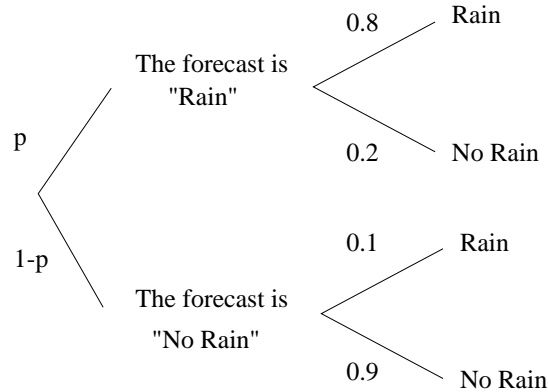MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Problem Set 2: Solutions
Due September 22, 2010

1. (a) The tree representation during the winter can be drawn as the following:

Let A be the event that the forecast was "Rain,"
let B be the event that it rained, and
let p be the probability that the forecast says "Rain." If it is in the winter, $p = 0.7$ and

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(B \mid A)\mathbf{P}(A)}{\mathbf{P}(B)} = \frac{(0.8)(0.7)}{(0.8)(0.7) + (0.1)(0.3)} = \frac{56}{59}.$$
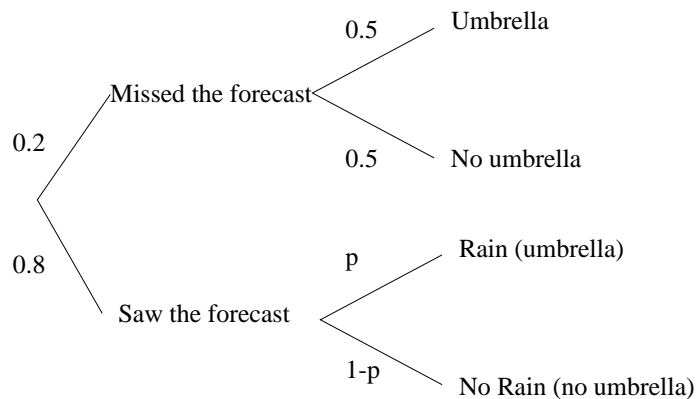
Similarly, if it is in the summer, $p = 0.2$ and

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(B \mid A)\mathbf{P}(A)}{\mathbf{P}(B)} = \frac{(0.8)(0.2)}{(0.8)(0.2) + (0.1)(0.8)} = \frac{2}{3}.$$

(b) Let C be the event that Victor is carrying an umbrella.
Let D be the event that the forecast is no rain.
The tree diagram in this case is:

$$\begin{aligned}
\mathbf{P}(D) &= 1 - p \\
\mathbf{P}(C) &= (0.8)p + (0.2)(0.5) = 0.8p + 0.1 \\
\mathbf{P}(C \mid D) &= (0.8)(0) + (0.2)(0.5) = 0.1
\end{aligned}$$

Therefore, $\mathbf{P}(C) = \mathbf{P}(C \mid D)$ if and only if $p = 0$. However, p can only be 0.7 or 0.2, which implies the events C and D can never be independent, and this result does not depend on the season.

(c) Let us first find the probability of rain if Victor missed the forecast.

$$\mathbf{P}(\text{actually rains} \mid \text{missed forecast}) = (0.8)p + (0.1)(1 - p) = 0.1 + 0.7p.$$

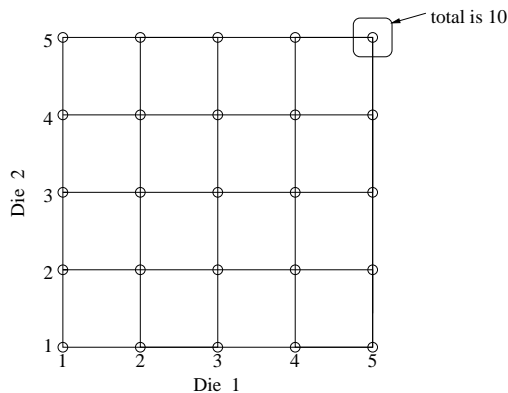Then, we can extend the tree in part (b) as follows:
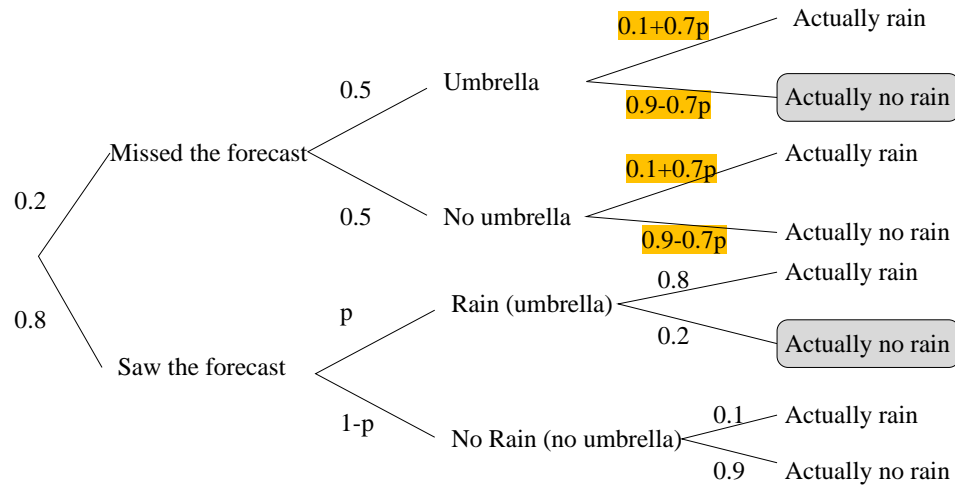
Therefore, given that Victor is carrying an umbrella and it is not raining, we are looking at the two shaded cases.

$$\mathbf{P}(\text{saw forecast} \mid \text{umbrella and not raining}) = \frac{(0.8)p(0.2)}{(0.8)p(0.2) + (0.2)(0.5)(0.9 - 0.7p)}$$

In fall and winter, $p = 0.7$, so the probability is $\frac{112}{153}$.
In summer and spring, $p = 0.2$, so the probability is $\frac{8}{27}$.

2. (a) i. $\boxed{\text{No}}$

Overall, there are 25 different outcomes in the sample space. For a total of 10, we should get a 5 on both rolls. Therefore $A \subset B$, and

$$\mathbf{P}(B|A) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A)} = \frac{\mathbf{P}(A)}{\mathbf{P}(A)} = 1$$

We observe that to get at least one 5 showing, we can have 5 on the first roll, 5 on the second roll, or 5 on both rolls, which corresponds to 9 distinct outcomes in the sample space. Therefore

$$\mathbf{P}(B) = \frac{9}{25} \neq \mathbf{P}(B|A)$$

ii. $\boxed{\text{No}}$ Given event A, we know that both roll outcomes must be 5. Therefore, we could not have event C occur, which would require at least one 1 showing. Formally, there are 9 outcomes in C, and

$$\mathbf{P}(C) = \frac{9}{25}$$

But

$$\mathbf{P}(C|A) = 0 \neq \mathbf{P}(C)$$

(b) i. $\boxed{\text{No}}$ Out of the total 25 outcomes, 5 outcomes correspond to equal numbers in the two rolls. In half of the remaining 20 outcomes, the second number is higher than the first one. In the other half, the first number is higher than the second. Therefore,

$$\mathbf{P}(F) = \frac{10}{25}$$

There are eight outcomes that belong to event E:

$$E = \{(1,2),(2,3),(3,4),(4,5),(2,1),(3,2),(4,3),(5,4)\}.$$

To find $\mathbf{P}(F|E)$, we need to compute the proportion of outcomes in E for which the second number is higher than the first one:

$$\mathbf{P}(F|E) = \frac{1}{2} \neq \mathbf{P}(F)$$

ii. $\boxed{\text{Yes}}$ Conditioning on event D reduces the sample space to just four outcomes

$$\{(2,5),(3,4),(4,3),(5,2)\}$$

which are all equally likely. It is easy to see that

$$\mathbf{P}(E|D) = \frac{2}{4} = \frac{1}{2}, \qquad \mathbf{P}(F|D) = \frac{2}{4} = \frac{1}{2}, \qquad \mathbf{P}(E \cap F|D) = \frac{1}{4} = \mathbf{P}(E|D)\mathbf{P}(F|D)$$

3. (a) Suppose we choose old widgets. Before we choose any widgets, there are $500 \cdot 0.15 = 75$ defective old widgets. The probability that we choose two defective widgets is

$$
\begin{aligned}
\mathbf{P}(\text{two defective}|\text{old}) &= \mathbf{P}(\text{first is defective}|\text{old}) \cdot \mathbf{P}(\text{second is defective}|\text{first is defective}, \text{old}) \\
&= \frac{75}{500}\frac{74}{499} = 0.02224
\end{aligned}
$$

Now let's consider the new widgets. Before we choose any widgets, there are $1500 \cdot 0.05 = 75$ defective old widgets. Similar to the calculations above,

$$
\begin{aligned}
\mathbf{P}(\text{two defective}|\text{new}) &= \mathbf{P}(\text{first is defective}|\text{new}) \cdot \mathbf{P}(\text{second is defective}|\text{first is defective}, \text{new}) \\
&= \frac{75}{1500}\frac{74}{1499} = 0.002568
\end{aligned}
$$

By the total probability law,

$$
\begin{aligned}
\mathbf{P}(\text{two defective}) &= \mathbf{P}(\text{old}) \cdot \mathbf{P}(\text{two defective}|\text{old}) \\
&\quad + \mathbf{P}(\text{new}) \cdot \mathbf{P}(\text{two defective}|\text{new}) \\
&= \frac{1}{2} \cdot 0.02224 + \frac{1}{2} \cdot 0.002568 = 0.01240.
\end{aligned}
$$

Note that this number is very close to what we would get if we ignored the effects of removing one defective widget before choosing the second widget:

$$
\begin{aligned}
\mathbf{P}(\text{two defective}) &= \mathbf{P}(\text{old}) \cdot \mathbf{P}(\text{two defective}|\text{old}) \\
&\quad + \mathbf{P}(\text{new}) \cdot \mathbf{P}(\text{two defective}|\text{new}) \\
&\approx \frac{1}{2} \cdot 0.15^2 + \frac{1}{2} \cdot 0.05^2 = 0.0125.
\end{aligned}
$$

(b) Using Bayes' rule,

$$
\begin{aligned}
\mathbf{P}(\text{old}|\text{two defective}) &= \frac{\mathbf{P}(\text{old}) \cdot \mathbf{P}(\text{two defective}|\text{old})}{\mathbf{P}(\text{old}) \cdot \mathbf{P}(\text{two defective}|\text{old}) + \mathbf{P}(\text{new}) \cdot \mathbf{P}(\text{two defective}|\text{new})} \\
&= \frac{\frac{1}{2} \cdot 0.02224}{\frac{1}{2} \cdot 0.02224 + \frac{1}{2} \cdot 0.002568} = 0.8965
\end{aligned}
$$

4. (a)

$$
\mathbf{P}(\text{find in A and in A}) = \mathbf{P}(\text{in A}) \cdot \mathbf{P}(\text{find in A}|\text{in A}) = 0.4 \cdot 0.25 = 0.1
$$
$$
\mathbf{P}(\text{find in B and in B}) = \mathbf{P}(\text{in B}) \cdot \mathbf{P}(\text{find in B}|\text{in B}) = 0.6 \cdot 0.15 = 0.09
$$

Oscar should search in Forest A first.

(b) Using Bayes' Rule,

$$
\begin{aligned}
\mathbf{P}(\text{in A}|\text{not find in A}) &= \frac{\mathbf{P}(\text{not find in A}|\text{in A}) \cdot \mathbf{P}(\text{in A})}{\mathbf{P}(\text{not find in A}|\text{in A}) \cdot \mathbf{P}(\text{in A}) + \mathbf{P}(\text{not find in A}|\text{in B}) \cdot \mathbf{P}(\text{in B})} \\
&= \frac{(0.75) \cdot (0.4)}{(0.4) \cdot (0.75) + (1) \cdot (0.6)} = \frac{1}{3}
\end{aligned}
$$

(c) Again, using Bayes' Rule,

$$
\begin{aligned}
\mathbf{P}(\text{looked in A}|\text{find dog}) &= \frac{\mathbf{P}(\text{find dog}|\text{looked in A}) \cdot \mathbf{P}(\text{looked in A})}{\mathbf{P}(\text{find dog})} \\
&= \frac{(0.25) \cdot (0.4) \cdot (0.5)}{(0.25) \cdot (0.4) \cdot (0.5) + (0.15) \cdot (0.6) \cdot (0.5)} = \frac{10}{19}
\end{aligned}
$$

(d) In order for Oscar to find the dog, it must be in Forest A, not found on the first day, alive, and found on the second day. Note that this calculation requires conditional independence of not finding the dog on different days and the dog staying alive.

$$
\begin{aligned}
\mathbf{P}(\text{find live dog in A day 2}) &= \mathbf{P}(\text{in A}) \cdot \mathbf{P}(\text{not find in A day 1}|\text{in A}) \\
&\quad \cdot \mathbf{P}(\text{alive day 2}) \cdot \mathbf{P}(\text{find day 2}|\text{in A}) \\
&= 0.4 \cdot 0.75 \cdot (1 - \frac{1}{3}) \cdot 0.25 = 0.05
\end{aligned}
$$

5. (a) We proceed as follows:

$$
\begin{aligned}
\mathbf{P}(A \cap (B \cup C)) &= \mathbf{P}((A \cap B) \cup (A \cap C)) \\
&= \mathbf{P}(A \cap B) + \mathbf{P}(A \cap C) - \mathbf{P}(A \cap B \cap C) \\
&\overset{*}{=} \mathbf{P}(A)\mathbf{P}(B) + \mathbf{P}(A)\mathbf{P}(C) - \mathbf{P}(A)\mathbf{P}(B)\mathbf{P}(C) \\
&= \mathbf{P}(A)\left[\mathbf{P}(B) + \mathbf{P}(C) - \mathbf{P}(B)\mathbf{P}(C)\right] \\
&= \mathbf{P}(A)\mathbf{P}(B \cup C),
\end{aligned}
$$

where the equality marked with $*$ follows from the independence of A, B, and C.

(b) <u>Proof 1:</u> If A and B are independent, then A^c and B^c are also independent (see Problem 1.43, page 63 for the proof).

For any two independent events U and V, DeMorgan's Law implies

$$
\begin{aligned}
\mathbf{P}(U \cup V) &= \mathbf{P}((U^c \cap V^c)^c) = 1 - \mathbf{P}(U^c \cap V^c) = 1 - \mathbf{P}(U^c) \cdot \mathbf{P}(V^c) \\
&= 1 - (1 - \mathbf{P}(U))(1 - \mathbf{P}(V)).
\end{aligned}
$$

We proceed to prove the statement by induction. Letting $U = A_1$ and $V = A_2$, the base case is proven above. Now we assume that the result holds for any n and show that it holds for $n+1$. For independent $\{A_1, \ldots, A_n, A_{n+1}\}$, let $B = \cup_{i=1}^{n} A_i$. It is easy to show that B and A_{n+1} are independent. Therefore,

$$
\begin{aligned}
\mathbf{P}(A_1 \cup A_2 \cup \ldots \cup A_{n+1}) &= 1 - (1 - \mathbf{P}(B)) \cdot (1 - \mathbf{P}(A_{n+1})) \\
&= 1 - \prod_{i=1}^{n+1} (1 - \mathbf{P}(A_i)),
\end{aligned}
$$

which completes the proof.

<u>Proof 2:</u> Alternatively, we can use the version of the DeMorgan's Law for n events:

$$
\begin{aligned}
\mathbf{P}(A_1 \cup A_2 \cup \ldots \cup A_n) &= \mathbf{P}((A_1^c \cap A_2^c \cap \ldots \cap A_n^c)^c) \\
&= 1 - \mathbf{P}(A_1^c \cap A_2^c \cap \ldots \cap A_n^c).
\end{aligned}
$$

But we know that $A_1^c, A_2^c, \ldots, A_n^c$ are independent. Therefore

$$
\begin{aligned}
\mathbf{P}(A_1 \cup A_2 \cup \ldots \cup A_n) &= 1 - \mathbf{P}(A_1^c)\mathbf{P}(A_2^c)\ldots\mathbf{P}(A_n^c) \\
&= 1 - \prod_{i=1}^{n} (1 - \mathbf{P}(A_i)).
\end{aligned}
$$

G1[†]. (a) The figure below describes the sample space via an infinite tree. The leaves of this tree are exactly all *finite* tournament histories; in addition, the two infinite paths represent the two *infinite* tournament histories that are possible. Note that the winner of the first game is either Alice or Bob; from then on, the winner of a game is either the winner of the previous game (in which case we have reached a leaf and the tournament has ended) or the player that sat out the previous game. The outcomes of the sample space correspond to the finite histories (which are identified with the leafs of the tree) and the two infinite histories: ACBACB... and BCABCA...
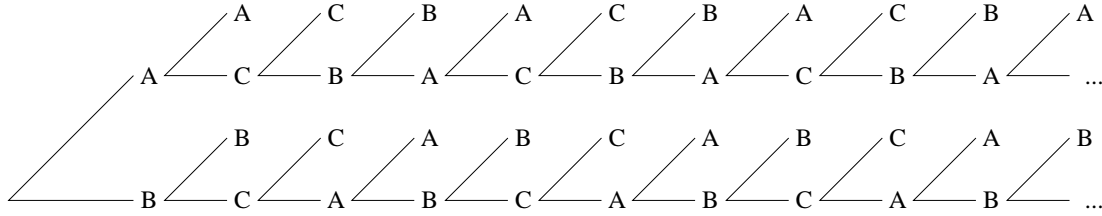
(b) The probability of an event is $1/2^k$ times the number of finite histories contained in the event. The probability of the event consisting of one or both infinite histories is 0. We have to show that this probability law satisfies the three probability axioms. It clearly satisfies nonnegativity and additivity. To check *normalization*, we have to verify that the probabilities of all tournament histories sum up to 1.

Start by noticing that two of the histories are infinite and have probability 0. Each one of the remaining histories has some finite length $k \geq 2$ (and hence is represented by one of the two leaves of the tree of the figure above at depth k) and probability $1/2^k$. Hence, summing all probabilities we get

$$2 \cdot 0 + \sum_{k=2}^{\infty} 2 \cdot \frac{1}{2^k} = \sum_{k=2}^{\infty} \frac{1}{2^{k-1}} = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} = \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{1}{2} \frac{1}{1 - 1/2} = 1.$$

(c) The probability that exactly 2 games will be played is the sum of the probabilities of the two leaves at depth 2; that is,

$$P(\text{exactly 2 games}) = \frac{1}{2^2} + \frac{1}{2^2} = \frac{1}{2}.$$

Similarly, the probability that exactly i games will be played, for $i = 3, 4, 5$, is

$$\begin{array}{rcl} P(\text{exactly 3 games}) & = & \frac{1}{2^3} + \frac{1}{2^3} = \frac{1}{4}, \\ P(\text{exactly 4 games}) & = & \frac{1}{2^4} + \frac{1}{2^4} = \frac{1}{8}, \\ P(\text{exactly 5 games}) & = & \frac{1}{2^5} + \frac{1}{2^5} = \frac{1}{16}. \end{array}$$

Hence, the probability that the tournament lasts no more than 5 games is

$$P(\text{at most 5 games}) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}.$$

Hence, it's pretty probable that the tournament will last at most that much.

The probability that Alice wins the tournament is the sum of the probabilities of the leaves of the tree that are labeled "A"; that is,

$$(\frac{1}{2^2} + \frac{1}{2^5} + \frac{1}{2^8} + \cdots) + (\frac{1}{2^4} + \frac{1}{2^7} + \frac{1}{2^{10}} + \cdots),$$

where the first summation includes all leaves from the upper part of the tree, while the second one takes care of the leaves on the lower part. Calculating, we have

$$\frac{1}{4}(1 + \frac{1}{2^3} + \frac{1}{2^6} + \cdots) + \frac{1}{16}(1 + \frac{1}{2^3} + \frac{1}{2^6} + \cdots) = \frac{5}{16} \sum_{j=0}^{\infty} \frac{1}{8^j} = \frac{5}{16} \frac{1}{1 - 1/8} = \frac{5}{14}.$$

By symmetry (note the correspondence between the histories where Alice wins and the histories where Bob does), Bob's probability of winning is $\frac{5}{14}$, as well. Then, since the outcomes where nobody wins (these are the two infinite tournament histories) have total probability 0, Carol wins with probability $1 - \frac{5}{14} - \frac{5}{14} = \frac{4}{14}$. Hence, by not participating in the first game, Carol enters the tournament with a disadvantage.

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Fall 2013