-y_2y_1')=-pW \tag{9.16}$$
$$\frac{W'}{W}=-p \implies W=ce^{-\int p(x)dx} \tag{9.17}$$
$$\frac{W}{y_1^2}=\frac{y_2'}{y_1}-y_2\frac{y_1'}{y_1^2}=\frac{d}{dx}\left[\frac{y_2}{y_1}\right]=\frac{y_2'}{y_1}+y_2\left(-\frac{1}{y_1^2}y_1'\right) \tag{9.18}$$
$$\frac{y_2}{y_1}=\int\frac{W}{y_1^2}dx=\int\frac{1}{y_1^2}e^{-\int p\,dx}dx \tag{9.19}$$
$$\implies y_2=y_1\int\frac{1}{y_1^2(x)}e^{-\int p\,dx}dx \tag{9.20}$$
$$y_1=\frac{x}{(1-x)^2}\to p=\frac{3}{x-1}\to e^{-\int p}=e^{-\int\frac{3}{x-1}dx}=e^{-3\ln(x-1)}=\frac{1}{(x-1)^3} \tag{9.21}$$

$$y_2 = \frac{x}{(1-x)^2} \int \frac{(1-x)^4}{x^2} \frac{1}{(z-1)^3} dx = \frac{x}{(1-x)^2} \int \frac{x-1}{x^2} dx = \int \left(\frac{1}{x} - \frac{1}{x^2} \right) \tag{9.22}$$

$$\implies y_2 = \frac{x}{(1-x)^2} \left[\ln(x) + \frac{1}{x} \right] \tag{9.23}$$

2. Derivative technique
   Consider when $\sigma_1 = \sigma_2$, indicial equation in $(\sigma - \sigma_1)^2 = 0$

$$y(x,\sigma) = x^\sigma \sum_{i=0}^{\infty} a_i(\sigma) x^i \tag{9.24}$$

$$\mathcal{L}_x[y(x,\sigma)] = (\sigma - \sigma_1)^2 x^\sigma \tag{9.25}$$

$$\frac{\partial}{\partial \sigma} \left(\mathcal{L}_x[y(x,\sigma)] \right) = 2(\sigma - \sigma_1) x^\sigma + (\sigma - \sigma_1)^2 \ln(x) x^\sigma \tag{9.26}$$

$$\mathcal{L}_x \left[\frac{\partial}{\partial \sigma} y(x,\sigma) \right] = 2(\sigma - \sigma_1) x^\sigma + (\sigma - \sigma_1)^2 \ln(x) x^\sigma \tag{9.27}$$

$$\sigma \to \sigma_1 \implies \mathcal{L}_x \left[\lim_{\sigma \to \sigma_1} \frac{\partial}{\partial \sigma} y(x,\sigma) \right] = 0 \tag{9.28}$$

$$y_2 = \lim_{\sigma \to \sigma_1} \left[\frac{\partial}{\partial \sigma} y(x,\sigma) \right] \tag{9.29}$$

   This is a solution.

Lecture 10

10.1 Special Functions

➤ Legendre equation:

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + l(l+1)y = 0 \tag{10.1}$$

➤ ∇^2 in polar coordinates $\to \theta$, $\cos\theta \equiv x$
➤ $x = 0$ regular point, $x = \pm 1$ regular singularities
➤ Can immediately get two solutions from:

$$a_{n+2} = \frac{n(n+1) - l(l+1)}{(n+1)(n+2)} a_n \tag{10.2}$$

(Convergence radius of $|x| < 1$)

1. $a_0 = 1, [a_{i,\text{odd}} \to 0]$

$$y_1 = 1 - \frac{l(l+1)}{2!}x^2 + \frac{(l-2)l(l+1)(l+3)}{4!}x^4 + \cdots \tag{10.3}$$

2. $a_0 = 0, [a_{i,\text{even}} \to 0, a_1 = 1]$

$$y_2 = x - \frac{(l-1)(l+2)}{3!}x^3 + \frac{(l-3)(l-1)(l+2)(l+4)}{5!}x^5 + \cdots \tag{10.4}$$

➤ if l is an integer,
- ➥ polynomial solution

$$P_l(x) = \begin{cases} l \text{ odd} & y_2 \\ l \text{ even} & \to y_1 \end{cases} \tag{10.5}$$

- ➥ non-polynomial solution

$$Q_l(x) = \begin{cases} l \text{ odd} & y_1 \\ l \text{ even} & \to y_2 \end{cases} \tag{10.6}$$

➤ $P_l(1) = 1$ - choice of parameterisation? $[P_l(-1) = (-1)^l]$
➤ Need the following in Quantum Mechanics, but not in this course:

$$Q_l = \begin{cases} l \text{ even} & \to \alpha_l y_2 \\ l \text{ odd} & \to \beta_l y_1 \end{cases} \tag{10.7}$$

$$\alpha_l = (-1)^{\frac{l}{2}} 2^l \frac{\left[\left(\frac{l}{2}\right)!\right]^2}{l!} \tag{10.8}$$

$$\beta_l = (-1)^{\frac{l+1}{2}} 2^{l-1} \frac{\left[\left(\frac{l-1}{2}\right)!\right]}{l!} \tag{10.9}$$

➤ Rodrigueis' Formula (solves Legendre's Equation)

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l \tag{10.10}$$

- ➥ Proof:

$$u = (x^2-1)^l \tag{10.11}$$

$$u' = 2xl(x^2-1)^{l-1} = \frac{2xlu}{x^2-1} \tag{10.12}$$

$$\implies (x^2-1)u' - 2xlu = 0 \tag{10.13}$$

Differentiate $l+1$ times

$$\frac{d^k}{dx^k}(a\cdot b)=\sum_{i=0}^{k}a^{(i)}b^{(k-i)}\frac{k!}{i!(k-i)!} \tag{10.14}$$

$$l=0\to ab^{(k)}\frac{k!}{0!(k)!}=ab^{(k)} \tag{10.15}$$

$$l=1\to a'b^{(k-1)}\frac{k!}{1!(k-1)!}=ka'b^{(k-1)} \tag{10.16}$$

$$l=2\to a''b^{(k-2)}\frac{k!}{2!(k-2)!}=\frac{1}{2}k(k-1)a''b^{(k-2)} \tag{10.17}$$

$$[(x^2-1)u^{(l+2)}+2x(l+1)u^{(l+1)}+u^{(l)}(l+1)l]-2l[xu^{(l+1)}+u^{(l)}(l+1)]=0 \tag{10.18}$$

$$(x^2-1)[u^{(l)}]''+2x[u^{(l)}]'-l(l+1)[u^{(l)}]=0 \tag{10.19}$$

This is the Legendre equation, and therefore Rodrigueis' formula is related to Legendre.

➤ Check normalisation, $x\to 1$:

$$\frac{d^k}{dx^k}(x^2-1)^k\Big|_{x=1}\to 2x(x^2-1)^{k-1}\to 2^k k! \tag{10.20}$$

$$\int_{-1}^{1}P_l(x)P_m(x)\,dx=\frac{2}{2l+1}\delta_{lm} \tag{10.21}$$

How to prove:

1.
$$\int_{-1}^{1}P_l(x)P_m(x)\,dx=\frac{2}{2l+1} \tag{10.22}$$
   (a) use Rodrigueis
   (b) integrate by parts

2.
$$\int_{-1}^{1}P_l(x)P_m(x)\,dx=0, l\neq m \tag{10.23}$$

$$(1-x^2)P_l''{}_2xP_l'+l(l+1)P_l=0 \tag{10.24}$$

$$[(1-x^2)P_l]'+l(l+1)P_l=0 \tag{10.25}$$

$$\int_{-1}^{1}P_m[(1-x^2)P_l']'dx=-l(l+1)\int_{-1}^{1}P_mP_l\,dx \tag{10.26}$$

$$[P_m(1-x^2)P_l']_{-1}^{1}-\int_{-1}^{1}P_m'P_l'(1-x^2)\,dx \tag{10.27}$$

$$=-l(l+1)\int_{-1}^{1}P_l(x)P_m(x)\,dx \tag{10.28}$$

$$\int_{-1}^{1}P_l'P_m'(1-x^2)\,dx=l(l+1)\int_{-1}^{1}P_l(x)P_m(x)\,dx \tag{10.29}$$

Symmetric under exchange of l and m
Also equal to

$$m(m+1)\int_{-1}^{1}P_l(x)P_m(x)\,dx \tag{10.30}$$

So

$$\underbrace{l(l+1)}_{N_1}\int_{-1}^{1}P_l(x)P_m(x)\,dx=\underbrace{m(m+1)}_{N_2}\int_{-1}^{1}P_l(x)P_m(x)\,dx \tag{10.31}$$

As $l\neq m, N_1\neq N_2$ so

$$\int_{-1}^{1}P_l(x)P_m(x)\,dx=0, l\neq m \tag{10.32}$$

This tells us any function between -1 and 1 can be expanded in Legendre polynomials:

$$\int_{-1}^{1} P_l(x)P_m(x)\,dx = \alpha_l\delta_{lm} \tag{10.33}$$

$$f(x),\ x \in [-1,1] \to f(x) = \sum_{l=0}^{\infty} k_l P_l(x) \tag{10.34}$$

$$\int_{-1}^{1} f(x)P_m(x) = \sum_{l=0}^{\infty}\int_{-1}^{1} k_l P_l P_m = k_m a_m \tag{10.35}$$

$$\implies k_m = \int_{-1}^{1} \frac{f(x)P_m(x)}{a_m} dx \tag{10.36}$$

Lecture 11

11.1 Legendre Equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + l(l+1)y = 0 \to P_l(x), P_l(x) = 1 \tag{11.1}$$

11.2 Generating Function

$$G(x,h) \equiv \frac{1}{\sqrt{1-2xh+h^2}} = \sum_{i=0}^{\infty} P_i(x)h^i \tag{11.2}$$

- contains all information about P_l
- can manipulate [∂] both sides to find nice properties

$$\partial_x G = (1-2xh+h^2)^{-3/2}h = \sum_{i=0}^{\infty} P_i'(x)h^i \tag{11.3}$$

$$\boxed{hG = \frac{h}{\sqrt{1-2xh+h^2}} = (1-2xh+h^2)\sum_{i=0}^{\infty} P_i'(x)h^i} \tag{11.4}$$

$$\boxed{h\sum_{i=0} P_i(x)h^i = (1-2xh+h^2)\sum_{i=0} P_i'(x)h^i} \tag{11.5}$$

$$\boxed{\partial_h G = \frac{x-h}{(1-2xh+h^2)^{3/2}} = \frac{x-h}{1-2xh+h^2}G(x,h) = \sum_{l=0}^{\infty} lP_l h^{l-1}} \tag{11.6}$$

$$\boxed{P_i = P_{i+1}' - 2xP_i' + P_{i-1}'} \tag{11.7}$$

$$\sum_{i=0}^{\infty} P_i' h^i = \frac{h}{1-2xh+h^2}G = \frac{h}{x-h}\sum_{i=0}^{\infty} iP_i h^{i-1} \tag{11.8}$$

$$\implies (x-h)\sum_{i=0}^{\infty} P_i' h^i = h\sum_{i=0}^{\infty} iP_i h^{i-1} \tag{11.9}$$

$$\boxed{iP_i = xP_i' - P_{i-1}'} \tag{11.10}$$

Substitute P_{i-1}':

$$(i+1)P_i = P_{i+1}' + xP_i', \ [l = i+1] \tag{11.11}$$

$$lP_{l-1} = P_l' - xP_{l-1}' \tag{11.12}$$

Remove P_{l-1}

$$\boxed{l(P_{l-1} - xP_l) = (1-x^2)P_l'} \tag{11.13}$$

Act with ∂_x:

$$l[P_{l-1}' - P_l - xP_l'] = (1-x^2)P_l'' - 2xP_l' \tag{11.14}$$

$$\boxed{-l(l+1)P_l = (1-x^2)P_l'' - 2xP_l'} \tag{11.15}$$

Check normal, $P_l(1) = 1$

$$G(1,h) = \frac{1}{\sqrt{1-2h+h^2}} = \frac{1}{1-h} \tag{11.16}$$

$$= \sum_{i=0}^{\infty} h^i = \sum_{i=0}^{\infty} P_i(1)h^i \tag{11.17}$$

$$\implies P_i(1) = 1 \tag{11.18}$$

11.3 Recursion Relation

$$\partial_h G \to (x-h)\sum_{l=0}^{\infty} P_l h^l = (1-2xh+h^2)\sum_{l=0}^{\infty} lP_l h^{l-1} \tag{11.19}$$

$$xP_l - P_{l-1} = (l+1)P_{l-1} - 2xlP_l + (l-1)P_{l-1} \tag{11.20}$$

$$(l+1)P_{l-1} = x(1+2l)P_l - lP_{l-1} \tag{11.21}$$

11.4 Spherical Harmonics

Associated Legendre Equation:

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left[l(l+1) - \frac{m^2}{1-x^2}\right]y = 0 \implies P_l^m(x),\ |m| < |l| \tag{11.22}$$

$$Y_{l,m}(\theta,\phi) = P_l^m(\cos\theta)e^{im\phi} \times N_{ml} \tag{11.23}$$

$$Y_{l,-m}(\theta,\phi) = (-1)^m Y_{l,m}^*(\theta,\phi) \tag{11.24}$$

$$\int_{-1}^{1} d\cos\theta \int_0^{2\pi} Y_{l,m}Y_{l',m'}^* d\phi = \delta_{ll'}\delta_{mm'} \tag{11.25}$$

Spherical harmonics are an orthonormal set over the spherical system

$$f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{l,m}Y_{l,m}(\theta,\phi) \tag{11.26}$$

$$a_{l,m} = \int_{-1}^{1} d\cos\theta \int_0^{2\pi} f(\theta,\phi)Y_{l,m}^* d\phi \tag{11.27}$$

Lecture 12

12.1 Bessel Functions

$$z^2\frac{d^2y}{dx^2}+z\frac{dy}{dx}+(z^2-v^2)y=0 \tag{12.1}$$

- v is a constant
- $z \to 0$ is a regular singular point

$$y=x^\sigma\sum_{i=0}^{\infty}a_ix^i \tag{12.2}$$

- Recursion relation:
  - $x^0 \to a_0[\sigma^2-v^2]=0$
  - $x^1 \to a_1[(\sigma+1)^2-v^2]=0$
  - $x^i \to a_i[(\sigma+i)^2-v^2]+a_{n-2}=0$
  - $\sigma=\pm v$
- $v \notin \mathbb{Z} \to v-[-v]=2v \notin \mathbb{Z} \to 2$ independent solutions
- Exception: $v=\frac{n}{2} \to v-(-v) \in \mathbb{Z} \to$ we may or may not find 2 solutions
-

$$\tilde{J}_{\pm v}=z^{\pm v}\left[1-\frac{z^2}{2(2\pm 2v)}+\frac{z^4}{2\cdot 4(2\pm 2v)(4\pm 2v)}+\cdots\right] \tag{12.3}$$

$$\tilde{J}_{\pm v}=z^{\pm v},\ J_{\pm v}=\tilde{J}_{\pm v}\frac{1}{2^{\pm v}\Gamma(1\pm v)} \tag{12.4}$$

  -

$$\Gamma(1+n)=n!,\ n\in\mathbb{N} \tag{12.5}$$

$$\Gamma(1+n)=\Gamma(n)n\forall n\in\mathbb{C} \tag{12.6}$$

-

$$J_v(z)=\sum_{i=0}^{\infty}\frac{(-1)^i}{i!\Gamma(v+i+1)}\left(\frac{z}{2}\right)^{v+2i} \tag{Bessel I}$$

- General solution

$$y=c_1J_{+v}+c_2J_{-v} \tag{12.7}$$

- If v is an integer:

$$J_{-v}(z)=(-1)^vJ_v(z) \tag{12.8}$$

- What about other solution? Define

$$y_v=\frac{J_v(z)\cos(\pi v)-J_{-v}(z)}{\sin(\pi v)} \tag{12.9}$$

  1. Not defined if $v\in\mathbb{Z}$
  2. Obviously, solution of Bessel
- If v is an integer, define y_v as limit $[v+\epsilon,\epsilon\to 0]$
- It turns out $\forall v, y_v$ and J_v are independent
- "y_v" - Bessel function of 2nd Kind
- $v>0$:
  - J_v is well defined in $[0,\infty]$
  - y_v is ill-defined in $z\to 0$ (not good)

12.1.1 Properties of Jv

- From definition:

$$\frac{d}{dz}[z^v J_v] = z^v J_{v-1} \tag{12.10}$$

- Orthonormal:

$$\int_a^b zJ_v(\lambda z)J_v(\mu z)\,dz = 0,\ \mu \neq \lambda \tag{12.11}$$

- $$f(z) = \sum_{i=0}^{\infty} c_i J_v(\lambda_i z),\ \lambda_i \text{ s.t. } J_v(\lambda_i a) = 0 \tag{12.12}$$

- Generating function

$$\exp\left[\frac{z}{2}\left(h - \frac{1}{h}\right)\right] = \sum_{i=0}^{\infty} J_i h^i \tag{12.13}$$

$$\implies J_{v-1} + J_{v+1} = \frac{2v}{2} J_v \tag{12.14}$$

12.2 Linear Partial Differential Equations

- Physics - 2nd Order
- For simplicity, mostly focus on 2 variable case, (x, y)
- $$f(x,y) \to \partial_x^2 f - x\partial_y^2 f + xy\partial_x\partial_y f = g(x,y) \tag{12.15}$$

Example: Classify functions of 2 variables

$$f_1 = x^4 + 4(x^2y + y^2 + 1) \qquad f_1(x^2 + 2y = p) = p^2 + 4 \tag{12.16}$$

$$f_2 = \sin(x^2 + 2y) \qquad f_2(x^2 + 2y = p) = \sin(p) \tag{12.17}$$

$$f_3 = \frac{x^2 + 2y + 2}{3x^2 + 6y + 5} \qquad f_3(x^2 + 2y = p) = \frac{p + 2}{2p + 5} \tag{12.18}$$

Differentiating:

$$\partial_x f_i = \partial_x f_i(p(x,y)) = \frac{df_i}{dp}\partial_x p \tag{12.19}$$

$$\partial_y f_i = \frac{df_i}{dp}\partial_y p \tag{12.20}$$

$$\partial_x f_i \frac{\partial p}{\partial y} = \partial_x f_i \frac{\partial p}{\partial x} \tag{12.21}$$

$\implies$ f_1, f_2, f_3 obey the same differential equation

- A single PDE admits infinite solutions
- Looking for solution - look for functional forms
- higher order:
  - 2nd order $\to$ 2 functional forms
  - nth order $\to$ n functional forms

12.2.1 First Order, 2 Variables

$$A\partial_x f + B\partial_y f + Cf = D \tag{12.22}$$

- A, B, C, D are functions of x, y
- $D = 0$ - "homogeneous"

- Technical definition:
  An equation is said to the homogeneous if f is a solution then λf is also a solution (λ constant)

-
$$A\partial_x f + B\partial_y f = 0 \tag{12.23}$$

Lecture 13

13.1 Linear PDEs continued

13.1.1 Homogeneous First Order, 2 Variables

$$A\partial_x f + B\partial_y f + Cf = 0 \tag{13.1}$$

- $f[p(x,y)]$ are solutions
- Start from $C = 0$:

$$A(x,y)\partial_x f + B(x,y)\partial_y f = 0 \tag{13.2}$$

  - The goal is to find the functional form, p:

$$f(x,y) = f(p), \;\; p = p(x,y) \tag{13.3}$$

$$\implies A\frac{df}{dx}\partial_x p + B\frac{df}{dp}\partial_y p = 0 \tag{13.4}$$

  - Find "surfaces" of constant $p \implies$

$$dp = 0 = \partial_x p\, dx + \partial_y p\, dy \tag{1}$$

$$\implies \frac{dy}{dx} + \frac{\partial_x p}{\partial_y p} = 0 \tag{2}$$

$$\implies \frac{df}{dp}\left[\frac{B}{A} + \frac{\partial_x p}{\partial_y p}\right] = 0 \tag{3}$$

$$(2) = (3) \to \frac{dy}{dx} = \frac{B}{A} \tag{4}$$

Example:
Solve:

$$x\partial_x f - 2y\partial_y f = 0 \tag{13.5}$$

$$A = x, B = -2y \tag{13.6}$$

$$\frac{dy}{dx} = \frac{-2y}{x} \implies \frac{dy}{y} = -2\frac{dx}{x} \tag{13.7}$$

$$\implies \ln(y) = -2\ln(x) + C \tag{13.8}$$

$$\implies y = \frac{\tilde{c}}{x^2} \implies x^2 y \text{ constant} \tag{13.9}$$

Generic solution is $f(p(x,y)) = f(x^2 y)$. Substitute in:

$$xf'2x - 2yf'x^2 = f'[2x^2 - 2x^2] = 0 \tag{13.10}$$

Now impose boundary conditions:

1. $f = 2y + 1$ on the line $x = 1$:

$$f(x^2 y) \to f(1\cdot y) = f(y) = 2y + 1 \tag{13.11}$$

$$f(\alpha) = 2\alpha + 1 \tag{13.12}$$

   General solution plus boundary condition:

$$f(x^2 y) = 2[x^2 y] + 1 \tag{13.13}$$

2. $f(1,1) - 4$:

$$f(x,y) = 4 + g(x^2 y), \;\; g(1) = 0 \tag{13.14}$$

   This is also a solution, but more arbitrary

- Add the C term back now:

$$A\partial_x f + B\partial_y f + Cf = 0 \tag{13.15}$$

- $f(p(x,y))$ does not work now, look for $f = h(x,y)\tilde{f}(p(x,y))$
- h must be any solution of differential equation
- take $f = h(x,y)\tilde{f}(p)$ and substitute:

$$\tilde{f}[A\partial_x h + B\partial_y h] + h[A\partial_x \tilde{f} + B\partial_y \tilde{f}] + Ch\tilde{f} = 0 \tag{13.16}$$

$$\implies \cancel{\tilde{f}[A\partial_x h + B\partial_y h + Ch]} + h[A\partial_x \tilde{f} + B\partial_y \tilde{f}] = 0 \tag{13.17}$$

Example:

$$x\partial_x u + 2\partial_y u - 2u = 0, \; u = h(x,y)f(p) \tag{13.18}$$

1. Solve:

$$A\partial_x f + B\partial_y f = 0 \tag{13.19}$$

$$\frac{dy}{dx} = \frac{B}{A} = \frac{2}{x} \tag{13.20}$$

$$\implies \frac{dy}{2} = \frac{dx}{x} \implies \frac{y}{2} = \ln(x) + c \tag{13.21}$$

$$\implies x = Ae^{\frac{y}{2}} \text{ - constant at } xe^{-\frac{y}{2}} \tag{13.22}$$

$$f = f\left(xe^{-\frac{y}{2}}\right) \tag{13.23}$$

2. Find any h that solve equation. Try $h = h(x)$:

$$xh' - 2h = 0 \implies h = x^2 \tag{13.24}$$

3. General solution is:

$$u = hf = x^2\left(xe^{-\frac{y}{2}}\right) \tag{13.25}$$

2. Let's find another h, e.g. look for $h = h(y)$:

$$2h' - 2h = 0 \implies h = e^y \tag{13.26}$$

$$\implies u = e^y f\left(xe^{-\frac{y}{2}}\right) \tag{13.27}$$

Warning: *"You should not get emotionally attached to what you call 'f'"*

13.1.2 Terminology

- "Homogeneous problem" -
  1. An equation is said to the homogeneous if f is a solution then λf is also a solution (λ constant)
  2. boundary is homogeneous - if f satisfies boundary conditions, λf also does
- Solution of inhomogeneous problem:

$$f = f^{\text{generic}}_{\text{homogeneous}} + g^{\text{particular}} \tag{13.28}$$

Example:

$$\partial_x u - x\partial_y u + u = f, \; u(0,y) = g(y) \tag{13.29}$$

1. Solve homogeneous problem
2. Find any particular solution which respects boundary conditions

Lecture 14

14.1 Second Order Linear PDEs

$$A(x,y)\partial_x^2 u + B(x,y)\partial_x\partial_y u + C(x,y)\partial_y^2 u + D(x,y)\partial_x u + E(x,y)\partial_y u + F(x,y)u = G(x,y) \tag{14.1}$$

- This is the most general problem - hard to deal with
- From now on, deal with much simpler cases:
  - $G = F = D = E = 0$
  - $A, B, C \to$ constants
- Notation

$$b^2 - 4ac \begin{cases} > 0 & \text{hyperbolic} \\ = 0 & \text{parabolic} \\ < 0 & \text{elliptic} \end{cases} \tag{14.2}$$

- First order $\to$ look for $u = u(p(x,y))$
- Same strategy here $\to$ look for $u(x,y) = u(p),\ p(x,y) = \alpha x + \beta y \implies p(x,y) = x + \lambda y$

$$A\partial_x^2 u + B\partial_x\partial_y u + C\partial_y^2 = 0 \tag{14.3}$$

$$u = u(p),\ p = x + \lambda y \tag{14.4}$$

$$\implies \partial_x u = \frac{du}{dp}\frac{\partial p}{\partial x} = \frac{du}{dp} \tag{14.5}$$

$$\implies \partial_x^2 u = \partial_x\left[\frac{du(p)}{dp}\right] = \frac{d^u}{dp^2}\partial_x p = \frac{d^u}{dp^2} \tag{14.6}$$

$$\implies \partial_y[\partial_x u] = \lambda\frac{d^u}{dp^2} \tag{14.7}$$

$$\implies \partial_y^2 u = \lambda^2\frac{d^u}{dp^2} \tag{14.8}$$

$$A\frac{d^2u}{dp^2} + B\lambda\frac{d^u}{dp^2} + C\lambda^2\frac{d^u}{dp^2} = 0 \tag{14.9}$$

$$\implies \left(\frac{d^u}{dp^2}\right)[A + B\lambda + C\lambda^2] = 0 \tag{14.10}$$

- Looking for non trivial solution: $\frac{d^u}{dp^2} \neq 0$
- Two solutions, λ_1, λ_2
- General solution of PDE - $u(x,y) = f(x + \lambda_1 y) + g(x + \lambda_2 y)$

Example: 1D Wave Equation

$$\partial_x^2 u - \frac{1}{c^2}\partial_t^2 u = 0 \tag{14.11}$$

- $A = 1, B = 0, C = \frac{1}{c^2} \implies B^2 - 4AC > 0 \implies$ hyperbolic

$$1 - \frac{1}{c^2}\lambda = 0 \implies \lambda = \pm c \tag{14.12}$$

$$\implies u(x,t) = f(x - ct) + g(x - ct) \tag{14.13}$$

Example: 2D Laplace Equation

$$\partial_x^2 u + \partial_y^2 u = 0 \tag{14.14}$$

- $A = C = 1, B = 0 \implies$ elliptic

$$\lambda = \pm\frac{\sqrt{-4}}{2} = \pm i \tag{14.15}$$

$$\implies u(x, y) = f(x + iy) + g(x - iy) \tag{14.16}$$

Example:

$$\partial_x^2 u + 2\partial_x\partial_y u + \partial_y^2 u = 0,\ \ A = 1, B = 2, C = 1 \tag{14.17}$$

$$\lambda = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{-2 \pm \sqrt{4 - 4}}{2} = -1 \tag{14.18}$$

$$\implies u = f(x - y) + g(?) \tag{14.19}$$

See if $xg(x - y)$ is solution:

$$\partial_x^2[xg(x - y)] = \partial_x[g(x - y) + xg'(x - y)] = g'(x - y) + g'(x - y) + xg''(x - y) \tag{14.20}$$

$$\partial_x\partial_y[xg(x - y)] = -g'(x - y) - xg'' \tag{14.21}$$

$$\partial_y^2[xg(x - y)] = xg'' \tag{14.22}$$

Plugging this into the equation shows it is a solution

$$u(x, y) = f(x - y) + xg(x - y) \tag{14.23}$$

14.1.1 The Wave Equation

The derivation is trivial, find online if needed
In one dimension:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{14.24}$$

Generally,

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \nabla^2 u \tag{14.25}$$

Must know the general form by heart

14.1.2 Diffusion Equation

- Density, ρ
- Thermal conductivity, κ
- Specific heat, s

How temperature field evolves, $u(x, t)$
Heat flux through surface, S: $\kappa(\underline{\nabla} \cdot u) \cdot \hat{n}$

$$\frac{dQ}{dt} = \kappa \int_S dS\,(\underline{\nabla} \cdot u) \cdot \hat{n} \tag{14.26}$$

$$\kappa \int_V \underline{\nabla} \cdot [\underline{\nabla} u]\, dV = \kappa \int_V \nabla^2 u\, dV \tag{14.27}$$

$$Q = \int \rho\, su(x, t)\, dV \implies \frac{dQ}{dt} = \int \frac{\partial u}{\partial t}\rho s\, dV \tag{14.28}$$

$$\frac{\partial u}{\partial t} = \frac{\kappa}{\rho s}\nabla^2 u \tag{14.29}$$

$$= \mathcal{K}\nabla^2 u \tag{14.30}$$

Lecture 15

15.1 Diffusion Equation

$$\frac{du}{dt} = \mathcal{K}\nabla^2 u \qquad \text{(3D Diffusion Relation)}$$
$$\frac{du}{dt} = \mathcal{K}\partial_x^2 u \qquad \text{(1D Diffusion Relation)}$$

Let's solve it:
$f(x+\lambda y)$ will not work, try to make a dimensionless variable using $x, t, \mathcal{K}$
$\eta = \frac{x^2}{\mathcal{K}t}$ - this is dimensionless from the 1D Diffusion Relation, since $\frac{1}{t} = \frac{\mathcal{K}}{x^2}$
Try $p = \frac{x^2}{\mathcal{K}t}$:

$$\partial_x u = \frac{du}{dp}\partial_x p = \frac{du}{dp}\left[\frac{2x}{\mathcal{K}t}\right] \tag{15.1}$$
$$\partial_x^2 u = \partial_x\left[\frac{du}{dp}\frac{2x}{\mathcal{K}t}\right] \tag{15.2}$$
$$= \left[\frac{d^u}{dp^2}\partial_x p\right]\frac{2x}{\mathcal{K}t} + \frac{du}{dp}\frac{2}{kt} \tag{15.3}$$
$$= \left(\frac{2x}{\mathcal{K}t}\right)^2\frac{d^2u}{dp^2} + \frac{2}{\mathcal{K}t}\frac{du}{dp} \tag{15.4}$$
$$\partial_t u = \frac{du}{dp}\partial_t p = \frac{du}{dp}\left[-\frac{x^2}{\mathcal{K}t^2}\right] \tag{15.5}$$
$$\implies 4f''\frac{x^2}{\mathcal{K}t^2} + f'\left[\frac{2}{t} + \frac{x^2}{\mathcal{K}t^2}\right] = 0 \tag{15.6}$$
$$4f''\eta + f'[2+\eta] = 0 \tag{15.7}$$
$$\frac{f''}{f'} = -\frac{1}{2\eta} - \frac{1}{4} \tag{15.8}$$
$$\frac{d\ln(f')}{d\eta} = -\frac{1}{2}\frac{d\ln(\eta)}{d\eta} - \frac{1}{4} \tag{15.9}$$
$$\implies \frac{d[\ln(\sqrt{\eta}f')]}{d\eta} = -\frac{1}{4} \tag{15.10}$$
$$\implies \ln(\sqrt{\eta}f') = -\frac{1}{4}\eta + c \tag{15.11}$$
$$f' = \frac{A}{\sqrt{\eta}}e^{-\frac{\eta}{4}} \tag{15.12}$$
$$f = A\int \frac{1}{\sqrt{\eta}}e^{-\frac{\eta}{4}}\,d\eta \tag{15.13}$$
$$\zeta = \frac{\sqrt{\eta}}{2} \implies d\zeta = \frac{1}{4}\left[\frac{1}{\sqrt{\eta}}d\eta\right] \tag{15.14}$$
$$\implies f(\zeta) = B\int_{\zeta_0}^{\zeta} e^{-\frac{\zeta'}{2}}\,d\zeta' \tag{15.15}$$
$$\implies \zeta = \frac{x}{2\sqrt{\mathcal{K}t}} \tag{15.16}$$

- For $t = 0, \zeta \to \infty, u = f(\zeta) = c \in \mathbb{R}$
- For $x = 0, \zeta = 0$ for any t - if we choose $\zeta_0 = 0, u(0,t) = 0$

Diffusion steady state:

$$\nabla^2 u = 0 \tag{Laplace Equation}$$
$$\nabla^2 u = \rho(\underline{x}) \tag{Poisson Equation}$$

For physics to work, the following must be positive:

$$\partial_t u = \mathcal{K}\nabla^2 u \tag{15.17}$$

Schrodinger apparently looks just like the diffusion equation:

$$-\hbar\partial_t\psi = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\tilde{x},t)\right]\psi \tag{15.18}$$

15.2 1D Wave Equation

$$\partial_x^2 u - \frac{1}{c^2}\partial_t^2 u = 0 \tag{15.19}$$
$$\implies u = f(x-ct) + g(x+ct) \tag{15.20}$$

What if $f = g$?
At $t = 0, f(x) = g(x) = A\cos(kx+\epsilon)$

$$u = A\sin(k[x-ct]+\epsilon) + A\sin(k[x+ct]+\epsilon) \tag{15.21}$$
$$= 2A\cos(kct)\cos(kx+\epsilon) \tag{15.22}$$

What about the general solution? boundary conditions?

$$u = f(x+ct) + g(x-ct) \tag{15.23}$$

At $t = 0$, typical boundary conditions would be position and velocity:

$$u(x,t)|_{t=0} = \phi(x) \tag{position}$$
$$\partial_t u(x,t)|_{t=0} = \psi(x) \tag{velocity}$$

Is this enough to completely find the solution? Yes.

$$u(x,t) = f(x+ct) + g(x-ct) \tag{15.24}$$
$$f(x) + g(x) = \phi(x) \tag{15.25}$$
$$cf'(x) - cg'(x) = \psi(x) \tag{15.26}$$
$$f(x) - g(x) = \frac{1}{c}\int_{x_0}^{x} dx'\,\psi(x') + k \tag{15.27}$$
$$(15.2) + (15.4) \implies 2f(x) = \phi(x) + \frac{1}{c}\int_{x_0}^{x} dx'\,\psi(x') + k \tag{15.28}$$
$$f(x) = \frac{\phi(x)}{2} + \frac{1}{2c}\int_{x_0}^{x+ct} dx'\,\psi(x') + \frac{k}{2} \tag{15.29}$$
$$g(x) = \frac{\phi(x)}{2} - \frac{1}{2c}\int_{x_0}^{x-ct} dx'\,\psi(x') - \frac{k}{2} \tag{15.30}$$
$$u(x,t) = f(x+ct) + g(x-ct) \tag{15.31}$$
$$= \frac{\phi(x+ct)}{2} + \frac{1}{2c}\int_{x_0}^{x} dx'\,\psi(x') + \frac{k}{2} + \frac{\phi(x-ct)}{2} - \frac{1}{2c}\int_{x_0}^{x} dx'\,\psi(x') - \frac{k}{2} \tag{15.32}$$
$$= \frac{\phi(x+ct)+\phi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} dx'\,\psi(x') \tag{15.33}$$

This is the general solution for the wave equation.

Lecture 16

16.1 First Order Boundary Conditions

Recall ODEs - if you know the expansion of the function in x_0 and its derivatives.

Same problem in PDEs:
Consider a boundary condition - $u(x,y) = \phi$ on the curve, C.
Spread the curve, must know how it changes with x and y.
Do we know $\partial_x u$ and $\partial_y u$?

Let us consider $A(x,y)\partial_x u + B(x,y)\partial_y u = F(x,y)$ - we need to know two boundary conditions, but only one equation.
We know how the function changes along C.

$$\frac{d\phi}{dS} = \frac{d}{dS}u(x,y)\Big|_{\text{on } C} = \partial_x u\frac{dx}{dS} + \partial_y u\frac{dy}{dS} \tag{16.1}$$

Now we have two equations and two unknowns.
We can find $\partial_x u$ and $\partial_y u$ unless the two equations are linearly dependent.

$$\underbrace{\begin{pmatrix} A(x,y) & B(x,y) \\ \frac{dx}{dS} & \frac{dy}{dS} \end{pmatrix}}_{M} \begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix} = \begin{pmatrix} F \\ \frac{d\phi}{dS} \end{pmatrix} \tag{16.2}$$

If M^{-1} exists then,

$$M \begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix} = \begin{pmatrix} F \\ \frac{d\phi}{dS} \end{pmatrix} \tag{16.3}$$

$$\cancel{M^{-1}M} \begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix} = M^{-1} \begin{pmatrix} F \\ \frac{d\phi}{dS} \end{pmatrix} \tag{16.4}$$

I can find $\partial_x u, \partial_y u \iff |M| \neq 0$

$$A\frac{dy}{dS} - B\frac{dx}{dS} = 0 \tag{16.5}$$

$$A\frac{dy}{dx} = B \tag{16.6}$$

$$\implies \frac{dy}{dx} = \frac{B}{A} \tag{16.7}$$

This is the equation for p - characteristic line along which information spreads.

Example:

$$x\partial_x u - 2y\partial_y u = 0 \tag{16.8}$$

$u = 2y + 1$ for $x = 1$ (or 2?) with $y \in [0,1]$

$$p \implies \frac{dy}{dx} = \frac{B}{A} = -\frac{2y}{x} \tag{16.9}$$

General solution: $u(x,y) = f(x^2 y)$
$x^2 y = c$ - characteristic

Last time, without the $y \in [0,1]$ restriction the solution was $u = 2x^2 + 1 + g(x^2 y)$ sch that $g(p) = 0$ for $p \in [0,1]$

If you have characteristics that a boundary condition crosses multiple times, there are no solutions - $u = 0$ everywhere.

16.2 Second Order Boundary Conditions

$$A\partial_x^2 u + B\partial_x\partial_y u + C\partial_y^2 u = F \tag{16.10}$$

In analogy with ODE: $u(x,y) = \phi(s)$ on c - Cauchy boundary condition.

$$\frac{\partial u}{\partial n} u(x,y) = \psi(n) \text{ on } c \tag{16.11}$$

- $u(x,y) = \phi \implies$ Dirichlet
- $\frac{\partial u}{\partial n} = \psi \implies$ Neumann

Consider Cauchy:

$$u|_c \phi, \partial_n u|_c = \psi \tag{16.12}$$

Can I find $\partial_x^2 u, \partial_y^2 u, \partial_x\partial_y u$?

$$\frac{\partial u}{\partial S} = \underline{\nabla} u \cdot \frac{d\underline{r}}{dS} = \partial_x u \frac{dx}{dS} + \partial_y u \frac{dy}{dS} = \phi' \tag{16.13}$$

$$\frac{\partial u}{\partial} = \underline{\nabla} \cdot \frac{d\hat{n}}{dS} = \partial_x u \frac{dy}{dS} - \partial_y u \frac{dx}{dS} = \psi \tag{16.14}$$

$$d\hat{r} = dx\hat{i} + dy\hat{j} \tag{16.15}$$

$$dS\hat{n}?(\hat{n})^2 = 1 \implies (dS\hat{n})^2 = dS^2 = dx^2 + dy^2 \tag{16.16}$$

$$\hat{n} \cdot d\underline{r} = 0 \because \hat{n} \perp d\underline{r} \tag{16.17}$$

$$\implies dS\hat{n} = dy\hat{i} - dx\hat{j} \tag{16.18}$$

Now we have two equations and two unknowns - $\partial_x u = K, \partial_y u = K'$
We want to find second derivatives so differentiate:

$$\frac{d}{dS}[\partial_x u] = \frac{dK}{dS} \tag{16.19}$$

$$\partial_x^2 u \frac{dx}{dS} + \partial_y \partial_x u \frac{dy}{dS} = \frac{dK}{dS} \tag{16.20}$$

$$\partial_x\partial_y u \frac{dx}{dS} + \partial_y^2 u \frac{dy}{dS} = \frac{dK'}{dS} \tag{16.21}$$

$$\begin{pmatrix} A & B & C \\ \frac{dx}{dS} & \frac{dy}{dS} & 0 \\ 0 & \frac{dx}{dS} & \frac{dy}{dS} \end{pmatrix} \begin{pmatrix} \partial_x^2 u \\ \partial_x\partial_y u \\ \partial_y^2 u \end{pmatrix} = \begin{pmatrix} F \\ \frac{dK}{dS} \\ \frac{dK'}{dS} \end{pmatrix} \tag{16.22}$$

Solution $\iff$ det $\neq 0$

$$A\left(\frac{dy}{dS}\right)^2 - B\frac{dx}{dS}\frac{dy}{dS} + C\left(\frac{dx}{dS}\right)^2 = 0 \tag{16.23}$$

$$A\left(\frac{dy}{dx}\right)^2 - B\frac{dx}{dS}\frac{dy}{dS} + C = 0 \tag{16.24}$$

Example:

$$\partial_x^2 u - \partial_t^2 u = 0 \to f(x+t) + g(x-t) \tag{16.25}$$

- Hyperbolic equation - Cauchy on open boundary
- Parabolic equation - Either Dirichlet or Neumann, open conditions
- Elliptic equation - Dirichlet or Neumann, closed

Lecture 17

17.1 Separation of Variables

$$u(t,x,y,z) = T(t)X(x)Y(y)Z(z) \tag{17.1}$$

This strategy does no always work.
When it does work it leads to enormous simplifications.
In many "physical" cases, this works:

1. Wave equation
2. Schrodinger equation
3. Diffusion

Example: 3D Wave Equation

$$\nabla^2 u = \frac{1}{c^2}\partial_t^2 u \to u = T(t)X(x)Y(y)Z(z) \tag{17.2}$$

$$TYZ\partial_x^2 X + TXZ\partial_y^2 Y + TXY\partial_z^2 Z = \frac{1}{c^2}XYZ\partial_t^2 T \tag{17.3}$$

$$\implies \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = \frac{1}{c^2}\frac{T''}{T} \tag{17.4}$$

$$\implies \frac{1}{c^2}\frac{T''}{T} = k \tag{17.5}$$

$$\implies \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = k \tag{17.6}$$

This is the separation constant, can be positive, negative, Real, or Complex.
For now, assume $k < 0, k = -u^2$:

$$\frac{1}{c^2}\frac{T''}{T} = -u^2 \tag{17.7}$$

$$\implies T'' + u^2c^2T = 0 \tag{17.8}$$

$$\implies T = Ae^{iuct} + Be^{-iuct} \tag{17.9}$$

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = -u^2 \tag{17.10}$$

$$\frac{X''}{X} = -u^2 - \frac{Y''}{Y} - \frac{Z''}{Z} \tag{17.11}$$

$$= k' = -l^2 \tag{17.12}$$

$$X'' + l^2X = 0 \tag{17.13}$$

$$X = Ce^{ilx} + De^{-ilx} \tag{17.14}$$

$$\implies \frac{Y''}{Y} = -m^2; \ \frac{Z''}{Z} = -n^2 \tag{17.15}$$

$$\implies -l^2 - m^2 - n^2 = -u^2 \tag{17.16}$$

$$\implies u = TXYZ = \left(Ae^{iuct} + Be^{-iuct}\right)\left(Ce^{ilx} + Be^{-ilx}\right)\left(Ee^{imy} + Fe^{-imy}\right)\left(Ge^{inx} + Fe^{-inx}\right) \tag{17.17}$$

Imagine the boundary conditions such that $A = D = F = H = 0$:

$$u = e^{i[lx+my+nz-uct]} = e^{i[\underline{k}\cdot\underline{r}-uct]} \tag{17.18}$$

Example: Diffusion in 1D

$$\partial_t u = \mathcal{K}\partial_x^2 u \tag{17.19}$$
$$u(x,t) = T(t)X(x) \tag{17.20}$$
$$XT' = \mathcal{K}TX'' \tag{17.21}$$
$$\frac{T'}{T} = \mathcal{K}\frac{X''}{X} \tag{17.22}$$
$$\frac{T'}{T} = \alpha \tag{17.23}$$
$$T = T_0 e^{\alpha t},\ \ \alpha < 0, \alpha = -\lambda^2 \tag{17.24}$$
$$\frac{T'}{T} = -\lambda^2 \tag{17.25}$$
$$T = T_0 e^{-\lambda^2 t} \tag{17.26}$$
$$\frac{T'}{T} = -\lambda^2;\ \frac{X''}{X} = -\frac{1}{\mathcal{K}}\lambda^2 \tag{17.27}$$
$$\implies X = A\sin\left(\frac{\lambda}{\sqrt{\mathcal{K}}}x\right) + B\cos\left(\frac{\lambda}{\mathcal{K}}x\right) \tag{17.28}$$
$$u_\lambda = e^{-\lambda^2 t}\left[\tilde{A}\sin\left(\frac{\lambda}{\sqrt{\mathcal{K}}}x\right) + \tilde{B}\cos\left(\frac{\lambda}{\mathcal{K}}x\right)\right] \tag{17.29}$$

"Please don't make a bomb" - Caola, 2018

Any choice of separation constant has a solution.
If the equation is linear, then a linear combination of solutions is a solution. Therefore, for linear equations:

$$u = \sum_\lambda c_\lambda u_\lambda \tag{17.30}$$

Example:
In plane, polar coordinates:

$$\nabla^2 u = 0 \tag{17.31}$$
$$\nabla^2 = \frac{1}{r}\partial_r[r\partial_r] + \frac{1}{r^2}\partial_\psi^2 \tag{17.32}$$
$$\nabla^2 u = \frac{\phi}{r}\partial_r[r\partial_r R] + \frac{R}{r^2}\partial_\psi^2\phi = 0 \tag{17.33}$$

- ➤ Separation in terms that only depend on R and terms that only depend on ϕ.
- ➤ Divide by ϕ:

$$\frac{1}{r}\partial_r[r\partial_r R] + \frac{R}{r^2}\frac{\partial_\psi^2\phi}{\phi} = 0 \tag{17.34}$$

- ➤ Divide by $\frac{R}{r^2}$:

$$\frac{r}{R}\partial_r[r\partial_r R] + \frac{\partial^2\psi\phi}{\phi} = 0 \tag{17.35}$$
$$\frac{r}{R}\partial_r[r\partial_r R] = k = n^2,\ \ n \in \mathbb{C} \tag{17.36}$$
$$n^2 + \frac{\partial^2\psi\phi}{\phi} = \implies \frac{\partial_\psi^2\phi}{\phi} = -n^2 \tag{17.37}$$
$$\phi = Ae^{in\psi} + Be^{-in\psi} \tag{17.38}$$
$$r\partial_r[r\partial_r R] - n^2 R = 0 \tag{17.39}$$
$$rR' + r^2R'' - n^2R = 0 \rightarrow \text{ - Euler equation} \tag{17.40}$$

- Try r^λ:

$$\lambda(\lambda-1)r^\lambda + \lambda r^\lambda - n^2 r^\lambda = 0 \tag{17.41}$$

$$\implies \lambda^2 = n^2 \implies \lambda = \pm n \tag{17.42}$$

$$\implies R = Ar^n + Br^{-n} \tag{17.43}$$

- General solution for separation constant, n^2:

$$u_n = \left(Ae^{in\psi} + Be^{-in\psi}\right)\left(Cr^n + Dr^{-n}\right) \tag{17.44}$$

- General solution:

$$u = \sum_n c_n u_n \tag{17.45}$$