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TP1 Optimisation

Part 1.

1.1.

$$\forall x \in \mathbb{R}^2, \qquad f(x) = 3{x_1}^2 + 0{,}5{x_2}^2 + 2x_2 + 2$$

$$\nabla f(x) = \begin{pmatrix} \frac{\partial}{\partial x_1} f(x) \\ \frac{\partial}{\partial x_2} f(x) \end{pmatrix} = \begin{pmatrix} 6x_1 \\ x_2 + 2 \end{pmatrix}$$

$$Among\ [1\ 2]^T, [0\ 3]^T, [1\ 0]^T, [0\ 0]^T, only\ [0\ 0]^T\ give$$

$$\forall d \in (\mathbb{R}^+)^2\ faisable\ direction\ d^T \nabla f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)\ positive$$

So according to FONC, which is not for any

$$x \in \{[1\ 2]^T, [0\ 3]^T, [1\ 0]^T\},$$

$$\boxed{[0\ 0]^T\ is\ a\ local\ minimum\ of\ f(.)}$$

1.2.

$$\forall x \in \mathbb{R}^2, \nabla f_1(x) = \begin{pmatrix} -x_1 - 1 \\ -4x_2 + 2 \end{pmatrix}\ and\ \ \nabla f_2(x) = \begin{pmatrix} 6{x_1}^2 + {x_2}^2 + 10x_1 \\ 2x_1x_2 + 2x_2 \end{pmatrix}$$

- $D^2 f_1(x) = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}\ which\ is\ symetric$

$$So\ since\ diagonal\ values\ are\ negatives, \qquad \forall x \in \mathbb{R}^2, D^2 f_1(x) < 0$$

$$and\ because\ \ \nabla f_1\left(\begin{pmatrix} -1 \\ \frac{1}{2} \end{pmatrix}\right) = 0,$$

$$then\ x^* = \begin{pmatrix} -1 \\ \frac{1}{2} \end{pmatrix}\ is\ a\ local\ minimiser$$

- $D^2 f_2(x) = \begin{bmatrix} 12x_1 + 10 & 2x_2 \\ 2x_2 & 2x_1 + 2 \end{bmatrix}\ which\ is\ symetric$

$$Since\ \{x^* \in \mathbb{R}^2 / \ \nabla f_2(x^*) = 0\ \} = \{(-1,-2), (-1,2), \left(-\frac{10}{6}, 0\right), (0,0)\}$$

We can now find cases for which one solution is a saddle points, a local minimum or a local maximum:

a) $For\ x^* = (-1,-2)$

$$D^2 f_1(x^*) = \begin{bmatrix} -2 & -4 \\ -4 & 0 \end{bmatrix} \ which\ is\ symetric$$

$$So\ since\ diagonal\ values\ are\ negatives,\ \ D^2 f_2(\,x^*) < 0$$

$$then\ x^* =\ (-1,-2)\ is\ a\ local\ minimum\ \text{of f(.)}$$

b) $For\ x^* = (-1,2)$

$$D^2 f_1(x^*) = \begin{bmatrix} -2 & 4 \\ 4 & 0 \end{bmatrix} \ which\ is\ symetric$$

$$SoSo\ since\ diagonal\ values\ are\ negatives,\ \ D^2 f_2(\,x^*) < 0$$

$$then\ x^* =\ (-1,2)\ is\ a\ local\ minimum\ \text{of f(.)}$$

c) $For\ x^* = \left(-\frac{10}{6},0\right),$

$$D^2 f_1(x^*) = \begin{bmatrix} -10 & 0 \\ 0 & -\frac{4}{3} \end{bmatrix} \ which\ is\ symetric$$

$$So\ since\ diagonal\ values\ are\ negatives,\ \ D^2 f_2(\,x^*) < 0$$

$$then\ x^* = \left(-\frac{10}{6},0\right)\ is\ a\ local\ minimum\ \text{of f(.)}$$

d) $For\ x^* = (0,0),$

$$D^2 f_1(x^*) = \begin{bmatrix} 10 & 0 \\ 0 & 2 \end{bmatrix} \ which\ is\ diagonal$$

$$So\ since\ diagonal\ values\ are\ positive,\qquad \forall x \in \mathbb{R}^2, D^2 f_1(x) > 0$$

$$then\ x^* =\ (-1,2)\ is\ a\ local\ maximum\ \text{of f(.)}$$

1.3.

$$for\ x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, for\ d = \begin{pmatrix} 2 \\ -1 \end{pmatrix},$$

$$\forall \alpha \in \mathbb{R}, f(x_0 + \alpha d) = (1+2\alpha)^2 + (1+2\alpha)(1-\alpha) - 4(-1+\alpha)^2 + 5$$

$$= 3 + 3\alpha - 2\alpha^2$$

$$Since\ f(x_0) = 1 + 1 - 4 + 5 = 3$$

$$then\ \frac{f(x_0+\alpha d) - f(x_0)}{\alpha} = -2\alpha + 3$$

$$so\ \frac{\partial}{\partial d} f(x_0) = \lim_{\alpha \to 0} \frac{f(x_0 + \alpha d) - f(x_0)}{\alpha} = 3 > 0$$

We conclude that $\boxed{d = \binom{2}{-1}\ is\ an\ increase\ direction.}$

Otherwise, $\forall x \in \mathbb{R}^2, \nabla f(x) = \binom{2x_1+x_2}{x_1-8x_2}$,

$$So\ \nabla f(x_0) = \begin{pmatrix} 3 \\ -7 \end{pmatrix}$$

$$Since\ \frac{\partial}{\partial d} f(x_0) = \nabla f(x_0)^T d = (3 \quad 7) \begin{pmatrix} 2 \\ -1 \end{pmatrix} = 13 > 0$$

$$We\ conclude\ that\ d\ is\ increase\ direction\ too.$$

1.4.

$$\nabla f(x) = \begin{pmatrix} 6x_1 - 2x_2 - 10 \\ 2x_1 + 6x_2 - 2 \end{pmatrix}$$

$$\nabla f(x^*) = 0\ if\ \begin{bmatrix} 6 & -2 \\ -2 & 6 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 10 \\ 2 \end{pmatrix}$$

$$Since\ \det\left(\begin{bmatrix} 6 & -2 \\ -2 & 6 \end{bmatrix}\right) = 32$$

So the problem is solvable and it exists one and only one solution:

$$x^* = \begin{pmatrix} 2 \\ 1 \end{pmatrix}\ works.$$

Moreover, $D^2 f(x^*) = \begin{bmatrix} 6 & -2 \\ -2 & 6 \end{bmatrix}$ $which\ is\ symmetric\ matrix$

According to Sylvester's Criterion, since $\Delta_1 = 6 > 0$ and $\Delta_2 = 32 > 0$

$$D^2 f(x^*)\ is\ positive\ definite.$$

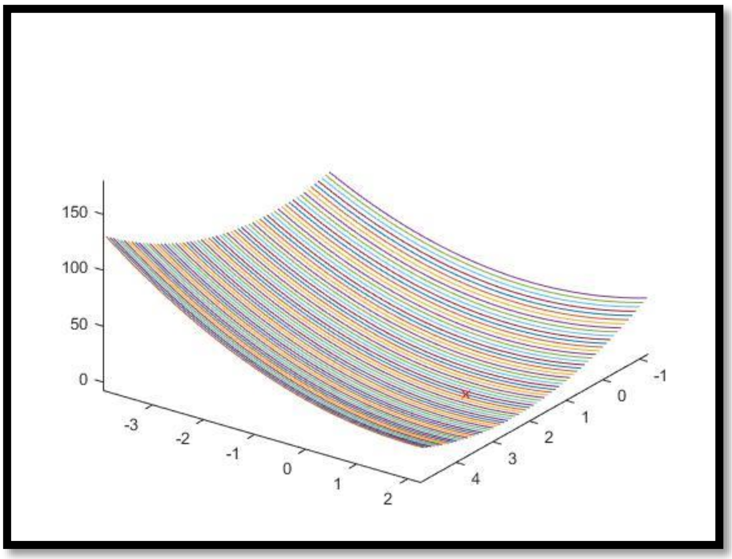
So $\boxed{x^* = \binom{2}{1}\ is\ the\ minimum\ for\ the\ programming\ problem}$

In order to verify my answer, I plot the graph of the cost function in the domain:

$$-5 \leq x_1 \leq 5$$
$$-5 \leq x_2 \leq 5$$

And it gives on Matlab script:

```
x1=[-5:0.1:5];
x2=x1;
[X,Y]=meshgrid(x1,x2);
Z=(Y+X-3).^2+2*(Y-X+1).^2;
plot3(X,Y,Z)
hold on
plot3(2,1,0,'rx')
```



Visually it gives us the the same solution:

$$x^* = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \; works.$$

And it reaches a minimum $f(x^*) = 0$

Part 2.

2.1.

I want to solve this problem:

$$min_x \; f(x) = x4 + 4x3 + 9x2 + 6x + 6$$
$$s.t. x \in [-2, 2]$$

So I write 2 Matlab scripts.

The first, *fgolden.m*, defines the cost function:

```
function [y] = fgolden(x)
y = (x^4)+4*(x^3)+9*(x^2)+6*x+6
```

The second, *Goldensection_part2_2_1*, use Golden section method with a tolerance $\varepsilon = 10^{-2}$.

```
a0=-2;
b0=2;
ro=(3-sqrt(5))/2;
e=b0-a0;
while e>(10^(-2))
   a1= ro*(b0-a0)+a0;
   b1= ro*(b0-a1)+a1;
   y=fgolden(b1);
   yb=y;
   y=fgolden(a1);
   ya=y;
```

```
    if yb>ya
        b0=b1;
    else
        a0=a1;
    end
    e=b0-a0
end
fprintf('a1 = %.2f\n',a1);
```

So it gaves me this result:

$$x^* = -0.45 \; with \; a \; precision \; of \; e = 0.0077$$

2.2.

2.2.1.

I want to solve this problem with 2 different methods:

$$min_x \, f(x) = 2x^4 - 5x^3 + 100x^2 + 30x - 75$$
$$s.t. \; x \in \mathbb{R}$$

So I write 5 Matlab scripts.

The 3 first (*fnewtsec.m, derivefnewtsec.m, derive2fnewtsec.m*) define the cost function and its first derivative and its second derivative:

```
function [y] = fnewtsec(x)
y = 2*(x^4)-5*(x^3)+100*(x^2)+30*x-75
```

```
function [y] = derivefnewtsec(x)
y = 8*(x^3)-15*(x^2)+200*x+30
```

```
function [y] = derive2fnewtsec(x)
y = 24*(x^2)-30*x+200
```

And the newton methods is then used in this routine *Newton_part2_2_1.m*:

```
x0=2;
y=derivefnewtsec(x0);
e=y;
while e>(10^(-4))
    y=derivefnewtsec(x0);
    df=y;
    y=derive2fnewtsec(x0);
    df2=y;
    xk=x0-df/df2;
    y=derivefnewtsec(xk);
    e=y;
    x0=xk;
end
fprintf('xk = %.2f\n',xk);
y=fnewtsec(xk);
```

Which give me this result:

$$x^* = -0.15$$

$$for\ f(x^*) = -77.2279$$

2.2.2

Secondly, the secant method is written in this routine:

```
x0=2;
xm1=2.1;
y=derivefnewtsec(x0);
e=y;
while e>(10^(-4))
   y=derivefnewtsec(x0);
   df=y;
   y=derivefnewtsec(xm1);
   dfm1=y;
   xk=x0-(x0-xm1)/(df-dfm1)*df;
   y=derivefnewtsec(xk);
   e=y;
   xm1=x0;
   x0=xk;
end
fprintf('xk = %.2f\n',xk);
y=fnewtsec(xk);
```

Which give me this result:

$$x^* = -0.15$$

$$for\ f(x^*) = -77.2324$$

Remarks: With a difference of 0.01 on the minimum value, we can say that the secant method is good alternative if the function f is not twice differentiable.

Part 3.

3.1.

$$min_x\ f(x) = 1 + 2x_1 e^{-x_1{}^2 - x_2{}^2}$$
$$s.t.\ x \in \mathbb{R}^2$$

$$\nabla f = \begin{pmatrix} 2exp(-x_1{}^2 - x_2{}^2)[1 - 2x_1{}^2] \\ -4x_2 x_1 exp(-x_1{}^2 - x_2{}^2) \end{pmatrix}$$

The MATLAB routine to solve this problem is:

```
x1 = x0-alpha(x0(1), x0(2))*[df1(x0(1), x0(2)) df2(x0(1), x0(2))];
ex = abs(df1(x1(1), x1 (2))^2 + df2(x1(1), x1(2))^2);
while (ex>=tol)
        x0 = x1 ;
        x1 = x0-alpha(x0(1) , x0(2))*[df1(x0(1), x0(2)) df2 (x0(1), x0(2))];
        ex = abs(df1(x1(1), x1(2))^2 + df2(x1(1), x1(2))^2 );
end
disp(x1) ;
```

3.2.

We check that the vector given is a global minimizer of the function.

$$\nabla f = \begin{pmatrix} 202x_1 - 200x_2 - 2 \\ 200{x_2}^2 - 200x_1 \end{pmatrix}$$

$$\nabla f \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$$

$$D^2 f(x) = \begin{pmatrix} 202 & -200 \\ -200 & 400x_2 \end{pmatrix}$$

$$D^2 f \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 202 & -200 \\ -200 & 400 \end{pmatrix}$$

The eigenvalues are $x_1 \approx 524 \qquad x_2 \approx 77$

which are both >0.

The SOSC are satisfied, so it is a global minimizer.

Is it the only one ?
Let a,b ∈ $\mathbb{R}$, if (a b) is a minimizer, we have to solve :

$$202a - 200b - 2 = 0 \text{ and } 200b^2 - 200a = 0$$

so $b = 1$ and $a = 1$ OR $b = -\frac{1}{101}$ and $a = b^2$

but $f\begin{pmatrix} a \\ b \end{pmatrix} \approx 1 > 0$

So, $\binom{1}{1}$ is the only global minimizer of the function f.

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3.3.

We calculate the conditions numbers of the different Q given. The smaller this number is, the faster the algorithm converges.
We found that $Q = \lambda I$ is the case that converges faster (the condition number is equal to 1).

Then, the case $Q = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ (the condition number is equal to 2).

Finally, the case $Q = \begin{pmatrix} 10 & 0 \\ 0 & 1 \end{pmatrix}$ is the slowest (the condition number is equal to 10).