CKM Matrix Elements

T. N. Pham

Centre de Physique Théorique, CNRS, Ecole Polytechnique, 91128 Palaiseau, Cedex, France
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The current analysis of the Cabibbo-Kobayashi-Maskawa(CKM) quark mixing matrix uses the standard parametrisation by 3 mixing angles and the CP-violating KM phase. However it would be more convenient to express these mixing angle parameters in terms of the known CKM matrix elements like V_{ud} , V_{us} , V_{ub} , V_{cb} and the CP-violating phase δ . The other CKM matrix elements are then expressed in terms of these known matrix elements instead of the standard mixing angles. In this paper, using V_{ud} , V_{us} , V_{ub} and V_{cb} from the current global fit, we show that the measured values for $|V_{td}/V_{ts}|$ and $\sin \beta$ imply $\gamma = (68 \pm 3)^{\circ}$ at which $\sin \beta$ and $\sin \alpha$ are near its maximum, consistent with the global fit.

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The current global fit of the CKM matrix elements in the Standard Model with imposed unitarity constraints[1, 2] seems to allow a rather precise determination of some of the less known CKM matrix elements like V_{cb} and V_{ub} . It would then be possible to directly express the mixing angle parameters in terms of the known CKM matrix elements like V_{ud} , V_{us} , V_{ub} , V_{cb} and the phase δ , rather than using a parametrization for the mixing angles as usually done in the current studies of the CKM matrix elements. The remaining CKM matrix elements V_{cd} , V_{cs} , V_{td} , V_{ts} and V_{tb} are thus completely determined directly in terms of these known quantities and the CP-violating phase δ , the main feature of our approach. Furthermore, the CP-violating phase δ can also be expressed in terms of the known CKM matrix elements and the angle γ , γ being one of the angle of the (db) unitarity triangle [2, 3] as shown in Fig. (1). The other angles β and α as well as the ratio $|V_{td}/V_{ts}|$ are then given as functions of γ . In this paper we shall present our calculation of ratio $|V_{td}/V_{ts}|$, $\sin \beta$ and $\sin \alpha$ in terms of γ using V_{ud} , V_{us} , V_{ub} , V_{cb} from the global fit. We find that the measured values for $|V_{td}/V_{ts}|$ and $\sin \beta$ imply that $\gamma = (68.6 \pm 3)^{\circ}$ and $\alpha \approx (89 \pm 3)^{\circ}$ consistent with the global fit values for these quantities.

We begin by writing down the Cabibbo-Kobayashi-Maskawa (CKM) [4, 5] quark mixing matrix as:

$$V_{\text{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

$$\tag{1}$$

To impose unitarity for the CKM matrix, we take the standard parametrization as used in [2] which is given as [6]:

$$V_{\text{CKM}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}\exp(-i\delta) \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}\exp(i\delta) & c_{12}c_{23} - s_{12}s_{23}s_{13}\exp(i\delta) & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}\exp(i\delta) & -c_{12}s_{23} - s_{12}c_{23}s_{13}\exp(i\delta) & c_{23}c_{13} \end{pmatrix}$$
(2)

where $s_{ij} = \sin(\theta_{ij}, c_{ij} = \cos(\theta_{ij})$ and δ is the CP-violating KM phase. The angles θ_{ij} can be chosen to be in the first quadrant, so s_{ij} and c_{ij} can be taken as positive [2]. The current global fit gives for the magnitudes of all 9 CKM matrix elements [1, 2]:

$$|V_{\text{CKM}}| = \begin{pmatrix} 0.97428 \pm 0.00015 & 0.2253 \pm 0.0007 & 0.00347^{+0.00016}_{-0.00012} \\ 0.2252 \pm 0.0007 & 0.97345^{+0.00015}_{-0.00016} & 0.0410^{+0.0011}_{-0.0007} \\ 0.00862^{+0.00026}_{-0.00020} & 0.0403^{+0.0011}_{-0.0007} & 0.999152^{+0.000030}_{-0.000045} \end{pmatrix}$$
(3)

Instead of using the current parametrization [2, 7], which is a form of Wolfenstein parametrization [8] and is unitary to all order in $s_{12} = \lambda$, we shall now express s_{ij} and c_{ij} in terms of V_{ud} , V_{us} , V_{ub} and V_{cb} . As these quantities are assumed to be positive, they can be obtained directly from the measured absolute values. For simplicity, we shall put $s_{13} = V_{ub}$ and the matrix element V_{ub} of the CKM matrix in Eq. (1) is now written with the phase δ taken out $(V_{ub} \to V_{ub} \exp(-i\delta))$. We have:

$$s_{12} = \frac{V_{us}}{\sqrt{(V_{ud}^2 + V_{us}^2)}}, \quad c_{12} = \frac{V_{ud}}{\sqrt{(V_{ud}^2 + V_{us}^2)}}$$

$$s_{13} = V_{ub}, \quad c_{13} = \sqrt{(1 - V_{ub}^2)}$$

$$s_{23} = \frac{V_{cb}}{\sqrt{(1 - V_{ub}^2)}}, \quad c_{23} = \frac{\sqrt{(1 - V_{ub}^2 - V_{cb}^2)}}{\sqrt{(1 - V_{ub}^2)}}.$$

$$(4)$$

Since $s_{12}^2 + c_{12}^2 = 1$, we have

$$V_{ud}^2 + V_{us}^2 = c_{13}^2 = 1 - V_{ub}^2 (5)$$

The matrix elements of the CKM matrix in Eq.(1) are then given by:

$$V_{us} = \frac{V_{us}\sqrt{(1-V_{ub}^2)}}{\sqrt{(V_{ud}^2+V_{us}^2)}}, \quad V_{ud} = \frac{V_{ud}\sqrt{(1-V_{ub}^2)}}{\sqrt{(V_{ud}^2+V_{us}^2)}}, \quad V_{ub} \to V_{ub} \exp(-i\delta)$$

$$V_{cd} = -\frac{V_{us}\sqrt{(1-V_{ub}^2-V_{cb}^2)}}{\sqrt{(V_{ud}^2+V_{us}^2)}\sqrt{(1-V_{ub}^2)}} - \frac{V_{ud}V_{cb}V_{ub} \exp(i\delta)}{\sqrt{(V_{ud}^2+V_{us}^2)}\sqrt{(1-V_{ub}^2)}},$$

$$V_{cs} = \frac{V_{ud}\sqrt{(1-V_{ub}^2-V_{cb}^2)}}{\sqrt{(V_{ud}^2+V_{us}^2)}\sqrt{(1-V_{ub}^2)}} - \frac{V_{us}V_{cb}V_{ub} \exp(i\delta)}{\sqrt{(V_{ud}^2+V_{us}^2)}\sqrt{(1-V_{ub}^2)}}, \quad V_{cb} \to V_{cb}$$

$$V_{td} = \frac{V_{us}V_{cb}}{\sqrt{(V_{ud}^2 + V_{us}^2)}\sqrt{(1 - V_{ub}^2)}} - \frac{V_{ud}\sqrt{(1 - V_{ub}^2 - V_{cb}^2)}V_{ub}\exp(i\delta)}{\sqrt{(V_{ud}^2 + V_{us}^2)}\sqrt{(1 - V_{ub}^2)}}$$

$$V_{ts} = -\frac{V_{ud}V_{cb}}{\sqrt{(V_{ud}^2 + V_{us}^2)}\sqrt{(1 - V_{ub}^2)}} - \frac{V_{us}\sqrt{(1 - V_{ub}^2 - V_{cb}^2)}V_{ub}\exp(i\delta)}{\sqrt{(V_{ud}^2 + V_{us}^2)}\sqrt{(1 - V_{ub}^2)}}$$

$$V_{tb} = \sqrt{(1 - V_{ub}^2 - V_{cb}^2)}$$

$$(6)$$

Since $s_{12}^2 + c_{12}^2 = 1$, we have

$$V_{ud}^2 + V_{us}^2 = c_{13}^2 = 1 - V_{ub}^2 (7)$$

We note that this expression could be used to obtain V_{ub} provided that V_{ud} and V_{us} could be measured with great precision which could be achieved in future experiments. With this relation, we recover the expressions for V_{ud} and V_{us} in Eq. (6), given here with the factor $\sqrt{(1-V_{ub}^2)}/\sqrt{(V_{ud}^2+V_{us}^2)}$ included so that unitarity for the CKM matrix is explicitly satisfied.

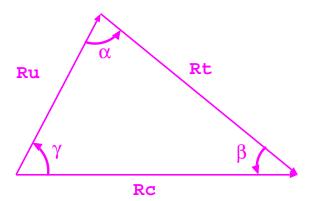


FIG. 1: The (db) unitarity triangle with the sides represent $R_u = |V_{ud}V_{ub}^*/V_{cd}V_{cb}^*|$, $R_t = |V_{td}V_{tb}^*/V_{cd}V_{cb}^*|$ and $R_c = 1$

The above expressions allow a direct determination of V_{cd} , V_{cs} , V_{td} , V_{ts} and V_{tb} in terms of V_{ud} , V_{us} , V_{ub} , V_{cb} . In the following we shall present expressions and results of our analysis for $|V_{td}/V_{ts}|$, $\sin \beta$, $\sin \alpha$ and $\sin \gamma$ in terms of the known CKM matrix elements provided by the global fit and the CP-violating phase δ .

As seen from the above expressions, the CP-violating phase δ originating from V_{ub} enters in the quantities V_{td} , V_{ts} and also in V_{cd} , V_{cs} and manifests itself in the CP-asymmetries in $B^0 - \bar{B}^0$ mixing and in charmless B decays, for example.

Consider now the quantity $|V_{td}/V_{ts}|$. We have from Eq. (6):

$$|V_{td}/V_{ts}| = \frac{V_{us}}{V_{ud}} \left(\frac{\sqrt{(1 - 2K_d \cos \delta + K_d^2)}}{\sqrt{(1 + 2K_s \cos \delta + K_s^2)}} \right)$$
(8)

with

$$K_d = \frac{V_{ud}V_{ub}}{V_{us}V_{cb}}\sqrt{(1 - V_{ub}^2 - V_{cb}^2)}, \quad K_s = \frac{V_{us}V_{ub}}{V_{ud}V_{cb}}\sqrt{(1 - V_{ub}^2 - V_{cb}^2)}, \tag{9}$$

For the (db) unitarity triangle, the angles α, β, γ are given by [2, 3]

$$\alpha = \arg(-V_{td}V_{tb}^*/V_{ud}V_{ub}^*), \quad \beta = \arg(-V_{cd}V_{cb}^*/V_{td}V_{tb}^*), \quad \gamma = \arg(-V_{ud}V_{ub}^*/V_{cd}V_{cb}^*)$$
(10)

with the sides:

$$R_u = |V_{ud}V_{ub}^*/V_{cd}V_{cb}^*|, \quad R_t = |V_{td}V_{tb}^*/V_{cd}V_{cb}^*|, \quad R_c = 1$$
(11)

Using the expressions in Eq. (6), we find:

$$\sin \alpha = \frac{\sin \delta}{\sqrt{(1 - 2K_t \cos \delta + K_t^2)}}, \quad \sin \gamma = \frac{\sin \delta}{\sqrt{(1 + 2K_c \cos \delta + K_c^2)}}$$

$$\sin \beta = \frac{(K_t + K_c) \sin \delta}{\sqrt{(1 - 2K_t \cos \delta + K_t^2)}\sqrt{(1 + 2K_c \cos \delta + K_c^2)}}$$
(12)

with

$$K_t = \frac{V_{ud}V_{ub}}{V_{us}V_{cb}}\sqrt{(1 - V_{ub}^2 - V_{cb}^2)}, \quad K_c = \frac{V_{ud}V_{ub}V_{cb}}{V_{us}\sqrt{(1 - V_{ub}^2 - V_{cb}^2)}},$$
(13)

$$(K_t + K_c = V_{ud}V_{ub}(1 - V_{ub}^2)/(V_{us}V_{cb}\sqrt{(1 - V_{ub}^2 - V_{cb}^2)}))$$

Similarly, we have:

$$R_{u} = \frac{(K_{c} + K_{t})}{\sqrt{(1 + 2 K_{c} \cos \delta + K_{c}^{2})}},$$

$$R_{t} = \frac{\sqrt{(1 - 2 K_{t} \cos \delta + K_{t}^{2})}}{\sqrt{(1 + 2 K_{c} \cos \delta + K_{c}^{2})}}, \quad R_{c} = 1.$$
(14)

From the expressions for $\sin \alpha$, $\sin \beta$ and $\sin \gamma$ in Eq. (12), we recover the relation

$$\sin \beta = R_u \sin \alpha, \quad \sin \gamma = R_t \sin \alpha \tag{15}$$

obtained previously in [7].

Eq. (12) for $\sin \gamma$ can be used to express δ in terms of γ . We have:

$$\cos \delta = \cos \gamma (\sqrt{(1 - K_c^2 \sin^2 \gamma)}) - K_c \sin^2 \gamma \tag{16}$$

As can be seen from Eq. (13), K_c is very much suppressed compared with K_t ($K_c = 0.006$, $K_t = 0.366$), $\sin \delta \approx \sin \gamma$ and $\cos \delta \approx \cos \gamma$ with $\sin^2 \gamma$ and $\cos \gamma$ correction terms of $O(10^{-4})$. Hence δ can be replaced by γ in the computed quantities with no great loss of accuracy. Thus, with $\sin \delta = \sin \gamma$ and $\cos \delta = \cos \gamma$, we obtain, with the global fit central value $V_{ub} = 0.00347$:

$$V_{cd} = -0.225112 - 0.000138 \exp(i\gamma), \quad V_{cs} = 0.973467 - 0.000032 \exp(i\gamma)$$
 (17)

$$V_{td} = 0.009237 - 0.003378 \exp(i\gamma), \quad V_{ts} = -0.039946 - 0.000781 \exp(i\gamma)$$
 (18)

We see that the contribution from the suppressed CKM matrix elements V_{cb} and V_{ub} produces only a small contribution to V_{cd} and V_{cs} . Thus unitarity of the CKM matrix in the standard model with 3 generations implies that V_{cd} and V_{cs} are rather well determined and therefore could be used in the semi-leptonic and non-leptonic decays of charmed mesons and charmed baryons. In particular the semi-leptonic D_s and D_s decays can be used to obtain the decays constants f_D and f_{D_s} .

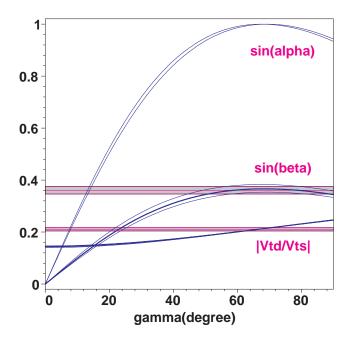


FIG. 2: $|V_{td}/V_{ts}|$, $\sin \beta$ and $\sin \alpha$ plotted against the angle γ . The middle curves represent the computed quantities for $V_{ub} = 0.00347$, the global fit central value; the upper and lower curves for for $V_{ub} = 0.00363$ and $V_{ub} = 0.00335$ respectively. The deviation from the central value for $|V_{td}/V_{ts}|$ and $\sin \alpha$ is barely visible. The solid straight lines are the measured values with experimental errors represented by the gray area.

The matrix element V_{td} , gets a large imaginary part from the phase δ in V_{ub} as given in Eq. (18). This large imaginary part is responsible for the large time-dependent CP-asymmetry in B

decays. For the $|V_{td}/V_{ts}|$, Eq. (8) gives, with δ replaced by γ :

$$|V_{td}/V_{ts}| = \frac{V_{us}}{V_{ud}} \left(\frac{\sqrt{(1 - 2K_d \cos \gamma + K_d^2)}}{\sqrt{(1 + 2K_s \cos \gamma + K_s^2)}} \right)$$
(19)

and by neglecting the suppressed K_c terms, from Eq. (15):

$$\sin \beta = \frac{B_u \sin \gamma}{\sqrt{(1 - 2K_t \cos \gamma + K_t^2)}}, \quad B_u = \frac{V_{ud}V_{ub}(1 - V_{ub}^2)}{V_{us}V_{cb}\sqrt{(1 - V_{ub}^2 - V_{cb}^2)}}$$

$$\sin \alpha = \frac{\sin \gamma}{\sqrt{(1 - 2K_t \cos \gamma + K_t^2)}}.$$
(20)

It is clear from Eq. (20) that $\sin \beta$ is a measure of γ , V_{ub} and V_{ub}/V_{cb} . Numerically, $B_u = 0.366$, $\sin \beta \approx 0.366 \sin \alpha$ as seen from the plot in Fig. (2). On the other hand, the curve for $|V_{td}/V_{ts}|$ in the plot implies that $\gamma = (68.6 \pm 3)^{\circ}$. The middle curve for $\sin \beta$ shows the measured $\sin \beta$ at almost the same value of γ . The lower curve for $\sin \beta$ is a bit below the measured value but consistent with experiment. The upper curve is slightly above the measured value. This indicates that a value for V_{ub} not too far from the global fit is favored.

In conclusion, by expressing the CKM matrix elements in terms of V_{ud} , V_{us} , V_{ub} , V_{cb} and the CP-violating phase δ instead of using the usual mixing angle parametrization, we have obtained analytical expressions for the CKM matrix elements and the angles of the (db) unitarity angle. Using the global fit for V_{ud} , V_{us} , V_{ub} , V_{cb} and the measured values for $|V_{td}/V_{ts}|$ and $\sin \beta$, we find $\gamma = (68.6 \pm 3)^{\circ}$ at which $\sin \beta$ and $\sin \alpha$ are near its maximum. Our simple and direct approach could be used in any further analysis of the CKM matrix elements with more precise measured values for V_{ub} and V_{cb} .

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