

Chapter 15

Quarkonium

15.1 Background

In the same way that an electron and proton are bound together by the electromagnetic force to make a hydrogen atom, a quark and antiquark can be bound together by the strong force to form a bound state called *quarkonium*.

Just as we can infer the Coulomb force between the electron and proton by studying the energy levels of the Hydrogen atom we can study the strong force between charm and bottom quarks by looking at the energy levels of the charmonium and bottomonium systems. By studying quarks that are heavy in comparison to their binding energy, we can apply methods from non-relativistic quantum mechanics that have been very successful in explaining the properties of the Hydrogen atom.

If we neglect spin effects the wavefunction for quarkonium satisfies the non-relativistic three-dimensional Schrödinger equation,

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi + [V(r) - E_{nl}]\psi = 0, \quad (15.1)$$

where μ is the reduced mass, $1/\mu = 1/m_1 + 1/m_2$, $m_{1,2}$ are the masses of the quarks and r is the distance between the quarks.

A suitable parameterisation of the interquark potential is

$$-\frac{4\alpha_S}{3r} + \beta r, \quad (15.2)$$

where α_S is the dimensionless strong coupling constant.¹ The second term represents a confining potential and is added to ensure that the quarks are bound and cannot exist as free particles.

The spherical symmetry of the potential ensures that

$$\psi = R_{nl}(r)Y_{\ell m}(\theta, \phi), \quad (15.3)$$

where n , ℓ and m represent the usual quantum numbers describing the different energy levels. Here $Y_{\ell m}(\theta, \phi)$ are the standard spherical harmonics and the radial wavefunction $u_{nl}(r) = rR_{nl}(r)$ satisfies

$$\frac{d^2u_{nl}}{dr^2} - \frac{l(l+1)}{r^2}u_{nl} + 2\mu(E_{nl} - V(r))u_{nl} = 0. \quad (15.4)$$

¹This is the equivalent of the fine-structure constant in electromagnetism for strong interactions.

In this project we will use natural units $\hbar = c = 1$ so that μ is measured in GeV/c^2 , $E_{n\ell}$ in GeV , r in GeV^{-1} and β in GeV^2 . The wavefunctions are normalised such that,

$$\int_0^\infty r^2 |R_{n\ell}(r)|^2 dr = \int_0^\infty |u_{n\ell}|^2 dr = 1. \quad (15.5)$$

The aim of this project is to study the properties of these quarkonium states using numerical calculations of the wavefunctions to obtain the energy levels. Once we have calculated the binding energy, the mass of the associated quarkonium states is

$$M_{n\ell} = m_1 + m_2 + E_{n\ell}. \quad (15.6)$$

A more detailed discussion of the physics of charmonium can be found in (eg.) Donoghue et al. 1992.

15.2 Numerical Solution

In order to solve Schrödinger's equation numerically we need to specify boundary conditions. In particular, as the wavefunction is well behaved at $r = 0$, we know that

$$u_{n\ell}(0) = 0. \quad (15.7)$$

We will also take

$$\frac{du_{n\ell}(0)}{dr} \neq 0, \quad (15.8)$$

so that if we do not care about the normalisation of the wavefunction, we can choose,

$$\frac{du_{n\ell}(0)}{dr} = 1. \quad (15.9)$$

Formally this is strictly only valid for $l = 0$ however in practice the loss of numerical accuracy due to this choice is small.

As $r \rightarrow \infty$

$$u_{n\ell}(r) \rightarrow 0, \quad \frac{du_{n\ell}(r)}{dr} \rightarrow 0. \quad (15.10)$$

These equations can be solved numerically using standard techniques for ordinary differential equations. The `scipy.integrate` module contains a good routine for solving a system of linear first order differential equations, `odeint`, which you can use. An example of how to use this function is given in the appendix. In order to use standard packages for solving differential equations we need to transform the 2nd order differential equation we wish to solve into two first order equations. This is easily achieved by defining $v_{n\ell} = \frac{du_{n\ell}}{dr}$, so that we have two equations

$$\frac{du_{n\ell}}{dr} = v_{n\ell}, \quad (15.11a)$$

$$\frac{dv_{n\ell}}{dr} = \frac{l(l+1)}{r^2} u_{n\ell} - 2\mu(E_{n\ell} - V(r))u_{n\ell}. \quad (15.11b)$$

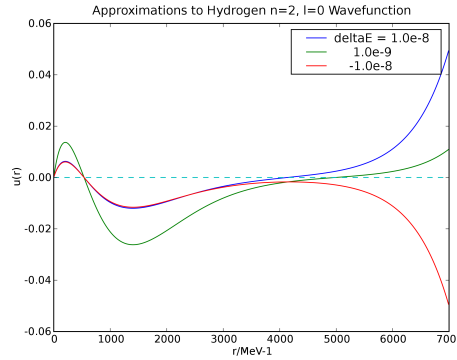


Figure 15.1: The effect of choosing an approximate energy on the function $u(r)$. When the energy is too high, the function crosses the axis twice. When it is too low, there is an extra turning point. For a very small offset in energy, the wavefunction is almost perfect, but just crosses the axis and diverges at large r . The exact solution will get close to the axis but never cross it.

You will need the solution out to a reasonable distance from the origin, $r_{\max} = 15 \text{ GeV}^{-1}$, should be sufficient but you should check the effect of this choice on your results.²

If we do not choose the correct energy eigenvalue the solution will not satisfy Equation 15.10, in particular it will diverge to either positive or negative infinity. We know this happens because the wavefunction will have either too many nodes (crossing the x-axis) or turning points. For $l = 0$, the number of nodes and turning points should be $n - 1$ and n , respectively. By counting the number of nodes and turning points at some trial values of the energy we can iterate towards the correct solution. If the trial energy is too large then the number of nodes will be too large. If the trial energy is too small, then number of nodes will be correct or too small, and there will be too many turning points. We can then guess a new energy and repeat the procedure. This is illustrated in Fig. 15.1.

Fortunately, we do not need to know the “correct” number of nodes and turning points at the outset. The best procedure to use is to use three test values for the energy $E_{1,2,3}$, with $E_2 = \frac{1}{2}(E_1 + E_3)$. We can then check the number of nodes and turning points at each value. If the number of nodes and turning points changes between E_1 and E_2 we can set $E_3 = E_2$ and repeat the procedure, or if these change between E_2 and E_3 we can set $E_1 = E_2$ and repeat. This will allow you to iterate until you have an accurate answer for the energy. It is important to check that $u_{n\ell}$ has the correct behaviour by plotting it. (Note that the solution will never be exact so you should not be surprised if $u_{n\ell}(r)$ diverges at large r). The value of β can be calculated in a similar way, but as it occurs with opposite sign to $E_{n\ell}$, if there are too many nodes then β is too small etc.. This approach requires that the correct value lies between E_1 and E_3 .

The wavefunction should be normalised as in Equation 15.5. You can achieve this by integrating your result, however you must truncate the integral at a suitable value of r if $u_{n\ell}(r)$ starts to diverge. You could simply sum-up an array of $|u(r)|^2$ values and multiply by the step size, or you could use the `simps` routine in the `scipy.integrate` package can be used to compute the numerical integral.

²In practice you will probably find it necessary to start a small distance away from the origin to avoid numerical instabilities for l states.

15.3 Work Plan

1. **Design your program.** The first task is to design your program. Use pseudocode techniques to design a program that you will use to investigate quarkonium systems.

As well as an outline of the code structure, you should include: (a) the names and descriptions of the functions you will use: what variables will they take as arguments and what will they return? (b) what data you will input to the program, how the program output will the results? (c) what data structures the program will use to store its internal data?

Your program will need to (1) solve the Schrödinger equation numerically for specific values of the quark masses, α_S , β and the energy; (2) calculate the number of nodes and turning points; (3) guess a new energy (or β) and iterate to find a bound solution. Once you have this basic building block you can write functions to find either the energy given β or β given the energy, and plot the resulting wavefunctions.

2. **Solve the Milestone Program.** It is extremely useful to check this program by using it to calculate the energy levels and wavefunctions of the hydrogen atom, where we have known analytic results. You can do this by making the following replacements $\mu = m_e$, $\frac{4}{3}\alpha_S \rightarrow \alpha = 1/137$, $\beta \rightarrow 0$. *You should take care with your units as the radius of the Hydrogen atom is very different to that of Charmonium.* You will find it easier to use MeV rather than GeV as your natural units, in this case. You should obtain the wavefunctions and energy levels of the $(n, \ell) = (1, 0)$, $(2, 0)$ and $(2, 1)$ states and check them against the known analytic results. In natural units $E_{n\ell} = -\mu\alpha^2/2n^2$ and the Bohr radius $a_0 = 1/(\mu\alpha)$. The wavefunctions can be found in many quantum mechanics books, for example, Appendix A of Sakurai (1994).

In your milestone interview, your program should calculate the energy of the $(2, 1)$ state and plot the corresponding radial probability density function, $|u_{n\ell}|^2$.

3. **Extend your program**

Now it is time to apply your program to study the properties of Charmonium. As a starting point, you should calculate β to 3 d.p. using the spin average Charmonium mass given in the appendix. You can then use this to give predictions for the energy of the $(n, \ell) = (1, 1)$ and $(2, 0)$ levels and produce plots of the normalised wavefunctions for the $(n, \ell) = (1, 0)$, $(1, 1)$ and $(2, 0)$ levels.

4. **Research with your program**

Once you have your basic program working for Charmonium there are a range of things you can study. You should not look into all of these issues, they are intended as a starting point for your own investigations. It is more important to investigate one point in depth than to cover a wide range of phenomena.

- The spectra of charmonium and bottomonium states can be studied and compared to the observed values. The first state containing a bottom and charm quark, the B_c meson, been confirmed at Fermilab with mass 6.276 GeV, you can look at the properties of these states using your program. Remember that the potential you are using is an approximation to the full QCD interactions of the quarks. You will find recent summary of known quarkonia transitions in Eichten et al 2007. The latest data is available on-line at www.pdg.gov.
- The choice we used for the potential is only one of a number which display the correct behaviour at both small and large distances. You can investigate the effect of different choices of potential

on your results. Another popular choice is

$$V(r) = cr^d \quad \begin{cases} c \approx 6.87 \text{ GeV}, \\ d \approx 0.1. \end{cases} \quad (15.12)$$

A discussion of the potentials for quarkonium can be found in Chapter XIII of Donoghue et al. (1992).

- The quarks have spin- $\frac{1}{2}$, therefore the s -wave ($l = 0$) states can be divided into spin-singlets and spin-triplets (as in the Helium atom). These states do not have exactly the same energies and are split by a hyperfine spin-spin interaction. This hyperfine splitting is

$$\Delta(n^3S_1 - n^1S_0) = \frac{8\alpha_S}{9m_q^2} |R_{n\ell}(0)|^2. \quad (15.13)$$

It thus depends on the amplitude of the wave function at the origin. In order to determine this, note that

$$\lim_{r \rightarrow 0} R_{n\ell}(r) = \lim_{r \rightarrow 0} \frac{u_{n\ell}(r)}{r} = \frac{du_{n\ell}(0)}{dr}. \quad (15.14)$$

The transition rates between the states provide deep insight into the nature of the strong force. More discussion can be found in Donoghue et al. (1992).

- Quarkonia have relatively short lifetimes. The lifetime of a state is largely determined by the probability that the quarks overlap, ie. $|\Phi(r=0)|^2$ (eg., Eichten et al. 2007). Your program allows you to determine the (ground state) wavefunction and hence to compare the decay rates of different systems. By analogy with the hydrogen atom you can compute the wave function and use this to compute other properties of quarkonia.

Appendix: Quarkonium Data

The data for charmonium is,

$$\begin{aligned} m_c &= 1.34 \text{ GeV}/c^2 \\ \alpha_S(4m_c^2) &= 0.40, \\ M_{n=1, l=0} &= 3.068 \text{ GeV}/c^2 \end{aligned} \quad (15.15)$$

The data for bottomonium is,

$$\begin{aligned} m_b &= 4.70 \text{ GeV}/c^2 \\ \alpha_S(4m_b^2) &= 0.28, \end{aligned} \quad (15.16)$$

Recent experimental values for the mass of quarkonium states can be found in Amsler (2008).

References

F. Halzen and A. D. Martin, “Quarks And Leptons: An Introductory Course In Modern Particle Physics, Chapter 2.13,” *Wiley* (1984).

J. J. Sakurai, “Modern Quantum Mechanics,” *Addison-Wiley* (1994).

C. Amsler *et al.* [Particle Data Group], “Review of particle physics,” *Phys. Lett. B* **667** (2008) 1. Available on line at <http://pdg.lbl.gov/>

J. F. Donoghue, E. Golowich and B. R. Holstein, “Dynamics Of The Standard Model,” *Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol.* **2**, 1 (1992). Available on google books. Pages 350-359 are relevant.

Rosenfeld R., 2013. http://www.ifsc.usp.br/lattice/Natal_1.pdf. A recent quick overview of the standard model and QCD.