Systems & Control Engg.
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SC 624, Spring 2017
Quiz 2
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Time: 90 min

Recall that $\mathbb{N}^*$ is the positive integers, $\mathbb{N} = \{0\} \cup \mathbb{N}^*$. For us $\mathrm{i} := \sqrt{-1}$.

Recall that a (non-empty) set X endowed with a maximal (smooth) atlas (a.k.a. a smoothness,) is a smooth manifold.

1. Let M be a smooth manifold of dimension n, and $\{(U_i, \phi_i)\}_{i \in \mathbb{N}}$ be an atlas of M compatible with its smoothness. Recall the topology that we endowed M with; A set $O \subset M$ is open iff $\phi_i(O \cap M_i)$ is an open subset of $\mathbb{R}^n$ for all $i \in \mathbb{N}$. Does there exist a countable base for this topology. If yes, find such a base. If no, prove why such a base cannot exist.

2. Endow the set $\mathbb{S}^n := \{(x_1, x_2, ..., x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\}$ with an atlas. Is the topology endowed on $\mathbb{S}^n$ by the atlas you found via the definition in the previous question the same as the subspace topology it inherits as a subspace of $\mathbb{R}^{n+1}$ with the standard topology.

3. Consider the map $F : \mathbb{S}^2 \ni (x_1, x_2, x_3) \to F(x_1, x_2, x_3) := (\frac{x_1}{2}, \frac{\sqrt{3}x_1}{2}, x_2, x_3) \in \mathbb{S}^3$.
   - Find the local representative of F and T_pF at $p = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ in the coordinates you found in question 2.
   - Define a curve $(-\frac{1}{2}, \frac{1}{2}) \ni t \mapsto \gamma_p(t) := (\sqrt{\frac{1-t^2}{2}}, \sqrt{\frac{1-t^2}{2}}, t) \in \mathbb{S}^2$. Find 2 distinct curves belonging to the equivalence class of $T_pF([\gamma_p(t)])$. Here, p is the point defined in the previous part.

4. Let $\mathbb{B}^n := \{(x_1, x_2, ..., x_{n+1}) \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i^2 < 1\}$. Is $\mathbb{B}^n$ diffeomorphic to $\mathbb{R}^n$? If yes, find such a diffeomorphism. If not, prove why they are not diffeomorphic.

5. Show that $T\mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$. Assume that $\mathbb{S}^1$ is endowed with any one of the smoothness demonstrated during the lectures.