Systems & Control Engg.	SC 624, Spring 2018	
IIT Bombay, Powai	***Quiz 3***	
Mumbai 400076	Date: 21 Feb 2018	
Maharashtra, India	Time: 90 min	

1. (15 points) Let $T^2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid a, b \in \mathbb{R},\ b < a,\ |x_3| \leq b,\ x_1^2 + x_2^2 + x_3^2 = a^2 \pm 2\sqrt{b^2 - x_3^2}\ \}$. Endow T^2 with a smoothness which makes it diffeomorphic to $S^1 \times S^1$.

2. (10 points) Let M, N be smooth manifolds such that $f : M \to N$ is a smooth diffeomorphism. If X and Y are smooth vector fields on M, Prove that $f_*[X, Y] = [f_*X, f_*Y]$.

3. (5 points) Let f, g be vector fields on $\mathbb{R}^n$ endowed with the standard smoothness, defined as :

$$\mathbb{R}^n \ni x \mapsto f(x) = Ax$$
$$\mathbb{R}^n \ni x \mapsto g(x) = (xx^T)x$$

   where $A \in M(n; \mathbb{R})$. Find $[f, g]$, the lie bracket of f and g.

4. (5 points) Let M be a smooth manifold of dimension n, and f a smooth vector field on M. Denote by γ_t^f the flow of the vector field, as defined in class. Prove that

$$\gamma_t^f \circ \gamma_s^f(x) = \gamma_{t+s}^f(x),\ \forall t, s \in \mathbb{R},\ x \in M \ \text{ such that the objects in the equation are all well defined.}$$

5. (5 points) Let $M = \mathbb{R}$ with the standard smoothness. Define a vector field f on M as

$$\mathbb{R} \ni x \mapsto f(x) = 1 + x^2 \in T_x\mathbb{R}$$

   Find the set $U \subset \mathbb{R} \times M$ such that $(t, x) \in U \iff \gamma_t^f(x)$ is well defined. Verify the property proved in the previous question for this particular vector field.