Systems & Control Engg. IIT Bombay, Powai Mumbai 400076 Maharashtra, India	SC 624, Spring 2017 ***Quiz 5*** Date: 28 March 2018 Due Date: 3 April 2018, 5:30 p.m	

Throughout this quiz, assume X is an open subset of $\mathbb{R}^n$.

1. (10 points) Fix a $p \in X$ and consider the curve $\alpha(t) = \gamma^g_{-t} \circ \gamma^f_{-t} \circ \gamma^g_t \circ \gamma^f_t(p)$. Show that $\alpha'(0) = 0$ and $\alpha''(0) = 2[f, g](p)$.

2. (10 points) Let $L_f := \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$ be the differential operator corresponding to a vector field f on X. Then prove that

$$L_f L_g - L_g L_f = L_{[f,g]} \quad \text{for all} \quad f, g \in \mathcal{X}(M).$$

3. (15 points) Let ϕ be a nonconstant smooth function on X, and g, f be vector fields on X such that $L_g\phi = L_g L_f \phi = \ldots = L_g L_f^{n-2}\phi = 0$. Show that $dL_f^j\phi(x_0), 0 \leqslant j \leqslant n-1$ are linearly independent iff $\mathrm{ad}_f^j g(x_0), 0 \leqslant j \leqslant n-1$ are linearly independent.
(Hint: Look at Witold Respondek's notes Eq.2.1 and the surrounding discussion to get more context.)

4. (5 points) Let f, g be smooth vector fields on X, and γ^f_t, γ^g_t denote their respective flows. Show that there exists a **unique** vector field h such that $L_f L_g - L_g L_f = L_h$. Also show that $h = [f, g]$.

5. (5 points) Is the distribution $\Delta : \mathbb{R}^3 \to \mathbb{R}^3$

$$\Delta(x) := \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ x_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 \\ 1 \end{pmatrix} \right\}$$

involutive?