Advanced Theoretical Physics

Dr Kendon and Prof Gardiner

Michaelmas Term 2019 - Epiphany Term 2020

Contents

Part I

Quantum Optics

Lecture 1

Syllabus:

- ➤ quantization of light
- ➤ creation and annihilation operators
- ➤ Hamiltonian of the E field
- ➤ number states
- ➤ coherent states
- ➤ squeezed states

- ➤ photon bunching and anti-bunching
- ➤ density operator
- ➤ pure states, mixed sates, entangled states
- ➤ decoherence
- ➤ atom-light interactions
- ➤ applications

```
                    slow      energy      fast

                 thermodynamics          special &
          large                       general relativity

                classical mechanics

          size ─────────────────────────────────────

                 quantum optics

          small
                                    quantum field theory
                statistical mechanics
```
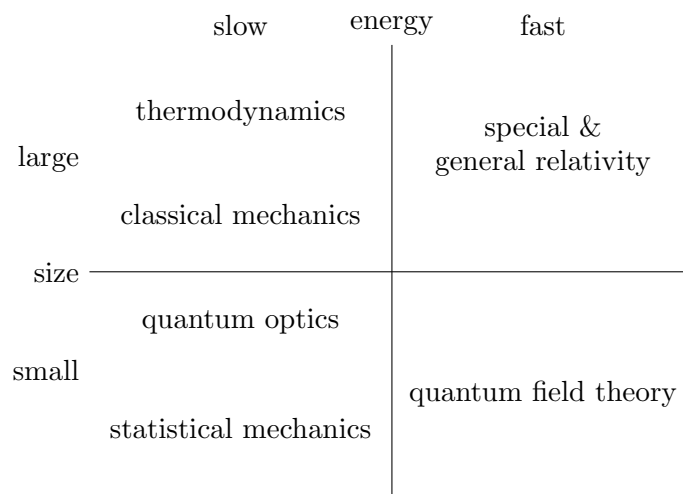
Ingredients:

- ➤ harmonic oscillators
- ➤ Gaussian integrals
- ➤ Hamiltonian mechanics (canonical variables q and p)
- ➤ maths of operators - adjoint, self-adjoint, Hermitian, commutation relations
- ➤ QM in both Schrodinger and Heisenberg pictures
- ➤ density matrices
- ➤ classical EM - Maxwell's equations in Coulomb gauge - especially plane waves and dipoles

Hanbury Brown and Tiss:

$$G(\tau) = I_A(t)I_B(t + \tau) \tag{1.1}$$

Lecture 2

2.1 Learning Outcomes

To be able to state, explain and apply the operator formalism of the quantum harmonic oscillator, including:

➤ the Hamiltonian in terms of the creation and annihilation operators
➤ the number operator and number states, eigenstates of the Hamiltonian
➤ definition of the creation and annihilation operators, commutation relations, and the adjoint and self-adjoint operators
➤ mathematical properties of the number states, completeness
➤ systems of two or more independent oscillators

2.2 Quantum Harmonic Oscillator

$$F = ma = m\ddot{x} \tag{2.1}$$
$$= -kx \tag{2.2}$$
$$x(t) = x_0 \sin \omega t \tag{2.3}$$
$$p_x(t) = p_0 \cos \omega t \tag{2.4}$$
$$V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2 \tag{2.5}$$
$$\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi \tag{2.6}$$
$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \tag{2.7}$$
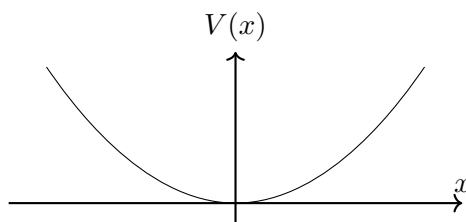
Start with writing the Hamiltonian, then turn everything into operators

$$H = \frac{p^2}{2m} + \underbrace{\frac{1}{2}m\omega^2 x^2}_{V(x)} \tag{2.8}$$
$$p \to \hat{p} = -i\hbar\frac{d}{dx}, \ x \to \hat{x} \tag{2.9}$$
$$[\hat{x}, \hat{p}] = i\hbar \tag{2.10}$$
$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \tag{2.11}$$
$$\hat{a} = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\hat{x} + i\hat{p}) \tag{2.12}$$
$$\hat{a}^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\hat{x} - i\hat{p}) \tag{2.13}$$

$$\hat{x} = \left(\frac{\hbar}{2m\omega}\right)^{1/2} (\hat{a} + \hat{a}^\dagger) \tag{2.14}$$

$$\hat{p} = -i \left(\frac{m\hbar\omega}{2}\right)^{1/2} (\hat{a} - \hat{a}^\dagger) \tag{2.15}$$

$$[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1 \tag{2.16}$$

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}) \tag{2.17}$$

$$\hat{a}^\dagger\hat{a} = \hat{n}, \ \hat{n}|n\rangle = n|n\rangle \tag{2.18}$$

$$\hat{H}|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle = E_n|n\rangle \tag{2.19}$$

How do the annihilation and creation operators, $\hat{a}$ and $\hat{a}^\dagger$ interact with the number states, $|n\rangle$?

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \tag{2.20}$$

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \tag{2.21}$$

Together, the creation and annihilation operators are known as the *ladder operators.* Ladder operators move the system up or down the energy levels of the harmonic potential.
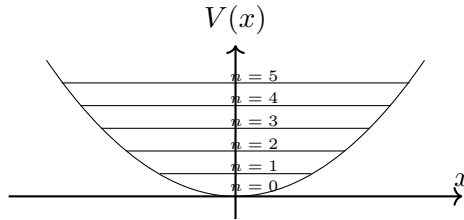
We now have a partly new mathematical representation. Notice that the potential still remains positive, it does not go negative. Therefore we must have:

$$\hat{a}|0\rangle = 0, \tag{2.22}$$

$$\hat{n} = \hat{a}^\dagger\hat{a}|0\rangle = 0. \tag{2.23}$$

$$\implies \hat{H}|0\rangle = E_0|0\rangle = \frac{1}{2}\hbar\omega|0\rangle \tag{2.24}$$

So the ground state is labelled '0' but does not have $E = 0$.
Now we introduce $\hat{O}^\dagger$ as the adjoint of $\hat{O}$ if

$$\langle\psi|\hat{O}|\phi\rangle = \langle\phi|\hat{O}^\dagger|\psi\rangle^* \ \forall\psi, \phi \tag{2.25}$$

A self-adjoint operator is equivalent to a Hermitian operator, i.e. $\hat{n}, \hat{H}$.
For adjoint operators:

$$(\hat{A} + \hat{B})^\dagger = \hat{A}^\dagger + \hat{B}^\dagger \tag{2.26}$$

$$(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger \tag{2.27}$$

$$(c\hat{A})^\dagger + c^*\hat{A}^\dagger \tag{2.28}$$

$$(\hat{A}^\dagger)^\dagger = \hat{A} \tag{2.29}$$

More on the number states:

➤ they are orthogonal

$$\langle n|n\rangle = 1 \tag{2.30}$$

$$\langle n|m\rangle = 0, \ n \neq m \tag{2.31}$$

$$\langle n|m\rangle = \delta_{n,m} \tag{2.32}$$

$$\tag{2.33}$$

➤ they form a basis (note: not mathematically a Hilbert space, but a Banah(?) space)

$$|\psi\rangle = \sum_n c_n |n\rangle \tag{2.34}$$

$$0 \le n \le \infty \tag{2.35}$$

2.3 Two Oscillators - independent

$$|\psi_0\rangle = \sum_n c_n |n\rangle_0 \tag{2.36}$$

$$|\psi_1\rangle = \sum_m c_m |m\rangle_1 \tag{2.37}$$

$$|\psi_{01}\rangle = \sum_{n,m} c_{n,m} |n\rangle_0 |m\rangle_1 \tag{2.38}$$

What we are doing is "tensoring" the Hilbert spaces: $\mathcal{H}_0 \otimes \mathcal{H}_1$:

$$|n\rangle_0 |m\rangle_1 \equiv |n\rangle_0 \otimes |m\rangle_1. \tag{2.39}$$

Now we have the operators, $\hat{a}_0, \hat{a}_0^\dagger, \hat{a}_1, \hat{a}_1^\dagger$:

$$\hat{a}_0 \otimes \mathbb{I}_1, \ \mathbb{I}_0 \otimes \hat{a}_1, \ldots \tag{2.40}$$

$$[\hat{a}_0, \hat{a}_1] = [\hat{a}_0, \hat{a}_1^\dagger] = 0 \tag{2.41}$$

$$\hat{H} = \hat{H}_0 \otimes \mathbb{I}_1 + \mathbb{I}_1 \otimes \hat{H}_1 \tag{2.42}$$

Note this is for non-interacting oscillators. For interacting,

$$\hat{H} = \hat{H}_0 \otimes \mathbb{I}_1 + \mathbb{I}_1 \otimes \hat{H}_1 + \mathcal{H}_{int}. \tag{2.43}$$

Lecture 3

3.1 <u>Learning Outcomes</u>

To be familiar with the route to quantisation of the electromagnetic field, in particular to:

- ➤ Explain and state the description of the electromagnetic field in terms of modes and polarization
- ➤ Be familiar with the equivalence between a mode of the field and a quantum harmonic oscillator
- ➤ To explain the form of (but not derive) expressions for the Hamiltonian of the electromagnetic field, and the electric and magnetic fields in terms of the creation and annihilation operatoes
- ➤ To recognise and explain the concepts of the Schrodinger and Heisenberg representations, and to explain which is being applied
- ➤ To explain and apply the concepts of adjoint and self-adjoint operators and their matrix elements

3.2 <u>Quantising the EM field</u>

Consider an EM scalar potential, $\phi = 0$ (no free charges), and a vector potential, $\underline{A}$.

$$\mathbf{E}(\underline{r}, t) = \frac{\partial}{\partial t}\underline{A} \qquad\qquad\qquad \mathbf{B}(\underline{r}, t) = \nabla \times \underline{A}(\underline{r}, t) \tag{3.1}$$

$$\underline{\nabla}\left[\underline{\nabla} \cdot \underline{A}\right] - \nabla^2\underline{A} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\underline{A} = 0 \tag{3.2}$$

Coulomb gauge, $\nabla \cdot \underline{A} = 0$.

$$\underline{A} = \sum_{\underline{k}}\left\{\underline{A}_{\underline{k}}\exp\left[i(\underline{k} \cdot \underline{r} - \omega_k t)\right] + \underline{A}_{\underline{k}}^*\exp\left[-i(\underline{k} \cdot \underline{r} - \omega_k t)\right]\right\} \tag{3.3}$$

$$\omega_k = c|k|, \ \underline{k} \cdot \underline{A}_k = 0 \tag{3.4}$$

Polarisation vectors, $\underline{e}_{k1}, \underline{e}_{k2}$ - orthonormal vectors perpendicular to $\underline{k}$.

$$\underline{A}_k = A_{k1}\underline{e}_{k1} + A_{k2}\underline{e}_{k2} \tag{3.5}$$

$$\underline{A} = \sum_{\underline{k},s}A_{\underline{k},s}\underline{e}_{\underline{k},s}\exp\left\{i(\underline{k} \cdot \underline{r} - \omega_k t)\right\} + A_{\underline{k},s}^*\underline{e}_{\underline{k},s}\exp\left\{-i(\underline{k} \cdot \underline{r} - \omega_k t)\right\} \tag{3.6}$$

The labels of the modes are $\underline{k}, s, s \in 1, 2$. They gives us the: direction; wavelength, $\frac{2\pi}{|\underline{k}|}$; and polarisation, s.

To quantise this classically:

$$H = \frac{1}{2}\epsilon_0\int\left(\mathbf{E} \cdot \mathbf{E} + c^2\mathbf{B} \cdot \mathbf{B}\right)\,dV \tag{3.7}$$

$$= 2\epsilon_0 V\sum_{\underline{k},s}\omega_k^2 A_{\underline{k},s}A_{\underline{k},s}^* \tag{3.8}$$

$$A_{\underline{k},s} = \frac{1}{2\omega_k\sqrt{\epsilon_0 V}}\left\{\omega_k q_{\underline{k},s} + ip_{\underline{k},s}\right\} \tag{3.9}$$

$$A_{\underline{k},s}^* = \frac{1}{2\omega_k\sqrt{\epsilon_0 V}}\left\{\omega_k q_{\underline{k},s} - ip_{\underline{k},s}\right\} \tag{3.10}$$

$q_{\underline{k},s}, p_{\underline{k},s}$ canonical coordinates (x, p).

$$H_{\underline{k},s} = \frac{1}{2}\left(p_{\underline{k},s}^2 + \omega_k^2 q_{\underline{k},s}^2\right) \tag{3.11}$$

Harmonic oscillator $m = 1$, $x \leftrightarrow p$. To transfer this from classical to quantum, you simply convert everything to its operator form. For a single mode:

$$\hat{H}_{\underline{k},s} = \left(\hat{a}_{\underline{k},s}^\dagger\hat{a}_{\underline{k},s} + \frac{1}{2}\right)\hbar\omega_k \tag{3.12}$$

$$[\hat{a}_{\underline{k},s}, \hat{a}^\dagger_{\underline{k},s}] = 1 \tag{3.13}$$

$$\hat{a}^\dagger_{\underline{k},s}\hat{a}_{\underline{k},s} = \hat{n}_{\underline{k},s} \tag{3.14}$$

Now we have eigenstates, $|n\rangle_{\underline{k},s}$. Note: modes are not always equal to photons, but you can have photons spread over several modes.

Going back on the substitution:

$$\hat{A}_{\underline{k},s} = \sqrt{\frac{\hbar}{2\omega_k\epsilon_0 V}}\hat{a}_{\underline{k},s} \qquad\qquad \hat{A}^\dagger_{\underline{k},s} = \sqrt{\frac{\hbar}{2\omega_k\epsilon_0 V}}\hat{a}^\dagger_{\underline{k},s} \tag{3.15}$$

From these, we can find the quantised electric and magnetic field expressions. We will mostly be concerned with the electric field throughout this course as it has a much stronger interaction with matter than the magnetic.

$$\hat{\underline{\mathbf{E}}}_{\underline{k},s}(\underline{r},t) = i\left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right)^{1/2}\underline{e}_{\underline{k},s}\left[\hat{a}_{\underline{k},s}\exp\{i(\underline{k}\cdot\underline{r} - \omega_k t)\} - \hat{a}^\dagger_{\underline{k},s}\exp\{-i(\underline{k}\cdot\underline{r} - \omega_k t)\}\right] \tag{3.16}$$

3.3   Multimode Fields

$$\hat{H}_{\underline{k},s} = \sum_{\underline{k},s}\hbar\omega_k\left(\hat{a}^\dagger_{\underline{k},s}\hat{a}_{\underline{k},s} + \frac{1}{2}\right) \tag{3.17}$$

So the modes are independent of each other, but will interact through matter. We have a basis of

$$|n_1 n_2 n_3 \dots\rangle \equiv |n_1\rangle_{\underline{k}1,s} \otimes |n_2\rangle_{\underline{k}2,s} \otimes \dots \tag{3.18}$$

Now we can write the electric field operator:

$$\hat{\underline{\mathbf{E}}}(\underline{r},t) = \sum_{\underline{k},s}\hat{\underline{\mathbf{E}}}_{\underline{k},s}(\underline{r},t) \tag{3.19}$$

$$= \sum_{\underline{k},s}i\left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right)^{1/2}\underline{e}_{\underline{k},s}\left\{\hat{a}_{\underline{k},s}\exp[i(\underline{k}\cdot\underline{r} - \omega_k t)] + \hat{a}^\dagger_{\underline{k},s}\exp[-i(\underline{k}\cdot\underline{r} - \omega_k t)]\right\} \tag{3.20}$$

This is written in the Heisenberg representation. Now if we look at the expectation value, for one mode of the electric field

$$_{\underline{k},s}\langle n|\hat{\underline{\mathbf{E}}}(\underline{r},t)|n'\rangle_{\underline{k},s} \tag{3.21}$$

This is time dependent as seen by the field operator and will oscillate in time through some means. As a reminder, consider an operator in the Heisenberg picture:

$$\hat{O}_H(t) = \hat{U}^\dagger(t,t_0)\hat{O}\hat{U}(t,t_0) \tag{3.22}$$

$$\hat{U}(t,t_0) = \exp\left[-i\frac{\hat{H}(t-t_0)}{\hbar}\right] \tag{3.23}$$

Lecture 4

4.1 Learning Outcomes

Quantum States of the EM field. We will cover three important families of quantum states of light: number states, coherent states, and squeezed states. After studying this part, you should:
- ➤ recognise each of these types of states, and be able to provide a simple definition
- ➤ be familiar with the expansion of coherent states in terms of number states
- ➤ be able to calculate expectation values and variances of physical quantities (e.g. electric field, photon number, quadratures, etc) for number states and coherent states
- ➤ be familiar with and able to define the quadrature operators
- ➤ be able to sketch phase space diagrams of coherent states and squeezed states
- ➤ to provide and explain the definition of a quadrature squeezed state and its key properties
- ➤ to describe the uncertainty relation and the concept of minimum uncertainty states

4.2 Single Mode Fields

$$\hat{n}_{\underline{k},s} = \hat{a}^{\dagger}_{\underline{k},s} \hat{a}_{\underline{k},s} \tag{4.1}$$

$$\hat{H}_{\underline{k},s} = \left(\hat{n}_{\underline{k},s} + \frac{1}{2} \right) \hbar\omega_k \tag{4.2}$$

The eigenstates are the number states, a.k.a Fock states, $|n\rangle_{\underline{k},s}$.

$$\hat{n}_{\underline{k},s}|n\rangle_{\underline{k},s} = n|n\rangle_{\underline{k},s} \tag{4.3}$$

$$\hat{H}_{\underline{k},s}|n\rangle_{\underline{k},s} = \hbar\omega_k \left(n + \frac{1}{2} \right) |n\rangle_{\underline{k},s} \tag{4.4}$$

It follows that the vacuum state has energy as well - $|0\rangle_{\underline{k},s}$, with energy $\frac{1}{2}\hbar\omega_k$.

$$\hat{a}^{\dagger}_{\underline{k},s}|0\rangle_{\underline{k},s} = |1\rangle_{\underline{k},s} \tag{4.5}$$

$$\hat{a}_{\underline{k},s}|1\rangle_{\underline{k},s} = |0\rangle_{\underline{k},s} \tag{4.6}$$

4.3 Multimode fields

For multimode states, we just do the sum of all of these states.

$$\hat{H} = \sum_{\underline{k},s} \left(\hat{n}_{\underline{k},s} + \frac{1}{2} \right) \hbar\omega_k \tag{4.7}$$

$$|n_0 n_1 n_2 \dots n_k \dots\rangle = |n_0\rangle_0 \otimes |n_1\rangle \otimes \cdots \otimes |n_k\rangle_k \otimes \dots \tag{4.8}$$

The modes cannot interact with each other, so are independent unless there is matter to interact with.

$$\left[\hat{a}^{\dagger}_{\underline{k},s}, \hat{a}_{\underline{k}',s'} \right] = \delta_{kk'}\delta_{ss'} \tag{4.9}$$

$$_{\underline{k},s}\langle n_k|n_{k'}\rangle_{\underline{k}',s} = 0, \ \forall k \neq k', s \neq s' \tag{4.10}$$

4.4 Electric Field of a Single Field Number State

What is the expectation value, $\langle \hat{\underline{\mathbf{E}}}_{\underline{k},s}(\underline{r},t)\rangle$?

$$_{\underline{k},s}\langle n|\hat{\underline{\mathbf{E}}}_{\underline{k},s}(\underline{r},t)|n\rangle_{\underline{k},s} = \underline{e}_{\underline{k},s}i \left(\frac{\hbar\omega_k}{2\epsilon_0 V} \right)^{1/2} \left[_{\underline{k},s}\langle n|\hat{a}_{\underline{k},s}|n\rangle_{\underline{k},s}e^{i(\underline{k}\cdot\underline{r}-\omega_k t)} - _{\underline{k},s}\langle n|\hat{a}^{\dagger}_{\underline{k},s}|n\rangle_{\underline{k},s}e^{-i(\underline{k}\cdot\underline{r}-\omega_k t)} \right] \tag{4.11}$$

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \implies \langle n|n-1\rangle = 0 \tag{4.12}$$

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \implies \langle n|n+1\rangle = 0 \tag{4.13}$$

$$\implies \langle \hat{\underline{\mathbf{E}}}_{\underline{k},s}(\underline{r},t)\rangle = 0, \ \forall n \tag{4.14}$$

I.e. the mean electric field is zero for number states. Now consider the expectation value for the square of the electric field, $\langle \hat{\underline{\mathbf{E}}}_{\underline{k},s}(\underline{r},t)^2 \rangle$.

$$\langle \hat{\underline{\mathbf{E}}}_{\underline{k},s}(\underline{r},t)^2 \rangle = 2\left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right)\left(n+\frac{1}{2}\right) \tag{4.15}$$

Now we can work out the variance:

$$\langle (\Delta\underline{\mathbf{E}})^2 \rangle = \langle \hat{\underline{\mathbf{E}}}^2 \rangle - \langle \hat{\underline{\mathbf{E}}} \rangle^2 = 2\left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right)\left(n+\frac{1}{2}\right) \tag{4.16}$$

$$H = \frac{1}{2}\epsilon_0 \int \left(\underline{\mathbf{E}}^2 + \frac{1}{c^2}\underline{\mathbf{B}}^2\right) dV \tag{4.17}$$

From analogy to the classical Hamiltonian, we can see how the variance would be related to the energy. The vacuum state fluctuates in its energy around zero, and this has observable effects.

4.5 Electric Field in Multimode Fields

For a multimode field's vacuum,

$$\langle (\Delta\underline{\mathbf{E}})^2 \rangle = \sum_{\underline{k},s} \left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right). \tag{4.18}$$

This term sums to infinity, which can be a problem, and leads to some effects:
- ➤ The Lamb shift in atoms' energy levels
- ➤ The Casimir effect causes a series of modes to form in between two plates that can be zero at the plates, while the modes outside the plates have no restrictions. The difference in the two areas of modes causes a force that pushes the plates together. The Casimir effect has a classical analogue when boats close to a harbor wall are pushed into the wall by the difference in waves from out in the open water and between the boat and the wall.

4.6 The Number States - A Summary

- ➤ Complete orthonormal basis
- ➤ Well defined photon number and energy
- ➤ Zero mean electric field - not at all classical
- ➤ E-field fluctuates, even for $|0\rangle$, and E-field fluctuations increase with n
- ➤ Most non-classical states you can get, and you can do experiments with them to show this using single or a few photons in the modes

Lecture 5

5.1   States with Classical Limits

$$E_x(z,t) \propto \sin(kx - \omega t) \tag{5.1}$$

$$\langle n|\hat{a}|n\rangle = \langle n|\hat{a}^\dagger|n\rangle = 0 \tag{5.2}$$

$$|\psi\rangle = \sum_n c_n|n\rangle \tag{5.3}$$

We want to look for eigenstates of $\hat{a}$:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \alpha \in \mathbb{C} \tag{5.4}$$

We did this because right eigenstates of $\hat{a}^\dagger$ do not exist.

$$\hat{a}^\dagger|\beta\rangle \neq \beta|\beta\rangle, \forall \beta \in \mathbb{C} \tag{5.5}$$

$$\langle \alpha|\hat{a}^\dagger = \alpha^*\langle\alpha| \tag{5.6}$$

However, $\hat{a}^\dagger$ does form eigenstates with left states.

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n|n\rangle \tag{5.7}$$

$$\hat{a}|\alpha\rangle = \sum_{n=1}^{\infty} c_n\sqrt{n}|n-1\rangle = \alpha\sum_{n=0}^{\infty} c_n|n\rangle \tag{5.8}$$

$$= \sum_{n=0}^{\infty} c_{n+1}\sqrt{n+1}|n\rangle = \alpha\sum_{n=0}^{\infty} c_n|n\rangle \tag{5.9}$$

$$c_{n+1} = \frac{\alpha}{\sqrt{n+1}}c_n \tag{5.10}$$

Use the fact that the states are normalised to find c_0:

$$\langle\alpha|\alpha\rangle = 1 \rightarrow c_0 \tag{5.11}$$

$$= |c_0|^2 \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha^*)^m}{\sqrt{m!}}\frac{(\alpha)^n}{\sqrt{n!}}\langle m|n\rangle \tag{5.12}$$

$$|c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = |c_0|^2 \exp|\alpha|^2 = 1 \tag{5.13}$$

If we take c_0 to be real and positive,

$$|\alpha\rangle = \exp\left\{-\frac{|\alpha|^2}{2}\right\}\sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{5.14}$$

These are the eigenstates of $\hat{a}$ with eigenvalue $\alpha \in \mathbb{C}$, $\alpha = 0 \implies |0\rangle$.

$$\langle\alpha|\hat{a}|\alpha\rangle = \alpha \tag{5.15}$$

$$\langle\alpha|\hat{a}^\dagger|\alpha\rangle = \alpha^* \tag{5.16}$$

The states $|\alpha\rangle$ we found are right eigenstates of $\hat{a}$ and left eigenstates of $\hat{a}^\dagger$,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \tag{5.17}$$

14

$$\langle\alpha|\hat{a}^\dagger = \alpha^*\langle\alpha| \tag{5.18}$$

So what is the mean photon number of these states, $\langle\alpha|\hat{n}|\alpha\rangle$?

$$\langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = |\alpha|^2 \tag{5.19}$$

So α behaves like a mean amplitude. What we want to calculate now is the variance of photon number, i.e. $\langle(\Delta n)^2\rangle = \langle n^2\rangle - \langle n\rangle^2$.

$$\langle n\rangle^2 = |\alpha|^4 \tag{5.20}$$

$$\langle\alpha|\hat{n}^2|\alpha\rangle = \langle\alpha|\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}|\alpha\rangle \tag{5.21}$$

We wish to normal order Eq (5.21) by switching the middle two operators. Using commutators, we have:

$$[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1 \tag{5.22}$$

$$\implies (5.21) = \langle\alpha|\hat{a}^\dagger(\hat{a}^\dagger\hat{a} + 1)\hat{a}|\alpha\rangle \tag{5.23}$$

$$= \langle\alpha|\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}|\alpha\rangle + \langle\alpha|\hat{a}^\dagger\hat{a}|\rangle \tag{5.24}$$

$$= |\alpha|^4 + |\alpha|^2 \tag{5.25}$$

$$\langle(\Delta n)^2\rangle = |\alpha|^2 \tag{5.26}$$

$$\langle n\rangle = \langle(\Delta n)^2\rangle \tag{5.27}$$

So the mean is equal to the variance. This is what is described in the Poisson distribution.

$$\frac{\Delta n}{n} = \langle n\rangle^{-1/2} \tag{5.28}$$

So this gets smaller as n gets larger, i.e. more classical for large n.

5.2 The electric field of a more classical state

$$\langle\hat{\underline{\mathbf{E}}}_x(z,t)\rangle = \langle\alpha|\hat{\underline{\mathbf{E}}}_x(z,t)|\alpha\rangle \tag{5.29}$$

$$= \langle\alpha|\left\{\left(\frac{i\hbar\omega_k}{2\epsilon_0 V}\right)^{1/2}\underline{e}_{\underline{k},x}\left[\hat{a}\exp(i(kz - \omega_k t)) - \hat{a}^\dagger\exp(-i(kz0\omega_k t))\right]\right\}|\alpha\rangle \tag{5.30}$$

$$= 2|\alpha|\left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right)^{1/2}\sin\{\omega t - kz - \theta\}\underline{e}_{\underline{k},x}, \quad \alpha = |\alpha|e^{i\theta} \tag{5.31}$$

$$\langle\hat{\underline{\mathbf{E}}}_x^2(z,t)\rangle = \frac{\hbar\omega_k}{2\epsilon_0 V}\left[1 + 4|\alpha|^2\sin^2(\omega t - kz - \theta)\right] \tag{5.32}$$

$$\langle(\Delta\hat{\underline{\mathbf{E}}}_x(z,t))^2\rangle = \frac{\hbar\omega_k}{2\epsilon_0 V} \tag{5.33}$$

This variance has no α in it, so the $|\alpha\rangle$ states only have vacuum fluctuations. The variance of coherent states = minimum possible, and doesn't depend on α or n. This looks like classical EM field, with $|\alpha|$ the amplitude, and $\langle n\rangle = |\alpha|^2$, but with an uncertainty fluctuation around the path.
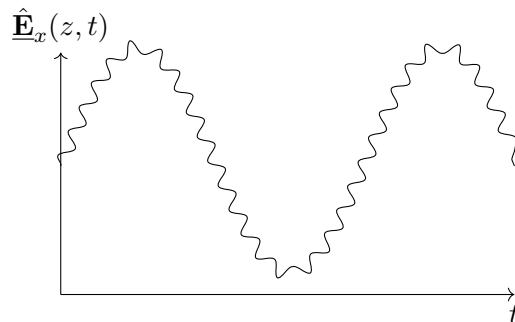
$$|\alpha\rangle = \exp\left\{-\frac{|\alpha|^2}{2}\right\} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{5.34}$$

$$|\beta\rangle = \exp\left\{-\frac{|\beta|^2}{2}\right\} \sum_{m=0}^{\infty} \frac{\beta^m}{\sqrt{m!}} |m\rangle \tag{5.35}$$

$$\langle\beta|\alpha\rangle = \exp\left\{-\frac{|\beta|^2 + |\alpha|^2}{2}\right\} \sum_{m,n} \frac{(\beta^*)^m (\alpha)^n}{\sqrt{m!n!}} \langle m|n\rangle \tag{5.36}$$

$$= \exp\left\{-\frac{|\beta|^2 + |\alpha|^2}{2}\right\} \sum_n \frac{(\beta^*\alpha)^n}{n!} \tag{5.37}$$

$$= \exp\left\{-\frac{|\beta|^2 + |\alpha|^2}{2}\right\} \exp\left\{\beta^*\alpha\right\} \tag{5.38}$$

$$\langle\beta|\alpha\rangle^2 = \exp\left\{-|\alpha - \beta|^2\right\} \tag{5.39}$$

So two states will depend on how much they differ from one another, they form an over-complete basis.

$$\frac{1}{\pi} \int |\alpha\rangle\langle\alpha| \, d^2\alpha = \sum_n |n\rangle\langle n| = \mathbb{I} \tag{5.40}$$

$$|\phi\rangle = \frac{1}{\pi} \int |\alpha\rangle\langle\alpha|\phi\rangle \, d^2\alpha \tag{5.41}$$

Lecture 6

$$\langle \hat{\underline{\mathbf{E}}}_x(z,t) \rangle = 2|\alpha| \left(\frac{\hbar \omega_k}{3\epsilon_0 V} \right)^{1/2} \sin \{\omega t - kz - \theta\} \, \underline{e}_{\underline{k},\alpha} \tag{6.1}$$

$$\alpha = |\alpha| e^{i\theta} \tag{6.2}$$

Now back to thinking about the harmonic oscillator, using $\hat{p}, \hat{q}$.

$$\hat{X}_1 = \frac{1}{2} \left(\hat{a} + \hat{a}^\dagger \right), \, \hat{q} \tag{6.3}$$

$$\hat{X}_2 = \frac{1}{2i} \left(\hat{a} - \hat{a}^\dagger \right), \, \hat{p} \tag{6.4}$$

$$[\hat{X}_1, \hat{X}_2] = \frac{i}{2} \tag{6.5}$$

$$\langle \hat{X}_1 \rangle_\alpha = \langle \alpha | \hat{X}_1 | \alpha \rangle = \frac{1}{2} \left\{ \langle \alpha | \hat{a} | \alpha \rangle + \langle \alpha | \hat{a}^\dagger | \alpha \rangle \right\} \tag{6.6}$$

$$= \frac{1}{2} \left\{ \alpha + \alpha^* \right\} = \mathrm{Re}(\alpha) \tag{6.7}$$

$$\langle \hat{X}_2 \rangle_\alpha = \frac{1}{2i} \left\{ \alpha - \alpha^* \right\} = \mathrm{Im}(\alpha) \tag{6.8}$$

$$\langle (\Delta \hat{X}_1)^2 \rangle = \langle (\Delta \hat{X}_2)^2 \rangle = \frac{1}{4} \tag{6.9}$$

Consider two Hermitian operators, $\hat{A}, \hat{B}$, then the modulus commutator is:

$$\Delta A \cdot \Delta B \geq \frac{1}{2} \left| \langle \psi | [\hat{A}, \hat{B}] | \psi \rangle \right| \tag{6.10}$$

$$(\Delta A)^2 = \langle (\Delta A)^2 \rangle = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2 \tag{6.11}$$

$$\langle (\Delta \hat{X}_1)^2 \rangle \langle (\Delta \hat{X}_2)^2 \rangle \geq \frac{1}{4} \left| [\hat{X}_1, \hat{X}_2] \right|^2 = \frac{1}{16} \tag{6.12}$$

For coherent states, Eq (6.12) is equal to $\frac{1}{16}$. So the coherent state is a minimum uncertainty state, where the remaining uncertainty is all in the vacuum fluctuations. We can represent the classical harmonic oscillator in phase space, but we want to do the same thing for a quantum harmonic oscillator, for a state $|\alpha\rangle$:
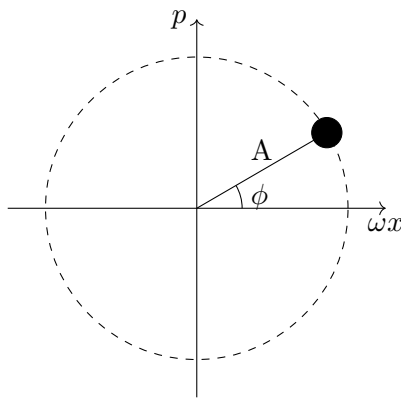
Classical HO
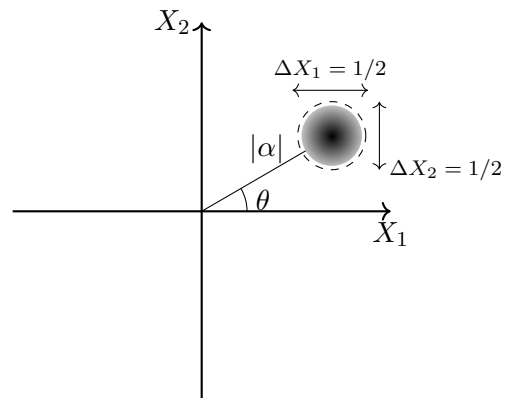
Quantum HO

How does the vacuum state look in this picture?

$$\langle (\Delta \hat{X}_1)^2 \rangle \langle (\Delta \hat{X}_2)^2 \rangle \geq \frac{1}{16} \tag{6.13}$$

So if you make one of these smaller than $\frac{1}{4}$, you can satisfy this equation by making the other larger - this forms squeezed states. You can also squeeze the vacuum state.
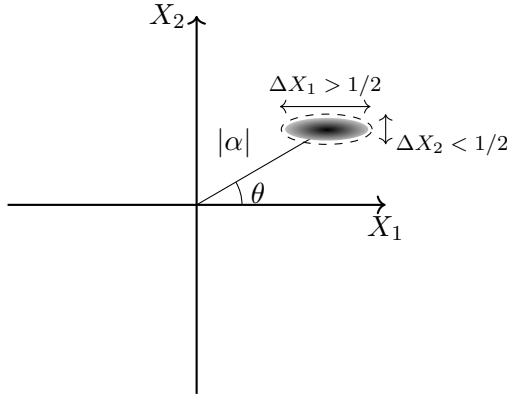
Squeezed quadrature state
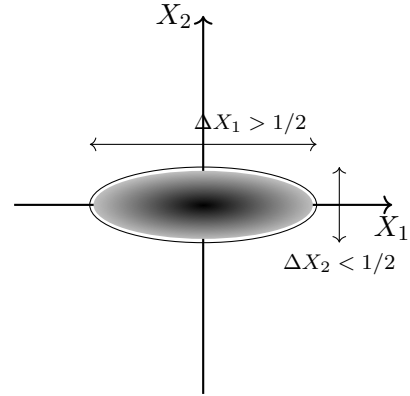
Squeezed vacuum state

You can squeeze the amplitude and make the phase more uncertain by showing the ellipse at an angle:
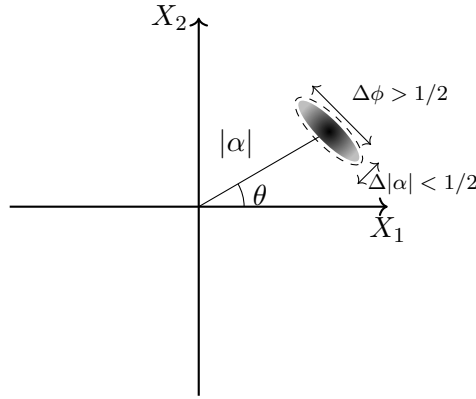
Squeezed phase state

To create these states, you need a new operator:

$$\hat{s}(\epsilon) = \exp\left[\frac{1}{2}\left(\epsilon^*\hat{a}^2 - \epsilon(\hat{a}^\dagger)^2\right)\right] \tag{6.14}$$

You can create this operator using:

➤ non-linear crystals
➤ atomic gases
➤ 4-wave mixing

$$\hat{H} = \underbrace{\hbar\omega\hat{a}^\dagger\hat{a}}_{\text{signal mode}} + \underbrace{\hbar\omega_p\hat{b}^\dagger\hat{b}}_{\text{pump mode}} + i\hbar\chi^{(2)}\left(\hat{a}^2\hat{b}^\dagger - (\hat{a}^\dagger)^2\hat{b}\right) \tag{6.15}$$

$\chi^{(2)}$ is known as the second order non-linear susceptibility. So we will treat the pump mode as almost classical,

$$\hat{b} \to \beta e^{i\omega_p t} \qquad\qquad \hat{b}^\dagger \to \beta^* e^{-i\omega_p t} \qquad\qquad \eta = \chi^{(2)}\beta \tag{6.16}$$

From this, we can write our Hamiltonian in the parametric approximation:

$$\hat{H}^{(PA)} = \hbar\omega\hat{a}^\dagger\hat{a} + i\hbar\left(\eta^*\hat{a}^2 e^{i\omega_p t} - \eta(\hat{a}^\dagger)^2 e^{-i\omega_p t}\right) \tag{6.17}$$

Then in the interaction picture,

$$\hat{H}_{int} = i\hbar \left(\eta^* \hat{a}^2 e^{i(\omega_p - 2\omega)t} - h.c. \right) \tag{6.18}$$

$$= i\hbar \left(\eta^* \hat{a}^2 - \eta(\hat{a}^\dagger)^2 \right), \ \omega_p = 2\omega \tag{6.19}$$

$$U = \exp\left(-\frac{i}{\hbar} \hat{H}_{int} \right) = \exp\left(\eta^* \hat{a}^2 - \eta(\hat{a}^\dagger)^2 \right) \tag{6.20}$$

Lecture 7

7.1 Learning Outcomes

Atom-light interaction and a model for photodetection. We will develop a fully quantum mechanical model for the interaction of light with atoms. After studying this part, you should:

➤ Be familiar with the atom-light interaction Hamiltonian in the electric dipole approximation, and explain this approximation.
➤ Be familiar with the Pauli operators as a quantum description of the two-level atom.
➤ Explain the physical basis of the Jaynes-Cummings model for atom-light interactions. Explain and use, but not derive, the resulting Hamiltonian.
➤ Be familiar with the concept of dressed state, and vacuum Rabi splitting in the context of the Jaynes Cummings Hamiltonian.
➤ Be able to perform simple calculations with combinations of atomic and field operators.
➤ Provide a simple explanation of a photodetector in the context of quantum optics.
➤ Be familiar with the operator expression for the photodetection probability, and understand the meaning of normal ordering in this context.

7.2 Atom-Light Interactions

$$\hat{H} = \hat{H}_{atom} + \hat{H}_{field} + \hat{H}_{int} \tag{7.1}$$

$$\hat{H}_{atom} = \frac{\hat{p}^2}{2m} + \hat{V}(r) \tag{7.2}$$

So for the EM field, classical version in something of $\hat{A}$. Interaction: force on e^- due to EM field,

$$\underline{F} - q\left[\mathbf{E}(\underline{r}, t) + \underline{v} \times \mathbf{B}(\underline{r}, t)\right] \tag{7.3}$$

where $\underline{F}$ is the Lorentz force. *We do not need to know the derivation for this, only to use it.* This is known as the minimal coupling Hamiltonian. Choose Coulomb gauge, $\phi = 0$, $\nabla \cdot A = 0$, but note that this is not Lorentz invariant. Now we apply the electric dipole approximation, where light $\gg$ atoms.

$$A_{\underline{k},s}(\underline{r}, t) = A_{\underline{k},s} \exp\left[i(\underline{k} \cdot \underline{r} 0 \omega_k t)\right] + c.c. \tag{7.4}$$

The length scale of the light in this is given by the wavevector, $\approx |\underline{k}| = 2\pi/\lambda$. For visible light, $\lambda \approx 400 - 700\,nm$; for microwaves, $\lambda \approx 1\,cm$. But for atoms, $\approx 0.1\,nm$, so as far as atoms are concerned, photons are huge. If $\underline{r}$ is the size of the atom, $\underline{k} \cdot \underline{r} \ll 1$. So $A(\underline{r}, t) \approx A(t)$ over the size of the atom.

$$\hat{H}_{int} = e\hat{\underline{r}} \cdot \hat{\mathbf{E}}(t) \tag{7.5}$$

The sign of e is positive, so e^- is $-e$. $e\hat{\underline{r}} = \hat{\underline{d}}$, the electric dipole operator.

$$\hat{H}_{int} = \hat{\underline{d}} \cdot \hat{\mathbf{E}}(t) \tag{7.6}$$

So we want a single mode field, and the best way to think about this is place the light into a cavity.

$$\hat{\mathbf{E}} = e_{\underline{k}} \left(\frac{\hbar \omega_k}{\epsilon_0 V}\right)^{1/2} (\hat{a} + \hat{a}^\dagger) \sin(kx) \tag{7.7}$$
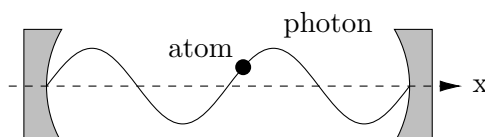
Two level atom - the two that match EM field.

$$\hat{H} = \hat{H}_{atom} + \hat{H}_{field} + \hat{H}_{int} \tag{7.8}$$

$$\hat{H}_{field} = \hbar\left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right) \tag{7.9}$$

The $\frac{1}{2}$ is the zero-point energy. A two-level atom is equivalent to a spin-$\frac{1}{2}$ particle.

$$\hat{\sigma}_+ = |e\rangle\langle g| \qquad\qquad \hat{\sigma}_- = |g\rangle\langle e| = \hat{\sigma}_+^\dagger \tag{7.10}$$

$$\hat{\sigma}_z = |e\rangle\langle e| - |g\rangle\langle g| \tag{7.11}$$

$$\hat{\sigma}_+ = \hat{\sigma}_x + i\hat{\sigma}_y \qquad\qquad \hat{\sigma}_- = \hat{\sigma}_x - \hat{\sigma}_y \tag{7.12}$$

Now we can write the Hamiltonian for the atom:

$$\hat{H}_{atom} = \frac{1}{2}(E_e - E_g)\hat{\sigma}_z = \frac{1}{2}\hbar\omega_0\hat{\sigma}_z \tag{7.13}$$

$$\hat{H}_{int} = -\underline{\hat{d}} \cdot \underline{\hat{\mathbf{E}}} = g\hat{d}(\hat{a} + \hat{a}^\dagger) \tag{7.14}$$

$$g = -\left(\frac{\hbar\omega}{\epsilon_0 V}\right)^{1/2}\sin(kx) \tag{7.15}$$

$$\langle g|\underline{\hat{d}}|g\rangle = \langle e|\underline{\hat{d}}|e\rangle = 0 \tag{7.16}$$

$$\underline{\hat{d}} = d|g\rangle\langle e| + d^*|e\rangle\langle g| \tag{7.17}$$

$$= d(\hat{\sigma}_+ + \hat{\sigma}_-) \tag{7.18}$$

$$\hat{H}_{int} = \hbar\left(\frac{dg}{\hbar}\right)(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a} + \hat{a}^\dagger) \tag{7.19}$$

$$\hat{H} = \frac{1}{2}\hbar\omega_0\hat{\sigma}_z + \hbar\omega_k\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \hbar\lambda(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a} + \hat{a}^\dagger) \tag{7.20}$$

Note that λ here is not a wavelength but just the constant from Eq (7.19).

7.3 Rotating Wave Approximation

So in the Heisenberg picture, we have

$$\hat{a}(t) = \hat{a}(0)e^{-i\omega t} \qquad\qquad \hat{a}^\dagger(t) = \hat{a}^\dagger(0)e^{+i\omega t} \tag{7.21}$$

$$\hat{\sigma}_+(t) = \hat{\sigma}_+(0)e^{+i\omega_0 t} \qquad\qquad \hat{\sigma}_-(t) = \hat{\sigma}_-(0)e^{-i\omega_0 t} \tag{7.22}$$

$\hat{H}_{int}$ has:

$$\hat{\sigma}_+\hat{a} \approx e^{i(\omega_0 - \omega)t} \qquad\qquad \hat{\sigma}_-\hat{a}^\dagger \approx e^{i(w - \omega_0)t} \tag{7.23}$$

$$\hat{\sigma}_+\hat{a}^\dagger \approx e^{i(\omega + \omega_0)t} \qquad\qquad \hat{\sigma}_-\hat{a} \approx e^{-i(\omega + \omega_0)t} \tag{7.24}$$

Eqs (7.23) oscillate slowly and conserve energy, while Eqs (7.24) oscillate fast and do not conserve energy. So we want to drop the fast oscillating terms as these are not useful to our picture of the Hamiltonian, and this is the rotating wave approximation. From this, we arrive at the Jaynes-Cummings Hamiltonian:

$$\hat{H}_{JC} = \frac{1}{2}\hbar\omega_0\hat{\sigma}_z + \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \hbar\lambda(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger) \tag{7.25}$$

This describes:

➤ energy of atom
➤ energy of photons/cavity mode (single mode)
➤ how energy can be exchanged between atom and cavity mode

$$\hat{\sigma}_+\hat{a}|g\rangle|n\rangle \implies |e\rangle|n-1\rangle \tag{7.26}$$

$$\hat{\sigma}_-\hat{a}^\dagger|e\rangle|n\rangle \implies |g\rangle|n+1\rangle \tag{7.27}$$

So these behave like raising and lowering operators as they shift the atom around the cavity.

So what assumptions did we use?

➤ electron can't go anywhere else outside the cavity
➤ number of photons is fixed, such that there were no other losses to outside the cavity
➤ only need $|g\rangle, |e\rangle$ for the two levels of the atom and $|n\rangle, |n+1\rangle$ for the states of the photons
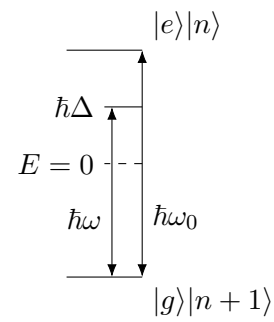
We can then form two basis states:

$$|\psi_1, n\rangle \equiv |e\rangle|n\rangle = |i\rangle \qquad\qquad |\psi_2, n\rangle \equiv |g\rangle|n+1\rangle = |f\rangle \qquad (7.28)$$

$$\hat{H}(n) = \hbar \begin{pmatrix} n\omega + \frac{1}{2}\omega_0 & \lambda\sqrt{n+1} \\ \lambda\sqrt{n+1} & (n+1)\omega - \frac{1}{2}\omega_0 \end{pmatrix} \qquad (7.29)$$

Matrix elements obtained from $\langle i|\hat{H}(n)|i\rangle$, etc.

Lecture 8

8.1   Rotating Wave Approx Ctd

We will take $\Delta = 0$, and for an initial state $|i\rangle = |e\rangle|n\rangle$,

$$|\psi(t)\rangle = c_i(t)|i\rangle + c_f(t)|f\rangle \tag{8.1}$$

$$E_i = \frac{1}{2}\hbar\omega + n\hbar\omega, \ E_f = -\frac{1}{2}\hbar\omega + (n+1)\hbar\omega \tag{8.2}$$

So the energies are equal, and conserved. We can then solve the Schrodinger equation for Eq (8.1), and find relations for the co-factors:

$$\dot{c}_i = -i\lambda\sqrt{n+1}c_f \qquad\qquad \ddot{c}_i + \lambda^2(n+1)c_i = 0 \tag{8.3}$$

$$c_i(t) = \cos\left(\lambda t\sqrt{n+1}\right) \qquad\qquad c_f(t) = i\sin\left(\lambda t\sqrt{n+1}\right) \tag{8.4}$$

$$P_i(t) = |c_i(t)|^2 = \cos^2\left(\lambda t\sqrt{n+1}\right) \tag{8.5}$$

To figure this out, we can go through many different solutions, but the simplest in this problem is to use what is called the **atomic inversion**:

$$\omega(t) = \langle\psi(t)|\hat{\sigma}_z|\psi(t)\rangle \tag{8.6}$$

$$= P_i(t) - P_f(t) = \cos\left(2\lambda t\sqrt{n+1}\right) \tag{8.7}$$

Now we will define the **Rabi frequency**, as

$$\Omega(n) = 2\lambda\sqrt{n+1} \tag{8.8}$$

Note that there are even Rabi oscillations for $n = 0$. This may seem a strange possibility at first, but there are many experiments to prove this is in fact the case, where it occurs in our cavity model as the vacuum of the single mode focused on the atom. This can appear quite classical at first, but in the more general case as you increase the number of states involved, it will seem more quantum. We can write general descriptions for our ground and excited states:

$$|\psi_g(t)\rangle = -i\sin_{n=0}^{\infty} c_n \sin\left(\frac{1}{2}\Omega(n)\right)|n+1\rangle \tag{8.9}$$

$$|\psi_e(t)\rangle = \sum_{n=0}^{\infty} c_n \cos\left(\frac{1}{2}\Omega(n)t\right)|n\rangle \tag{8.10}$$

$$\omega(t) = \sum_{n=0}^{\infty} |c_n|^2 \cos\left(\Omega(n)t\right), \ c_n = \frac{\alpha^n}{\sqrt{n!}}e^{-|\alpha|^2/2} \tag{8.11}$$

For a single number state, P_i oscillates between ground and excited in a classical sinusoidal appearance, but when we consider a coherent state with average $\langle n\rangle$, it drops down into being a superposition of being ground and excited, then after a while excites into partial oscillations again, and continues this interesting pattern. *input graphs*

Now, we want to analyse the Jaynes-Cummings Hamiltonian through the calculation of the matrix elements instead, so we will start with our same two pairs of states, this time with $\Delta \neq 0$.

$$|i\rangle = |e\rangle|n\rangle \qquad\qquad |f\rangle = |g\rangle|n+1\rangle \tag{8.12}$$

$$\hat{H}(n) = \hbar\begin{pmatrix} n\omega + \frac{1}{2}\omega_0 & \lambda\sqrt{n+1} \\ \lambda\sqrt{n+1} & (n+1)\omega - \frac{1}{2}\omega_0 \end{pmatrix} \tag{8.13}$$

This has eigenenergies with the Rabi frequency, which is the same as before if $\Delta = 0$.

$$E_\pm = \hbar \left\{ \left(n + \frac{1}{2} \right) \omega \pm \frac{1}{2} \Omega_n(\Delta) \right\} \tag{8.14}$$

$$\Omega_n(\Delta) = \left[\Delta^2 + 4\lambda^2(n+1) \right]^{1/2} \tag{8.15}$$

We then have eigenstates,

$$|n,+\rangle = \cos\left(\frac{1}{2}\Phi_n\right)|i\rangle + \sin\left(\frac{1}{2}\Phi_n\right)|f\rangle \qquad |n,-\rangle = -\sin\left(\frac{1}{2}\Phi_n\right)|i\rangle + \cos\left(\frac{1}{2}\Phi_n\right)|f\rangle \tag{8.16}$$

We have used a new angle here, Φ_n, where this is defined as

$$\Phi_n = \arctan\left(\frac{2\lambda\sqrt{n+1}}{\Delta}\right) = \arctan\left(\frac{\Omega_n(0)}{\Delta}\right). \tag{8.17}$$

We are happy with this definition even in the limit of $\Delta \to 0$, as tans have known angles even at ∞. The eigenstates above are known as *dressed states* of the atom, so when the atom is in a single-mode field like this, the field changes the eigenstates so they become eigenstates of the whole system. This looks like the energies of the eigenstates being changed by the application of the field. The original eigenstates we considered, $|e\rangle|n\rangle$ and $|g\rangle|n+1\rangle$, are called *bare states*. These bare states do not have the same energy, unless $\Delta = 0$.
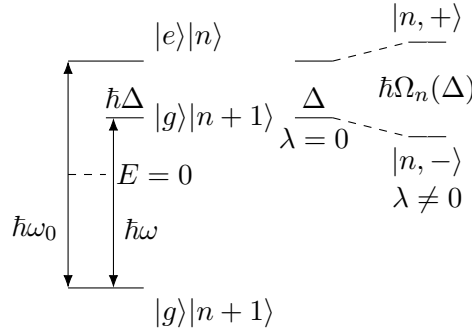
So if we define $\Delta = \omega_0 - \omega$, we can see that now Δ is the energy splitting between the states $|e\rangle|n\rangle$ and $|g\rangle|n+1\rangle$, but this is for $\lambda = 0$ and we can see the dependence on λ on these energy states through the Rabi frequency. It is then clear that this energy splitting Δ, will increase for the eigenenergies, $|n,-\rangle$ and $n,+\rangle$. This is known as a *Stark Shift*, since it is interacting through the E-field of light. Specifically, it is known as the AC or dynamic Stark shift, as the source of the electric field is not static, but dynamic.

So as $\Delta \to 0$, we are on resonance, and the bare states are degenerate, but the dressed states are still split.

$$\arctan\left(\frac{2\lambda\sqrt{n+1}}{\Delta \to 0}\right) \to \frac{\pi}{2} \qquad\qquad \cos\left(\frac{1}{2}\Phi_n\right) = \sin\left(\frac{1}{2}\Phi_n\right) \to \frac{1}{\sqrt{2}} \tag{8.18}$$

$$|n,+\rangle = \frac{1}{\sqrt{2}}\left(|e\rangle|n\rangle + |g\rangle|n+1\rangle\right) \qquad\qquad |n,-\rangle = \frac{1}{\sqrt{2}}\left(-|e\rangle|n\rangle + |g\rangle|n+1\rangle\right) \tag{8.19}$$
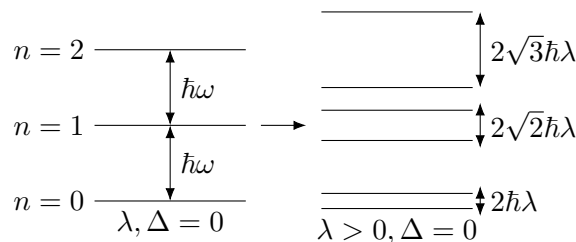
The phenomenom shown above is known as vacuum Rabi splitting, where we get anharmonic energy levels due to coupling as $\lambda \neq 0$. This is the underlying physics that allows us to engineer quantum technology by compressing waves down several orders of magnitude. This all begins as the light and atom when allowed to interact begin to behave as a composite system.

Lecture 9

9.1 Photodetection

We are interested in detecting low numbers of photons, so we will make use of the physics previously discussed, in terms of Hamiltonians

$$\hat{H} = \hat{H}_{field} + \hat{H}_{atom} + \hat{H}_{int} \tag{9.1}$$

$$\hat{H}_{field} = \sum_{\underline{k},s} \hbar\omega_k \left(\hat{a}^\dagger_{\underline{k},s}\hat{a}_{\underline{k},s} + \frac{1}{2} \right) \tag{9.2}$$

$$\hat{H}_{atom} = \hbar\omega_0\hat{\sigma}_z, \ |k| = \frac{2\pi}{\lambda} \tag{9.3}$$

$$\tag{9.4}$$

We then want to use the electric dipole approximation to build our interaction Hamiltonian,

$$\hat{H}_{int} = e\hat{\underline{r}} \cdot \hat{\underline{\mathbf{E}}} = -\hat{d} \cdot \hat{\underline{\mathbf{E}}}(\underline{R}, t) \tag{9.5}$$

We are aiming to ionise the atom, because then we have a free electron to detect as well. Once we have one electron freed, we can pull many more until we have multiple electrons to form a current, which we can detect. Of course to ionise the atom, we require the photon to add enough energy to free the electron.

Two types of photodetectors:

➤ Photomultiplier tube (PMT):
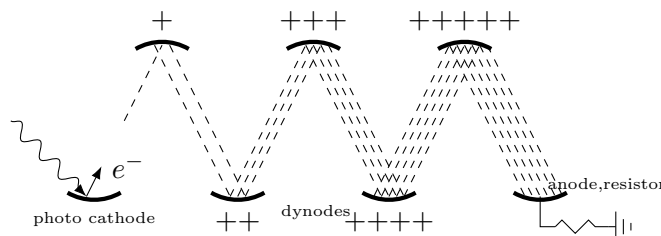
It is in a tube so it has a high vacuum. This tube can result in a gain $\approx 10^8$.

➤ Avalanche Photo Detector (APD)
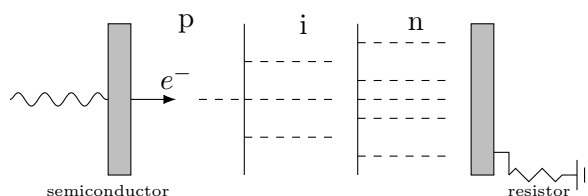
We do not have as high a gain here, but using a semiconductor instead of a high vacuum is more practical and easier to make.

For a free electron,

$$|e\rangle = \frac{1}{\sqrt{V}}e^{i\underline{q}\cdot\underline{r}} \tag{9.6}$$

So for our interaction Hamiltonian then,

$$\hat{H}_{int} = -\hat{d} \cdot \underline{\hat{\mathbf{E}}}(\underline{r},t) \tag{9.7}$$

$$\underline{\hat{\mathbf{E}}}(\underline{r},t) = i\sum_{\underline{k},s}\left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right)^{1/2}\underline{e}_{\underline{k},s}\left[\hat{a}^{\dagger}_{\underline{k},s}(t)e^{i\underline{k}\cdot\underline{r}} - \hat{a}_{\underline{k},s}(t)e^{-i\underline{k}\cdot\underline{r}}\right] \tag{9.8}$$

This is clearly in the Heisenberg picture as we have time-dependent operators. Here, $\underline{e}$ is the polarisation vector.

$$\hat{a}_{\underline{k},s}(t) = \hat{a}_{\underline{k},s}e^{-i\omega_k t} \qquad\qquad \hat{a}^{\dagger}_{\underline{k},s}(t) = \hat{a}^{\dagger}_{\underline{k},s}e^{i\omega_k t} \tag{9.9}$$

We want to split the electric field into two parts for absorption and emission:

$$\underline{\hat{\mathbf{E}}}_{+}(\underline{r},t) \equiv i\sum_{\underline{k},s}\left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right)^{1/2}\underline{e}_{\underline{k},s}\hat{a}_{\underline{k},s}(t)e^{i\underline{k}\cdot\underline{r}} \tag{9.10}$$

$$\underline{\hat{\mathbf{E}}}_{-}(\underline{r},t) \equiv -i\sum_{\underline{k},s}\left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right)^{1/2}\underline{e}_{\underline{k},s}\hat{a}^{\dagger}_{\underline{k},s}(t)e^{-i\underline{k}\cdot\underline{r}} \tag{9.11}$$

$$\tag{9.12}$$

These have been defined such that $\underline{\hat{\mathbf{E}}}_{-} \equiv \underline{\hat{\mathbf{E}}}^{\dagger}_{+}$ and vice versa; these operators have taken on the properties of the creation and annihilation operators. We only want to use $\underline{\hat{\mathbf{E}}}_{+}$ as this deals with absorption, which is what we're interested in.

$$\langle F|\hat{H}|I\rangle \approx -\underbrace{\langle e|\hat{d}|g\rangle}_{\text{atom}}\underbrace{\langle f|\underline{\hat{\mathbf{E}}}_{+}(\underline{r},t)|i\rangle}_{\text{field}} \tag{9.13}$$

The atom term above is equal to d, but the field term isn't so clear. We must sum over all possible final states to find this expectation value, using the **broadband detector assumption.**

$$P_f = |\langle F|\hat{H}|I\rangle|^2 \propto |\langle f|\underline{\hat{\mathbf{E}}}_{+}(\underline{r},t)|i\rangle|^2 \tag{9.14}$$

$$P_i \propto \sum_f |\langle f|\underline{\hat{\mathbf{E}}}_{+}(\underline{r},t)|i\rangle|^2 \tag{9.15}$$

$$\propto \sum_f \langle i|\underline{\hat{\mathbf{E}}}_{-}(\underline{r},t)|f\rangle\langle f|\underline{\hat{\mathbf{E}}}_{+}(\underline{r},t)|i\rangle \tag{9.16}$$

If this sum is over all possible f and doesn't depend on any of the other terms, then the $|f\rangle\langle f|$ will sum to the identity.

$$P_i \propto \langle i|\underline{\hat{\mathbf{E}}}_{-}(\underline{r},t)\underline{\hat{\mathbf{E}}}_{+}(\underline{r},t)|i\rangle \tag{9.17}$$

$$\underline{\hat{\mathbf{E}}}_{+}(\underline{r},t) \equiv i\sum_{\underline{k},s}\left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right)^{1/2}\underline{e}_{\underline{k},s}\hat{a}_{\underline{k},s}(t)e^{i\underline{k}\cdot\underline{r}} \tag{9.18}$$

Depending on what our state is, wrapping it around the electric field operators can produce some fairly non-trivial outcomes. Note that the ordering of these operators is important; if the operators were the other way around, they would be acting on the f bra/kets instead of the i bra/kets, and we would not have been able to sum and cancel the fs.

Reminding ourselves that we are trying to count photons, we can see that $\underline{\hat{\mathbf{E}}}_{-}\underline{\hat{\mathbf{E}}}_{+} \propto \hat{a}^{\dagger}\hat{a} = \hat{n}$, the number operator. Therefore our current result makes sense as it will indeed count photons, for discrete states or coherent states (detecting average number of photons instead). We will see more on this when we discuss correlation functions in lecture 11, but we first need to understand beamsplitters in lecture 10.

9.2   Coherence/correlation functions

First, we will remind ourselves of the classical functions. An early example is Young's slits experiment:
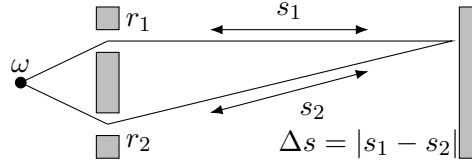
We need

$$\Delta s \leq \frac{c}{\Delta \omega} \tag{9.19}$$

$$\Delta t = \frac{\Delta s}{c} = \frac{1}{\Delta \omega} \tag{9.20}$$

$\frac{c}{\Delta \omega}$ is the coherence length, and Δt is the coherence time. Now what about the fiel at the detector?

$$t_1 = t - \frac{s_1}{c}, \; t_2 = t - \frac{s_2}{c} \tag{9.21}$$

$$\underline{\mathbf{E}}(\underline{r}, t) = k_1 \underline{\mathbf{E}}(\underline{r}_1, t_1) + k_2 \underline{\mathbf{E}}(\underline{r}_2, t_2) \tag{9.22}$$

$$I(\underline{r}) = \langle |\underline{\mathbf{E}}(\underline{r}, t)|^2 \rangle_t \tag{9.23}$$

$$= |k_1|^2 \langle |\underline{\mathbf{E}}(\underline{r}_1, t_1)|^2 \rangle + |k_2|^2 \langle |\underline{\mathbf{E}}(\underline{r}_2, t)|^2 \rangle + 2\mathrm{Re}\left[k_1^* k_2 \langle \underline{\mathbf{E}}^*(\underline{r}_1, t_1) \underline{\mathbf{E}}(\underline{r}_2, t_2) \rangle \right] \tag{9.24}$$

$$= I_1 + I_2 + 2\mathrm{Re}[\text{interference}] \tag{9.25}$$

So we introduce $\gamma^{(1)}(x_1, x_2)$. This is the first order normalised mutual coherence function.

$$\gamma^{(1)}(x_1, x_2) = \frac{\langle \underline{\mathbf{E}}^*(\underline{r}_1, t_1) \underline{\mathbf{E}}(\underline{r}_2, t_2) \rangle}{\{\langle |\underline{\mathbf{E}}(\underline{r}_1, t_1)|^2 \rangle \langle |\underline{\mathbf{E}}(\underline{r}_2, t_2)|^2 \rangle\}^{1/2}} \tag{9.26}$$

$$I(\underline{r}) = I_1 + I_2 + 2\sqrt{I_1 I_2} \mathrm{Re}\left[k_1^* k_2 \gamma^{(1)}(x_1, x_2)\right] \tag{9.27}$$

Now we set $k_j = |k_j| e^{i\phi_j}$.

$$I(\underline{r}) = I_1 + I_2 + 2\sqrt{I_1 I_2} |\gamma^{(1)}(x_1, x_2)| \cos(\phi_1 - \phi_2) \tag{9.28}$$

$$\tag{9.29}$$

The cos term in this is the phase difference from path difference. The value of $\gamma^{(1)}$ sets the amount of coherence:

$$\gamma^{(1)} \begin{cases} = 1 & \text{full coherence} \\ > 0, < 1 & \text{partial coherence} \\ = 0 & \text{no coherence} \end{cases} \tag{9.30}$$

Lecture 10

10.1 Learning Outcomes

We will develop the quantum mechanical theory of the beamsplitter. After studying this section, you should

➤ Be familiar with the clasical description of a lossless beamsplitter.
➤ Be able to derive equations that relate the creation and annihilation operators for a lossless beamsplitter.
➤ Understand the meaning of coincidence in the context of the beamsplitter.
➤ Understand the meaning of unitary transforms in the context of the beamsplitter.
➤ Be familiar with the significance of a beamsplitter for single photon states, and the results when the two input modes each contain a single photon (Hong-Ou-Mandel effect).
➤ Be familiar with the operation of a beamsplitter with coherent states.
➤ Be able to calculate the joint output state of a beamsplitter for any number state or coherent state input, or combinations thereof.

10.2 The beam splitter

A general beamsplitter will look like the following:
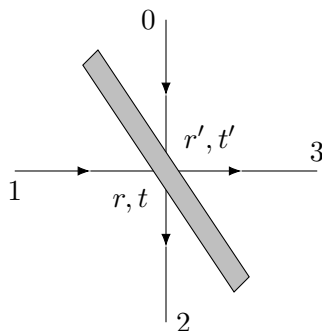
r, t, r', t' are complex, so they introduce phases. We will only deal with **lossless beamsplitters:** $|r| = |r'|$, $|t| = |t'|$. Beamsplitters mix modes, so 2 and 3 are both linear combinations of 0 and 1. In a quantum beamsplitter, there will **always** be two inputs; if we do not put two in, then the second will be the vacuum, which must still be considered. For now, we will not consider polarisation dependence, but this will be addressed later on. Classically,

$$\begin{pmatrix} a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} t' & r \\ r' & t \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \tag{10.1}$$

For the quantum case,

$$\begin{pmatrix} \hat{a}_2 \\ \hat{a}_3 \end{pmatrix} = \begin{pmatrix} t' & r \\ r' & t \end{pmatrix} \begin{pmatrix} \hat{a}_0 \\ \hat{a}_1 \end{pmatrix} \tag{10.2}$$

and similarly for $\hat{a}^\dagger$ but with the complex conjugates of t, r etc. We shall fix the phase shift so reflected vs transmitted $\pm i$, $\exp\left(\pm i\frac{\pi}{2}\right)$. Using a 50:50 beam splitter, $|r| = |t| = \frac{1}{\sqrt{2}}$.

$$\begin{pmatrix} \hat{a}_2 \\ \hat{a}_3 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} \hat{a}_0^\dagger \\ \hat{a}_1^\dagger \end{pmatrix} \tag{10.3}$$

This is the balanced Hadamard operator used in Quantum information.
We want to calculate $\langle \hat{n}_2 \rangle$. So for the input, we have

$$|0\rangle_0 |1\rangle_1 \equiv |0_0 1_1\rangle \tag{10.4}$$

$$\langle \hat{n}_2 \rangle = \langle 0_0 1_1 | \hat{a}_2^\dagger \hat{a}_2 | 0_0 1_1 \rangle \tag{10.5}$$

So how do we define the output operators?

$$\hat{a}_2 = \frac{1}{\sqrt{2}} \left(\hat{a}_0 + i\hat{a}_1 \right) \qquad\qquad \hat{a}_3 = \frac{1}{\sqrt{2}} \left(i\hat{a}_0 + \hat{a}_1 \right) \tag{10.6}$$

$$\hat{a}_2^\dagger = \frac{1}{\sqrt{2}} \left(\hat{a}_0^\dagger - i\hat{a}_1^\dagger \right) \qquad\qquad \hat{a}_3^\dagger = \frac{1}{\sqrt{2}} \left(-i\hat{a}_0^\dagger + \hat{a}_1^\dagger \right) \tag{10.7}$$

So we can input these into the expectation value equation:

$$\langle \hat{n}_2 \rangle = \langle 0_0 1_1 | \frac{1}{2} \left(\hat{a}_0^\dagger - i\hat{a}_1^\dagger \right) \left(\hat{a}_0 + i\hat{a}_1 \right) |0_0 1_1\rangle \tag{10.8}$$

$$= \langle 0_0 1_1 | \frac{1}{2} \left\{ \hat{a}_0^\dagger \hat{a}_0 - i\hat{a}_1^\dagger \hat{a}_0 + i\hat{a}_0^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_1 \right\} |0_0 1_1\rangle \tag{10.9}$$

$$= \langle 0_0 1_1 | \frac{1}{2} \left\{ 0 + 0 + 0 + 1 \right\} |0_0 1_1\rangle = \frac{1}{2} = \langle \hat{n}_3 \rangle \tag{10.10}$$

Naturally, this is exactly what we expected from a 50:50 beamsplitter - $\hat{n}_2$ and $\hat{n}_3$ have an even split. However, if we consider

$$\langle 0_0 1_1 | \hat{n}_2 \hat{n}_3 | 0_0 1_1 \rangle = 0 \neq \langle \hat{n}_2 \rangle \langle \hat{n}_3 \rangle \tag{10.11}$$

The photons will only ever come out at one port at a single time, so on average it is 50:50, but will alternate between the ports, so one port is always at 0 - **photon anti-bunching.** This is not what we would expect classically, where $\langle I_2 I_3 \rangle = \langle I_2 \rangle \langle I_3 \rangle$. So putting single photons through a beam splitter is something very non-classical.

We can approach this problem in another way:

$$|0\rangle_0 |1\rangle_1 = \hat{a}_1^\dagger |0\rangle_0 |0\rangle_1 \tag{10.12}$$

So this time we start with two vacuum states and use the creation operator to add photons.

$$\hat{a}_2^\dagger = \frac{1}{\sqrt{2}} \left\{ \hat{a}_0^\dagger - i\hat{a}_1^\dagger \right\} \qquad\qquad \hat{a}_3^\dagger = \frac{1}{\sqrt{2}} \left\{ -i\hat{a}_0^\dagger + \hat{a}_1^\dagger \right\} \tag{10.13}$$

$$\hat{a}_1^\dagger = \frac{1}{\sqrt{2}} \left\{ i\hat{a}_2^\dagger + \hat{a}_3^\dagger \right\} \qquad\qquad \hat{a}_0^\dagger = \frac{1}{\sqrt{2}} \left\{ \hat{a}_2^\dagger + i\hat{a}_3^\dagger \right\} \tag{10.14}$$

Note that vacuum in means vacuum out, i.e. $|0_0 0_1\rangle \rightarrow |0_2 0_3\rangle$. Now let's apply $\hat{a}_1^\dagger$, which means adding a photon to port 1.

$$|0_0 1_1\rangle \rightarrow \frac{1}{\sqrt{2}} \left\{ i\hat{a}_2^\dagger + \hat{a}_3^\dagger \right\} |0_2 0_3\rangle \tag{10.15}$$

$$= \frac{1}{\sqrt{2}} \left\{ i |1_2 0_3\rangle + |0_2 1_3\rangle \right\} \tag{10.16}$$

$$\langle \hat{n}_2 \rangle = \langle \hat{n}_3 \rangle = \frac{1}{2}, \ \langle \hat{n}_2 \hat{n}_3 \rangle \qquad\qquad\qquad = 0 \tag{10.17}$$

So this method reproduces the same results as before. Note that we cannot write Eq (10.16) as a product state, i.e. $|1_2 0_3\rangle \neq |1_2\rangle |0_3\rangle$. Instead, this is known as an entangled state.

Now, what happens if we put one photon into each input at once? We now have four equally likely possibilities:

(a)

(b)

(c)

(d)
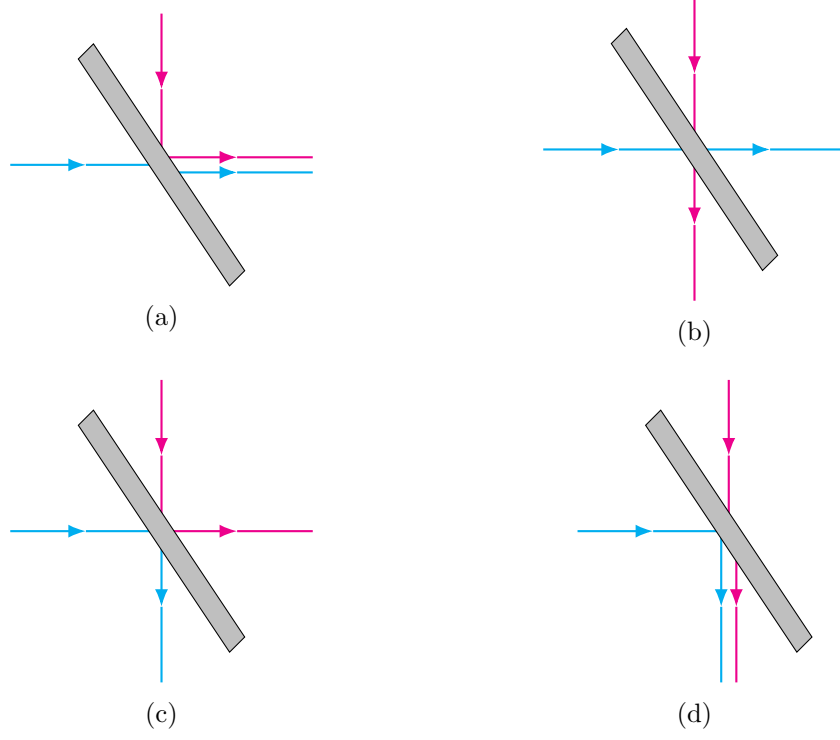
$$|1_0 1_1\rangle = \hat{a}_0^\dagger \hat{a}_1^\dagger |0_0 0_1\rangle \rightarrow \frac{1}{2}\left(\hat{a}_2^\dagger + i\hat{a}_3^\dagger\right)\left(i\hat{a}_2^\dagger + \hat{a}_3^\dagger\right)|0_2 0_3\rangle \tag{10.18}$$

$$\rightarrow \frac{1}{2}\left\{i\hat{a}_2^\dagger \hat{a}_2^\dagger - \hat{a}_3^\dagger \hat{a}_2^\dagger + \hat{a}_3^\dagger \hat{a}_2^\dagger + i\hat{a}_3^\dagger \hat{a}_3^\dagger\right\}|0_2 0_3\rangle \tag{10.19}$$

$$\rightarrow \frac{i}{2}\left\{\sqrt{2}|2_2 0_3\rangle + \sqrt{2}|0_2 2_3\rangle\right\} \tag{10.20}$$

10.3   Coherent States and Beamsplitters
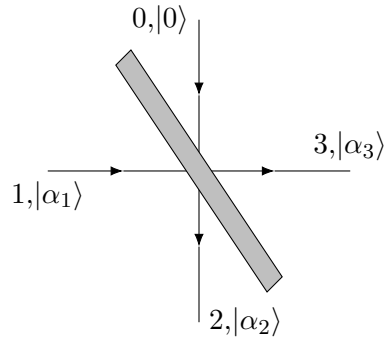
$$|0\rangle_0 |\alpha_1\rangle_1 \rightarrow |\alpha_2\rangle_2 |\alpha_3\rangle_3 \tag{10.21}$$

So inputting one coherent state with a vacuum, yields coherent states in both outputs. For 50:50,

$$\alpha_2 = \frac{i\alpha_1}{\sqrt{2}} \qquad\qquad \alpha_3 = \frac{\alpha_1}{\sqrt{2}} \qquad\qquad BS = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \tag{10.22}$$

Now what if we have

$$|\alpha_0\rangle_0 |\alpha_1\rangle_1 \rightarrow |\alpha_2\rangle_2 |\alpha_3\rangle_3 \tag{10.23}$$

31

We want to have the same amplitude going into both 1 and 0, but with a phase difference, so $|\alpha_0| = |\alpha_1| = |\alpha|$, with $\alpha_1 = |\alpha|\epsilon^{i\theta}$ and $\alpha_0 = |\alpha|$. We will find that

$$P_2 \propto \sin^2 \frac{\theta}{2} \tag{10.24}$$

This can be compared to the coherent state results in Lecture 6.

Lecture 11

11.1   Learning Outcomes

We will discuss interference, and both the classical and quantum definitions of first and second order coherence. After studying this section, you should

➤ Be able to calculate the output state of an interferomater with number states or coherent states in the input modes.
➤ Understand and be able to apply the definition of first-order coherence, and be familiar with its link to the visibility of interference fringes.
➤ Be familiar with, and able to apply, the definition of second-order coherence.
➤ In each of the above cases, to be familiar with and able to apply the quantum definitions in terms of correlation functions involving electric field operators.
➤ To understand, and be able to apply the terms incoherent, partially coherent, and fully coherent in the context of first-order coherence.
➤ To understand and be able to apply the terms Poissonian, sub-Poissonian, bunching, and anti-bunching in the context of second-order coherence.
➤ To be familiar with the first- and second-order coherence properties of single-mode number states, coherent states, and thermal states.

11.2   Interferometers

The Mach-Zehnder interferometer is the simplest and cleanest interferometer:
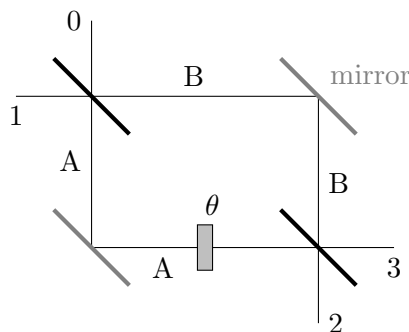
Let's work through it. Going through the first beamsplitter, we get:

$$|0\rangle_0|1\rangle_1 \rightarrow \frac{1}{\sqrt{2}} \left(|0\rangle_A|1\rangle_B + i|1\rangle_A|0\rangle_B\right) \tag{11.1}$$

The phase from the mirrors is $e^{i\pi/2}$ on both, but we don't need to know it.
Now, we need to put in a phase shift, $e^{i\theta}$. We can only phase shift the second term, as we can't phase shift the vacuum - it has no well defined phase to shift.

$$|0\rangle_0|1\rangle_1 \rightarrow \frac{1}{\sqrt{2}} \left(|0\rangle_A|1\rangle_B + i|1\rangle_A|0\rangle_B\right) \tag{11.2}$$

$$\rightarrow \frac{1}{\sqrt{2}} \left(|0\rangle_A|1\rangle_B + ie^{i\theta}|1\rangle_A|0\rangle_B\right) \tag{11.3}$$

Then, we have the second beamsplitter, which transforms as:

$$|0\rangle_A|1\rangle_B \rightarrow \frac{1}{\sqrt{2}} \left(|1\rangle_2|0\rangle_3 + i|0\rangle_3|1\rangle_3\right) \tag{11.4}$$

$$|1\rangle_A|0\rangle_B \rightarrow \frac{1}{\sqrt{2}} \left(|0\rangle_2|1\rangle_3 + i|1\rangle_2|0\rangle_3\right) \tag{11.5}$$

So we can then put both beamsplitters together for the full transform:

$$|0\rangle_0|\rangle_1 \rightarrow \frac{1}{2}\left(i(1+e^{i\theta})|0\rangle_2|1\rangle_3 + (1-e^{i\theta})|1\rangle_2|0\rangle_3\right) \tag{11.6}$$

$$P_2 = \frac{1}{4}(1-e^{i\theta})^2 = \frac{1}{2}(1-\cos\theta) \tag{11.7}$$

$$P_3 = \frac{1}{2}(1+\cos\theta) \tag{11.8}$$

Single photons $\rightarrow$ wavelike interference, averaging over many.

What we had before interferometers was double beam experiments:
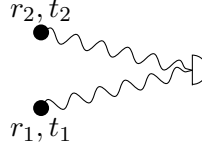
$$\underline{\mathbf{E}}(vr, t) = k_1\underline{\mathbf{E}}(\underline{r}_1, t_1) + k_2\underline{\mathbf{E}}(\underline{r}_2, t_2) \tag{11.9}$$

From this, we found the first-order normalised mutual coherence function:

$$\gamma^{(1)}(\underline{r}_1, t_1, \underline{r}_2, t_2) = \frac{\langle\underline{\mathbf{E}}^*(\underline{r}_1, t_1)\underline{\mathbf{E}}(\underline{r}_2, t_2)\rangle_t}{\left\{\langle|\underline{\mathbf{E}}(\underline{r}_1, t_1)|^2\rangle_t\langle|\underline{\mathbf{E}}(\underline{r}_2, t_2)|^2\rangle_t\right\}^{1/2}} \tag{11.10}$$

$$I(\underline{r}) = I_1 + I_2 + \sqrt{I_1 I_2}|\gamma^{(1)}(\underline{r}_1, t_1, \underline{r}_2, t_2)|\cos(\phi_1 - \phi_2) \tag{11.11}$$

$$\gamma^{(1)}\begin{cases}= 1 & \text{fully coherent} \\ < 1 & \text{partially coherent} \\ = 0 & \text{fully incoherent}\end{cases} \tag{11.12}$$
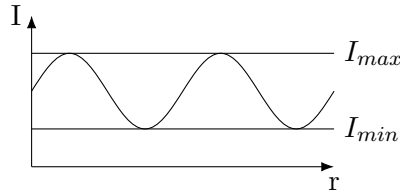
Rayleigh fringe visibility is defined as:

$$\nu = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} \tag{11.13}$$

$$\nu = \begin{cases}1 & \text{fully coherent} \\ 0 & \text{fully incoherent}\end{cases} \tag{11.14}$$

Reminder that our first order coherence depends on $\langle\underline{\mathbf{E}}^*(\underline{r}_1, t_1)\underline{\mathbf{E}}(\underline{r}_2, t_2)\rangle_t$, but

$$P_i \propto \langle i|\hat{\underline{\mathbf{E}}}_-(\underline{r}, t)\hat{\underline{\mathbf{E}}}_+(\underline{r}, t)|i\rangle \propto \hat{a}^\dagger\hat{a} = \hat{n} \tag{11.15}$$

This is the quantum version of the first order coherence. What about our **quantum** first-order normalised mutual coherence function?

$$g^{(1)}(\underline{r}_1, t_1, \underline{r}_2, t_2) = \frac{\langle\hat{\underline{\mathbf{E}}}_-(\underline{r}_1, t_1)\hat{\underline{\mathbf{E}}}_+(\underline{r}_2, t_2)\rangle}{\left\{\langle\hat{\underline{\mathbf{E}}}_-\hat{\underline{\mathbf{E}}}_+(\underline{r}_1, t_1)\rangle\langle\hat{\underline{\mathbf{E}}}_-\hat{\underline{\mathbf{E}}}_+(\underline{r}_2, t_2)\rangle\right\}^{1/2}} \tag{11.16}$$

We find that both number and coherent states can be fully coherent, i.e. $g^{(1)} = 1$. The first order coherent function tells you how coherent you are and how visible your fringes will be.

11.3   Second Order Coherence
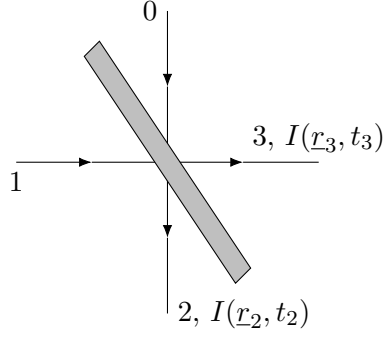
Let's consider a classical second-order normalised mutual coherence function:

$$\gamma^{(2)}(\underline{r}_3, t_3, \underline{r}_2, t_2) = \frac{\langle I(\underline{r}_3, t_3) I(\underline{r}_2, t_2) \rangle_t}{\langle I(\underline{r}_3, t_3) \rangle \langle I(\underline{r}_2, t_2) \rangle_t} \tag{11.17}$$

This is second order as $I \propto \underline{\mathbf{E}}^* \underline{\mathbf{E}}$, so each term has to $\underline{\mathbf{E}}$-fields, instead of one as previously. Now let's look at the quantum version again:

$$g^{(2)}(\underline{r}_3, t_3, \underline{r}_2, t_2) = \frac{\langle \hat{\underline{\mathbf{E}}}_-(\underline{r}, t_3) \hat{\underline{\mathbf{E}}}_-(\underline{r}_2, t_2) \hat{\underline{\mathbf{E}}}_+(\underline{r}_2, t_2) \hat{\underline{\mathbf{E}}}_+(\underline{r}_3, t_3) \rangle}{\langle \hat{\underline{\mathbf{E}}}_- \hat{\underline{\mathbf{E}}}_+ (\underline{r}_3, t_3) \rangle \langle (\hat{\underline{\mathbf{E}}}_- \hat{\underline{\mathbf{E}}}_+ (\underline{r}_2, t_2) \rangle} \tag{11.18}$$

Lecture 12

For the Mach-Zehnder interferometer, what if $\theta = 0$? If for a single photon going through one beamsplitter, we have the Hadamard operator,

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \tag{12.1}$$

then, for two beamsplitters, as seen in the interferometer, without a phase shift, we would have the Hadamard operator squared,

$$H^2 = \frac{1}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = I \tag{12.2}$$

which makes the identity matrix and the one photon will simply come out of a single port again.

12.1 Single mode field (no beamsplitter)

The only difference here is that everything that commutes can be cancelled out, and we are left with the dagger operators:

$$g_{\text{single}}^{(2)} = \frac{\langle \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{a}^\dagger \hat{a} \rangle} \tag{12.3}$$

$$= \frac{\langle \hat{n}(\hat{n} - 1) \rangle}{\langle \hat{n} \rangle^2} \tag{12.4}$$

Where we use the ladder operators' commutation relation to redefine.

1. Coherent state:

$$\langle \alpha | \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} | \alpha \rangle = |\alpha|^4 \tag{12.5}$$

$$\langle \alpha | \hat{n} | \alpha \rangle = |\alpha|^2 \tag{12.6}$$

$$g^{(2)} = 1 \tag{12.7}$$

2. Number states:

$$\langle 1 | \hat{n}(\hat{n} - 1) | 1 \rangle = 0 \tag{12.8}$$

$$\frac{\langle n | \hat{n}(\hat{n} - 1) | n \rangle}{\langle \hat{n} \rangle^2} = \frac{n^2 - n}{n^2} = 1 - \frac{1}{n} \tag{12.9}$$

For $g^{(2)}$, number states with more photons look more classical. For a $g^{(2)} < 1$, we call it sub-Poissonian, because for a coherent state with $g^{(2)} = 1$, we have a Poisson distribution.

12.2 Two modes

The Hanburg, Brown, and Twiss experiment (didn't quite manage this), actually Jeff Kimble and Herbert Waller.
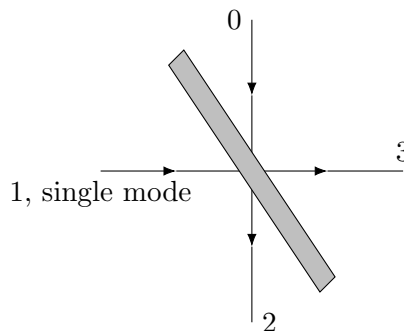
$$|0_0 1_1\rangle \rightarrow \frac{1}{\sqrt{2}} \{i|1_2 0_3\rangle + |0_2 1_3\rangle\} \tag{12.10}$$

So single time, two modes.

$$g^{(2)}(t_2 = t_3) = \frac{\langle \hat{a}_3^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_3 \rangle}{\langle \hat{a}_3^\dagger \hat{a}_3 \rangle \langle \hat{a}_2^\dagger \hat{a}_2^\dagger \rangle} = \frac{\langle \hat{n}_3 \hat{n}_2 \rangle}{\langle \hat{n}_3 \rangle \langle \hat{n}_2 \rangle} = 0 \tag{12.11}$$

But if we take it at different times, then the input is now $|0_0 1_1\rangle_{t_2}|0_0 1_1\rangle_{t_3}$, so now $\langle \hat{n}_2 \hat{n}_3 \rangle = \frac{1}{4}$, and

$$g^{(2)}(t_2 - t_3) = 1 \tag{12.12}$$

Now if we put light into both inputs:

$$|1_0 1_1\rangle \rightarrow \frac{i}{\sqrt{2}} \{|0_2 2_3\rangle + |2_2 0_3\rangle\} \tag{12.13}$$

$$g^{(2)} = \frac{\langle \hat{n}_2 \hat{n}_3 \rangle}{\langle \hat{n}_2 \rangle \langle \hat{n}_3 \rangle} = 0 \tag{12.14}$$

So this is definitely non-classical.

12.3 Coherent states in beamsplitter

$$|0_1 \alpha_1\rangle \rightarrow \frac{i}{\sqrt{2}}|\alpha_2, \alpha_3\rangle \tag{12.15}$$

$$\langle \hat{n}_2 \rangle \hat{n}_3 \rangle = \langle \hat{n}_2 \hat{n}_3 \rangle \implies g^{(2)} = 1 \tag{12.16}$$

12.4 Summary

For first order, we have $\gamma^{(1)}$ for classical, and $g^{(1)}$ for quantum. We are looking at amplitudes:

$$\frac{\langle \hat{\underline{\mathbf{E}}}_+^{(1)} \hat{\underline{\mathbf{E}}}_-^{(2)} \rangle}{(\langle \hat{\underline{\mathbf{E}}}_+^{(1)} \hat{\underline{\mathbf{E}}}_-^{(1)} \rangle \langle \hat{\underline{\mathbf{E}}}_+^{(2)} \hat{\underline{\mathbf{E}}}_-^{(2)} \rangle)^{1/2}} \tag{12.17}$$

Amplitudes of interference fringes - fringe visibility.
How coherent the EM fields are:

$$0 \leq \gamma^{(1)}, \text{ (incoherent)} \tag{12.18}$$
$$g^{(1)} \leq 1, \text{ (coherent)} \tag{12.19}$$

For second order, we have $\gamma^{(2)}$ for classical, and $g^{(2)}$ for quantum.

$$1 \leq \gamma^{(2)} < \infty \tag{12.20}$$
$$0 \leq g^{(2)} < 1 \tag{12.21}$$

We looked at small number states, where

$$g^{(2)} = 1 - \frac{1}{n} \tag{12.22}$$

We also looked at coherent states, where we have:

$$g^{(2)} = 1, \text{ (Poissonian)} \tag{12.23}$$
$$g^{(2)} < 1, \text{ (sub-Poissonian)} \tag{12.24}$$

Lecture 13

13.1   Learning Outcomes

We will discuss polarisation in terms of two-mode states and projective measurements. After studying this section, you will:

➤ Be able to calculate the probability of detecting a photon after a linear polariser, for any two-mode input state (number, coherent) and any combination of polariser angle and initial polarisation.

➤ Understand the action of a linear polariser in terms of projective measurements; be familiar with the use of the projection operator, and be able to disucss the state before and after the measurement.

13.2   Polarisation

$$\hat{\underline{\mathbf{E}}}(\underline{r}, t) = i \sum_{\underline{k}, s} \omega_k \underline{e}_{\underline{k}, s} \left[A_{\underline{k}, s} \exp\left\{ i(\underline{k} \cdot \underline{r} - \omega_k t) \right\} + c.c. \right] \tag{13.1}$$

We are now concerned with $\underline{e}_{\underline{k}, s}$, the polarisation vector of the field. There are two polarisations to make up a complete basis, contained in the index s. Above is what we will see classically.

Consider two modes with same $\underline{k}, \omega_k$, where $\underline{k} \propto \hat{z}$, so we have $\underline{e}_x, \underline{e}_y$.

$$\hat{\underline{\mathbf{E}}}(\underline{r}, t) = i \omega_k \left[(A_x \underline{e}_x + A_y \underline{e}_y) \exp\left\{ i(\underline{k} \cdot \underline{r} - \omega_k t) \right\} + c.c. \right] \tag{13.2}$$
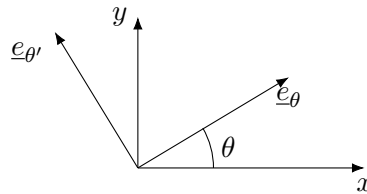
$$\underline{e}_\theta = \cos\theta \underline{e}_x + \sin\theta \underline{e}_y \tag{13.3}$$

$$\underline{e}_{\theta'} = -\sin\theta \underline{e}_x + \cos\theta \underline{e}_y \tag{13.4}$$

Hnece, for the amplitudes, $A_x, A_y \to A_\theta, A_{\theta'}$:

$$\begin{pmatrix} A_\theta \\ A_{\theta'} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix} \tag{13.5}$$

Now we must have our polarising beamsplitters:
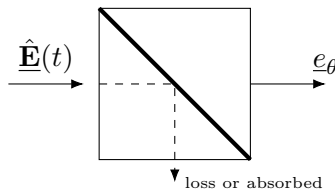
$$A_y = 0 \tag{13.6}$$

$$A_\theta = A_x \cos\theta \tag{13.7}$$

$$I_\theta \propto |A_x|^2 \cos^2\theta \tag{13.8}$$

What about the quantum setting? We have our operator that we will split into the different cartesian polarisations, $\hat{a}_{\underline{k}, s} \to \hat{a}_x, \hat{a}_y$, and we have a rotation matrix, $\hat{a}_\theta, \hat{a}_{\theta'}$. Note that

$$[\hat{a}_\theta, \hat{a}_{\theta'}] = 0 \tag{13.9}$$

$$[\hat{a}_\theta, \hat{a}_\theta^\dagger] = 1 \tag{13.10}$$

So these are just the creation and annihilation operators in a specific orientation.

Example: Number state with one photon polarised along θ

We have a two mode field (with $|0\rangle$ the single mode vacuum):

$$|1_\theta, 0_{\theta'}\rangle = \hat{a}_\theta^\dagger |0, 0\rangle \tag{13.11}$$

$$= (\hat{a}_x^\dagger \cos\theta + \hat{a}_y^\dagger \sin\theta)|0, 0\rangle \tag{13.12}$$

$$= \cos\theta |1_x, 0_y\rangle + \sin\theta |0_x, 1_y\rangle \tag{13.13}$$

We can create from the vacuum state in any way we want to, so we don't have to worry about what we are using for this, and whether we use a number or coherent state.

We can show that

$$\hat{n}|1_\theta, 0_{\theta'}\rangle = \sum_{\underline{k}, s} \hat{n}_{\underline{k}, s}|1_\theta, 0_{\theta'}\rangle \tag{13.14}$$

$$= |1_\theta, 0_{\theta'}\rangle \tag{13.15}$$

$$\langle\hat{n}\rangle = 1 \tag{13.16}$$
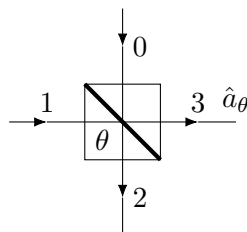
$$\text{Prob} \propto \langle\hat{n}\rangle = \langle\psi|\hat{a}_\theta^\dagger \hat{a}_\theta|\psi\rangle \tag{13.17}$$

So what if we have a state $|\psi\rangle$ of one photon, but we don't know polarisation. Depending on this polarisation, we will be likelier to get different measurements. In a single measurement, we record 0 or 1.

$$P_0 = |\langle 0, 0|\psi\rangle|^2 \qquad\qquad P_1 = |\langle 1_{\theta P}|\psi\rangle|^2 \tag{13.18}$$

All we can do with one measurement is if we detect photon, i.e. 1. So then we know something: that is definitely not polarised in θ' direction, because then it definitely wouldn't have come through. To determine θ, we need to make many measurements, or many copies, of $|\psi\rangle$ - this need gives rise to the concept of ensembles, where the ratio of 0 to 1 lets you work out what θ is.

We can now try another experiment:
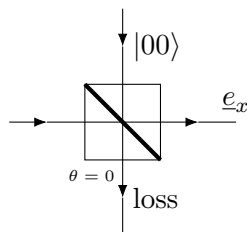
$$|\psi\rangle = |1_{\pi/4}, 0_{3\pi/4}\rangle \tag{13.19}$$

So we have one photon, polarised at 45 degrees. The probability of detecting one photon is then

$$P_1 = |\langle 1_x, 0_x|1_{\pi/4}, 0_{3\pi/4}\rangle|^2 = \frac{1}{2}, \ \left(\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}\right) \tag{13.20}$$

Now let's compare with a random mixture of two states, so half the time we have $|1_x, 0_y\rangle$, and half the time we have $|0_x, 1_y\rangle$. Again however, we get $P_1 = \frac{1}{2}$. How do we tell the difference between the two cases?

If we set the polariser to $\theta_p = \frac{\pi}{4}$, $|1_{\pi/4}, 0_{3\pi/4}\rangle \implies P_1 = 1$, but the fifty-fifty mixture still yields $P_1 = 1$.

What we have been writing here are the projection operators for these states:

$$|\psi\rangle = \sum_j c_j |j\rangle \tag{13.21}$$

where $\{j\}$ is a complete orthonormal basis.

$$P_j = |\langle j|\psi\rangle|^2 = |c_j|^2 \tag{13.22}$$
$$= \langle\psi|j\rangle\langle j|\psi\rangle \tag{13.23}$$
$$= \langle\psi|\hat{P}_j|\psi\rangle, \ \hat{P}_j = |j\rangle\langle j| \tag{13.24}$$
$$\hat{P}_j|\psi\rangle = |j\rangle\langle j|\psi\rangle = c_j|j\rangle \tag{13.25}$$

So this projection operator, $\hat{P}_j$, is projecting the state $|\psi\rangle$ onto $|j\rangle$ - however this is not normalised as it is. So we need as set of these so the normalisation sums to one.

$$P_x = |\langle 1_x 0_y|\psi\rangle|^2 \tag{13.26}$$
$$|\psi\rangle = \frac{1}{\sqrt{2}}\{|1_x 0_y\rangle + |0_x 1_y\rangle\} = |1_{\pi/4}, 0_{3\pi/4}\rangle \tag{13.27}$$
$$P_x = \frac{1}{2}, \hat{P}_x = |1_x 0_y\rangle\langle 1_x 0_y| \tag{13.28}$$
$$\hat{P}_x|\psi\rangle = \frac{1}{\sqrt{2}}|1_x 0_y\rangle \tag{13.29}$$

If we apply $\hat{P}_x$ but "fail", i.e. no photon, then we must have $|0_x 1_y\rangle$, so we have then define

$$\hat{P}_y = |0_x 1_y\rangle\langle 0_x 1_y| \tag{13.30}$$
$$\hat{P}_y|\psi\rangle = \frac{1}{\sqrt{2}}|0_x 1_y\rangle \tag{13.31}$$

Again, this is not normalised by itself.

So for measurements there will be at least two outcomes (if only one, then there is clearly no need to make the measurement!), and we need a complete set of projectors which sum to identity, $\left\{\hat{P}_x, \hat{P}_y\right\}$.

$$\hat{P}_x + \hat{P}_y = \mathbb{I}^2 = |1_x 0_y\rangle\langle 1_x 0_y| + |0_x 1_y\rangle\langle 0_x 1_y| \tag{13.32}$$
$$\sum_j \hat{P}_j = \mathbb{I} \tag{13.33}$$

So we have out complete set of projective measurements. *Other types of measurements are possible but these are beyond the current scope.*

Lecture 14

14.1 Learning Outcomes

Density matrices, pure states, mixed states, and entangled states. To be familiar with the concepts of pure states, mixed states, and the density matrix. To apply them to entangled states and product states, and use the partial trace. After studying this section you will be able to:

➤ Explain and state the definitions of pure states, mixed states, and the density matrix.
➤ Perform calculations using the density operator and the trace (e.g. trace of the density operator, expectation values) for pure states and mixed states.
➤ Explain and state the definitions of entangled states and product states for bi-partite systems in a pure state.
➤ To be familiar with the definition and properties of the partial trace and its application to the joint density matrix.
➤ To be familiar with and apply the partial trace over a sub-system of a bipartite density matrix, and understand how the outcome differs for product states and entangled states.

14.2 The Density Matrix

Thus far, we have been working with pure states, $|\psi\rangle$, which represent everything about a particular quantum state.

$$|\psi\rangle = \sum_j c_j |j\rangle \tag{14.1}$$

So we can form a density matrix for a pure state, $\hat{\rho}$ - it is the projector of the state onto itself.

$$\hat{\rho} = |\psi\rangle\langle\psi| = \sum_j c_j |j\rangle \sum_m c_m^* \langle m| \tag{14.2}$$

$$= \sum_{j,m} c_j c_m^* |j\rangle\langle m| \tag{14.3}$$

Example: Horizontal and Vertical polarisation

$$|L\rangle = \frac{1}{\sqrt{2}} \{|1_H, 0_V\rangle + i|0_H, 1_V\rangle\} \tag{14.4}$$

$$\hat{\rho}_L = |L\rangle\langle L| = \frac{1}{2} \{|H\rangle\langle H| + i|V\rangle\langle H| - |H\rangle\langle V| + |V\rangle\langle V|\} \tag{14.5}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \tag{14.6}$$

Note: this is Hermitian, and the diagonal terms give likelihoods of being in H or V, while the off-diagonals form coherences, i.e. the relative phases between the different basis states.
For a pure state, the Schrodinger equation can be written as

$$\frac{d}{dt}\hat{\rho} = \frac{i}{\hbar}[\hat{\rho}, \hat{H}]. \tag{14.7}$$

This is known as the van-Neumann equation.

So let's go back to the example of source giving horizontal or vertical polarisations, but we don't know which one we were getting. We can write this in terms of a density matrix,

$$\hat{\rho}_{H/V} = \frac{1}{2}|H\rangle\langle H| + \frac{1}{2}|V\rangle\langle V| = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{14.8}$$

The difference here is there is no off-diagonal terms, so there is no coherences in this ensemble, i.e. a mixed state. Above, we just had the projector of one state $|L\rangle$, but now we have the sum of two projectors. So the coherences will depend on whether we have a mixed or pure state.

More generally, we can have

$$\hat{\rho} = \sum_j p_j |\psi_j\rangle\langle\psi_j|. \tag{14.9}$$

This will work for any (not necessarily orthogonal) set of states $|\psi_j\rangle$. In general, this decomposition is not unique, unless you know it is from a particular set.

Example: Pure or Mixed?

$$\hat{\rho} = \frac{1}{2}|L\rangle\langle L| + \frac{1}{2}|H\rangle\langle H| \tag{14.10}$$

$$= \frac{1}{2} \cdot \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{14.11}$$

$$= \frac{1}{4}\begin{pmatrix} 3 & i \\ -i & 1 \end{pmatrix} \tag{14.12}$$

There is an easy way to test if this state is pure or mixed. We square $\hat{\rho}$, or $\hat{\rho}^\dagger\hat{\rho}$, but $\hat{\rho}$ is Hermitian, so it is just $\hat{\rho}^2$. For a pure state, $\hat{\rho}_L$,

$$\hat{\rho}_L = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \qquad \hat{\rho}_L^2 = \frac{1}{4}\begin{pmatrix} 2 & 2i \\ -2i & 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = \hat{\rho}_L \tag{14.13}$$

So we want $\hat{\rho} = \hat{\rho}^2$ for a pure state. Let's check for the state above:

$$\rho = \frac{1}{4}\begin{pmatrix} 3 & i \\ -i & 1 \end{pmatrix} \qquad \hat{\rho}^2 = \frac{1}{16}\begin{pmatrix} 4 & 4i \\ -4i & 2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix} \neq \hat{\rho} \tag{14.14}$$

So the above is not a pure state, but a mixed state.

It is very easy to see why this is the case for a pure state generally, as

$$\hat{\rho} = |\psi\rangle\langle\psi| \qquad\qquad \hat{\rho}^2 = |\psi\rangle\underbrace{\langle\psi|\psi\rangle}_{1}\langle\psi| = \hat{\rho} \tag{14.15}$$

If the diagonal elements of $\hat{\rho}$ describe the likelihoods of being in whichever state, then we have to look at the trace - the sum of the diagonal entries. When performing in terms of an operator,

$$\text{Tr}[\hat{A}] = \sum_j \langle j|\hat{A}|j\rangle \tag{14.16}$$

for some orthonormal basis $\{|j\rangle\}$. This is invariant of basis, so any basis will give the same trace.

We can find the traces of a multiplication of operators too:

$$\text{Tr}[\hat{\rho} \cdot \hat{P}_n] = \langle n|\hat{\rho}|n\rangle, \ \hat{P}_n = |n\rangle\langle n| \tag{14.17}$$

So the second operator will "wrap" itself around the first essentially.

What about the expectation value of $\hat{A}$ on $\hat{\rho}$?

$$\text{Tr}[\hat{\rho} \cdot \hat{A}] = \langle\hat{A}\rangle_{\hat{\rho}} \tag{14.18}$$

For a pure state, we can see this is true easily from $\langle\psi|\hat{A}|\psi\rangle$ and $\hat{\rho} = |\psi\rangle\langle\psi|$, and by linearity, it can also be proven for mixed states.

14.3   Useful Properties of Trace

➤ $\mathrm{Tr}[\hat{A}\hat{B}] = \mathrm{Tr}[\hat{B}\hat{A}]$, even if $[\hat{A}, \hat{B}] \neq 0$. This comes from cyclic permutation: $\mathrm{Tr}[\hat{A}\hat{B}\hat{C}] = \mathrm{Tr}[\hat{B}\hat{C}\hat{A}] = \mathrm{Tr}[\hat{C}\hat{A}\hat{B}]$.

➤ $\mathrm{Tr}[\hat{A} + \hat{B}] = \mathrm{Tr}[\hat{A}] + \mathrm{Tr}[\hat{B}]$.

➤ The condition for $\hat{\rho}$ to be a density operator comes from conservation of probability and is (for pure and mixed states): $\mathrm{Tr}[\hat{\rho}] = 1$.
Whereas $\mathrm{Tr}[\hat{\rho}^2] = \mathrm{Tr}[\hat{\rho}]$ **only for pure states.**

14.4   Entangled States

Entangled states are quantum correlation, where these are more correlated than it is possibly to get in classical physics.
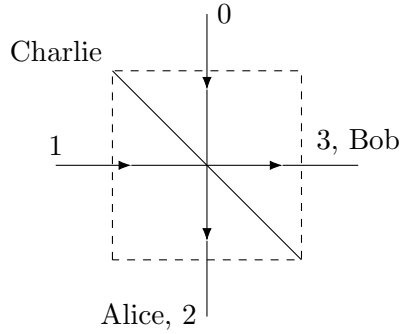
Charlie has 0 and 1 and makes the states. So we will input $|+\rangle$ states, where

$$|+\rangle = \frac{1}{\sqrt{2}}|H + V\rangle, \tag{14.19}$$

with a polariser of $\frac{\pi}{4}$.

$$|+_0\rangle|+_1\rangle = \frac{1}{2}\left\{(H_0 + V_0)(H_1 + V_1)\right\} \tag{14.20}$$

$$= \frac{1}{2}\left\{|H_0 H_1\rangle + |H_0 V_1\rangle + |V_0 H_1\rangle + |V_0 V_1\rangle\right\} \tag{14.21}$$

$$= \frac{1}{2}\left\{-|H_2 H_3\rangle + i|H_3 V_3\rangle + i|V_2 H_2\rangle + |V_2 V_3\rangle\right\} \tag{14.22}$$

$$= \frac{1}{2}\left\{|V_2 V_3\rangle - |H_2 H_3\rangle + i(|H_3 V_3\rangle + |H_2 V_2\rangle)\right\} \tag{14.23}$$

We will discard the imaginary term in what is called post selection and renormalise, by considering when both outputs get one photon, but we don't know which. The imaginary term describes when either state gets both photons so we ignore it.

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|V_2 V_3 - H_2 H_3\rangle\right) \tag{14.24}$$

What happens if we measure if it has horizontal or vertical polarised photons? We send into another beamsplitter and see where it comes out.
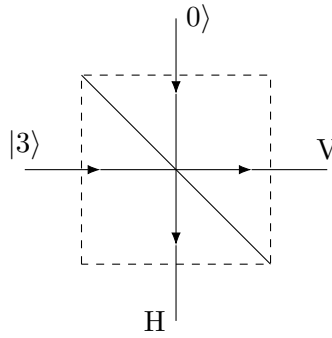
But how do we write the state $|3\rangle$ that we are putting into the polarising beamsplitter?

$$|\psi_{AB}\rangle\langle\psi_{AB}| = \hat{\rho}_{AB} \tag{14.25}$$

$$\mathrm{Tr}_B[\hat{\rho}_{AB} \equiv \mathrm{Tr}_B[\hat{\rho}_{AB}\mathbb{I}_B] \tag{14.26}$$

$$\hat{\rho}_{AB} = \frac{1}{2}\left(|V_2V_3\rangle\langle V_2V_3| - |V_2V_3\rangle\langle H_2H_3| - |H_2H_3\rangle\langle V_2V_3| + |H_2H_3\rangle\langle H_2H_3|\right) \tag{14.27}$$

$$\mathbb{I}_B = |H_3\rangle\langle H_3| + |V_3\rangle\langle V_3| \tag{14.28}$$

Note the projector is not normalised. We are going to measure then ignore the outcome.

$$\hat{\rho}_A = \mathrm{Tr}_B[\hat{\rho}_{AB}] = \frac{1}{2}\left(|H_2\rangle\langle H_2| + |V_2\rangle\langle V_2|\right) \tag{14.29}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{14.30}$$

So for either Alice or Bob it looks like what they're getting is a stream of photons which are randomly either horizontally or vertically polarised but there are no coherences here. This is one of the signatures of entanglement - *you start with a pure state, you trace out one of the parties, and the other one will see something mixed.*

Now let's see what happens when they do the measurements. So Charlie is sending a stream of photons like this, and Alice and Bob can see the following:

Alice	0	0	H	2	V	H	2	2	V	V
Bob	2	2	H	0	V	H	0	0	V	V

So we get a whole stream of different outcomes, but the key thing is is that whenever they get a single photon each, they will get the same outcomes for these photons and know what each gets. What makes this Quantum is that you can do this in any pair of bases and you will always get this level of correlation, regardless of the polarising angle of the beamsplitter (as long as it is a shared angle) and this will not work on classical states.

This example is the first half of how you do quantum key distribution for quantum cryptography - the other half is how to stop somebody intercepting the photons and cheating on you.

Part II

Quantum Theory In Condensed Matter

Lecture 1   Bose-Einstein Statistics

1.1   Statistical Ensembles

In statistical mechanics, one frequently formally considers N copies of the system under consideration. Such an assembly is a *statistical ensemble.*

➤ In the *microcanonical ensemble*, each copy has the same total poarticle number N and total energy U.

➤ In the *canonical ensemble*, each copy has the same total particle number N, however the total energy may fluctuate. The system is considered to be in equilibrium with an external heat bath (maintaining a constant temperature T.

➤ In the *grand canonical ensemble*, the total particle number and the total energy may fluctuate. The system is considered to be in equilibrium with an external heat bath, and a particle bath (maintaining a constant chemical potential μ).

We are generally interested in macroscopic systems of an effectively infinite number of particles, and may take the *thermodynamic limit* $V \to \infty$ in which the particle density $n = N/V$ is held constant. The thermodynamic predictions of the three ensembles are then identical, and it is usually more convenient to use the grand canonical ensemble. If the N-body quantum states of N quantum particles have energy $E_n^{(N)}$ for $n = 1, 2, \ldots$, then in the grand canonical ensemble, each state occurs with probability

$$P_n^{(N)} = \frac{1}{Z} \exp\left(-\frac{1}{k_B T}\left[E_n^{(N)} - \mu N\right]\right), \text{ where } Z = \sum_{N,k} \exp\left(-\frac{1}{k_B T}\left[E_n^{(N)} - \mu N\right]\right) \quad (1.1)$$

is the *grand partition function.*

1.2   Particle with Periodic Boundary Conditions

Consider a point particle of mass m moving within a volume $V = L_x L_y L_z$, where L_i are lengths in the $i = x, y, z$ directions. The Hamiltonian

$$\hat{H} = \frac{1}{2m}\left(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2\right), \quad (1.2)$$

and the eigenstates of the momentum operator $\hat{p} = (\hat{p}_x, \hat{p}_y, \hat{p}_z)$ are also eigenstates of $\hat{H}$ (subject to periodic boundary conditions). In the *position representation*, they take the form of plane waves:

$$\psi_k(\underline{r}) = \frac{1}{\sqrt{V}}e^{i\underline{k}\cdot\underline{r}}, \text{ where } \underline{k} = \left(\frac{2\pi n_x}{L_x}, \frac{2\pi n_y}{L_y}, \frac{2\pi n_z}{L_z}\right). \quad (1.3)$$

The differences in k between neighbouring eigenstates are $2\pi/L_i$, in the $i = x, y, z$ directions. The k-space density of the available states is therefore

$$\rho(\underline{k}) = \frac{L_x L_y L_z}{(2\pi)^3} = \frac{V}{(2\pi)^3}. \quad (1.4)$$

1.3   Spherical Shells in k

We divide up k-space into thin concentric shells. A shell of radius k_S and thickness δk_S has volume $4\pi k_S^2 \delta k_S$, and therefore contains M_S states, where

$$M_S = 4\pi k_S^2 \delta k_S \rho(\underline{k}) = 4\pi k_S^2 \delta k_S \frac{V}{(2\pi)^3}. \quad (1.5)$$

Each such quantum state has energy $\epsilon_S = \hbar^2 k_S^2/2m$, from which we can determine an energy increment $\delta\epsilon_S = (\hbar^2/m)k_S \delta k_S$. Hence, the number of available states between energies ϵ_S and $\epsilon_S + \delta\epsilon_S$ is

$$M_S = V g(\epsilon_S)\delta\epsilon_S, \text{ where } g(\epsilon) = \frac{m^{3/2}}{\sqrt{2}\pi^2\hbar^3}\epsilon^{1/2} \quad (1.6)$$

is the (energy) density of states per unit volume. *All states on a given shell have the same energy.*

1.4 Thermal Equilibrium and the Bose-Einstein Distribution

Consider an *ideal Bose gas*, i.e. a system of many identical, non-interacting bosons, e.g. in a box subject to periodic boundary conditions. In *thermal equilibrium*, the particles are distributed so that the occupation numbers N_S in each shell of energy ϵ_S maximise the total entropy. This yields the *Bose-Einstein distribution:*

$$f_{BE}(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/k_B T} - 1}. \tag{1.7}$$

1.5 Particle Density in the Thermodynamic Limit

Using the Bose-Einstein distribution in Eq (1.7), the total number of particles in a volume V (subject to periodic boundary conditions) is

$$N = \sum_{\underline{k}} \frac{1}{e^{(\epsilon_{\underline{k}}-\mu)/k_B T} - 1}. \tag{1.8}$$

Taking the thermodynamic limit $V \to \infty$, the possible $\underline{k}$ values tend to a continuum, and we replace the sum $\sum_{\underline{k}}$ with an integral $\int \rho(\underline{k}) \, dk_x \, dk_y \, dk_z \equiv \int \rho(\underline{k}) \, d^3k = [V/(2\pi)^3] \int d^3k$. Hence

$$N = \frac{V}{(2\pi)^3} \int \frac{d^3k}{e^{(\epsilon_{\underline{k}}-\mu)/k_B T} - 1} \implies n \equiv \frac{N}{V} = \frac{1}{(2\pi)^3} \int \frac{d^3k}{e^{(\epsilon_{\underline{k}}-\mu)k_B T} - 1}. \tag{1.9}$$

Recalling $V/(2\pi)^3 \, d^3k = [V/(2\pi)^3]4\pi k^2 \, dk = V g(\epsilon) \, d\epsilon$, then in terms of the density of states per unit volume, $g(\epsilon)$,

$$n = \int \frac{g(\epsilon) \, d\epsilon}{e^{(\epsilon_{\underline{k}}-\mu)/k_B T} - 1}. \tag{1.10}$$

Lecture 2 Bose-Einstein Condensation

2.1 Particle Density in Terms of the Fugacity

Using the fugacity $z \equiv e^{\mu/k_B T}$ and $x \equiv \epsilon/k_B T$, we can rewrite the particle density n as

$$n = \frac{(mk_B T)^{3/2}}{\sqrt{2}\pi^2 \hbar^3} \int_0^\infty \frac{ze^{-x}}{1 - ze^{-x}} x^{1/2}\, dx. \tag{2.1}$$

Taylor expanding $(1 - ze^{-x})^{-1}$ around $ze^{-x} = 0$, we deduce the identity

$$ze^{-x}(1 - ze^{-x})^{-1} = ze^{-x}\left(1 + ze^{-x} + z^2 e^{-2x} + \dots\right) = \sum_{j=1}^\infty z^j e^{-jx}. \tag{2.2}$$

As $x > 0$, this expansion is convergent for $z < 1$, and we can evaluate the integral over x using:

$$\int_0^\infty e^{-jx} x^{1/2}\, dx = \frac{1}{j^{3/2}} \underbrace{\int_0^\infty e^{-y} y^{1/2}\, dy}_{\equiv \Gamma(3/2)} = \frac{1}{j^{3/2}} \frac{\sqrt{\pi}}{2}. \tag{2.3}$$

This implies

$$n = \left(\frac{mk_B T}{2\pi\hbar^2}\right)^{3/2} g_{3/2}(z), \text{ where } g_{3/2}(z) = \sum_{j=1}^\infty \frac{z^j}{j^{3/2}}. \tag{2.4}$$

We therefore need to know something of the properties of $g_{3/2}(z)$ to evaluate the particle density n.

2.2 Properties of g

The series defining $g_{3/2}(z)$ converges when $|z| < 1$, and diverges when $|z| > 1$. At $z = 1$, it is just convergent:

$$g_{3/2}(1) = \zeta(3/2) = 2.612, \text{ where } \zeta(s) \equiv \sum_{j=1}^\infty \frac{1}{j^s} \tag{2.5}$$

is the Riemann zeta function. The derivative of $g_{3/2}(z)$, however, is infinite at $z = 1$, since

$$\left.\frac{d}{dz} g_{3/2}(z)\right|_{z=1} = \frac{1}{z}\sum_{j=1}^\infty \frac{z^j}{j^{1/2}}\bigg|_{z=1} = \sum_{j=1}^\infty \frac{1}{j_{1/2}} \tag{2.6}$$

diverges. With these limiting values, we can make a sketch of $g_{3/2}(z)$ for $0 \le z \le 1$:
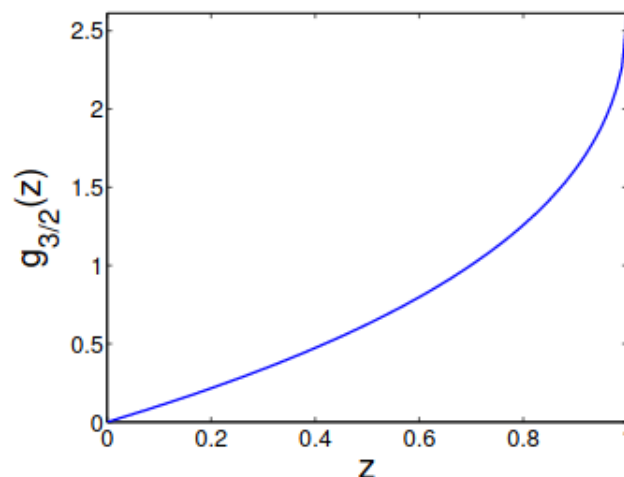
2.3   Critical Temperature

We can invert Eq (2.4) to get

$$g_{3/2}(z) = \left(\frac{2\pi\hbar^2}{mk_BT}\right)^{3/2} n. \tag{2.7}$$

If we are in a regime of high T or low n, it follows that $g_{3/2}(z)$ must take a small value. From the plot of $g_{3/2}(z)$ above, we see that this implies small z, which from the definition of $g_{3/2}(z)$ means we can approximate $g_{3/2}(z) \approx z$. Hence, recalling $z = e^{\mu/k_BT}$,

$$\mu \approx -\frac{3}{2}k_BT\ln\left(\frac{mk_BT}{2\pi\hbar^2 n^{2/3}}\right), \tag{2.8}$$

i.e., the chemical potential is negative.

As n is fixed, it follows from Eq (2.4) that as T is lowered (but assumed $T > 0$), z must increase, with maximum finite value of $z = 1$, i.e. $\mu = 0$. It straightforwardly follows that this occurs at a *critical temperature* (recalling $g_{3/2}(1) = 2.612$) of

$$T_C = \frac{2\pi\hbar^2}{k_Bm}\left(\frac{n}{2.612}\right)^{2/3}. \tag{2.9}$$

2.4   Macroscopic Occupation of the Ground State

Recall that the Bose-Einstein distribution from Eq (1.7) predicts the occupation of $\epsilon = \epsilon_{\underline{k}=0} = 0$ state to be

$$N_0 = \frac{1}{e^{-\mu/k_BT} - 1}. \tag{2.10}$$

Hence, $\mu = 0 \implies N_0 \to \infty$ (Bose-Einstein condensation). Recalling that we are in the thermodynamic limit ($N, V \to \infty$), what this actually means is that $n_0 = N_0/V$ is some finite fraction of n. Rewriting Eq (2.10) for large N_0, we obtain

$$\mu = -k_BT\ln\left(1 + \frac{1}{N_0}\right) \approx -\frac{k_BT}{N_0}. \tag{2.11}$$

Hence, as $N_0 \to \infty$, $\mu \to 0$. Therefore, below T_C, the chemical potential μ is effectively 0.

2.5   Below the Critical Temperature

Below T_C, we must take $\underline{k} = 0$ point into account separately, and so we set $\mu = 0$ and modify Eq (1.8) to

$$N = N_0 + \sum_{\underline{k}\neq 0}^{\infty} \frac{1}{e^{\epsilon_{\underline{k}}/k_BT} - 1}. \tag{2.12}$$

Again, replacing the sum by an integral (excluding $\underline{k} = 0$), the particle density is given by

$$n = n_0 + \frac{(mk_BT)^{3/2}}{\sqrt{2}\pi^2\hbar^3}\int_0^\infty \frac{e^{-x}}{1 - e^{-x}}x^{1/2}\,dx. \tag{2.13}$$

The integral is identical to that evaluated in Sec (2.1), and so for $T < T_C$,

$$n = n_0 + 2.612\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2}. \tag{2.14}$$

One speaks of a *condensate density* n_0, and a *normal density* n_n. Using the definition of T_C in Eq (2.9), the *condensate fraction* can be compactly written as

$$\frac{n_0}{n} = 1 - \left(\frac{T}{T_C}\right)^{3/2}. \tag{2.15}$$

Hence, at $T = 0$, all particles are in the ground state, but the proportion decreases to zero as $T \to T_C$ from below, and is zero above T_C.

2.6   Discontinuity in Heat Capacity Implies Phase Transition

The heat capacity can be obtained by differentiating the internal energy per particle u while keeping the density n constant:

$$C_V = \frac{\partial u}{\partial T}. \tag{2.16}$$

The total internal energy U can be determined by multiplying the density of states $g(\epsilon)$ by the volume V and the Bose-Einstein distribution $f_{BE}(\epsilon)$, and then integrating the resulting total energy distribution, multiplied over ϵ, over all values of ϵ:

$$U = V \int_0^\infty \frac{\epsilon g(\epsilon)\, d\epsilon}{e^{(\epsilon - \mu)/k_B T} - 1} = \frac{V(k_B T)^{5/2} m^{3/2}}{\sqrt{2}\pi^2 \hbar^3} \int_0^\infty \frac{z e^{-x}}{1 - z e^{-x}} x^{3/2}\, dx. \tag{2.17}$$

To calculate the average energy per particle,

$$u = \frac{U}{N} \equiv \frac{U/V}{N/V}, \tag{2.18}$$

we use the expression for n in Eq (2.4) to determine (by similar methods as in Sec 2.1), for $T > T_C$,

$$\frac{U}{V} = \frac{(k_B T)^{5/2} m^{3/2}}{\sqrt{2}\pi^2 \hbar^3} \underbrace{\sum_{j=1}^\infty \frac{z^j}{j^{5/2}}}_{\equiv g_{5/2}(z)} \underbrace{\int_0^\infty e^{-y} y^{3/2}\, dy}_{\equiv \Gamma(5/2)}, \tag{2.19}$$

$$u = k_B T \frac{\Gamma(5/2) g_{5/2}(z)}{\Gamma(3/2) g_{3/2}(z)} = \frac{3 k_B T}{2} \frac{g_{5/2}(z)}{g_{3/2}(z)}, \tag{2.20}$$

where we have used the identity $\Gamma(t) = (t-1)\Gamma(t-1)$. Note that $g_{5/2}(z)$ and $g_{3/2}(z) \to z$ as $z \to 0$, and so $\mu \approx \frac{3}{2} k_B T$ when $T \gg T_C$, i.e. the classical result.
For $T < T_C$, we use n as given by Eq (2.9), and determine

$$u = \frac{3 k_B}{2} \frac{T^{5/2}}{T_C^{3/2}} \frac{g_{5/2}(1)}{g_{3/2}(1)}, \tag{2.21}$$

where $g_{5/2}(1) = \zeta(5/2) = 1.342$.
For $T \gg T_C$, we obtain the classical result

$$C_V = \frac{\partial u}{\partial T} \approx \frac{3}{2} k_B; \tag{2.22}$$

while for $T < T_C$,

$$C_V + \frac{15}{4} \frac{g_{5/2}(1)}{g_{3/2}(1)} \left(\frac{T}{T_C} \right)^{3/2} k_B. \tag{2.23}$$

When plotted as a function of T, the curves for $T < T_C$ and $T > T_C$ meet at a cusp (discontinuity in slope). This implies that the free energy is not analytic at T_C, and so BEC is a true thermodynamic *phase transition*.

doodle plot from visualiser

Lecture 3 BEC in Ultracold Dilute Gases of Alkali Atoms

3.1 Bosonic Alkali Atomic Gases

Alkali metals are group 1 elements, and have a single valence electron in the outermost s-orbital: $2s$ for Li, $3s$ for Na, $4s$ for K, $5s$ for Rb. The other electrons are in fully occupied shells, and therefore have total orbital and spin angular momentum $= 0$. If the isotope is such that the nucleus is composed of an odd total number of protons and neutrons, the nucleus will have a net half-integer spin ($= \frac{3}{2}$ for ^{87}Rb, ^{23}Na, ^{7}Li). The *total spin* will then take an integer value ($= 1, 2$ for ^{87}Rb, ^{23}Na, ^{7}Li). It is possible to prepare the gas such that only one of these states is present, which can be considered a gas of identical particles having integer spin, i.e. *identical bosons.*

3.2 Critical Temperature

Densities of trapped atomic gases are typically $\approx 10^{11} - 10^{15}$cm^{-3}. Using such values and appropriate atomic masses to determine T_C for the *ideal, homogeneous* Bose gase yield T_C between 10 nanoKelvin and 1 microKelvin. Such temperatures can be achieved relatively straightforwardly by methods of laser and evaporative cooling; BEC was first observed in 1995 in magnetically trapped gases of ^{87}Rb and ^{23}Na (Nobel prize in physics 2001).

3.3 Atom-Atom Interactions

The atoms interact with each other, and so we do not have an ideal Bose gas (interactions are in fact essential to establish thermal equilibrium during cooling). Interactions are dominated by elastic two-body collisions (in such dilute gases, the probability of 3 atoms meeting at the same point in space is extremely low). No binding takes place in elastic collisions, and so atomic clusters do not tend to form. At the very cold temperatures under consideration, the collisions are very low energy, and the pair-interaction may be approximated by

$$V(\underline{r} - \underline{r}') \approx g\delta(\underline{r} - \underline{r}'), \text{ where } g = \frac{4\pi\alpha_s\hbar^2}{m}, \tag{3.1}$$

and α_s is the s-wave scattering length.

3.4 Mean-Field Potential

Usually α_s is positive, implying a net repulsive interaction. On average this can be represented by an additional potential, proportional to the *atomic density $n(\underline{r})$*, felt by each particle due to its interaction with all other particles:

$$V_{\text{eff}}(\underline{r}) = gn(\underline{r}). \tag{3.2}$$

In this approximation, the trapped atoms obey an effective Schrodinger equation,

$$H_{\text{eff}}\psi(\underline{r}) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{trap}}(\underline{r}) + V_{\text{eff}}(\underline{r}) \right]\psi(\underline{r}) = \epsilon\psi(\underline{r}), \tag{3.3}$$

where the trapping potential $V_{\text{trap}}(\underline{r})$ can usually be considered harmonic. Finding the steady states $\psi(\underline{r})$ and their corresponding "eigenvalues" ϵ is not simply a matter of diagonalising H_{eff} however, as $V_{\text{eff}}(\underline{r})$ is dependent on the state of the system, i.e. the effective Schrodinger equation is *non-linear.*

3.5 Gross-Pitaevskii Equation

An appropriate thermodynamic limit is to let $N \to \infty$ while relaxing the trap. If

$$V_{\text{trap}}(\underline{r}) = \frac{m\omega^2 r^2}{2}, \tag{3.4}$$

mathematically this corresponds to keeping $N\alpha_s/\alpha_h$ constant, as while letting $N \to \infty$,

$$\alpha_h \equiv \sqrt{\frac{\hbar}{m\omega}} \to \infty. \tag{3.5}$$

Below a critical temperature T_C, a transition to a "macroscopically occupied" quantum state occurs, but the mode (i.e. the spatical dependency of this state) changes with varying $T < T_C$ due to the combination of $V_{\text{trap}}(\underline{r})$ and atom-atom interactions. In the thermodynamic limit and as $T \to 0$, all but a vanishingly small proportion of atoms enter a "ground state" mode $\psi_0(\underline{r})$, which obeys the *Gross-Pitaevski equation*:

$$\left[-\frac{\hbar^2}{2m} + V_{\text{trap}}(\underline{r}) + gN|\psi_0(\underline{r})|^2 \right] \psi_0(\underline{r}) = \epsilon_0 \psi_0(\underline{r}). \tag{3.6}$$

This is a very useful equation for typical atom numbers ($\approx 10^3 - 10^8$).

3.6 Thomas-Fermi Limit

Consider $V_{\text{trap}}(\underline{r}) = m\omega^2 r^2/2$. All other things being equal, of gN is allowed to become very large (this corresponds to letting $N\alpha_s/\alpha_h \to \infty$), the kinetic energy is completely dominated over most of the mode bby the potential energy. Hence,

$$\left[\frac{m\omega^2 r^2}{2} + gN|\psi_0(\underline{r})|^2 \right] \approx \epsilon_0. \tag{3.7}$$

We can therefore determine a limiting spatial dependency of $\psi_0(\underline{r})$:

$$\psi_0(\underline{r}) = \begin{cases} \sqrt{\frac{\epsilon_0 - m\omega^2 r^2/2}{gN}} & \text{if } r \leq \sqrt{\frac{2\epsilon_0}{m\omega^2}}, \\ 0 & \text{if } r > \sqrt{\frac{2\epsilon_0}{m\omega^2}}, \end{cases} \tag{3.8}$$

and knowing the norm of $\psi_0(\underline{r})$ must be 1, one can show that in this limit

$$\epsilon_0 = \left(\frac{15gN}{8\pi} \right)^{2/5} \left(\frac{m\omega^2}{2} \right)^{3/5}. \tag{3.9}$$

3.7 Hydrodynamic Formulation

To a good approximation, the atomic condensate dynamics can frequently be described by the *time dependent* Gross-Pitaevskii equation:

$$i\hbar \frac{\partial}{\partial t} \psi(\underline{r}) = \left[-\frac{\hbar^2}{2m} \nabla^2 + V_{\text{trap}}(\underline{r}) + gN|\psi(\underline{r})|^2 \right] \psi(\underline{r}). \tag{3.10}$$

Defining the probability density $\rho(\underline{r}) = |\psi(\underline{r})|^2$ and a velocity field

$$\underline{v}(\underline{r}) = \frac{\hbar}{2im\rho(\underline{r})} \left[\psi^*(\underline{r}) \nabla \psi(\underline{r}) - \psi(\underline{r}) \nabla \psi^*(\underline{r}) \right], \tag{3.11}$$

it is possible to formulate a hydrodynamic description. The (coupled) equations of motion are the *continuity equation*,

$$\frac{\partial}{\partial t} \rho(\underline{r}) = -\nabla \cdot \left[\rho(\underline{r}) \underline{v}(\underline{r}) \right], \tag{3.12}$$

and the equation of motion for the velocity field,

$$m \frac{\partial}{\partial t} \underline{v}(\underline{r}) + -\nabla \left[\frac{m}{2} \underline{v}(\underline{r})^2 + V_{\text{trap}}(\underline{r}) + gN\rho(\underline{r}) - \frac{\hbar^2}{2m\sqrt{\rho(\underline{r})}} \nabla^2 \sqrt{\rho(\underline{r})} \right]. \tag{3.13}$$

If the density is relatively smooth, the final term above can be neglected. In this case, together with the continuity equation, we have a dynamical description of a fluid with *zero viscosity*, i.e. *a superfluid*. When considering a fluid, one would typically not have a $V_{\text{trap}}(\underline{r})$ term, and it would also be more usual to write the equations in terms of $n(\underline{r}) = N\rho(\underline{r})$, i.e. the condensate number density, in the present context.

Lecture 4   Classical and Quantum Fluids

4.1   Characteristic Properties of Classical and Quantum Fluids

A *fluid* freely deforms or *flows*, i.e. a gas or liquid. In normal (classical) fluids, their physical properties are determined from classical statistical mechanics. A *quantum fluid* remains fluid at sufficiently low temperatures for the effects of quantum mechanics to have a dominant role.

4.2   Statistical Mechanics of Classical Interacting Many-Body Systems

The classical Hamiltonian for N mass m interacting identical point particles is

$$H(\underline{p}_1, \ldots, \underline{p}_N, \underline{r}_1, \ldots, \underline{r}_N) = \frac{1}{2m} \sum_{k=1}^{N} \underline{p}_k^2 + \frac{1}{2} \sum_{j \neq k=1}^{N} V(\underline{r}_j - \underline{r}_k). \tag{4.1}$$

The probability of each possible configuration is given by the Boltzmann probability density for N particles and temperature T,

$$P(\underline{p}_1, \ldots, \underline{p}_N, \underline{r}_1, \ldots, \underline{r}_N) = \frac{1}{Z_N} e^{-H/k_B T}, \tag{4.2}$$

$$\text{where } Z_N = \left(\frac{1}{N!}\right) \int e^{-H/k_B T} \, d^3 p_1 \ldots d^3 p_N \, d^3 r_1 \ldots d^3 r_N \tag{4.3}$$

is the *classical partition function*. The $\frac{1}{N!}$ accounts for the particles' indistinguishability. Z_N factorises into position- and momentum-dependent terms:

$$Z_N = \left(\prod_{k=1}^{N} \int e^{-p_k^2/2mk_B T} \, d^3 p_k\right) Q_N = (2\pi m k_B T)^{3N/2} Q_N, \tag{4.4}$$

$$\text{where } Q_N = \left(\frac{1}{N!}\right) \int \exp\left(-\sum_{j \neq k=1}^{N} \frac{V(\underline{r}_j - \underline{r}_k)}{2k_B T}\right) d^3 r_1 \ldots d^3 r_N. \tag{4.5}$$

The momentum part of Z_N is a product of independent individual particle terms. Hence, individual particle momenta are statistically independent, and a particle has momentum which is in a region $d^3 p$ of momentum space with probability

$$P(\underline{p}) \, d^3 p = \frac{e^{-p^2/2mk_B T}}{(2\pi m k_B T)^{3/2}} \, dp. \tag{4.6}$$

The fraction of particles with momentum *magnitude* between p and $p + dp$ is

$$P_{MB}(p) \, dp = \frac{4\pi p^2 e^{-p^2/2mk_B T}}{(2\pi m k_B T)^{3/2}} \, dp, \tag{4.7}$$

where $P_{MB}(p)$ is the *Maxwell-Boltzmann* momentum distribution.

4.3   Interactions in Helium and the Noble Gases

The atoms interact predominantly via a pairwise interaction potential $V(r)$, where r is the *relative coordinate* describing the distance between atoms. $V(r)$ contains a short-ranged repulsion and a weak, long-range van der Waals attraction, and it can be sufficient to consider a *Lennard-Jones* model potential:

$$V(r) = \epsilon_0 \left[\left(\frac{r_0}{r}\right)^{12} - 2\left(\frac{r_0}{r}\right)^6\right], \tag{4.8}$$

where ϵ_0 is the potential depth, and r_0 is the position of the potential minimum.

4.4 Thermal de Broglie Wavelength

The de Broglie wavelength is defined as

$$\lambda = \frac{2\pi\hbar}{p}. \tag{4.9}$$

For a finite temperature system, a characteristic length scale is set by the *thermal de Broglie wavelength*,

$$\lambda_{dB} = \frac{2\pi\hbar}{\sqrt{2\pi m k_B T}} = \sqrt{\frac{2\pi\hbar^2}{m k_B T}}. \tag{4.10}$$

λ_{dB} corresponds to a momentum $\sqrt{2\pi m k_B T}$, which is distinct from both the *mean* momentum $\sqrt{8 m k_B T/\pi}$ and the *most probable* momentum $\sqrt{2 m k_B T}$. Quantum effects should be significant once λ_{dB} is comparable to other typical length scales in the fluid.

4.5 Significance of Quantum Effects in the Liquid Phase(s)

At standard atmospheric pressure, Neon (atomic mass 20) liquifies at $\approx 27\,\text{K}$ and freezes at $\approx 24\,\text{K}$. The $\lambda_{dB} \approx 0.07\,\text{nm}$ is much less than $r_0 = 0.296\,\text{nm}$, and quantum effects are unimportant throughout the gas and liquid phases. Helium 4 (atomic mass 4) liquifies at $4\,\text{K}$. There are two liquid phases (He I and He II) and no solid phase at standard atmospheric pressure. Hence $\lambda_{dB} \approx 0.4\,\text{nm}$ is comparable to $r_0 = 0.264\,\text{nm}$, ad quantum effects are significant.

4.6 Phenomenological Low Temperature Model of a Solid Phase

Assume each atom vibrates with angular frequency ω_0 around its equilibrium position in a crystal lattice as an independent quantum harmonic oscillator, with zero-point energy per atom,

$$E_0 = \frac{3}{2}\hbar\omega_0. \tag{4.11}$$

Assume a face-centred cubic lattice structure (12 nearest neighbours, equivalent to 4 springs of spring constant k acting in each of the x, y, and z directions). The spring constant k (and hence $\omega_0 = \sqrt{4k/m}$) can be taken from the second-order coefficient from a Taylor expansion of a Lennard-Jones potential about the equilibrium position r_0:

$$k = \frac{1}{2}\left.\frac{d^2V(r)}{dr^2}\right|_{r=r_0} = \frac{36\epsilon_0}{r_0^2}. \tag{4.12}$$

4.7 Absence of a Solid Phase in Helium

From this simple model, for Neon ($\epsilon_0 = 3.94\,\text{meV}$, $r_0 = 0.296\,\text{nm}$), the zero-point energy

$$E_0 \equiv 3\hbar\sqrt{\frac{36\epsilon_0}{mr_0^2}} \approx 4\,\text{meV}, \tag{4.13}$$

which is comparable to ϵ_0. Thus Neon supports a solid phase. For Helium ($\epsilon_0 = 1.03\,\text{meV}$, $r_0 = 0.265\,\text{nm}$), this yields a zero-point energy $E_0 \approx 7\,\text{meV}$ which is must greater than the potential well depth ϵ_0, and there is no solid phase.

4.8 Singularity in Heat Capcity/Specific Heat as a Function of Temperature

Plotting C_V as a function of T, at the boundary between the He I and He II phases ($T = T_C = 2.17\,\text{K}$) one observes a singularity, known as the *lambda point* (due to the alleged resemblance of such a plot near T_C to the letter λ). At low T, $C_V \propto T^3$ (unlike BEC). Near T_C, C_V has a weak power law behaviour:

$$C_V = \begin{cases} C(T) + A_+|T - T_C|^{-\alpha}, & T > T_C, \\ C(T) + A_-|T - T_C|^{-\alpha}, & T < T_C, \end{cases} \tag{4.14}$$

where $C(T)$ is a smooth function of T near T_C. Within the theory of phase transitions, measured values of the *critical exponent* $\alpha(=-0.009)$, A_+, and A_- are consistent with this phase transition belonging to a *universality class* known as the three-dimensional XY-model. The *order* of such systems is described by a two-dimensional unit vector at every point in space $\underline{n}(\underline{r})$. In He I, this vector is spatially random; in He II, there is a spatial ordering, like the ordering in a ferromagnet.

4.9   The Macroscopic Wavefunction

A two-dimensional unit vector can be equivalently described by the phase of a *complex number*, such as the Gross-Pitaevskii wavefunction, motivating the introduction of a similar macroscopic wavefunction $\psi_0(\underline{r})$ here. The wavefunction corresponds to a *condensate*, or macroscopic number of particles. We can normalise $\psi_0(\underline{r})$ such that

$$n_0 = |\psi_0|^2 \text{ is the condensate density,} \tag{4.15}$$

$$N_0 = n_0 V = \int |\psi_0(\underline{r})|^2 \, d^3r \text{ is the number of condensed particles, and} \tag{4.16}$$

$$\psi_0(\underline{r}) = \sqrt{n_0(\underline{r})}e^{i\theta(\underline{r})}. \tag{4.17}$$

If $|\psi_0(\underline{r})| = 0$, the phase $\theta(\underline{r})$ is undefinedl if $|\psi_0(\underline{r})| \neq 0$, the phase $\theta(\underline{r})$ is a natural parameter of the system. $\psi_0(\underline{r})$ is the *order parameter* of the He II phase.

Lecture 5   Superfluiditity in Helium II

5.1   Superfluid Viscosity

Assuming a macroscopic wavefunction $\psi_0(\underline{r}) = \sqrt{n_0}e^{i\theta}$, we define the *superfluid velocity* $\underline{v}_s$, which defines a *current density* (particle flow) $\underline{j}_0 = n_0\underline{v}_s$.

5.2   Superflow and Viscosity

A normal fluid's flow viscosity v through a (length L, radius R) capillary depends on the fluid's viscosity η and the pressure difference $\Delta P = P_1 - P_2$, such that

$$\frac{\Delta P}{L} \approx \eta \frac{v}{R^2}. \tag{5.1}$$

Piotr Kapitza found that with He II, ΔP was always 0 whatever the value of v. *Superflow* is therefore fluid flow with *zero viscosity*.

Alternatively, consider a disc suspended in fluid and made to undergo torsional oscillations about its axis. The oscillation frequency depends on the torsional stiffness of the support, and the moment of inertia of the disc. In a viscous fluid, a thin layer of fluid is dragged along with the disc's motion, effectively increasing the intertia of the disc. With a stack of closely spaced discs, all fluid between the discs should be dragged along, contributing to the inertia. Immersing a stack of discs in He II, **Elepter Andronikashvili** showed that a temperature-dependent fraction of He II contributed to the intertia but a fraction did not. This motivated a *two fluid* model of He II.

5.3   Two Fluid Model

The total particle density of the fluid is $n = n_s + n_n$, and the *normal component* n_n acts like a conventional viscous fluid. It contributes to the inertia of the rotating discs, but friction with the walls prevents flow down the capillary. The superfluid component n_s flows with zero viscosity. It does not contribute to the inertia of the rotating discs, but flows freely down the capillary. The two fluid components move relative to each other without friction.

5.4   Temperature Dependence of the Superfluid and Normal Components

As $T \to 0$, almost all fluid is superfluid, i.e. $n_s \approx n, n_n \approx 0$, and near $T = 0$ the temperature dependence is $n_s(T) \approx n - AT^4$ (A is a constant). Near the critical temperature T_C nearly all the fluid is normal, and it is observed experimentally that n_s vanishes like a power law:

$$n_s \approx \begin{cases} B(T_C - T)^a & T < T_C, \\ 0 & T > T_C, \end{cases} \tag{5.2}$$

where the critical exponent $a \approx 0.67$ (again consistent with predictions based on the three-dimensional XY model) and B is another constant.

5.5   Two Fluid Hydrodynamical Quantities

The *superfluid density* is the mass density of the superfluid component $\rho_s = mn_s$. There are two types of current flow, so that the total current density $\underline{j} = \underline{j}_s + \underline{j}_n$, where

$$\underline{j}_s = n_s\underline{v}_s, \qquad\qquad\qquad \underline{j}_n = n_n\underline{v}_n, \tag{5.3}$$

and $\underline{v}_s, \underline{v}_n$ are the velocities of the two fluid components.

5.6   Flow Quantisation

From the definition of superfluid velocity, we have $\underline{v}_s = \frac{\hbar}{m}\underline{\nabla}\theta$. Hence,

$$\underline{\nabla} \times \underline{v}_s \equiv \frac{\hbar}{m}\underline{\nabla} \times \underline{\nabla}\theta = 0, \tag{5.4}$$

i.e. the flow is *irrotational*. Consider a flow around the inside of a torus, in particular a closed path going all the way around the torus exactly once. The flow circulation κ is defined by the integral

$$\kappa = \oint \underline{v}_s \cdot d\underline{r} = \frac{\hbar}{m} \oint \underline{\nabla}\theta \cdot d\underline{r} = \frac{\hbar}{m}\Delta\theta, \tag{5.5}$$

where $\Delta\theta$ is the phase change after having traversed the path. The value of any such line integral of $\underline{v}_s$ should be *independent* of the (topologically similar) path between two endpoints. Hence, $\Delta\theta$ is the same for any path wrapping around the torus exactly once. If $\psi_0(\underline{r})$ is to be single-valued, we must have $\psi_0(\underline{r}) = \psi_0(\underline{r})e^{i\Delta\theta}$, i.e. $\Delta\theta = 2\pi j$ where j is an integer. Hence $\kappa = \frac{h}{m}j$ is quantised in units of $\frac{h}{m}$.

5.7   Phase Slips and Persistent Currents

Consider liquid Helium, at rest ($\kappa = 0 \implies j = 0$) and trapped between two concentric cylinders. Rotating the cylinders accelerates the normal component until it has the same angular velocity as the apparatus, but leaves the superfluid component at rest. If the system is then cooled slowly, particles from the normal fluid will gradually be added to the superfluid component. Hence, their angular momentum will be transferred to the superfluid component, which will begin to rotate. The circulation κ is observed to increase in sudden jumps of $\frac{h}{m}$, called *phase slip* events. These correspond to changes in the quantum number j, and are therefore analogous to transitions between quantum states in an atom. Conversely, cooling an initially rotating normal fluid below T_C yields a circulating superflow. Without viscosity, in principle this flow continues indefinitely.

5.8   Irrotational Flow and Vortices

Consider a cylindrical container of liquid Helium. In this case (topological considerations would appear to be absent), the irrotational flow condition $\underline{\nabla} \times \underline{v}_s = 0 \implies \oint \underline{v}_s \cdot d\underline{r} = 0$ around any closed path. A *vortex*, however, can satisfy $\underline{\nabla} \times \underline{v}_s = 0$ almost everywhere and still allow a net rotation. In cylindrical polar coordinates,

$$\underline{\nabla} \times \underline{v}_s = \frac{1}{r}\begin{vmatrix} \underline{e}_r & r\underline{e}_\phi & \underline{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ v_r & rv_\phi & v_z \end{vmatrix}. \tag{5.6}$$

A circulating flow with cylindrical symmetry will obey $\underline{\nabla} \times \underline{v}_s = 0$ if $\frac{1}{r}\frac{\partial rv_\phi}{\partial r} = 0$. The flow velocity around the vortex must therefore be of the form

$$\underline{v}_s = \frac{\kappa}{2\pi r}\underline{e}_\phi, \tag{5.7}$$

where κ is the (quantised) net circulation. In practice only the lowest energy ($j = \pm 1$) vortices are observed.

5.9   Vortex Core

Eq (5.7) satisfies $\underline{\nabla} \times \underline{v}_s = 0$ except at $r = 0$, the *vortex core* (in Helium, the core region is ≈ 1Å). In the vortex core $\psi_0(0) = 0$, and so θ is undefined there. Hence $n_s \equiv |\psi_0(0)|^2 = 0$ at the core, and also the superfluid current density $\underline{j}_s \equiv n_s\underline{v}_s = 0$. This avoids any singularity in the supercurrent at the centre of the vortex (topological considerations are thus not entirely absent). Superfluid Helium inside a rotating cylinder manifests a centrifugally curved free surface, implying circulation. The *net circulation provided by many vortices*, circumventing a naive interpretation of the irrotationality conditions, explains this.

5.10 Vortex Density of a Superfluid in a Rotating Cylinder

If the cylinder has radius R and angular velocity ω, the total circulation around the cylinder boundary will be the velocity at the boundary multiplied by the diameter, i.e. taking the integration contour around the cylinder boundary,

$$\kappa = \oint \underline{v}_s \cdot d\underline{r} = (\omega R)(2\pi R). \tag{5.8}$$

The circulation is quantised; if there are N_v vortices with $j = 1$, then it follows that $\kappa = \frac{h}{m} N_v$. Hence, the number of vortices per unit area is

$$\frac{N_v}{\pi R^2} = \frac{2m\omega}{h}. \tag{5.9}$$

At low rotation rates, vortices from ordered triangular arrays.

Lecture 6   Interactions and Excitations in Helium II

6.1   Interacting Many-Body Quantum Systems

Considering pairwise interactions only, the quantum-mechanical Hamiltonian for N interacting particles is given in the position representation by

$$\mathcal{H} = -\sum_{j=1}^{N} \frac{\hbar^2}{2m} \underline{\nabla}_j^2 + \frac{1}{2} \sum_{j \neq k=1}^{N} V(\underline{r}_j - \underline{r}_k). \tag{6.1}$$

$\mathcal{H}$ acts on N-particle wavefunctions of the form $\Psi(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N)$. If the particles are bosons, the wavefunction must be *symmetric* under particle exchange, i.e.

$$\Psi(\ldots, \underline{r}_j, \ldots, \underline{r}_k, \ldots) = \Psi(\ldots, \underline{r}_k, \ldots, \underline{r}_j, \ldots). \tag{6.2}$$

In general, a basis of N-body energy eigenfunctions $\Psi_n^{(N)}(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N)$ exists, such that

$$\mathcal{H}\Psi(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N) = E_n^{(N)} \Psi_n^{(N)}(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N). \tag{6.3}$$

6.2   Statistical Mechanics of Interacting Many-Body Quantum Systems

At $T = 0$, the system is in the lowest energy N-particle eigenfunction. At finite T, the quantum state with energy eigenvalue $E_n^{(N)}$ has Boltzmann probability

$$P_n = \frac{1}{\mathcal{Z}} e^{-(E_n^{(N)} - \mu N)/k_B T}, \qquad\qquad \mathcal{Z} = \sum_{N,n} e^{-(E_n^{(N)} - \mu N)/k_B T}, \tag{6.4}$$

where $\mathcal{Z}$ is the grand canonical partition function. All thermodynamic quantities can be calculated from $\mathcal{Z}$, e.g.

$$\langle N \rangle = k_B T \frac{\partial \ln \mathcal{Z}}{\partial \mu}, \qquad\qquad U = \langle \mathcal{H} \rangle = \mu \langle N \rangle - \frac{\partial \ln \mathcal{Z}}{\partial (k_B T)^{-1}}. \tag{6.5}$$

In principle, this provides a direct way to calculate observable properties of interacting many-particle systems; in practice, this is generally a formidable task. Perturbation theory often produces nearly exact results in the weakly-interacting Bose gas, but in a strongly-interacting system such as liquid Helium, predictions will be qualitative at best. Quantum Monte Carlo (QMC) methods can be used to directly calculate observables of many-particle bosonic systems; one can extrapolate to the $N \to \infty$ thermodynamic limit by varying N.

6.3   One-Particle Density Matrix

It is helpful to define the *one-particle density matrix* (for a pure state):

$$\rho_1(\underline{r}_1', \underline{r}_1) = N \int \Psi(\underline{r}_1', \underline{r}_2, \ldots, \underline{r}_N)\Psi^*(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N) \, d^3r_2, \ldots, d^3r_N, \tag{6.6}$$

i.e. a correlation function of the wavefunction between coordinates $\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N$ and $\underline{r}_1', \underline{r}_2, \ldots, \underline{r}_N$, averaged over the configurations of all particles except the first. The choice of coordinate $\underline{r}_1$ is arbitrary. As the wavefunction must be symmetric under particle exchange, we say

$$\rho_1(\underline{r}', \underline{r}) \equiv \rho_1(\underline{r}_1', \underline{r}_1) = \rho_1(\underline{r}_2', \underline{r}_2) = \cdots = \rho_1(\underline{r}_N', \underline{r}_N). \tag{6.7}$$

For a homogeneous system, the translational symmetry of any equilibrium state implies $\rho_1(\underline{r}', \underline{r}) = \rho_1(\underline{r} - \underline{r}')$. Hence, the unit norm of $\Psi(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N)$ then means

$$\int \rho_1(0) \, d^3r = N \implies \rho_1(0) = \frac{N}{V} = n, \tag{6.8}$$

i.e. the particle number density.

6.4 Long-Range Correlations

In the absence of interactions, at $T = 0$ all particles occupy a single one-particle state, i.e.

$$\Psi(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N) = \phi_0(\underline{r}_1)\phi_0(\underline{r}_2)\ldots\phi_0(\underline{r}_N) \implies \rho_1(\underline{r} - \underline{r}') = N\phi_0(\underline{r}')\phi_0^*(\underline{r}). \tag{6.9}$$

The state $\phi_0(\underline{r})$ is normally $e^{i\underline{k}\cdot\underline{r}}/\sqrt{V}$ where $\underline{k} = 0$, hence

$$\rho_1(\underline{r} - \underline{r}') = \frac{N}{V} = n. \tag{6.10}$$

In the presence of interactions, we define the condensate density n_0 to be

$$n_0 = \lim_{|\underline{r}-\underline{r}'|\to\infty} \rho_1(\underline{r} - \underline{r}'). \tag{6.11}$$

We can then define the macroscopic wavefunction $\phi_0(\underline{r})$ through

$$\rho_1(\underline{r} - \underline{r}') \approx \psi_0(\underline{r}')\psi_0^*(\underline{r}), \text{ for large } |\underline{r} - \underline{r}'|. \tag{6.12}$$

Hence $\psi_0(\underline{r})$ has norm n_0, as desired. For $T < T_C$, $\rho_1(\underline{r} - \underline{r}')$ approaches a temperature-dependent constant value $n_0(T)$ for large $|\underline{r} - \underline{r}'|$. As $T \to T_C$, $n_0(T) \to 0$, and at T_C and above, n_0 vanishes.

6.5 The Momentum Distribution

The *quantal* probability $P(\underline{p})$ that any given particle has momentum in the region d^3p of momentum space is defined (for a pure state, i.e. at $T = 0$) through

$$P(\underline{p})\, d^3p = \frac{V\, d^3p}{(2\pi\hbar)^3}\langle\Psi|\hat{n}_{\underline{k}}|\Psi\rangle, \tag{6.13}$$

where $|\Psi\rangle$ is the many-particle wavefunction in Dirac form, $\hat{n}_{\underline{k}}$ is the number operator for the state with momentum $\underline{p} = \hbar\underline{k}$, and $V\, d^3p/(2\pi\hbar)^3$ is the number of quantum state in the region d^3p of momentum space. The expectation value is written as

$$\langle\Psi|\hat{n}_{\underline{k}}|\Psi\rangle = \int \rho_1(\underline{r})e^{i\underline{k}\cdot\underline{r}}d^3r, \quad \rho_1(\underline{r}) = n_0 + \Delta\rho_1(\underline{r}), \tag{6.14}$$

where $\underline{r}$ is now a relative coordinate, and $\Delta\rho_1(\underline{r}) \to 0$ for large $\underline{r}$. Carrying out the Fourier transform,

$$\langle\Psi|\hat{n}_{\underline{k}}|\Psi\rangle = \int n_0 e^{i\underline{k}\cdot\underline{r}}d^3r + \int \Delta\rho_1(\underline{r})e^{i\underline{k}\cdot\underline{r}}d^3r = n_0 V\delta_{\underline{k}0} + f(\underline{k}), \tag{6.15}$$

$$\implies P(\underline{p}) = n_0 V\delta(\underline{p}) + \frac{V}{(2\pi\hbar)^3}f\left(\frac{\underline{p}}{\hbar}\right). \tag{6.16}$$

Hence at $T = 0$, there is a condensate and a contribution not unlike a Maxwell-Boltzmann distribution due to zero-point energies. Raising T to T_C, $n_0 \to 0$, and what remains tends to a genuine (thermal) Maxwell-Boltzmann distribution. Neutron scattering experiments and QMC integrations both indicate that in liquid Helium as $T \to 0$, the condensate component n_0 is only $\approx 0.1n$. (Recall that at $T = 0$, the superfluid component $n_s = n$).

6.6 Galilean Invariance

If $\Psi_0(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N)$ is the ground state wavefunction for a superfluid at rest, a uniformly moving (with velocity $\underline{v} = \hbar\underline{q}/m$) trial wavefunction is $e^{i\underline{q}\cdot(\underline{r}_1+\underline{r}_2+\cdots+\underline{r}_N)}\Psi_0(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_N)$. If $\underline{q}$ is very small ($\approx 1/L$, where L is the macroscopic sample length), this trial function is a near-exact eigenstate of the (Galilean invariant) Hamiltonian $\mathcal{H}$. The total momentum is then $\langle\hat{P}\rangle = N\hbar q$, where $\hat{P} = \sum_{j=1}^{N}\hat{p}_j$, and the total energy

$$\langle\mathcal{H}\rangle \approx E_0 + N\frac{\hbar^2 q^2}{2m} = E_0 + \frac{1}{2M}\langle\hat{P}\rangle^2, \text{ where } M = Nm. \tag{6.17}$$

$\langle\mathcal{H}\rangle$ is then a classical Hamiltonian for an object of mass M and momentum $\langle\hat{P}\rangle$, i.e. velocity $\underline{v}_s = \langle\hat{P}\rangle/M = \hbar q/m$. The implication is that at $T = 0$, the whole mass M of the fluid contributes to the superflow, not just the condensate fraction (*rigidity* of the ground state).

6.7 Elementary Excitations in a Normal Fluid

Consider fluid flowing in a narrow tube. Friction and viscosity arise because fluid particles experience random scattering events from the atomically rough walls, transferring momentum to the walls and providing the viscous friction force. In a moving-with-the-fluid reference frame, it is the rough walls which move, providing time-dependent perturbations. Potential energy due to defects in the wall is equivalent to a time-dependent potential $V(\underline{r} + \underline{v}t)$ in the moving frame. According to Fermi's golden rule for time-dependent perturbation theory, a single quantum particle with initial $\underline{p}_i, \epsilon_i$ can be scattered elastically to final $\underline{p}_f, \epsilon_f$ only if

$$\epsilon_f = \epsilon_i - \underline{v} \cdot (\underline{p}_i - \underline{p}_f). \tag{6.18}$$

For a particle initially in-condensate ($\underline{p} = 0$), we create an *elementary excitation* only if $\epsilon(\underline{p}) = \underline{v} \cdot \underline{p}$. In a normal fluid $\epsilon(\underline{p}) = \frac{p^2}{2m}$, and the condition $\frac{p^2}{2m} = \underline{v} \cdot \underline{p}$ is obeyed on a cone described by

$$|\underline{p}| = 2mv \cos \phi, \tag{6.19}$$

where ϕ is the angle to the direction of $\underline{v}$. The rough walls can therefore always impart momentum to a normal fluid, leading to viscous friction. This applies to classical fluids, strongly-interacting normal quantum fluid (He I), as well as the ideal Bose gas, which is therefore *never superfluid.*

6.8 Helium II Quasiparticle Spectrum

The elementary excitation or *quasiparticle* spectrum for a superfluid must be very different, such that it is impossible to satisfy $\epsilon(\underline{p}) = \underline{v} \cdot \underline{p}$. **Lev Landau** postulated an "N-shaped" spectrum, which has been confirmed by many experiments. At small $\underline{p}$, the energy is approximately linear: $\epsilon(\underline{p}) = c|\underline{p}|$. as is typical of phonons in solids (*phonon branch*). A single He atom couples strongly to the many-particle condensate, leading to near rigid-body motion, and a phonon-like spectrum. At very large $\underline{p}$, the spectrum approaches that of a conventional liquid: $\epsilon(\underline{p}) = \frac{p^2}{2m^*}$, where m^* is the *effective mass* which is used as the particles interact strongly. A single atom can then move relatively independently, hence a *free particle*-like spectrum. Between these two extremes, the spectrum has a local minimum (*roton branch*):

$$\epsilon(\underline{p}) = \Delta + \frac{(p - p_0)^2}{2\mu}. \tag{6.20}$$

Neighbours of a moving atom move out of the way in a circular backflow, i.e. one particle moves forwards, accompanied by a ring of particles rotating backwards.

6.9 Quasiparticles and Superfluidity

Consider $c_{\min}$, the slope of the line joining $p = 0, \epsilon = 0$ to the local minimum in the roton branch of the quasiparticle spectrum. It is clear that $\epsilon(\underline{p}) > c_{\min}|\underline{p}|$. Consequently, a low energy quasiparticle starting with near-zero initial momentum cannot be scattered into any possible final states (which process must satisfy $\epsilon(\underline{p}) = \underline{v} \cdot \underline{p}$) when $|\underline{v}_s| < c_{\min}$. The superfluid therefore flows without dissipation due to scattering of quasiparticles, provided that the flow velocity is less than the ideal critical velocity $c_{\min}$. Experimentally, the actual critical velocity is rather less than this ideal limit.

Lecture 7   Superconductivity Phenomena

7.1   Thermal Occupation of Energy Bands in Crystalline Solids

The wavefunction of electrons in crystalline solids obey *Bloch's theorem,*

$$\psi_{q\underline{k}} = u_{q\underline{k}}(\underline{r})e^{i\underline{k}\cdot\underline{r}}, \tag{7.1}$$

where $u_{q\underline{k}}(\underline{r})$ is a periodic function, $\hbar\underline{k}$ is the crystal momentum or *quasimomentum*, and $\underline{k}$ takes values in the first Brillouin zone of the reciprocal lattice. The energies of the Bloch wave states give the *energy bands, $\epsilon_{q\underline{k}}$.* A state with energy ϵ is occupied according to the *Fermi-Dirac* distribution:

$$f(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/k_BT}+1}, \text{ where } \frac{N}{V} = \frac{2}{(2\pi)^3}\sum_q \int \frac{d^3k}{e^{(\epsilon_{q\underline{k}}-\mu)k_BT}} \tag{7.2}$$

(the net electron density per unit volume) sets the chemical potential μ, and T is the temperature. The factor 2 is due to the two electron spin states. At very low temperatures $(k_BT \ll \mu)$, the Fermi gas assumes a highly degenerate state, where $f(\epsilon_{q\underline{k}})$ is nearly equal to unity 'inside' the Fermi surface, and equal to zero outside. The Fermi surface can be defined by $\epsilon_{q\underline{k}} = \epsilon_F$, where $\epsilon_F = \mu$ is the Fermi energy. We shall generally assume there to be only one conduction band at the Fermi surface, and ignore the band index q. The density of *conduction electrons* is then

$$n = \frac{2}{(2\pi)^3}\int \frac{d^3k}{e^{(\epsilon_{\underline{k}}-\mu)/k_BT}+1}, \tag{7.3}$$

where $\epsilon_{\underline{k}}$ is the nergy of the single band which crosses the Fermi surface.

7.2   Fermi Gas Description of Electrons in Metals

Metallic conduction is dominated by the thin shell of quantum states with energies $\epsilon_F - k_BT < \epsilon < \epsilon_F + k_BT$, i.e. the only states that can be thermally excited at temperature T (dilute gas of electrons excited above ϵ_F and holes below ϵ_F). Electrical conductivity is given in the *Drude theory* of conduction as

$$\sigma = \frac{ne^2\tau}{m}, \tag{7.4}$$

where m is the effective mass of the conduction electrons, $-e$ is the electron charge, and τ is the average time between collisions with impurities or other electrons. The conductivity is defined by the *constitutive equaiton,*

$$\underline{j} = \sigma\underline{\mathbf{E}}, \tag{7.5}$$

where $\underline{j}$ is the electrical current density flowing in response to the external electric field $\underline{\mathbf{E}}$. The resistivity $\rho \equiv \frac{1}{\sigma}$, and is thus proportional to the conduction electron *scattering rate* τ^{-1}:

$$\rho = \frac{m}{ne^2}\tau^{-1}. \tag{7.6}$$

7.3   Scattering Processes, Temperature, and Resistivity

Conductivity depends on temperature mainly through: scattering by impurities (rate τ_i^{-1}); electron -electron interactions (τ_{ee}^{-1}); and electron-phonon collisions (τ_{ep}^{-1}). Hence,

$$\rho = \frac{m}{ne^2}\left(\tau_i^{-1} + \tau_{ee}^{-1} + \tau_{ep}^{-1}\right). \tag{7.7}$$

➤ τ_i^{-1} is essentially independent of T for nonmagnetic impurities.

62

➤ τ_{ee}^{-1} is proportional to T^2.
➤ At low temperatures, τ_{ep}^{-1} is proportional to T^5.

Hence, at very low T, we expect

$$\rho = \rho_0 + aT^2 + \cdots, \tag{7.8}$$

where the residual resistivity ρ_0 depends only on the impurities. For a *superconductor* upon cooling, the resistivity ρ first follows the typical behaviour, and then *suddenly vanishes* at a critical temperature T_C. **Heike Kamerlingh Onnes** first observed superconductivity in mercury ($T_C = 4.1\,\mathrm{K}$) in 1911. Other metal elements also exhibit superconductivity at standard pressures, with niobium ($T_C = 9.3\,\mathrm{K}$) having the highest critical temperature.

7.4   Zero Resistivity

Within superconductors, $\rho = 0$, which implies σ is infinite below T_C. Consistency with the constitutive equation and a finite current density j demands $\underline{\mathbf{E}} = 0$. Note that all superconductors also have a critical current I_C above which the superconductivity is destroyed and the resistance becomes finite again. The sharp change from finite to zero resistivity implies a thermodynamic phase transition from a *normal state* to a *superconducting state*.

7.5   Magnetic Field Flux through the Centre of a Superconducting Ring

Flux through the centre of a ring of superconducting wire is defined by the surface integral

$$\Phi = \int \underline{\mathbf{B}} \cdot \underline{dS}, \tag{7.9}$$

where $\underline{dS}$ is oriented perpendicular to the plane of the ring. We use the Maxwell equation and Stoke's theorem, respectively given as

$$\nabla \times \underline{\mathbf{E}} = -\frac{\partial \underline{\mathbf{B}}}{\partial t}, \qquad\qquad \int (\nabla \times \underline{\mathbf{E}}) \cdot \underline{dS} = \oint \underline{\mathbf{E}} \cdot d\underline{r}. \tag{7.10}$$

Then, considering a closed line integral around the inside of the ring,

$$-\frac{d\Phi}{dt} = \oint \underline{\mathbf{E}} \cdot d\underline{r}. \tag{7.11}$$

As $\underline{\mathbf{E}} = 0$ within a superconductor, $\frac{d\Phi}{dt} = 0$, so the flux does not change with time.

7.6   Persistent Currents

A current I set up to circulate in a ring of superconducting wire, because there is no dissipatoin of energy due to finite resistance, *never* (up to years) decays. To set up such a current, we apply an external magnetic field $\underline{\mathbf{B}}_{ext}$ to the wire ring at $T > T_C$. The superconductor is in its normal state, and $\underline{\mathbf{B}}_{ext}$ passes easily through it. Now cool the system below T_C. If we now set $\underline{\mathbf{B}}_{ext} = 0$, the flux Φ can remain constant only if the superconductor generates its own magnetic field $\underline{\mathbf{B}}$ through the centre of the ring, by having a constant circulating current I around it.

7.7   The Meissner-Ochsenfeld Effect

Consider a sample cooled below T_C, and then slowly turn on a small external magnetic field $\underline{\mathbf{B}}_{ext}$. As $\underline{\mathbf{E}} = 0$ inside the superconducting smaple, the Maxwell equation

$$\nabla \times \underline{\mathbf{E}} = -\frac{\partial \underline{\mathbf{B}}}{\partial t} \implies \frac{\partial \underline{\mathbf{B}}}{\partial t} = 0. \tag{7.12}$$

Hence, $\underline{\mathbf{B}}$ inside the sample cannot change, i.e. $\underline{\mathbf{B}}_{ext}$ *does not penetrate* the sample. Turning on $\underline{\mathbf{B}}_{ext}$ when the sample is above T_C, the field penetrates the sample; upon cooling below T_C, the magnetic field is *expelled*. This is the *Meissner-Ochsenfeld effect*, seen as *definitive* of superconductivity. It is technically simpler to demonstrate flux expulsion (Meissner-Ochsenfeld effect) than zero resistivity (requiring attached of electric leads). More fundamentally, it is a property of *thermal equilibrium* rather than a *non-equilibrium transport effect*.

7.8 Application of Maxwell's Equations in a Magnetic Medium

To maintain $\underline{\mathbf{B}} = 0$ inside the sample whatever (small) external fields are imposed, there must be screening currents flowing around the edges of the sample producing a magnetic field equal and opposite to the applied. We split $\underline{j}$ into components from *externally applied currents* (e.g. in coils producing the external field) and *internal screening currents*:

$$\underline{j} = \underline{j}_{ext} + \underline{j}_{int}. \tag{7.13}$$

The *screening currents* produce a magnetisation $\underline{M}$ per unit volume in the sample, through

$$\nabla \times \underline{M} = \underline{j}_{int}. \tag{7.14}$$

We also define a magnetic field $\underline{H}$ in terms of external currents only through

$$\nabla \times \underline{H} = \underline{j}_{ext}. \tag{7.15}$$

Put together, these two equations yield the familiar phrase

$$\underline{\mathbf{B}} = \mu_0(\underline{H} + \underline{M}). \tag{7.16}$$

7.9 Perfect Diamagnetism

Consider an infinitely long cylindrical sample with an external current flowing around it in solenoidal coils.

$$\underline{H} = I\frac{N}{L}\underline{e}_z \tag{7.17}$$

(N coils per unit length L) is then uniform within it. Imposing the *Meissner condition* ($\underline{\mathbf{B}} = 0$) yields the magnetisation $\underline{M} = -\underline{H}$. The *magnetic susceptibility* χ is defined by

$$\chi = \left.\frac{dM}{dH}\right|_{H=0}. \tag{7.18}$$

For superconductors, $\chi = -1$. Solids with negative χ are *diamagnets* (for positive χ, paramagnets), i.e. they screen out part of the external magnetic field. Superconductors are *perfect diamagnets*, and $\chi = -1$ below T_C implies the Meissner-Ochsenfeld effect, hence superconductivity.

7.10 Type I and Type II Superconductors

χ is defined in the limit of very weak $\underline{H}$. If $\underline{H}$ is increased for a *type I superconductor*, the $\underline{\mathbf{B}}$ field remains zero inside the sample (i.e. $\underline{M} = -\underline{H}$) until suddenly at a critical field magnitude H_C, the superconductivity is destroyed. With *type II superconductors*, there is a lower critical H_{C1} and an upper critical field H_{C2}. Increasing H from zero, initially $\underline{M} = -\underline{H} \implies \underline{\mathbf{B}} = 0$. Above H_{C1}, magnetic flux starts to enter the sample ($\underline{\mathbf{B}} \neq 0$) and $\underline{M}$ starts to decrease until at H_{C1}, $\underline{M} = 0$ and the superconductivity is destroyed. **Alexei Abrikosov** showed the magnetic field can enter a type II superconductor because of *vortices* - regions of circulating supercurrent around small normal metal cores. The magnetic field passes through the vortex cores, and the circulating currents screen the rest of the superconductor from the magnetic field.