Hyperfine splitting of quarkonium

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Using a QCD-motivated model we show that $[M(^3S_1)]^2-[M(^1S_0)]^2$ is nearly constant for all quarkonium states. We find that the acceptable range of the QCD scale parameter to explain the weak quark-mass dependence of $[M(^3S_1)]^2-[M(^1S_0)]^2$ is $\Lambda=140\pm60$ MeV.

It has been known[1] for some time that the relation

$$[M(^3S_1)]^2-[M(^1S_0)]^2\cong\text{constant} \tag{1}$$

holds empirically for quarkonium states which contain at least one light quark. In this article we show that this empirical rule can be understood qualitatively in terms of QCD-motivated models. We shall point out further that this is not exactly constant but depends slightly on the masses of constituent quarks. This leads us to predict mass differences between $M(^3S_1)$ and $M(^1S_0)$ in a reliable way.

Assuming that the potential is the superposition of a scalar potential $V_c(R)$ and the fourth component of the vector potential $V_v(R)$, the hyperfine splitting of the ground-state quarkonium $q_i\bar{q}_j$ is given by

$$H_{\mathrm{HFS}}=\frac{2\langle \mathbf{S}_i\cdot\mathbf{S}_j\rangle}{3m_im_j}\nabla^2[V_v(R)] . \tag{2}$$

Here m_i and $\mathbf{S}_i$ are the mass and spin of the quark with flavor i. If $V_v(R)$ is given by the one-gluon-exchange term with a fixed α_s,

$$V(R)=-4\alpha_s/(3R)+V_c(R) , \tag{3}$$

Eq. (2) leads to a familiar form[2]

$$M_{ij}\cong m_i+m_j+\epsilon_{ij}+\xi_{ij}\langle \mathbf{S}_i\cdot\mathbf{S}_j\rangle/m_im_j \tag{4}$$

with

$$\xi_{ij}=(8\pi/3)\,|\psi_{ij}(0)|^2(4\alpha_s/3) .$$

ϵ_{ij} is the nonrelativistic binding energy and $\langle\ \rangle$ denotes the spin-spin expectation value. The scalar confining potential was first proposed[3] from the analysis of the ^3P_J data in $c\bar{c}$.

Then we find, instead of Eq. (1),

$$\begin{aligned}[M_{ij}(^3S_1)]^2-[M_{ij}(^1S_0)]^2 &=\frac{\xi_{ij}}{m_im_j}\left[2(m_i+m_j)-\frac{\xi_{ij}}{2m_im_j}\right]\\ &=\xi_{ij}\left[\frac{2}{\mu}-\frac{\xi_{ij}}{2\mu^2(m_i+m_j)^2}\right] ,\end{aligned} \tag{5}$$

where μ is the reduced mass

$$\mu=m_im_j/(m_i+m_j) \tag{6}$$

and ϵ_{ij} is neglected since this term is unimportant for the hyperfine mass splitting. The second term on the right-hand side of Eq. (5) is small compared to the first term. For example, it contributes only 1% and 0.2% for D and B mesons, respectively. For π and ρ mesons the second term becomes 17% of the first term. However, for π and ρ mesons, the difference between two particles is so large that the perturbative method is not appropriate. The large spin-spin force induces the 1S-2S mixing and the $^3S_1$-$^1S_0$ mass splitting becomes larger[4] than that predicted by the perturbative method. We have made an explicit calculation following Ref. 4 and found that the $^3S_1$-$^1S_0$ splitting increases roughly with the same amount as the decrease due to the second term in Eq. (5). Thus, the second term and the deviation from the perturbative calculation approximately cancel each other accidentally. For heavier quarkonium systems both are small. Therefore, we can neglect both for all quarkonium states.

Thus, we have

$$[M_{ij}(^3S_1)]^2-[M_{ij}(^1S_0)]^2\cong2\xi_{ij}/\mu=64\pi\alpha_s\,|\psi_{ij}(0)|^2/9\mu . \tag{7}$$

If $\alpha_s\,|\psi_{ij}(0)|^2/\mu$ is approximately constant, we are immediately led to the phenomenological relation (1). The running coupling constant $\alpha_s(Q^2)$ in QCD is given by

$$\alpha_s(Q^2)=12\pi/[(33-2n_f)\ln(Q^2/\Lambda^2)] . \tag{8}$$

Here we may take Q to be the particle mass which is approximately twice the quark mass for quarkonium with equal quark masses. For those with unequal quark masses it is natural to assume that Q is related to the reduced mass since Q is the momentum transfer and has nothing to do with the c.m. motion or m_i+m_j. The most natural assumption is to take

$$Q\equiv4\mu=4m_im_j/(m_i+m_j) ,$$

which becomes $2m_i$ for the equal-mass case. The value of n_f is the number of fermions, and we assume

$$n_f=\begin{cases}3 \text{ for } q\bar{q},q\bar{s},q\bar{c},q\bar{b},s\bar{c},s\bar{b}\ (q\equiv u \text{ or } d),\\ 4 \text{ for } c\bar{c},c\bar{b},\\ 5 \text{ for } b\bar{b} .\end{cases} \tag{9}$$

In practice the n_f dependence of $\alpha_s(Q^2)$ is not so relevant in the following discussions.

In order to determine the values of $|\psi_{ij}(0)|^2$ for various quarkonium states, let us use potential models and find wave functions by solving wave equations. These values can be compared with those computed from experimental decay rates such as ${}^3S_1 \to e^+e^-$, and the purely phenomenological scaling law,

$$|\psi(0)|^2 \propto \begin{cases} \mu^{1.89\pm0.15} & [\text{Jackson (Ref.5)}], \\ \mu^{1.9} & [\text{Hara (Ref.6)}], \end{cases} \tag{10}$$

has been found.

In the following we compute $|\psi(0)|^2$ using potential models and show that the obtained $|\psi(0)|^2$ has the scaling behavior similar to Eq. (10). We use the Schrödinger equation to compute the wave function since we found many encouraging results in terms of the nonrelativistic Schrödinger-model approach.

(i) Phenomenologically, the Schrödinger model has been extremely successful, i.e., meson and baryon spectra are reproduced in a consistent way,[7] and the electromagnetic transition rates are improved by including the spin-spin force.[8] The correct masses of F and F^* mesons were predicted (F, $F^*=1963$, 2099 MeV) *before* the experimental values changed substantially (from 2018, 2130 MeV to 1970, $\sim$2110 MeV). This means that the agreement with the data is really not accidental.

(ii) If we assume that the Coulomb term is the fourth component of a vector potential and the rest (including confining potential) is a scalar potential, we can show[9] that relativistic effects cancel each other, and the spectra and wave functions become very similar to those for the Schrödinger theory even for the light-quarkonium systems.

Let us now use a typical quarkonium potential[10] (hereafter we call this the potential A),

$$V(R)=V_0(R)-be^{-R/c}+V_0\ , \qquad V_0(R)=\begin{cases} -\alpha/R+d & \text{for } R<R_1, \\ aR & \text{for } R\geq R_1\ . \end{cases} \tag{11}$$

$\alpha=\frac{4}{3}\times 0.31$, $a=0.787$ GeV/fm, $b=0.956$ GeV, $c=2.05$ GeV^{-1}, and $V_0=-0.823$ GeV.

The parameters R_1 and d are chosen so that the potential is connected smoothly (i.e., $R_1=\sqrt{\alpha/a}$, $d=aR_1+\alpha/R_1$). This model works not only for $c\bar{c}$ and $b\bar{b}$ states but also for the $q\bar{q}$, $s\bar{s}$, $q\bar{s}$, $q\bar{c}$, $s\bar{c}$, and $q\bar{b}$ states with quark masses

$$(m_u, m_s, m_c, m_b)=(0.336,\ 0.630,\ 1.9,\ 5.29\ \text{GeV})\ . \tag{12}$$

Using this potential we compute $|\psi(0)|^2$ for various quarkonium states. As shown in Fig. 1, $|\psi(0)|^2$ is approximately given by

$$|\psi(0)|^2 \propto \mu^{1.52}\ , \tag{13}$$

which is not very far from the ones which were found by various decay rates. Thus we can approximately reproduce the phenomenological relation (10). We believe that the small discrepancy between Eq. (10) and Eq. (13) is within the theoretical ambiguity for the following reasons.

(i) The $|\psi(0)|^2$ for ${}^3S_1$ and that for ${}^1S_0$ are different due to the spin-spin force.[7] Equation (10) is derived by neglecting this, while Eq. (13) is derived after cutting off the spin-spin force.

(ii) Equation (10) is obtained by neglecting the higher-order QCD corrections, e.g., a factor $(1-16\alpha_s/3\pi+\cdots)$ in $Q\bar{Q}\to e^+e^-$ decay rate, which might be substantially different from one.

The relation (13) is rather insensitive to the choice of the precise shape of the potential and quark masses. For example, if we reduce all quark masses (12) by 50 MeV, then we obtain $|\psi(0)|^2 \propto \mu^{1.50}$. Less singular examples are provided by potentials which for short distances approach the asymptotic behavior predicted by a perturbative two-loop calculation.[11] The Kühn-Ono potential[12] is one such example (we call this the potential T),

$$V(R)=-\frac{16\pi}{25R\ln f(\Lambda_{\overline{\mathrm{MS}}}R)}\left[1+\frac{(2\gamma_E+\frac{53}{75})}{\ln f(\Lambda_{\overline{\mathrm{MS}}}R)}-\frac{462\ln\ln f(\Lambda_{\overline{\mathrm{MS}}}R)}{625\ln f(\Lambda_{\overline{\mathrm{MS}}}R)}\right]+a\sqrt{R}+c\ , \tag{14}$$

with $a=0.66$ GeV$^{3/2}$, $b=20$, $c=-1.45$, $\Lambda_{\overline{\mathrm{MS}}}=140$ MeV ($\overline{\mathrm{MS}}$ denotes the modified minimal-subtraction scheme), $(m_u,\ m_s,\ m_c,\ m_b)=(0.336,\ 0.62,\ 1.9,\ 5.27\ \text{GeV})$, which reproduces the $b\bar{b}$, $c\bar{c}$, $c\bar{u}$, $b\bar{u}$, and $u\bar{u}$ spectra, where γ_E is Euler's constant. We find $|\psi(0)|^2 \propto \mu^{1.43}$ for this potential. Thus one can conclude that the μ dependence of $|\psi(0)|^2$ is fairly model independent and roughly consistent with the phenomenological one, Eq. (10).

On the other hand, we have theoretical ambiguities of a factor of 3 for the absolute value of $|\psi(0)|^2$ even for $c\bar{c}$. Let us show some examples: $|\psi_{c\bar{c}}(0)|^2=183$, 145, 127, 63 fm^{-3} for potential A, T, the Martin potential,[13] and the Bahnot-Rudaz potential,[14] respectively. Thus we use the formula

$$[M({}^3S_1)]^2-[M({}^1S_0)]^2=\kappa[64\pi\alpha_s(\mu)\,|\psi(0)|^2/9\mu]\ , \tag{15}$$

where κ is a fitting parameter which absorbs ambiguities in the absolute value of $|\psi(0)|^2$ and anomalous color magnetic moment. We will show numerically that κ is not far from one.

Equations (2) and (7) do not hold if the potential differs from Eq. (3). There is no $\delta(R)$ term for potentials which do not contain a $1/R$ term, e.g., the Martin potential[13] or the potential T. However, Martin still assumed[13] that the spin-spin force is proportional to $|\psi(0)|^2/m_im_j$ since the potential is a phenomenologocial one whose higher derivatives may be very different from those of the real potential. Thus if we use Martin's assumption, Eq. (15) should

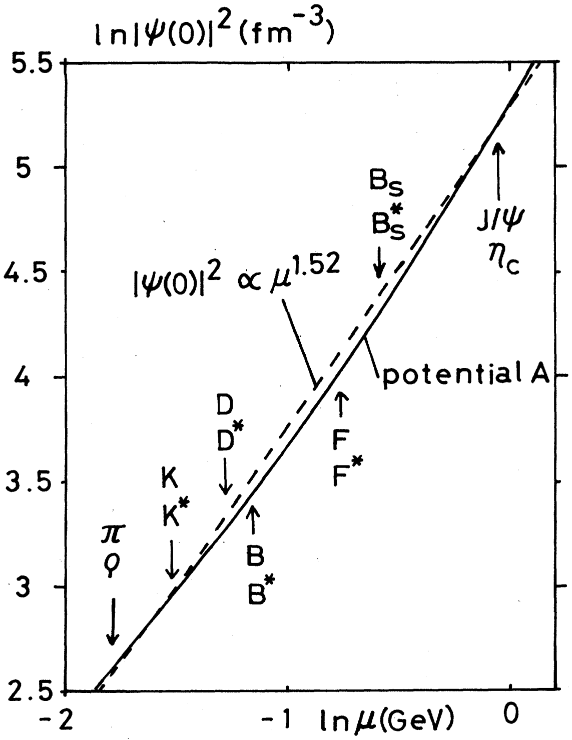

FIG. 1. The $|\psi(0)|^2$ predicted in model A is plotted as a function of reduced mass μ. For comparison the line $|\psi(0)|^2 \propto \mu^{1.52}$ is plotted.

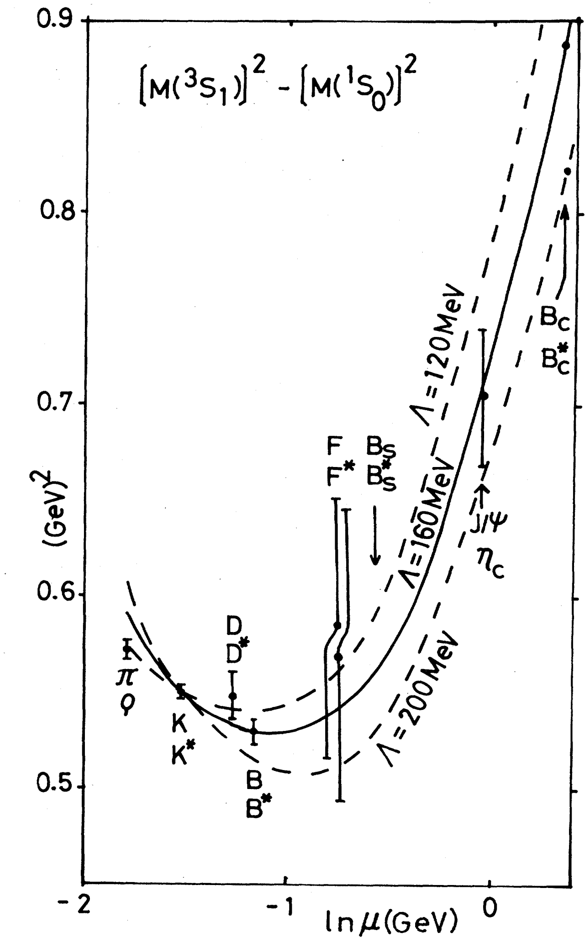

FIG. 2. The quantity constant $\times\alpha_s|\psi(0)|^2/\mu$ predicted in model A is plotted as a function of the reduced mass μ. The experimental value for $K^{*2}-K^2$ is used as an input. Three curves which correspond to $\Lambda=120$, 160, and 200 MeV are shown.

hold for any potential. Since $\alpha_s(\mu)$ is fixed by the relation (8) with $Q=4\mu$ only, model dependence comes from the μ dependence of $|\psi(0)|^2$.

We first chose here the potential A and compute the $\alpha_s(\mu)|\psi(0)|^2/\mu$ as a function of μ. Results are shown in Fig. 2. The experimental datum for K^*-K is used to fix the value of κ. The obtained κ for each Λ is not far from one, i.e., $\kappa=1.34$, 1.14, and 0.99 for $\Lambda=120$, 160, and 200 MeV. From Fig. 2 we find that the acceptable QCD scale parameter is

$$\Lambda=160\pm40 \text{ MeV} . \tag{16}$$

In Table I we show $[M(^3S_1)]^2-[M(^1S_0)]^2$ and $M(^3S_1)-M(^1S_0)$ for $\Lambda=120$, 160, and 200 MeV. An interesting result is

$$F^*-F=\begin{cases}132\pm6 \text{ MeV, our model,}\\ 139.5\pm18 \text{ MeV, PEP4 data (Ref. 15),}\\ 144\pm16 \text{ MeV, ARGUS data (Ref. 16) .}\end{cases} \tag{17}$$

More precise experimental data can check our prediction.

In order to see the model dependence we have repeated the analysis by making use of the less singular potential T. We briefly summarize the results.

(i) To fit the data we need smaller Λ of order 100 MeV. Thus we should enlarge the allowed range of Λ from Eq. (16) to

$$\Lambda=140\pm60 \text{ MeV} . \tag{18}$$

(ii) The value of κ necessary to fit the data becomes large (~ 1.6).

(iii) The allowed ranges for F^*-F and for other hyperfine splittings are not much different from those for the potential A except for $\Upsilon-\eta_b$.

(iv) The predicted splitting for $\Upsilon-\eta_b$ is around 60 MeV, which is smaller than that for the potential A (87 ± 8 MeV). Thus the predicted splitting in our model is

$$\Upsilon-\eta_b=70\pm20 \text{ MeV} . \tag{19}$$

For singular potentials, such as the potential A, the $|\psi(0)|^2$ increases rapidly as the reduced mass increases. Therefore, it is understandable that we find smaller splitting for the potential T than that for the potential A.

In summary, we have used the Breit-Fermi Hamiltonian and the QCD-motivated potential models to compute the hyperfine splittings of quarkonium states. We are led to the following conclusions.

(i) We have derived a relation

$$[M(^3S_1)]^2-[M(^1S_0)]^2=\kappa(64\pi/9)\alpha_s(\mu)|\psi(0)|^2/\mu ,$$

which works for any quarkonium states.

TABLE I. Hyperfine splitting of quarkonium predicted by the potential model A. $\Lambda = 160$ MeV corresponds to our best fit.

$1S$ $[M(^3S_1)]^2-[M(^1S_0)]^2$ (GeV^2)	π ρ	K K^*	$s\bar{s}$	D D^*	F F^*	B B^*	B_s B_s^*	B_c B_c^*	η_c J/ψ	η_b Υ
$\Lambda = 120$ MeV	0.576	0.550 (input)	0.540	0.540	0.563	0.540	0.585	0.959	0.766	1.798
$\Lambda = 160$ MeV	0.592	0.550 (input)	0.527	0.530	0.538	0.527	0.555	0.888	0.715	1.646
$\Lambda = 200$ MeV	0.609	0.550 (input)	0.513	0.520	0.514	0.513	0.527	0.823	0.667	1.510
$M(^3S_1)-M(^1S_0)$ (MeV)										
$1S$										
$\Lambda = 120$ MeV	636		310	139	138	50.9	54.7	76.0	128	95.5
$\Lambda = 160$ MeV	652	input	302	137	132	49.7	51.4	70.4	118	87.4
$\Lambda = 200$ MeV	672		292	134	126	48.4	49.3	65.2	110	80.1
experiment	625±5	397±2		141.5±3	139.5±18 144±16	50±6			115.9±6	
$2S$										
$\Lambda = 160$ MeV	242	133	116	72	69	32	31	37	62	37
experiment	290±120								92±5?	
$3S$										
$\Lambda = 160$ MeV	103	88	78	54	50	26	25	27	44	26

(ii) Since $\alpha_s(\mu)\,|\,\psi(0)\,|^2/\mu$ is approximately constant for all quarkonium states which include at least one light quark, we can reproduce the well-known empirical relation

$$\rho^2-\pi^2=K^{*2}-K^2=D^{*2}-D^2$$
$$=B^{*2}-B^2\approx\text{constant}\ .$$

(iii) We can even explain the small deviation of the experimental values from this constant especially for $c\bar{c}$ assuming the QCD scale parameter

$$\Lambda=140\pm60\ \text{MeV}\ .$$

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