▷ Decision Version of a problem.

Eg: Given a graph $G = (V, E)$,
s, t are two nodes in V, and
integer K. Is there exists a path from s to t of length $\leq K$?

▷ Optimization version of the same problem.

▷ Find shortest path from s to t for given graph $G = (V, E)$

$\boxed{\text{Decision version of Problem P}}$

{Two Answer → Yes / → No}

Instance x of P → Algo for P → Yes / → No

[Yes Instance / NO instance of Problem P]

▷ Is the given list sorted?

P = { decision problems solvable by an algorithm in time polynomial in input size }

$\rightarrow O(n^c)$

(not a formal definition)
[You will learn in 5th semester in T.O.C]

[non-deterministic polynomial]

NP = { decision problem for which Yes answer has a polynomial length certificate and a polytime verification algorithm }

[$\triangleright$ $P \subseteq NP$] ??

Given G, s, t, k, $\exists$ a path length $\leq k$ from s and t ? $\longrightarrow$ P

" " " $\geq k$ " " " ? $\longrightarrow$ NP

Eulerian Cycle → visits every edge exactly once = Does there exist EC?

↳ $\in P$

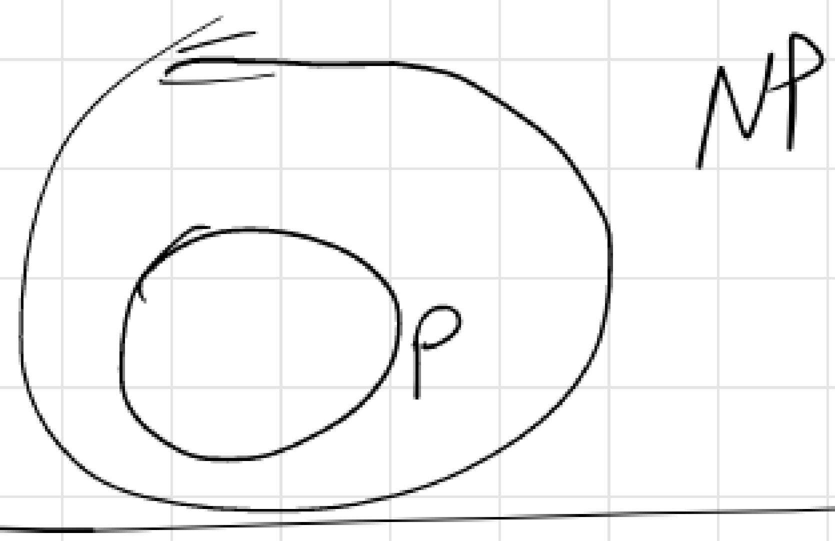
Ham Cycle → visits every vertex exactly once.

Does there exist HC?

↳ $\in NP$

We know $P \subseteq NP$

$[NP \subseteq ?]$ → open problem "10 Million \$\$"



$X \in NP$

$\forall\, Y \in NP \quad Y \leq_p X$

X is NP-hard

Reduction

$\Pi_1 \leq_p \Pi_2$

[We can solve Π_1 in poly time using a polytime algorithm for Π_2.]

[If there exists polytime algorithm A that converts every instance of x of Π_1 to $A(x)$ of Π_2 s.t.

$x \in$ Yes-Instance iff

$A(x) \in$ Yes-Instance]

NP-hard $\equiv$ Every problem in NP reduces to it

$\left.\Pi \in NP \text{ \& } \Pi \in NP\text{-hard}\right] \rightarrow \Pi = NP\text{Complete}$

$$\left\{ \left. \begin{array}{l} \text{If } \Pi \in P \\ \Pi \in NP\text{-Complete} \end{array} \right\} \Rightarrow NP \subseteq P \right\}$$

(But we don't know at this moment any problem satisfying both properties together)

Π is NP-Complete

1) Π is NP

2) $\Pi' \leq_p \Pi$ for all $\Pi' \in NP$

Lemma1 if $\Pi_1 \leq_p \Pi_2$ and $\Pi_2 \in P$,

then $\Pi_1 \in P$

Lemma2. If Π is NP-complete and Π is in P, $NP \subseteq P$

Lemma3 if Π is NP-complete and $\Pi \notin P$ then no NP-complete problem is in P

First NP-Complete problem

1971 Cook - Levin

"$SAT \in NPC$"

$SAT \in NP$ ✓

Take any $\Pi \in NP \xrightarrow{\leq_p} SAT$

To prove $Y \in$ NP-Complete

- → $Y \in NP$
- → Choose an NP-Complete problem X
- → $X \leq_p Y$
- → Y is in NP-Complete

If $X \in NPC$ and $Y \in NP$, $X \leq_p Y$ then $Y \in NPC$

→ Take any $W \in NP$

$W \leq_p X$ ($\because X$ is NPC)

$X \leq_p Y$ (given)

$W \leq_p Y$ (by transitivity)

Hence $Y \in$ NP-C

SAT ∈ NPC

SAT: Given a CNF ϕ, does it have a satisfying truth assignment?

CNF: Conjuction of clauses $C_1 \wedge C_2 \wedge C_3 \cdots$

Clauses: disjunction of literals $C_j = (x_i \vee \bar{x}_j \vee x_k)$

literal: boolean variable x_i or $\bar{x}_i$

1972 <Richard Karp> proved 21 problems ∈ NPC, using above result and reduction..

3-SAT: SAT where each clause contains exactly 3 literals. (and each literal corresponds to different variable)

Eg: $\phi = (\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3)$
$\wedge (\bar{x}_1 \vee x_2 \vee x_4)$

ϕ = Yes Instance
x_1 = True x_2 = True x_3 = False
x_4 = False

$3SAT \in NPC$

$3SAT \in NP$ ✓

$[3SAT \in NP\text{-hard}]$

Reduce from some known problem $\in NPC$

$SAT \equiv$ $\underbrace{(x_1 \vee x_2 \cdots \vee x_\ell)}$ $\ell \geq 4$

ϕ

3-SAT

$$(x_1 \vee x_2 \vee y) \wedge (\bar{y} \vee x_3 \vee y_2) \wedge$$
$$(\bar{y}_2 \vee x_4 \vee y_3) \wedge \cdots$$
$$(y_{\ell-3} \vee x_{\ell-1} \vee x_\ell)$$

Converted to ϕ' } in polynomial time

claim ϕ is satisfiable iff ϕ' is satisfiable.

$\Rightarrow \phi$ is satisfiable

Each clause in ϕ is satisfiable.

<u>W.L.O.G</u>

$$C = (x_1 \vee x_2 \vee x_3 \vee x_4 \vee x_5 \vee x_6 \vee x_7)$$

$C' =$

$$(x_1 \vee x_2 \vee y_1) \wedge (\bar{y}_1 \vee x_3 \vee y_2)$$
$$\wedge (\bar{y}_2 \vee x_4 \vee y_3) \wedge (\bar{y}_3 \vee \underbrace{x_5}_{\text{True}} \vee y_4)$$
$$\wedge (\bar{y}_4 \vee x_6 \vee x_7)$$

Set $y_3 = T$ $\quad y_2 = T$ $\quad y_1 = T$

Set $y_4 = F$

$(\Leftarrow)$ if ϕ' is satisfiable then ϕ is satisfiable

<u>H.W</u> (Explaination on board)

Independent Set (IS) $\in$ NPC

Does G has IS of size $\geq k$?

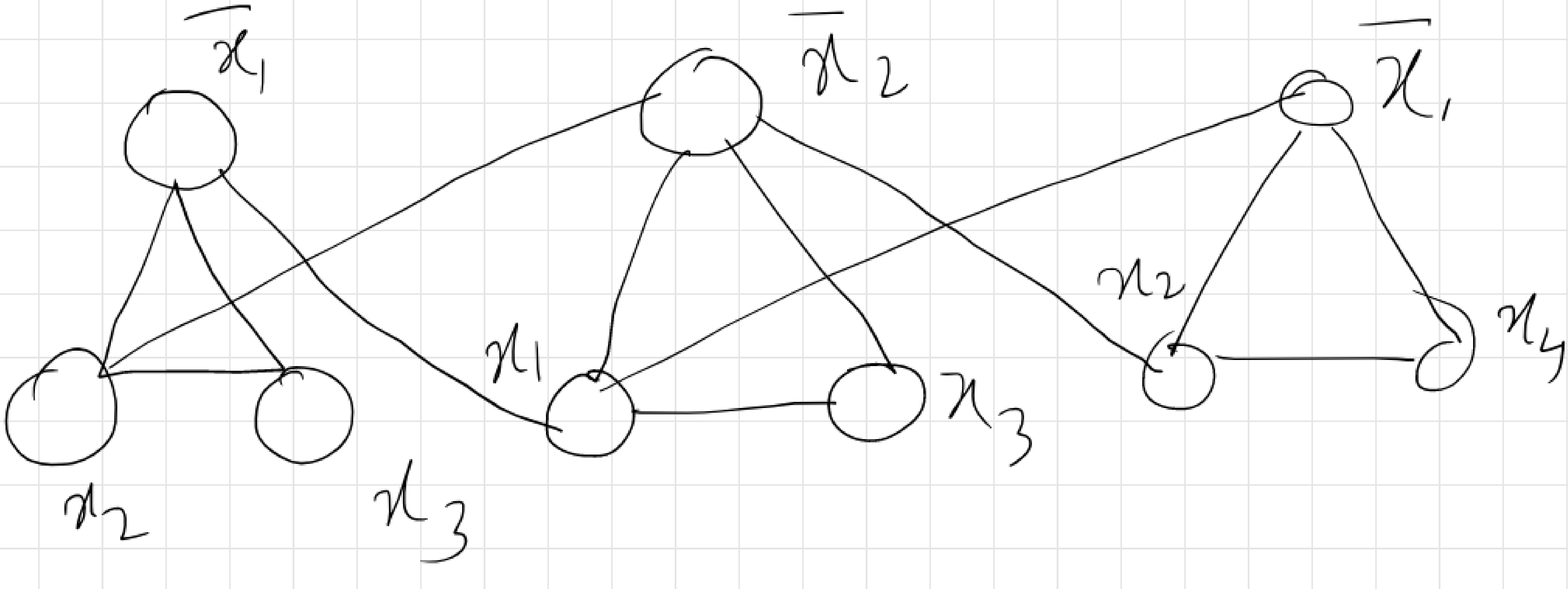
IS $\in$ NP ✓

IS $\in$ NP-hard ??

3SAT with k clauses

$\phi \longrightarrow G$

$(\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \wedge$
$(\bar{x}_1 \vee x_2 \vee x_4)$

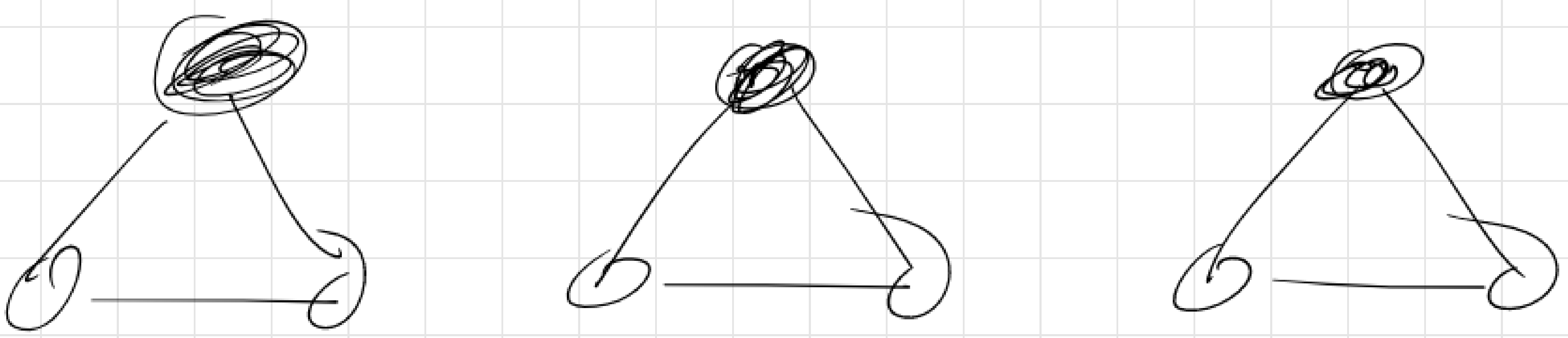


Lemma ϕ is satisfiable iff IS of size K.

Proof

$\Rightarrow$ take any satisfying assignment ϕ

$\rightarrow$ Select one true literal from each clause (triangle)



Independenset of size $\phi = K$)

$(\Leftarrow)$ Let S be independent set of size K.

So S must contain exactly one node from each $\triangle$.

set all these literals to true.

All clauses in ϕ satisfied.

[IS to Clique]

IS: Given G and k

$$(G, k) \longrightarrow (G^c, k)$$

G has an independent set of size at least k ~~the~~ iff G has a Clique of size at least k.