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Stars and Galaxies

Stars

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Lecture 1

see DUO for slides Black body emission curve:

- LHS from peak lambda is Rayleigh Jeans tail
- RHS from peak is Wien tail

$$\lambda_{max} = \frac{2.9 \times 10^{-3}}{T} m$$
$$\lambda_{max.\,Bet} = 8.3 \times 10^{-7} m \implies T \approx 3500\, K$$
$$\lambda_{max,Sun} = 5.5 \times 10^{-7} m \implies T \approx 5300\, K$$
$$\lambda_{max,Bel} = 3.0 \times 10^{-7} m \implies T \approx 9400\, K$$

Lecture 2

2.1 Excitation Energies

- Bohr model
- page 8 on slides
- n denotes the orbitals/electron shells
- $n = 1$ is the ground state

$$E = E_{high} - E_{low} = \frac{hc}{\lambda} = -13.6\Big(\frac{1}{n_{high}^2} - \frac{1}{n_{low}^2}\Big)$$
$$n = 2 \rightarrow 4$$
$$E = 2.55\, eV \implies \lambda = 486.1\, nm \implies H\beta$$

- this was absorption
- $H\beta$ is shorthand for Balmer series β
- Optical light

$$n = 2 \rightarrow 1$$
$$E = 10.2\, eV \implies \lambda = 121.6\, nm \implies Ly\alpha$$

- this was emission
- $Ly\alpha$ is shorthand for Lyman series α
  - UV light
- Photons emitted from de-excitation in random direction
  - statistics means we probably won't see this

Lecture 3 Ratios of Excitation Levels

$$
\begin{aligned}
n &= 2 \to 1 \\
\frac{N_2}{N_1} &= \frac{g_2}{g_1} e^{-\frac{(E_2 - E_1)}{kT}} \\
g_1 &= 2 \ ; \ g_2 = 8 \ ; \ T = 5800\,K \\
\frac{N_2}{N_1} &= 5.1 \times 10^{-9}
\end{aligned}
$$

- 1 billionth of H atoms in first excited state, negligible

3.1 Ionisation Energies

- χ is the ionisation energy

$$
\begin{aligned}
\frac{N_{i+1}}{N_i} &= \frac{2Z_{i+1}}{n_e Z_i}\left(\frac{2\pi m_e kT}{h^2}\right)^{\frac{3}{2}} e^{-\frac{\chi}{kT}} E \qquad > -13.6\left(\frac{1}{\infty^2} - \frac{1}{n_{low}^2}\right) eV \\
n = 1 \to \infty &\implies E > 13.6\,eV \\
n = 2 \to \infty &\implies E > 3.4\,eV
\end{aligned}
$$

Lecture 4

4.1 Binary Star Systems

- slide 8, binary system
- look at the semi-major axes of the orbits of the two stars around the centre of mass of the system
  - a_1 and a_2 for m_1 and m_2

$$
\begin{aligned}
P^2 &= \frac{4\pi^2 a^3}{G(m_1 + m_2)} \\
a &= a_1 + a_2
\end{aligned}
$$

- Smaller semi-major axis means larger mass
- similar to see-saw

$$
m_1 a_1 = m_2 a_2 \implies \frac{m_1}{m_2} = \frac{a_2}{a_1}
$$

- ratio of the semi-major axes gives ratio of masses
- actually measure α, angle of separation:
  - for d, distance from us

$$
\alpha_n = \frac{a_n}{d} \implies \frac{m_1}{m_2} = \frac{\alpha_2}{\alpha_1}
$$

4.2 Visual Binary Systems

Normal Example

- $d = 10\,pc$; $P = 200$ days
- $\alpha_1 = 0.02$" ; $\alpha_2 = 0.08$"

$$\begin{aligned} a_1 &= \alpha_1 d = 0.2\,Au \;;\; a_2 = a_2 = \alpha_2 d = 0.8\,Au \\ a &= a_1 + a_2 = 1\,Au \\ m_1 + m_2 &= \frac{4\pi^2 a^3}{GP^2} = 3.4M_\odot = M_{tot} \\ \frac{m_1}{m_2} &= \frac{\alpha_2}{\alpha_1} = \frac{a_2}{a_1} = 4.0 = M_{rot} \\ m_1 &= \left[\frac{M_{rot}}{1+M_{rot}}\right] M_{tot} = 2.72M_\odot \\ m_2 &= \left[\frac{1}{1+M_{rot}}\right] M_{tot} = 0.68M_\odot \end{aligned}$$

Inclination Example

- For angled systems that aren't flat against our observations:

$$\begin{aligned} \hat{\alpha}_n &= \alpha_n \cos i \\ m_1 + m_2 &= \frac{4\pi^2}{G}\left(\frac{d}{\cos i}\right)\frac{\hat{\alpha}^3}{P^2} \\ \hat{\alpha} &= \hat{\alpha}_1 + \hat{\alpha}_2 \end{aligned}$$

- Has no effect on mass ratios observed - cos cancels
- Above equation means the actual masses will be affected by the inclination

4.3 Spectroscopic Binaries

- Correcting for inclination:

$$v_{nr}^{max} = v_n \sin i$$

- Assume $e << 1$

$$v_n = \frac{2\pi a_n}{P}\frac{m_1}{m_2} \qquad\qquad = \frac{v_2}{v_1}$$

- Same sort of stuff as visual binaries, but sin instead of cos basically

Special Case: Eclipsing Spectroscopic Binaries

- $i \approx 90°$
- don't need any corrections etc

<u>**Lecture 5**</u>

$$P = \underbrace{\frac{\rho k T}{\mu m_H}}_{\text{ideal gas law}} + \frac{1}{3}aT^4$$

- Hydrostatic Equilibrium:

➥ Pressure force = Gravitational force

$$\begin{aligned}
P\,on\,dA &= [P(r+dr) - P(r)]dA \\
&= dP\,dA \\
Gravitational &= g\,\underbrace{\underbrace{dA\,dr}_{volume}\,\rho}_{mass},\;\; g = \frac{GM_r}{r^2} \\
dP\,dA &= -g\rho\,dA\,dr \\
\frac{dP}{dr} &= -\frac{GM_r\rho}{r^2} \\
\frac{dM_r}{dr} &= 4\pi r^2\rho \\
M_r &= \frac{4}{3}\pi r^2\rho \\
\frac{dP}{dr} &= -G\frac{4}{3}\pi r\rho^2 \\
\int_{P_s}^{P_c} dP &= -\frac{4}{3}\pi G\rho^2\int_R^0 r\,dr \\
P_c &= \frac{2}{3}\pi G\rho^2 r^2,\;\; P_s = 0\,at\,r = R \\
&= \frac{2}{3}\pi G r^2\left[\frac{3}{4}\frac{M}{\pi r^3}\right]^2 \\
&= \frac{3}{8\pi}\frac{GM^2}{R^4}
\end{aligned}$$

➤ Example for our sun:

$$\begin{aligned}
M &= 2\times 10^{30}kg\;;\;\; R \approx 7\times 10^8 m \\
P_c &\approx 10^{14} N\,m^{-2} \\
P_{c,\,true} &\approx 2\times 10^{16} N\,m^{-2}
\end{aligned}$$

➤ out as assumed uniform density

<u>Lecture 6</u>

6.1 Virial Theorem

$$\begin{aligned}
\frac{dP}{dr} &= -\frac{GM\rho}{r^2}\times V = \frac{4}{3}\pi r^3 \\
V\frac{dP}{dr} &= -\frac{GM\rho}{r^2}\frac{4}{3}\pi r^3 \\
&- \text{plug in } \frac{dm}{dr} = 4\pi r^2\rho \\
V\frac{dP}{dr} &= \frac{1}{3}\frac{GM}{r}\frac{dm}{dr} \\
\int_0^{P(R)} V\,dP &= -\frac{1}{3}\underbrace{\int_0^M \frac{GM}{r}dm}_{\text{Total GPE}=U} \\
LHS: \int U\,dV &= UV - \int V\,dU \\
\int_0^{P(R)} V\,dP &= \underbrace{[PV]_0^{R_0}}_{=0} - \int_0^{V(R)} P\,dV = -\frac{1}{3}U
\end{aligned}$$

$$-3\int_0^{V(R)} P\,dV = U,\ dV = \frac{dm}{\rho} \implies$$
$$-3\int_0^{M} \frac{P}{\rho} dm = U \ \text{ - generalised form of Virial Theorem}$$
$$\text{Ideal Gas: } P = nkT = \frac{\rho kT}{\mu m_H}$$
$$\text{Average KE: } = \frac{3}{2}kT$$
$$\text{KE per kilo: } = \frac{3}{2}\frac{kT}{\mu m_H}$$
$$E_{KE} = \frac{3}{2}\frac{kT}{\mu m_H} = \frac{3}{2}\frac{P}{\rho}$$
$$-3\int_0^{M} \frac{P}{\rho} dm = U,\ \frac{P}{\rho} = \frac{2}{3}E_{KE}$$
$$\underbrace{\int_0^M E_{KE}\,dm}_{\text{Total KE, assume ideal gas}} = -\frac{1}{2}U$$
$$\implies K = -\frac{1}{2}U$$

6.1.1 Energy from Gravitational Collapse

$$dU_{g,i} = -\frac{GM_r dm_i}{r} \text{ - GPE of point mass}$$
$$\text{Consider shells of material}$$
$$dm = 4\pi r^2 \rho dr$$
$$dU_g = -\frac{GM_r 4\pi r^2 \rho}{r} dr \text{ - GPE of a shell}$$
$$U_g = -4\pi G \int_0^R M_r \rho_r dr$$
$$M_r = \frac{4}{3}\pi r^3 \bar{\rho} \text{ - avg density isn't too bad here}$$
$$U_g = -\frac{16}{3}\pi^2 G \bar{\rho}^2 \int_0^R r^4 dr$$
$$= -\frac{16}{15}\pi^2 G \bar{\rho}^2 R^5$$
$$\text{Convert back to mass}$$
$$U = -\frac{9}{15}\frac{GM^2}{R} \text{ - GPE of the star}$$
$$K = -\frac{1}{2}U$$
$$\implies E = \frac{3}{10}\frac{GM^2}{R}$$
$$E \approx \frac{3}{10}GM^2\Big[\frac{1}{R} - \frac{1}{R_{initial}}\Big]$$
$$= \frac{3}{10}\frac{GM^2}{R} \iff R << R_{initial}$$

Lecture 7

7.1 Binding Energies of Fusion

$$
\begin{aligned}
E_b(Z,N) &= \Delta mc^2 = [Zm_p + Nm_n - m(Z,N)]c^2 \\
E_b(4,0) &= [4m_p - m_{He,4}]c^2 = 26.731\,MeV \\
\frac{4m_p}{m_{He,4}} &= 1.007 \implies e = 0.7\% \\
E_\odot &= (0.1 \times M_\odot) \times 0.007 \times c^2 \\
&= 1.3 \times 10^{44}\,J \\
t &\approx \frac{E_\odot}{L_\odot} = 10^{10} yr
\end{aligned}
$$

7.2 Coulomb Barrier

- looking at probability that two particles are close enough for nuclear force to be important
- see figure on page 7 of slides
- using classical physics, we get

$$
\begin{aligned}
E &= \frac{1}{2}mv^2 = \frac{3}{2}kT = \frac{1}{4\pi\epsilon_0}\frac{Z_1Z_2e^2}{r} \\
T &= \frac{1}{6\pi\epsilon_0}\frac{Z_1Z_2e^2}{rk} = \underbrace{1.1 \times 10^{10}K}_{r=10^{-15}m\ ;\ Z_1=Z_2=1}
\end{aligned}
$$

- too high for our Sun
- use deBroglie wavelength and consider quantum effects

$$
\begin{aligned}
\lambda &= \frac{h}{p},\ p = mv\ [m = \mu_m] \\
E &= \frac{1}{2}mv^2\ ;\ v^2 = \frac{p^2}{m^2} \\
E &= \frac{p^2}{2m} \\
p^2 &= \left(\frac{h}{\lambda}\right)^2 \\
E &= \frac{(\frac{h}{\lambda})^2}{2m} = \frac{h^2}{\lambda^2}\frac{1}{2m} \\
&= \frac{1}{4\pi\epsilon_0}\frac{Z_1Z_2e^2}{\lambda} = \frac{h^2}{\lambda^2}\frac{1}{2m} \\
\frac{1}{\lambda} &= \frac{2}{4\pi\epsilon_0}\frac{Z_1Z_2e^2m}{h^2} \\
\text{replace } \frac{1}{r} &\text{ with } \frac{1}{\lambda} \\
T &= \frac{1}{12\pi^2\epsilon_0^2}\frac{Z_1^2Z_2^2e^4m}{kh^2} = 9.8 \times 10^6 K
\end{aligned}
$$

- this happens due to quantum tunneling

7.3 Probability of Nuclear Reactions

- see graph on page 13 of slides
- nuclear reaction probability is the product of Maxwell-Boltzmann and Tunneling Probability

Lecture 8

8.1 Nuclear Conservation Rules

1. electric charge must be conserved
2. nucleon umber must be conserved
   - $p, n = +1$
3. lepton number must be conserved
   - $e^{\mp} = \pm 1$
   - $\nu_e^{\mp} = \pm 1$

$${}^{A}_{Z}X$$

- A - atomic number for element X (nucleon number)
- Z - number of protons (electric charge)

8.2 Proton-Proton Chains

$$\begin{aligned} {}^{1}_{1}H + {}^{1}_{1}H &\rightarrow {}^{2}_{1}H + e^{+} + \nu_e \\ {}^{2}_{1}H + {}^{1}_{1}H &\rightarrow {}^{3}_{2}He + \gamma \\ {}^{3}_{2}He + {}^{3}_{2}He &\rightarrow {}^{4}_{2}He + {}^{1}_{1}H + {}^{1}_{1}H \\ \implies 4{}^{1}_{1}H &\rightarrow {}^{4}_{2}He + \underbrace{2e^{+} + 2\nu_e + 2\gamma}_{26.7\,MeV} \end{aligned}$$

8.3 CNO Cycle

$$\begin{aligned} {}^{12}_{6}C + {}^{1}_{1}H &\rightarrow {}^{13}_{7}N + \gamma \\ {}^{13}_{7}N &\rightarrow \underbrace{{}^{13}_{6}C + e^{+} + \nu_e}_{\beta \text{ decay}} \\ {}^{13}_{6}C + {}^{1}_{1}H &\rightarrow {}^{14}_{7}N + \gamma \\ {}^{14}_{7}N + {}^{1}_{1}H &\rightarrow {}^{15}_{8}O + \gamma \\ {}^{15}_{8}O &\rightarrow \underbrace{{}^{15}_{7}N + e^{+}\nu_e}_{\beta \text{ decay}} \\ {}^{15}_{7}N + {}^{1}_{1}H &\rightarrow {}^{12}_{6}C + {}^{4}_{2}He \\ \text{Total: } 4{}^{1}_{1}H &\rightarrow {}^{4}_{2}He + \underbrace{2e^{+} + 2\nu_e + 3\gamma}_{E=26.7\,MeV} \end{aligned}$$

Lecture 9

9.1 Energy produced in Stars

$$\begin{aligned} dL &= \epsilon\, dm \quad [W] \\ \epsilon_{i,X} &= \epsilon_0 X_i X_X \rho^{\alpha} T^{\beta} \quad [W\, kg^{-1}] \\ dm &= 4\pi r^2 \rho\, dr \\ \implies \frac{dL}{dr} &= 4\pi r^2 \rho \epsilon \end{aligned}$$

Slide 5 diagram

- Solid line just to do with fusion then no fusion
- Dashed line has that shape as volume increase so dL/dr does but then temperature starts falling so fusion decreases

9.2 Energy Seen on Earth

- Electrons lose energy travelling through sun

$$\frac{\lambda_{surface}}{\lambda_{core}} \approx 3 \times 10^6$$

9.3 Mean Free Paths

- vt - distance travelled
- n - particles per unit volume
- nvt - particle per unit area
- $n\sigma vt$ - number of interactions

$$\begin{aligned} l &= \frac{vt}{n\sigma vt} \\ &= \frac{1}{n\sigma} \end{aligned}$$

- This is the mean distance before a collision

$$\begin{aligned} d &= \sum_i l_i \\ d^2 &= d \cdot d \\ &= \sum_j \sum_i l_i \cdot l_j \end{aligned}$$

- When $i \neq j$, $l_i \cdot l_j = 0$

$$\begin{aligned} d^2 &= Nl^2 \\ \implies N &= \left(\frac{d}{l}\right)^2 \end{aligned}$$

- Use ideal gas law to help in questions of this

$$\begin{aligned} t_{total} &= t_{travel} + Nt_{scatter} \\ &= \frac{Nl}{c} + N \times 10^8 \\ &= 5700\, yrs + \cdots = 10^6\, yrs \end{aligned}$$

9.4 Radiation

$$\begin{aligned} P &= \frac{1}{3}aT^4 \\ \frac{dP}{P}dr &= \frac{dP}{dT}\frac{dT}{dr} \\ \frac{dP}{dr} &= \frac{4}{3}aT^3\frac{dT}{dr} \\ \frac{dP}{dr} &= -\frac{\kappa\rho}{c}F_{rad} \end{aligned}$$

$$\kappa rho = n\sigma$$
$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\kappa\rho F_{rad}}{T^3}$$
$$L = 4\pi r^2 F_{rad}$$
$$\frac{dT}{dr} = -\frac{3}{16\pi ac}\frac{\kappa\rho L_r}{T^3 r^2}$$

Lecture 10

10.1 Opacity

$$dI_\lambda = -\kappa_\lambda \rho I_\lambda ds$$
$$\int_{I_{\lambda,0}}^{I_{\lambda,f}} \frac{dI_\lambda}{I_\lambda} = -\int \kappa_\lambda \rho ds$$
$$\implies I_{\lambda,f} = I_{\lambda,0} e^{-\int_0^s \kappa_\lambda \rho ds}$$
$$I_{\lambda,f} = I_{\lambda,0} \quad \underbrace{e^{-\kappa_\lambda \rho s}}_{\text{optical depth, } \tau}$$
$$= I_{\lambda,0} e^{-\tau}, \ \tau = \kappa_\lambda \rho s$$

- $\tau < 1$ - optically thin
- $\tau > 1$ - optically thick

Different sources of Opacity

- Two classes of opacity:
  1. Absorption - photon energy lost of KE of gas or degraded
  2. Scattering - photon reemitted at different direction, sometimes degraded

1. Bound-Bound transitions
   - typical temperature roughly $\leq 10^5$K
   - most effective for neutral gas
   - scattering and absorption
2. Bound-free transitions
   - typical temperature of $10^4 \to 10^6$K
   - partially ionised gas
   - absorption
3. Free-free emission
   - typical temperature of $10^4 \to 10^6$K
   - partially ionised gas
   - absorption
4. Electron scattering
   - dominant at roughly $\geq 10^6$K
   - fully ionised gas
   - scattering

Lecture 11 Lecture 10

11.1 Schwarzchild Criterion for Convection

- slide 4 - 9

$$\gamma = \frac{C_p}{C_V} = \frac{s+2}{s}$$

- s is degrees of freedom

$$
\begin{aligned}
P &= k_a \rho^\gamma \\
\frac{dP}{P} &= \frac{\gamma d\rho}{\rho} \\
\gamma &= \frac{\rho}{P}\frac{dP}{d\rho}
\end{aligned}
$$

Surrounding gas

$$
\begin{aligned}
P &= nkT = \frac{\rho k T}{\mu m_H} \\
\frac{dP}{P} &= \frac{d\rho}{\rho} + \frac{dT}{T} \\
\frac{d\rho}{\rho} &= \frac{dP}{P} - \frac{dT}{T} \\
\frac{dP}{d\rho}_{sur} &> \frac{dP}{d\rho}_{adiab} \Bigg[\times \frac{\rho}{P} \\
\frac{\rho}{P}\frac{dP}{d\rho}_{sur} &> \frac{\rho}{P}\frac{dP}{d\rho}_{adiab} \\
\frac{\rho}{P}\frac{dP}{d\rho}_{sur} &> \gamma_{ad} \\
\frac{P}{dP}\Big(\frac{dP}{P} - \frac{dT}{T}\Big)_{sur} &< \frac{1}{\gamma_{adiab}} \\
\frac{P}{dP}\frac{dP}{P} - \frac{P}{dP}\frac{dT}{T} &< \frac{1}{\gamma_{adiab}} \\
1 - \Big(\frac{P}{dP}\frac{dT}{T}\Big)_{sur} &< \frac{1}{\gamma_{adiab}} \\
\frac{T}{P}\Big(\frac{dP}{dT}\Big)_{sur} &< \frac{\gamma_{adiab}}{\gamma_{adiab} - 1} \\
\Big|\frac{dT}{dr}\Big|_{sur} &> \Big(\frac{\gamma_{adiab} - 1}{\gamma_{adiab}}\Big)\frac{T}{P}\Big|\frac{dP}{dr}\Big|_{sur}
\end{aligned}
$$

Convection in the Sun For the sun:

$$
\begin{aligned}
-\frac{3}{16\pi ac}\frac{k\rho L_r}{T^3 r^2} &> \Big(\frac{\gamma - 1}{\gamma}\Big)\frac{T}{P}\frac{dP}{dr} \\
\frac{dP}{dr} &= -\frac{GM_r\rho}{r^2} \\
\frac{L_r}{M_r} &> \frac{16\pi acG}{\kappa\rho}\frac{aT^4}{3}\frac{\gamma - 1}{\gamma} \\
&> \frac{16\pi acG}{\kappa\rho}P_{rad}\frac{\gamma - 1}{\gamma} \\
&> 1.9 \times 10^{-3}\, W\, kg^{-1}
\end{aligned}
$$

Mixing length

$$
l = \alpha Hp
$$

$$
\frac{dP}{dr} = -\frac{GM_r\rho}{r^2} \implies \frac{1}{Hp} = -\frac{1}{P}\frac{dP}{dr}
$$

$$
Hp = \frac{Pr^2}{GM_r\rho}
$$

$$l = \frac{\alpha P r^2}{G M_r \rho}$$

Lecture 12

12.1 Cepheid Variables

$$\log\left(\frac{L}{L_\odot}\right) = 1.15\log_{10}\Pi^d + 2.47$$
$$\Pi^d = 10\,\text{days} \implies L = 4200\,L_\odot$$
$$\text{observed} < f > = 10^{-15} W\, m^{-2}$$
$$L = 4\pi d^2 < f >$$
$$d = \sqrt{\frac{L}{4\pi < f >}}$$

12.2 Stellar Pulsation

$$V_s = \sqrt{\frac{\gamma P}{\rho}},\ \gamma = \frac{C_p}{C_V}$$
$$\Pi = 2\int_0^R \frac{dr}{V_s}$$
$$\frac{dP}{dr} = -\frac{G M_r \rho}{r^2}$$
$$\text{const } p \implies \mu = \frac{4}{3}\pi r^3 \rho$$
$$\frac{dP}{dr} = -\frac{4}{3}G\pi r \rho^2$$
$$dP = -\frac{4}{3}G\pi\rho^2 \int_0^R r\,dr$$
$$P(r) = \frac{4}{3}G\pi\rho^2\left[\frac{R^2}{2} - \frac{r^2}{2}\right]$$
$$\Pi = 2\int_0^R \frac{dr}{V_s}$$
$$= 2\int_0^R \frac{dr}{\sqrt{\frac{2}{3}\gamma G\rho(R^2 - r^2)}}$$
$$= 2\sqrt{\frac{3}{2\gamma\pi G\rho}}\left[\sin^{-1}\left(\frac{r}{R}\right)\right]_0^R$$
$$= \sqrt{\frac{3\pi}{2G\rho\gamma}}$$

Lecture 13

13.1 Jeans Mass

➤ For the gravitational collapse of a gas cloud:

$$
\begin{aligned}
GE = U &= -\frac{3}{5}\frac{GM^2}{R} \\
KE = K &= \frac{3}{2}NkT \\
&= \frac{3}{2}\frac{M_c}{\mu m_H}kT \\
2K &< |U| \\
2\left(\frac{3}{2}\frac{M_c kT}{\mu m_H}\right) &< \frac{3}{5}\frac{GM_c^2}{R_c} \\
R_c &= \left(\frac{3}{4}\frac{M_c}{\pi\rho_0}\right)^{\frac{1}{3}} \\
2\left(\frac{3}{2}\frac{M_c kT}{\mu m_H}\right) &< \frac{3}{5}GM_c^2\left(\frac{4}{3}\frac{\pi\rho_0}{M_c}\right)^{\frac{1}{3}} \\
\frac{5M_c kT}{\mu mHG} &< M_c^2\left(\frac{4}{3}\frac{\pi\rho_0}{M_c}\right)^{\frac{1}{3}} \\
M_c &< M_J \\
M_J &\approx \left(\frac{5kT}{G\mu m_H}\right)^{\frac{3}{2}}\left(\frac{3}{4\pi\rho_0}\right)^{\frac{1}{2}}
\end{aligned}
$$

13.2 Free-fall gravitational collapse

1. $M_c > M_J$
   - free fall collapse
   - optically thin
   - pressure increase
   - temperature constant
2. Fragmentation
   - optically thin
   - individual regions exceed local M_J
3. M_J minimised: Protostar
   - optically thick
   - pressure increase
   - temperature increase
   - Slow contraction (Kelvin-Helmholtz timescale)

Lecture 14

14.1 Stellar Evolution

1. Increase in μ (mean molecular mass) with time:

$$P = nkT = \frac{\rho kT}{\mu m_H}$$

As μ increases, ρ and T also increase for the pressure to remain constant. Recall:

$$\epsilon_{i,X} = \epsilon_0 X_i X_X \rho^\alpha T^\beta, \alpha \approx 1$$

For proton-proton chain, $\beta \approx 4$. For CNO, $\beta \approx 17$. Luminosity increases with time.

14.1.1 Lifetime of Nuclear Fusion

$$
\begin{aligned}
t &= \frac{E_{tot}}{L} = \frac{X\zeta M c^2}{L} \\
\zeta_{pp} &= \frac{4m_p - m_{He}}{m_{He}} \approx 0.007 \\
t_{\odot} &= 10^{10} \text{ yrs} \\
L_{ms} &= L_{\odot}\left(\frac{M_{\odot}}{M}\right)^{\alpha} \\
t_{ms} &= \frac{X\zeta M c^2}{L_{\odot}}\left(\frac{M_{\odot}}{M}\right)^{\alpha} \\
&= 10^{10}\frac{M}{M_{\odot}}\left(\frac{M_{\odot}}{M}\right)^{\alpha} \\
\therefore t_{ms} &= 10^{10}\left(\frac{M_{\odot}}{M}\right)^{\alpha-1}
\end{aligned}
$$

Lecture 15

15.1 Eddington Limit

$$
\begin{aligned}
L_{Edd} &= \frac{4\pi c G M}{\kappa}, M = 100M_{\odot}, \kappa = \kappa_{es} = 0.04\, kg\, m^{-2} \\
&= 3 \times 10^6 L_{\odot}
\end{aligned}
$$

15.2 Photodisintegration

$$
\begin{aligned}
\lambda_{max} &= \frac{2.9 \times 10^{-3}}{T},\ E = \frac{hc}{\lambda} \\
T_c \geq 3 \times 10^9 K &\implies E \geq 1\, MeV
\end{aligned}
$$

Last Days of Fusion

- Shell fusion
- Silicon to Iron in Core
- P_{core} = high

Endothermic Release

- Iron breaking down into Helium and Helium breaking down in protons and neutrons
- still shell fusion ongoing
- P_{core} = rapidly decreasing

Electron capture

- very high density
- shell fusion
- $p + e^- \implies n + \nu_e$
- P_{core} = rapidly decreasing
- neutrino burst

Rapid core collapse

- shell fusion
- $P_{core} \approx 0$

Core rebound

- shell fusion
- $\rho > 8 \times 10^{18}\,kg\,m^{-3}$
- the strong force repels collapse and rebounds outwards

Supernova

- previous step drives supernova
- strong force drives high energy pushing
- generates a shock wave - more photodisintegration
- electron capture repeats and another neutrino burst
- nuclear synthesis of heavier elements, including beyond iron (endothermic)

<u>**Lecture 16**</u> Lecture 16

16.1 Electron Degeneracy Pressure

$$
\begin{aligned}
\Delta x \Delta p_x &\approx \hbar \\
p_m in &\approx \Delta p_x \approx \frac{\hbar}{\Delta x} \\
P &\approx \frac{1}{2} n_e p v \\
n_e &= \frac{\#e}{vol} = \frac{Z}{A}\frac{\rho}{m_H} \\
p_x &= \Delta p_x = \frac{\hbar}{\Delta x} \\
\Delta x &= n_e^{-1/3} \implies p_x = \hbar n_e^{1/3} \\
p^2 &= p_x^2 + p_y^2 + p_z^2 = 3p_x^2 \\
\implies p &= \sqrt{3} p_x = \sqrt{3}\hbar n_e^{1/3} \\
p &= mv = m_e v \\
\implies v &= \frac{p}{m_e} = \frac{\sqrt{3}}{m_e}\hbar n_e^{1/3} \\
P &= \frac{1}{3} n_e p v \\
p &= \sqrt{3}\hbar \left[\left(\frac{Z}{A}\right)\frac{\rho}{m_H}\right]^{1/3} \\
v &= \frac{\sqrt{3}}{m_e}\hbar \left[\left(\frac{Z}{A}\right)\frac{\rho}{m_H}\right]^{1/3} \\
\therefore P &= \frac{\hbar^2}{m_e} \left[\left(\frac{Z}{A}\right)\frac{\rho}{m_H}\right]^{5/3}
\end{aligned}
$$

16.2 White Dwarf Cooling

$$
t_{cool} = \frac{E_{WD}}{L_{WD}} = \left(\frac{3kT_{c,WD}}{2}\right)\left(\frac{M_{WD}}{Am_H}\right)\left(\frac{1}{L_{WD}}\right)
$$

Lecture 17

17.1 Rotation Period of Pulsars

$$
\begin{aligned}
\text{Centripetal Acceleration} &= \text{Gravitational Acceleration} \\
\omega_{max}^2 R &= \frac{GM}{R} \\
M &= \frac{4}{3}\pi R^3 \rho \\
\omega_{max}^2 R &= G\frac{4}{3}\pi R \rho \\
\omega &= 2\pi f = \frac{2\pi}{P} \\
\frac{4\pi^2}{P^2} R &= \frac{4}{3} G\pi R \rho \\
P_{min} &= \left(\frac{3\pi}{G\rho}\right)^{1/2}
\end{aligned}
$$

17.2 Stellar Core Rotation

Conservation of angular momentum:

$$
\begin{aligned}
I_i \omega_i &= I_f \omega_f, \; I = CMR^2 \\
CMR_i^2 \omega_i &= CMR_f^2 \omega_f, \; \omega = \frac{2\pi}{P} \\
\frac{2\pi}{P_f} &= \frac{2\pi}{P_i}\left(\frac{R_i}{R_f}\right)^2 \\
P_f &= P_i \left(\frac{R_f}{R_i}\right)^2
\end{aligned}
$$