Hyperdimensional Turing Machine

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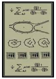

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What are Hypervectors

A Hypervector space, H, is a set of high dimensional vectors (10,000+ dim) with the following:

1. Similarity $d : H \times H \to \mathbb{R}$
   - $d(x, x) \geq d(y, z)$
   - $x \sim y \iff d(x, y) > t$, where t is some threshold.
   - $x \not\sim y$ with high probability, x, y chosen at random.
2. Bind (*) : $H \times H \to H$
   - $a * a = 1$
   - $a * b \not\sim a$
   - $a * b \not\sim b$
3. Bundle $(+)$: $H \times H \to H$
   - $a + b \sim a$
   - $a + b \sim b$
   - $a * (b + c) = a * b + a * c$
4. Permute $r : H \to H$
   - $r(a) \not\sim a$
   - $r^{-1}(r(a)) = a$

What do hypervectors do?

- Initialize a, b, c, d randomly
- Let $r = a * b + c * d$
- $r \not\sim a$, $r \not\sim b$, $r \not\sim c$, $r \not\sim d$
- $r * a = a * a * b + a * c * d = b + a * c * d$
- $a * c * d \not\sim a, b, c, d$
- $r * a \sim b$, but
- $r * a \not\sim a$, $r * a \not\sim c$, $r * a \not\sim d$
- Similarly, for $r * b, r * c, r * d$

What do hypervectors do?

- r represents a function
  $R: \{a, b, c, d\} \subset H \to \{a, b, c, d\} \subset H$
- $R(x) = \underset{y \in \{a,b,c,d\}}{\arg\max}\ d(r * x, y)$
- $R(x) \begin{cases} a \mapsto b \\ b \mapsto a \\ c \mapsto d \\ d \mapsto c \end{cases}$
- We can encode general "switch" statements in this form

An Example of Hypervectors: Binary Bipolar

- (10,000-dim+) vectors with elements initialized randomly with each element as -1 or 1
- $d(a, b) = \frac{a \cdot b}{|a||b|}$: cosine similarity
- $a + b$: element-wise addition

  (Can also be select each element from one of the vectors at random)
- $a * b$: element-wise multiplication
- $r(a) = r(a_1, a_2, \ldots, a_n) = (a_n, a_1, \ldots, a_{n-1})$: re-indexing

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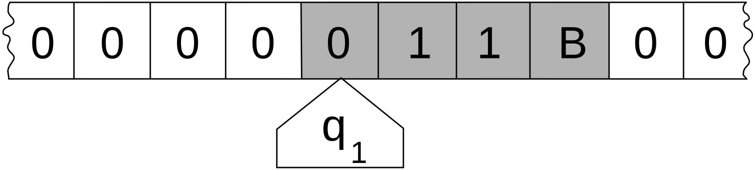
Recap of Turing Machines

- Head on an infinite tape
- Finite State space: Q
- Finite Symbol space: Γ
- Head $= \delta : \; Q \times \Gamma \to Q \times \Gamma \times \{\text{Left}, \text{Right}\}$
- Tape $= \Gamma^*$
- Since Q and Γ have finitely many symbols/states, we can encode each of them as a random hypervector.



Turing tape graphic

Basic Turing Counter

- As a starter task, we used binary counting
- $Q = \{w, r\}$, $\Gamma = \{0, 1, x\}$
- δ is defined as follows:

Γ / Q	w	r
0	(1, r, Right)	(0, r, Right)
1	(0, w, Left)	(1, r, Right)
x	Impossible	(x, w, Left)

- The tape is initialized as $\ldots 00000000\underset{\wedge}{x}$
- The state is initialized as r
- The number is read from the tape whenever the head reaches the x at the end

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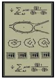
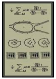


Basic Hypervector Turing Counter

- We create the following random bypolar hypervectors:
  - One for reading each state/symbol: {Read0, Read1, ReadX, ReadR, ReadW}
  - One for setting each state/symbol: {Write0, Write1, WriteX, WriteR, WriteW}
  - One for moving in each direction: {Left, Right}
- We construct our rule hypervector as follows:

$$
\begin{aligned}
T = {} & Read0 * ReadW * (Write1 + WriteR + Right) \\
& + Read0 * ReadR * (Write0 + WriteR + Right) \\
& + Read1 * ReadW * (Write0 + WriteW + Left) \\
& + Read1 * ReadR * (Write1 + WriteR + Right) \\
& + ReadX * ReadR * (WriteX + WriteW + Left)
\end{aligned}
$$

- At each step, we take $T *$ Tape[i] $*$ State and compute the most similar state, symbol, and direction



Less Basic Hypervector Turing Counter

- Having separate symbols for reading and writing states feels inelegant
- We need them in the previous architecture to avoid this: $0 * r * (0 + r + \text{Right}) = 0 * 0 * r + 0 * r * r + 0 * r * \text{Right} = 0 + r + 0 * r * \text{Right}$, which is similar to both 0 and r.
- We can eliminate this need by introducing a new operation: Stack ($\wedge$), and its inverse Unstack ($\vee$)
  - $a \wedge b = a * r(b)$
  - $a \wedge b \not\sim a$, $a \wedge b \not\sim b$
  - $a \vee b = r^{-1}(a * b)$
  - $(a \wedge b) \vee a = b$
  - $(a \wedge (b \wedge c)) \vee (a \wedge b) = c$
  - Stack is not associative, and should be read from right to left

Less Basic Hypervector Turing Counter

- We now only create the following random bypolar hypervectors:
  - One for each state: {R, W}
  - One for each symbol: {0, 1, X}
  - One for moving in each direction: {Left, Right}
- We construct our new rule hypervector as follows:

$$
\begin{aligned}
T = {} & 0 \wedge W \wedge (1 + R + Right) \\
& + 0 \wedge R \wedge (0 + R + Right) \\
& + 1 \wedge W \wedge (0 + W + Left) \\
& + 1 \wedge R \wedge (1 + R + Right) \\
& + X \wedge R \wedge (X + W + Left)
\end{aligned}
$$

- At each step, we take $T \vee (\text{Tape[i]} \wedge \text{State})$ and compute the most similar state, symbol, and direction

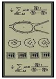
A Few Further Improvements:

- If we do not wish to restrict our induced function to 3 subsets of our tape:
- We can create 3 additional random hypervectors: {Symbol, State, Move}
- We can then modify each piece of our rule from the form $0 \wedge W \wedge (1 + R + Right)$ to the form $0 \wedge W \wedge (Symbol * 1 + State * R + Move * Right)$
- Now, to get the new symbol, state, and move, we perform $T \vee (\text{Tape[i]} \wedge \text{CurrentState}) * Symbol$, $T \vee (\text{Tape[i]} \wedge \text{CurrentState}) * State$, and $T \vee (\text{Tape[i]} \wedge \text{CurrentState}) * Move$ respectively, and find the most similar item of the entire known set.

A Few Further Improvements

- For small enough tapes (and high enough dimensions), we can encode the entire tape in a single hypervector.
- We create a new random hypervector called TapeIndex
- We can now define the hypervector

  $$\text{Tape} = \mathit{TapeIndex} * X + \sum_{n=1}^{N} r^n(\text{TapeIndex}) * 0$$
- We initialize the hypervector $\text{CurrentTapeIndex} = \text{TapeIndex}$
- To read the tape, we compute the most similar vector to $\text{Tape} * \text{CurrentTapeIndex}$
- To move the tape left, we set $\text{CurrentTapeIndex} = r(\text{CurrentTapeIndex})$, and to move right we set $\text{CurrentTapeIndex} = r^{-1}(\text{CurrentTapeIndex})$



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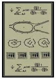
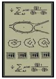


Hypervectors for Learnable Computing

- Many constructions of hypervector operations, like the ones used, are differentiable
- This allows "learnable" hypervector architectures using methods like SGD
- Hypervectors support various other optimization/search algorithms as well
- These architectures may also funtion as "Neural Turing Machines"



Hypervectors for Robust Computing

- Since hypervectors are built with random noise, they are robust to perturbations
- In preliminary testing, the second architecture can successfully count to 1024 with 10% noise added each step
- This can also be considered a proof of concept for more practical general computing architectures with hypervectors
- Dr. Hahn and I are currently working on a hypervector RAM machine architecture, which should be equally robust
- Fast hardware implementations of hypervectors are being developed

Thank You!

Questions?