Practice 1. Introduction to Matlab

1. What is Matlab?

The first version of Matlab dates from the 70's, and was designed as a support tool for courses of *Theory of Matrices*, *Linear Algebra* and *Numerical Analysis*. The name Matlab is an acronym: " Matrix Laboratory." Today, Matlab is a very powerful program, with a manageable environment, which includes scientific tools, technical calculations, graphical display and a high level programming language.

2. Elementary operations and variables

The way of representing numbers and of operating in Matlab is the same as in pocket calculators. For example:

`3`　`-99`　`.001`　`9.63`　`1.62e-20`

Note that the decimal separator (point) is used instead of a comma. The usual operations are performed with the same symbols and in the same sequence as in calculators. Note that e-20 means 10^{-20}.

addition	subtraction	multiplication	division	power
a+b	a-b	a*b	a/b	a^b

In order to execute a command, you must press the key **Intro** ↩ . For example, to calculate the value of $3 + 5 \times 2 + 1$, the following instruction is carried out

```
>> 3+5*2+1
```

and the following answer (ans) is obtained

```
ans =
14
```

This means that the result is saved in the variable `ans`. In contrast,

```
>> s=(3+5)*2+1
```

indicates to Matlab that the result of this operation is to be saved in the variable `s`. Notice the difference with the previous case, the parentheses alter the value.

2.1. Rules to name variables

- The variable name can have up to 63 characters (31 in previous versions), which can be letters, numbers and the hyphen to underscore (_).
- The first character must be a letter. `lado2` is a valid name, but `2lado` is not.

- Matlab is case sensitive: the variable `Base` is different from the variable `base`.
- A variable name cannot have blank spaces: `side1` is valid, but `side 1` is not.
- There are names that should be avoided because they have their own meaning in MATLAB: `ans`, `pi`, `Inf`, ...

2.2. Punctuation marks and cursor movements

- You can define multiple variables in one line if separated by commas. For example

```
>> base = 2, height = 3, area = base * height
```

- The variables can also be separated by semicolons. In that case the command is carried out but the result does not appear in the command window:

```
>> base = 5; height = 2; area = base * height
```

- The percent symbol `%` is used to write comments in the same line. The comments do not change the results and are usually written in scripts (programs) to remind us of what we are doing.
- The up and down arrow keys allow us to recover previous commands. The left and right arrow keys allow us to move to left and right in the same line without modifications. The Inicio and Fin keys can also be used for moving to the beginning and end of a line respectively.

3. The desktop

The window of MATLAB displays a desktop divided into several parts:

- **Command Window** is the window where you enter commands
- This window can be toggled with the window **Workspace** which provides information about the current variables.
- You can also see the window **Current Folder** that displays the contents of the current directory.
- All the commands are recorded in the **Command History**.

If we want to clear the command window (**Command Window**) we can do this by using `clc`. It should be borne in mind that this does not affect variables that are already in use.

If you double-click a line of **Command History** the line is carried out in the command window. If we click and drag the line with the mouse, we can correct it before it runs.

The command **whos** allows us to see the different variables and their corresponding class and number of bytes.

4. How to find help

The commands `doc` and `helpwin` or `help` are used to get information about a given subject. For instance,

```
>> doc ans
>> help ans
```

provide information about `ans`.

If you do not know the exact command on which you wish further information, you can simply type `helpwin` to open a help window **Help** in which a list of topics, an index of terms and a word search appear, among other things. You can also write `help sqrt` to obtain information about `sqrt`, the square root, (or any other function) directly on the command window. Try `help elfun` to obtain a list of elementary functions. Note that `exp(1)` is the number e=2.71828... and any exponential can be written in the same way, not as $e^\wedge$ (the argument). The natural logarithm is `log`. For instance

```
>> exp(1)
```

```
>> ans=2.7183
```

```
>> log(ans)
```

```
>> ans=1
```

```
>> log10(100)
```

```
>> ans=2
```

5. Formats

When MATLAB presents the results, it chooses a default format with a maximum of 3 digits for the integer part and 4 for the decimal part. If the number needs to be shown with more digits, Matlab uses the *exponential notation*. This is the option *short* of the command `format`. For example:

```
>> format short
>> pi
ans =
3.1416
```

Try

```
>> 10*pi
>> 100*pi
>> 1000*pi
```

Note that the letter **e** between two numbers means that the first number is multiplied by 10 raised to the second:

```
>> 2e3
>> 2e-3
```

The most used formats are:

`format short`	3 digits is the maximum for the integer part and 4 for the decimal part; if the number is greater it is converted to the exponential notation.
`format long`	2 digits for the integer part and 15 for the decimal; if the number is greater it is converted to the exponential notations.

You can consult other forms of presentation of results with `doc format` . Regardless of the format in which a calculation is shown on screen, the computer performs all calculations with 16 significant figures (double precision).

vpa(x) uses variable-precision floating-point arithmetic (VPA) to evaluate each element of the symbolic input x to at least d significant digits, where d is the value of the digits function. The default value of digits is 32. For instance

```
>> vpa(pi,20)
ans=3.1415926535897932385
```

one integer (3) plus 19 decimal digits.

6. Some mathematical functions

MATLAB has a wide range of functions — a complete list can be obtained with the command `doc elfun` — corresponding to the most used mathematical functions. Examples of these functions are:

Scientific Notation	Name in MATLAB	Meaning
$\|x\|$	`abs(x)`	Absolute value of x
$\sin x$	`sin(x)`	sine of x
$\cos x$	`cos(x)`	cosine of x
$\tan x$	`tan(x)`	tangent of x
$\sin^{-1} x$	`asin(x)`	arcsine of x
$\cos^{-1} x$	`acos(x)`	arccosine of x
$\tan^{-1} x$	`atan(x)`	arctangent of x
e^x	`exp(x)`	exponential of x
$\ln x$	`log(x)`	logarithm with base e of x
$\sqrt{x}$	`sqrt(x)`	square root of x

In the trigonometric functions of this list, the angle is always expressed in radians. It is possible to express the angle in degrees (See for example the command `sind`).

For instance, to calculate $sin(\pi/2)$, $sin(\pi)$, e^4 and $\sqrt{2}$ we write:

```
>> sin(pi/2)
>> sin(pi)
>> exp(4)
>> sqrt(2)
```

Why is sin(pi) different from zero ? Try sin(sym(pi)).

7. Symbolic expressions

The capabilities of MATLAB can be expanded by installing various modules (*toolboxes*). One, called SYMBOLIC MATH TOOLBOX, allows symbolic computation, i.e. enables us to manipulate variables without using their numerical approximations.

To use the symbolic calculation module SYMBOLIC MATH TOOLBOX is necessary to establish *symbolic objects* representing the symbolic variables. For *misuse of language* , the *symbolic objects* of MATLAB are also called *symbolic variables*.

Among others, the module SYMBOLIC MATH TOOLBOX allows you to perform the following tasks:

`syms x y z`	Creates the symbolic variables `x`, `y`, `z`.
`solve(Expr)`	Calculates the *zeros* of `Expr`.
`solve(Expr,z)`	Calculates the values of `z` that make `Expr` equal to zero.
`subs(S,x,a)`	Substitutes in `S` the variable `a` for `x`.
`pretty(S)`	Presents the expression S in elegant form.
`double(S)`	Calculates the numerical value of a symbolic expression.
`expand(S)`	Expands the expression `S` as the product of its factors.
`factor(S)`	Factorizes, if possible, the expression `S`.
`simplify(S)`	Simplifies a symbolic expression.

7.1. Example

1. Solve the equation $x^3 + 3x^2 - 4 = 0$.

   We have to calculate the *zeros* of $p(x) = x^3 + 3x^2 - 4$.

```
>> syms x
>> p = x^3+3*x^2-4;
>> solve(p,x)
```

2. Use `factor` to factorize the polynomial p of the previous exercise.

```
>> factor(p)
```

   Did you expect this result?

3. Calculate the value of the polynomial p at $x = \sqrt{2}$.

```
>> s1 = subs(p,x,sqrt(2))
```

   However, if we want a numerical expression for the answer, we have to use the command `double`, that writes the result with double precision

```
>> double(s1)
ans =

   4.828427124746190
```

4. We can use a `function handle` to represent a function. It has some advantages and enables us to manipulate the functions in more general ways. Compare with the previous example

```
>> clear
>> syms x
>> p = @(x) x^3+3*x^2-4;
>> solve(p,x)
```

```
>> factor(p,x)
```

```
>> s1=p(sqrt(2))
```

Now we could try to locate the zero of this function in a given interval. For example in $[0\,,2]$

```
>> fzero(p,[0 2])

ans=1
```

However, what happens in $[-3,-1]$?

7.2. Example

Solve the quadratic equation

$$ax^2 + bx + c = 0, \quad \text{with } a, b, c \in \mathbb{R}.$$

```
>> syms x a b c
>> E = a*x^2+b*x+c
>> s=solve(E,x)       % produces the two solutions
>> pretty(s)          % shows the expression in a clearer way
```

Note the importance of making clear what is the variable that should be isolated:

```
>> s=solve(E,b)       % Take care. b is isolated by this command
```

7.3. Example

Expand the polynomial $p(x) = 2(x-1) - 2(x-1)^2 + (x-1)^3$ in powers of x. Factorize the polynomial and find its roots.

```
>> syms x
>> p = 2*(x-1) - 2*(x-1)^2 + (x-1)^3
>> pretty(p)
>> p = expand(p)
>> factor(p)
>> solve(p)
   ans =
      1
      2+i
      2-i
```

This means that p has a real root 1 and the complex roots $2+i$ y $2-i$ (they are conjugate).

7.4. Example

Given $f(x) = x^2 + 1$ and $g(x) = x^3 + 2x - 3$. Calculate $f + g$, $f - g$, $f \cdot g$, f/g, f^{-1}, $g \circ f$ y $f \circ g$.

```
>> syms x
>> f=x^2+1;
>> g=x^3+2*x-3;
>> f+g
>> f-g
>> f*g
>> f/g
>> finverse(f) % calculates the inverse function of f
>> compose(g,f) % calculates g composed with f
>> compose(f,g) % calculates f composed with g
```

Note that when we calculate the inverse of f, we only obtain the root with positive sign (f is not one-to-one). You cannot carry out these operations for function handles.

7.5. Example

Given $f = xe^{x^2-1}$. Calculate $f(2)$, $f(-5)$ y $f(2) \times f(-5)$.

We can do the example using symbolic variables (left) or numerical variables (right)

```
>> syms x
>> f=x*exp(x^2-1);
>> a=subs(f,x,2)
>> b=subs(f,x,-5)
>> a*b
```

```
>> x=2;
>> a=x*exp(x^2-1)
>> x=-5;
>> b=x*exp(x^2-1)
>> a*b
```

8. Exercises

1. Do the following operations:

   a) $\dfrac{10000}{400 + 6 \cdot 500}$ *Sol:* 2.9412

   b) $270^{\frac{1}{3}}(690 + 876)$ *Sol:* 10121.53

   c) $\dfrac{500\,(645 + 7843)}{45 + 9}$ *Sol:* 78592.59

   d) $\dfrac{21 + 78}{43^{\frac{1}{2}} + 80^3}$ *Sol:* 0.00019336

2. Calculate the value of these functions at the given points:

   a) $f(x) = \dfrac{x^2}{6x + x^3}$ at $x = 1$ and $x = -0.5$ *Sol:* $0.1429,\ -0.0800$

   b) $f(x) = \dfrac{\sin x}{1 + \cos x}$ at $x = 2$ and $x = 2^\circ$ (2 degrees). *Sol:* $1.5574,\ 0.0175$

   c) $f(x) = \ln|x + 2|$ at $x = 4$, $x = -2$ and $x = -10$. *Sol:* $1.7918, NaN,\ 2.0794$

   d) $f(x) = \dfrac{e^x}{e^{2x+1}}$ at $x = 5$ and $x = -2$. *Sol:* $0.0025,\ 2.7183$

3. Solve the equations:

   a) $x^3 - 3x + 2 = 0$ *Sol:* $1, 1, -2$

b) $x^4 - 2x + 1 = 0$ *Sol:* $1, 0.5437, -0.7718 + 1.1151i, -0.7718 - 1.1151i$

c) $\ln(x^2 - 1) = 1$ *Sol:* $-1.9283, 1.9283$

d) $\sin x = -1$ *Sol:* $\{-\pi/2 + 2k\pi; k \in \mathbb{Z}\}$

4. Let $f(x) = x \sin x$, $g(x) = x^2 - 1$ y $h(x) = e^{x+3}$. Calculate:

a) $h \circ g \circ f$ *Sol:* $e^{x^2 \sin^2 x + 2}$

b) $f \circ g \circ h$ *Sol:* $(e^{2(x+3)} - 1) \sin(e^{2(x+3)} - 1)$

c) $h^{-1} \circ h$ *Sol:* x

d) $(f + g) \circ h$ *Sol:* $e^{(x+3)} \sin e^{(x+3)} + e^{2(x+3)} - 1$

e) $f \circ (g + h)$ *Sol:* $(x^2 - 1 + e^{x+3}) \sin(x^2 - 1 + e^{x+3})$

f) $f(2) \times g(3)$ *Sol:* 14.5488

g) $(f(1) + g(4)) \times h(4)$ *Sol:* 17372.28

Practice 2. Functions of a real variable: Graphing, limits and continuity

1. Vectors

One of the most remarkable aspects of MATLAB is the way it manipulates and operates on vectors and matrices.

1.1. Row vectors

- In general, vectors are introduced writing between brackets each of their coordinates separated by a space or a comma. For example:

```
>> v=[4 -6 5]
>> v=[4,-6,5]
```

- You can also enter them by specifying the value of each coordinate in the desired order:

```
>> w(2)=-6, w(1)=4, w(3)=5
>> w
```

- Other commands for particular cases :

`v=[a:h:b]`	Defines a row vector whose first coordinate is `a` and the other coordinates are obtained by adding `h` successively without exceeding `b`. If not specified `h` takes, by default, the value 1. It can also be written without the brackets.
`v=linspace(a,b,n)`	Defines a row vector of `n` coordinates, whose first coordinate is `a` and whose last coordinate is `b`, with a constant difference between consecutive coordinates. If not specified `n` takes, by default, the value 100.

```
>> u=linspace(-4,7,6)
>> u=linspace(1,3,5)
>> u=linspace(1,100)
>> v=[-4:2:7]
>> v=-4:2:7   %  it can be written without the brackets
>>v=linspace(-4,7,6)
>> v=1:10
```

Notice the difference between $v = [-4 : 2 : 7]$ and linspace(-4,7,6), the former command uses a step of 2, produces six elements and finishes at 6, the latter finishes at 7 and also creates six elements, but the step is $(7 + 4)/5 = 2.2$

1.2. Column Vectors

In general, they are entered as row vectors, but separating the elements with a semicolon, we can enter them as column vectors:

```
>> u=[0;1;-5]
>> w=[1 2 3 4]'
```

Besides, the command `u'` calculates the transpose of a given vector `u` (as well as the transpose of any matrix).

1.3. Operations with vectors

Let v and u be vectors with n coordinates and α a scalar:

α*v	multiplies α by v
v/α	multiplies $1/\alpha$ by v
u+v	sums u and v
u-v	subtracts v from v

```
>> u=[1,5,6];
>> v=[2,6,-1];
>> u+v
>> 3*u
>> u/2
>> u-v
```

In addition to the mathematical operations with vectors the program MATLAB allows other *operations* that are performed coordinate by coordinate .

u.*v	multiplies u by v coordinate by coordinate
u./v	divides u by v coordinate by coordinate
u.^v	raises u to v coordinate by coordinate
α+v	sums α to each coordinate of v
v-α	subtracts α from each coordinate of v
v.^α	raises each coordinate of v to α
α.^v	raises α to each coordinate of v
α./v	divides α by each coordinate of v

It is important to remember that the "." that appears in some of the previous operations is part of the operator syntax. It indicates that we work with the vectors coordinate by coordinate.

```
>> u=[1,2,1];
>> v=[3,2,0];
>> u.*v
>> u.^v
>> v.^u
>> u.^2
>> 2.^u
>> u+3
>> u-5
>> 1./u
```

Many of the functions defined in MATLAB can be applied directly to vectors (coordinate by coordinate) and the result is again a vector.

```
>> u=[1,0,2];
>> exp(u)
>> cos(u)
>> sqrt(u)
```

2. Plotting Graphs

2.1. The plot command

The command `plot` is one of the basic commands of MATLAB to draw the graph of a function.

- `plot(x,y)`: draws the set of points (x_i, y_i), where x and y are row vectors with the same number of coordinates. To draw the graph of $y = f(x)$ we must specify the points $(x_i, f(x_i))$.

  For instance, to plot the function $y = x^2$ on the interval $[-3, 3]$ we can type:

  ```
  >> x=[-3:0.01:3];
  >> y=x.^2;
  >> plot(x,y)
  ```

  The graph features (color, line, axis, title, texts, etc.) can be changed using the Toolbar in the graph window.

- `plot(x,y,S)`: means the same as `plot(x,y)` but with the options specified in `S`. The types of line, color and marking are described in `S` in quotes. The codes of lines, colors and markings can be found using `doc plot`.

  To draw $y = x^3 - 1$ in the interval $[0, 5]$ with a dashed red line and asterisks on the points:

  ```
  >> x=[0:0.1:5];
  >> y=x.^3-1;
  >> plot(x,y,'--r*')
  ```

- `plot(x1,y1,S1,x2,y2,S2,...)`: presents on the same axes the graphs defined by the triples `xi, yi, Si`. This command allows you to represent several functions in the same graph.

  For instance, to plot the function cosine in green color and the function sine in red color on $[-\pi, \pi]$:

```
>> x=[-pi:0.01:pi];
>> y1=cos(x);
>> y2=sin(x);
>> plot(x,y1,'g',x,y2,'r')
>> axis([-pi pi -1 1])
```

We have redefined the axes with x in $[-\pi, \pi]$ and y in $[-1, 1]$.

2.1.1. Example

We have the following data of the decay of a radioactive substance:

Time (hours)	Mass (mg)
0	102.9
1	75.8
2	56.1
3	42.2
4	31.1
5	23.6

Verify graphically that the exponential curve $y = 102.04e^{-0.29x}$ fits roughly the previous data. *Solution.*- We represent simultaneously the curve and the points described in the table.(Use `*`, red for points in the table, and light blue for the exponential curve):

```
>> x1=[0:5];
>> y1=[102.9, 75.8, 56.1, 42.2, 31.1, 23.6];
>> x2=[0:0.01:5];
>> y2=102.04*exp(-0.29*x2);
>> plot(x1,y1,'r*',x2,y2,'c')
```

2.2. The command ezplot

The command `ezplot(function,[xmin xmax])` plots the function defined symbolically on the interval `[xmin,xmax]`.

For instance, to plot the function $y = x^2$ on the interval $[-3, 3]$

```
>> syms x
>> ezplot(x^2,[-3,3])
```

You can also use `fplot`. There is a slight difference, can you see it?

2.3. Other important commands

Each time you run the command `plot` and `ezplot`, MATLAB creates a graphical window and removes the previous window. Sometimes it is interesting to draw two or more functions on the same window or have multiple graphical windows:

`hold on`	The graph remains fixed so that subsequent graphing commands are applied on it.
`hold off`	Resets all properties of a graph to their default values
`figure(n)`	Selects the graphic window `Figure n` as the active window; if it does not exist, it is created.
`close all`	Close all the graphic windows.
`grid`	To use a grid on the plot, `grid on` to create it, `grid off` to remove it.

Besides, we can modify the appearance and scaling of the axes of a graph with the command `axis` and its options . For example (the list is not exhaustive)

`axis([x1 x2 y1 y2])`	Determines the visible intervals for the axis OX and OY.
`axis equal`	Sets the same scale for each axis.

2.4. Example

Represent graphically on the interval $[-5,5]$ the function

$$f(x) = \begin{cases} \dfrac{2x^2+3}{5} & \text{si} \quad x \le 1, \\ 6-5x & \text{si} \quad 1 < x < 3, \\ x-3 & \text{si} \quad x \ge 3. \end{cases}$$

Solution.- We will solve it with `plot` on the left and with `ezplot` on the right. We plot simultaneously with `plot` the three sections of the function on the intervals $[-5,1]$, $[1,3]$ and $[3,5]$

```
>> x1=[-5:0.01:1];
>> x2=[1:0.01:3];
>> x3=[3:0.01:5];
>> y1=(2*x1.^2+3)/5;
>> y2=6-5*x2;
>> y3=x3-3;
>> plot(x1,y1,x2,y2,x3,y3)
```

```
>> syms x
>> f1=(2*x^2+3)/5;
>> f2=6-5*x;
>> f3=x-3;
>> figure(2)
>> ezplot(f1,[-5,1])
>> hold on
>> ezplot(f2,[1,3])
>> ezplot(f3,[3,5])
>> axis([-5,5,-10 12])
```

We mus redefine the axes-the x-axis from -5 to 5 and the y-axis approximately from the minimum to the maximum of the function. Of course, we can edit the graph by inserting labels and legends. You can avoid this redefinition by using fplot from the beginning. Close the figures con `close all`.

2.5. Example

Plot the function

$$f(x) = \frac{1}{1+2^{1/x}}, \quad x \in [-1,1], \; x \neq 0.$$

- We can use the command `ezplot` taking the interval $[-1,1]$:

```
>> syms x
>> f=1/(1+2^(1/x))
>> pretty(f)
>> ezplot(f,[-1 1])
```

Is the graph correct? What happens at $x = 0$?

- We plot it on $[-1, 0)$ y en $(0, 1]$, entering the following commands:

```
>> figure(2)
>> ezplot(f,[-1 0])
>> hold on
>> ezplot(f,[0 1])
```

Do you get the expected result? Compare the intervals of the axes of the graph with those of the previous section. Do they agree? If they do not, modify them to obtain a similar graph by adding the command

```
>> axis([-1,1,-0.2,1.2])
```

2.6. Example

Let $f(x) = e^{-x}$ and $g(x) = x^2$. Calculate and show in different windows and on the interval $[-3, 3]$ the functions $f \circ g$ and $g \circ f$.

```
>> syms x
>> f=exp(-x)
>> g=x^2
>> h1=compose(g,f)
>> h2=compose(f,g)
>> ezplot(h1,[-3,3])
>> figure(2)
>> ezplot(h2,[-3,3])
```

3. Limits and continuity

As outlined below, the symbolic computation module Symbolic Math Toolbox provides Matlab with the ability to perform some of the fundamental operations of mathematical analysis of functions of one variable. We begin with the calculation of limits and the analysis of the continuity of a function.

`limit(f,x,a)`	Calculates the limit of f as x approaches a: $\lim\limits_{x\to a} f(x)$. If the variable is not specified, its default value is x. If a is not specified, its default value is 0
`limit(f,x,a,'right')`	Calculates the limit from the right : $\lim\limits_{x\to a^+} f(x)$
`limit(f,x,a,'left')`	Calculates the limit from the left : $\lim\limits_{x\to a^-} f(x)$
`limit(f,x,inf)`	Calculates the limit of f at *plus infinity*: $\lim\limits_{x\to+\infty} f(x)$
`limit(f,x,-inf)`	Calculates the limit of f at *minus infinity*: $\lim\limits_{x\to-\infty} f(x)$

For instance, calculate: $\lim\limits_{x\to 0}\dfrac{\sin x}{x}$, $\lim\limits_{x\to 2}\dfrac{x-2}{x^2-4}$, $\lim\limits_{x\to 0}\dfrac{1}{x}$, $\lim\limits_{x\to 0^+}\dfrac{1}{x}$, $\lim\limits_{x\to 0^-}\dfrac{1}{x}$, $\lim\limits_{x\to +\infty}\dfrac{1}{x}$, $\lim\limits_{x\to -\infty}\dfrac{1}{x}$

```
>> syms x
>> limit(sin(x)/x,x,0)
>> limit((x-2)/(x^2-4),x,2)
>> limit(1/x,x,0)
>> limit(1/x,x,0,'right')
>> limit(1/x,x,0,'left')
>> limit(1/x,x,+inf)
>> limit(1/x,x,-inf)
```

3.1. Example

Calculate the following limits

$$\lim_{x\to 0}\frac{\sin 5x}{\ln(1+4x)} \qquad \lim_{x\to 0}\frac{\ln(1+\sin 4x)}{e^{\sin 5x}-1} \qquad \lim_{x\to 0}(\cos x)^{\frac{1}{\sin x}}$$

```
>> syms x
>> f=sin(5*x)/log(1+4*x);pretty(f)
>> limit(f,x,0)
>> g=log(1+sin(4*x))/(exp(sin(5*x))-1);pretty(g)
>> limit(g,x,0)
>> h=cos(x)^(1/sin(x));pretty(h)
>> limit(h,x,0)
```

3.2. Example

Analyze the continuity of the function

$$f(x)=\begin{cases} \dfrac{2x^2+3}{5} & \text{si} \quad x\le 1,\\ 6-5x & \text{si} \quad 1<x<3,\\ x-3 & \text{si} \quad x\ge 3. \end{cases}$$

Solution.- On the open intervals: $(-\infty,1)$, $(1,3)$ y $(3,+\infty)$ it is continuous because it is the result of basic operations with continuous functions. Now we have to study what happens at the points $x=1$ and $x=3$.

Analysis at the point $x=1$:

```
>> syms x
>> f1=(2*x^2+3)/5;
>> limit(f1,x,1,'left')
ans =
1
>> f2=6-5*x;
>> limit(f2,x,1,'right')
ans =
1
>> subs(f1,x,1)
ans =
     1
```

Since $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$, f is continuous at $x = 1$.

Study at the point $x = 3$:

```
>> limit(f2,x,3,'left')
ans =
-9
>> f3=x-3;
>> limit(f3,x,3,'right')
ans =
0
```

Since $\lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$, $\lim_{x \to 3} f(x)$ does not exist, therefore f is not continuous at $x = 3$.

Thus, the function f is continuous on $\mathbb{R} - \{3\}$.

4. Exercises

1. Plot the functions:

   a) $f(x) = \dfrac{x^2+2}{x-3}$ *b)* $f(x) = \sqrt{x^2+2}$

   c) $f(x) = x^2 e^{-x}$ *d)* $f(x) = \dfrac{\ln x^2}{x}$

2. Study the continuity of the following function and graph it on the interval $[-5, 5]$.

$$f(x) = \begin{cases} -2x+1 & \text{si} \quad x \leq -1 \\ x^2 & \text{si} \quad -1 < x < 0 \\ \sin x & \text{si} \quad x \geq 0 \end{cases}$$

Sol: f is continuous on $\mathbb{R} - \{-1\}$.

3. Calculate the following limits:

   a) $\lim_{x \to 0} \dfrac{e^{-x}-1}{x}$ *b)* $\lim_{x \to 0} (\cos x)^{\frac{1}{x}}$ *c)* $\lim_{x \to +\infty} \left(\dfrac{2x+3}{2x+1}\right)^{x+1}$ *d)* $\lim_{x \to \pi/2} e^{\tan x}$

Sol.: *a)* -1, *b)* 1, *c)* e, *d)* it does not exist, infinity from the left, zero from the right.

Practice 3. Functions of a real variable: Derivatives

1. Calculation of derivatives

To calculate the *derivative function* of a function, MATLAB has the command `diff`:

`diff(f,x)`	Calculates the derivative of the symbolic expression f with respect to the variable x. If the variable is not specified, MATLAB will choose one by default.
`diff(f,x,n)`	Calculates the n-order derivative of f with respect to the variable x.

For example: (1) For $f(x) = x^2$, calculate $f'(x)$, that is, $\dfrac{df}{dx}$.

(2) For $z = \cos(xy)$, calculate $\dfrac{dz}{dx}$ and $\dfrac{dz}{dy}$.

(3) For $g(x) = x \sin x$, calculate $g'''(x)$.

(4) For $h(x) = \ln x$, calculate $h''(x)$.

(5) For $f(x) = \dfrac{1}{x}$, calculate $f'(x)$, $f''(x)$ y $f'''(x)$.

```
>> syms x y
>> diff(x^2,x)
>> diff(cos(x*y),x)
>> diff(cos(x*y),y)
>> diff(x*sin(x),x,3)
>> diff(log(x),x,2)
>> f=1/x
>> diff(f,x), diff(f,x,2), diff(f,x,3)
```

1.1. Example

Plot the graphs of the functions

$$f(x) = \frac{x}{1+x^2},$$

of its derivative and of its second derivative in the interval $[-3, 3]$ (in the same graph). Notice the relation between the extrema and the inflection points in the graphs of the three functions.

Solution.- First, we calculate the first and the second derivative of f.

```
>> syms x
>> f=x/(1+x^2); pretty(f)
>> df=diff(f,x)
>> pretty(df)
>> simplify(df)
>> pretty(ans)
>> d2f=diff(f,x,2), pretty(d2f)
>> d2f=simplify(d2f), pretty(d2f)
```

We plot the function and its derivatives on the interval $[-3, 3]$. After having graphed the three functions, we could change the color of the different lines by using the buttons on the graphic window. In order to distinguish the three functions, we can add a legend with the command `legend` .

```
>> ezplot(f,[-3,3])
>> hold  on
>> ezplot(df,[-3,3])
>> ezplot(d2f,[-3,3])
>> grid on
>> legend('f','df','d2f')
```

1.2. Example

Identify the intervals on which the following function is increasing and on which it is decreasing. Analyze its concavity

$$f(x) = \frac{x^2 - 4}{x^3}.$$

Calculate its asymptotes and the intercepts with the axes.

Solution.- We calculate the first derivative of f and the points at which this derivative is zero (the critical points):

```
>> syms x
>> f=(x^2-4)/x^3; pretty(f)
>> df=diff(f)
>>crit=solve(df,x)
crit =
-2*3^(1/2)
 2*3^(1/2)
```

Thus, the monotonicity intervals of the function are $(-\infty, -2\sqrt{3})$, $(-2\sqrt{3}, 0)$, $(0, 2\sqrt{3})$, $(2\sqrt{3}, +\infty)$ (Note that the function is not defined at 0). We study the sign of the first derivative on the monotonicity intervals. To evaluate a symbolic expression at a point we use the command `subs`:

```
>> subs(df,x,-4)
>> subs(df,x,-1)
```

We obtain $f'(-4) = -0.0156 < 0$ and $f'(-1) = 11 > 0$, therefore, the function is decreasing on $(-\infty, -2\sqrt{3})$ and increasing on $(-2\sqrt{3}, 0)$, besides, since f is odd, it is increasing on $(0, 2\sqrt{3})$ and decreasing on $(2\sqrt{3}, +\infty)$. In this case, the function is odd and by studying its monotonicity on the negative semiaxis, we can deduce its behaviour on the positive semiaxis. If there were no symmetries, we should carry out similar calculations for the positive semiaxis. We see that the function has a local minimum at the first critical point and a local maximum at the second. Why ?

We obtain the points at which the second derivative is zero (possible inflection points):

```
>> d2f=diff(f,2)
>> infl=solve(d2f,x)
infl =
2*6^(1/2)
-2*6^(1/2)
```

Thus, The concavity intervals are $(-\infty, -2\sqrt{6})$, $(-2\sqrt{6}, 0)$, $(0, 2\sqrt{6})$, $(2\sqrt{6}, +\infty)$. We study the sign of the second derivative on these intervals:

```
>> subs(d2f,x,-5)
>> subs(d2f,x,-1)
```

We obtain that $f''(-5) = -6.4000 \cdot 10^{-4} < 0$ and $f''(-1) = 46 > 0$, therefore, the function is concave down on $(-\infty, -2\sqrt{6})$ and $(0, 2\sqrt{6})$ and concave up on $(-2\sqrt{6}, 0)$ and $(2\sqrt{6}, +\infty)$ (using again the symmetry of the function). Both points where the second derivative is zero are inflection points in this case. Why ?

Now we graph the function and verify that the results previously obtained agree with the graphic representation.

```
>> ezplot(f,[-8,8])
>> grid on
```

To calculate the asymptotes:

```
>> limit(f,x,0,'right')
ans =
-Inf
>> limit(f,x,0,'left')
ans =
Inf
```

Then, the line $x = 0$ is a vertical asymptote.

To calculate the oblique asymptotes, first we calculate the slope m and, if $m \in \mathbb{R}$, then we calculate n:

```
>> m=limit(f/x,x,inf)
m =
0
>> n=limit(f-m*x,x,inf)
n =
0
```

Therefore, the line $y = 0$ is a horizontal asymptote as x approaches $+\infty$, and by symmetry , it is also a horizontal asymptote as x approaches $-\infty$.

To calculate the intercepts of the graph of f with the axis x, we solve the equation $f(x) = 0$:

```
>> solve(f,x)
ans =
 -2
  2
```

Therefore, it intersects the axis x at points $(2, 0)$ and $(-2, 0)$. Note that there is no intersection with the y axis, because the function is not defined at the point $x = 0$.

1.3. Example

Calculate the point of the line $2x + y = 1$ closest to the point $(2, 1)$.

Solution.- We must calculate the minimum of the euclidean distance on the plane

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

We can ignore the square root, because if the distance (as a function of x and y) has a minimum at a point, its square (squared distance function) also attains a minimum at the same point. Thus, the problem is equivalent to finding a minimum of the function

$$F(x, y) = (x-2)^2 + (y-1)^2,$$

where x e y must satisfy the equation

$$2x + y = 1.$$

Substituting this expression in F we obtain a function of one variable:

$$f(x) = (x-2)^2 + (1-2x-1)^2 = 5x^2 - 4x + 4.$$

We search for the critical points of f:

```
>> syms x y
>> f=(x-2)^2+(y-1)^2
>>f=subs(f,y,1-2*x)
f=(x - 2)^2 + 4*x^2
>> df=diff(f)
df =
10*x-4
>> solve(df)
ans =
2/5
```

We verify that the function has a minimum value at 2/5:

```
>> d2f=diff(f,2)
d2f =
10
```

The second derivative of f is the constant function $f''(x) = 10$. Thus the value of f'' at 2/5 is positive and the function has an absolute minimum at 2/5.

The calculation of the second derivative can be complicated in some cases. To see if a function has a maximum or a minimum value at a point, we can use the criterion of the first derivative. The calculations — that are very simple in this case — can be carried out by using the command `subs`.

```
>> subs(df,x,0)
ans =
-4
>> subs(df,x,3)
ans =
26
```

The function is decreasing on $(-\infty, 2/5)$ and increasing on $(2/5, +\infty)$, then it has an absolute minimum at 2/5.

Now we compute the corresponding value of y:

```
>> y=1-2*2/5
y =
1/5
```

The requested point is $(2/5, 1/5)$.

We check the result graphically. To do this, we calculate the line perpendicular to $2x + y = 1$ that passes through the point $(2, 1)$ and obtain

$$y = x/2.$$

Finally we represent in the same graph the line $2x + y = 1$, its perpendicular $y = x/2$, and the points $(2, 1)$ and $(2/5, 1/5)$.

```
>> ezplot(1-2*x,[-1,4])
>> hold on
>> ezplot(x/2,[-1,4])
>> axis equal
>> axis([-1,4,-1,2])
>> plot(2,1,'r*')      % draws the point (2,1) in red
>> plot(2/5,1/5,'m*') %  draws (2/5,1/5) in magenta
```

We have fixed the same scale for the axes (axes equal) so that both lines look perpendicular.

2. Taylor polynomials

To find the Taylor polynomial of a given function we use the command:

`taylor(f,x,a,'order',n)`	Calculates the Taylor polynomial of $f(x)$ of order $n-1$ at the point a.
`taylortool`	Is an interactive calculator of Taylor polynomials

For example, for the functions $f(x) = e^x$ and $g(x) = \cos x$, we are going to calculate:

(1) the Taylor polynomial of f of order 4 at the point 0, (or the MacLaurin polynomial of f of order 4)

(2) the Taylor polynomial of f of order 6 at the point 0, (or the MacLaurin polynomial of f of order 6)

(3) the Taylor polynomial of f of order 3 at the point 2,

(4) the Taylor polynomial of g of order 4 at the point $\pi/4$,

```
>> syms x
>> f=exp(x), g=cos(x)
>> taylor(f,x,0,'order',5)
>> taylor(f,x,0,'order',7)
>> taylor(f,x,2,'order',4)
>> taylor(g,x,pi/4,'order',5)
```

Note that, in this last case, we obtain the polynomial:

$$T(x) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2 + \frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{48}\left(x - \frac{\pi}{4}\right)^4.$$

2.1. Example

Represent the function $f(x) = \sin x$ and its MacLaurin polynomials of order 1, 3 and 5 on the interval $[-\pi, \pi]$.

Solution.- First we calculate the polynomials:

```
>> syms x
>> f=sin(x);
>> tf1=taylor(f,x,0,'order',2), tf3=taylor(f,x,0,'order',4), tf5=taylor(f,x,0,'order',6)
tf1 =
x
tf3 =
x-1/6*x^3
tf5 =
x-1/6*x^3+1/120*x^5
```

Now we make the graphical representation of f and its Taylor polynomials. Once the functions are plotted, we could change the color, types, etc. of the individual curves. We add a legend to distinguish them.

```
>> ezplot(f,[-pi,pi])
>> hold on
>> ezplot(tf1,[-pi,pi])
>> ezplot(tf3,[-pi,pi])
>> ezplot(tf5,[-pi,pi])
>> grid on
>> legend('f','tf1','tf3','tf5')
```

3. Exercises

1. Find the intervals of monotonicity, extreme points and concavity of the function:

$$f(x) = x + \frac{1}{x-2}$$

   Check the obtained results graphically. Calculate the asymptotes and the intercepts.

   Sol.: f is increasing on $(-\infty, 1)$ and $(3, +\infty)$, and decreasing on $(1, 2)$ and $(2, 3)$, local maximum at $x = 1$, local minimum at $x = 3$. f is concave down on $(-\infty, 2)$ and concave up on $(2, +\infty)$. Asymptotes: $x = 2$, $y = x$. Intercepts: $(1, 0)$, $(0, -1/2)$

2. Let f be a function defined as:

$$f(t) = \frac{e^{\left(\frac{1}{2}-\frac{1}{t}\right)}}{t}, \quad t > 0,$$

   that describes the evolution with time of the concentration of a chemical compound in a certain chemical reaction. Calculate the time at which the concentration is maximum and this maximum value. Study the intervals of monotonicity and concavity of the function. What is the limit of $f(t)$ as t approaches 0 from the right ?. Sketch the graph.

   Sol: The maximum value is attained at $t = 1$ and the maximum concentration is $e^{-\frac{1}{2}}$. The function is increasing on $(0, 1)$ and decreasing on $(1, \infty)$. It is concave up on $(0, 1 - \sqrt{2}/2)$, concave down on $(1 - \sqrt{2}/2, 1 + \sqrt{2}/2)$ and concave up on $(1 + \sqrt{2}/2, \infty)$. The limit is zero.

3. Find the dimensions of the rectangle whose perimeter is 12 meters and has the shortest diagonal.

   Sol: Square whose side is 3 meters.

4. Identify the intervals on which the function $f(x) = \dfrac{1}{x^2+1}$ is increasing and on which it is decreasing. Analyze its concavity. Approximate the function by a parabola at $x = 0$. Plot the function and the parabola in the same graph.

   Sol: The function is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. It is concave up on $(-\infty, -1/\sqrt{3})$, concave down on $(-1/\sqrt{3}, 1/\sqrt{3})$ and concave up again on $(1/\sqrt{3}, \infty)$. The approximating parabola is $y = 1 - x^2$.

5. We consider the function f defined as:

$$f(x) = \begin{cases} x^5 \ln x & \text{if} \quad x \in (0, +\infty) \\ a & \text{if} \quad x = 0 \end{cases}$$

a) Calculate a so that the function is continuous on $[0, +\infty)$.

b) Find the extrema of f in $[0, +\infty)$.

c) Study the monotonicity and concavity of the function. Plot the function.

Sol: a) $a = 0$. b) f has its absolute minimum at $e^{-\frac{1}{5}}$ and its relative maximum at 0. c) f is decreasing on $(0, e^{-\frac{1}{5}})$ and increasing on $(e^{-\frac{1}{5}}, \infty)$, f is concave down on $(0, e^{-\frac{9}{20}})$ and concave up on $(e^{-\frac{9}{20}}, \infty)$

6. Calculate the McLaurin polynomials of order 2, 4, 6 y 8 of the function $f(x) = x \sin x$ and plot them together with the function in the same graph

Sol: $P_2(x) = x^2$

$P_4(x) = x^2 - 1/6x^4$

$P_6(x) = x^2 - 1/6x^4 + 1/120x^6$

$P_8(x) = x^2 - 1/6x^4 + 1/120x^6 - 1/5040x^8$

Practice 4. Functions of a real variable: integration

1. Integral Calculus

To calculate indefinite integrals and definite integrals, Matlab has the command `int`:

`int(f,x)`	Calculates the indefinite integral $\int f(x)\,dx$, without the additive constant, that is, it calculates an antiderivative of f. If the variable is not specified, MATLAB will choose one by default, always giving preference to the variable x.
`int(f,x,a,b)`	Calculates the definite integral $\int_a^b f(x)\,dx$.

For instance, to calculate the following integrals:

$$\int xe^x\,dx\ ,\quad \int_0^1 xe^x\,dx\ ,\quad \int \cos(xy)\,dx\ ,\quad \int \cos(xy)\,dy\ ,\quad \int_a^b \cos x\,dx\ ,\quad \int_1^{+\infty} \frac{dx}{x^2}\ ,\quad \int_1^{+\infty} \frac{dx}{x}$$

```
>> syms x y
>> int(x*exp(x))
>> int(x*exp(x),0,1)
>> int(cos(x*y))
>> int(cos(x*y),y)
>> syms a b
>> int(cos(x),a,b)
>> int(1/x^2,1,inf)
>> int(1/x,1,inf)
```

obtaining the following results:

$$\int xe^x\,dx = e^x(x-1)+c\ ,\quad \int_0^1 xe^x\,dx = 1\ ,\quad \int \cos(xy)\ dx = \frac{\sin(xy)}{y}+c\ ,\quad \int \cos(xy)\ dy = \frac{\sin(xy)}{x}+c$$
$$\int_a^b \cos x\ dx = \sin b - \sin a\ ,\quad \int_1^{+\infty}\frac{dx}{x^2} = 1\ ,\quad \int_1^{+\infty}\frac{dx}{x} = \infty.$$

1.1. Example

Calculate the area of the region bounded by the curve of equation $y = 2x^3$ and the line $y = 8x$, performing the following steps:

1. Calculate the intersection between the curve and the line with the command `solve`.
2. Plot the curve and the line on an interval which contains the intersection points.
3. Calculate the requested area.

Solution.-

1. Calculate the intersection points between the curve and the line, calling f the expression for the curve and g the expression for the line:

```
>> syms x
>> f=2*x^3
>> g=8*x
>> solve(f-g,x)
ans =
-2
 0
 2
```

2. We graph f and the line on the interval $[-2, 2]$ (this interval contains all the roots):

```
>> ezplot(f,[-2,2])
>> hold on
>> ezplot(g,[-2,2])
>> grid on
>> hold off
```

3. In the above graph we can see that f is above the line in $[-2, 0]$ and below the line in $[0, 2]$. To find the area we need to solve the integral

$$\int_{-2}^{2} |f(x) - g(x)| dx.$$

If we solve the exercise with pencil and paper we need to perform two integrals to calculate the area, whose calculation with MATLAB is:

```
>> a1=int(f-g,x,-2,0)
a1 =
8
>> a2=int(g-f,x,0,2)
a2 =
8
>> area=a1+a2
area =
16
```

Since MATLAB has the absolute value function we can calculate the area with a single integral:

```
>> area=int(abs(f-g),x,-2,2)
area =
16
```

4. We can fill the area with a color (red) writing the following commands

```
>> x1=-2:0.01:2;
>> y1=8*x1;
>> x2=2:-0.01:-2;
>> y2=2*x2.^3;
>> x=[x1 x2];
>> y=[y1 y2];
>> patch(x,y,'r')
```

1.2. Example

Find the area of the region enclosed by the graph of the function $f(x) = \dfrac{x^2-1}{x^2+1}$ and its horizontal asymptote, carrying out the following steps:

1. Find the horizontal asymptote, calculating the limits at infinity.
2. Analyze if there exists intersection between f and the horizontal asymptote.
3. Determine graphically the relative position of f and its asymptote.
4. Calculate the requested area.

Solution.-

1. We calculate the horizontal asymptote at $+\infty$ and at $-\infty$ (they could be different):

```
>> syms x
>> f = (x^2-1)/(x^2+1); pretty(f)
>> limit(f,x,inf)
ans =
1
>> limit(f,x,-inf)
ans =
1
```

   The horizontal asymptote (from the left and from the right) is $y = 1$.

2. We study whether the function intersects the asymptote by solving the equation $f(x) = 1$:

```
>> solve(f-1,x)
ans =

Empty sym: 0-by-1
```

   Note that the answer is that there is no solution of the equation $f(x) - 1 = 0$, then the function f does not intersect the asymptote.

3. We graph f and the asymptote, choosing a suitable interval for the function, for example $[-5, 5]$:

```
>> ezplot(f,[-5,5])
>> hold on
>> ezplot('1',[-5,5])
>> grid on
>> hold off
```

4. We plot the domain in blue

```
>> clear
>> x1=-5:0.01:5;
>> x2=5:-0.01:-5;
>> y1=(x1.^2-1)./(x1.^2+1);
>> y2=ones(size(x2));
>> x=[x1 x2];
>> y=[y1 y2];
>> patch(x,y,'b')
```

5. Finally, we calculate the area

```
>> area = int(1-f,x,-inf,inf)
area =
2*pi
```

1.3. Example

Find the area of the region bounded above by the function

$$f(x) = (-x^2 + x + 3)\ln x$$

and bounded below by the x-axis. Represent graphically this region.

Solution.- First, we calculate the intersection points of f with the x-axis

```
>> syms x
>> f=(-x^2+x+3)*log(x);  pretty(f)
>> sol=solve(f,x)
sol =
1
1/2+1/2*13^(1/2)
1/2-1/2*13^(1/2)
>> sol=double(sol)
sol =
1.0000
2.3028
-1.3028
>>sol=sort(sol)
sol =
-1.3028
1.0000
2.3028
```

The root, -1.3028, is not valid since f is not defined for negative values.

Now we represent graphically the function f on the interval $[1, 2.3028]$

```
>> ezplot(f,[sol(2),sol(3)])
>> grid on
```

Since the function is positive on the interval, we calculate the area by performing the integral:

```
>> int(f,x,sol(2),sol(3))
>> double(ans)
ans =
0.8404
```

We can also calculate the volume of the solid generated by rotating this region around the x-axis. The formula we must use is

$$V_x = \pi \int_a^b f^2\,dx$$

where a and b are the endpoints of the region on the x-axis. If the region is rotated around the y-axis, the formula is

$$V_y = 2\pi \int_a^b xf\,dx$$

. Applying both expressions, we obtain

```
>> double(pi*int(f^2,sol(2),sol(3)))

ans =

    2.0408

>> double(2*pi*int(x*f,sol(2),sol(3)))

ans =

    8.7232
```

1.4. Example

The antiderivatives of the function $f(x) = e^{-x^2}$ cannot be expressed by elementary functions. Then, the integral

$$\int_0^1 e^{-x^2}\,dx$$

has to be calculated by approximations. We will approximate its value by using Taylor polynomials.

First, we will calculate the value of the definite integral with the command `int`, and, to see more digits, we choose `format long`:

```
>> format long
>> syms x
>> f=exp(-x^2)
>> int(f,0,1)
ans =
1/2*erf(1)*pi^(1/2)
>> double(ans)
ans =
0.746824132812427
```

We calculate MacLaurin polynomials of f of order 2, 4, 6, 10 y 14.

Remember that the command `taylor(f,x,a,'order',n)` calculates the Taylor polynomial of f of order $n-1$ at the point a.

```
>> p2=taylor(f,x,0,'order',3),p4=taylor(f,x,0,'order',5),p6=taylor(f,x,0,'order',7),
p10=taylor(f,x,0,'order',11),p14=taylor(f,x,0,'order',15)
p2 =
1-x^2
p4 =
1-x^2+1/2*x^4
p6 =
1-x^2+1/2*x^4-1/6*x^6
p10 =
1-x^2+1/2*x^4-1/6*x^6+1/24*x^8-1/120*x^10
p14 =
1-x^2+1/2*x^4-1/6*x^6+1/24*x^8-1/120*x^10+1/720*x^12-1/5040*x^14
```

Let us now calculate the definite integrals of these polynomials on the interval $[0,1]$. Notice how the approximation improves as we consider Taylor polynomials of a higher order:

```
>> double(int(p2,0,1))
ans =
   0.666666666666667
>> double(int(p4,0,1))
ans =
   0.766666666666667
>> double(int(p6,0,1))
ans =
   0.742857142857143
>> double(int(p10,0,1))
ans =
   0.746729196729197
>> double(int(p14,0,1))
ans =
   0.746822806822807
```

2. Exercises

1. Calculate the following antiderivatives:

$$a)\ \int \frac{dx}{1+e^x} \qquad b)\ \int \sec x\, dx \qquad c)\ \int e^{ax}\sin bx\, dx$$

$$d)\ \int x^3 \ln x\, dx \qquad e)\ \int \sin^{-1} x\, dx \qquad f)\ \int x\tan^{-1}\sqrt{x^2-1}\, dx$$

Sol.:

a) $x - \ln(1+e^x) + c$ *b)* $\ln|secx + tanx| + c$

c) $\dfrac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$ *d)* $x^4\left(\dfrac{\ln x}{4} - \dfrac{1}{16}\right) + c$

e) $x\sin^{-1}x + \sqrt{1-x^2} + c$ *f)* $\dfrac{1}{2}\left(x^2\tan^{-1}\sqrt{x^2-1} - \sqrt{x^2-1}\right) + c$

2. Calculate the value of the following improper integrals.

a) $\displaystyle\int_2^{+\infty}\frac{dx}{x^2-1}$ *b)* $\displaystyle\int_e^{+\infty}\frac{dx}{x\ln^2 x}$ *c)* $\displaystyle\int_{-\infty}^{0} xe^x\,dx$

d) $\displaystyle\int_{-\infty}^{+\infty}\frac{dx}{x^2+1}$ *e)* $\displaystyle\int_3^{5}\frac{xdx}{\sqrt{x^2-9}}$ *f)* $\displaystyle\int_{-\infty}^{+\infty}\sin x\,dx$

Sol.: *a)* 0.5493 *b)* 1 *c)* -1 *d)* π *e)* 4 *f)* It does not exist.

3. Represent graphically and find the area of the region bounded by the graph of the function

$$f(x) = \frac{x+1}{x^2+x+1}$$

and the x-axis between 0 and 1.

Sol.: 0.8516

4. Represent graphically the region enclosed by the graph of the function $f(x) = \dfrac{x-1}{(x+1)^2}$ and the lines $y = x,\ x = 0$ and $x = 5$. Calculate the area of the given region.

Sol: 12.3749

5. Given the curve $f(x) = \sin x + \cos x$

   a) Calculate the volume of the solid generated by revolving the region between the curve and the x-axis on the interval $[0, \pi/2]$, about the x-axis.
   b) Calculate the volume of the solid generated by revolving the region between the curve and the x-axis on the interval $[0, \pi/2]$, about the y-axis.

Sol.: *a)* $\pi\left(\frac{\pi}{2}+1\right)$ *b)* π^2

6. Let us consider the function

$$f(x) = \frac{x^2}{x^2-1}.$$

   a) Graph the function f.
   b) Approximate the function f by a parabola in a neighbourhood of 0.
   c) Find the area of the region bounded by the graph of f and the x-axis between $-1/2$ y $1/2$.
   d) Find the area of the region enclosed by the graph of f and the parabola between $-1/2$ y $1/2$.

Sol.: *b)* $P(x) = -x^2$ *c)* area = 0.0986 *d)* area = 0.0153

7. Let $f(x) = \dfrac{1}{x^2 - 4}$.

a) Graph f. Choose a suitable interval to plot the function.

b) Calculate the area of the enclosure bounded by the graph of f and the x-axis on the interval $[3, +\infty)$.

c) Calculate the volume generated by revolving this region about the x-axis.

d) Calculate the volume generated by revolving the same region about the y-axis.

Sol.: *b*) 0.4024 *c*) 0.0776 *d*) Infinite.

Practice 5. Multivariable functions: Graphing. Partial derivatives.

1. Symbolic expressions involving several variables

Symbolic expressions that depend on more than one variable are constructed similarly to those of a single variable, therefore we must define — with `syms` — each of the variables involved in the expression.

To define the function $f(x,y) = xe^{-x^2-y^2}$ we must write:

```
>> syms x y
>> f=x*exp(-x^2-y^2)
```

To calculate the function at a given point, we write the command `subs`, as in previous practices:

`subs(S,x,a)`	In the symbolic expression `S`, substitutes the variable `a` for `x`.
`subs(S,{x,y},{a,b})`	In the symbolic expression `S`, substitutes the variables `a,b` for `x,y`.

To calculate the function $f(x,y) = xe^{-x^2-y^2}$ at the point $(1,2)$:

```
>> subs(f,{x,y},{1,2})
```

2. 3D graphs

To plot points and curves in 3D we carry out a command similar to `plot`, `plot3`:

`plot3(x,y,z)`	Draws the point with coordinates (x,y,z). If x, y, z are vectors of the same size, this command plots the lines joining the given points.

For example:

```
>> plot3(2,3,1)                        % plots the point (2,3,1)
>> plot3(2,3,1,'*r')                   % plots the point (2,3,1) with a red asterisk
>> plot3([2,3,4],[3,0,-1],[1,-2,5])        % lines joining (2,3,1),(3,0,-2) y (4,-1,5)
>> plot3([2,3,4],[3,0,-1],[1,-2,5],'g*')  % draws (2,3,1),(3,0,-2) and (4,-1,5) with
green* at the points.
>> plot3([2,3,4],[3,0,-1],[1,-2,5],'g*-') % also draws the lines
```

Each time a command is used to draw: `ezplot`, `plot`, `plot3`,... MATLAB creates and activates a graphic window to which the name Figure n is assigned.

Sometimes it is interesting to draw two or more functions on the same window or to open several graphic windows:

`hold on`	Activates the `hold` command, and from that moment on all the new graphs are added to the last open window
`hold off`	Deactivates the `hold` command.
`figure(n)`	Selects the graphic window `Figure n` as an active window; if it does not exist, it is created.
`close all`	Closes all the graphic windows.
`subplot(m,n,p)`	Divides the graphic window in a table of $m \times n$ subwindows and places the graph in the p-th from left to right and from top to bottom.

2.1. Example

Plot, with the command `plot3`, the circular helix defined by the equations:

$$\begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases} \qquad t \in [0, 10\pi].$$

Solution:

```
>> t=linspace(0,10*pi,1000);
>> x=cos(t);
>> y=sin(t);
>> z=t;
>> plot3(x,y,z)
```

Repeat the exercise using only 20 points with the command `linspace` and look at the plot obtained.

2.2. A surface step by step. The `meshgrid` command

Our goal is to plot a function that depends on two variables, for example $f(x,y) = x + y$, whose domain is the whole $\mathbb{R}^2$. Given that the computer has a limited processing capacity, we have to decide on which part of the domain we are going to plot the function. In our case, we will graph the function on the rectangle of $\mathbb{R}^2$:

$$D = [-1,2] \times [2,6] = \{(x,y) \in \mathbb{R}^2 / -1 \leq x \leq 2,\ 2 \leq y \leq 6\}$$

To be able to represent the part of the graph of f on D

$$graph(f) = \{(x,y,z) \in \mathbb{R}^3 / (x,y) \in D,\ z = f(x,y)\}$$

we must discretize the domain D, i.e., choose a finite set of points on which we work. We perform it in the following way: From two vectors `x` and `y`, the grid defined by two matrices X e Y is generated with the command `[X Y]=meshgrid(x,y)` .

Once we have constructed the grid, we calculate the values that the function of two variables f takes on that grid with $Z = f(X,Y)$. Then, we graph the function with the command `surf(X,Y,Z)` . For example,

```
>> x=[-1 0 1 2] , y=[2 4 6]
x =
    -1     0     1     2
y =
     2     4     6
```

```
>> [X,Y]=meshgrid(x,y)
X =
    -1     0     1     2
    -1     0     1     2
    -1     0     1     2
Y =
     2     2     2     2
     4     4     4     4
     6     6     6     6
>> Z=X+Y
Z =
     1     2     3     4
     3     4     5     6
     5     6     7     8
>> surf(X,Y,Z)  % draws the plane z=x+y
```

To understand what the `meshgrid` command does, we draw the grid points with the `plot` command:

```
>> plot(X,Y,'.r')
```

We can obtain different presentations of the surface with the instructions:

`surf(X,Y,Z), shading interp`	Draws the surface $z = f(x, y)$ with a smooth transition of colours.
`surfc(X,Y,Z)`	The same as `surf`, adding the level curves of the surface projected on the plane xy.
`mesh(X,Y,Z)`	Draws the surface $z = f(x, y)$ with coloured lines and white surface.
`meshc(X,Y,Z)`	The same as `mesh`, adding the level curves of the surface projected on the plane xy.
`contour(X,Y,Z)`	Draws the level curves of the surface. The number of curves is selected automatically.
`contour(X,Y,Z,n)`	As `contour`, but drawing n curves.
`contour3(X,Y,Z)`	Draws the level curves on the surface. The number of curves is selected automatically.
`contour3(X,Y,Z,n)`	As `contour3`, but drawing n curves.

Observe the plots obtained for the above plane with the following commands:

```
>> figure(2), surfc(X,Y,Z), shading interp
>> figure(3), mesh(X,Y,Z)
>> figure(4), meshc(X,Y,Z)
>> figure(5), contour(X,Y,Z)
>> figure(6), contour3(X,Y,Z)
```

2.3. Example

Plot the paraboloid of equation $z = x^2 + y^2$, with $-2 \leq x \leq 2$, $-3 \leq y \leq 3$.

In the other figures, draw 100 level curves on the plane and plot 100 level curves on the surface . Draw different plots carrying out the above commands.

Solution:

```
>> clear                                              % clears all the variables
>> close all                                          % clears all the graphic windows
>> x=-2:0.1:2; y=-3:0.1:3; [x,y]=meshgrid(x,y);
>> z=x.^2+y.^2;                          % we operate with the matrices element by element
>> surf(x,y,z), shading interp
>> figure(2), contour(x,y,z,100)     %  draws 100 level curves on the plane
>> figure(3), contour3(x,y,z,100)    %  draws 100 level curves on the surface
```

2.4. Exercise

Plot the functions

$$f(x,y) = \frac{xy}{x^2+y^2} \quad \textit{with} \quad -2 \le x \le 2, \quad -2 \le y \le 2.$$

$$g(x,y) = \frac{x^2 y}{x^4+y^2} \quad \textit{with} \quad -4 \le x \le 4, \quad -4 \le y \le 4.$$

$$h(x,y) = y\frac{x^2-y^2}{x^2+y^2} \quad \textit{with} \quad -4 \le x \le 4, \quad -4 \le y \le 4.$$

Rotate the images to observe the behaviour of the functions at the point $(x,y) = (0,0)$.

2.5. Example

Plot the following functions on the given domain. Use `subplot(2,3,p)` to draw each function and below, 20 level curves on the plane.

(a) $z = x\,y$, in $[-1,1] \times [-1,1]$.

(b) $z = x^2 - y^2$, in $[-1,1] \times [-1,1]$.

(c) $z = \sqrt{x^2+y^2}$, in $[-1,1] \times [-1,1]$.

Solution:

```
>> x=linspace(-1,1,30); y=x; [x,y]=meshgrid(x,y);
>> z=x.*y; subplot(2,3,1), surf(x,y,z), shading interp
>> subplot(2,3,4), contour(x,y,z,20)
>> z=x.^2-y.^2; subplot(2,3,2), surf(x,y,z), shading interp
>> subplot(2,3,5), contour(x,y,z,20)
>> z=sqrt(x.^2+y.^2); subplot(2,3,3), surf(x,y,z), shading interp
>> subplot(2,3,6), contour(x,y,z,20)
```

2.6. Example

Plot the graph and the level curves of the function $z = \ x \cdot e^{-(x^2+y^2)}$ on the region $[-2,2] \times [-2,2]$.

Plot in another figure: on the left, the surfaces, in the center, the level curves on the surface, and, on the right, the level curves.

Solution:

```
>> clear
>> close all
>> x=-2:0.1:2; y=x; [x,y]=meshgrid(x,y);
>> z=x.*exp(-(x.^2+y.^2));
>> surfc(x,y,z), shading interp
>> figure(2), subplot(1,3,1), surf(x,y,z), shading interp
>> subplot(1,3,2), contour3(x,y,z)
>> subplot(1,3,3), contour(x,y,z)
```

3. Partial derivatives

To calculate the partial derivatives of a multivariable function f defined by a symbolic expression, MATLAB has the command `diff`:

`diff(f,x)`	Calculates the partial derivative of f with respect to the variable x.
`diff(f,x,n)`	Calculates the n-th partial derivative of f with respect to the variable x.

We are going to calculate the partial derivatives of the function $f(x, y) = x^2 \cos y + xy^2$:

```
>> syms x y
>> f=x^2*cos(y)+x*y^2;
>> diff(f,x)
>> diff(f,y)
```

3.1. Example

Let $f(x, y) = \tan^{-1} \dfrac{y}{x}$. Calculate the value of the expression

$$E(x, y) = \frac{\partial^2 f(x, y)}{\partial x^2} + \frac{\partial^2 f(x, y)}{\partial y^2}.$$

We find the value of the second partial derivatives of f and we add them:

```
>> syms x y
>> f=atan(y/x);
>> E=diff(f,x,2)+diff(f,y,2)
E =
2*y/x^3/(1+y^2/x^2)-2*y^3/x^5/(1+y^2/x^2)^2-2/x^3/(1+y^2/x^2)^2*y
>> simplify(E)
```

3.2. Example

Let $f(x, y) = x^2 + \sin xy$. Calculate $\dfrac{\partial^3 f(x, y)}{\partial x^2 \partial y}$.

We calculate the third partial derivative:

```
>> syms x y
>> f=x^2+sin(x*y);
>> diff(diff(f,y),x,2)
ans =
-cos(x*y)*y^2*x-2*sin(x*y)*y
```

3.3. Gradient

We can obtain the gradient vector $\nabla f(x, y)$ of the above function at the point (x, y):

```
>> grad=[diff(f,x),diff(f,y)]
```

If we want to calculate the gradient of f at the point $(1, -2)$:

```
>> subs(grad,{x,y},{1,-2})
```

4. Directional derivatives

If f is differentiable at the point (x_0, y_0), the directional derivative of f at the point (x_0, y_0) in the direction of a unit vector $\vec{u}$ is defined by

$$f'_{\vec{u}}(x_0, y_0) = df(x_0, y_0)(\vec{u}) = \nabla f(x_0, y_0) \cdot \vec{u}.$$

If the vector is not a unit vector, we must *normalize* it; for this we can use the `norm` command that calculates the modulus of a vector.

4.1. Example

Calculate the directional derivative of the function

$$f(x, y) = \sqrt{x^2 + 2y^2}$$

at the point $(-1, 2)$ in the direction of the vector $\vec{v} = (-2, 3)$.

First, we calculate the gradient of f at the point $(-1, 2)$:

```
>> syms x y
>> f=sqrt(x^2+2*y^2);
>> grad=[diff(f,x),diff(f,y)]
grad =
[ 1/(x^2+2*y^2)^(1/2)*x, 2/(x^2+2*y^2)^(1/2)*y]
>> gradp=subs(grad,{x,y},{-1,2})
gradp =
-0.3333    1.3333
```

Second, we normalize the vector $\vec{v}$:

```
>> v=[-2,3];
>> u=v/norm(v)
u =
-0.5547    0.8321
```

Finally, we calculate the directional derivative of f at $(-1, 2)$ in the direction of $\vec{v} = (1, 3)$. To calculate the scalar product of the gradient and the vector $\vec{u}$, we multiply the gradient (as a row vector) by $\vec{u}$ (as a column vector). To convert a row vector into a column vector(or viceversa) we must add (**'**) to the corresponding vector.

```
>> gradp*u'
ans =
1.2943
```

5. Exercises

1. Represent graphically the following surfaces with the corresponding level curves on the rectangle $[-4,4] \times [-4,4]$.

$$f(x,y) = \frac{8y}{1+x^2+y^2} \qquad f(x,y) = \ln\left(\frac{1+x^2}{1+y^2}\right) \qquad f(x,y) = \sin(x^2+y^2)$$

2. Let $f(x,y) = \cos(x+y^2)$. Calculate

$$\frac{\partial^2 f(x,y)}{\partial y \partial x} - \frac{\partial^3 f(x,y)}{\partial x \partial y^2}.$$

Sol.: $2(1-y)\cos(x+y^2) - 4y^2\sin(x+y^2)$.

3. Find the directional derivative of the function $f(x,y) = x^3 + y^4$ at the point (1,1) in the direction of the vector $\vec{v} = (-1,3)$.

Sol.: 2.8460

4. A team of oceanographers is developing a map of the sea to try to recover the wrecks (remains of ships that have sunk). By using the sonar they construct the model

$$f(x,y) = 0.1(xy^2 + x^2y) \qquad (x,y) \in [0,2] \times [0,2],$$

where x and y are the coordinates on the plane and f is the depth (all the distances are measured in kilometers).

a) Represent graphically the depth f on the given rectangle.

b) Represent graphically the seabed $(-f)$ on the given rectangle.

c) How deep is the wreck if it is at $(1, 3/2)$? Represent this point on the plotted surface.

d) If we move the shipwreck to the east, will we raise it or will we sink it further ? What will happen if we move it to the south?

Sol.: *c*) 0.3750 *d*) It sinks in the east direction and it rises in the south direction.