▷ Decision Version of a problem.

Eg: Given a graph $G = (V, E)$,
s, t are two nodes in V, and
integer K. Is there exists a
path from s to t of length $\leq K$?

▷ Optimization version of the same
Problem.

▷ Find shortest path from s to t
for given graph $G = (V, E)$

Decision version of Problem P
$\left\{ \text{Two Answer} \begin{array}{c} \rightarrow \text{Yes} \\ \rightarrow \text{No} \end{array} \right\}$

Instance $\rightarrow$ | Algo for | $\rightarrow$ Yes
X of P | P | $\rightarrow$ No

[Yes Instance / No instance of Problem P]
▷ Is the given list sorted ?

P= { decision problems solvable by an algorithm in time polynomial in input size }

$\rightarrow O(n^c)$

(not a formal definition)
[you will learn in 5th semester in T.O.C]

→[non-deterministic polynomial]

NP= { decision problem for which Yes answer has a polynomial length certificate and a polytime verification algorithm }

[▷ P ⊆ NP] ??

Given G, S, t, k, ∃ a path length ≤ k from s and t? ⟶ P

" " " " " ≥ k

" " " " " ? ⟶ NP

Eulerian Cycle → visits every edge exactly
  ↳ ∈ P once = Does there exist EC?
Ham Cycle → visits every vertex exactly
       once.
       Does there exist HC?
        ↳ ∈ NP

We know $P \subseteq NP$

$[NP \subseteq ?] \longrightarrow$ open problem
        "10 Million $$"

NP

P

$X \in NP$

$\forall Y \in NP \qquad Y \leq_p X$

X is NP-hard

Reduction

$$\Pi_1 \leq_P \Pi_2$$

$\Big[$ We can solve Π_1 in polytime using a polytime algorithm for Π_2. $\Big]$

$\Big[$ If there exists polytime algorithm A that converts every instance of x of Π_1 to $A(x)$ of Π_2 s.t

$x \in$ Yes-Instance iff

$A(x) \in$ Yes-Instance $\Big]$

NP-hard $\equiv$ Every problem in NP reduces to it

$\Pi \in NP$ & $\Pi \in NP$ hard $\Big] \rightarrow \Pi = NP$ Complete

---

$\Bigg\{ \begin{array}{l} \text{If } \Pi \in P \\ \Pi \in NP\text{-Complete} \end{array} \Bigg\} \Rightarrow NP \subseteq P \Bigg\}$

(But we don't know at this moment any problem satisfying both properties together)

Π is NP-Complete

1) $\overline{\Pi}$ is NP

2) $\overline{\Pi}' \leq_p \Pi$ for all $\Pi' \in$ NP

Lemma1 if $\Pi_1 \leq_p \Pi_2$ and $\Pi_2 \in P$,

then $\Pi_1 \in P$

Lemma2 If Π is NP-Complete and Π is in P, $NP \leq P$

Lemma3 if Π is NP-complete and $\Pi \notin P$ then no NP-complete problem is in P

First NP-Complete problem

1971 Cook - Levin

"$\text{SAT} \in \text{NPC}$"

$\text{SAT} \in \text{NP}$ ✓

Take any $\Pi \in NP \xrightarrow{\leq_p} \text{SAT}$

To prove $Y \in$ NP-Complete

→ $Y \in$ NP

→ Choose an NP-Complete problem X

→ $X \leq_p Y$

→ Y is in NP-Complete

If $X \in$ NPC and $Y \in$ NP, $X \leq_p Y$ then

$Y \in$ NPC

→ Take any $W \in$ NP

$W \leq_p X$ ($\because X$ is NPC)

$X \leq_p Y$ (given)

$W \leq_p Y$ (by transitivity)

Hence $Y \in$ NP-C

SAT ∈ NPC

SAT: Given a CNF ϕ, does it have a satisfying truth assignment?

CNF: Conjunction of clauses $C_1 \wedge C_2 \wedge C_3 \cdots$

Clauses: disjunction of literals $C_j = (x_i \vee \bar{x}_j \vee x_k)$

literal   boolean variable x_i or $\bar{x}_i$

1972 ⟨Richard Karp⟩ proved 21 problems ∈ NPC, using above result and reduction.

3-SAT: SAT where each clause contains exactly 3 literals. (and each literal corresponds to different variable)

Eg: $\phi = (\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3)$
$$\wedge (\bar{x}_1 \vee x_2 \vee x_4)$$

$\begin{cases} \phi = \text{Yes Instance} \\ x_1 = \text{True} \quad x_2 = \text{True} \quad x_3 = \text{False} \\ x_4 = \text{False} \end{cases}$

$3SAT \in NPC$

$3SAT \in NP$ ✓

$[3SAT \in NP\text{-hard}]$

Reduce from some known problem $\in NPC$

Thus SAT < 3 SAT

$SAT = \emptyset$ $\underbrace{(x_1 \vee x_2 \cdots \vee x_\ell)}$ $\ell \geq 4$

3-SAT
$$
\begin{cases}
(x_1 \vee x_2 \vee y) \wedge (\bar{y} \vee x_3 \vee y_2) \wedge \\
(\bar{y_2} \vee x_4 \vee y_3) \wedge \cdots \\
(y_{\ell-3} \vee x_{\ell-1} \vee x_\ell)
\end{cases}
$$

Converted to
$\emptyset'$ $\Big\}$ in polynomial time

claim $\emptyset$ is satisfiable iff
$\emptyset'$ is satisfiable

$\Rightarrow \emptyset$ is satisfiable

Each clause in $\emptyset$ is satisfiable.

W.L.O.G

$$C = (x_1 \lor x_2 \lor x_3 \lor x_4 \lor \underline{\underline{x_5}} \lor x_6 \lor x_7)$$

$$C' =$$
$$(x_1 \lor x_2 \lor y_1) \land (\overline{y_1} \lor x_3 \lor y_2)$$
$$\land (\overline{y_2} \lor x_4 \lor y_3) \land (\overline{y_3} \lor \underline{\underline{x_5}} \lor y_4)$$
$$\land (\overline{y_4} \lor x_6 \lor x_7)$$

$$\underline{True}$$

$$\begin{cases} \text{Set } y_3 = T \qquad y_2 = T \qquad y_1 = T \\ \text{Set } \quad y_4 = F \end{cases}$$

$(\Leftarrow)$ if $\emptyset'$ is satisfiable then $\emptyset$ is satisfiable

$\underline{H.W}$ (Explaination on board)

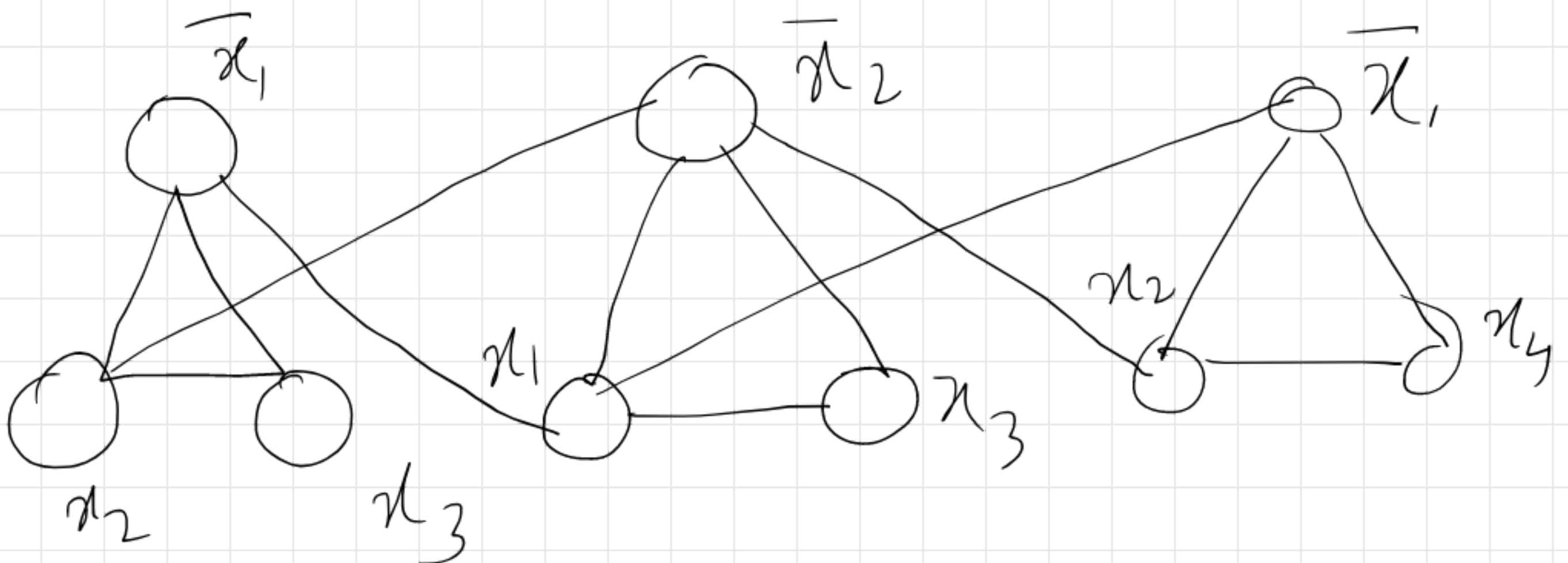# Independent Set(IS) ∈ NPC

Does G has IS of size $\geq K$?

IS ∈ NP ✓

IS ∈ NP-hard ??

3SAT with k clause

$$\emptyset \longrightarrow G$$

$$(\bar{x}_1 \lor x_2 \lor x_3) \land (x_1 \lor \bar{x}_2 \lor x_3) \land$$
$$(\bar{x}_1 \lor x_2 \lor x_4)$$
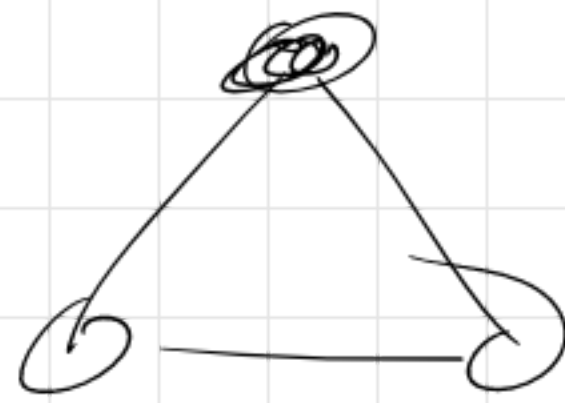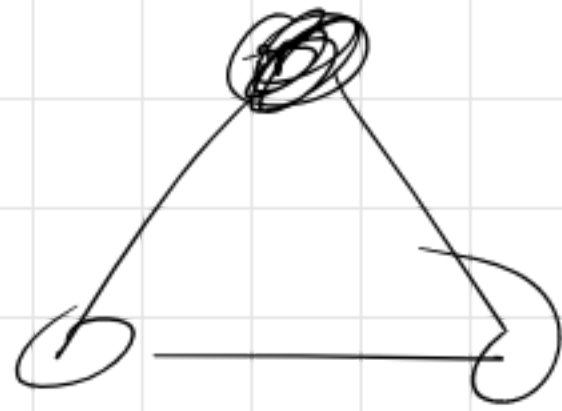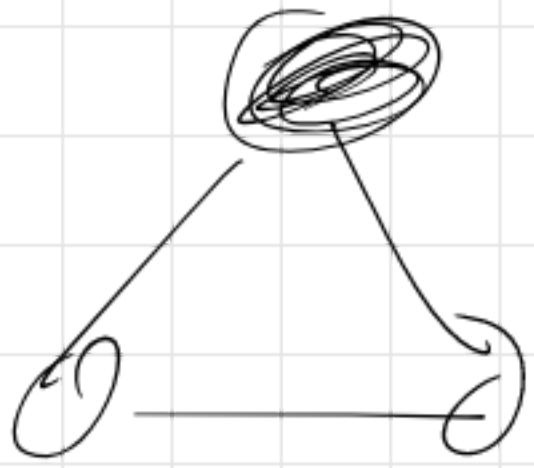
Lemma $\emptyset$ is satisfiable iff IS of size k.

Proof

$\Rightarrow$ take any satisfying assignment $\emptyset$

$\rightarrow$ Select one true literal from each clause (triangle)

Independent set of size $\emptyset = \cancel{k}$)

($\Leftarrow$) Let S be independent set of size k.

S must contain exactly one node from each $\triangle$;

set all these literals to true.

All clauses in $\emptyset$ satisfied.

[IS to Clique]

IS: Given G and k

$$(G, k) \longrightarrow (G^c, k)$$

G has an independent set of size at least k the iff G has a clique of size at least k.