Optimization

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Lecture 5: Linear Programming

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Outline of the talk

Modeling: a simple example

A diet problem

Choose the quantities x_1 x_2 of two types of food for a daily need of a person.

- *One unit of food i costs c_i euros.*
- *One unit of food i contains p_i units of protein and v_i units of vitamins.*
- *A healthy diet requires at least p_{min} units of protein and v_{min} units of vitamins.*

How to choose x_1 and x_2 in order to obtain an economic but healthy diet ?

$$\begin{aligned} \min_{x} \quad & c_1x_1 + c_2x_2 \\ s.t. \quad & p_1x_1 + p_2x_2 \geq p_{min} \\ & v_1x_1 + v_2x_2 \geq v_{min} \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

Consider the following parameters: $c_1 = 1$, $c_2 = 1$, $p_1 = 2$, $p_2 = 4$, $v_1 = 5$, $v_2 = 2$, $p_{min} = 8$, $v_{min} = 10$.

Modeling: a simple example

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How to choose x_1 and x_2 in order to obtain an economic but healthy diet ?

$$
\begin{aligned}
\min_{\mathbf{x}} \quad & c_1 x_1 + c_2 x_2 \\
s.t. \quad & p_1 x_1 + p_2 x_2 \geq p_{min} \\
& v_1 x_1 + v_2 x_2 \geq v_{min} \\
& x_1 \geq 0, x_2 \geq 0
\end{aligned}
$$

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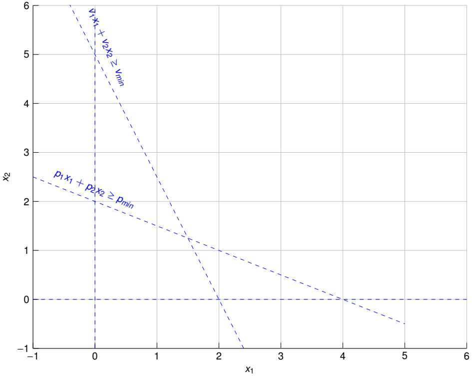
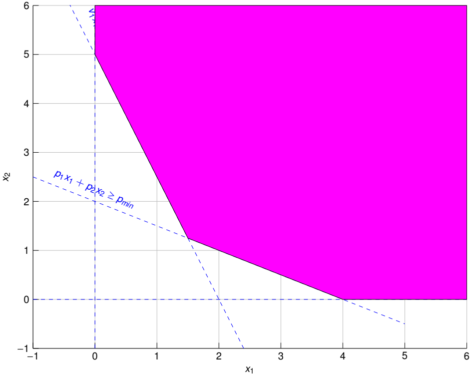
Modeling: a simple example



The feasible domain.

Modeling: a simple example



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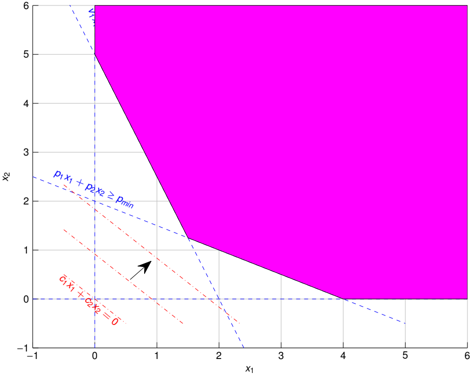
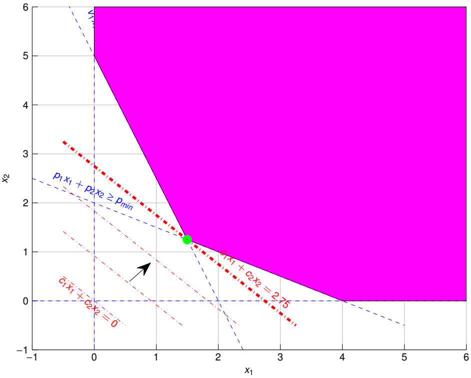


Cost function level sets.

Modeling: a simple example



The optimal solution.

Exercise: repeat the same question for $c_1 = 2$, $c_2 = 4$.

Modeling: another diet problem

A diet problem

Choose the quantities $\boldsymbol{x} = [x_1 \dots x_5]^T$ *of 5 types of food for a daily need of a person. The quantities should cover the following needs: 2000cal, 55g of Protein and 800g of Calcium, with the least cost possible.*

Food	Unit	Energy(cal)	Protein(g)	Calcium(mg)	Price(Centimes)	Quantity
Cereal	28g	110	4	2	30	x_1
Meat	100g	200	23	12	100	x_2
Eggs	1	80	6	26	20	x_3
Milk	250cl	160	8	285	50	x_4
Vegetables	250g	260	14	80	15	x_5

Can not be solved graphically.

Modeling: another diet problem

A diet problem

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Milk	250cl	160	8	285	50	x_4
Vegetables	250g	260	14	80	15	x_5

$$\begin{aligned}
\min_{\mathbf{x}} \quad & 30x_1 + 100x_2 + 20x_3 + 50x_4 + 15x_5 \\
\text{s.t.} \quad & 110x_1 + 200x_2 + 80x_3 + 160x_4 + 260x_5 \geq 2000 \\
& 4x_1 + 23x_2 + 6x_3 + 8x_4 + 14x_5 \geq 55 \\
& 2x_1 + 12x_2 + 26x_3 + 285x_4 + 80x_5 \geq 800 \\
& x_1 \geq 0, \dots, x_5 \geq 0
\end{aligned}$$

Can not be solved graphically.

1. Introduction
2. Standard Form
3. Fundamental Theorems
4. The Simplex Method
5. The two-phase simplex method

The Standard Form

Standard Form

Consider the linear program of the form

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

*where $\mathbf{b} \geq \mathbf{0}$, $A \in \mathbb{R}^{m \times n}$ with $m < n$ and $\operatorname{rank}(A) = m$. In this case, we say that the linear program is in **standard form**.*

Techniques to solve linear programs are often presented for the standard form

No loss of generality: we can always write linear programs in the standard form

The Standard Form

How to transform a linear program to the standard form ?

- *If an element of* $\boldsymbol{b}$ *is negative* $b_i < 0 \Rightarrow$ *we multiply the* i^{th} *line of* A *and* b_i *by* -1.

- $\max \boldsymbol{c}^T \boldsymbol{x} \Rightarrow \min -\boldsymbol{c}^T \boldsymbol{x}$

- $\alpha_0 x_0 + \cdots + \alpha_n x_n \leq \beta$ *with* $x_0 \geq 0, \cdots x_n \geq 0 \Rightarrow$
  $\alpha_0 x_0 + \cdots + \alpha_n x_n + y = \beta$ *with* $x_0 \geq 0, \cdots x_n \geq 0$ *and* $y \geq 0$

- $\alpha_0 x_0 + \cdots + \alpha_n x_n \geq \beta$ *with* $x_0 \geq 0, \cdots x_n \geq 0 \Rightarrow$
  $\alpha_0 x_0 + \cdots + \alpha_n x_n - y = \beta$ *with* $x_0 \geq 0, \cdots x_n \geq 0$ *and* $y \geq 0$

- $x \geq x_{min}$ *(no sign on* x*)* $\Rightarrow x - y = x_{min}$ *with* $y \geq 0 \Rightarrow u - v - y = x_{min}$ *with* $u, v, y \geq 0$

- $x \leq x_{max} \Rightarrow$ *(no sign on* x*)* $x + y = x_{max}$ *with* $y \geq 0 \Rightarrow u - v + y = x_{max}$ *with* $u, v, y \geq 0$

The Standard Form

Example

convert the following problem into a standard form

$$\begin{aligned} \min_{x_1} \quad & x_1 \\ s.t. \quad & x_1 \leq 5 \\ & x_1 \geq 0 \end{aligned}$$

Example

$$\begin{aligned} \max_{\mathbf{x}} \quad & 2x_2 - x_1 \\ s.t. \quad & 4x_1 = 2x_2 - 5 \\ & x_2 \geq -2 \\ & x_2 \leq 2 \\ & x_1 \leq 0 \end{aligned}$$

Definitions

Consider $B \in \mathbb{R}^{n\times n}$ a square matrix whose columns are n linearly independent columns of A. We can suppose that $A = [B\ D]$ where $D \in \mathbb{R}^{n\times(n-m)}$ is a matrix containing the remaining $m - n$ columns of A.

Basic solution

If $\mathbf{x}_B$ is a solution to $B\mathbf{x}_B = \mathbf{b}$ that is $\mathbf{x}_B = B^{-1}\mathbf{b}$, then

$$X = \begin{bmatrix} \mathbf{x}_B \\ \mathbf{0} \end{bmatrix}$$

is a solution to $A\mathbf{x} = \mathbf{b}$ that is $[B\ D]\mathbf{x} = \mathbf{b}$.

- *In this case $\mathbf{x}$ is called a basic solution with respect to the basis B.*
- *If in addition $\mathbf{x} \geq \mathbf{0}$, then it is called a basic feasible solution (BFS)*
- *If some elements of $\mathbf{x}_B$ are zero, then it is called a degenerate basic feasible solution*
- *Finally, if $\mathbf{x}_B > \mathbf{0}$, then it is called a non-degenerate basic feasible solution*

*Note that we call an $\mathbf{x}$ satisfying $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ a **feasible solution***

Fundamental Theorems

Theorem 1

Consider a linear programming problem.

- *If there exists a feasible solution, then there exists a basic feasible solution*
- *If there exists an optimal feasible solution, then there exists an optimal basic feasible solution*

Theorem 2

Consider $A \in \mathbb{R}^n$, $m < n$. Let $\mathcal{S}$ be the (convex) set of all feasible solutions, that is

$$\mathcal{S} = \{\boldsymbol{x} \in \mathbb{R}^n : \ A\boldsymbol{x} = \boldsymbol{b}, \quad \boldsymbol{x} \geq \boldsymbol{0}\}$$

*Then, $\boldsymbol{x}$ is an **extreme point** of $\mathcal{S}$ if and only if it is a basic feasible solution*

Idea: *In order to solve a linear programming under standard form, we can test all the extreme points, that is all the basic feasible solutions ! However the number of possible solutions is* $\binom{n}{m} = \frac{n!}{m!(n-m)!}$

Testing all basic feasible solutions

Example

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 1 & -2 & -1 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$

- *For* $B = [\mathbf{a}_1\ \mathbf{a}_2]$ *we get* $\mathbf{x}_B = [6\ 2]^T$
  thus $\mathbf{x} = [6\ 2\ 0\ 0]^T$ *is a non-degenerate basic feasible solution*
- *For* $B = [\mathbf{a}_1\ \mathbf{a}_4]^T$ *or* $B = [\mathbf{a}_2\ \mathbf{a}_4]^T$ *or* $B = [\mathbf{a}_3\ \mathbf{a}_4]^T$ *we get* $\mathbf{x}_B = [0\ 2]^T$
  thus $\mathbf{x} = [0\ 0\ 0\ 2]^T$ *is a degenerate basic feasible solution*
- *For* $B = [\mathbf{a}_2\ \mathbf{a}_3]^T$ *we get* $\mathbf{x}_B = [2\ \ -6]^T$
  thus $\mathbf{x} = [0\ 2\ \ -6\ 0]^T$ *is a basic but infeasible solution* $\mathbf{x} \not\geq \mathbf{0}$
- *Note that* $\mathbf{x} = [3\ 1\ 0\ 1]^T$ *is feasible but not basic*

Moving from an extreme point to a better one

It is possible to move from one extreme point to an adjacent extreme point

Example

Consider the problem

$$\begin{aligned} \max_{\mathbf{x}} \quad & 3x_1 + 5x_2 \\ s.t. \quad & x_1 + 5x_2 \leq 40 \\ & 2x_1 + x_2 \leq 20 \\ & x_1 + x_2 \leq 12 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

Moving from an extreme point to a better one

This linear programming problem can be written using the standard form

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & -3x_1 - 5x_2 \\ \text{s.t.} \quad & x_1 + 5x_2 + x_3 = 40 \\ & 2x_1 + x_2 + x_4 = 20 \\ & x_1 + x_2 + x_5 = 12 \\ & x_1 \geq 0, \ldots, x_5 \geq 0 \end{aligned}$$

That is, we get a standard form problem

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

with

$$A = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{a}_5 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} 40 \\ 20 \\ 12 \end{bmatrix} \quad \boldsymbol{c} = \begin{bmatrix} -3 \\ -5 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Moving from an extreme point to a better one

Note that $\mathbf{x} = [0\ 0\ 40\ 20\ 12]^T$ *is a basic feasible solution in the basis* $B = [\mathbf{a}_3\ \mathbf{a}_4\ \mathbf{a}_5]$*. it is easy to see that this solution corresponds to*

$$0\mathbf{a}_1 + 0\mathbf{a}_2 + 40\mathbf{a}_3 + 20\mathbf{a}_4 + 12\mathbf{a}_5 = \mathbf{b} \tag{1}$$

The corresponding cost in this case is

$$z_0 = \mathbf{c}^T\mathbf{x} = [c_1\ c_2\ c_3\ c_4\ c_5]\begin{bmatrix}0\\0\\40\\20\\12\end{bmatrix} = 40c_3 + 20c_4 + 12c_5\ (= 0) \tag{2}$$

If we want to move to another BFS, we need to choose between $\mathbf{a}_1$ *and* $\mathbf{a}_2$ *to include into the basis. We also have to remove one of the vectors* $\mathbf{a}_3$, $\mathbf{a}_4$, $\mathbf{a}_5$ *from the basis.*

Moving from an extreme point to another

If we choose to include $\boldsymbol{a}_1$ *into the basis, we will need to remove one of the vectors* $\boldsymbol{a}_3$, $\boldsymbol{a}_4$, $\boldsymbol{a}_5$ *from the basis. First, note that*

$$\boldsymbol{a}_1 = \boldsymbol{a}_3 + 2\boldsymbol{a}_4 + \boldsymbol{a}_5 \quad \Rightarrow \epsilon \boldsymbol{a}_1 - \epsilon \boldsymbol{a}_3 - 2\epsilon \boldsymbol{a}_4 - \epsilon \boldsymbol{a}_5 = 0 \tag{3}$$

for $\epsilon > 0$. *Adding* (3) *to* (1) *we get*

$$\epsilon \boldsymbol{a}_1 + (40 - \epsilon)\boldsymbol{a}_3 + (20 - 2\epsilon)\boldsymbol{a}_4 + (12 - \epsilon)\boldsymbol{a}_5 = \boldsymbol{b}$$

that is we obtain a solution

$$\boldsymbol{x}_1 = \begin{bmatrix} \epsilon \\ 0 \\ 40 - \epsilon \\ 20 - 2\epsilon \\ 12 - \epsilon \end{bmatrix}$$

with new cost

$$\begin{aligned} \boldsymbol{c}^T \boldsymbol{x}_1 &= \epsilon c_1 + (40 - \epsilon)c_3 + (20 - 2\epsilon)c_4 + (12 - \epsilon)c_5 \\ &= \underbrace{(40c_3 + 20c_4 + 12c_5)}_{z_0} + \epsilon\Big(c_1 - \underbrace{(c_3 + 2c_4 + c_5)}_{z_1}\Big) = z_0 + \epsilon r_1 \end{aligned}$$

Moving from an extreme point to another

We want the coefficient corresponding to one of the vectors $\boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5$ *to be zero. Also, we want all the coefficients to be non negative (to obtain a BFS). Clearly,* $\epsilon = 10$ *will lead to*
Thus, we get another basic feasible solution with respect to the basis $B = [\boldsymbol{a}_1 \ \boldsymbol{a}_3 \ \boldsymbol{a}_5]$ *:*

$$\boldsymbol{x}_1 = \begin{bmatrix} 10 \\ 0 \\ 30 \\ 0 \\ 2 \end{bmatrix}$$

Moving from an extreme point to another

If we choose to include $\boldsymbol{a}_2$ *into the basis, we will need to remove one of the vectors* $\boldsymbol{a}_3$, $\boldsymbol{a}_4$, $\boldsymbol{a}_5$ *from the basis. First, note that*

$$\boldsymbol{a}_2 = 5\boldsymbol{a}_3 + \boldsymbol{a}_4 + \boldsymbol{a}_5 \quad \Rightarrow \epsilon\boldsymbol{a}_2 - 5\epsilon\boldsymbol{a}_3 - \epsilon\boldsymbol{a}_4 - \epsilon\boldsymbol{a}_5 = 0 \tag{4}$$

for $\epsilon > 0$. *Adding* (4) *to* (1) *we get*

$$\epsilon\boldsymbol{a}_2 + (40 - 5\epsilon)\boldsymbol{a}_3 + (20 - \epsilon)\boldsymbol{a}_4 + (12 - \epsilon)\boldsymbol{a}_5 = \boldsymbol{b}$$

that is we obtain a solution

$$\boldsymbol{x}_2 = \begin{bmatrix} 0 \\ \epsilon \\ 40 - 5\epsilon \\ 20 - \epsilon \\ 12 - \epsilon \end{bmatrix}$$

with new cost

$$\begin{aligned} \boldsymbol{c}^T\boldsymbol{x}_2 &= \epsilon c_2 + (40 - 5\epsilon)c_3 + (20 - \epsilon)c_4 + (12 - \epsilon)c_5 \\ &= \underbrace{(40c_3 + 20c_4 + 12c_5)}_{z_0} + \epsilon\Big(c_2 - \underbrace{(5c_3 + c_4 + c_5)}_{z_2}\Big) = z_0 + \epsilon r_2 \end{aligned}$$

Moving from an extreme point to another

We want the coefficient of one of the vectors $\boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5$ *to be zero. Also, we want all the coefficients to be non negative (to obtain a BFS). Clearly,* $\epsilon = 8$ *will lead to*

$$8\boldsymbol{a}_2 + 30\boldsymbol{a}_3 + 2\boldsymbol{a}_5 = \boldsymbol{b}$$

Thus, we get another basic feasible solution with respect to the basis $B = [\boldsymbol{a}_2 \ \boldsymbol{a}_4 \ \boldsymbol{a}_5]$ *:*

$$\begin{bmatrix} 10 & 0 & 30 & 0 & 2 \end{bmatrix}^T$$

$$\boldsymbol{x}_2 = \begin{bmatrix} 0 \\ 8 \\ 0 \\ 12 \\ 4 \end{bmatrix}$$

Elementary row operations

Elementary row operations can be used to solve linear algebraic equations $C\mathbf{x} = \mathbf{d}$ and calculate the inverse of a matrix C :

- *Interchanging any two rows*
- *Multiplying a row by a real nonzero number*
- *Adding a scalar multiple of a row to another*

To find the solution to $C\mathbf{x} = \mathbf{d}$ we form $[C\ \mathbf{d}]$ and perform elementary row operations to get $E_p \cdots E_1[C\ \mathbf{d}] = [I\ \mathbf{x}^\star]$. Clearly, $E = E_p \cdots E_1 = C^{-1}$ and $E\mathbf{d} = \mathbf{x}^\star$, thus $C^{-1}\mathbf{d} = \mathbf{x}^\star$

To find the inverse of C we form $[C\ I]$ and perform elementary row operations to get $E_p \cdots E_1[C\ I] = [I\ D]$. Clearly, $E = E_p \cdots E_1 = C^{-1}$ and $E = D$, thus $C^{-1} = D$

Example

Multiplying a matrix by

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Permutation of the first and the third lines

Elementary row operations

Example

Multiplying a matrix by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Multiplying second line by α

Example

Multiplying a matrix by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

Add $2\times$ *first line to forth line*

Finding a basic solution

Consider the standard form

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad A\boldsymbol{x} = \boldsymbol{b}$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

with $A \in \mathbb{R}^{m\times n}$ with $m < n$ and rank$(A) = m$.
Without loss of generality, we consider the invertible $B = [\boldsymbol{a}_1 \ \ldots \ \boldsymbol{a}_m] \in \mathbb{R}^{m\times m}$

- *Form an augmented matrix $[A \ \boldsymbol{b}]$*
- *Apply elementary row operations*

$$E[A \ \boldsymbol{b}] = E[B \ D \ \boldsymbol{b}] = [I \ \tilde{D} \ \tilde{\boldsymbol{b}}]$$

- *Consider $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_B \\ \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} \tilde{\boldsymbol{b}} \\ \boldsymbol{0} \end{bmatrix}$ clearly*

$$EA\boldsymbol{x} = EA\begin{bmatrix} \tilde{\boldsymbol{b}} \\ \boldsymbol{0} \end{bmatrix} = [I \ \tilde{D}]\begin{bmatrix} \tilde{\boldsymbol{b}} \\ \boldsymbol{0} \end{bmatrix} = \tilde{\boldsymbol{b}} = E\boldsymbol{b}$$

thus

$$EA\boldsymbol{x} = E\boldsymbol{b} \quad \Rightarrow A\boldsymbol{x} = \boldsymbol{b}$$

Thus $\boldsymbol{x}$ is a basic solution with respect to the basis $B = [\boldsymbol{a}_1 \ \ldots \ \boldsymbol{a}_m]$ and $E = B^{-1}$

Finding a basic solution

In fact, for any $\mathbf{x}_D$ *it is easy to see that* $\mathbf{x} = \begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_D \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{b}} - \tilde{D}\mathbf{x}_D \\ \mathbf{x}_D \end{bmatrix}$ *is a solution to* $EA\mathbf{x} = E\mathbf{b}$ *thus to* $A\mathbf{x} = \mathbf{b}$

Also, any solution to $A\mathbf{x} = \mathbf{b}$ *has the form* $\mathbf{x} = \begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_D \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{b}} - \tilde{D}\mathbf{x}_D \\ \mathbf{x}_D \end{bmatrix}$ *for some* $\mathbf{x}_D$

Changing the basic solution

Note that we denote the columns of A by $\boldsymbol{a}_j$ $(1 \le j \le n)$
*By elementary row operations we find the **Canonical Augmented Matrix** :*

$$[A\ \boldsymbol{b}] \ \underset{\rightarrow}{E}\ [I\ \tilde{D}\ \tilde{\boldsymbol{b}}] = \begin{bmatrix} 1 & 0 & \cdots & 0 & y_{1\,m+1} & \cdots & y_{1\,n} & y_{1\,0} \\ 0 & 1 & & 0 & y_{2\,m+1} & \cdots & y_{2\,n} & y_{2\,0} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & y_{m\,m+1} & \cdots & y_{m\,n} & y_{m\,0} \end{bmatrix}$$

we get the basic solution $\boldsymbol{x} = \begin{bmatrix} \tilde{\boldsymbol{b}} \\ \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} y_{1\,0} \\ y_{2\,0} \\ \vdots \\ y_{m\,0} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ *with respect to the basis* $B = [\boldsymbol{a}_1\ \ldots\ \boldsymbol{a}_m]$

$$A\boldsymbol{x} = \boldsymbol{b} \Leftrightarrow y_{1\,0}\,\boldsymbol{a}_1 + y_{2\,0}\,\boldsymbol{a}_2 + \cdots + y_{m\,0}\,\boldsymbol{a}_m = \boldsymbol{b}$$

and $[I\ \tilde{D}] = EA$ *thus*

$$E^{-1}[I\ \tilde{D}] = B[I\ \tilde{D}] = A \Leftrightarrow \boldsymbol{a}_j = y_{1\,j}\,\boldsymbol{a}_1 + y_{2\,j}\,\boldsymbol{a}_2 + \cdots + y_{m\,j}\,\boldsymbol{a}_j \qquad \forall m < j \le n$$

Changing the basic solution

Changing the basic solution $\Leftrightarrow$ changing the basis

We want to replace $\boldsymbol{a}_p$ $1 \leq p \leq m$ *by* $\boldsymbol{a}_q$ $m < q \leq n$ *in the basis* B

In the old basis

$$\boldsymbol{a}_q = \sum_{i=1}^{m} y_{i\,q}\,\boldsymbol{a}_i = \sum_{i=1,\ i\neq p}^{m} y_{i\,q}\,\boldsymbol{a}_i + y_{p\,q}\,\boldsymbol{a}_p$$

In the new basis If $y_{p\,q} \neq 0$ *which is equivalent to linear interdependency of* $\{\boldsymbol{a}_i\}_{1\leq i\leq m\ \ i\neq p} \cup \boldsymbol{a}_q$

$$\boldsymbol{a}_p = \frac{1}{y_{p\,q}}\boldsymbol{a}_q - \sum_{i=1,\ i\neq p}^{m} \frac{y_{i\,q}}{y_{p\,q}}\,\boldsymbol{a}_i$$

In the old basis

$$\boldsymbol{a}_j = \sum_{i=1}^{m} y_{i\,j}\,\boldsymbol{a}_i = \sum_{i=1,\ i\neq p}^{m} y_{i\,j}\,\boldsymbol{a}_i + y_{p\,j}\boldsymbol{a}_p$$

In the new basis

$$\boldsymbol{a}_j = \sum_{i=1,\ i\neq p}^{m} (y_{i\,j} - \frac{y_{i\,q}}{y_{p\,q}})\boldsymbol{a}_i + \frac{y_{p\,j}}{y_{p\,q}}\boldsymbol{a}_q$$

Changing the basic solution

In the old basis

$$\begin{aligned} y_{ij} &\rightarrow y_{ij} - \frac{y_{iq}}{y_{pq}} y_{pj} \qquad i \neq p, \ \forall j \\ y_{pj} &\rightarrow \frac{y_{pj}}{y_{pq}} \end{aligned}$$

This operation is called ***pivoting about the*** pq***th element****.*

This will result in a matrix whose qth column has zeros every where except the (p, q) *entry which is* 1*.*

We will see that the pivoting can be expressed easily using elementary row operations.

The Simplex Algorithm

The simplex algorithm

The idea behind the simplex algorithm is to go from a BFS to another till finding the optimal solution

Two questions are still open :

- *What vector $\mathbf{a}_p$ to take out of the the basis ?*
- *What new vector $\mathbf{a}_q$ to bring in to the the basis ?*

Consider a BFS $\mathbf{x} = \begin{bmatrix} \mathbf{x}_B \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{b}} \\ \mathbf{0} \end{bmatrix}$ *remember this solution is feasible if* $\mathbf{x} \geq \mathbf{0}$. *Not all basic solutions are feasible.*

We consider first the case of non-degenerate FBS.

The Simplex Algorithm

What vector $\boldsymbol{a}_p$ to take out of the the basis ?

Suppose that $\boldsymbol{a}_q$ $q > m$ enters the basis, in the old basis we have :

$$y_{1\,q}\,\boldsymbol{a}_1 + \cdots + y_{m\,q}\,\boldsymbol{a}_m = \boldsymbol{a}_q$$

and

$$y_{1\,0}\,\boldsymbol{a}_1 + \cdots + y_{m\,0}\,\boldsymbol{a}_m = \boldsymbol{b}$$

thus

$$(y_{1\,0} - \alpha y_{1\,q})\,\boldsymbol{a}_1 + \cdots + (y_{m\,0} - \alpha y_{m\,q})\,\boldsymbol{a}_m + \alpha \boldsymbol{a}_q = \boldsymbol{b}$$

we want to make one coefficient equals 0 while keeping the others positive (to guarantee the feasibility), this can be guaranteed by choosing

$$p = arg\ \min_i \{\frac{y_{i\,0}}{y_{i\,q}}, \quad y_{i\,q} > 0\} \qquad \alpha = \frac{y_{p\,0}}{y_{p\,q}}$$

If the minimum is achieved for more that one index we get a degenerate new solution. In this case, we choose the smallest p possible. Finally, if all y_{iq} are negative, the set of feasible solutions is unbounded.

The Simplex Algorithm

What new vector $\boldsymbol{a}_q$ to bring in to the the basis ?

Consider the current BFS $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_B \\ \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} \tilde{\boldsymbol{b}} \\ \boldsymbol{0} \end{bmatrix}$. *The cost function is*

$$z = \boldsymbol{c}^T \boldsymbol{x} = c_1 x_1 + \cdots + c_n x_n$$

The initial cost function

$$z = [\boldsymbol{c}_B^T \ \boldsymbol{c}_D^T]\boldsymbol{x} = \boldsymbol{c}_B^T \tilde{\boldsymbol{b}} = c_1 y_{1\,0} + \cdots + c_m y_{m\,0} = z_0$$

$\boldsymbol{a}_q$ *$q > m$ enters the basis and* $\boldsymbol{a}_p$ *leave the basis*

$$p = arg\ \min_i \{\frac{y_{i\,0}}{y_{i\,q}}, \quad y_{i\,q} > 0\} \qquad \alpha = \frac{y_{p\,0}}{y_{p\,q}}$$

$$(y_{1\,0} - \alpha y_{1\,q})\,\boldsymbol{a}_1 + \cdots + (y_{m\,0} - \alpha y_{m\,q})\,\boldsymbol{a}_m + \alpha \boldsymbol{a}_q = \boldsymbol{b}$$

thus

$$z = (y_{1\,0} - \alpha y_{1\,q})\,c_1 + \cdots + (y_{m\,0} - \alpha y_{m\,q})\,c_m + \alpha c_q$$

The Simplex Algorithm

$$z = z_0 + \alpha(c_q - (c_1 y_{1\,q} + \cdots + c_m y_{m\,q}))$$

Consider

$$z_q = c_1 y_{1\,q} + \cdots + c_m y_{m\,q}$$

then

$$z = z_0 + \alpha(c_q - z_q) \tag{5}$$

*If a **reduced cost coefficient** is negative $r_q = c_q - z_q < 0$ then by entering a_q it is possible to decrease the cost function.*

*When obtaining several $r_q < 0$, then a common practice is to take the smallest one. If all $r_q = c_q - z_q \geq 0$ then the current BFS can be shown to be optimal. This defines the **stopping criterion**.*

This can be shown by noting that any solution to $A\mathbf{x} = \mathbf{b}$ has the form $\mathbf{x} = \begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_D \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{b}} - \tilde{D}\mathbf{x}_D \\ \mathbf{x}_D \end{bmatrix}$ for some $\mathbf{x}_D = [x_{m+1}, \ldots, x_n]$ and showing that $z = \mathbf{c}^T\mathbf{x} = z_0 + \sum_{i=m+1}^{n} r_i x_i$.
Clearly, for a feasible solution ($x_i \geq 0$) $z = \mathbf{c}^T\mathbf{x}$ will be larger or equal to z_0.

The Simplex Algorithm

Simplex Algorithm:

- *Form a canonical augmented matrix corresponding to an initial basic feasible solution.*
- *Calculate r_j the reduced cost coefficients corresponding to the nonbasic variables*
- *If they are all non negative stop — the current basic feasible solution is optimal*
- *Select the smallest $r_q < 0$*
- *If no $y_{iq} > 0$, stop—the problem is unbounded;*
  else, calculate $p = \arg\min_i \{\frac{y_{i0}}{y_{iq}}, \quad y_{iq} > 0\}$ (If more than one index i minimizes $\frac{y_{i0}}{y_{iq}}$, we let p be the smallest such index.)
- *Update the canonical augmented matrix by pivoting about the (p, q)th element.*
- *Go to step 2.*

An example

Example

Solve the following linear programming problem

$$\begin{aligned} \max_{\boldsymbol{x}} \quad & 3x_1 + 5x_2 \\ s.t. \quad & x_1 + 5x_2 \leq 40 \\ & 2x_1 + x_2 \leq 20 \\ & x_1 + x_2 \leq 12 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

The matrix form

Consider the standard form

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

where $A = [B\ D]$ *with* $B = [\boldsymbol{a}_1, \cdots, \boldsymbol{a}_m]$, $D = [\boldsymbol{a}_{m+1}, \cdots, \boldsymbol{a}_n]$, $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_B \\ \boldsymbol{x}_D \end{bmatrix}$ $\boldsymbol{c}^T = [\boldsymbol{c}_B^T\ \boldsymbol{c}_D^T]$.

The problem is equivalent to

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}_B^T \boldsymbol{x}_B + \boldsymbol{c}_D^T \boldsymbol{x}_D \\ \text{s.t.} \quad & B\boldsymbol{x}_B + D\boldsymbol{x}_D = \boldsymbol{b} \\ & \boldsymbol{x}_B \geq \boldsymbol{0}, \boldsymbol{x}_D \geq \boldsymbol{0} \end{aligned}$$

The matrix form

If $\mathbf{x}_D = \mathbf{0}$ *then* $\mathbf{x} = \begin{bmatrix} B^{-1}\mathbf{b} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_B \\ \mathbf{0} \end{bmatrix}$ *is a BFS corresponding to* B*. In this case, the cost function value is*

$$z_0 = \mathbf{c}_B^T B^{-1}\mathbf{b}$$

If $\mathbf{x}_D \neq \mathbf{0}$*, in this case the solution is not basic. In this case,* $\mathbf{x}_B = B^{-1}\mathbf{b} - B^{-1}D\mathbf{x}_D$*. In this case, the cost function value is*

$$\begin{aligned} z &= \mathbf{c}_B^T\mathbf{x}_B + \mathbf{c}_D^T\mathbf{x}_D \\ &= \mathbf{c}_B^T B^{-1}\mathbf{b} - \mathbf{c}_B^T B^{-1}D\mathbf{x}_D + \mathbf{c}_D^T\mathbf{x}_D \\ &= \mathbf{c}_B^T B^{-1}\mathbf{b} + \underbrace{(\mathbf{c}_D^T - \mathbf{c}_B^T B^{-1}D)}_{\mathbf{r}_D^T}\mathbf{x}_D \end{aligned}$$

and we obtain

$$z = z_0 + \mathbf{r}_D^T\mathbf{x}_D$$

If $\mathbf{r}_D \geq \mathbf{0}$ *and since* $\mathbf{x}_D \geq \mathbf{0}$*, then the corresponding FBS* $B^{-1}\mathbf{b}$ *is optimal. If* $\mathbf{r}_D$ *has a negative component, then it is possible to reduce the cost function by increasing the corresponding components of* $\mathbf{x}_D$*. That is, by changing the basis !*

The matrix form

*Let us now form the following matrix called the **tableau** of the LP problem*

$$\begin{bmatrix} A & \mathbf{b} \\ \mathbf{c}^T & \mathbf{0} \end{bmatrix} = \begin{bmatrix} B & D & \mathbf{b} \\ \mathbf{c}_B^T & \mathbf{c}_D^T & 0 \end{bmatrix}$$

By performing some elementary row operations, or equivalently by the following matrix multiplying we get

$$\begin{bmatrix} B^{-1} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} A & \mathbf{b} \\ \mathbf{c}^T & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_m & B^{-1}D & B^{-1}\mathbf{b} \\ \mathbf{c}_B^T & \mathbf{c}_D^T & 0 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{I}_m & \mathbf{0} \\ -\mathbf{c}_B^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I}_m & B^{-1}D & B^{-1}\mathbf{b} \\ \mathbf{c}_B^T & \mathbf{c}_D^T & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_m & B^{-1}D & \overbrace{B^{-1}\mathbf{b}}^{\mathbf{x}_B} \\ \mathbf{0} & \underbrace{\mathbf{c}_D^T - \mathbf{c}_B^T B^{-1} D}_{\mathbf{r}_D} & \underbrace{-\mathbf{c}_B^T B^{-1}\mathbf{b}}_{-z_0} \end{bmatrix}$$

*The last matrix is called the **canonical tableau** corresponding to B. Once the tableau is calculated, it is possible to calculate the the tableau corresponding to a different basis using elementary row operations, such as the case for the basic simplex method.*

An example

Example

Solve the following linear programming problem, using the matrix form

$$\begin{aligned} \max_{\mathbf{x}} \quad & 3x_1 + 5x_2 \\ s.t. \quad & x_1 + 5x_2 \leq 40 \\ & 2x_1 + x_2 \leq 20 \\ & x_1 + x_2 \leq 12 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

The initialization of the simplex method

Till now we supposed that we start with a BFS. This is not always possible. Starting with the standard form

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ s.t. \quad & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

we need a method to find a BFS permitting to start the algorithm. This can be done by solving the following artificial problem

$$\begin{aligned} \min_{[\boldsymbol{x}^T\ \boldsymbol{y}^T]^T} \quad & y_1 + y_2 + \cdots + y_m \\ s.t. \quad & [A\ \boldsymbol{I}_m] \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \boldsymbol{b} \\ & \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} \geq \boldsymbol{0} \end{aligned}$$

The two-phase simplex method

Note that the artificial problem has a trivial BFS $\begin{bmatrix} \mathbf{0} \\ \mathbf{b} \end{bmatrix}$*, thus we can solve it using using the simplex method.*

Theorem 3

The original problem has a BFS iff the artificial problem has an optimal feasible solution with zero cost function, that is with $\mathbf{y} = \mathbf{0}$*.*

Te above theorem can be used to initiate the simplex algorithm using the resulting solution. The resulting first n components form a FBS to the original problem.

This is done by deleting the columns corresponding to the artificial variables, and revert back to the original objective function.

This is called the ***two-phase simplex method****.*

The initialization of the simplex method

Example

Solve the following linear programming problem, using the matrix form

$$\begin{aligned} \min_{\mathbf{x}} \quad & 2x_1 + 3x_2 \\ s.t. \quad & 4x_1 + 2x_2 \geq 12 \\ & x_1 + 4x_2 \geq 6 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

The above linear programming problem can be written using the standard form

$$\begin{aligned} \min_{\mathbf{x}} \quad & 2x_1 + 3x_2 \\ s.t. \quad & 4x_1 + 2x_2 - x_3 \qquad = 12 \\ & x_1 + 4x_2 \qquad - x_4 = 6 \\ & x_1 \geq 0, \ldots, x_4 \geq 0 \end{aligned}$$

The initialization of the simplex method

That is, we get a standard form problem

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T \mathbf{x} \\ s.t. \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

with

$$A = \begin{bmatrix} 4 & 2 & -1 & 0 \\ 1 & 4 & 0 & -1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 12 \\ 6 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}$$

It is not direct to find a BFS for this problem. To do so, we can solve

$$\begin{aligned} \min_{[\mathbf{x}^T \ \mathbf{y}^T]^T} \quad & y_1 + y_2 \\ s.t. \quad & [A \ \ I_2] \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0}, \end{aligned}$$

with

The initialization of the simplex method

Phase 1 :

$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{a}_6$	$\boldsymbol{b}$
4	2	-1	0	1	0	12
1	4	0	-1	0	1	6
0	0	0	0	1	1	0

We should update the last row to transform it into a canonical tableau

$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{a}_6$	$\boldsymbol{b}$
4	2	-1	0	1	0	12
1	$\boxed{4}$	0	-1	0	1	6
-5	-6	1	1	0	0	-18

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 12 \\ 6 \end{bmatrix}, \qquad B = [\boldsymbol{a}_5 \ \boldsymbol{a}_6]$$

$q = 2$, $\boldsymbol{a}_2$ *enters the basis (because of the* -6*).*

$p = arg\ \min_i \{\frac{y_{i0}}{y_{iq}}, \quad y_{iq} > 0\} = 2$ *thus* $\boldsymbol{a}_6$ *exits the basis.*

The initialization of the simplex method

$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	$\mathbf{a}_6$	$\mathbf{b}$
$\boxed{3.5}$	0	-1	0.5	1	-0.5	9
0.25	1	0	-0.25	0	0.25	1.5
-3.5	0	1	-0.5	0	1.5	-9

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} 0 \\ 1.5 \\ 0 \\ 0 \\ 9 \\ 0 \end{bmatrix}, \qquad B = [\mathbf{a}_5 \ \mathbf{a}_2]$$

$q = 1$, $\mathbf{a}_1$ *enters the basis (because of the* -3.5*).*
$p = 1$ *thus* $\mathbf{a}_5$ *exits the basis.*

$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	$\mathbf{a}_6$	$\mathbf{b}$
1	0	-2/7	1/7	2/7	-1/7	18/7
0	1	1/14	-2/7	-1/14	2/7	6/7
0	0	0	0	1	1	0

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} 18/7 \\ 6/7 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad B = [\mathbf{a}_1 \ \mathbf{a}_2]$$

The initialization of the simplex method

Phase 2:

We start by deleting the columns corresponding to the artificial variables, and reverting back to the original objective function :

	$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{b}$
	1	0	-2/7	1/7	18/7
	0	1	1/14	-2/7	6/7
$\boldsymbol{c}^T \rightarrow$	2	3	0	0	0

$$\boldsymbol{x} = \begin{bmatrix} 18/7 \\ 6/7 \\ 0 \\ 0 \end{bmatrix}, \qquad B = [\boldsymbol{a}_1 \ \boldsymbol{a}_2]$$

We update the last row to transform the tableau into a canonical one

$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{b}$
1	0	-2/7	1/7	18/7
0	1	1/14	-2/7	6/7
0	0	5/14	4/7	-54/7

$$\boldsymbol{x} = \begin{bmatrix} 18/7 \\ 6/7 \\ 0 \\ 0 \end{bmatrix}, \qquad B = [\boldsymbol{a}_1 \ \boldsymbol{a}_2]$$

The initialization of the simplex method

Thus, the optimizer is ${x_1}^\star = 18/7$, ${x_2}^\star = 6/7$ *and the optimal cost is* $54/7$ *which can be found in the tableau or can be calculated from* $2{x_1}^\star + 3{x_2}^\star$.