Mathematical Methods in Physics

Warming up exercises - Solution

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1 Lecture 1

1.1

The angles are $\theta_1 = 90°$, $\theta_2 \sim 50.77°$, $\theta_3 \sim 39.23°$.

1.2

The scalar triple product is zero for three coplanar vectors.

1.3

a) $a_{j1}x_1 + a_{j2}x_2 + a_{j3}x_3$; b) $a_i x_i x_3$; c) δ_{ik}.

2 Lecture 2

2.1

They are perpendicular if their normal vectors are perpendicular.

2.2

a) Yes; b) No. For instance: $\alpha(x_1, x_2, 1)^T$ is not in V_2 if $\alpha \neq 1$, the zero and inverse elements are missing.

3 Lecture 3

3.1

a) Linearly independent; b) Linearly dependent.

3.2

This is the conventional scalar product used in $\mathbb{C}^3$.

4 Lecture 4

4.1

a) 180; b) 0; c) 80.

4.2

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -17/3 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

4.3

$$C^{-1} = -\frac{1}{369} \begin{pmatrix} 32 & 45 & -17 \\ -72 & -9 & -54 \\ 1 & 36 & 11 \end{pmatrix}$$

5 Lecture 5

5.1

a) Eigenvalues: $\lambda_1 = 2$ and $\lambda_2 = 3$. Eigenvector forms: $\mathbf{x}_1^T = (x, 2x)$, $\mathbf{x}_2^T = (x, x)$ with x arbitrary. A possible choice is $\mathbf{x}_1^T = (1, 2)$, $\mathbf{x}_2^T = (1, 1)$.

b) Eigenvalues: $\lambda_1 = 4$ and $\lambda_2 = -1$. Eigenvector forms: $\mathbf{x}_1^T = (x, x)$, $\mathbf{x}_2^T = (x, -2x/3)$ with x arbitrary. A possible choice is $\mathbf{x}_1^T = (1, 1)$, $\mathbf{x}_2^T = (3, -2)$.

6 Lecture 6

6.1

$$D = S^\dagger A S \text{ with } D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } S = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & \sqrt{2} & \sqrt{3} \\ 1 & \sqrt{2} & -\sqrt{3} \\ -2 & \sqrt{2} & 0 \end{pmatrix}.$$

Note that your result for S could be different since it depends on your eigenvector choice.

7 Lecture 7

7.1

a) Yes; b) Yes; c) No: infinite discontinuity at $x = 0$; d) Yes; e) No: infinite discontinuity at $x = \pi/2$; f) No: two valued.

7.2

1.

8 Lecture 8

8.1

$a_0 = \frac{2}{\pi}; \quad a_1 = 0, \quad a_r = -\frac{1+(-1)^r}{(r^2-1)\pi}$ for $r = 2, 3, 4...,$ $\quad b_1 = \frac{1}{2}, \quad b_r = 0$ for $r = 2, 3, 4...$

9 Lecture 9

9.1

$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \frac{1}{\alpha + i\omega}.$

9.2

a) $\frac{e^{i2\omega/a}}{a} \hat{f}(\omega/a),$ b) $2\cos(4\omega)\, \hat{f}(\omega).$

10 Lecture 10

10.1

a) $f(8),$ b) $-1,$ c) $0.$

10.2

$\omega = \pm\sqrt{k/m}.$

11 Lecture 11

11.1

a) $\delta(x-3)/6 + \delta(x+3)/6,$ *b)* $\delta(2\,x) = \delta(x)/2,$ *c)* $\delta((x+1)x)) = \delta(x) + \delta(x+1).$

11.2

a) $H(2-t),$ *b)* $\cos t(H(t) - H(t-\pi)),$ *c)* $tH(t) + (2-2t)H(t-1) - (2-t)H(t-2).$

12 Lecture 12

12.1

a) $\bar{f}_1(t) = e^{-3s}/3, \quad$ b) $\bar{f}_2(s) = 8e^{-2s}, \quad$ c) $\bar{f}_3(s) = 12/(s^2 - 9), \ s > 3.$

12.2

a) $f_1(t) = t\, e^{-3t}, \quad$ b) $f_2(t) = \cos 5t, \quad$ c) $f_3(t) = 3\, e^{-t} \sin t - 3e^t.$

13 Lecture 13

13.1

$d\mathbf{a} = (5u^4 du)\, \mathbf{i} + (4\, v\, e^{4u} du + e^{4u} dv)\, \mathbf{j}.$

13.2

a) $\mathbf{a}(u) = (1 - 2u)\, \mathbf{i} + 2\, \mathbf{j} + u\, \mathbf{k}$, with $-1 \leq u \leq 3$,
b) $\mathbf{a}(u) = u\, \mathbf{i} + 2u\, \mathbf{j} + 3u\, \mathbf{k}$, with $0 \leq u \leq 1$.

14 Lecture 14

14.1

As expected, the radius of curvature is $\rho = 2$.

14.2

$d\mathbf{S} = a\, \sin\theta\, \mathbf{r}\, d\theta d\phi, \quad dS = a^2\, \sin\theta\, d\theta d\phi.$

15 Lecture 15

15.1

a) $\nabla \cdot \mathbf{a} = z + 2y + xy, \quad$ b) $\nabla \times \mathbf{b} = y/x\, \mathbf{i} + (yz/x^2 + 2x/y^3)\, \mathbf{k}$,
c) $\nabla^2 \phi = 2z^3/y^4 + 20x^2 z^3/y^6 + 6x^2 z/y^4$.

15.2

15.3

For instance: $\mathbf{r}(t) = (1 + t)\, \mathbf{i} + 4t\, \mathbf{j} + (1 - 3t)\, \mathbf{k}$ with $0 \leq t \leq 1. \quad I = -35/3.$

16 Lecture 16

16.1

$\nabla \times \mathbf{a} = 0$. The potential is $\phi = x^2 z + y^2 z^2 - z + c$, where c is a constant. $I = 0$.

16.2

$I = \pi$.

17 Lecture 17

17.1

$I = 20\,\pi$.

17.2

$dV = \rho\, d\rho\, d\phi\, dz$,
$d\mathbf{S}_\rho = \rho\,(\cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}) d\phi\, dz, \quad d\mathbf{S}_\phi = (-\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}) d\rho\, dz, \quad d\mathbf{S}_z = \rho\,\mathbf{k}\, d\rho\, d\phi.$

18 Lecture 18

18.1

The boundary is a a rectangle with sides $AB : -3 \le x \le 3$, $y = z = 0$, $BC : 0 \le z \le 4$, $y = 0$, $x = 3$, $CD : -3 \le x \le 3$, $y = 0$, $z = 4$, $DA : 0 \le z \le 4$, $y = 0$, $x = -3$. The orientation is from A to B to C to D.

18.2

$I = \pi a^3 \cos\alpha \sin\alpha$.

19 Lecture 19

19.1

$\hat{\mathbf{e}}_r = \cos\phi \sin\theta\,\mathbf{i} + \sin\phi \sin\theta\,\mathbf{j} + \cos\theta\,\mathbf{k}$, $\hat{\mathbf{e}}_\phi = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}$,
$\hat{\mathbf{e}}_\theta = \cos\phi \cos\theta\,\mathbf{i} + \sin\phi \cos\theta\,\mathbf{j} - \sin\theta\,\mathbf{k}$, $h_r = 1$, $h_\phi = r\,\sin\theta$, $h_\theta = r$, $\quad \nabla \times \mathbf{a} = \hat{\mathbf{e}}_r / r^2$.

19.2

$\hat{\mathbf{e}}_u = (v\cos\phi\mathbf{i} + v\sin\phi\mathbf{j} + u\mathbf{k})/(u^2+v^2)^{1/2}$, $\quad \hat{\mathbf{e}}_v = (u\cos\phi\mathbf{i} + u\sin\phi\mathbf{j} - v\mathbf{k})/(u^2+v^2)^{1/2}$,
$\hat{\mathbf{e}}_\phi = -\sin\phi\mathbf{i} + \cos\phi\mathbf{j}$, $\quad h_u = h_v = (u^2+v^2)^{1/2}$, $h_\phi = uv$.