Systems & Control Engg.
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SC 624, Spring 2017
Quiz 7
Date: 21 Apr 2018
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Recall that $\mathbb{N}^*$ is the positive integers, $\mathbb{N} = \{0\} \cup \mathbb{N}^*$. For us $\mathrm{i} := \sqrt{-1}$.

Recall that a (non-empty) set X endowed with a maximal (smooth) atlas (a.k.a. a smoothness,) is a smooth manifold.

1. (15 points) Endow the set $\mathbb{S}^n := \left\{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=1}^{n+1} x_i^2 = 1\right\}$ with an atlas. We have seen in class how given a smooth atlas on a set, we can endow it with a topology using the charts and chart maps. In the case of $\mathbb{S}^n$, is this topology obtained via the atlas you found the same as the subspace topology it inherits as a subspace of $\mathbb{R}^{n+1}$with the standard topology?

2. (15 points) Let $T^2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid R, r \in \mathbb{R},\ r < R,\ |x_3| \leq r, \left(\sqrt{x_1^2 + x_2^2} - R\right)^2 + x_3^2 = r^2\ \}$. Endow T^2 with a smoothness which makes it diffeomorphic to $S^1 \times S^1$.

3. (15 points) Find the Tangent space at a point x and the Tangent map of the left action and inverse at a point x corresponding to the following Lie groups:

   - The special Euclidean group
   $$SE(n) := \left\{ \begin{pmatrix} R & V \\ 0 & 1 \end{pmatrix} \;\middle|\; R^\top R = I, R \in \mathbb{R}^{n\times n}, V \in \mathbb{R}^n \right\}.$$
   - The n-dimensional euclidean space $\mathbb{R}^n$ equipped with the standard group operation.

4. (30 points) Let f be a smooth vector field on the smooth manifold X, and let $y \in X$ be a stationary point of f.

   - Show that if there exists a Lyapunov function from f at y, then y is stable.
   - A lyapunov function V at y is called a strict lyapunov function if it satisfies $\dot{V}(x) < 0$ for all $x \in U \setminus \{y\}$. Show that if there exists a strict Lyapunov function at y, then y is asymptotically stable.