SC 624, Spring 2018
Quiz 4
Date: 27 February 2018
Time: 120 min

1. (5 points) Given X a square matrix, recall the matrix exponential:

$$\mathrm{e}^X := \sum_{i=0}^{\infty} \frac{X^i}{i!}.$$

   - Consider the vector field f on $\mathbb{R}^n$ given by

   $$\mathbb{R}^n \ni x \to f(x) := Ax \in \mathbb{R}^n,\ A \in \mathbb{R}^{n\times n}$$

   Show that the flow of f, $\gamma_t^f(x) = \mathrm{e}^{At}\, x$.
   - Suppose the $n \times n$ matrices A and M satisfy

   $$AM + MA^\top = 0.$$

   (Note that none of the matrices is assumed to be non-singular.) Show that

   $$\mathrm{e}^{(At)}\, M\, \mathrm{e}^{(A^\top t)} = M \quad \text{for all } t.$$

2. (6 points) Let $M = \mathbb{R}^2$. f and g are vector fields on M defined as :

$$f(x,y) = (x,0)$$
$$g(x,y) = (0,y)$$

   - Find the lie bracket of f and g, $[f,g]$.
   - Find the flows corresponding to the vector fields f and g, γ_t^f and γ_t^g.
   - Show that the flows of f and g commute. i.e $\gamma_t^f \circ \gamma_s^g = \gamma_s^g \circ \gamma_t^f, \forall t, s \in \mathbb{R}$

3. (6 points) The special Euclidean group is defined as

$$SE(n) := \left\{ \begin{pmatrix} R & V \\ 0 & 1 \end{pmatrix} \,\middle|\, R^\top R = I, R \in \mathbb{R}^{n\times n}, V \in \mathbb{R}^n \right\}.$$

   - Show that $SE(n)$ is a group when endowed with the binary operation of matrix multiplication in the usual sense.
   - The Hadamard (or Schur) product of two matrices $A, B \in \mathbb{R}^{n\times n}$ is a matrix $A \times B \in \mathbb{R}^{n\times n}$ defined as $(A \times B)_{i,j} = A_{i,j}B_{i,j}$ i.e the matrix obtained by entry-wise multiplication. Is $SE(n)$ endowed with the Hadamard product a group.
   - Find a subset of $\mathbb{R}^{n\times n}$ that is a group when endowed with the Hadamard product.

4. (8 points) Consider the map F given by

$$\mathbb{R}^2 \ni x \to F(x) := Rx \in \mathbb{R}^2,\ R \in \mathbb{R}^{2\times 2}, R^T R = I$$

   Let g be a vector field on $\mathbb{R}^2$ defined as

$$g : \mathbb{R}^2 \ni (x_1, x_2) \to g((x_1, x_2)) := (x_1, 0) \in \mathbb{R}^2$$

- Show that F is a diffeomorphism.
- Find F^*g. Remember, $F^*g(x) = DF(F^{-1}(x)).g(F^{-1}(x))$.
- Find the flows of g and F^*g, γ_t^g and $\gamma_t^{F^*g}$.
- Show that $\gamma_t^{F^*g} = F \circ \gamma_t^g \circ F^{-1}$.