# Stars and Galaxies

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# Chapter 1

# Observational

*other stuff in notebook*

## Lecture 3

- Parts of atmosphere are opaque due to water vapour, $O_3$, etc
- Correcting for atmospheric absorption:
  - **GET IMAGES FROM SLIDES**

$$X = 1 \text{ airmass}$$

$$X = \sec(z) \text{ airmasses}$$

$$-\int_{I_C}^{I_O} \frac{dI}{I} = \int_0^X k\,dX$$

$$\ln \frac{I_{obs}}{I_{corr}} = kX + c$$

$$\frac{I_{obs}}{I_{corr}} = e^{-kX}$$

$$m_{obs} - m_{corr} = -2.5 \log \frac{I_{obs}}{I_{corr}}$$

$$m_{obs} - m_{corr} = -2.5 \log e^{-kX}$$

$$= 2.5kX \log e$$

$$m_{corr} = m_o bs - A_\lambda(z = 0) \sec z$$

- Atmospheric refraction
  - **MATHS AND PICS IN SLIDES**
  - plane parallel atmosphere
  - apply laws of refraction
  - basic trig stuff
  - always in small angle approx range
  - $r = (n - 1) \tan(z_0)$
- Refractive index also has wavelength dep
- atmos ref turns into an atmos dispersion
- disperses more for smaller wavelength

- 3 or 4 arcsecs
  - a lot
- Every object appears as a spectrum as colors separate
- atmos emission
  - fluorescent emission
    * air glow
  - emits thermal radiation for TE
  - Most emission is from OH molecules in upper atmos
    * vibrational and rotational movement
- want to try and stay away from regions with lots of this emission
- Other sources of emission:
  - light pollution
    * from ground
    * from satellites and aircraft
  - zodiacal light
    * light scattered from interplanetery dust
    * in plane of the Solar System
  - scattered light
    * e.g. from the moon
    * telescope scheduling to dark, grat, and bright time
- more diffcult observations at longer wavelengths
  - more background issues
- dust causes lots of interference
  - at longer wavelengths, interaction between dust and photons is smaller
  - interaction cross-section
- easier to see through dust a lot easier and see other galaxies etc at longer wavelengths
- Atmospheric turbulence
  - *twinkle twinkle little star*
  - Stars twinkle due to light getting bounced around in atmos
- Angular resolution of telescope limited by Fraunhofer Diffraction
  - *see last year*
  - Airy disk
  - assume stars as point sources
  - large telescope $\implies$ small airy disk
  - small telescope $\implies$ large airy disk
  - how close before two stars are seen as one?
- Characterise resolution with Rayleigh criterion
  - at some point the principle maximum of one star overlays with the principle minimum of the second
    * *diffraction limit*
  - $\theta_{dl} = 1.22\frac{\lambda}{D}$
    * integrate round a cylinder using Bessel fns to get this
    * covered sort of later on in other module
- Atmos is constantly moving
  - changing size, density, and temperature causes different path lengths ever dt for stars
  - sum up over lots of dt for observing
    * causes blurring though
  - no longer airy disk, severely blurry
- for atmos turbulence, the seeing is defined as minimum angle between two stars that can just be resolved
  - typically in arcsec
  - 50x worse than the diffraction limit
- Detectors
  - Charged Coupled Device
  - little silicon micro-circuits

  - little ray of capacitors
  - discrete energy bands
    - conduction band and valence band
    - difference of $\approx 1.1$eV
  - upper cut-off wavelengths governed by band gap voltage difference
  - lower wavelengths cut-off by absorption of photons into the silion
  - excellent Quantum efficiency
    - $> 90\%$
  - high dynamic range
  - excellent linearity
  - excellent stability
  - still not enough pixels

# Lecture 4

### Back to CCDs:

- Well Depth
  - how many electrons can be stored in the upper state, usually 100s of thousands
- use binary for how many levels for the signal
  - i.e. 8 bit $= 2^8 = 256$ levels
- System Gain
  - how many photo-electrons are required for digital output of 1
  - small gain means reduced saturation signal

### Photometry

- Process of obtaining quatative (numerical) values of the birghtness of celestial objects
- CCD gives output prop to number of photons incident on each pixel
- Photometry takes raw data and corrects for noise from other sources
- Noise is just any interference for the image
- SNR (signal to noise ratio) defined as ratio of useful to non-useful data
- Poisson stats
  - arrival of photons governed by this
  - studied for how cameras observe sky stuff
  - see stats last year
  - Hughes and Hase and labs stuff

$$P(n, N) = \frac{\exp(-N) N^n}{n!}$$

- High means approximates Gaussian stats
- mean is N
  - also Variance
  - std dev is $\sqrt{N}$
- Telescope experiments can take eight hours or so
  - so use Poisson errors for easy error in counts
- Small error associated with read out

### Basic Data Reduction to Correct for Background in CCD

- Bias
  - a zero second readout whihc results in a constant offset

  - allows for understanding of the "noise" quantity
- Dark
  - CCD band stuff
  - CCD will be in TE so will promote thermal photons
  - thermal photons can hit detector and skew results
  - this will increase in time
- Flat Field
  - variations in sensitivity
  - varied energy ever so slightly across CCD
  - quantum efficiency
  - slight changes across the CCD in efficiency causes a non-uniform field across CCD
- Also have sky background counts
  - these are often the most significant contributor

$$\text{Final Frame} = \frac{\text{Object Frame} - (\text{dark+bias})}{\text{Flat Field} - (\text{dark+bias})}$$

$$\text{Final Frame} = \frac{\text{Object Frame} - (\text{dark+bias})}{\text{Flat Field} - (\text{dark+bias})} - \frac{\text{Sky Frame} - (\text{dark+bias})}{\text{Flat Field} - (\text{dark+bias})}$$

$$\implies \text{Final Frame} = \frac{\text{Object Frame} - \text{Sky Frame}}{\text{Flat Field} - (\text{dark+bias})}$$

**Noise Sources**

- Basic sources of noise are:
  1. Readout noise, $\sigma_{rd}$ electrons (Gaussian)
  2. Photon noise on the signl from the object (Poisson)
     - $= \sqrt{f_{abj}t}$
  3. Photon noise on the signal from the sky background (Poisson)
     - $= \sqrt{f_{bg}t}$
  4. Photon noise on the dark current (Poisson)
     - $\sqrt{dt}$
- Uncorrelated noise sources can be added in quadrature
  - $\sigma_{\text{total}} = \sqrt{\sigma_1^2 + \sigma_2^2}$
- Signal/Noise

$$SNR = \frac{S}{\sqrt{S + D + B + \sigma_{rd}^2}}$$

- S - signal
- B - background
- D - dark
- $\sigma_{rd}$ - read error
- Prev equation assumes all the terms are in photo-electrons
- Will need to be accounted for if in ADU
- counts in number of photons
- gain can be set to more than 1
  - confuses simple SNR eqn and changes what you plug in

**SNR Approximations**

- Common approximations:
  1. Photon noise limited on the object

- signal dominates so can ignore other terms for SNR
2. Sky Limited
   - sky background dominates, only count background
3. Read Noise Limited
   - read background dominates, only count read term

# Lecture 5

## Spectroscopy

- Most useful tool in astro
- measurement of intensity of a light source
  - function of wavelength
- Different specta:
  1. light from source straight to detector
     - continous spectrum
  2. light from source travels through a cloud of gas straight to detector
     - continuous spectrum with dark lines
  3. light from source travels into cloud and scatters through it to detector
     - bright line spectrum on black background
- Types of spectrograph
  1. Refraction (prisms)
  2. Diffraction gratings
  3. Interference (Fabry-Perot interferometer)
  - focus on diffraction grating
- Diffraction grating
  1. Slit
     - need this to focus light from source of interest and block everything else
  2. Collimating lens
     - make sure light lands parallel to diffraction grating
  3. Diffraction grating
  4. Camera
- Condition for constructive interference:

$$n\lambda = d\sin\theta$$
$$\frac{d\theta}{d\lambda} = \frac{n}{d\cos\theta}$$

- $\frac{d\theta}{d\lambda}$ is known as angular dispersion (rad/nm)
  - higher dispersions from higher spectral orders and smaller line spacings
  - more convenient for Reciprocal Linear dispersion ($\frac{d\lambda}{dx}$)
  - measuring wavelength per unit x at detector (nm/mm)
  - multiply $\frac{d\theta}{d\lambda}$ by plate scale $\frac{d\theta}{dx} = \frac{1}{f_{cam}}$

$$\frac{d\lambda}{dx} = \frac{d\lambda}{d\theta}\frac{d\theta}{dx} = \frac{d}{f_{cam}n}\cos\theta$$

### Grating Equation

- For angles of incidence to grating
- For diffraction grating or reflection

$$n\lambda = d(\sin\alpha + \sin\beta)$$

$$n\lambda\rho = \sin\alpha + \sin\beta \ ; \ \rho = \frac{1}{d}$$

**Resolving Power**

- Recall angle for blurred star

$$\theta = 1.22\frac{\lambda}{D}$$

- Resolving power of a spectrograph is wavelength over band pass:
  - $\lambda$ is the wavelength
  - $\Delta\lambda$ is the minimum discernible difference in $\lambda$

$$R = \frac{\lambda}{\Delta\lambda} = nN$$

$$R = \frac{n\rho\lambda W}{\chi D_T}$$

- Where
  - n is diffraction order#
  - N is number of lines
  - $\rho$ is the ruling density (lines/mm)
  - $\lambda$ is the wavelength
  - W is the grating size
  - $\chi$ is the angular size of the image of a star on slit
  - $D_T$ is the telescope size
- Don't want too narrow a slit
  - optimise width of slit for photons from star
  - spectral resolution gets blurred
- Second equation above is for a practical spectrograph
  - At most wavelengths, this value of R is much less than that given by $nN$

**CDs, DVDs, and Blu-Rays**

- basically diffractions gratings
- DVDs store more info than CDs based on diffraction types
- Blu-Rays need UV light to make sense

# Lecture 6

## Measuring Stars

- Black body radiation

$$E(\lambda, T) = \frac{2hc^2}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda kT}} - 1}$$

- Characteristic temperature is where $\frac{dE}{d\lambda} = 0$, bump at top of curve

- Colours of stars depends on plot, nearest colour to peak is visible colour

$$L = 4\pi R^2 \sigma T^4$$

- Calc distance to star?
  - use parallax
  - define 1 parsec as distance corresponding to parallax of $\theta = 1$"
  - 1 psc = $206265AU$

## Interferometry

- Combines light from two telescopes
  - makes it possible to measure stars
  - interfere the light and measure phase difference
  - diffraction limit: $1.22\frac{\lambda}{D}$
- As star tracks across sky, path length changes
  - phase will shift in and out of phase with movement
  - more complicated for two light sources
  - get a more complex fringe pattern
    - modulated by $\frac{\lambda}{D}$ for each telescope
- Moving telescopes apart changes fringe pattern
  - at some point apart, the fringe pattern will disappear and will resolve the star
  - can then use maths to find $\theta$ and find the radius using that and the distance away
  - VLT uses more than two telescopes
- Aperture synthesis
  - a trick we need for observations
  - path length will not change between two telescopes, if they come over parallel
  - Will have a 'y' pattern of telescope arrays so that path length will always be changing no matter what way it is passing over the sky

# Lecture 7

- Zero-point mag gives one count
- **See example sheet from Lecture 6 for some good notes**

## Multi-Wavelength Techniques

- Missing a huge fraction of images outside visual
  - how do we see the rest of it?
- X-ray radiations
  - electrons wizzing around
  - Accelerated to high energies in plasma state
  - effecitvely in about a million K
  - protons will make electrons change path, and emit energy
  - accretion disks generate some of this
- Difficulties
  - X-rays have too high energies
  - mirrors absorb it and don't work
  - very shallow angle mirrors focus instead
  - Grazing incidence

- UV radiation
  - temperatures of around $50\,kK$
  - massive stars
  - clumpy as all around clumps of new big stars forming in groups
- Difficulties
  - CCDs have lower QE for these lower energies
  - hard to move energy level difference in CCDs to measure UV accurately
  - swamped by other photons
  - use a blocking filter to try and filter visual photons away and just get UV
- Infra-red radiation
  - begin to suffer from sky background here
  - to do it accurately, you need to be in space
  - see a 'fuzz' tracing spiral arms on galaxies
    - hot dust in the interstellar medium being heated by stars
    - emission from cooler stars
    - globular clusters of old stars
- Sub-millimeter radiation
  - looking at $T = 3 \to 10\,K$
  - challenging to detect such low energies
  - very sensitive thermometers
  - liquid helium at a few micro-Kelvin
  - changes resistance and allows current to flow for a second
- Why
  - Pillars of Creation
  - lots of dusty regions
    - actively forming stars in the dust clouds
    - carbonaceous material - graphite, diamonds etc
    - silicates
    - ices
  - optical photons increases dust temperature slightly, still around 10 K though
    - emits 100 micron wavelength photons to lose temperature
  - looking at Pillars in sub-millimeter shows clouds glowing now
  - can observe nebulae very differently in sub-millimeter
- Radio radiation
  - 3 components
    - local thunderstorms
    - distant thunderstorms - radio waves bounce round atmosphere
    - constant hiss with period of 23 hours 56 minutes and 4.1 seconds
    - sidereal day
  - This hiss is the galactic emission
  - surface of telescopes need to be 'smooth'
  - smoothness isn't as necessary for radios
    - easy to build big telescopes for radio without this concern
  - very difficult to get a high resolution radio telescope

# Lecture 8

## Radios Ctd

- Biggest telescope is FAST
  - 500m diameter
- Why observe in radio?

- 21cm
  - Neutral H emission
- electron can have parallel or anti-parallel spin
  - two sub ground states
- anti-parallel is lower energy than parallel so will eventually flip to this one
  - very small energy difference
  - hyper-fine energy splitting
  - this takes a few millions years though
- lots of H in galaxies
  - probability adds up to observe this
  - pointing radio telescopes sees this

## Telescope Tech

- 'Twinkling star'
  - caused by atmosphere moving around and bumping image around
  - break it up into sub-images
    - speckles
  - whole image will also move around
- Fried parameter
  - $r_0 \approx 10\, cm$
  - size of turbulent cells
  - coherence time
    - $t_0 = \frac{r_0}{v}$
    - v is wind speed
    - this means that a star will only be stable for about $10\, ms$
- Correcting this
  - light comes in normally
  - hits third mirror that can change angle with actuators
  - then hits a beam splitter
    - 50% to computer analyser
    - 50% to somewhere else
  - computer constantly measures image and changes actuators to correct image for turbulence
    - uses fast Fourier transforms to get back to real image
    - happens every millisecond or so
  - this requires bright star though
  - shine lasers up to $15\, km$ into atmosphere to focus
    - this creates a fake star for corrections - 'natural guide star'

## Exoplanets

- How do we observe planets against photon noise of stars?
  - observe stellar spectrum and planet spectrum for comparison
  - heavier molecules are more difficult to observe as they're lower down
    - refraction issues
  - detecting $O_3$ would be a key trigger for life
    - not able to do it yet

# Chapter 2

# Stars

*see DUO for pdf slides*

## Lecture 1

- Black body emission curve
  - LHS from peak lambda is Rayleigh Jeans tail
  - RHS from peak is Wien tail

$$\lambda_{max} = \frac{2.9 \times 10^{-3}}{T}\, m$$

$$\lambda_{max.\,Bet} = 8.3 \times 10^{-7} m \implies T \approx 3500\, K$$

$$\lambda_{max,Sun} = 5.5 \times 10^{-7} m \implies T \approx 5300\, K$$

$$\lambda_{max,Bel} = 3.0 \times 10^{-7} m \implies T \approx 9400\, K$$

## Lecture 2

**Excitation Energies**

- Bohr model
- page 8 on slides
- n denotes the orbitals/electron shells
- $n = 1$ is the ground state

$$E = E_{high} - E_{low} = \frac{hc}{\lambda} = -13.6\Big(\frac{1}{n_{high}^2} - \frac{1}{n_{low}^2}\Big)$$

$$n = 2 \to 4$$

$$E = 2.55\, eV \implies \lambda = 486.1\, nm \implies H\beta$$

- this was absorption
- $H\beta$ is shorthand for Balmer series $\beta$
  - Optical light

$$n = 2 \to 1$$
$$E = 10.2\,eV \implies \lambda = 121.6\,nm \implies Ly\alpha$$

- this was emission
- $Ly\alpha$ is shorthand for Lyman series $\alpha$
  - UV light
- Photons emitted from de-excitation in random direction
  - statistics means we probably won't see this

**Ratios of Excitation Levels**

$$n = 2 \to 1$$
$$\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-\frac{(E_2 - E_1)}{kT}}$$
$$g_1 = 2\ ;\ g_2 = 8\ ;\ T = 5800\,K$$
$$\frac{N_2}{N_1} = 5.1 \times 10^{-9}$$

- 1 billionth of H atoms in first excited state, negligible

**Ionisation Energies**

- $\chi$ is the ionisation energy

$$\frac{N_{i+1}}{N_i} = \frac{2Z_{i+1}}{n_e Z_i} \Big(\frac{2\pi m_e kT}{h^2}\Big)^{\frac{3}{2}} e^{-\frac{\chi}{kT}}$$
$$E > -13.6 \Big(\frac{1}{\infty^2} - \frac{1}{n_{low}^2}\Big)\,eV$$
$$n = 1 \to \infty \implies E > 13.6\,eV$$
$$n = 2 \to \infty \implies E > 3.4\,eV$$

# Lecture 3

## Binary Star Systems

- slide 8, binary system
- look at the semi-major axes of the orbits of the two stars around the centre of mass of the system
  - $a_1$ and $a_2$ for $m_1$ and $m_2$

$$P^2 = \frac{4\pi^2 a^3}{G(m_1 + m_2)}$$
$$a = a_1 + a_2$$

- Smaller semi-major axis means larger mass
- similar to see-saw

$$m_1 a_1 = m_2 a_2 \implies \frac{m_1}{m_2} = \frac{a_2}{a_1}$$

- ratio of the semi-major axes gives ratio of masses
- actually measure $\alpha$, angle of separation:
  - for d, distance from us

$$\alpha_n = \frac{a_n}{d} \implies \frac{m_1}{m_2} = \frac{\alpha_2}{\alpha_1}$$

**Visual Binary Systems**

**Normal Example**

- $d = 10\,pc$ ; $P = 200$ days
- $\alpha_1 = 0.02$" ; $\alpha_2 = 0.08$"

$$a_1 = \alpha_1 d = 0.2\,Au \ ; \ a_2 = a_2 = \alpha_2 d = 0.8\,Au$$
$$a = a_1 + a_2 = 1\,Au$$
$$m_1 + m_2 = \frac{4\pi^2 a^3}{GP^2} = 3.4 M_\odot = M_{tot}$$
$$\frac{m_1}{m_2} = \frac{\alpha_2}{\alpha_1} = \frac{a_2}{a_1} = 4.0 = M_{rot}$$
$$m_1 = \left[\frac{M_{rot}}{1 + M_{rot}}\right] M_{tot} = 2.72 M_\odot$$
$$m_2 = \left[\frac{1}{1 + M_{rot}}\right] M_{tot} = 0.68 M_\odot$$

**Inclination Example**

- For angled systems that aren't flat against our observations:

$$\hat{\alpha}_n = \alpha_n \cos i$$
$$m_1 + m_2 = \frac{4\pi^2}{G}\left(\frac{d}{\cos i}\right)\frac{\hat{\alpha}^3}{P^2}$$
$$\hat{\alpha} = \hat{\alpha}_1 + \hat{\alpha}_2$$

- Has no effect on mass ratios observed - cos cancels
- Above equation means the actual masses will be affected by the inclination

**Spectroscopic Binaries**

- Correcting for inclination:

$$v_{nr}^{max} = v_n \sin i$$

- Assume $e << 1$

$$v_n = \frac{2\pi a_n}{P}$$
$$\frac{m_1}{m_2} = \frac{v_2}{v_1}$$

- Same sort of stuff as visual binaries, but sin instead of cos basically

**Special Case: Eclipsing Spectroscopic Binaries**

- $i \approx 90°$
- don't need any corrections etc

# Lecture 4

$$P = \underbrace{\frac{\rho k T}{\mu m_H}}_{\text{ideal gas law}} + \frac{1}{3} a T^4$$

- Hydrostatic Equilibrium:
  - Pressure force = Gravitational force

$$\begin{aligned}
P\,on\,dA &= [P(r+dr) - P(r)]dA \\
&= dP\,dA \\
Gravitational &= g\,\underbrace{\underbrace{dA\,dr}_{volume}\,\rho}_{mass},\;\; g = \frac{GM_r}{r^2} \\
dP\,dA &= -g\rho\,dA\,dr \\
\frac{dP}{dr} &= -\frac{GM_r\rho}{r^2} \\
\frac{dM_r}{dr} &= 4\pi r^2 \rho \\
M_r &= \frac{4}{3}\pi r^2 \rho \\
\frac{dP}{dr} &= -G\frac{4}{3}\pi r \rho^2 \\
\int_{P_s}^{P_c} dP &= -\frac{4}{3}\pi G \rho^2 \int_R^0 r\,dr \\
P_c &= \frac{2}{3}\pi G \rho^2 r^2,\;\; P_s = 0\,at\,r = R \\
&= \frac{2}{3}\pi G r^2 \Big[\frac{3}{4}\frac{M}{\pi r^3}\Big]^2 \\
&= \frac{3}{8\pi}\frac{GM^2}{R^4}
\end{aligned}$$

- Example for our sun:

$$M = 2\times 10^{30}kg\;;\;\; R \approx 7\times 10^8 m$$

$$P_c \approx 10^{14} N\,m^{-2}$$

$$P_{c,\,true} \approx 2\times 10^{16} N\,m^{-2}$$

- out as assumed uniform density

# Lecture 5

## Virial Theorem

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2} \times V = \frac{4}{3}\pi r^3$$

$$V\frac{dP}{dr} = -\frac{GM\rho}{r^2}\frac{4}{3}\pi r^3$$

$$-\text{ plug in } \frac{dm}{dr} = 4\pi r^2\rho$$

$$V\frac{dP}{dr} = \frac{1}{3}\frac{GM}{r}\frac{dm}{dr}$$

$$\int_0^{P(R)} V\,dP = -\frac{1}{3}\underbrace{\int_0^M \frac{GM}{r}dm}_{\text{Total GPE}=U}$$

$$LHS: \int U\,dV = UV - \int V\,dU$$

$$\int_0^{P(R)} V\,dP = \underbrace{[PV]_0^{R_0}}_{=0} - \int_0^{V(R)} P\,dV = -\frac{1}{3}U$$

$$-3\int_0^{V(R)} P\,dV = U,\ dV = \frac{dm}{\rho} \implies$$

$$-3\int_0^M \frac{P}{\rho}dm = U \text{ - generalised form of Virial Theorem}$$

$$\text{Ideal Gas: } P = nkT = \frac{\rho kT}{\mu m_H}$$

$$\text{Average KE: } = \frac{3}{2}kT$$

$$\text{KE per kilo: } = \frac{3}{2}\frac{kT}{\mu m_H}$$

$$E_{KE} = \frac{3}{2}\frac{kT}{\mu m_H} = \frac{3}{2}\frac{P}{\rho}$$

$$-3\int_0^M \frac{P}{\rho}dm = U,\ \frac{P}{\rho} = \frac{2}{3}E_{KE}$$

$$\underbrace{\int_0^M E_{KE}\,dm}_{\text{Total KE, assume ideal gas}} = -\frac{1}{2}U$$

$$\implies K = -\frac{1}{2}U$$

## Energy from Gravitational Collapse

$$
\begin{aligned}
dU_{g,i} &= -\frac{GM_r dm_i}{r} \text{ - GPE of point mass} \\
&\text{Consider shells of material} \\
dm &= 4\pi r^2 \rho dr \\
dU_g &= -\frac{GM_r 4\pi r^2 \rho}{r} dr \text{ - GPE of a shell} \\
U_g &= -4\pi G \int_0^R M_r \rho_r dr \\
M_r &= \frac{4}{3}\pi r^3 \bar{\rho} \text{ - avg density isn't too bad here} \\
U_g &= -\frac{16}{3}\pi^2 G \bar{\rho}^2 \int_0^R r^4 dr \\
&= -\frac{16}{15}\pi^2 G \bar{\rho}^2 R^5 \\
&\text{Convert back to mass} \\
U &= -\frac{9}{15}\frac{GM^2}{R} \text{ - GPE of the star} \\
K &= -\frac{1}{2}U \\
\implies E &= \frac{3}{10}\frac{GM^2}{R} \\
E &\approx \frac{3}{10}GM^2\Big[\frac{1}{R} - \frac{1}{R_{initial}}\Big] \\
&= \frac{3}{10}\frac{GM^2}{R} \iff R << R_{initial}
\end{aligned}
$$

# Lecture 6

## Binding Energies of Fusion

$$
\begin{aligned}
E_b(Z, N) &= \Delta mc^2 = [Zm_p + Nm_n - m(Z, N)]c^2 \\
E_b(4, 0) &= [4m_p - m_{He,4}]c^2 = 26.731\, MeV \\
\frac{4m_p}{m_{He,4}} &= 1.007 \implies e = 0.7\% \\
E_\odot &= (0.1 \times M_\odot) \times 0.007 \times c^2 \\
&= 1.3 \times 10^{44} J \\
t &\approx \frac{E_\odot}{L_\odot} = 10^{10} yr
\end{aligned}
$$

## Coulomb Barrier

- looking at probability that two particles are close enough for nuclear force to be important
- see figure on page 7 of slides
- using classical physics, we get

$$E = \frac{1}{2}mv^2 = \frac{3}{2}kT = \frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{r}$$
$$T = \frac{1}{6\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{rk} = \underbrace{1.1 \times 10^{10} K}_{r=10^{-15}m \ ; \ Z_1=Z_2=1}$$

- too high for our Sun
- use deBroglie wavelength and consider quantum effects

$$\lambda = \frac{h}{p}, \ p = mv \ [m = \mu_m]$$
$$E = \frac{1}{2}mv^2 \ ; \ v^2 = \frac{p^2}{m^2}$$
$$E = \frac{p^2}{2m}$$
$$p^2 = \left(\frac{h}{\lambda}\right)^2$$
$$E = \frac{(\frac{h}{\lambda})^2}{2m} = \frac{h^2}{\lambda^2}\frac{1}{2m}$$
$$= \frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{\lambda} = \frac{h^2}{\lambda^2}\frac{1}{2m}$$
$$\frac{1}{\lambda} = \frac{2}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2 m}{h^2}$$
$$\text{replace } \frac{1}{r} \text{ with } \frac{1}{\lambda}$$
$$T = \frac{1}{12\pi^2\epsilon_0^2}\frac{Z_1^2 Z_2^2 e^4 m}{kh^2} = 9.8 \times 10^6 K$$

- this happens due to quantum tunneling

### Probability of Nuclear Reactions

- see graph on page 13 of slides
- nuclear reaction probability is the product of Maxwell-Boltzmann and Tunneling Probability

## Lecture 7

### Nuclear Conservation Rules

1. electric charge must be conserved
2. nucleon umber must be conserved
    - $p, n = +1$
3. lepton number must be conserved
    - $e^{\mp} = \pm 1$
    - $\nu_e^{\mp} = \pm 1$

$${}^{A}_{Z}X$$

- A - atomic number for element X (nucleon number)
- Z - number of protons (electric charge)

## Proton-Proton Chains

$$\begin{aligned}
&{}^1_1H + {}^1_1H \to {}^2_1H + e^+ + \nu_e \\
&{}^2_1H + {}^1_1H \to {}^3_2He + \gamma \\
&{}^3_2He + {}^3_2He \to {}^4_2He + {}^1_1H + {}^1_1H \\
\implies &4{}^1_1H \to {}^4_2He + \underbrace{2e^+ + 2\nu_e + 2\gamma}_{26.7\,MeV}
\end{aligned}$$

## CNO Cycle

$$\begin{aligned}
&{}^{12}_6C + {}^1_1H \to {}^{13}_7N + \gamma \\
&{}^{13}_7N \to \underbrace{{}^{13}_6C + e^+ + \nu_e}_{\beta \text{ decay}} \\
&{}^{13}_6C + {}^1_1H \to {}^{14}_7N + \gamma \\
&{}^{14}_7N + {}^1_1H \to {}^{15}_8O + \gamma \\
&{}^{15}_8O \to \underbrace{{}^{15}_7N + e^+\nu_e}_{\beta \text{ decay}} \\
&{}^{15}_7N + {}^1_1H \to {}^{12}_6C + {}^4_2He \\
\text{Total: } &4{}^1_1H \to {}^4_2He + \underbrace{2e^+ + 2\nu_e + 3\gamma}_{E=26.7\,MeV}
\end{aligned}$$