International Series on Actuarial Science

Actuarial Mathematics for Life Contingent Risks

THIRD EDITION

David C. M. Dickson, Mary R. Hardy and Howard R. Waters

NEW! Full coverage SOA LTAM

Institute and Faculty of Actuaries

CAMBRIDGE

Actuarial Mathematics for Life Contingent Risks

The substantially updated third edition of the popular *Actuarial Mathematics for Life Contingent Risks* is suitable for advanced undergraduate and graduate students of actuarial science, for trainee actuaries preparing for professional actuarial examinations, and for life insurance practitioners who wish to increase or update their technical knowledge. The authors provide intuitive explanations alongside mathematical theory, equipping readers to understand the material in sufficient depth to apply it in real world situations and to adapt their results in a changing insurance environment. Topics include modern actuarial paradigms, such as multiple state models, cash flow projection methods and option theory, all of which are required for managing the increasingly complex range of contemporary long-term insurance products.

Numerous exam-style questions allow readers to prepare for traditional professional actuarial exams, and extensive use of Excel ensures that readers are ready for modern, Excel-based exams and for the actuarial work environment. The Solutions Manual (ISBN 9781108747615), available for separate purchase, provides detailed solutions to the text's exercises.

DAVID C. M. DICKSON holds a PhD in Actuarial Science from Heriot-Watt University and is a Fellow of the Institute and Faculty of Actuaries and the Institute of Actuaries of Australia. David lectured for seven years at Heriot-Watt University before moving to the University of Melbourne in 1993. In 2000 David was appointed to the Chair in Actuarial Studies in Melbourne. He was Head of the Department of Economics from 2016 to 2018. He has twice been awarded the HM Jackson Prize of the Institute of Actuaries of Australia, most recently for his book *Insurance Risk and Ruin*.

MARY R. HARDY is Professor of Actuarial Science at the University of Waterloo. She earned her PhD in Actuarial Science from Heriot-Watt University, where she lectured for 11 years before moving to the University of Waterloo in 1997. She is a Fellow of the Institute and Faculty of Actuaries, and of the Society of Actuaries. In 2007 she was awarded the Chartered Enterprise Risk Analyst designation of the Society of Actuaries, through their thought-leader recognition program. In 2013 she was awarded the Finlaison Medal of the Institute and Faculty of Actuaries for services to the actuarial profession, in research, teaching and governance.

HOWARD R. WATERS is Professor Emeritus at the Department of Actuarial Mathematics and Statistics at Heriot-Watt University. He holds a DPhil in mathematics from Oxford University, and worked as a consulting actuary for several years before joining Heriot-Watt University. He is a Fellow of the Institute and Faculty of Actuaries. He was awarded the Finlaison medal of the Institute of Actuaries in 2006 for services to actuarial research and education.

INTERNATIONAL SERIES ON ACTUARIAL SCIENCE

Editorial Board
Christopher Daykin (Independent Consultant and Actuary)
Angus Macdonald (Heriot-Watt University)

The *International Series on Actuarial Science*, published by Cambridge University Press in conjunction with the Institute and Faculty of Actuaries, contains textbooks for students taking courses in or related to actuarial science, as well as more advanced works designed for continuing professional development or for describing and synthesizing research. The series is a vehicle for publishing books that reflect changes and developments in the curriculum, that encourage the introduction of courses on actuarial science in universities, and that show how actuarial science can be used in all areas where there is long-term financial risk.

A complete list of books in the series can be found at www.cambridge.org/isas. Recent titles include the following:

Modelling Mortality with Actuarial Applications
Angus S. Macdonald, Stephen J. Richards & Iain D. Currie

Claims Reserving in General Insurance
David Hindley

Financial Enterprise Risk Management (2nd Edition)
Paul Sweeting

Insurance Risk and Ruin (2nd Edition)
David C.M. Dickson

Predictive Modeling Applications in Actuarial Science, Volume 2: Case Studies in Insurance
Edited by Edward W. Frees, Richard A. Derrig & Glenn Meyers

Predictive Modeling Applications in Actuarial Science, Volume 1: Predictive Modeling Techniques
Edited by Edward W. Frees, Richard A. Derrig & Glenn Meyers

Computation and Modelling in Insurance and Finance
Erik Bølviken

ACTUARIAL MATHEMATICS FOR LIFE CONTINGENT RISKS

THIRD EDITION

DAVID C. M. DICKSON
University of Melbourne

MARY R. HARDY
University of Waterloo, Ontario

HOWARD R. WATERS
Heriot-Watt University, Edinburgh

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre,
New Delhi – 110025, India

103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781108478083
DOI: 10.1017/9781108784184

© David C. M. Dickson, Mary R. Hardy and Howard R. Waters 2009, 2013, 2020

This publication is in copyright. Subject to statutory exception
and to the provisions of relevant collective licensing agreements,
no reproduction of any part may take place without the written
permission of Cambridge University Press.

First published 2009
Second edition 2013
Third edition 2020

Printed in the United Kingdom by TJ International Ltd. Padstow Cornwall

A catalogue record for this publication is available from the British Library

Library of Congress Cataloging-in-Publication Data
Names: Dickson, D. C. M. (David C. M.), 1959- author. | Hardy, Mary, 1958-
author. | Waters, H. R. (Howard Richard), author.
Title: Actuarial mathematics for life contingent risks / David C. M.
Dickson, University of Melbourne, Mary R. Hardy, University of Waterloo,
Ontario, Howard R. Waters, Heriot-Watt University, Edinburgh.
Description: Third edition. | Cambridge ; New York : Cambridge University
Press, [2020] | Includes bibliographical references and index.
Identifiers: LCCN 2019042023 (print) | LCCN 2019042024 (ebook) |
ISBN 9781108478083 (hardback) | ISBN 9781108784184 (epub)
Subjects: LCSH: Insurance–Mathematics. | Risk (Insurance)–Mathematics.
Classification: LCC HG8781 .D528 2020 (print) | LCC HG8781 (ebook) |
DDC 368/.01–dc23
LC record available at https://lccn.loc.gov/2019042023
LC ebook record available at https://lccn.loc.gov/2019042024

ISBN 978-1-108-47808-3 Hardback

Cambridge University Press has no responsibility for the persistence or accuracy of URLs for external or third-party internet websites referred to in this publication and does not guarantee that any content on such websites is, or will remain, accurate or appropriate.

To Carolann, Vivien and Phelim

Contents

Preface to the third edition

Life insurance has undergone enormous change in the last two to three decades. New and innovative products have been developed at the same time as we have seen vast increases in computational power. In addition, the field of finance has experienced a revolution in the development of a mathematical theory of options and financial guarantees, first pioneered in the work of Black, Scholes and Merton, and actuaries have come to realize the importance of that work to risk management in actuarial contexts.

In this book we have adapted the traditional approach to the mathematics of life contingent risks to account for the products, science and technology that are relevant to current and future actuaries, taking into consideration both demographic and financial uncertainty. The material is presented with a certain level of mathematical rigour; we want readers to understand the principles involved, rather than to memorize methods or formulae. The reason is that a rigorous approach will prove more useful in the long run than a short-term utilitarian outlook, as theory can be adapted to changing products and technology in ways that techniques, without scientific support, cannot. However, this is a very practical text. The models and techniques presented are versions, a little simplified in parts, of the models and techniques in use by actuaries in the forefront of modern actuarial management.

The first seven chapters set the context for the material, and cover traditional actuarial models and theory of life contingencies, with modern computational techniques integrated throughout, and with an emphasis on the practical context for the survival models and valuation methods presented. Through the focus on realistic contracts and assumptions, we aim to foster a general business awareness in the life insurance context, at the same time as we develop the mathematical tools for risk management in that context.

From Chapter 8, we move into more modern theory and methods.

In Chapter 8 we introduce multiple state models, which generalize the life–death contingency structure of previous chapters. Using multiple state models allows a single framework for a wide range of insurance, including income replacement insurance, where benefits and premiums depend on the health status of the policyholder, and critical illness insurance, which pays a benefit on diagnosis of certain serious medical disorders. We also present other applications of multiple state models, including long-term care, continuing care retirement communities and structured settlements.

In Chapter 9 we consider a particular type of multiple state model, namely the multiple decrement model, which occurs frequently in actuarial applications, a notable example being in pension plan valuation.

In Chapter 10 we apply the models and results from multiple state models to insurance involving two lives, typically domestic partners. It is increasingly common for partners to purchase life insurance cover or annuity income products where the benefits depend on both lives, not on a single insured life.

In Chapter 11 we apply the theory developed in the earlier chapters to problems involving pension benefits. Pension mathematics has some specialized concepts, particularly in funding principles, but in general this chapter is an application of the theory in the preceding chapters.

In Chapter 12 we move to a more sophisticated view of interest rate models and interest rate risk. In this chapter we explore the crucially important difference between diversifiable and non-diversifiable risk.

In Chapter 13 we introduce a general algorithm for projecting the emerging surplus of insurance policies, by considering the year-to-year net cash flows. One of the liberating aspects of the computer revolution for actuaries is that we are no longer required to summarize complex benefits in a single actuarial value; we can go much further in projecting the cash flows to see how and when surplus will emerge. This is much richer information that the actuary can use to assess profitability and to better manage portfolio assets and liabilities. In life insurance contexts, the emerging cash flow projection is often called ‘profit-testing’.

In Chapter 14 we follow up on the cash flow projections of Chapter 13 to show how profit testing can be used to analyse Universal Life insurance, which is very popular in North America.

In Chapter 15 we use the emerging cash flow approach to assess equity-linked contracts, where a financial guarantee is commonly part of the contingent benefit. The real risks for such products can only be assessed taking the random variation in potential outcomes into consideration, and we demonstrate this with Monte Carlo simulation of the emerging cash flows.

The products that are explored in Chapter 15 contain financial guarantees embedded in the life contingent benefits. Option theory is the mathematics of valuation and risk management of financial guarantees. In Chapter 16 we introduce the fundamental assumptions and results of option theory.

In Chapter 17 we apply option theory to the embedded options of financial guarantees in insurance products. The theory can be used for pricing and for determining appropriate reserves, as well as for assessing profitability.

In Chapter 18 we move into a different aspect of actuarial work and discuss some of the techniques that are used to estimate the survival models that appear in earlier chapters.

In Chapter 19 we present a very brief introduction to the important practical topic of modelling longevity through stochastic mortality models.

The material in this book is designed for undergraduate and graduate programmes in actuarial science, for those self-studying for professional actuarial exams and for practitioners interested in updating their skill set. The content has been designed primarily to prepare readers for practical actuarial work in life insurance and pension funding and valuation. The text covers all of the most recent syllabus requirements for the LTAM exam of the Society of Actuaries and for the CM1 exam of the UK Institute and Faculty of Actuaries. Some of the topics in this book are not currently covered by those professional exams, and many of the topics that are in the exams are covered in significantly more depth in the text, particularly where we believe the content will be valuable beyond the exams.

Students and other readers should have sufficient background in probability to be able to calculate moments of functions of one or two random variables, and to handle conditional expectations and variances. We assume familiarity with the binomial, uniform, exponential, normal and lognormal distributions. Some of the more important results are reviewed in Appendix A. Readers are also assumed to have a knowledge of maximum likelihood estimation, also reviewed in Appendix A. We also assume that readers have completed an introductory level course in the mathematics of finance, and are aware of the actuarial notation for interest, discount and annuities-certain.

Throughout, we have opted to use examples that liberally call on spreadsheet-style software. Spreadsheets are ubiquitous tools in actuarial practice, and it is natural to use them throughout, allowing us to use more realistic examples, rather than having to simplify for the sake of mathematical tractability. Other software could be used equally effectively, but spreadsheets represent a fairly universal language that is easily accessible. To keep the computation requirements reasonable, we have ensured that all but one of the examples and exercises can be completed in Microsoft Excel, without needing any VBA code or macros. Readers who have sufficient familiarity to write their own code may find more efficient solutions than those that we have presented, but our principle is that no reader should need to know more than the basic Excel functions and applications. It will be very useful for anyone working through the material of this book to construct their own spreadsheet tables as they work through the first seven chapters, to generate mortality and actuarial functions for a range of mortality models and interest rates. In the worked examples in the text, we have worked with greater accuracy than we record, so there will be some differences from rounding when working with intermediate figures.

One of the advantages of spreadsheets is the ease of implementation of numerical integration algorithms. We assume that students are aware of the principles of numerical integration, and we give some of the most useful algorithms in Appendix B.

The material in this book is appropriate for three one-semester courses. The first six chapters form a fairly traditional basis, and would reasonably constitute a first course. Chapters 7–12 introduce more contemporary material, and could be used for the second course. Chapter 11, on pension mathematics, is not required for subsequent chapters, and could be omitted if a single focus on life insurance is preferred. Chapters 13–17 form a coherent, cash-flow-based coverage of variable insurance, which could be the basis of the third, more advanced course. Chapter 18 can reasonably be covered at any point after Chapter 8, and Chapter 19 at any point after Chapter 5.

Changes from the second edition

The text has been updated to reflect changes in insurance and pension benefits since the first edition was published in 2009. In particular, we illustrate how the methods and models covered can be applied to a wide range of newer insurance contracts contingent on morbidity rather than just mortality. Examples include critical illness and long-term care insurance. We have expanded the pension valuation material to give more detail on career average earnings plans, which have become much more popular in the past decade. We have included a chapter (Chapter 18) on how the models that we use throughout the book are developed from demographic data. Finally, we have included discussion of issues around changing mortality, considering deterministic and stochastic models of mortality improvement.

- The first chapter has been significantly expanded to provide readers with greater background about life insurance practice and products. New material has been included on topics such as health insurance, continuing care retirement communities and structured settlements.

- Chapter 3 includes new material about deterministic modelling of mortality improvement and the construction of mortality improvement scales.
- Chapter 8 contains new material on state-dependent annuity and insurance functions, as well as recursions for state-dependent policy values. We now also show how multiple state models can be applied to topics such as critical illness insurance, long-term care and structured settlements.
- Chapter 9 contains material on multiple decrement models that was in Chapter 8 of the second edition. As a result, Chapters 10–16 of the second edition appear as Chapters 11–17 in this edition.
- Chapter 11 has been expanded to include updated material on the valuation and funding of pension plan benefits and new content on the valuation and funding of retiree health benefits.
- Chapters 13 and 14 have been rearranged. Chapter 14 now covers Universal Life insurance. Some of the material on participating insurance that was previously in this chapter has been moved to Chapter 13.
- Chapter 18 is a new chapter dealing with estimation for lifetime distributions and multiple state transition intensities. Consequently, Appendix A has been expanded to include a review of the key points about maximum likelihood estimation.
- Chapter 19 is a new chapter which provides an introduction to the key ideas about the Lee–Carter and Cairns–Blake–Dowd stochastic mortality models.
- The end-of-chapter exercises have been reorganized as short, long and Excel-based questions. We have also added new exercises to almost all chapters.
- In a number of places, particularly Chapters 6 and 8, we have changed exercises to make them more useful for examination preparation, in particular using tables of insurance functions, rather than assuming that readers can access the required functions using an Excel workbook.

Acknowledgements

We acknowledge all the colleagues and students who provided comment and feedback during the writing of each edition of this text. Special thanks go to Jeff Beckley, Carole Bernard, Ken Bonvallet, Phelim Boyle, Jessica Dang, Chris Groendyke, Johnny Siu-Hang Li, Ana Maria Mera, Gary Parker, Kok Keng Siaw, Steve White, Matthew Till and Mike Zhu.

We are grateful to the Society of Actuaries for permission to reproduce questions from their MLC and LTAM exams, for which they own copyright. The relevant questions are noted in the text.

The authors gratefully acknowledge the contribution of the Departments of Statistics and Actuarial Science, University of Waterloo, Actuarial Mathematics and Statistics, Heriot-Watt University, and the Department of Economics,

University of Melbourne, in welcoming the non-resident authors for short visits to work on this book.

Finally, thanks to Carolann Waters, Vivien Dickson and Phelim Boyle, to whom this book is dedicated, for their unstinting support and generosity.

David Dickson
University of Melbourne

Mary Hardy
University of Waterloo

Howard Waters
Heriot-Watt University

1
Introduction to life and long-term health insurance

1.1 Summary

In this chapter we lay out the context for the mathematics of later chapters, by describing some of the background to modern actuarial practice, as it pertains to long-term, life contingent payments. We describe the major types of life insurance products that are sold in developed insurance markets, and discuss how these products have evolved over the recent past. We also consider long-term insurance that is dependent on the health status of the insured life, rather than simply survival or death. Finally, we describe some common pension designs.

We give examples of the actuarial questions arising from the risk management of these contracts. How to answer such questions, and solve the resulting problems, is the subject of the following chapters.

1.2 Background

The first actuaries were employed by life insurance companies in the early eighteenth century to provide a scientific basis for managing the companies' assets and liabilities. The liabilities depended on the number of deaths occurring amongst the insured lives each year. The modelling of mortality became a topic of both commercial and general scientific interest, and it attracted many significant scientists and mathematicians to actuarial problems, with the result that much of the early work in the field of probability was closely connected with the development of solutions to actuarial problems.

The earliest life insurance policies were annual contracts; the purchaser would pay an amount, called the **premium**, to the insurer, nominating an individual whose life was insured under the contract. The insured life might be the purchaser, but could also be a third party. If the insured life died during the year that the contract was in force, the insurer would pay a predetermined lump sum, the **sum insured**, to the policyholder or his or her estate. Each year the premium would increase as the probability of death increased. If the insured life became very ill at the renewal date, the insurance might not be renewed, in which case no benefit would be paid on the insured life's subsequent death. Over a large number of contracts, the premium income each year should approximately match the claims outgo. This method of matching income and outgo annually, with no attempt to smooth or balance the premiums over the years, is called **assessmentism**. This method is still used for group life insurance, where an employer purchases life insurance cover for its employees on a year-to-year basis.

The radical development in the later eighteenth century was the level premium contract. The problem with assessmentism was that the annual increases in premiums discouraged policyholders from renewing their contracts. The level premium policy offered the policyholder the option to lock in a regular premium, payable perhaps weekly, monthly, or annually, which was fixed for the term of the contract. This was much more popular with policyholders, as they would not be priced out of the insurance contract just when it might be most needed. For the insurer, the attraction of the longer contract was a greater likelihood of the policyholder paying premiums for a longer period. However, a problem for the insurer was that the longer contracts were more complex to model, and offered more financial risk. For these contracts actuarial techniques had to develop beyond the year-to-year modelling of mortality probabilities. In particular, it became necessary to incorporate financial considerations into the modelling of income and outgo. Over a one-year contract, the time value of money is not a critical aspect. Over, say, a 30-year contract, it becomes a very important part of the modelling and management of risk.

Another development in life insurance in the nineteenth century was the concept of **insurable interest**. This was a requirement in law that the person contracting to pay the life insurance premiums should face a financial loss on the death of the insured life – an insurance payout should not leave the beneficiary financially better off than if the insured life had not died. The insurable interest requirement ended the practice where individuals would insure persons (often public figures) with no connection to the purchaser, as a form of gambling. It also, importantly, removed the incentive for a policyholder to hasten the death of the insured life. Subsequently, insurance policies tended to be purchased by the insured life, and in the rest of this book we use the convention that the policyholder who pays the premiums is also the life insured, whose survival or death triggers the payment of the sum insured under the conditions of the contract.

The earliest studies of mortality include life tables constructed by John Graunt and Edmund Halley. A life table summarizes a survival model by specifying the proportion of lives that are expected to survive to each age. Using London mortality data from the early seventeenth century, Graunt proposed, for example, that each new life had a probability of 40% of surviving to age 16, and a probability of 1% of surviving to age 76. Edmund Halley, famous for his astronomical calculations, used mortality data from the city of Breslau in the late seventeenth century as the basis for his life table, which, like Graunt's, was constructed by proposing the average ('medium' in Halley's phrase) proportion of survivors to each age from an arbitrary number of births. Halley took the work two steps further. First, he used the table to draw inference about the conditional survival probabilities at intermediate ages. That is, given the probability that a newborn life survives to each subsequent age, it is possible to infer the probability that a life aged, say, 20, will survive to each subsequent age, using the condition that a life aged zero survives to age 20. The second major innovation was that Halley combined the mortality data with an assumption about interest rates to find the value of a whole life annuity at different ages. A whole life annuity is a contract paying a level sum at regular intervals while the named life (the annuitant) is still alive. The calculations in Halley's paper bear a remarkable similarity to some of the work still used by actuaries in pensions and life insurance.

This book continues in the tradition of combining models of mortality with models in finance to develop a framework for pricing and risk management of long-term policies in life and health insurance. Many of the same techniques are relevant also in pensions mathematics. However, there have been many changes since the first long-term policies of the late eighteenth century.

We usually use the term **insurance** to refer to a contract under which the benefit is paid as a single lump sum, either on the death of the policyholder or on survival to a predetermined **maturity date**. (In the UK it is common to use the term **assurance** for insurance contracts involving lives, and 'insurance' for contracts involving property.) An **annuity** is a benefit in the form of a regular series of payments, usually conditional on the survival of the policyholder.

1.3 Traditional life insurance contracts

1.3.1 Introduction

The three traditional forms of life insurance are term, whole life and endowment insurance. These policies dominated insurance markets until the 1980s, and in some countries are still popular today. However, the design of life insurance has broadened significantly in the past few decades. In this section we describe features of traditional life insurance policies. In the next section we review more modern developments.

1.3.2 Term insurance

Term (or temporary) insurance pays a lump sum benefit on the death of the policyholder, provided death occurs before the end of a specified term. Typical contract terms range from 10 to 30 years.

The premiums for term insurance are usually very small relative to the sum insured, because the insurer has to pay a death benefit on only a small proportion of the policies issued. If a policyholder who is aged 40 purchases 10-year term insurance, then the probability that the insurer will pay any death benefit on the policy (which is just the probability that the life dies before age 50) might be around 2%. So, around 98% of such policies will expire with no death benefit payable, and the premiums from these policies subsidize the benefits for the 2% for which the death benefit must be paid.

A term insurance policyholder may choose to lapse their policy, which means the policyholder would cease paying premiums. In this case, their insurance cover would also cease, and there would be no further payments by the policyholder or the insurer.

The main purpose of term insurance is for family protection. For a relatively low monthly cost, term insurance can protect the policyholder's spouse and children against financial hardship in the event of the policyholder's death.

Another use of term insurance is to protect businesses against losses arising from the deaths of key employees. In this case the business pays the premiums and receives the sum insured if the insured life dies during the term. The business must demonstrate insurable interest. This type of insurance used to be called **Key Person Insurance**, but is now more commonly called **COLI** for Company Owned Life Insurance.

Most term insurance policies offer a level sum insured, funded by level monthly or annual premiums. A variant is **decreasing term insurance** where the death benefit decreases over the term of the policy. Decreasing term insurance may be used in conjunction with a home loan; if the policyholder dies, the outstanding loan is paid from the term insurance proceeds, sparing the policyholder's family from the potential difficulty, expense and distress of struggling to make the loan payments after the policyholder's death. The death benefit under the policy can be set to match the outstanding loan on the home in each year of the contract.

Renewable term insurance offers the policyholder the option to renew the policy at the end of the original term, without further evidence of the policyholder's health status, up to a maximum age. For example, if the maximum age is 85, a policyholder purchasing a 20-year renewable term policy at age 45 would have the option to renew the policy at age 65 for another 20 years. The premium for the renewed contract would be greater than the original, because the probability of paying the death benefit increases, but insurability is guaranteed, which means that renewal does not depend at all on the health status of the policyholder at that time. In **yearly renewable term insurance** each individual contract has just a one-year term.

Convertible term insurance offers the policyholder the option to convert to a whole life insurance at the end of the original term, or, for a renewable term policy, at the end of the first or second renewal. The premium would be recalculated for the new whole life policy, depending on the age at conversion. Conversion does not depend on or require evidence of the policyholder's health status at conversion, but there may be a maximum age at which conversion is permitted (typically, around age 75).

1.3.3 Whole life insurance

Whole life, or **permanent**, **insurance** pays a lump sum benefit on the death of the policyholder whenever it occurs. For regular premium contracts, the premium is often payable only up to some maximum age, typically 80–90 years old. The point of whole life insurance is that it should pay the death benefit whenever the policyholder dies, and it would not meet this objective if policies lapsed at older ages through non-payment of premiums, whether because of financial strain or through decreasing ability of the policyholder to manage their affairs in older age.

In general, whole life insurance will be significantly more expensive than term insurance, relative to the death benefit, as the probability of paying the death benefit (ignoring lapses) is 100%.

Traditionally, if a whole life policyholder decides to discontinue the policy after an initial period, they would be eligible for a **cash value** or **surrender value**, representing the investment part of the accumulated premiums. In the early years of a whole life policy, the cash values tend to be very low. In later years they may be substantial, though typically very much less than the sum insured. Recently it has become common in some countries to offer whole life policies with no cash value payable on surrender.

Life insurance works by combining premiums with the investment income (earned by investing the premiums), such that, by the time the policyholder dies, the premiums plus the investment income are sufficient, on average, to pay the sum insured. For short-term policies, premiums cover most of the sum insured. For long-term contracts the investment income becomes a much more significant component. The term of a whole life contract may be very long – a policy sold to a life aged 40 may still be in force over 50 years later – so the contribution of investment income is much more significant than for most term insurance. However, predicting investment returns over very long terms is very difficult, so insurers tend to calculate premiums using very conservative (that is, low side) assumptions about investment returns, resulting in relatively high premiums. This means that most policyholders (the ones who do not die very early) earn quite low rates of return on their premiums, compared with, for example, simply paying the same amount into a mutual fund investment over the same period.

Suppose an insurer is pricing a whole life policy at a time when long-term interest rates of 7% per year are available, for a 20-year investment. The insurer may calculate the premium assuming 6% per year interest, allowing the difference of 1%, which is called the **interest spread**, to cover profit and allow a margin for adverse experience. The risk to the insurer is that interest rates could fall below 6% per year at some point during the contract, in which case the interest earned on investments made at that time would fall short of the amount required to pay the sum insured. On the other hand, if the insurer is more cautious, perhaps assuming only 3% per year interest, the investment part of the policy will look quite unattractive for the customer, compared with the 7% per year available from direct investment.

One solution to this problem is to charge the higher premium, but promise to pay back to the policyholder some share of the profits if the investments do well. If the investment experience is very poor, there will be no profits, and no profit-sharing. This is the principle behind **participating** insurance, where 'participating' refers to the policyholders' participation in the distribution of profits. Participating insurance is also called **par** insurance for short, and is more commonly called **with-profit** outside North America.

The policyholders' share of profits is called **dividends** in North America, and **bonuses** elsewhere. We use the term 'dividend' when the profit share is distributed in the form of cash (or cash equivalent, such as a reduction in premium), and 'bonus' when the profit share is distributed in the form of additional insurance. In fact, the form of distribution is an important design feature for participating insurance, with different jurisdictions favouring different distribution methods, with the following being the most common.

Cash refunds may be distributed at regular (e.g. annual) intervals, based on the profit emerging in the preceding year. This is common for North American participating insurance.

Premium reductions work very similarly to cash refunds. The profit allocated to the policyholder for the year may be used to reduce the premiums due in the period to the subsequent allocation date.

Increased death benefits are determined by using the emerging profit to increase the death benefit.

There are many variants of these methods. It is common in North America for policyholders to be given some choice about the distribution – for example, offering a cash dividend as a standard benefit, but with options to convert to additional death benefit.

In the UK, profits were invariably distributed in the form of benefit increases. Bonuses would be awarded in two stages. **Reversionary bonuses** are applied to contracts in force, increasing the benefits by a specified percentage. There are three variations:

- *simple reversionary bonus* means the bonus rate is applied to the original sum insured only;
- *compound reversionary bonus* means the bonus rate is applied to the total of the sum insured and previous reversionary bonuses;
- *super-compound reversionary bonus* is a method with two bonus rates each year, the first applying to the original sum insured, and the second applying to the total of previous bonus declarations.

Terminal bonuses are used to top up the sum insured when the benefit is finally paid. Separating the profit share into reversionary and terminal bonuses allows insurers to take a more cautious attitude to distributing unrealized capital gains.

It is important to note that for all traditional participating whole life insurance, dividends and bonuses are never negative. Only profits are shared, not losses.

Profit distribution methods for participating insurance have an important impact on actuarial management, and on the techniques for pricing and marketing of policies. We note some of the more important considerations here.

- Cash dividends are attractive to policyholders; they are easy to understand, and offer flexibility. If a policyholder is in financial difficulty, the cash may enable the policyholder to maintain the policy longer, as it can be used to offset premiums. If the policyholder wants to increase their death cover, the cash bonus can be used to buy more insurance – but at greater cost, generally, than the reversionary bonus, as it constitutes a new policy and therefore incurs new policy expense charges.
- Cash dividends may be taxable. If the policyholder has no need for the cash, it is not likely to be a tax-efficient asset.
- Reversionary bonuses are more complex for policyholders to understand, but do offer a tax efficient distribution, that is also consistent with the purpose of the policy – to provide a death benefit over long terms.
- Insurers may offer a limited share of profits for policies that are surrendered. This can be particularly unfair when profit is distributed as reversionary bonus, as policyholders who contributed to the profits each year may only receive a small proportion of them on surrender. If profits are shared through

cash dividends, then, at most, the policyholder would lose one-year's profit share on surrender.

- Cash dividends require the insurer to liquidate assets, which may not be in the best interests of maximizing return. Reversionary bonus means that profits remain under the insurer's management, and so provides more potential for future profit for the insurer.
- Generally, insurers prefer to offer smooth bonuses and dividends, that is, with little variation from year to year. This is generally easier with reversionary and terminal bonus, as the actual payment is delayed until the policy matures.
- Cash dividends are expensive to operate, if every policyholder is paid a dividend each year.

Traditional participating whole life insurance is still popular in North America, but is no longer widely available in the UK or Australia, though some non-participating whole life policies are still marketed. One reason for the relatively greater success of the product in North America is that insurers there offered larger and more predictable cash values, so that policyholders could achieve a reasonable return on their premiums, even if they surrendered the contract.

Whole life insurance may be used by policyholders in a number of ways, for example as follows.

- For older lives, simplified whole life insurance may be used to cover funeral expenses. These policies have a relatively low sum insured, tend to be non-participating, and do not offer any cash value on surrender.
- For older, higher-wealth lives, whole life insurance may be used to reduce inheritance taxes, if the proceeds from insurance are taxed at a lower marginal rate than directly inherited wealth.
- For younger lives, participating whole life insurance can provide a simple, passive long-term investment opportunity, with the advantage of substantial death benefit (compared with premiums) in the event of early death.

As mentioned above, for some policies, particularly those designed for older policyholders (for tax planning or funeral expenses), cash values may not be offered on surrender of the contract. This can reduce the premiums, as the excess funds from lapsed contracts subsidize the remaining policies. This is called **lapse-supported** insurance. However, if policyholders have the ability to sell their policies to a third party, then the lapsation profits may be very low. This has led to the rise of Stranger Owned Life Insurance, or **STOLI**, where an investment firm (usually specializing in this business) makes a cash payment to a policyholder who wants to surrender their policy. The investment firm then takes over the payment of premiums as long as the original policyholder survives, and receives the sum insured on their death. If the value of the cash settlement to the original policyholder plus the cost of premiums after the sale of the policy is less than the value of the sum insured, then the investment firm makes a profit. If the original policyholder survives longer than expected, then the firm may make a loss. Often the policy is exchanged for a very deep discount on the sum insured, even for quite elderly policyholders, allowing the investment firm to make very significant profits on a large proportion of their policies.

1.3.4 Endowment insurance

Endowment insurance offers a lump sum benefit paid either on the death of the policyholder or at the end of a specified term, whichever occurs first. This is a hybrid of a term insurance and a fixed term investment. If the policyholder dies, the sum insured is paid, just as it would be under a term insurance; if the policyholder survives, the sum insured is paid at the end of the contract. Similarly to whole life insurance, the probability of a payout on an endowment insurance, ignoring lapses, is 100%. For similar reasons to the whole life case, endowment insurance typically offers cash values on early surrender, and may be issued in participating or non-participating forms.

Endowment insurance is no longer offered through mainstream insurers in North America or the UK. The main purpose of endowment insurance is as an investment, but the low returns offered and lack of flexibility meant that the contract could not compete with an increasing variety of pure investment options that became widely available in the latter part of the twentieth century. Traditional endowment insurance then evolved into modern insurance/investment hybrids such as the Universal Life or Unit Linked policies described later in this chapter.

It is interesting to note, however, that traditional endowment insurance policies are increasing in popularity in developing nations, notably for microinsurance, where the amounts involved are small. In this context endowment insurance policies may be used in conjunction with microfinance, to support small sum lending to individuals and small businesses who may not have access to traditional banking services.

1.3.5 Options and variations on traditional insurance

Insurance riders are optional benefits that a policyholder can select at the issue of a contract. In this section we describe some common riders and other variations associated with traditional insurance.

Joint life insurance: For term and whole life insurance, policies issued on 'joint lives' have premiums and benefits that depend on the survival of two people, typically spouses. The most common format is a first-to-die policy, where the death benefit is paid on the first death of the couple, provided (for term insurance) that the death occurs within the policy term.

Joint life policies are increasingly popular as households increasingly rely on the earnings of two partners, not just one.

Multiple life insurance: Similarly to joint life policies, multiple life term insurance policies offer a benefit payable on the first death, or on each death, within a specified group of individuals, provided death occurs within the term of the policy. This feature is commonly used to insure business partners.

Guaranteed cash values: As discussed above, whole life and endowment insurance policies usually offer cash values on surrender. The policyholder may be able to lock in guaranteed cash values by paying an additional premium. In some jurisdictions, guaranteed cash values are required by law.

Policy loans: For policies that offer cash values, policyholders may be able to borrow money from the insurer, using the cash value of the policy as collateral. A common use for this is to pay premiums when the policyholder cannot otherwise raise the necessary funds. When the policy with an outstanding policy loan attached becomes a claim (or is surrendered), the sum insured (or cash value) is reduced by the amount of the outstanding loan and interest.

Allowing policy loans increases the chance that the policyholder will continue with the policy, rather than surrender. Since, in most cases, continuing with the policy offers a better opportunity for the insurer to make profits, it is to the insurer's advantage to facilitate policy loans. Consequently, the interest rates charged on policy loans may be quite low, relative to market rates.

Accelerated benefits due to terminal illness: Under this rider, the death benefit will be paid early if the policyholder can provide medical documentation that they are suffering from a terminal illness, and are not expected to live more than one year.

The early payment of benefit on terminal illness is called an **accelerated death benefit**. For the insurer, paying the claim slightly early reduces the incentive for the policyholder to sell the policy on to a third party, in a special kind of end-of-life STOLI called a **viatical settlement**.

Accidental death benefit: For a small additional premium, policyholders can choose to have an increased sum insured payable if the cause of death is accidental, rather than through natural causes. Surprisingly, this concept provided the inspiration for the Hollywood feature film *Double Indemnity*, which was released in 1944.

Premium waiver on disability: This policy rider allows policyholders to suspend paying premiums during periods of severe illness or disability. The premium waiver would require medical evidence, and would be limited to premiums due during the policyholder's normal working lifetime, typically up to age 65.

Family income benefit (FIB): One of the purposes of term insurance is to provide funds to bridge the policyholder's family through the financial strain following his or her death. The FIB rider offers a specified amount to be paid at regular intervals between the policyholder's death and the end of the original contract term. For example, consider a policyholder who takes out a 20-year term insurance, with an FIB rider with benefit \$10 000 per year. Now suppose the policyholder dies 15 years into the policy term. Then, in addition to the regular term insurance death benefit, the FIB would pay the policyholder's family \$10 000 per year for the remaining five years of the original contract.

Critical illness insurance: A benefit is paid on diagnosis of one of a specified set of critical illnesses or disabilities, typically including most cancers, stroke and heart disease. Critical illness cover is discussed in more detail in Section 1.7.

1.4 Modern insurance contracts

1.4.1 Why innovate?

Compared with traditional policies, modern insurance is more complex, apparently more transparent in terms of costs and benefits, with flexibility in premiums and variability in benefits. We explain some reasons below.

1. **Competition with mutual funds and banks for policyholders' savings**
   Insurers developed combined insurance/investment hybrid products to attract savings away from other investment options such as mutual funds. This gives insurers more assets under management, and more chance to generate profits for their shareholders.
2. **Changing demographics and lifecycles impact insurance design**
   Nowadays, many jobs are more short term; as policyholders move in and out of work, insurance needs to offer increased flexibility, to meet policyholders' needs in both good times and bad.

3. **Developments in science and technology**
   The science of financial risk management has developed significantly in the past 30 years, offering insurers the possibility of designing valuable guarantees that can be safely risk managed. Also, more powerful computational facilities allow more complex modelling and prediction.
4. **Better informed customers**
   Potential policyholders are better informed about how products work, as financial advice has become more reliable (though there are still very many examples of bad financial advice), and is more freely available, through newspapers, or through social media. This has reduced (but, unfortunately, not eliminated) the creation of insurance products that are very profitable for the insurers and the sales intermediaries, but were not really suitable for the individuals persuaded to purchase them.

1.4.2 Universal life insurance

Universal life insurance is a very important product, particularly in North America. It is generally issued as a whole life contract, but with transparent cash values, so that policyholders can view the policy as a form of savings account with built-in life insurance, rather than a whole life contract. Given that most policyholders will surrender their policies for a significant cash value, perhaps when they retire, the Universal Life contract can be seen as an updated and more flexible version of traditional endowment insurance.

Policyholders choose a level of death benefit, which may be fixed, or may increase as the invested premiums earn interest. Premiums are deposited into a notional account (notional, as the assets are not actually separate from the general funds of the insurer). The insurer shares investment profits through the **credited interest rate** which is declared and applied by the insurer at regular intervals, typically monthly, and which reflects (perhaps indirectly) the investment performance of the underlying assets. The credited interest rate cannot be less than 0%, so the account cannot lose value. In some cases, there may be a guaranteed minimum rate which is greater than 0%. Unlike traditional insurance, which uses fixed premiums, Universal Life premiums are quite flexible. Provided there are sufficient funds in the policyholder's account to cover costs, the policyholder may reduce or even skip paying premiums for a period.

The notional account, made up of the premiums and credited interest, is subject to monthly deductions; there is a charge for the cost of life insurance cover, and a separate charge to cover expenses. The **account balance** or **account value** is the balance of funds in the policyholder's account. The account value represents the cash value for a surrendering policyholder, after an initial period (typically 7–10 years) during which surrender charges are applied to ensure recovery of the costs incurred by the insurer in issuing the policy.

1.4.3 Unitized with-profit

Unitized with-profit (UWP) is an evolution of traditional with-profit insurance which was popular for a time in the UK and Australia. It is similar to Universal Life insurance, except that in place of account values, policyholders' funds are expressed in terms of *units*, which are shares in a notional asset portfolio. The units increase in value, through the performance of the underlying investments. Bonuses may be awarded by adding additional units to the account. On death or maturity an additional terminal bonus may be added. On surrender, policyholders receive the cash value of their units, with a surrender penalty applied in the early period of the policy.

After some poor publicity surrounding with-profit business and, by association, unitized with-profit business, these product designs were largely withdrawn from the UK and Australian markets in the early 2000s. However, they will remain important for many years as many companies carry large portfolios of UWP policies issued during the 1980s and 1990s.

1.4.4 Equity-linked insurance

Equity-linked life insurance has an endowment insurance structure, with a fixed term, and with benefits paid on the earlier of the policyholder's death and the end of the contract term. Policyholders who surrender their contracts before the end of the term will generally receive a cash surrender value at that time. The death, surrender and maturity benefits are linked to the performance of a specified investment fund.

So far this sounds similar to the unitized with-profit policy, but there are two important differences.

- For equity-linked insurance, the fund that determines the return on invested premiums is a real fund, not a notional collection of assets within the insurer's general account, as for the UWP contract.
- Equity-linked insurance benefits may increase or decrease over time, in line with the underlying fund. The UWP and Universal Life benefits will only increase (or stay the same); they cannot decrease.

There are several different varieties of equity-linked insurance, but they operate in similar ways.

1. The policyholder's premiums are invested in an open-ended, mutual fund style account.
2. On death before the maturity date, the death benefit will be at least the value of the accumulated premiums; often there will be a Guaranteed Minimum Death Benefit, or GMDB, that will increase the payout if the underlying investments have performed poorly over the term of a contract.

3. On early surrender, the policyholder receives the value of the accumulated premiums, with a surrender penalty deducted at early durations.
4. On survival to the end of the contract, the policyholder receives at least the value of their accumulated premiums, possibly more if the policy offers a Guaranteed Minimum Maturity Benefit, or GMMB.

Unit-linked insurance is a form of equity-linked insurance sold outside North America. Like the UWP policies, policyholders' funds are expressed in units (or shares) of the underlying assets. Unit-linked policies generally do not offer a GMMB. The death benefit is often a multiple of the value of the policyholder's units at the time of death.

Variable annuities, also known as **segregated funds**, are equity-linked insurance policies sold in North America, which are becoming increasingly popular in other areas. Despite the name, the benefit under a Variable Annuity is a lump sum, not an annuity, although the policies carry the option to convert the proceeds to an annuity at maturity. A Variable Annuity policy will offer a GMDB and a GMMB, with additional guarantees available at additional cost.

1.5 Marketing, pricing and issuing life insurance

1.5.1 Insurance distribution methods

Most people find insurance dauntingly complex. Brokers who connect individuals to an appropriate insurance product have, since the earliest times, played an important role in the market. There is an old saying amongst actuaries that '*insurance is sold, not bought*', which means that the role of an intermediary in persuading potential policyholders to take out an insurance policy is crucial in maintaining an adequate volume of new business. Brokers and other financial advisors are often remunerated through a **commission system**. The commission would be specified as a percentage of the premium paid. Typically, there is a higher percentage paid on the first premium than on subsequent premiums. This is referred to as a **front-end load**. Some advisors may be remunerated on a fixed fee basis, or may be employed by one or more insurance companies on a salary basis. Face-to-face insurance sales focus on higher-wealth individuals who are already connected with financial advisors. For other customers, banks may act as intermediaries, but the rising trend is for **direct marketing**. This covers insurance sold through television advertising, but the more recent developments involve online sales.

The nature of the business sold by direct marketing methods tends to differ from the broker-sold business, as the target audience is likely to be less wealthy. Television advertising is used, for example, for **pre-need** insurance, which is aimed at older lives and covers funeral costs. Term insurance is a relatively straightforward contract, and as long as the sum insured is not too high, is highly suited to online marketing and issue.

1.5.2 Underwriting

It is important in modelling life insurance liabilities to consider what happens when a life insurance policy is purchased. Selling life insurance policies is a competitive business and life insurance companies are constantly considering ways in which to change their procedures so that they can improve the service to their customers and gain a commercial advantage over their competitors. The account given below of how policies are sold covers some essential points but is necessarily a simplified version of what actually happens.

For a given type of policy, such as a 10-year term insurance, the insurer will have a schedule of premium rates. These rates will depend on the size of the policy and some other factors known as **rating factors**. An applicant's risk level is assessed by asking them to complete a **proposal form** giving information on relevant rating factors, generally including their age, gender (where legislation permits), smoking habits, occupation, any dangerous hobbies, and personal and family health history. The insurer may ask for permission to contact the applicant's doctor to enquire about their medical history. In some cases, particularly for very large sums insured, the life insurer may require that the applicant's health be checked by a doctor employed by the insurer.

The process of collecting and evaluating this information is called **underwriting**. The purpose of underwriting is, first, to classify potential policyholders into broadly homogeneous risk categories, and secondly to assess what additional premium would be appropriate for applicants whose risk factors indicate that standard premium rates would be too low.

On the basis of the application and supporting medical information, potential life insurance policyholders will generally be categorized into one of the following groups.

- **Preferred lives** have very low mortality risk based on the standard information. The preferred applicant would have no recent record of smoking; no evidence of drug or alcohol abuse; no high-risk hobbies or occupations; no family history of disease known to have a strong genetic component; no adverse medical indicators such as high blood pressure or cholesterol level or body mass index.

  The preferred life category is commonly used in North America, but has not yet caught on elsewhere. In other areas there is no separation of preferred and normal lives.

- **Normal lives** may have some higher-rated risk factors than preferred lives (where this category exists), but are still insurable at standard rates. Most applicants fall into this category.
- **Rated lives** have one or more risk factors at raised levels and so are not acceptable at standard premium rates. However, they can be insured for a higher premium. An example might be someone having a family history of heart disease. These lives might be individually assessed for the appropriate additional premium to be charged. This category would also include lives with hazardous jobs or hobbies which put them at increased risk.
- **Uninsurable lives** have such significant risk that the insurer will not enter an insurance contract at any price.

Within the first three groups, applicants would be further categorized according to the relative values of the various risk factors, with the most fundamental being age, gender and smoking status. Note, however, that gender-based premiums are no longer permitted in some jurisdictions, including the European Union countries.

Most applicants (around 95% for traditional life insurance) will be accepted at preferred or standard rates for the relevant risk category. Another 2%–3% may be accepted at non-standard rates because of an impairment, or a dangerous occupation, leaving around 2%–3% who will be refused insurance.

The rigour of the underwriting process will depend on the type of insurance being purchased, on the sum insured and on the distribution process of the insurance company. Term insurance, particularly if the sum insured is very large, is generally more strictly underwritten than whole life insurance, as the risk taken by the insurer is greater. If the underwriting is not strict there is a risk of **adverse selection** by policyholders. Adverse selection (also called anti-selection) in insurance arises when policyholders use information about their own individual risk profile to make choices that will benefit them, with a potential adverse outcome for the insurer. So, we would expect very high-risk individuals to apply for insurance with larger death benefits than low-risk individuals. Since the risk to the insurer rises with the sum insured, applications involving very large sums insured would generally trigger more rigorous underwriting to counter the adverse selection risk.

The distribution method also affects the level of underwriting. Often, direct marketed contracts are sold with relatively low benefit levels, and with the attraction that no medical evidence will be sought beyond a standard questionnaire. The insurer may assume relatively heavy mortality for these lives to compensate for potential adverse selection. By keeping the underwriting relatively light, the expenses of writing new business, termed **acquisition expenses**, can be kept low, which is an attraction for high-volume, low-sum-insured contracts.

It is interesting to note that with no third party medical evidence the insurer is placing a lot of weight on the veracity of the policyholder. Insurers have a phrase for this – that both insurer and policyholder may assume ‘utmost good faith’ or ‘*uberrima fides*’ on the part of the other side of the contract. In practice, in the event of the death of the insured life, the insurer may investigate whether any pertinent information was withheld from the application. If it appears that the policyholder held back information, or submitted false or misleading information, the insurer may not pay the full sum insured.

1.5.3 Premiums

A life insurance policy may involve a single premium, payable at the outset of the contract, or a regular series of premiums payable provided the policyholder survives, perhaps with a fixed end date. In traditional contracts the regular premium is generally a level amount throughout the term of the contract; in more modern contracts the premium might be variable, at the policyholder's discretion for investment products such as equity-linked insurance, or at the insurer's discretion for certain types of renewable term insurance.

Regular premiums may be paid annually, semi-annually, quarterly, monthly or weekly. Monthly premiums are common as it is convenient for policyholders to have their outgoings payable with approximately the same frequency as their income.

An important feature of all premiums is that they are paid at the start of each period. Suppose a policyholder contracts to pay annual premiums for a 10-year insurance contract. The premiums will be paid at the start of the contract, and then at the start of each subsequent year provided the policyholder is alive. So, if we count time in years from $t = 0$ at the start of the contract, the first premium is paid at $t = 0$, the second is paid at $t = 1$, and so on, to the tenth premium paid at $t = 9$. Similarly, if the premiums are monthly, then the first monthly instalment will be paid at $t = 0$, and the final premium will be paid at the start of the final month at $t = 9\frac{11}{12}$ years. (Throughout this book we assume that all months are equal in length, at $\frac{1}{12}$ years.)

1.6 Life annuities

Annuity contracts offer a regular series of payments. When an annuity depends on the survival of the recipient, it is called a ‘life annuity’. The recipient is called an annuitant. If the annuity continues until the death of the annuitant, it is called a **whole life annuity**. If the annuity is paid for some maximum period, provided the annuitant survives that period, it is called a **term life annuity**.

Annuities are often purchased by older lives to provide income in retirement. Buying a whole life annuity guarantees that the income will not run out before the annuitant dies.

Annuities cannot be surrendered; there is no cash value once the annuity payments commence. The main reason is that allowing surrenders would create unmanageable risk of adverse selection – the lives who are most unwell are most likely to surrender. Annuity pricing assumes that on the annuitant's death, any excess funds built up from investing the premiums are then used to offset the costs of annuities for surviving annuitants.

Types of annuities that may be issued include the following.

- **Single Premium Deferred Annuity (SPDA):** Under an SPDA contract, the policyholder pays a single premium in return for an annuity which commences payment at some future, specified date. The annuity is 'life contingent', by which we mean the annuity is paid only if the policyholder survives to the payment dates. If the policyholder dies before the annuity commences, there may be a death benefit due. If the policyholder dies soon after the annuity commences, there may be some minimum payment period, called the guarantee period, and the balance would be paid to the policyholder's estate.
- **Single Premium Immediate Annuity (SPIA):** This contract is the same as the SPDA, except that the annuity commences as soon as the contract is effected. This might, for example, be used to convert a lump sum retirement benefit into a life annuity to supplement a pension. As with the SPDA, there may be a guarantee period applying in the event of the early death of the annuitant.
- **Regular Premium Deferred Annuity (RPDA):** The RPDA offers a deferred life annuity with premiums paid through the deferred period. It is otherwise the same as the SPDA.
- **Joint life annuity:** A joint life annuity is issued on two lives, typically a couple (that is, married or cohabiting). The annuity, which may be single premium or regular premium, immediate or deferred, continues while both lives survive, and ceases on the first death of the couple.
- **Last survivor annuity:** A last survivor annuity is similar to the joint life annuity, except that payment continues while at least one of the lives survives, and ceases on the second death of the couple.
- **Reversionary annuity:** A reversionary annuity is contingent on two lives, usually a couple. One is designated as the annuitant, and one the insured. No annuity benefit is paid while the insured life survives. On the death of the insured life, if the annuitant is still alive, the annuitant receives an annuity for the remainder of their life.
- **Guaranteed annuity:** A guaranteed annuity is paid for a minimum period, regardless of the survival or death of the annuitant. After the guarantee period, if the annuitant is still alive, the annuity is paid for the remainder of their lifetime.

Annuity sales methods are similar to life insurance, with individual brokers playing an important role for higher-wealth individuals. In addition, as annuities are often used to convert retirement savings into retirement income, pension plan managers may work with retirees on annuity purchase.

There is no underwriting for regular annuities. The risk to the insurer (or annuity provider) is that the annuitant lives longer than expected; it is not considered feasible to seek health evidence that potential annuitants are too long-lived.

1.7 Long-term coverages in health insurance

1.7.1 Disability income insurance

Disability income insurance, also known as income protection insurance, is designed to replace income for individuals who cannot work, or cannot work to full capacity due to sickness or disability. Typically, level premiums are payable at regular intervals through the term of the policy, but are suspended during periods of disability. Benefits are paid at regular intervals during periods of disability. The benefits are usually related to the policyholder's salary, but, to encourage the policyholder to return to work as soon as possible, the payments are often capped at 50–70% of the salary that is being replaced. The policy could continue until the insured person reaches retirement age.

Common features or options of disability income insurance include the following.

- The **waiting period** or **elimination period** is the time between the beginning of a period of disability and the beginning of the benefit payments. Policyholders select a waiting period from a list offered by the insurer, with typical periods being 30, 60, 180 or 365 days.
- The payment of benefits based on **total disability** requires the policyholder to be unable to work at their usual job, and to be not working at a different job. Medical evidence of the disability is also required by the insurer at intervals.
- If the policyholder can do some work, but not at the full earning capacity established before the period of disability, they may be eligible for a lower benefit based on **partial disability**.
- The amount of disability benefits payable may be reduced if the policyholder receives disability-related income from other sources, for example from workers' compensation or from a government benefit programme.
- The benefit payment term is selected by the policyholder from a list of options. Typical terms are two years, five years, or up to age 65. Once the disability benefit comes into payment, it will continue to the earlier of the recovery of the policyholder to full health, or the end of the selected benefit

term, or the death of the policyholder. If the policyholder moves from full disability to partial disability, then the benefit payments may be decreased, but the total term of benefit payment (covering the full and partial benefit periods) could be fixed.

For shorter benefit payment terms, the policy covers each separate period of disability, so even if the full benefit term of, say, two years has expired, if the policyholder later becomes disabled again, provided sufficient time has elapsed between periods of disability, the benefits would be payable again for another period of two years.

- When two periods of disability occur with only a short interval between them, they may be treated as a single period of disability for determining the benefit payment term. The **off period** determines the required interval for two periods of disability to be considered separately rather than together, and it is set by the insurer.

  For example, suppose a policyholder purchases disability income insurance with a two-year benefit term, monthly benefit payments and a two-month waiting period. The insurer sets the off period at six months. The policyholder becomes sick on 1 January 2017, and remains sick until 30 June 2017. She returns to work but suffers a recurrence of the sickness on 1 September 2017.

  The first benefit payment would be made at the end of the elimination period, on 1 March 2017, and would continue through to 30 June. Since the recurrence occurs within the six-month off period, the second period of sickness would be treated as a continuation of the first. That means that the policyholder would not have to wait another two months to receive the next payment, and it also means that on 1 September, four months of the 24-month benefit term would have expired, and the benefits would continue for another 20 months, or until earlier recovery.
- **Own job** or **any job**: the definition of total disability may be based on the policyholder's inability to perform their own job, or on their ability to perform any job that is reasonable given the policyholder's qualifications and experience. A policy that pays benefits only if the policyholder is unable to perform any job requires the policyholder to be very ill before any payments are made. On the other hand, the policy that pays out when the policyholder is unable to do her/his own job, even if they can undertake paid work that is less demanding then their own job, will pay out more often, and will therefore be more costly.
- Disability income insurance may be purchased as a group insurance by an employer, to offset the costs of paying long-term disability benefits to the employees. Group insurance rates (assuming employees cannot opt out) may be lower than the equivalent rates for individuals, because the group policies

carry less risk from adverse selection. There are also economies of scale, and less risk of non-payment of premiums from group policies.

- Long-term disability benefits may be increased in line with inflation.
- Policies often include additional benefits such as **return to work assistance**, which offsets costs associated with returning to work after a period of disability; for example, the policyholder may need some re-training, or it may be appropriate for the policyholder to phase their return to work by working part-time initially. It is in the insurer's interests to ensure that the return to work is as smooth and as successful as possible for the policyholder.

1.7.2 Long-term care insurance

In a typical North American long-term care (LTC) contract, premiums are paid regularly while the policyholder is well. When the policyholder requires care, based on the benefit triggers defined in the policy, there is a **waiting period**, similar to the elimination period for disability income insurance; 90 days is typical. After this, the policy will pay benefits as long as the need for care continues, or until the end of the selected benefit payment period.

Common features or options associated with LTC insurance in the USA and Canada include the following.

- The trigger for the payment of benefits is usually described in terms of the Activities of Daily Living, or ADLs. There are six ADLs in common use:
  - Bathing
  - Dressing
  - Eating (does not include cooking)
  - Toileting (ability to use the toilet and manage personal hygiene)
  - Continence (ability to control bladder and bowel functions)
  - Transferring (getting in and out of a bed or chair)

  If the policyholder requires assistance to perform two or more of the ADLs, based on certification by a medical practitioner, then the LTC benefit is triggered, and the waiting period, if any, commences.
- There is often an alternative trigger based on severe cognitive impairment of the policyholder.
- Although the most common policy design uses two ADLs for the benefit trigger, some policies use three.
- At issue, the policyholder may select a definite term benefit period (typical options are between two years and five years), or may select an indefinite period, under which benefit payments continue as long as the trigger conditions apply.
- The benefit payments may be based on a reimbursement approach, under which the benefits are paid directly to the caregiving organization, and

cover the cost of providing appropriate care, up to a daily or monthly limit. Alternatively, the benefit may be based on a fixed annuity payable during the benefit period. The policyholder may have the flexibility to apply the benefit to whatever form of care is most suitable, but there is no guarantee that the annuity would be sufficient for the level of care required.

- The insurer may offer the option to have the payments, or payment limits, increase with inflation.
- Similarly to disability income insurance, an off period, typically six months, is used to determine whether two successive periods of care are treated separately or as a single continuous period.
- Hybrid LTC and life insurance policies are becoming popular. There are different ways to combine the benefits.
  - Under the **return of premium** approach, if the benefits paid under the LTC insurance are less than the total of the premiums paid, the balance is returned as part of the death benefit under the life insurance policy.
  - Under the **accelerated benefit** approach, the sum insured under the life insurance policy is used to pay LTC benefits. If the policyholder dies before the full sum insured has been paid in LTC benefits, the balance is paid as a death benefit.
  - The policyholder may add an **extension of benefits** option to the hybrid insurance, which would provide for continuation of the LTC benefits for a pre-determined period after the original sum insured has been exhausted. Typically, extension periods offered are in the range of two to five years.
- Premiums are designed to be level throughout the policy term, but insurers may retain the right to increase premiums for all policyholders if the experience is sufficiently adverse. Generally, insurers must obtain approval from the regulating body for such rate increases. In this circumstance, policyholders may be given the option to maintain the same premiums for a lower benefit level.

LTC insurance in other countries is generally similar to the North American design, with some variation that we describe briefly here.

Policies in France, where LTC insurance is very popular, are simpler and cheaper than in the USA; with average premiums of around 25% of those in the USA. Benefits are paid as a fixed or inflation-indexed annuity. The policyholder may choose a policy based on ‘mild or severe dependency’ or one based on ‘severe dependency’ only, which is the cheaper and more popular option. Severe dependency is defined as *bed- or chair-bound, requiring assistance several times a day* or *cognitive impairment requiring constant monitoring*. Mild dependency refers to cases where the individual needs help with eating, bathing and/or some mobility, but is not bed- or chair-bound.

Reasons for the lower premiums, relative to the North American model, include (i) lower average benefits; (ii) lower risk of payment, as the ‘severe dependency’ requirement is more stringent than the US ADL requirements; (iii) policies are often purchased through group plans facilitated by employers, reducing the expenses; and (iv) individuals in France tend to purchase their policies at younger ages than in the USA.

In Germany, basic LTC costs are covered under the government-provided social health insurance. Individuals can top up the government benefit with private LTC insurance, or can opt out of the state benefit (and thereby opt out of the tax supporting the benefit) and use LTC insurance instead. The benefits are fixed annuities.

In Japan, LTC insurance is offered on a stand-alone basis or as a rider on a whole life policy. The benefit is triggered when the policyholder reaches a specified level of dependency, and additional benefits may be added when the level of dependency increases.

In the UK, regular premium LTC policies are no longer offered, as they never reached the necessary level of popularity for the business to be sustained. In their place is a different kind of pre-funding, called an **immediate needs annuity**. This is a single premium immediate annuity that is purchased as the individual is about to move permanently into residential long-term care. The benefit is paid as a regular fixed annuity, but is paid directly to the care home, saving the policyholder from having to pay income tax on the proceeds. Because the lives are assumed to be somewhat impaired, and the insurer’s exposure to adverse selection with respect to longevity is reduced, the benefit amount per unit of single premium may be significantly greater than a regular single premium life annuity at the same age.

1.7.3 Critical illness insurance

Critical illness insurance pays a lump sum benefit on diagnosis of one of a list of specified diseases and conditions. Different policies and insurers may cover slightly different illnesses, but virtually all include heart attack, stroke, major organ failure and most forms of cancer. Policies may be whole life or for a definite term. Unlike disability income insurance or LTC insurance, once the claim arises, the benefit is paid and the policy expires. A second critical illness diagnosis would not be covered. Some policies offer a partial return of premium if the policy expires or lapses without a critical illness diagnosis.

Level premiums are typically paid monthly throughout the term, though they may cease at, say, 75 for a long-term policy.

Critical illness cover may be added to a life insurance policy as an **accelerated benefit rider**. In this case, the critical illness diagnosis triggers the payment of some or all of the death benefit under the life insurance. Where the full benefit is accelerated, the policy expires on the critical illness diagnosis. If only part of the benefit is accelerated, then the remainder is paid out when the policyholder dies.

1.7.4 Chronic illness insurance

Chronic illness insurance pays a benefit on diagnosis of a chronic illness, defined as one from which the policyholder will not recover, although the illness does not necessarily need to be terminal. The illness must be sufficiently severe that the policyholder is no longer able to perform two or more of the ADLs listed in the LTC insurance section. The benefit under a chronic illness policy is paid as a lump sum or as an annuity.

Chronic illness insurance is typically added to a standard life insurance policy as an accelerated benefit rider, similar to the critical illness case.

1.8 Mutual and proprietary insurers

A **mutual** insurance company is one that has no shareholders. The insurer is owned by the with-profit policyholders. All profits are distributed to the with-profit policyholders through dividends or bonuses.

A **proprietary** insurance company has shareholders, and usually has with-profit policyholders as well. The participating policyholders are not owners, but have a specified right to some of the profits. Thus, in a proprietary insurer, the profits must be shared in some predetermined proportion between the shareholders and the with-profit policyholders.

Many early life insurance companies were formed as mutual companies. More recently, in the UK, Canada and the USA, there has been a trend towards demutualization, which means the transition of a mutual company to a proprietary company, through issuing shares (or cash) to the with-profit policyholders. Although it would appear that a mutual insurer would have marketing advantages, as participating policyholders receive all the profits and other benefits of ownership, the advantages cited by companies who have demutualized include increased ability to raise capital, clearer corporate structure and improved efficiency.

1.9 Other life contingent contracts

In the following sections we discuss benefit and payment streams which, like the life and health insurance premiums and benefits described above, are life contingent, and are subject to actuarial valuation and risk management, but which are not insurance contracts.

1.9.1 Continuing care retirement communities

Continuing care retirement communities (CCRCs) are residential facilities for seniors, with different levels of medical and personal support designed to adapt to the residents as they age. Many CCRCs offer funding packages where the costs of future care are covered by a combination of an entry fee and a monthly charge. The description below, and examples used in subsequent chapters of this book, follow the US industry standard definitions and systems.

There are generally three or four of the following categories of residence in a CCRC.

Independent living units (ILUs) represent the first stage of residence in a CCRC. These are apartments with fairly minimal external care provided (for example, housekeeping, emergency call buttons, transport to shopping).

Assisted living units (ALUs) allow more individual support for residents who need help with at least one, and commonly several, of the activities of daily living. Most of the support at this level is non-medical – help with bathing, dressing, preparation of meals, etc.

The **skilled nursing facility** (SNF) is for residents who need ongoing medical care. The SNF often looks more like a hospital facility.

Memory care units (MCU) offer a separate, more secure facility for residents with severe dementia or other cognitive impairment.

The industry has developed different forms of funding for CCRCs. Not every CCRC will offer all funding options, and some will offer variants that are not described here, but these are the major forms in current use.

- Residents can choose to pay a large upfront fee, and monthly payments which are level, or which are only increasing with cost of living adjustments. The resident is guaranteed that all residential, personal assistance and health care needs will be covered without further cost. This is called a **full life care** contract, or life care contract.
- Under a **modified life care** contract, residents pay lower monthly fees, and possibly a lower entry fee, but will have to pay additional costs for some services if they need them. For example, residents may be charged a higher monthly fee as they move into the ALU, with further increases on entry to the SNF or the MCU. Typically, the increases would be less than the full market cost of the additional care, meaning that the costs are partially pre-funded through the entry fee and regular monthly payments.
- **Fee-for-service** contracts involve little or no pre-funding of health care. Residents pay for the health care they receive at the current market rates.

Fee-for-service contracts have the lowest entry fee and monthly payments, as these only cover the accommodation costs.

- Prospective residents entering under full life care or modified life care must be sufficiently well to live independently when they enter the CCRC, and a medical examination is generally required. Entrants who are already sufficiently disabled to need more care are eligible only for fee-for-service contracts.
- Under full or modified life care contracts, the CCRC may offer a partial refund of the entry fee on the resident's death or when the resident moves out. This may involve some options, for example, the resident can choose a higher entry fee with a partial refund, or a lower entry fee with no refund.
- There are some CCRCs that offer (partial) ownership of the ILU, in place of some or all of the entry fee. When the resident moves out of independent living permanently, or dies, the unit is sold, with the proceeds shared between the resident (or her estate) and the CCRC.
- It is common for couples to purchase CCRC membership jointly, and different payment schedules may be applied to couples in comparison with schedules for single residents entering the CCRC.

The average age at entry to a CCRC in the USA is around 80, with full life care entrants generally being younger than modified life care entrants, who are younger than fee-for-service entrants, on average.

The full life care and (to a lesser extent) modified life care contracts transfer the risk of increasing health care costs from the resident to the CCRC, and therefore are a form of insurance.

1.9.2 Structured settlements

When a person is injured because of a negligent or criminal act committed by another person, or by an institution, legal processes will determine a suitable amount of compensation paid to the injured party (IP) by the person or institution who caused the injury (the responsible person, RP). Often cases are solved outside of the formal court system, but, if the issue is settled through a court case, the IP might be referred to as the plaintiff, and the RP as the defendant.

The compensation may be paid as a lump sum, but in some jurisdictions it is more common for the payment to be paid as an annuity, or as a combination of a lump sum and an annuity. If the injury is very serious, such as paralysis, loss of limbs, or permanent brain damage, the settlement will be a whole life annuity. Less severe injuries may be compensated with a term life annuity, extending to the point where the individual is expected to be recovered. Annuity payments may increase from time to time to offset the effects of inflation. The reason for using an annuity format rather than a lump sum is that the annuity better replicates the losses of the IP, in the form of lost wages and/or ongoing expenses associated with medical care or additional needs arising from the injury. Rehabilitation costs and any expenses associated with re-training for the workplace would also be covered through the settlement.

A **structured settlement** is the payment schedule agreed between the IP and the RP, usually through their lawyers, or through an insurer when the RP's liability is covered by an insurance policy. The annuity part may be funded with a single premium immediate annuity purchased from an insurer or from a firm that specializes in structured settlements.

Structured settlements are often used for payments under **workers' compensation** insurance. Workers' compensation (also known as Workers' Comp, or Employer's Liability) is a type of insurance purchased by employers to fund the costs of compensating employees who are injured at work. Structured settlements are also commonly used in medical malpractice cases, and for other personal injury claims, such as from motor vehicle accidents.

Replacement of income will normally be at less than 100% of pre-injury earnings, and there are several reasons for this.

- In some countries (including the USA and the UK) income from a structured settlement annuity is not taxed. Hence, less annuity is required to support the IP's pre-injury lifestyle.
- The insurer wants to ensure that the IP has a strong incentive to return to work.
- The amount of compensation may be reduced if the IP is determined to be partially at fault in the incident.

The annuity will typically include some allowance for inflation. This may be a fixed annual increase, or the annuity may be fully indexed to inflation.

In cases of potentially severe injury, there is often a period of uncertainty as to the extent of damage and long-term prognosis for the IP. For example, it may take a year of treatment and rehabilitation to determine the level of permanent damage from a spinal cord injury. In such cases there may be an interim arrangement of benefit until the time of **maximum medical improvement**, at which point the final structured settlement will be determined.

Structured settlements evolved from a system where the entire compensation was in a lump sum form, but paying compensation as a lump sum requires the IP to manage a potentially very large amount of money. There is a strong temptation for the IP to overspend; research indicates that 80%–90% of recipients spend their entire lump sum compensation within five years. Even a fairly prudent individual who invests the award in stocks and bonds could lose 30% of their funds in a stock market crash. An annuity relieves the IP from investment risk and from **dissipation risk**, which is the risk of overspending, leading to subsequent financial hardship. The move from lump sum to annuities in structured settlements has led to two different approaches to determining the payments.

The **top-down approach** starts with determination of an appropriate lump sum compensation, and then converts that to an annuity.

The **bottom-up approach** starts with a suitable income stream, and then converts that to a capital value.

Because the purpose of the settlement is to restore the IP to their former financial position, as far as possible, the bottom-up method seems most appropriate.

In some areas of the USA the IP may transfer their annuity to a specialist firm in exchange for a lump sum, under a '**structured settlement buy-out**'. After concerns that the buy-out firms were making excessive profits on these transactions, the market has become more regulated, with buy-outs in many areas prohibited or at least requiring court approval. Structured settlement buy-outs are not permitted in Canada, where the structured settlement provider must ensure that the payments are going directly to the IP.

1.10 Pensions

Many actuaries work in the area of employer-sponsored pension plan design, valuation and risk management. Pension plans typically offer employees (also called plan members) lump sum and annuity benefits (or a combination of these) when the employee retires. Some plans also offer benefits if the employee dies while still employed. Pension benefits therefore depend on the survival and employment status of the member, and are quite similar in nature to life insurance benefits – that is, they involve investment of contributions long into the future to pay for future life contingent benefits. In this section we give a slightly more detailed description of the different types of pension plan that actuaries typically work with.

1.10.1 Defined Benefit pensions

Defined Benefit (DB) pension plans provide members with lifetime retirement income, with the amount of annual pension determined using a formula that depends on the member's salary and period of service. The pension plan may also offer a lump sum retirement benefit, usually a multiple of the annual pension.

The benefits are funded by contributions paid by the employer and (usually) the employee over the working lifetime of the employee. The contributions are invested, and the accumulated contributions must be enough, on average, to pay the pensions when they become due.

The annual pension benefit in a defined benefit pension plan, which we denote as B, is typically determined from the formula

$$B = \alpha\, S\, n \tag{1.1}$$

where α is the plan **accrual rate**, typically around 1%–2%; n is the number of years of employment within the plan (n does not need to be an integer) and S is one of the following measures of the retiree's pensionable salary, depending on the plan type:

Final Salary Pension Plan: S is the retiree's average salary over the last few years of employment (typically three to five years). This type of plan is also sometimes (and more accurately) called a Final Average Salary plan.

Career Average Earnings Pension Plan: S is the average salary earned by the retiree over their entire period of employment within the plan.

Career Average Revalued Earnings Pension Plan: S is the average salary earned by the retiree over their entire career, but with all salaries adjusted for inflation to values at retirement.

The interpretation of the benefit formula is that during each year of pensionable employment, the employee accrues $\alpha\, S$ of annual retirement pension.

DB plans may also offer **withdrawal benefits** for employees who leave before retirement age. A typical benefit would be a pension based on the benefit formula above, but with the start date deferred until the employee reaches the normal retirement age. Employees who leave to move to a new employer may have the option of taking a lump sum with the same value as the deferred pension, which can be invested in the pension plan of the new employer.

Some pension plans also offer **death in service** benefits, for employees who die during their period of employment. The benefit might be a lump sum payment, where the amount depends on the salary at the time of death, and a pension for the employee's spouse, based again on formula (1.1), but with a different accrual rate.

1.10.2 Defined Contribution

Defined Contribution (DC) pensions work more like a bank account than an insurance or annuity contract. Employees and their employer pay a predetermined contribution (usually a fixed percentage of salary) into a fund, and the fund earns interest. When the employee leaves or retires, the proceeds are available to them as a lump sum. The employee may use the proceeds to buy an annuity. Alternatively, they may live on the funds without purchasing an annuity, drawing down some amount each year until the retiree dies, or the funds are exhausted.

Using the DC funds to purchase a life annuity offers the security of lifetime income, but takes away the flexibility provided by having the money readily available. If the retiree does not buy an annuity, they run the risk that their funds will expire before they do. This is another example of dissipation risk.

1.11 Typical problems

We are concerned in this book with developing the mathematical models and techniques used by actuaries working in long-term insurance and pensions. The primary responsibility of the life or health insurance actuary is to maintain the solvency and profitability of the insurer. Premiums must be sufficient to pay benefits; the assets held must be sufficient to pay the contingent liabilities; bonuses and benefits payable to policyholders should be fair.

Consider, for example, a whole life insurance contract issued to a life aged 50. The sum insured may not be paid for 40 years or more. The premiums paid over the period will be invested by the insurer to earn significant interest; the accumulated premiums must be sufficient to pay the benefits, on average. To ensure this, the actuary needs to model the survival probabilities of the policyholder, the investment returns likely to be earned and the expenses likely to be incurred in maintaining the policy. The actuary may take into consideration the probability that the policyholder decides to terminate the contract early. The actuary may also consider the profitability requirements for the contract. Then, when all of these factors have been modelled, the actuary must use the results to set an appropriate premium.

Subsequently, at regular intervals over the term of the policy, the actuary must determine how much money the insurer should hold to ensure that, with very high probability, the funds will be sufficient to cover future benefits and expenses. This is called the valuation process. For with-profit insurance, the actuary must also determine a suitable level of bonus or dividend.

The problems are rather more complex if the insurance also covers morbidity (sickness) risk, or involves several lives. All of these topics are covered in the following chapters.

The actuary may also be involved in decisions about how the premiums are invested. It is vitally important that the insurer remains solvent, as the contracts are very long-term and individual policyholders rely on the insurer for their future financial security. The selection and management of investments can increase or mitigate the risk of insolvency.

The pensions actuary working with defined benefit pensions must determine contribution rates which will be sufficient to meet the benefits promised, allowing for investment proceeds, and using models that allow for the working patterns of the employees. Sometimes, the employer may want to change the benefit structure, and the actuary is responsible for assessing the potential cost and impact. When one company with a pension plan takes over another, the actuary will assist with determining the best way to allocate the assets from the two plans, and perhaps how to merge the benefits. On a smaller scale, when a pension plan member divorces, an actuary may be involved in assessing a fair division of the pension assets.

1.12 Notes and further reading

A number of essays describing actuarial practice can be found in Renn (1998). This book also provides both historical and more contemporary contexts for life contingencies.

The original papers of Graunt and Halley are available online (and any search engine will find them). Anyone interested in the history of probability and actuarial science will find these interesting, and remarkably modern.

Charles *et al.* (2000) gives more information on the behaviour of recipients of compensation under structured settlements.

1.13 Exercises

Shorter exercises

Exercise 1.1 Explain why premiums are payable in advance, so that the first premium is due now rather than in one year's time.

Exercise 1.2 It is common for insurers to design whole life contracts with premiums payable only up to age 80. Why?

Exercise 1.3 Is term insurance lapse-supported? Justify your answer.

Exercise 1.4 Explain with reasons which of the following contract types will have the highest initial fees for a healthy life entering an independent living unit:

(A) Full life care,
(B) Modified life care,
(C) Fee-for-service.

Longer exercises

Exercise 1.5 (a) Why do insurers generally require evidence of health from a person applying for life insurance but not for an annuity?

(b) Explain why an insurer might demand more rigorous evidence of a prospective policyholder's health status for a term insurance than for a whole life insurance.

Exercise 1.6 Lenders offering mortgages to home owners may require the borrower to purchase life insurance to cover the outstanding loan on the death of the borrower, even though the mortgaged property is the loan collateral.

(a) Explain why the lender might require term insurance in this circumstance.
(b) Describe how this term insurance might differ from the standard term insurance described in Section 1.3.
(c) Can you see any problems with lenders demanding term insurance from borrowers?

Exercise 1.7 Describe the difference between a cash bonus and a reversionary bonus for participating whole life insurance. What are the advantages and disadvantages of each for (a) the insurer and (b) the policyholder?

Exercise 1.8 A policyholder purchases a participating whole life policy, with sum insured \$50 000. Profits are distributed through reversionary bonuses.

Calculate the total sum insured plus bonus in each of the first five years of the contract, under each of the following assumptions.

(a) Simple reversionary bonuses of 5% are declared at the start of each year after the first.
(b) Compound reversionary bonuses of 5% are declared at the start of each year after the first.
(c) Super-compound reversionary bonuses of 5% of the initial sum insured, plus 10% of past bonuses, are declared at the start of each year after the first.

Exercise 1.9 Mungo purchases disability income insurance. The benefit is a payment of \$2000 per month during qualifying periods of disability. The term for each period of benefit is five years. The waiting period is one year, and the off period is six months. The policy expires after 10 years.

Determine the amounts and times of benefit payments under the scenario described in the following table. Time is measured from the inception of the policy, and 'sick' indicates that, subject to the waiting period and benefit term constraints, Mungo is sufficiently unable to work to qualify for the insurance benefits.

Time interval from inception	Sick or Healthy	Time interval from inception	Sick or Healthy
0.00–1.00	Healthy	3.50–3.75	Healthy
1.00–1.25	Sick	3.75–8.00	Sick
1.25–2.00	Healthy	8.00–8.75	Healthy
2.00–3.50	Sick	8.75–10.00	Sick

Exercise 1.10 Andrew is retired. He has no pension, but has capital of \$500 000. He is considering the following options for using the money:

(a) Purchase an annuity from an insurance company that will pay a level amount for the rest of his life.
(b) Purchase an annuity from an insurance company that will pay an amount that increases with the cost of living for the rest of his life.
(c) Purchase a 20-year annuity certain.
(d) Invest the capital and live on the interest income.
(e) Invest the capital and draw \$40 000 per year to live on.

What are the advantages and disadvantages of each option?

Answers to selected exercises

1.4 (A) Full life care

1.8 The total sum insured and bonus are as follows:

Year	Simple	Compound	Super-Compound
1	50 000	50 000	50 000
2	52 500	52 500	52 500
3	55 000	55 125	55 250
4	57 500	57 881	58 275
5	60 000	60 775	61 603

1.9 Benefit payments are as follows:

Time from inception	Benefit payments
3.00–3.50	6 months at \$2000 per month
3.75–8.00	51 months at \$2000 per month
9.75–10.00	3 months at \$2000 per month

2
Survival models

2.1 Summary

In this chapter we represent the future lifetime of an individual as a random variable, and show how probabilities of death or survival can be calculated under this framework. We then define the force of mortality, which is a fundamental quantity in mortality modelling. We introduce some actuarial notation, and discuss properties of the distribution of future lifetime. We introduce the curtate future lifetime random variable, which represents the number of complete years of future life, and is a function of the future lifetime random variable. We explain why this function is useful and derive its probability distribution.

2.2 The future lifetime random variable

In Chapter 1 we saw that many insurance policies provide a benefit on the death of the policyholder. When an insurance company issues such a policy, the policyholder's date of death is unknown, so the insurer does not know exactly when the death benefit will be payable. In order to estimate the time at which a death benefit is payable, the insurer needs a model of human mortality, from which probabilities of death at particular ages can be calculated, and this is the topic of this chapter.

We start with some notation. Let (x) denote a life aged x, where $x \geq 0$. The death of (x) can occur at any age greater than x, and we model the future lifetime of (x) by a continuous random variable which we denote by T_x. This means that $x + T_x$ represents the age-at-death random variable for (x). Let F_x be the distribution function of T_x, so that

$$F_x(t) = \Pr[T_x \leq t].$$

Then $F_x(t)$ represents the probability that (x) does not survive beyond age $x + t$, and we refer to F_x as the **lifetime distribution** from age x. In many life insurance problems we are interested in the probability of survival rather than death, and so we define S_x as

$$S_x(t) = 1 - F_x(t) = \Pr[T_x > t].$$

Thus, $S_x(t)$ represents the probability that (x) survives for at least t years, and S_x is known as the **survival function**.

Given our interpretation of the collection of random variables $\{T_x\}_{x \geq 0}$ as the future lifetimes of individuals, we need a connection between any pair of them. To see this, consider T_0 and T_x for an individual who is now aged x. The random variable T_0 represented the future lifetime at birth for this individual, so that, at birth, the individual's age at death would have been represented by T_0. This individual could have died before reaching age x – the probability of this was $\Pr[T_0 < x]$ – but has survived. Now that the individual has survived to age x, so that we know that $T_0 > x$, her future lifetime is represented by T_x and her age at death is now $x + T_x$. If she dies within t years from now, then $T_x \leq t$ and $T_0 \leq x + t$. Loosely speaking, we require the events $[T_x \leq t]$ and $[T_0 \leq x + t]$ to be equivalent, given that the individual survives to age x. We achieve this by making the following assumption for all $x \geq 0$ and for all $t > 0$

$$\boxed{\Pr[T_x \leq t] = \Pr[T_0 \leq x + t | T_0 > x].} \tag{2.1}$$

This is an important relationship.

Now, recall from probability theory that for two events A and B

$$\Pr[A|B] = \frac{\Pr[A \text{ and } B]}{\Pr[B]},$$

so, interpreting $[T_0 \leq x + t]$ as event A, and $[T_0 > x]$ as event B, we can rearrange the right-hand side of (2.1) to give

$$\Pr[T_x \leq t] = \frac{\Pr[x < T_0 \leq x + t]}{\Pr[T_0 > x]},$$

that is,

$$F_x(t) = \frac{F_0(x + t) - F_0(x)}{S_0(x)}. \tag{2.2}$$

Also, using $S_x(t) = 1 - F_x(t)$,

$$\boxed{S_x(t) = \frac{S_0(x + t)}{S_0(x)},} \tag{2.3}$$

which can be written as

$$\boxed{S_0(x + t) = S_0(x)\, S_x(t).} \tag{2.4}$$

This is a very important result. It shows that we can interpret the probability of survival from birth to age $x+t$ as the product of

(1) the probability of survival to age x from birth, and
(2) the probability, having survived to age x, of further surviving to age $x+t$.

Note that $S_x(t)$ can be thought of as the probability that (0) survives to at least age $x+t$ given that (0) survives to age x, so this result can be derived from the standard probability relationship

$$\Pr[A \text{ and } B] = \Pr[A|B] \times \Pr[B]$$

where the events here are $A = [T_0 > x+t]$ and $B = [T_0 > x]$, so that

$$\Pr[A|B] = \Pr[T_0 > x+t | T_0 > x],$$

which we know from equation (2.1) is equal to $\Pr[T_x > t]$.

Similarly, any survival probability for (x), for, say, $t+u$ years can be split into the probability of surviving the first t years, and then, given survival to age $x+t$, subsequently surviving another u years. That is,

$$\begin{aligned} S_x(t+u) &= \frac{S_0(x+t+u)}{S_0(x)} \\ \Rightarrow S_x(t+u) &= \frac{S_0(x+t)}{S_0(x)} \frac{S_0(x+t+u)}{S_0(x+t)} \\ \Rightarrow S_x(t+u) &= S_x(t)\, S_{x+t}(u). \end{aligned} \tag{2.5}$$

We have already seen that if we know survival probabilities from birth, then, using formula (2.4), we also know survival probabilities for our individual from any future age x. Formula (2.5) takes this a stage further. It shows that if we know survival probabilities from any age x (≥ 0), then we also know survival probabilities from any future age $x+t$ ($\geq x$).

Any survival function for a lifetime distribution must satisfy the following conditions to be valid.

Condition 1 $S_x(0) = 1$*; that is, the probability that a life currently aged x survives 0 years is 1.*

Condition 2 $\lim_{t\to\infty} S_x(t) = 0$*; that is, all lives eventually die.*

Condition 3 *The survival function must be a non-increasing function of t; that is, it cannot be more likely that* (x) *survives, say 10.5 years than 10 years, because in order to survive 10.5 years,* (x) *must first survive 10 years.*

These conditions are both necessary and sufficient, so that any function S_x which satisfies these three conditions as a function of t (≥ 0), for a fixed

x (≥ 0), defines a lifetime distribution from age x, and, using formula (2.5), for all ages greater than x.

For all the distributions used in this book, we make three additional assumptions:

Assumption 2.1 *$S_x(t)$ is differentiable for all $t > 0$.*

Note that together with Condition 3 above, this means that $\frac{d}{dt}S_x(t) \leq 0$ for all $t > 0$.

Assumption 2.2 $\lim_{t\to\infty} t\,S_x(t) = 0.$

Assumption 2.3 $\lim_{t\to\infty} t^2\,S_x(t) = 0.$

These last two assumptions ensure that the mean and variance of the distribution of T_x exist. These are not particularly restrictive constraints – we do not need to worry about distributions with infinite mean or variance in the context of individuals' future lifetimes. These three extra assumptions are valid for all distributions that are feasible for human lifetime modelling.

Example 2.1 Let

$$F_0(t) = \begin{cases} 1 - (1 - t/120)^{1/6} & \text{for } 0 \leq t < 120, \\ 1 & \text{for } t \geq 120. \end{cases}$$

Calculate the probability that

(a) a newborn life survives beyond age 30,
(b) a life aged 30 dies before age 50, and
(c) a life aged 40 survives beyond age 65.

Solution 2.1 (a) The required probability is

$$S_0(30) = 1 - F_0(30) = (1 - 30/120)^{1/6} = 0.9532.$$

(b) From formula (2.2), the required probability is

$$F_{30}(20) = \frac{F_0(50) - F_0(30)}{1 - F_0(30)} = 0.0410.$$

(c) From formula (2.3), the required probability is

$$S_{40}(25) = \frac{S_0(65)}{S_0(40)} = 0.9395.$$

□

We remark that in the above example, $S_0(120) = 0$, which means that under this model, survival beyond age 120 is not possible. In this case we refer to 120 as the **limiting age** of the model. In general, if there is a limiting age, we use the Greek letter ω to denote it. In models where there is no limiting age, it is often practical to introduce a limiting age in calculations, at some point where the probability of surviving longer is negligible. We will see examples later in this chapter.

2.3 The force of mortality

The force of mortality is an important and fundamental concept in modelling future lifetime. We denote the force of mortality at age x by μ_x and define it as

$$\mu_x = \lim_{dx \to 0^+} \frac{1}{dx} \Pr[T_0 \le x + dx \mid T_0 > x]. \tag{2.6}$$

From equation (2.1) we see that an equivalent way of defining μ_x is

$$\mu_x = \lim_{dx \to 0^+} \frac{1}{dx} \Pr[T_x \le dx],$$

which can be written in terms of the survival function S_x as

$$\mu_x = \lim_{dx \to 0^+} \frac{1}{dx} \left(1 - S_x(dx)\right). \tag{2.7}$$

Note that the force of mortality depends, numerically, on the unit of time; if we are measuring time in years, then μ_x is measured per year.

The force of mortality is best understood by noting that for very small dx, formula (2.6) gives the approximation

$$\mu_x \, dx \approx \Pr[T_0 \le x + dx \mid T_0 > x]. \tag{2.8}$$

Thus, for very small dx, we can interpret $\mu_x \, dx$ as the probability that a life who has attained age x dies before attaining age $x + dx$. For example, suppose we have a life aged exactly 50, and that the force of mortality at age 50 is 0.0044 per year. A small value of dx might be a single day, or 0.00274 years. Then the approximate probability that the life dies on his 50th birthday is $0.0044 \times 0.00274 = 1.2 \times 10^{-5}$.

We can relate the force of mortality at age x to the survival function from birth, S_0. As $S_x(dx) = \frac{S_0(x+dx)}{S_0(x)}$, formula (2.7) gives

$$\begin{aligned}
\mu_x &= \frac{1}{S_0(x)} \lim_{dx \to 0^+} \frac{S_0(x) - S_0(x + dx)}{dx} \\
&= \frac{1}{S_0(x)} \left(-\frac{d}{dx} S_0(x) \right).
\end{aligned}$$

Thus,

$$\boxed{\mu_x = \frac{-1}{S_0(x)} \frac{d}{dx} S_0(x).} \tag{2.9}$$

From standard results in probability theory, we know that the probability density function for the random variable T_x, which we denote f_x, is related to the distribution function F_x and the survival function S_x by

$$f_x(t) = \frac{d}{dt}F_x(t) = -\frac{d}{dt}S_x(t).$$

So, it follows from equation (2.9) that

$$\boxed{\mu_x = \frac{f_0(x)}{S_0(x)}.}$$

We can also relate the force of mortality function at any age $x + t$, $t > 0$, to the lifetime distribution of T_x. Assume x is fixed and t is variable. Then $d(x + t) = dt$ and so

$$\begin{aligned}
\mu_{x+t} &= -\frac{1}{S_0(x+t)}\frac{d}{d(x+t)}S_0(x+t) \\
&= -\frac{1}{S_0(x+t)}\frac{d}{dt}S_0(x+t) \\
&= -\frac{1}{S_0(x+t)}\frac{d}{dt}(S_0(x)S_x(t)) \\
&= -\frac{S_0(x)}{S_0(x+t)}\frac{d}{dt}S_x(t) \\
&= \frac{-1}{S_x(t)}\frac{d}{dt}S_x(t).
\end{aligned}$$

Hence

$$\mu_{x+t} = \frac{f_x(t)}{S_x(t)}. \tag{2.10}$$

This relationship gives a way of finding μ_{x+t} given the survival function S_x.

We can also use equation (2.9) to develop a formula for $S_x(t)$ in terms of the force of mortality function, μ_{x+s}, for $0 \le s \le t$. We use the fact that for a function h whose derivative exists, and where $h(x) > 0$ for all x,

$$\frac{d}{dx}\log h(x) = \frac{1}{h(x)}\frac{d}{dx}h(x),$$

so, from equation (2.9), we have

$$\mu_x = -\frac{d}{dx}\log S_0(x),$$

and integrating this identity over $(0, y)$ yields

$$\int_0^y \mu_x \, dx = -\big(\log S_0(y) - \log S_0(0)\big).$$

As $\log S_0(0) = \log \Pr[T_0 > 0] = \log 1 = 0$, we obtain

$$S_0(y) = \exp\left\{ -\int_0^y \mu_x \, dx \right\},$$

from which it follows that

$$\boxed{S_x(t) = \frac{S_0(x+t)}{S_0(x)} = \exp\left\{ -\int_x^{x+t} \mu_r \, dr \right\} = \exp\left\{ -\int_0^t \mu_{x+s} \, ds \right\}.} \tag{2.11}$$

This means that if we know μ_y for all $y \geq 0$, then we can calculate all the survival probabilities $S_x(t)$, for any x and t. In other words, the force of mortality function fully describes the lifetime distribution, just as the function S_0 does. In fact, it is often more convenient to describe the lifetime distribution using the force of mortality function than the survival function.

Example 2.2 As in Example 2.1, let

$$F_0(x) = 1 - (1 - x/120)^{1/6}$$

for $0 \leq x \leq 120$. Derive an expression for μ_x.

Solution 2.2 As $S_0(x) = (1 - x/120)^{1/6}$, it follows that

$$\frac{d}{dx} S_0(x) = \tfrac{1}{6}(1 - x/120)^{-5/6} \left(-\tfrac{1}{120}\right),$$

and so

$$\mu_x = \frac{-1}{S_0(x)} \frac{d}{dx} S_0(x) = \tfrac{1}{720}(1 - x/120)^{-1} = \frac{1}{720 - 6x}.$$

As an alternative, we could use the relationship

$$\mu_x = -\frac{d}{dx} \log S_0(x) = -\frac{d}{dx}\left(\frac{1}{6}\log(1 - x/120)\right) = \frac{1}{720(1 - x/120)}$$
$$= \frac{1}{720 - 6x}.$$

□

2.3.1 Mortality laws

We saw in equation (2.11) that the full distribution of T_x can be determined if we know the force of mortality μ_y for all $y \geq x$. This has led to several important distributions for future lifetime being derived by assuming a mathematical function for the force of mortality; historically, these are called mortality laws.

Two of the most useful mortality laws are **Gompertz' law**, given by

$$\mu_x = B\,c^x \quad \text{where } B > 0,\ c > 1,$$

and **Makeham's law**, which is a generalization of Gompertz' law, given by

$$\mu_x = A + B\,c^x \quad \text{where } A,\ B > 0,\ c > 1.$$

The force of mortality under Gompertz' law increases exponentially with age since $c > 1$; we have $B > 0$ since the force of mortality must be positive.

The force of mortality under Makeham's law adds a constant term which was designed to reflect the risk of accidental death. This term has more impact at younger ages, when the age-related force of mortality is very small. At older ages, the exponential term is the dominant one. These two formulae are very similar (and a simple way to remember which is which is that the letter 'a' appears in both Makeham's name and his mortality law).

We will see in the next chapter that the force of mortality for most populations is not an increasing function of age over the entire age range. Nevertheless, both models often provide a good fit to mortality data over certain age ranges, particularly from middle age to early old age.

Example 2.3 Derive expressions for $S_x(t)$ for (a) Gompertz' law, and (b) Makeham's law.

Solution 2.3 (a) For Gompertz' law, using equation (2.11), we have

$$S_x(t) = \exp\left\{ -\int_x^{x+t} Bc^r\, dr \right\}.$$

Writing c^r as $\exp\{r \log c\}$,

$$\begin{aligned}
\int_x^{x+t} Bc^r dr &= B \int_x^{x+t} \exp\{r \log c\}\, dr \\
&= \frac{B}{\log c} \exp\{r \log c\} \Big|_x^{x+t} \\
&= \frac{B}{\log c} \left(c^{x+t} - c^x \right),
\end{aligned}$$

giving

$$S_x(t) = \exp\left\{-\frac{B}{\log c}c^x(c^t - 1)\right\}.$$

(b) Similarly to (a), we have

$$\int_x^{x+t} (A + Bc^r)dr = At + B\int_x^{x+t} \exp\{r \log c\}\, dr$$
$$= At + \frac{B}{\log c}\left(c^{x+t} - c^x\right),$$

giving

$$S_x(t) = \exp\left\{-At - \frac{B}{\log c}c^x(c^t - 1)\right\}. \tag{2.12}$$

We remark that this is often written as

$$S_x(t) = s^t g^{c^x(c^t-1)},$$

where $s = e^{-A}$ and $g = \exp\{-B/\log c\}$. $\square$

One of the earliest mortality laws proposed was **De Moivre's law** which states that $\mu_x = 1/(\omega - x)$ for $0 \leq x < \omega$. This is a very unrealistic model, and therefore impractical for human populations. Under De Moivre's law, T_x is uniformly distributed on the interval $(0, \omega{-}x)$, which means that the probability that a life currently aged x dies between ages $x + t$ and $x + t + dt$ is the same for all $t > 0$, as long as $x + t < \omega$.

The **generalized De Moivre's law** states that $\mu_x = \alpha/(\omega - x)$ for some $\alpha > 0$, and for $0 \leq x < \omega$. We have already met an example of this mortality law in Example 2.2 where $\omega = 120$ and $\alpha = 1/6$. Again, it does not in any way represent human mortality, and so is not useful in practice. Under the generalized De Moivre's law, T_x has a beta distribution on $(0, \omega - x)$ for $0 \leq x < \omega$.

Another simple mortality law that is very unrealistic for modelling human mortality is the **constant force of mortality** assumption which states $\mu_x = \mu$ for all $x \geq 0$. Under this model, T_x has an exponential distribution. (See Exercise 2.7.)

Although the De Moivre and constant force models are not useful for overall mortality for humans, they may be used in other contexts, such as modelling the failure time of machine components. In addition, in Chapter 3, we use these models in a limited sense, to model mortality between integer ages.

Example 2.4 Calculate the survival function and probability density function for T_x using Gompertz' law of mortality, with $B = 0.0003$ and $c = 1.07$, for $x = 20, x = 50$ and $x = 80$. Plot the results and comment on the features of the graphs.

Solution 2.4 For $x = 20$, the force of mortality is $\mu_{20+t} = Bc^{20+t}$ and the survival function is

$$S_{20}(t) = \exp\left\{-\frac{B}{\log c}c^{20}(c^t - 1)\right\}.$$

The probability density function is found from (2.10):

$$\begin{aligned}\mu_{20+t} &= \frac{f_{20}(t)}{S_{20}(t)} \\ \Rightarrow f_{20}(t) &= \mu_{20+t}\, S_{20}(t) = Bc^{20+t}\exp\left\{-\frac{B}{\log c}c^{20}(c^t - 1)\right\}.\end{aligned}$$

Figure 2.1 shows the survival functions for ages 20, 50 and 80, and Figure 2.2 shows the corresponding probability density functions. These figures illustrate some general points about lifetime distributions.

First, we see an effective limiting age, even though, in principle there is no age at which the survival probability is exactly zero. Looking at Figure 2.1, we see that although $S_x(t) > 0$ for all combinations of x and t, survival beyond age 120 is very unlikely.

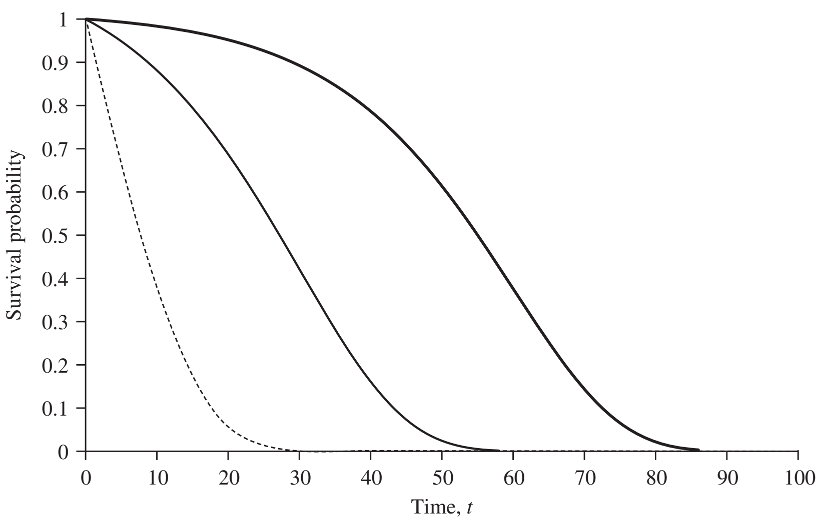


Figure 2.1 $S_x(t)$ for $x = 20$ (bold), 50 (solid) and 80 (dotted).

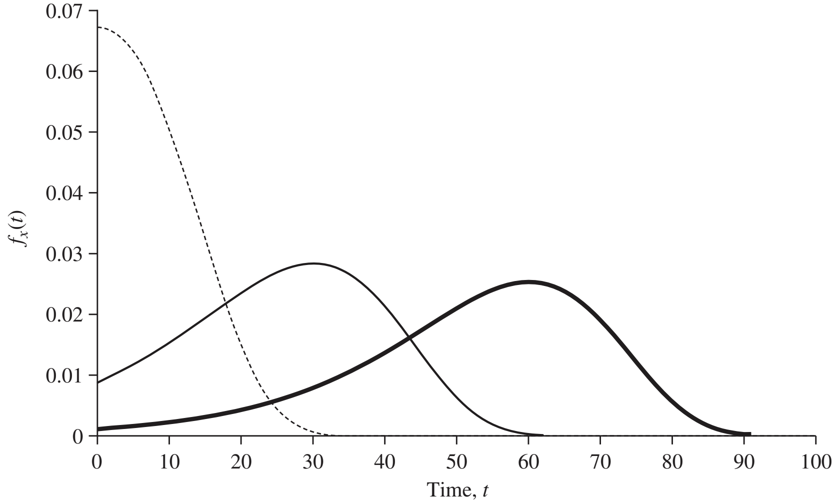


Figure 2.2 $f_x(t)$ for $x = 20$ (bold), 50 (solid) and 80 (dotted).

Second, we note that the survival functions are ordered according to age, with the probability of survival for any given value of t being highest for age 20 and lowest for age 80. For survival functions that give a more realistic representation of human mortality, this ordering can be violated, but it usually holds at ages of interest to insurers. An example of the violation of this ordering is that $S_0(1)$ may be smaller than $S_x(1)$ for $x \geq 1$, as a result of perinatal mortality.

Looking at Figure 2.2, we see that the densities for ages 20 and 50 have similar shapes, but the density for age 80 has a quite different shape. For ages 20 and 50, the densities have their respective maximums at (approximately) $t = 60$ and $t = 30$, indicating that death is most likely to occur around age 80. The decreasing form of the density for age 80 also indicates that death is more likely to occur at age 80 than at any other age for a life now aged 80. A further point to note about these density functions is that although each density function is defined on $(0, \infty)$, the spread of values of $f_x(t)$ is much greater for $x = 20$ than for $x = 50$, which, as we will see in Table 2.1, results in a greater variance of future lifetime for $x = 20$ than for $x = 50$. □

2.4 Actuarial notation

The notation used in the previous sections, $S_x(t)$, $F_x(t)$ and $f_x(t)$, is standard in statistics. Actuarial science has developed its own notation, **International Actuarial Notation**, that encapsulates the probabilities and functions of greatest interest and usefulness to actuaries. The force of mortality notation, μ_x, comes from International Actuarial Notation. We summarize the relevant actuarial notation in this section, and rewrite the important results developed so far in this chapter in terms of actuarial functions.

The actuarial notation for survival and mortality probabilities is

$$\boxed{{}_tp_x = \Pr[T_x > t] = S_x(t),} \tag{2.13}$$

$$\boxed{{}_tq_x = \Pr[T_x \le t] = 1 - S_x(t) = F_x(t),} \tag{2.14}$$

$$\boxed{{}_{u|t}q_x = \Pr[u < T_x \le u + t] = S_x(u) - S_x(u+t).} \tag{2.15}$$

That is

$\boldsymbol{{}_tp_x}$ is the probability that (x) survives to at least age $x + t$,

$\boldsymbol{{}_tq_x}$ is the probability that (x) dies before age $x + t$,

$\boldsymbol{{}_{u|t}q_x}$ is the probability that (x) survives u years, and then dies in the subsequent t years, that is, between ages $x + u$ and $x + u + t$.

We may drop the subscript t if its value is 1, so that p_x represents the probability that (x) survives to at least age $x + 1$. Similarly, q_x is the probability that (x) dies before age $x + 1$. In actuarial terminology q_x is called the **mortality rate** at age x. We call ${}_{u|t}q_x$ a **deferred mortality probability**, because it is the probability that death occurs in the interval of t years, following a deferred period of u years.

The relationships below follow immediately from the definitions above and the previous results in this chapter:

$${}_tp_x + {}_tq_x = 1,$$

$${}_{u|t}q_x = {}_up_x - {}_{u+t}p_x,$$

$${}_{t+u}p_x = {}_tp_x \, {}_up_{x+t} \quad \text{(from (2.5))}, \tag{2.16}$$

$$\mu_x = -\frac{1}{{}_xp_0}\frac{d}{dx}\,{}_xp_0 \quad \text{(from (2.9))}. \tag{2.17}$$

Similarly,

$$\mu_{x+t} = -\frac{1}{{}_tp_x}\frac{d}{dt}\,{}_tp_x \Rightarrow \frac{d}{dt}\,{}_tp_x = -{}_tp_x\,\mu_{x+t}, \tag{2.18}$$

$$\mu_{x+t} = \frac{f_x(t)}{S_x(t)} \Rightarrow f_x(t) = {}_tp_x\,\mu_{x+t} \quad \text{(from (2.10))}, \tag{2.19}$$

$${}_tp_x = \exp\left\{-\int_0^t \mu_{x+s}\,ds\right\} \quad \text{(from (2.11))}. \tag{2.20}$$

As F_x is a distribution function and f_x is its density function, it follows that

$$F_x(t) = \int_0^t f_x(s)\, ds,$$

which can be written in actuarial notation as

$$_tq_x = \int_0^t {_sp_x}\, \mu_{x+s}\, ds. \tag{2.21}$$

This is an important formula, which can be interpreted as follows. Consider time s, where $0 \le s < t$. The probability that (x) is alive at time s is $_sp_x$, and the probability that (x) dies between ages $x+s$ and $x+s+ds$, having survived to age $x+s$, is (loosely) $\mu_{x+s}ds$, provided that ds is very small. Thus $_sp_x\, \mu_{x+s}ds$ can be interpreted as the probability that (x) dies between ages $x+s$ and $x+s+ds$. Now, we can sum over all the possible death intervals s to $s+ds$ – which requires integrating because these are infinitesimal intervals – to obtain the probability of death before age $x+t$.

We illustrate this event sequence using the time-line diagram shown in Figure 2.3.

This type of interpretation is important as it can be applied to more complicated situations, and we will employ the time-line again in later chapters.

In the special case when $t = 1$, formula (2.21) becomes

$$q_x = \int_0^1 {_sp_x}\, \mu_{x+s}\, ds.$$

When q_x is small, it follows that p_x is close to 1, and hence $_sp_x$ is close to 1 for $0 \le s < 1$. Thus

$$q_x \approx \int_0^1 \mu_{x+s}\, ds \approx \mu_{x+1/2},$$

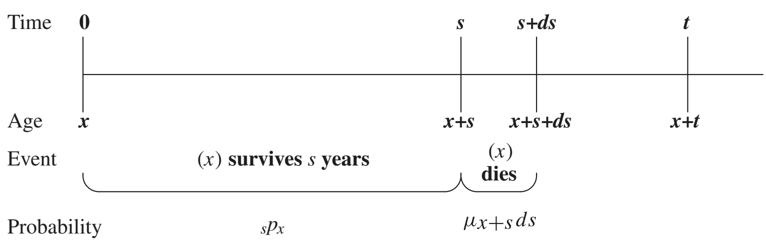


Figure 2.3 Time-line diagram for $_tq_x$

where the second relationship follows by the mid-point rule for numerical integration.

Example 2.5 As in Example 2.1, let

$$F_0(x) = 1 - (1 - x/120)^{1/6}$$

for $0 \le x \le 120$. Calculate both q_x and $\mu_{x+1/2}$ for $x = 20$ and for $x = 110$, and comment on these values.

Solution 2.5 We have

$$p_x = \frac{S_0(x+1)}{S_0(x)} = \left(1 - \frac{1}{120 - x}\right)^{1/6},$$

giving $q_{20} = 0.00167$ and $q_{110} = 0.01741$, and from the solution to Example 2.2, $\mu_{20\frac{1}{2}} = 0.00168$ and $\mu_{110\frac{1}{2}} = 0.01754$. We see that $\mu_{x+1/2}$ is a good approximation to q_x when the mortality rate is small, but is not such a good approximation, at least in absolute terms, when the mortality rate is not close to 0. □

2.5 Mean and standard deviation of T_x

Next, we consider the expected future lifetime of (x), $\mathrm{E}[T_x]$, denoted in actuarial notation by $\overset{\circ}{e}_x$. We call this the **complete expectation of life**. We note from formulae (2.18) and (2.19) that

$$f_x(t) = {}_tp_x\, \mu_{x+t} = -\frac{d}{dt}\, {}_tp_x. \tag{2.22}$$

From the definition of an expected value, we have

$$\begin{aligned} \overset{\circ}{e}_x &= \int_0^\infty t f_x(t)\, dt \\ &= \int_0^\infty t\, {}_tp_x\, \mu_{x+t}\, dt. \end{aligned}$$

We now use (2.22) to evaluate this integral, using integration by parts as

$$\begin{aligned} \overset{\circ}{e}_x &= -\int_0^\infty t \left(\frac{d}{dt}\, {}_tp_x\right) dt \\ &= -\left(t\, {}_tp_x \Big|_0^\infty - \int_0^\infty {}_tp_x\, dt \right). \end{aligned}$$

Now recall that in Section 2.2, we stated (in Assumption 2.2) that we will assume throughout that $\lim_{t\to\infty} t\,{}_tp_x = 0$, which means that

$$\boxed{\overset{\circ}{e}_x = \int_0^\infty {}_tp_x\,dt.} \tag{2.23}$$

Similarly, for $\mathrm{E}[T_x^2]$, we have

$$\begin{aligned}
\mathrm{E}[T_x^2] &= \int_0^\infty t^2\,{}_tp_x\,\mu_{x+t}\,dt \\
&= -\int_0^\infty t^2 \left(\frac{d}{dt}{}_tp_x\right) dt \\
&= -\left(t^2\,{}_tp_x\Big|_0^\infty - \int_0^\infty {}_tp_x\,2t\,dt \right) \\
&= 2\int_0^\infty t\,{}_tp_x\,dt. \qquad (2.24)
\end{aligned}$$

So we have integral expressions for $\mathrm{E}[T_x]$ and $\mathrm{E}[T_x^2]$. For some lifetime distributions (though not the useful ones) we are able to integrate directly. In other cases we have to use numerical integration techniques to evaluate the integrals in (2.23) and (2.24). The variance of T_x can then be calculated as

$$\mathrm{V}[T_x] = \mathrm{E}\left[T_x^2\right] - \left(\overset{\circ}{e}_x\right)^2.$$

Example 2.6 As in Example 2.1, let

$$F_0(x) = 1 - (1 - x/120)^{1/6}$$

for $0 \le x \le 120$. Calculate $\overset{\circ}{e}_x$ and $\mathrm{V}[T_x]$ for (a) $x = 30$ and (b) $x = 80$.

Solution 2.6 As $S_0(x) = (1 - x/120)^{1/6}$, we have

$${}_tp_x = \frac{S_0(x+t)}{S_0(x)} = \left(1 - \frac{t}{120-x}\right)^{1/6}.$$

Now recall that this formula is valid for $0 \le t \le 120 - x$, because, under this model, survival beyond age 120 is impossible. Technically, we have

$${}_tp_x = \begin{cases} \left(1 - \frac{t}{120-x}\right)^{1/6} & \text{for } x + t \le 120, \\ 0 & \text{for } x + t > 120. \end{cases}$$

So the upper limit of integration in equation (2.23) is $120-x$, and

$$\overset{\circ}{e}_x = \int_0^{120-x} \left(1 - \frac{t}{120-x}\right)^{1/6} dt.$$

We make the substitution $y = 1 - t/(120-x)$, so that $t = (120-x)(1-y)$, giving

$$\begin{aligned}\overset{\circ}{e}_x &= (120-x)\int_0^1 y^{1/6}dy \\ &= \tfrac{6}{7}(120-x).\end{aligned}$$

So $\overset{\circ}{e}_{30} = 77.143$ and $\overset{\circ}{e}_{80} = 34.286$. We note that under this model the expectation of life at any age x is 6/7 of the time to age 120.

For the variance we require $\mathrm{E}[T_x^2]$. Using equation (2.24) we have

$$E\left[T_x^2\right] = 2\int_0^{120-x} t\,{}_tp_x\,dt = 2\int_0^{120-x} t\left(1 - \frac{t}{120-x}\right)^{1/6} dt.$$

Again, we substitute $y = 1 - t/(120-x)$ giving

$$\begin{aligned}E\left[T_x^2\right] &= 2(120-x)^2\int_0^1 (y^{1/6} - y^{7/6})\,dy \\ &= 2(120-x)^2\left(\frac{6}{7} - \frac{6}{13}\right).\end{aligned}$$

Then

$$\begin{aligned}\mathrm{V}[T_x] = \mathrm{E}[T_x^2] - \left(\overset{\circ}{e}_x\right)^2 &= (120-x)^2\left(2(6/7 - 6/13) - (6/7)^2\right) \\ &= (120-x)^2\,(0.056515) = ((120-x)\,(0.23773))^2.\end{aligned}$$

So $\mathrm{V}[T_{30}] = 21.396^2$ and $\mathrm{V}[T_{80}] = 9.509^2$.

Since we know under this model that all lives will die before age 120, it makes sense that the uncertainty in the future lifetime should be greater for younger lives than for older lives. □

A feature of the model used in Example 2.6 is that we can obtain analytic formulae for quantities of interest such as $\overset{\circ}{e}_x$, but for many models this is not possible. For example, when we model mortality using Gompertz' law, there is no explicit formula for $\overset{\circ}{e}_x$ and we must use numerical integration to calculate moments of T_x. In Appendix B we describe in detail how to do this.

Table 2.1 *Values of* $\overset{\circ}{e}_x$*,* $\mathrm{SD}[T_x]$ *and expected age at death for the Gompertz model with* $B = 0.0003$ *and* $c = 1.07$.

x	$\overset{\circ}{e}_x$	$\mathrm{SD}[T_x]$	$x + \overset{\circ}{e}_x$
0	71.938	18.074	71.938
10	62.223	17.579	72.223
20	52.703	16.857	72.703
30	43.492	15.841	73.492
40	34.752	14.477	74.752
50	26.691	12.746	76.691
60	19.550	10.693	79.550
70	13.555	8.449	83.555
80	8.848	6.224	88.848
90	5.433	4.246	95.433
100	3.152	2.682	103.152

Table 2.1 shows values of $\overset{\circ}{e}_x$ and the standard deviation of T_x (denoted $\mathrm{SD}[T_x]$) for a range of values of x using Gompertz' law, $\mu_x = Bc^x$, where $B = 0.0003$ and $c = 1.07$. For this survival model, ${}_{130}p_0 = 1.9 \times 10^{-13}$, so that using 130 as the maximum attainable age in our numerical integration is accurate enough for practical purposes.

We see that $\overset{\circ}{e}_x$ is a decreasing function of x, as it was in Example 2.6, but in that example $\overset{\circ}{e}_x$ was a linear function of x, which is not the case here.

We are sometimes interested in the future lifetime random variable subject to a cap of n years, which is represented by the random variable $\min(T_x, n)$. For example, suppose that (x) is entitled to a benefit payable continuously for a maximum of n years, conditional on survival. Then $\min(T_x, n)$ would represent the payment period for the benefit. We derive the mean and variance of this random variable, using a similar approach to the derivation of the mean and variance of T_x. The expected value of $\min(T_x, n)$ is called the **term expectation of life**, is denoted $\overset{\circ}{e}_{x:\overline{n}|}$, and is found as

$$
\begin{aligned}
\mathrm{E}[\min(T_x,\, n)] = \overset{\circ}{e}_{x:\overline{n}|} &= \int_0^n t\, {}_tp_x\, \mu_{x+t}\, dt + \int_n^\infty n\, {}_tp_x\, \mu_{x+t}\, dt \\
&= \int_0^n t \left(-\frac{d}{dt}\, {}_tp_x \right) dt + n\, {}_np_x \\
&= -\left(t\, {}_tp_x \Big|_0^n - \int_0^n {}_tp_x\, dt \right) + n\, {}_np_x \\
\Rightarrow \overset{\circ}{e}_{x:\overline{n}|} &= \int_0^n {}_tp_x\, dt.
\end{aligned}
$$

The $\overline{n}|$ notation is used to denote a period n-years (just as in annuity-certain notation), and is used extensively in later chapters.

2.6 Curtate future lifetime

2.6.1 K_x and e_x

In many insurance applications we are interested not only in the future lifetime of an individual, but also in what is known as the individual's curtate future lifetime. The **curtate future lifetime** random variable is defined as the integer part of future lifetime, and is denoted by K_x for a life aged x. If we let $\lfloor\ \rfloor$ denote the floor function, we have

$$K_x = \lfloor T_x \rfloor.$$

We can think of the curtate future lifetime K_x as the number of complete years lived in the future by (x). As an illustration of the importance of curtate future lifetime, consider the situation where a life aged x at time 0 is entitled to payments of 1 at times $1, 2, 3, \ldots$ provided that (x) is alive at these times. Then the number of payments made equals the number of complete years lived after time 0 by (x). This is the curtate future lifetime.

For $k = 0, 1, 2, \ldots$, we note that $K_x = k$ if and only if (x) dies between the ages of $x + k$ and $x + k + 1$. Thus

$$\begin{aligned}
\Pr[K_x = k] &= \Pr[k \le T_x < k + 1] \\
&= {}_{k|}q_x \\
&= {}_kp_x - {}_{k+1}p_x \\
&= {}_kp_x - {}_kp_x\, p_{x+k} \\
&= {}_kp_x\, q_{x+k}.
\end{aligned}$$

The expected value of K_x is denoted by e_x, and is referred to as the curtate expectation of life (even though it represents the expected curtate lifetime). So

$$\begin{aligned}
\mathrm{E}[K_x] = e_x &= \sum_{k=0}^{\infty} k \Pr[K_x = k] \\
&= \sum_{k=0}^{\infty} k\,({}_kp_x - {}_{k+1}p_x) \\
&= ({}_1p_x - {}_2p_x) + 2({}_2p_x - {}_3p_x) + 3({}_3p_x - {}_4p_x) + \cdots \\
&= \sum_{k=1}^{\infty} {}_kp_x. \qquad (2.25)
\end{aligned}$$

Note that the lower limit of summation is $k = 1$.

Similarly,

$$\begin{aligned}
\mathrm{E}[K_x^2] &= \sum_{k=0}^{\infty} k^2 \left({}_kp_x - {}_{k+1}p_x \right) \\
&= ({}_1p_x - {}_2p_x) + 4({}_2p_x - {}_3p_x) + 9({}_3p_x - {}_4p_x) + 16({}_4p_x - {}_5p_x) + \cdots \\
&= 2\sum_{k=1}^{\infty} k \, {}_kp_x - \sum_{k=1}^{\infty} {}_kp_x \\
&= 2\sum_{k=1}^{\infty} k \, {}_kp_x - e_x .
\end{aligned}$$

As with the complete expectation of life, there are few lifetime distributions that allow $\mathrm{E}[K_x]$ and $\mathrm{E}[K_x^2]$ to be calculated analytically. For more realistic models, such as Gompertz or Makeham, we can calculate the values easily using Excel or other suitable software, and by setting an effective limiting age, as we did for Table 2.1.

Analogous to the random variable $\min(T_x, n)$ we have the random variable $\min(K_x, n)$. For example, if a life aged x is entitled to payments of 1 at times $1, 2, 3, \ldots, n$, where n is an integer, then $\min(K_x, n)$ represents the number of payments made. An important difference between these two random variables is that $\min(T_x, n)$ is a mixed random variable (with a density over $(0, n)$ and a mass of probability at n), whereas $\min(K_x, n)$ is a discrete random variable since K_x is a discrete random variable. The expected value of $\min(K_x, n)$ is denoted $e_{x:\overline{n}|}$, and when n is an integer is given by

$$e_{x:\overline{n}|} = \sum_{k=1}^{n} {}_kp_x .$$

The proof of this result is set as Exercise 2.5.

2.6.2 Comparing $\mathring{e}_x$ and e_x

As the curtate future lifetime is the integer part of future lifetime, it is natural to ask if there is a simple relationship between $\mathring{e}_x$ and e_x. We can obtain an approximate relationship by writing

$$\mathring{e}_x = \int_0^{\infty} {}_tp_x \, dt = \sum_{j=0}^{\infty} \int_j^{j+1} {}_tp_x \, dt .$$

If we approximate each integral using the trapezium rule for numerical integration (see Appendix B), we obtain

Table 2.2 *Values of e_x and $\overset{\circ}{e}_x$ for Gompertz' law with $B = 0.0003$ and $c = 1.07$.*

x	e_x	$\overset{\circ}{e}_x$
0	71.438	71.938
10	61.723	62.223
20	52.203	52.703
30	42.992	43.492
40	34.252	34.752
50	26.192	26.691
60	19.052	19.550
70	13.058	13.555
80	8.354	8.848
90	4.944	5.433
100	2.673	3.152

$$\int_j^{j+1} {}_tp_x\,dt \approx \tfrac{1}{2}\left({}_jp_x + {}_{j+1}p_x\right),$$

and hence

$$\overset{\circ}{e}_x \approx \sum_{j=0}^{\infty} \tfrac{1}{2}\left({}_jp_x + {}_{j+1}p_x\right) = \tfrac{1}{2} + \sum_{j=1}^{\infty} {}_jp_x.$$

Thus, we have an approximation that is frequently applied in practice, namely

$$\overset{\circ}{e}_x \approx e_x + \tfrac{1}{2}. \tag{2.26}$$

In Chapter 5 we will meet a refined version of this approximation. Table 2.2 shows values of $\overset{\circ}{e}_x$ and e_x for a range of values of x when the survival model is Gompertz' law, with $B = 0.0003$ and $c = 1.07$. Values of e_x were calculated by applying formula (2.25) with a limiting age of 130, and values of $\overset{\circ}{e}_x$ are as in Table 2.1. Table 2.2 illustrates that in this particular case, formula (2.26) is a very good approximation for younger ages, but is less accurate at very old ages. This observation is true for most realistic survival models.

2.7 Notes and further reading

In recent times, the Gompertz–Makeham approach has been generalized further to give the GM(r, s) (Gompertz–Makeham) formula,

$$\mu_x = h_r^1(x) + \exp\{h_s^2(x)\},$$

where h_r^1 and h_s^2 are polynomials in x of degree r and s respectively. A discussion of this formula can be found in Forfar *et al.* (1988). Both Gompertz' law and Makeham's law are special cases of the GM formula. A more comprehensive list of mortality laws is given in Macdonald *et al.* (2018).

In Section 2.3, we noted the importance of the force of mortality. A further significant point is that when mortality data are analysed, the force of mortality is a natural quantity to estimate, whereas the lifetime distribution is not. This is discussed more in Chapter 18, and in more specialized texts, such as Macdonald *et al.* (2018).

For more general distributions, the quantity $f_0(x)/S_0(x)$, which actuaries call the force of mortality at age x, is known as the **hazard rate** in survival analysis and the **failure rate** in reliability theory.

2.8 Exercises

Shorter exercises

Exercise 2.1 You are given that

$$p_x = 0.99, \quad p_{x+1} = 0.985, \quad {}_3p_{x+1} = 0.95, \text{ and } q_{x+3} = 0.02.$$

Calculate (a) p_{x+3} , (b) ${}_2p_x$, (c) ${}_2p_{x+1}$, (d) ${}_3p_x$, (e) ${}_{1|2}q_x$.

Exercise 2.2 Show that $e_x = p_x(1 + e_{x+1})$, and hence calculate ${}_3p_{60}$ from the following table.

x	e_x
60	15.96
61	15.27
62	14.60
63	13.94

Exercise 2.3 Suppose that Gompertz' law applies with $B = 0.00013$ and $c = 1.03$. Calculate (a) ${}_{10}p_{40}$ and (b) $\frac{d}{dt}\,{}_tp_{40}$ at $t = 10$.

Exercise 2.4 Let $\mu_{x+t} = 0.002 + 0.001t$ for $0 \le t \le 1$. Calculate q_x.

Exercise 2.5 Show that for integer n,

$$e_{x:\overline{n}|} = \sum_{k=1}^{n} {}_kp_x.$$

Longer exercises

Exercise 2.6 The function

$$G(x) = \frac{18\,000 - 110x - x^2}{18\,000}$$

has been proposed as the survival function $S_0(x)$ for a mortality model.

(a) What is the implied limiting age ω?
(b) Verify that the function G satisfies the criteria for a survival function.
(c) Calculate ${}_{20}p_0$.
(d) Determine the survival function for a life aged 20.
(e) Calculate ${}_{10|10}q_{20}$.
(f) Calculate μ_{50}.

Exercise 2.7 Let $F_0(t) = 1 - e^{-\lambda t}$, where $\lambda > 0$.

(a) Show that $S_x(t) = e^{-\lambda t}$.
(b) Show that $\mu_x = \lambda$.
(c) Show that $e_x = (e^{\lambda} - 1)^{-1}$.
(d) What conclusions do you draw about using this lifetime distribution to model human mortality?

Exercise 2.8 You are given that $S_0(x) = e^{-0.001\,x^2}$ for $x \geq 0$.

(a) Derive a formula for $f_0(x)$.
(b) Derive a formula for μ_x.
(c) Calculate ${}_{5|15}q_{65}$.

Exercise 2.9 Show that

$$\frac{d}{dx}\,{}_tp_x = {}_tp_x\,\left(\mu_x - \mu_{x+t}\right).$$

Exercise 2.10 You are given that mortality follows Makeham's law, and that ${}_{10}p_{50} = 0.974054$, ${}_{10}p_{60} = 0.935938$ and ${}_{10}p_{70} = 0.839838$. Calculate c.

Exercise 2.11 (a) Show that $\overset{\circ}{e}_x \leq \overset{\circ}{e}_{x+1} + 1$.
(b) Show that $\overset{\circ}{e}_x \geq e_x$.
(c) Explain (in words) why $\overset{\circ}{e}_x \approx e_x + \frac{1}{2}$.
(d) Is $\overset{\circ}{e}_x$ always a non-increasing function of x?

Exercise 2.12 (a) Show that

$$\overset{\circ}{e}_x = \frac{1}{S_0(x)} \int_x^{\infty} S_0(t)\,dt.$$

(b) Hence, or otherwise, prove that $\frac{d}{dx}\overset{\circ}{e}_x = \mu_x\,\overset{\circ}{e}_x - 1$.

Hint: $\dfrac{d}{dx}\left\{\int_a^x g(t)\,dt\right\} = g(x)$. What about $\dfrac{d}{dx}\left\{\int_x^a g(t)\,dt\right\}$?

(c) Deduce that $x + \mathring{e}_x$ is an increasing function of x, and explain this result intuitively.

Exercise 2.13 Let the random variable T denote the time lived by a life (x) after the age of $x + n$.

(a) Write down the distribution function of T in terms of the lifetime distribution function F_x.
(b) Use your answer to part (a) to write down an integral expression for E[T] in terms of the probability density function of T_x, and then simplify this expression as far as possible.

Exercise 2.14 (a) Show that under De Moivre's law (i.e. $\mu_x = 1/(\omega - x)$), T_x is uniformly distributed on $(0, \omega - x)$.
(b) Calculate the difference between $\mathring{e}_x$ and e_x, assuming that the future lifetime of (x) follows De Moivre's law and that both x and ω are integers.

Exercise 2.15 A new machine component is assumed to fail at time T, where the force of failure is

$$\mu_t = \frac{1}{4(10 - t)} \quad \text{for} \quad 0 \le t < 10.$$

(a) Derive the density function, $f(t)$, and the survival function, $S(t)$, for this distribution.
(b) Calculate the mean and standard deviation of the time to failure.

Excel-based exercises

Exercise 2.16 Let $S_0(x) = \exp\left\{-\left(Ax + \frac{1}{2}Bx^2 + \frac{C}{\log D}D^x - \frac{C}{\log D}\right)\right\}$, where A, B, C and D are all positive.

(a) Show that the function S_0 is a survival function.
(b) Derive a formula for $S_x(t)$.
(c) Derive a formula for μ_x.
(d) Now suppose that

$$A = 0.00005, \quad B = 0.0000005, \quad C = 0.0003, \quad D = 1.07.$$

(i) Calculate ${}_{20|10}q_{30}$.
(ii) Calculate e_{70}.
(iii) Calculate $\mathring{e}_{70}$ using numerical integration.

Exercise 2.17 (a) Construct a table of p_x for Makeham's law with parameters $A = 0.0001$, $B = 0.00035$ and $c = 1.075$, for integer x from age 0 to age 130, using Excel. You should set the parameters so that they can be easily changed, and you should keep the table, as many exercises and examples in future chapters will use Makeham's law.

(b) Use the table to determine the age last birthday at which a life currently aged 70 is most likely to die.

(c) Use the table to calculate e_{70}.

(d) Using a numerical approach, calculate $\overset{\circ}{e}_{70}$.

Exercise 2.18 A life insurer assumes that the force of mortality of smokers at all ages is twice the force of mortality of non-smokers.

(a) Show that, if * represents smokers' mortality, and the 'unstarred' function represents non-smokers' mortality, then

$$ {}_tp_x^* = ({}_tp_x)^2 . $$

(b) Calculate the difference between the life expectancy of smokers and non-smokers aged 50, assuming that non-smokers' mortality follows Gompertz' law, with $B = 0.0005$ and $c = 1.07$.

(c) Calculate the variance of the future lifetime for a non-smoker aged 50 and for a smoker aged 50 under Gompertz' law.

Hint: You will need to use numerical integration for parts (b) and (c).

Answers to selected exercises

2.1 (a) 0.98 (b) 0.97515 (c) 0.96939 (d) 0.95969 (e) 0.03031

2.2 0.93834

2.3 (a) 0.995078 (b) −0.000567

2.4 0.00250

2.6 (a) 90 (c) 0.8556 (d) $1 - 3x/308 - x^2/15\,400$ (e) 0.1169 (f) 0.021

2.8 (c) 0.45937

2.10 1.105

2.15 (b) The mean is 8 and the standard deviation is $2\frac{2}{3}$

2.16 (d) (i) 0.1082 (ii) 13.046 (iii) 13.544

2.17 (b) 73 (c) 9.339 (d) 9.834

2.18 (b) 6.432 (c) 125.89 (non-smokers), 80.11 (smokers)

3
Life tables and selection

3.1 Summary

In this chapter we define a life table. For a life table tabulated at integer ages only, we show, using fractional age assumptions, how to calculate survival probabilities for all ages and durations.

We discuss some features of national life tables from Australia, England & Wales and the United States.

We then consider life tables appropriate to individuals who have purchased particular types of life insurance policy and discuss why the survival probabilities differ from those in the corresponding national life table. We consider the effect of 'selection' of lives for insurance policies, for example through medical underwriting. We define a select survival model and we derive some formulae for such a model.

We consider heterogeneity in populations, exploring how combining lives with different underlying mortality impacts the mortality experience of the group as a whole.

Finally, we present some methods for constructing survival models which allow for trends in underlying population mortality rates.

3.2 Life tables

Given a survival model, with survival probabilities ${}_tp_x$, we construct the **life table** for the model, from some initial age x_0 to a maximum or limiting age ω, using a function $\{l_x\}, x_0 \leq x \leq \omega$, where l_{x_0} is an arbitrary positive number (called the **radix** of the table) and, for $0 \leq t \leq \omega - x_0$,

$$l_{x_0+t} = l_{x_0}\ {}_tp_{x_0}.$$

From this definition we see that, for $x_o \le x \le x+t \le \omega$,

$$\begin{aligned} l_{x+t} &= l_{x_o}\ {}_{x+t-x_o}p_{x_o} \\ &= l_{x_o}\ {}_{x-x_o}p_{x_o}\ {}_tp_x \\ &= l_x\ {}_tp_x\,, \end{aligned}$$

so that

$$\boxed{{}_tp_x = l_{x+t}/l_x.} \tag{3.1}$$

For any $x \ge x_o$, we can interpret l_{x+t} as the expected number of survivors at age $x+t$ from l_x independent lives aged x. This interpretation is more natural if l_x is an integer, in which case the number of survivors to age $x+t$, denoted by $L_{x,t}$, is a binomial random variable, with parameters l_x and ${}_tp_x$. That is, suppose we have l_x independent lives aged x, and each life has a probability ${}_tp_x$ of surviving to age $x+t$. Then the number of survivors to age $x+t$ is a binomial random variable, $L_{x,t} \sim B(l_x,\ {}_tp_x)$. The expected value of the number of survivors is $\mathrm{E}[L_{x,t}]$, which is

$$\mathrm{E}[L_{x,t}] = l_x\ {}_tp_x = l_{x+t}.$$

We always use the life table in the form l_y/l_x which is why the radix of the table is arbitrary – it would make no difference to the survival model if all the l_x values were multiplied by 100, for example.

From (3.1), if we only have the l_x values, we can use them to calculate survival and mortality probabilities. For example,

$$q_{30} = 1 - \frac{l_{31}}{l_{30}} = \frac{l_{30} - l_{31}}{l_{30}} \tag{3.2}$$

and

$${}_{15|30}q_{40} = {}_{15}p_{40}\ {}_{30}q_{55} = \frac{l_{55}}{l_{40}}\left(1 - \frac{l_{85}}{l_{55}}\right) = \frac{l_{55} - l_{85}}{l_{40}}. \tag{3.3}$$

In principle, a life table may be defined for all x from the initial age, x_o, to the limiting age, ω. In practice, it is very common for a life table to be presented, and in some cases even defined, at integer ages only. In this form, the life table is a useful way of summarizing a lifetime distribution since, with a single column of numbers, it allows us to calculate probabilities of surviving or dying over integer numbers of years starting from an integer age.

In some cases, a life table tabulated at integer ages also shows values of d_x, where

$$\boxed{d_x = l_x - l_{x+1},} \tag{3.4}$$

Table 3.1 *Extract from a life table.*

x	l_x	d_x
30	10 000.00	34.78
31	9 965.22	38.10
32	9 927.12	41.76
33	9 885.35	45.81
34	9 839.55	50.26
35	9 789.29	55.17
36	9 734.12	60.56
37	9 673.56	66.49
38	9 607.07	72.99
39	9 534.08	80.11

as these are used to compute q_x. From (3.4) we have

$$d_x = l_x\left(1 - \frac{l_{x+1}}{l_x}\right) = l_x(1-p_x) = l_x\,q_x.$$

We can also arrive at this relationship if we consider the random variable D_x, representing the number of deaths between ages x and $x+1$, from l_x lives aged x. This is a binomial random variable – that is $D_x \sim B(l_x, q_x)$. Then $d_x = \mathrm{E}[D_x]$ is the expected number of deaths in the year of age from x to $x+1$, from a group of l_x lives aged exactly x, and

$$\boxed{d_x = l_x\,q_x\,.} \tag{3.5}$$

Example 3.1 Table 3.1 gives an extract from a life table. Calculate

$$\text{(a) } l_{40}\,, \quad \text{(b) } {}_{10}p_{30}\,, \quad \text{(c) } q_{35}\,, \quad \text{(d) } {}_{5}q_{30}\,, \quad \text{(e) } {}_{5|}q_{30}.$$

Solution 3.1 (a) From equation (3.4), $l_{40} = l_{39} - d_{39} = 9\,453.97$.

(b) From equation (3.1), ${}_{10}p_{30} = \frac{l_{40}}{l_{30}} = \frac{9\,453.97}{10\,000} = 0.94540$.

(c) From equation (3.5), $q_{35} = \frac{d_{35}}{l_{35}} = \frac{55.17}{9\,789.29} = 0.00564$.

(d) Following equation (3.2), ${}_{5}q_{30} = \frac{l_{30}-l_{35}}{l_{30}} = 0.02107$.

(e) Following equation (3.3), ${}_{5|}q_{30} = \frac{l_{35}-l_{36}}{l_{30}} = \frac{d_{35}}{l_{30}} = 0.00552$.

□

3.3 Fractional age assumptions

A life table $\{l_x\}_{x\geq x_0}$ provides exactly the same information as the corresponding survival distribution, S_{x_0}. However, we commonly use the term 'life table' to mean the l_x function tabulated at integer ages only. This does not contain all the information in the corresponding survival model, as it is not sufficient for calculating probabilities involving non-integer ages or durations. For example, the life table gives us ${}_1p_{30} = l_{31}/l_{30}$, but not ${}_1p_{30.5}$ (non-integer age) or ${}_{0.75}p_{30}$ (non-integer duration), or ${}_{0.75}p_{30.5}$ (non-integer age and duration). So, if we only have values of l_x at integer ages from the life table, we need an additional assumption, or some further information, to calculate probabilities involving non-integer ages and durations. Specifically, we need to make some assumption about the distribution of the future lifetime random variable between integer ages.

We use the term **fractional age assumption** to describe such an assumption. It may be specified in terms of the force of mortality function or the survival or mortality probabilities.

In this section we assume that a life table is specified at integer ages only and we describe the two most useful fractional age assumptions.

3.3.1 Uniform distribution of deaths

The uniform distribution of deaths (UDD) assumption is the most common fractional age assumption. It can be formulated in two different, but equivalent, ways as follows.

UDD1 For integer x, and for $0 \leq s < 1$, assume that

$$\boxed{{}_sq_x = s\,q_x\,.} \tag{3.6}$$

That is, for any integer x, the mortality probability over $s < 1$ years is s times the one-year mortality probability.

UDD2 For a life (x), where x is an integer, with future lifetime random variable, T_x, and curtate future lifetime random variable, K_x, define a new random variable R_x to represent the fractional part of the future lifetime of (x) lived in the year of death, so that $T_x = K_x + R_x$. We assume

$$\boxed{R_x \sim U(0,1), \text{ independent of } K_x.}$$

So this assumption states that R_x has a uniform distribution on (0,1), regardless of the distribution of K_x. Recall that if $X \sim U(0,1)$, then $\Pr(X \leq u) = u$ for $0 \leq u \leq 1$ (see Appendix A).

The equivalence of these two assumptions is demonstrated as follows. First, assume that UDD1 is true. Then for integer x, and for $0 \leq s < 1$,

$$\begin{aligned}
\Pr[R_x \le s] &= \sum_{k=0}^{\infty} \Pr[R_x \le s \text{ and } K_x = k] \\
&= \sum_{k=0}^{\infty} \Pr[k \le T_x \le k+s] \\
&= \sum_{k=0}^{\infty} {}_kp_x \, {}_sq_{x+k} \\
&= \sum_{k=0}^{\infty} {}_kp_x \, s\,(q_{x+k}) \quad \text{using UDD1} \\
&= s \sum_{k=0}^{\infty} {}_kp_x \, q_{x+k} \\
&= s \sum_{k=0}^{\infty} \Pr[K_x = k] \\
&= s.
\end{aligned}$$

This proves that $R_x \sim U(0,1)$. To prove the independence of R_x and K_x, note that

$$\begin{aligned}
\Pr[R_x \le s \text{ and } K_x = k] &= \Pr[k \le T_x \le k+s] \\
&= {}_kp_x \, {}_sq_{x+k} \\
&= s \, {}_kp_x \, q_{x+k} \\
&= \Pr[R_x \le s] \Pr[K_x = k]
\end{aligned}$$

since $R_x \sim U(0,1)$. This proves that UDD1 implies UDD2.

To prove the reverse implication, assume that UDD2 is true. Then for integer x, and for $0 \le s < 1$,

$$\begin{aligned}
{}_sq_x &= \Pr[T_x \le s] \\
&= \Pr[K_x = 0 \text{ and } R_x \le s] \\
&= \Pr[R_x \le s] \; \Pr[K_x = 0]
\end{aligned}$$

as K_x and R_x are assumed independent. Thus,

$$ {}_sq_x = s\,q_x. \tag{3.7}$$

The UDD2 derivation of the result explains why this assumption is called the Uniform Distribution of Deaths, but in practical applications of this assumption, formulation UDD1 is the more useful of the two.

An immediate consequence is that

$$\boxed{l_{x+s} = l_x - s\,d_x \quad 0 \le s < 1.} \tag{3.8}$$

This follows because for $0 \le s < 1$

$$_sq_x = \frac{l_x - l_{x+s}}{l_x}$$

and substituting $s\,q_x = s\,d_x/l_x$ for $_sq_x$, we have

$$s\,\frac{d_x}{l_x} = \frac{l_x - l_{x+s}}{l_x}.$$

Hence

$$l_{x+s} = l_x - s\,d_x \qquad \text{for } 0 \le s \le 1.$$

Thus, UDD implies that l_{x+s} is a linearly decreasing function of s between integer ages. In practice this can be very useful, as it allows us to use linear interpolation to calculate survival probabilities for non-integer terms, provided the starting age is an integer. Thus, if t and x are both integers, and $0 < s < 1$, then under UDD

$$\begin{aligned} {}_{t+s}p_x &= \frac{l_{x+t+s}}{l_x} \\ &= \frac{l_{x+t} - s\,d_{x+t}}{l_x} \\ &= \frac{l_{x+t} - s\,(l_{x+t} - l_{x+t+1})}{l_x} \\ &= \frac{(1-s)\,l_{x+t} + s\,l_{x+t+1}}{l_x} \end{aligned} \tag{3.9}$$

$$\Rightarrow {}_{t+s}p_x = (1-s) \times {}_tp_x + s \times {}_{t+1}p_x. \tag{3.10}$$

Note that both (3.9) and (3.10) are linear interpolations.

Differentiating equation (3.6) with respect to s, we obtain

$$\frac{d}{ds}\,{}_sq_x = \frac{d}{ds}s\,q_x = q_x, \quad 0 \le s < 1.$$

We also know that

$$\frac{d}{ds}\,{}_sq_x = {}_sp_x\,\mu_{x+s}, \quad s > 0,$$

because the left-hand side is the derivative of the distribution function for T_x, which is equal to the density function on the right-hand side. So, between integer ages, the density function is constant, and more specifically,

$$\boxed{{}_sp_x\,\mu_{x+s} = q_x \quad \text{for } 0 \le s < 1.} \tag{3.11}$$

Since q_x is constant with respect to s, and ${}_sp_x$ is a decreasing function of s, we can see that μ_{x+s} is an increasing function of s, which is appropriate for ages of interest to insurers. However, if we apply the approximation over successive ages, we obtain a discontinuous function for the force of mortality, with discontinuities occurring at integer ages, as we illustrate in Example 3.4. Although this is undesirable, it is not a serious drawback in practice.

Example 3.2 Given that $p_{40} = 0.999473$, calculate ${}_{0.4}q_{40.2}$ under the assumption of a uniform distribution of deaths.

Solution 3.2 It is important to remember that the UDD result, that ${}_sq_x = s\,q_x$, requires x to be in integer and requires $0 < s < 1$. So the first step in applying UDD to this problem is to restate the required probability in terms that involve survival probabilities from age 40 as

$$ {}_{0.4}q_{40.2} = 1 - {}_{0.4}p_{40.2} = 1 - \frac{{}_{0.6}p_{40}}{{}_{0.2}p_{40}} = 1 - \frac{1 - {}_{0.6}q_{40}}{1 - {}_{0.2}q_{40}}. $$

Now we can apply UDD to the numerator and denominator giving

$$ {}_{0.4}q_{40.2} = 1 - \frac{1 - 0.6q_{40}}{1 - 0.2q_{40}} = 0.000211. $$

□

Example 3.3 Use the life table extract in Table 3.1, with the UDD assumption, calculate (a) ${}_{1.7}q_{33}$ and (b) ${}_{1.7}q_{33.5}$.

Solution 3.3 (a) Because the starting age is an integer, we can use the linear interpolation approach from equation (3.9), with $x = 33, t = 1$, and $s = 0.7$, giving

$$ \begin{aligned} {}_{1.7}q_{33} &= 1 - {}_{1.7}p_{33} = 1 - \frac{0.3\,l_{34} + 0.7\,l_{35}}{l_{33}} = 1 - 0.991808 \\ &= 0.008192. \end{aligned} $$

(b) To calculate ${}_{1.7}q_{33.5}$, we first need to express it in terms of survival probabilities from age 33, as

$$ \begin{aligned} {}_{1.7}q_{33.5} &= 1 - {}_{1.7}p_{33.5} = 1 - \frac{{}_{2.2}p_{33}}{{}_{0.5}p_{33}} = 1 - \frac{0.8l_{35} + 0.2l_{36}}{0.5l_{33} + 0.5l_{34}} \\ &= 0.008537. \end{aligned} $$

Alternatively, using (3.8),

$$ \begin{aligned} {}_{1.7}q_{33.5} &= 1 - {}_{1.7}p_{33.5} = 1 - \frac{l_{35.2}}{l_{33.5}} = 1 - \frac{l_{35} - 0.2d_{35}}{l_{33} - 0.5d_{33}} \\ &= 0.008537. \end{aligned} $$

□

Example 3.4 Under the assumption of a uniform distribution of deaths, calculate $\lim_{t\to 1-} \mu_{40+t}$ using $p_{40} = 0.999473$, and calculate $\lim_{t\to 0+} \mu_{41+t}$ using $p_{41} = 0.999429$.

Solution 3.4 From formula (3.11), we have $\mu_{x+t} = q_x/{}_tp_x$ for $0 \le t < 1$.

Setting $x = 40$ yields

$$\lim_{t\to 1-} \mu_{40+t} = q_{40}/p_{40} = 5.27 \times 10^{-4},$$

while setting $x = 41$ yields

$$\lim_{t\to 0+} \mu_{41+t} = q_{41} = 5.71 \times 10^{-4}.$$

□

3.3.2 Constant force of mortality

A second fractional age assumption is that the force of mortality is constant between integer ages. Thus, for integer x and $0 \le s < 1$, we assume that μ_{x+s} does not depend on s, and we denote it μ_x^*. We can obtain the value of μ_x^* from the life table by using the fact that

$$p_x = \exp\left\{-\int_0^1 \mu_{x+s}\,ds\right\}.$$

So, if $\mu_{x+s} = \mu_x^*$ for $0 \le s < 1$ then $p_x = e^{-\mu_x^*}$ and $\mu_x^* = -\log p_x$. Further, given μ_x^*, and $r < 1$, we have

$${}_rp_x = \exp\left\{-\int_0^r \mu_x^*\,ds\right\} = e^{-r\mu_x^*} = (p_x)^r.$$

Similarly, for $r, t > 0$ and $r + t < 1$,

$$\boxed{{}_rp_{x+t} = \exp\left\{-\int_0^r \mu_x^*\,ds\right\} = (p_x)^r.} \tag{3.12}$$

Thus, under the constant force assumption, the probability of surviving for a period of $r < 1$ years from age $x + t$ is independent of t, provided that $r + t < 1$.

The assumption of a constant force of mortality between integer ages leads to a step function for the force of mortality over successive years of age, whereas we would expect the force of mortality to increase smoothly. However, if the true force of mortality increases slowly over the year of age, the constant force of mortality assumption is reasonable.

Example 3.5 Given that $p_{40} = 0.999473$, calculate ${}_{0.4}q_{40.2}$ under the assumption of a constant force of mortality.

Solution 3.5 We have ${}_{0.4}q_{40.2} = 1 - {}_{0.4}p_{40.2} = 1 - (p_{40})^{0.4} = 2.108 \times 10^{-4}$. □

Example 3.6 Given that $q_{70} = 0.010413$ and $q_{71} = 0.011670$, calculate ${}_{0.7}q_{70.6}$ under the assumption of a constant force of mortality.

Solution 3.6 We will work from the survival probability. We need to separate the survival probabilities applying in different age-years to use (3.12). That is

$$\begin{aligned} {}_{0.7}p_{70.6} &= {}_{0.4}p_{70.6} \times {}_{0.3}p_{71} \\ &= p_{70}^{0.4} \times p_{71}^{0.3} = 0.989587^{0.4} \times 0.988330^{0.3} \\ &= 0.992321 \\ \Rightarrow {}_{0.7}q_{70.6} &= 0.007679. \end{aligned}$$

□

Example 3.7 Using the life table extract in Table 3.1, with the constant force of mortality assumption, calculate (a) ${}_{1.7}q_{33}$ and (b) ${}_{1.7}q_{33.5}$.

Solution 3.7 (a) We write ${}_{1.7}q_{33}$ in terms of survival probabilities over whole and fractional years as

$$\begin{aligned} {}_{1.7}q_{33} &= 1 - {}_{1.7}p_{33} = 1 - p_{33}\ {}_{0.7}p_{34} = 1 - p_{33}\, p_{34}^{0.7} = 1 - 0.991805 \\ &= 0.008195. \end{aligned}$$

(b) Similarly,

$$\begin{aligned} {}_{1.7}q_{33.5} &= 1 - {}_{1.7}p_{33.5} = 1 - \frac{{}_{2.2}p_{33}}{{}_{0.5}p_{33}} = 1 - \frac{{}_{2}p_{33}\ {}_{0.2}p_{35}}{{}_{0.5}p_{33}} \\ &= 1 - \frac{{}_{2}p_{33}\, p_{35}^{0.2}}{p_{33}^{0.5}} \\ &= 0.008537. \end{aligned}$$

□

Note that in Examples 3.2 and 3.5, and in Examples 3.3 and 3.7, we have used two different methods to solve the same problems, and the solutions agree to at least five decimal places. It is generally true that the assumptions of a uniform distribution of deaths and a constant force of mortality produce very similar solutions to problems. The reason for this is that under the constant force of mortality assumption

$$q_x = 1 - e^{-\mu_x^*} \approx \mu_x^*$$

provided that μ_x^* is small, and for $0 < t < 1$,

$$ {}_tq_x = 1 - e^{-t\mu_x^*} \approx t\mu_x^*. $$

In other words, the approximation to ${}_tq_x$ is t times the approximation to q_x, which is what we obtain under the UDD assumption.

3.4 National life tables

Life tables based on the mortality experience of the whole population of a country are regularly produced for countries around the world. Separate life tables are usually produced for males and for females and possibly for some other groups of individuals, for example by race or socio-economic group.

Table 3.2 shows values of $q_x \times 10^5$, where q_x is the probability of dying within one year, for selected ages x, separately for males and females, for the populations of Australia, England & Wales and the United States. These tables are constructed using records of deaths and census data. The relevant years are indicated in the column headings.

Figure 3.1 shows the US 2013 Social Security mortality rates for males and females; the national life table graphs for England & Wales and for Australia are similar. Note that we have plotted these on a logarithmic scale in order to highlight the main features. Also, we have plotted a continuous line although the data are by integer age only.

Table 3.2 *Values of* $q_x \times 10^5$ *from some national life tables.*

	Australian Life Tables 2010–12		English Life Table 17 2010–2012		US Life Tables 2013	
x	Males	Females	Males	Females	Males	Females
0	412	335	476	381	651	537
1	35	27	31	24	46	38
2	21	17	21	18	29	22
10	9	7	9	7	9	9
20	61	26	50	20	104	38
30	83	34	73	36	147	67
40	134	75	147	86	210	130
50	287	172	310	214	509	321
60	660	384	802	533	1 126	660
70	1 675	980	2 069	1 330	2 283	1 533
80	5 189	3 150	5 740	4 070	5 830	4 255
90	16 121	12 825	16 814	13 509	16 504	13 102
100	31 255	35 130	36 107	32 134	35 354	30 467

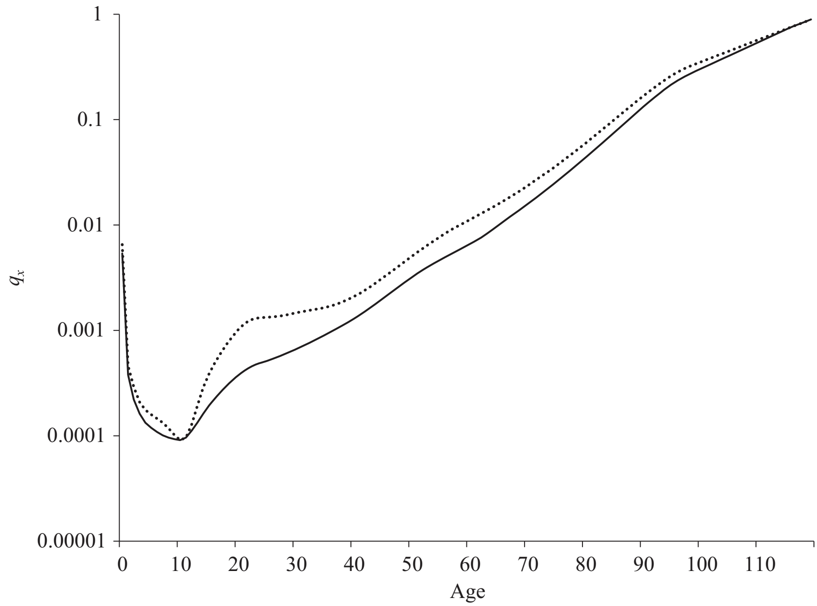


Figure 3.1 US 2013 mortality rates, male (dotted) and female (solid).

For all three national life tables, and for both males and females, the values of q_x follow very similar patterns relative to age x. We note the following points from Table 3.2 and Figure 3.1.

- The value of q_0 is relatively high. Mortality rates immediately following birth, termed **perinatal mortality**, are high, largely due to the low survival rates for babies born with serious congenital disabilities. The value of q_x does not reach this level again until about age 55. This can be seen from Figure 3.1.
- The rate of mortality is much lower after the first year, less than 10% of its level in the first year, and declines until around age 10.
- In Figure 3.1 we see that the pattern of male and female mortality in the late teenage years diverges significantly, with a steeper incline in male mortality. Not only is this feature of mortality for young adult males common for different populations around the world, it is also a feature of historical populations in countries such as the UK where mortality data has been collected for several centuries. It is sometimes called **the accident hump**, as many of the deaths causing the 'hump' are from accidental rather than natural causes.
- Mortality rates increase from age 10, with the accident hump creating a relatively large increase between ages 10 and 20 for males, a more modest increase from ages 20 to 40, and then steady increases from age 40.

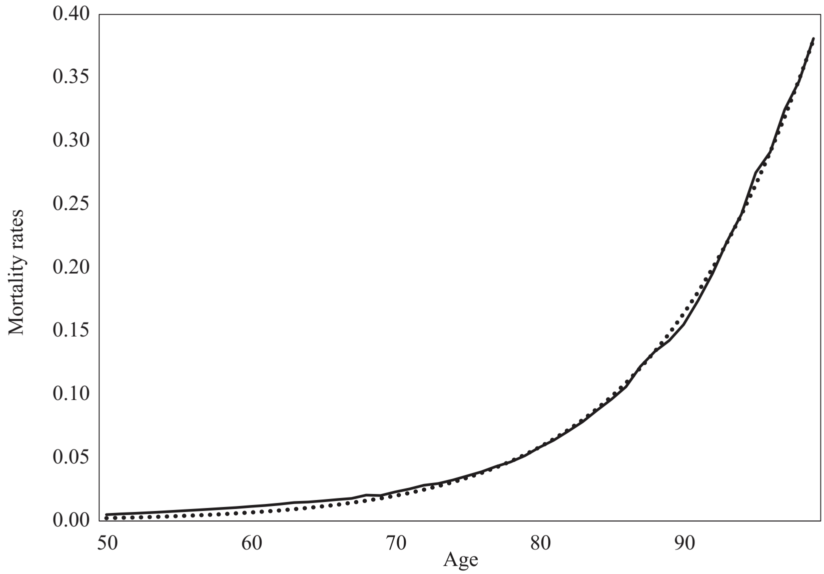


Figure 3.2 US 2015 male mortality rates (solid), with fitted Gompertz mortality rates (dotted).

- For each age, all six values of q_x are broadly comparable, with, for each country, female mortality rates less than male rates, sometimes very significantly.
- The Gompertz model introduced in Chapter 2 is relatively simple, in that it requires only two parameters and has a force of mortality with a straightforward exponential functional form, $\mu_x = Bc^x$. We stated in Chapter 2 that this model does not provide a good fit across all ages. We can see from Figure 3.1 that the model cannot fit the perinatal mortality, nor the accident hump. However, the mortality rates at later ages are rather better behaved, and the Gompertz model often proves useful over older age ranges. Figure 3.2 shows a Gompertz curve fitted to the US male population mortality rates from 2015, for ages 50–100. The Gompertz curve provides a pretty close fit – which is a particularly impressive feat, considering that Gompertz proposed the model in 1825.

A final point about Table 3.2 is that we have compared three national life tables using values of the probability of dying within one year, q_x, rather than the force of mortality, μ_x. This is because values of μ_x are not published for any ages for the US Life Tables. Also, values of μ_x are not published for age 0 for the other two life tables – there are technical difficulties in the estimation of μ_x within a year in which the force of mortality is changing rapidly, as it does between ages 0 and 1.

3.5 Survival models for life insurance policyholders

Suppose we have to choose a survival model appropriate for a man who is currently aged 50 and living in the UK, and who has just purchased a 10-year term insurance policy. We could use a national life table, such as English Life Table 17, so that, for example, we could assume that the probability this man dies before age 51 is 0.00310 as shown in Table 3.2. However, in the UK, as in other countries with well-developed life insurance markets, the mortality experience of people who purchase life insurance policies tends to be different from the population as a whole. The mortality of different types of life insurance policyholders is investigated separately, and life tables appropriate for these groups are published.

Table 3.3 shows values of the force of mortality ($\times 10^5$) at two-year intervals from age 50 to age 60 taken from English Life Table 17, Males (ELTM 17), and from a life table prepared from data relating to term insurance policyholders in the UK in 1999–2002 and which assumes the policyholders *purchased their policies at age 50*. This second set of values comes from Table A14 of a 2006 working paper of the Continuous Mortality Investigation in the UK. Hereafter we refer to this working paper as CMI; further details are given at the end of this chapter. The values of the force of mortality for ELTM 17 correspond to the values of q_x shown in Table 3.2.

The striking feature of Table 3.3 is the difference between the two sets of values. The values from the CMI table are very much lower than those from ELTM 17. There are at least three reasons why we would expect these probabilities to be different.

(1) The data on which the two life tables are based relate to different calendar years; 2010–2012 in the case of ELTM 17 and 1999–2002 in the case of CMI. Mortality rates in the UK, as in many other countries, have been decreasing for some years so we might expect rates based on more recent data to be lower. (See Section 3.11 for more discussion of mortality trends). However, this would mean that, all else being equal, the ELTM 17

Table 3.3 *Values of the force of mortality* $\times 10^5$.

x	ELTM 17	CMI A14
50	297	78
52	361	152
54	437	240
56	532	360
58	643	454
60	768	573

probabilities should be *lower* than the CMI rates. Since they are actually significantly higher, other explanations for the differences are needed.

(2) An important difference is that ELTM 17 is a life table based on the *whole male population* of England & Wales, whereas CMI Table A14 is based on the experience of males who are *term insurance policyholders*. Within any large group, there are likely to be variations in mortality rates between subgroups. This is true in the case of the population of England and Wales, where socio-economic classification, usually defined in terms of occupation, has a significant effect on mortality. Put simply, the better your job, and hence the wealthier you are, the lower your mortality rates. Given that people who purchase term insurance policies are likely to be among the better paid people in the population, we have an explanation for a large part of the difference between the values in Table 3.3.

(3) The third reason, which is the most significant, arises from the underwriting process which policyholders must complete before the insurer will issue the insurance policy. Underwriting ensures that people who purchase life insurance are healthy at the time of purchase, so the CMI figures apply to lives who were all healthy at age 50, when the insurance was purchased. The English Life Tables, on the other hand, are based on whole population data, which is a mixture of healthy and unhealthy lives. This is an example of **selection**, and we discuss it in more detail in the following section.

3.6 Life insurance underwriting

The values of the force of mortality in Table 3.3 are based on data for males who purchased term insurance at age 50. CMI Table A14 gives values for different ages at the purchase of the policy ranging from 17 to 90. Values for ages at purchase 50, 52, 54 and 56 are shown in Table 3.4.

Table 3.4 *Values of the force of mortality* $\times 10^5$ *from CMI Table A14.*

	Age at purchase of policy			
x	50	52	54	56
50	78	—	—	—
52	152	94	—	—
54	240	186	113	—
56	360	295	227	136
58	454	454	364	278
60	573	573	573	448
62	725	725	725	725
64	917	917	917	917
66	1159	1159	1159	1159

There are two significant features of the values in Table 3.4, which can be seen by looking at the rows of values for ages 56 and 62.

(1) Consider the row of values for age 56. Each of the four values in this row is the force of mortality at age 56 based on data from the UK over the period 1999–2002 for males who are term insurance policyholders. The only difference is that they purchased their policies at different ages. The more recently the policy was purchased, the lower the force of mortality. For example, for a person who purchased their policy at age 56, the value is 0.00136, whereas for someone of the same age who purchased his policy at age 50, the value is 0.00360.
(2) Now consider the row of values for age 62. These values, all equal to 0.00725, do not depend on whether the policy was purchased at age 50, 52, 54 or 56.

These features are due to life insurance underwriting, which we described in Chapter 1. Recall that the life insurance underwriting process evaluates medical and lifestyle information to assess whether the policyholder is insurable at normal rates.

The important point for this discussion is that the mortality rates in the CMI tables are based on individuals accepted for insurance at normal premium rates, that is, individuals who have passed the required health checks. This means, for example, that a man aged 50 who has just purchased a term insurance at the normal premium rate is known to be in good health, (assuming the health checks are effective) and so is likely to be much healthier, and hence have a lower mortality rate, than a man of age 50 picked randomly from the population. When this man reaches age 56, we can no longer be certain he is in good health – all we know is that he was in good health six years ago. Hence, his mortality rate at age 56 is higher than that of a man of the same age who has just passed the health checks and been permitted to buy a term insurance policy at normal rates. This explains the differences between the values of the force of mortality at age 56 in Table 3.4.

The effect of passing the health checks at issue eventually wears off, so that at age 62, the force of mortality does not depend on whether the policy was purchased at age 50, 52, 54 or 56. This is point (2) above. However, note that these rates, 0.00725, are still much lower than μ_{62} $(= 0.0091)$ from ELTM 17, despite the fact that the ELTM values are from more recent data. This is a combination of the residual selection effect from underwriting, and the lighter mortality experience of the socio-economic subgroup of the population who tend to buy insurance.

In the USA, there has been much growth in marketing insurance for 'preferred lives', who are assumed to be even healthier than the standard insured population. These preferred lives tend to be from the highest socio-economic groups. Mortality and wealth are closely linked.

3.7 Select and ultimate survival models

A feature of the survival models studied in Chapter 2 is that probabilities of future survival depend only on the individual's current age. For example, for a given survival model and a given term t, ${}_tp_x$, the probability that an individual currently aged x will survive to age $x + t$, depends only on the current age x. Such survival models are called **aggregate survival models**, because lives are all aggregated together.

The difference between an aggregate survival model and the survival model for term insurance policyholders discussed in Section 3.6 is that, in the latter case, probabilities of future survival depend not only on current age but also on how long ago the policy was purchased, which was when the policyholder joined the group of insured lives.

This leads us to the following definition. The mortality of a group of individuals is described by a **select and ultimate survival model**, usually shortened to **select survival model**, if the following statements are true.

(a) Future survival probabilities for an individual in the group depend on the individual's current age *and* on the age at which the individual joined the group.
(b) There is a positive number (generally an integer), which we denote by d, such that if an individual joined the group more than d years ago, future survival probabilities depend only on current age. The initial selection effect is assumed to have worn off after d years.

We use the following terminology for a select survival model. An individual who enters the group at, say, age x, is said to be **selected**, or just **select**, at age x. The period d after which the age at selection has no effect on future survival probabilities is called the **select period** for the model. The mortality that applies to lives after the select period is complete is called **ultimate** mortality, so that the complete model comprises a select period followed by the ultimate period.

Going back to the term insurance policyholders in Section 3.6, we can identify the 'group' as male term insurance policyholders in the UK. A select survival model is appropriate in this case because passing the health checks at age x indicates that the individual is in good health and so has lower mortality rates, on average, than someone of the same age who passed these checks some years ago. There are indications in Table 3.4 that the select period, d, for this group is less than or equal to six years. In fact, the select period is five years for this particular model. Select periods typically range from one year to 15 years for life insurance mortality models.

For the term insurance policyholders in Section 3.6, selection leads to lower mortality at early durations. However, selection can occur in many different ways and does not always lead to lower mortality rates, as Example 3.8 shows.

Example 3.8 Consider men who need to undergo surgery because they are suffering from a particular disease. The surgery is complicated and there is a probability of only 50% that they will survive for a year following surgery. If they do survive for a year, then they are fully cured and their future mortality follows the Australian Life Tables 2010–12, Males, from which you are given the following values:

$$l_{60} = 91\,649, \quad l_{61} = 91\,044, \quad l_{70} = 82\,577.$$

Calculate

(a) the probability that a man aged 60 who is just about to have surgery will be alive at age 70,
(b) the probability that a man aged 60 who had surgery at age 59 will be alive at age 70, and
(c) the probability that a man aged 60 who had surgery at age 58 will be alive at age 70.

Solution 3.8 In this example, the 'group' is all men who have had the operation. Being selected at age x means having surgery at age x. The select period of the survival model for this group is one year, since if they survive for one year after being 'selected', their future mortality depends only on their current age.

(a) The probability of surviving to age 61 is 0.5. Given that he survives to age 61, the probability of surviving to age 70 is

$$l_{70}/l_{61} = 82\,577/91\,044 = 0.9070.$$

Hence, the probability that this individual survives from age 60 to age 70 is

$$0.5 \times 0.9070 = 0.4535.$$

(b) Since this individual has already survived for one year following surgery, his mortality follows the Australian Life Tables 2010–12, Males. Hence, his probability of surviving to age 70 is

$$l_{70}/l_{60} = 82\,577/91\,649 = 0.9010.$$

(c) Since this individual's surgery was more than one year ago, his future mortality is exactly the same, probabilistically, as the individual in part (b). Hence, his probability of surviving to age 70 is 0.9010. □

Selection is not a feature of national life tables since, ignoring immigration, an individual can enter the population only at age zero. It is an important feature of many survival models based on data from, and hence appropriate to, life insurance policyholders. We can see from Tables 3.3 and 3.4 that its effect on the force of mortality can be considerable. For these reasons, select survival models are important in life insurance mathematics.

The select period will be different for different contexts. For CMI Table A14, which relates to term insurance policyholders, it is five years, as noted above; for CMI Table A2, which relates to whole life and endowment policyholders, the select period is two years. Generally, the underwriting process is more rigorous for term insurance, as the risk of adverse selection is greater, which leads to a longer selection impact for term insurance.

In the next section we introduce notation and develop some formulae for select survival models.

3.8 Notation and formulae for select survival models

A select survival model represents an extension of the ultimate survival model studied in Chapter 2. In Chapter 2, survival probabilities depended only on the current age of the individual. For a select survival model, probabilities of survival depend on current age and (within the select period) age at selection, i.e. age at joining the group. However, the survival model for those individuals all selected at the same age, say x, depends only on their current age and so fits the assumptions of Chapter 2. This means that, provided we fix and specify the age at selection, we can adapt the notation and formulae developed in Chapter 2 to a select survival model. This leads to the following definitions:

${}_tp_{[x]+s} = \Pr[\ (x{+}s)$, who was select at age x, survives to age $x+s+t]$,

${}_tq_{[x]+s} = \Pr[(x{+}s)$, who was select at age x, dies before age $x+s+t]$,

$\mu_{[x]+s}$ is the force of mortality at age $x+s$ for an individual who was select at age x,

$$\mu_{[x]+s} = \lim_{h\to 0^+}\left(\frac{1-{}_hp_{[x]+s}}{h}\right).$$

From these definitions we can derive the following formula

$${}_tp_{[x]+s} = \exp\left\{-\int_0^t \mu_{[x]+s+u}\,du\right\}.$$

This formula is derived precisely as in Chapter 2. It is only the notation that has changed.

For a select survival model with a select period d and for $t \geq d$, (that is, for durations at or beyond the select period) the values of $\mu_{[x-t]+t}$, ${}_sp_{[x-t]+t}$ and ${}_{u|s}q_{[x-t]+t}$ do not depend on t, they depend only on the current age x. So, for $t \geq d$ we drop the selection information and just write functions in terms of the attained age, i.e. μ_x, ${}_sp_x$ and ${}_{u|s}q_x$. For values of $t < d$, we refer to, for example, $\mu_{[x-t]+t}$ as being in the **select** part of the survival model and for $t \geq d$ we refer to $\mu_{[x-t]+t}$ $(\equiv \mu_x)$ as being in the **ultimate** part of the survival model.

3.9 Select life tables

For an ultimate survival model the life table $\{l_x\}$ is useful since it can be used to calculate probabilities such as ${}_{t|u}q_x$ for values of t, u and x. We can construct a **select life table** in a similar way but we need the table to reflect duration since selection, as well as age, during the select period. Suppose we wish to construct this table for a select survival model for ages at selection from, say, x_0 (≥ 0). Let d denote the select period, assumed to be an integer number of years.

The construction in this section is for a select life table specified at all ages and not just at integer ages. However, select life tables are usually presented at integer ages only, as for the life tables introduced earlier in this chapter.

First we consider the survival probabilities of those individuals who were selected at least d years ago and hence are now subject to the ultimate part of the model. The minimum age of these lives is $x_0 + d$. For these people, future survival probabilities depend only on their current age and so, as in Chapter 2, we can construct an ultimate life table, $\{l_y\}$, for them from which we can calculate probabilities of surviving to any future age.

Let l_{x_0+d} be an arbitrary positive number. For $y > x \geq x_0 + d$, we define

$$l_y = {}_{y-x}p_x \; l_x. \tag{3.13}$$

Formula (3.13) defines the life table within the ultimate part of the model, which applies after the select period has expired. Within the ultimate part of the model we can interpret l_y as the expected number of survivors to age y out of l_x lives currently aged x $(< y)$, who were select at least d years previously.

Next, we define the life table within the select period. We do this for a life who was select at age x by working backwards from the value of l_{x+d}. For $x \geq x_0$ and for $0 \leq t \leq d$, we define $l_{[x]+t}$ as

$${}_{d-t}p_{[x]+t} = \frac{l_{x+d}}{l_{[x]+t}} \Rightarrow l_{[x]+t} = \frac{l_{x+d}}{{}_{d-t}p_{[x]+t}}, \tag{3.14}$$

which means that if we had $l_{[x]+t}$ lives aged $x+t$, who were select t years ago, then the expected number of survivors to age $x+d$ is l_{x+d}.

Example 3.9 A life currently aged 52 was select at age 50. The survival model has a select period of 10 years. Write down the following probabilities in terms of $l_{[x]+t}$ and l_y for appropriate x, t, and y:

(a) the probability that the life survives to age 58, and
(b) the probability that the life survives to age 62.

Solution 3.9 (a) As the select period is 10 years, we are interested in the life surviving within the select period from $[50]+2$ to $[50]+8$, so the required probability is

$$_6p_{[50]+2} = \frac{l_{[50]+8}}{l_{[50]+2}}.$$

(b) We are now interested in the life surviving from $[50] + 2$ for a period of 10 years, which takes the life beyond the end of the select period, so the required probability is

$$_{10}p_{[50]+2} = \frac{l_{[50]+12}}{l_{[50]+2}} = \frac{l_{62}}{l_{[50]+2}}.$$

□

The construction of the select life table preserves the interpretation of the l_x and $l_{[x]+t}$ functions as expected numbers of survivors within the ultimate and the select parts of the model respectively. For example, suppose we have $l_{[x]+t}$ individuals currently aged $x + t$ who were select at age x.

For u such that $u + t \geq d$, we have

$$l_{x+t+u} = {}_up_{[x]+t}\ l_{[x]+t}$$

demonstrating that l_{x+t+u} is the expected number of survivors to age $x+t+u$ from the $l_{[x]+t}$ lives aged $x + t$ who were select at age x.

For u such that $u + t < d$, we have

$$l_{[x]+t+u} = {}_up_{[x]+t}\ l_{[x]+t}$$

demonstrating that $l_{[x]+t+u}$ is the expected number of survivors to age $x+t+u$ out of $l_{[x]+t}$ lives currently aged $x + t$, who were select at age x.

Example 3.10 Write an expression for ${}_{2|6}q_{[30]+2}$ in terms of $l_{[x]+t}$ and l_y for appropriate x, t, and y, assuming a select period of five years.

Solution 3.10 Note that ${}_{2|6}q_{[30]+2}$ is the probability that a life currently aged 32, who was select at age 30, will die between ages 34 and 40. We can write this probability as the product of the probabilities of the following events:

- a life aged 32, who was select at age 30, will survive to age 34, and,
- a life aged 34, who was select at age 30, will die before age 40.

Table 3.5 *Extract from 2002 US Life Tables, Females.*

x	l_x
70	80 556
71	79 026
72	77 410
73	75 666
74	73 802
75	71 800

Hence,

$$\begin{aligned} {}_{2|6}q_{[30]+2} &= {}_2p_{[30]+2}\ {}_6q_{[30]+4} \\ &= \frac{l_{[30]+4}}{l_{[30]+2}}\left(1 - \frac{l_{[30]+10}}{l_{[30]+4}}\right) \\ &= \frac{l_{[30]+4} - l_{40}}{l_{[30]+2}}. \end{aligned}$$

Note that $l_{[30]+10} \equiv l_{40}$ since 10 years is longer than the select period for this survival model. □

Example 3.11 A select survival model has a select period of three years. Its ultimate mortality is equivalent to the US Life Tables, 2002, Females. Some l_x values for this table are shown in Table 3.5.

You are given that for all ages $x \geq 65$,

$$p_{[x]} = 0.999, \quad p_{[x-1]+1} = 0.998, \quad p_{[x-2]+2} = 0.997.$$

Calculate the probability that a woman currently aged 70 will survive to age 75 given that

(a) she was select at age 67,
(b) she was select at age 68,
(c) she was select at age 69, and
(d) she is select at age 70.

Solution 3.11 (a) Since the woman was select three years ago and the select period for this model is three years, she is now subject to the ultimate part of the survival model. Hence the probability she survives to age 75 is l_{75}/l_{70}, where the l_x values are taken from US Life Tables, 2002, Females. The required probability is

$${}_5p_{70} = 71\,800/80\,556 = 0.8913.$$

(b) We have

$$ {}_5p_{[68]+2} = \frac{l_{[68]+2+5}}{l_{[68]+2}} = \frac{l_{75}}{l_{[68]+2}} = \frac{71\,800}{l_{[68]+2}}. $$

We calculate $l_{[68]+2}$ by noting that

$$ l_{[68]+2} \times p_{[68]+2} = l_{[68]+3} = l_{71} = 79\,026. $$

We are given that $p_{[68]+2} = 0.997$. Hence, $l_{[68]+2} = 79\,264$ and so

$$ {}_5p_{[68]+2} = 0.9058. $$

(c) We have

$$ {}_5p_{[69]+1} = \frac{l_{[69]+1+5}}{l_{[69]+1}} = \frac{l_{75}}{l_{[69]+1}} = \frac{71\,800}{l_{[69]+1}}. $$

We calculate $l_{[69]+1}$ by noting that

$$ l_{[69]+1} \times p_{[69]+1} \times p_{[69]+2} = l_{[69]+3} = l_{72} = 77\,410. $$

We are given that $p_{[69]+1} = 0.998$ and $p_{[69]+2} = 0.997$. Hence, $l_{[69]+1} = 77\,799$ and so ${}_5p_{[69]+1} = 0.9229$.

(d) We have

$$ {}_5p_{[70]} = \frac{l_{[70]+5}}{l_{[70]}} = \frac{l_{75}}{l_{[70]}} = \frac{71\,800}{l_{[70]}}. $$

Proceeding as in parts (b) and (c),

$$ \begin{aligned} l_{[70]} \times p_{[70]} \times p_{[70]+1} \times p_{[70]+2} &= l_{[70]+3} = l_{73} = 75\,666 \\ \Rightarrow l_{[70]} &= 75\,666/(0.997 \times 0.998 \times 0.999) = 76\,122 \\ \Rightarrow {}_5p_{[70]} &= 0.9432. \end{aligned} $$

□

Example 3.12 An extract from CMI Table A5 is given in Table 3.6. This table is based on UK data from 1999 to 2002 for male non-smokers who are whole life or endowment insurance policyholders. It has a select period of two years.

(a) Use the table to calculate the following probabilities:
(i) ${}_4p_{[70]}$, (ii) ${}_3q_{[60]+1}$, (iii) ${}_{2|}q_{73}$.

(b) You are given that $e_{72} = 13.3122$.
(i) Calculate $e_{[70]}$.
(ii) Calculate $\mathring{e}_{[70]}$ assuming UDD between integer ages.

Solution 3.12 CMI Table A5 gives values of $q_{[x-t]+t}$ for $t = 0$ and $t = 1$ and also for $t \geq 2$. Since the select period is two years $q_{[x-t]+t} \equiv q_x$ for $t \geq 2$.

Table 3.6 *CMI Table A5: male non–smokers who have whole life or endowment policies.*

Age, x	Duration 0 $q_{[x]}$	Duration 1 $q_{[x-1]+1}$	Duration 2+ q_x
60	0.003469	0.004539	0.004760
61	0.003856	0.005059	0.005351
62	0.004291	0.005644	0.006021
63	0.004779	0.006304	0.006781
⋮	⋮	⋮	⋮
70	0.010519	0.014068	0.015786
71	0.011858	0.015868	0.017832
72	0.013401	0.017931	0.020145
73	0.015184	0.020302	0.022759
74	0.017253	0.023034	0.025712
75	0.019664	0.026196	0.029048

Note also that each row of the table relates to a life *currently* aged x, where x is given in the first column. Select life tables, tabulated at integer ages, can be set out in different ways – for example, each row could relate to a fixed age at selection – so care needs to be taken when using such tables.

(a) (i) We calculate ${}_4p_{[70]}$ as

$$\begin{aligned}
{}_4p_{[70]} &= p_{[70]} \times p_{[70]+1} \times p_{[70]+2} \times p_{[70]+3} \\
&= p_{[70]} \times p_{[70]+1} \times p_{72} \times p_{73} \\
&= (1 - q_{[70]}) \times (1 - q_{[70]+1}) \times (1 - q_{72}) \times (1 - q_{73}) \\
&= 0.989481 \times 0.984132 \times 0.979855 \times 0.977241 \\
&= 0.932447.
\end{aligned}$$

(ii) We calculate ${}_3q_{[60]+1}$ as

$$\begin{aligned}
{}_3q_{[60]+1} &= q_{[60]+1} + p_{[60]+1}\, q_{62} + p_{[60]+1}\, p_{62}\, q_{63} \\
&= q_{[60]+1} + (1 - q_{[60]+1})\, q_{62} + (1 - q_{[60]+1})\,(1 - q_{62})\, q_{63} \\
&= 0.017756.
\end{aligned}$$

(iii) We calculate ${}_{2|}q_{73}$ as

$$\begin{aligned}
{}_{2|}q_{73} &= {}_2p_{73}\, q_{75} \\
&= (1 - q_{73})\,(1 - q_{74})\, q_{75} \\
&= 0.027657.
\end{aligned}$$

(b) (i) Recall from Exercise 2.2 that $e_x = p_x(1 + e_{x+1})$.
Similarly, $e_{[70]} = p_{[70]}(1 + e_{[70]+1})$ and $e_{[70]+1} = p_{[70]+1}(1 + e_{72})$, giving

$$\begin{aligned} e_{[70]} &= p_{[70]} + {}_2p_{[70]}\,(1 + e_{72}) \\ &= 0.989481 + 0.973780(1 + 13.3122) \\ &= 14.9264. \end{aligned}$$

(ii) We want to calculate

$$\mathring{e}_{[70]} = \int_0^\infty {}_tp_{[70]}\,dt,$$

and as UDD allows us to calculate survival probabilities from integer ages for non-integer terms, we restate this integral in terms of a sum of one-year integrals as

$$\mathring{e}_{[70]} = \int_0^1 {}_tp_{[70]}\,dt + \int_1^2 {}_tp_{[70]}\,dt + \int_2^3 {}_tp_{[70]}\,dt + \cdots .$$

Now consider each term in this sum:

$$\begin{aligned} \int_0^1 {}_tp_{[70]}\,dt &= \int_0^1 (1 - {}_tq_{[70]})\,dt = \int_0^1 (1 - t\,q_{[70]})\,dt = 1 - \tfrac{1}{2}q_{[70]} \\ &= 1 - \tfrac{1}{2}(1 - p_{[70]}) = \tfrac{1}{2}(1 + p_{[70]}), \\ \int_1^2 {}_tp_{[70]}dt &= p_{[70]} \int_0^1 {}_tp_{[70]+1}\,dt = p_{[70]}\,\tfrac{1}{2}(1 + p_{[70]+1}) \\ &= \tfrac{1}{2}(p_{[70]} + {}_2p_{[70]}), \\ \int_2^3 {}_tp_{[70]}dt &= {}_2p_{[70]} \int_0^1 {}_tp_{72}\,dt = {}_2p_{[70]}\,\tfrac{1}{2}(1 + p_{72}) = \tfrac{1}{2}({}_2p_{[70]} + {}_3p_{[70]}), \end{aligned}$$

and so on. Collecting the terms together gives

$$\begin{aligned} \mathring{e}_{[70]} &= \tfrac{1}{2}\big(1 + p_{[70]} + p_{[70]} + {}_2p_{[70]} + {}_2p_{[70]} \\ &\qquad + {}_3p_{[70]} + {}_3p_{[70]} + {}_4p_{[70]} + \cdots\big) \\ &= \tfrac{1}{2} + p_{[70]} + {}_2p_{[70]} + {}_3p_{[70]} + \cdots \\ &= e_{[70]} + \tfrac{1}{2} \\ &= 15.4264. \end{aligned}$$

□

Example 3.13 A select survival model has a two-year select period and is specified as follows. The ultimate part of the model follows Makeham's law, so that

$$\mu_x = A + Bc^x$$

where $A = 0.00022, B = 2.7 \times 10^{-6}$ and $c = 1.124$. The select part of the model is such that for $0 \le s \le 2$,

$$\mu_{[x]+s} = 0.9^{2-s} \mu_{x+s}.$$

Starting with $l_{20} = 100\,000$, calculate values of

(a) l_x for $x = 21, 22, \ldots, 82$,
(b) $l_{[x]+1}$ for $x = 20, 21, \ldots, 80$, and,
(c) $l_{[x]}$ for $x = 20, 21, \ldots, 80$.

Solution 3.13 First, note that

$${}_tp_x = \exp\left\{-At - \frac{B}{\log\, c} c^x (c^t - 1)\right\}$$

and for $0 \le t \le 2$,

$$\begin{aligned} {}_tp_{[x]} &= \exp\left\{-\int_0^t \mu_{[x]+s} ds\right\} \\ &= \exp\left\{0.9^{2-t}\left(\frac{1-0.9^t}{\log(0.9)}A + \frac{c^t - 0.9^t}{\log(0.9/c)}Bc^x\right)\right\}. \qquad (3.15) \end{aligned}$$

(a) Values of l_x can be calculated recursively from

$$l_x = p_{x-1} l_{x-1} \quad \text{for} \quad x = 21, 22, \ldots, 82.$$

(b) Values of $l_{[x]+1}$ can be calculated from

$$l_{[x]+1} = l_{x+2}/p_{[x]+1} \quad \text{for} \quad x = 20, 21, \ldots, 80.$$

(c) Values of $l_{[x]}$ can be calculated from

$$l_{[x]} = l_{x+2}/{}_2p_{[x]} \quad \text{for} \quad x = 20, 21, \ldots, 80.$$

Sample values are shown in Table 3.7. The full table up to age 100 is given in Table D.1 in Appendix D. □

This model is used extensively throughout this book for examples and exercises. We call it the **Standard Select Survival Model** in future chapters.

The ultimate part of the model, which is a Makeham model with $A = 0.00022, B = 2.7 \times 10^{-6}$ and $c = 1.124$, is also used in many examples and exercises where a select model is not required. We call this the **Standard Ultimate Survival Model**.

Table 3.7 *Select life table with a two-year select period, Example 3.13.*

x	$l_{[x]}$	$l_{[x]+1}$	l_{x+2}	$x+2$	x	$l_{[x]}$	$l_{[x]+1}$	l_{x+2}	$x+2$
			100 000.00	20	50	98 552.51	98 450.67	98 326.19	52
			99 975.04	21	51	98 430.98	98 318.95	98 181.77	53
20	99 995.08	99 973.75	99 949.71	22	52	98 297.24	98 173.79	98 022.38	54
21	99 970.04	99 948.40	99 923.98	23	53	98 149.81	98 013.56	97 846.20	55
22	99 944.63	99 922.65	99 897.79	24	54	97 987.03	97 836.44	97 651.21	56
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
47	98 856.38	98 778.94	98 684.88	49	79	77 465.70	75 531.88	73 186.31	81
48	98 764.09	98 679.44	98 576.37	50	80	75 153.97	73 050.22	70 507.19	82
49	98 663.15	98 570.40	98 457.24	51					

3.10 Some comments on heterogeneity in mortality

We noted in Section 3.5 the significant difference between the mortality of the population as a whole, and the mortality of insured lives. It is worth noting, further, that there is also considerable variability when we look at the mortality experience of different groups of insurance company customers and pension plan members. Of course, male and female mortality differs significantly, in shape and level. Actuaries will generally use separate survival models for men and women when this does not breach discrimination laws. Smoker and non-smoker mortality differences are very important in whole life and term insurance; smoker mortality is substantially higher at all ages for both sexes, and separate smoker/non-smoker mortality tables are in common use.

In addition, insurers will generally use product-specific mortality tables for different types of contracts. Individuals who purchase immediate or deferred annuities may have different mortality to those purchasing term insurance. Insurance is sometimes purchased under group contracts, for example by an employer to provide death in service insurance for employees. The mortality experience from these contracts will generally be different to the experience of policyholders holding individual contracts. The mortality experience of pension plan members may differ from the experience of lives who purchase individual pension policies from an insurance company. Interestingly, the differences in mortality experience between these groups will depend significantly on country. Studies of mortality have shown, though, that the following principles apply quite generally.

- ◇ Wealthier lives experience lighter mortality overall than less wealthy lives.
- ◇ There will be some impact on the mortality experience from self-selection; an individual will only purchase an annuity if he or she is confident of living long enough to benefit; conversely, an individual who has some reason to anticipate heavier mortality is more likely to purchase term

insurance. While underwriting can identify some selective factors, there may be other information that cannot be gleaned from the underwriting process (at least not without excessive cost). So those buying term insurance might be expected to have slightly heavier mortality than those buying whole life insurance, and those buying annuities might be expected to have lighter mortality. This is often called **adverse selection** or **anti-selection** by actuaries, as the selection effect acts to make the insurance or annuity benefits more valuable to the purchaser.

◇ The more rigorous the underwriting, the lighter the resulting mortality experience. For group insurance, there will be minimal underwriting. Each person hired by the employer will be covered by the insurance policy almost immediately; the insurer does not get to accept or reject the additional employee, and will rarely be given information sufficient for underwriting decisions. However, the employee must be healthy enough to be hired, which gives some selection information.

All of these factors may be confounded by tax or legislative systems that encourage or require certain types of contracts. In the UK, it is very common for retirement savings proceeds to be converted to life annuities. In other countries, including the USA, this is much less common. Consequently, the type of person who buys an annuity in the USA might be quite a different (and more self-select) customer than the typical individual buying an annuity in the UK.

Example 3.14 Three-year term insurance policies are sold to a group of independent lives each aged 65. At the start of the term, 35% of the policyholders are smokers and 65% are non-smokers. The survival probabilities for the non-smokers are $p_{65} = p_{66} = p_{67} = 0.95$. The force of mortality of the smokers is 3.5 times that of non-smokers at all ages. Calculate the proportion of smokers amongst the policyholders expected to die during the three-year term.

Solution 3.14 Let p_x^s and μ_x^s respectively denote the one-year survival probability and force of mortality at age x for the smokers. Functions with no superscript are used for the non-smokers. The one-year survival probability for smokers is

$$p_x^s = \exp\left\{-\int_0^1 \mu_{x+t}^s\,dt\right\} = \exp\left\{-3.5\int_0^1 \mu_{x+t}\,dt\right\} = p_x^{3.5}.$$

So we have $p_x^s = 0.95^{3.5}$ for $x = 65,\ 66,\ 67$.

Suppose the initial group size is N. Then, since ${}_3p_{65} = 0.95^3$, the expected number of deaths from the non-smoker group is

$$0.65N\left(1 - 0.95^3\right) = 0.09271N,$$

and the expected number of deaths from the smoker group is

$$0.35N(1 - (0.95^{3.5})^3) = 0.14575N.$$

So the proportion of deaths expected to come from the smoker group is

$$\frac{0.14575N}{N(0.14575 + 0.09271)} = 61.1\%.$$

Even though the smokers comprise only 35% of the population, they are expected to account for over 60% of the deaths. □

3.11 Mortality improvement modelling

A challenge in developing and using survival models is that survival probabilities are not constant. Commonly, mortality experience gets lighter over time; each generation, on average, lives longer than the previous generation. This can be explained by advances in health care and by improved standards of living. Of course, there are exceptions, such as mortality shocks from war or from disease, or declining life expectancy in countries where access to health care worsens, often because of civil upheaval.

The changes in mortality over time are sometimes separated into three components: trend, shock and idiosyncratic. The trend describes the gradual reduction in mortality rates over time. We often refer to this as the longevity trend. The shock describes short-term jumps in mortality rates, often caused by war or pandemic disease. The idiosyncratic component describes year-to-year random variation that does not come from trend or shock, though it is often difficult to distinguish.

While the shock and idiosyncratic components are inherently unpredictable, we can identify trends by examining aggregate mortality patterns over a number of years. We can then allow for mortality improvement by using a survival model which depends on both age and calendar year. So, for example, we expect the mortality rate for lives who are aged 50 in 2015 to be different from the mortality rate for lives who are aged 50 in 2025; a life table that depends on both age and calendar year can be used to capture this. In this section we present some models and methods for integrating mortality improvement into actuarial analysis for life contingent risks.

First, it might be valuable to demonstrate what we mean by mortality or longevity improvement. In Figure 3.3 we show raw (that is, with no smoothing) mortality rates for US males aged 30–44 from 1960–2015, and for US females aged 50–69 for the same period, obtained from the Human Mortality Database (HMD). In each figure the higher lines are for the older ages, and the lower lines for the younger ages.

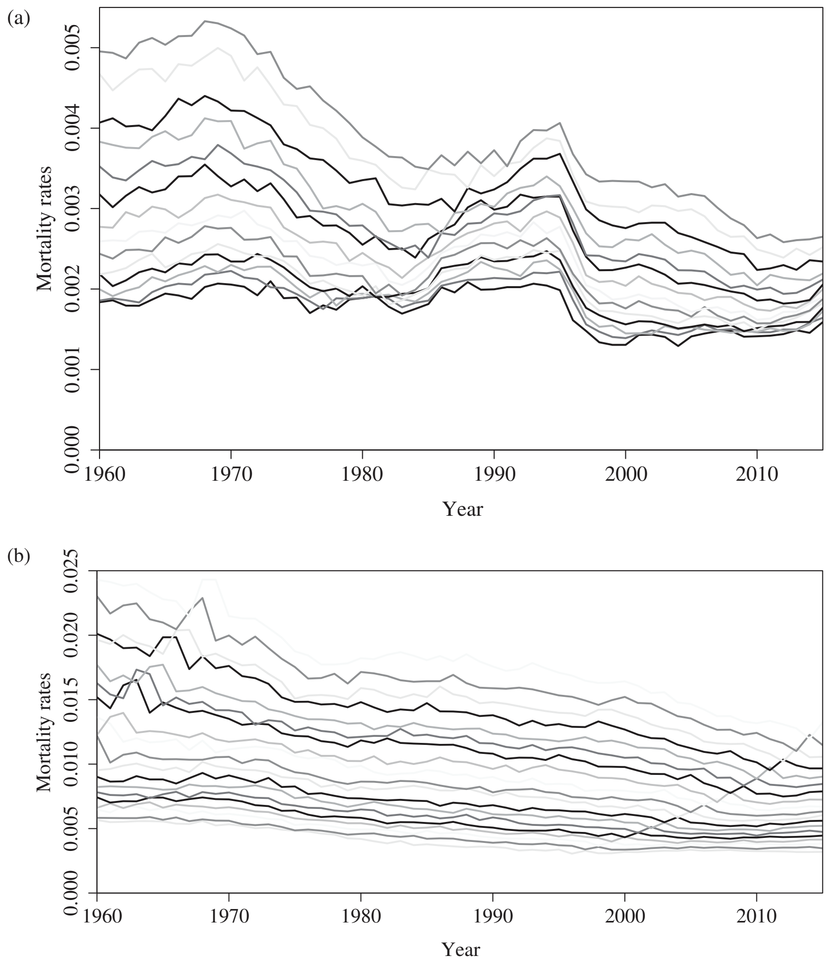


Figure 3.3 US mortality experience 1960–2015 (from HMD): (a) males aged 30–44 and (b) females aged 50–69.

Overall, we see that, for each age, mortality rates are generally declining over time, although there are exceptional periods where the rates shift upwards.

We also note that the rates are not very smooth. There appears to be some random variation around the general trends.

When modelling mortality we generally smooth the raw data to reduce the impact of sampling variability. It is also common in longevity modelling to use heatmaps of mortality improvement to illustrate the two-dimensional data, rather than the age curves of mortality rates in Figure 3.3.

In Figure 3.4 we show a plot of smoothed mortality improvement factors for US data, for 1951–2007. The mortality improvement factor is the percentage

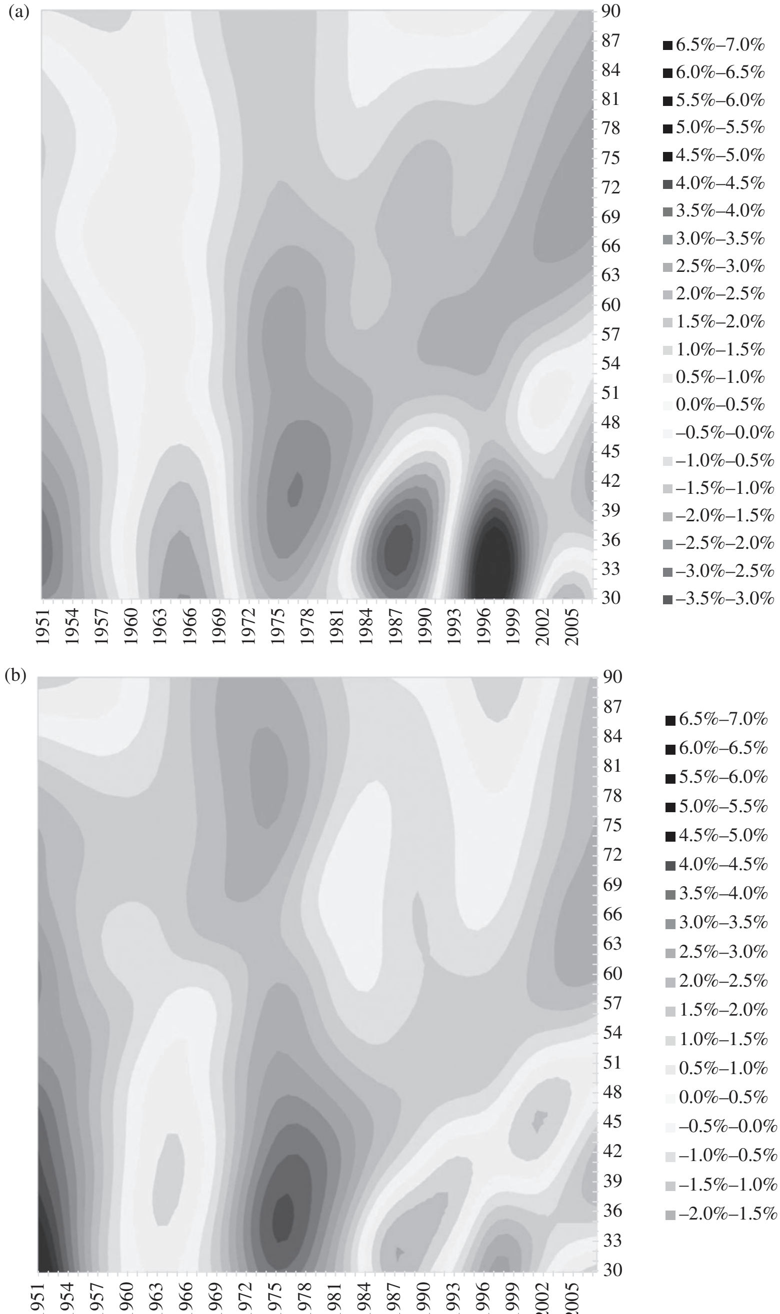


Figure 3.4 US smoothed mortality improvement heatmaps, 1951–2007, for (a) males and (b) females.

reduction in the mortality rate for each age over each successive calendar year. That is, if the smoothed mortality rate for age x in year t is $\tilde{q}(x,t)$ then the smoothed **mortality improvement factor** at age x and year t is

$$\varphi(x,t) = 1 - \frac{\tilde{q}(x,t)}{\tilde{q}(x,t-1)}\,.$$

The heatmaps and the curves in Figure 3.3 illustrate the following three effects.

Year effects

Calendar year effects are identified in the heatmaps with vertical patterns. For example, look at the years 1958–1970 in Figure 3.4(a). The vertical lighter column for those years indicates that longevity improvement was paused or reversed for all ages in those years, though the impact was different for different age groups; the same phenomenon is apparent in the raw data in Figure 3.3(a), where we see gently rising mortality rates over most ages between 1960 and 1970. The next vertical section of the graph shows mortality improving again, with improvement more marked for younger lives (illustrated with the darker tones) than for older.

In Figures 3.3(a) and 3.4(a) we also see a very clear and severe deterioration in mortality between 1984 and 1991 most strongly affecting younger males. This area illustrates the impact of the HIV/AIDS epidemic on younger male mortality in the USA. In the following period, around 1993–2000, mortality in the same age range showed very strong improvement, as medical and social management of HIV/AIDS produced an extraordinary turnaround in the impact of the disease on population mortality.

Age effects

Age effects in the heatmaps are evident from horizontal patterns; in Figure 3.4 there is little evidence of pure age effects that are protracted across the whole period. The most obvious impact of age in the heatmaps is in the way that different age groups are impacted differently by the calendar year effects. For example, the mortality improvement experienced by US females in the 1970s was more significant for people below 45 years old than for older lives.

In both the heatmaps, we see less intense patterns of improvement or decline at older ages. It is common to assume that we will not see any significant mortality improvement at the very oldest ages, say, beyond age 95. The idea is that, although more people are living to older ages, there is not much evidence that the oldest attainable age is increasing. This phenomenon is referred to as the **rectangularization** of mortality, from the fact that the trend in longevity is generating more rectangular-looking survival curves (i.e. curves of ${}_tp_0$ for values of t from age 0 to, say, age 120 years) without significantly shifting the right tail of the survival curve.

Cohort effects

Cohort effects refer to patterns of mortality that are consistent for lives born in the same year. Cohort effects can be seen in the individual age rates in Figure 3.3 as spikes or troughs that move up diagonally across the curves, as the lives who are, say, aged 40 in 1951, if they survive, become the lives who are aged 41 in 1952, and so on. Cohort effects are seen in the heatmaps as diagonal patterns from lower left to upper right. In Figure 3.4(a) there is a diagonal band of higher improvement applying to lives born around 1935–1942, and there is a similar band in 3.4(b), but for lives born a few years later, in the period from around 1940–1945.

Cohort mortality effects are not observed in all populations, and there is still substantial uncertainty as to why they occur.

If mortality rates are generally declining over time, it may not be suitable to assume the same rate of mortality in actuarial calculations regardless of how far ahead we are looking. Advances in medical science, and improvements in other social determinants of longevity, such as nutrition and access to health care, lead us to expect that, in general, mortality rates will continue to decline.

To model mortality as a function of both age and time, we replace the age-based mortality rate q_x with a rate based on the attained age x and on the calendar year that the age is attained, t. We let $q(x,t)$ denote the mortality rate applying to lives who attain age x in year t, and $p(x,t) = 1 - q(x,t)$. We may measure t relative to some base year, or it may be used to indicate the full calendar year.

There are two approaches to modelling mortality trends. The first is a deterministic approach, where we model $q(x,t)$ as a fixed, known function, using a deterministic **mortality improvement scale** function. The second is a stochastic approach, where we treat future values of $q(x,t)$ as a series of random variables. In this chapter we consider some deterministic methods that are commonly used. In Chapter 19, we discuss some stochastic models of longevity that have been adopted by actuaries and others.

3.12 Mortality improvement scales

Broadly, the deterministic approach to mortality improvement models uses a two-step process.

Step 1 Choose a base year and construct tables of mortality rates for lives attaining each integer age in the base year. This gives the values $q(x,0)$.

Step 2 Construct a scale function that can be applied to the base mortality rates to generate appropriate rates for future years.

3.12.1 Single-factor mortality improvement scales

The simplest scale functions depend only on age. If we denote the improvement factor for age x as φ_x, then, for $t = 1, 2, 3, \ldots$,

$$q(x,t) = q(x,0)(1-\varphi_x)^t \,.$$

So, if ${}_rp(x,t)$ denotes the probability that a life who is aged x at time t survives r years, using the single-factor improvement model we have

$$\begin{aligned}
{}_rp(x,t) &= p(x,\ t)\ p(x{+}1,\ t{+}1)\ p(x{+}2,\ t{+}2)\ \cdots\ p(x{+}r{-}1,\ t{+}r{-}1) \\
&= \big(1 - q(x,\ t)\big)\big(1 - q(x{+}1,\ t{+}1)\big)\cdots\big(1 - q(x{+}r{-}1,\ t{+}r{-}1)\big) \\
&= \big(1 - q(x,0)(1{-}\varphi_x)^t\big)\big(1 - q(x{+}1,0)(1{-}\varphi_{x+1})^{t+1}\big) \\
&\quad \ldots\big(1 - q(x{+}r{-}1,0)(1{-}\varphi_{x+r-1})^{t+r-1}\big).
\end{aligned}$$

In Figure 3.5, we show a set of age-based improvement factors published by the Society of Actuaries in 1994, known as Scale AA.

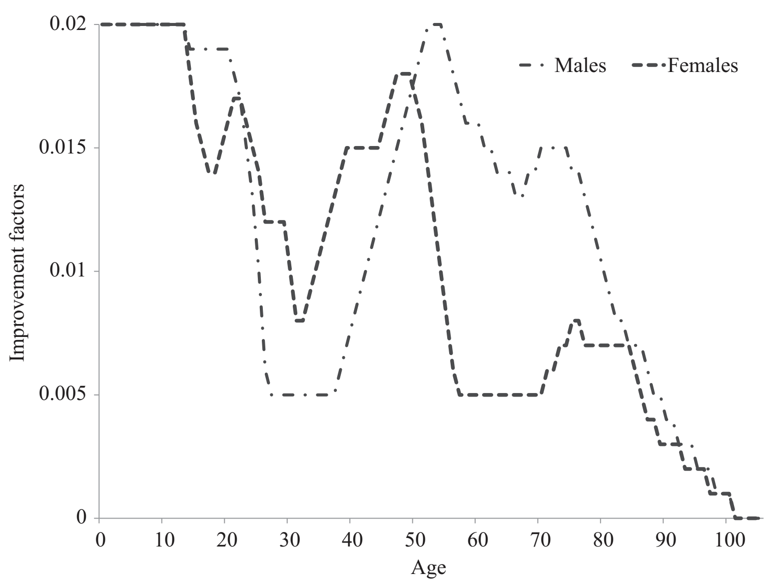


Figure 3.5 Scale AA mortality improvement factors.

Table 3.8 *RP2000 Male healthy annuitant mortality rates, with Scale AA improvement factors.*

Age x	$q(x, 0)$	φ_x
60	0.008196	0.016
61	0.009001	0.015
62	0.009915	0.015
63	0.010951	0.014
64	0.012117	0.014
65	0.013419	0.014
66	0.014868	0.013
67	0.016460	0.013
68	0.018200	0.014
69	0.020105	0.014
70	0.022206	0.015

Example 3.15 In Table 3.8 we show base mortality rates for males in the year 2000, and we show the Scale AA mortality improvement factors, denoted φ_x, for the same age range.

Calculate the 10-year survival probability for a life aged 60, with and without the mortality improvement scale.

Solution 3.15 Without mortality improvement we have

$$ {}_{10}p_{60} = \prod_{t=0}^{9} (1 - q(60 + t, 0)) = 0.87441. $$

With mortality improvement we have

$$ \begin{aligned} {}_{10}p_{60} &= \prod_{t=0}^{9} (1 - q(60 + t, t)) \\ &= \prod_{t=0}^{9} \left(1 - q(60 + t, 0)(1 - \varphi_{60+t})^t\right) = 0.88277. \end{aligned} $$

As we expect, the survival probability is a little higher when we allow for mortality improvement. □

The one-factor mortality improvement scales have proven too simplistic. The AA scale predicts that mortality at age 70 would improve by 1.5% per year indefinitely, but the heatmap shows improvement rates of around 2.75% in the mid 2000s. On the other hand, the heatmap shows that the higher values of the improvement factors might not persist for later cohorts.

3.12.2 Two-factor mortality improvement scales

A more robust approach to deterministic mortality improvement scales uses improvement factors that are a function of both age and calendar year. This approach is used in the MP2014 tables of the Society of Actuaries as well as in the CPM scales of the Canadian Institute of Actuaries. The two-factor mortality improvement function is $\varphi(x,t)$, where the mortality rate for a life aged x in year t, for $t = 1,2,3,\ldots$, is determined from the base mortality rates as

$$q(x,t) = q(x,0)\prod_{k=1}^{t}(1-\varphi(x,k)) . \tag{3.16}$$

This means that the r-year survival probability, ${}_rp(x,t)$, is now

$$\begin{aligned}
{}_rp(x,t) &= \big(1-q(x,\,t)\big)\big(1-q(x+1,\,t+1)\big)\cdots\big(1-q(x+r-1,\,t+r-1)\big)\\
&= \Big(1-\big(q(x,0)(1-\varphi(x,1))(1-\varphi(x,2))\cdots(1-\varphi(x,t))\big)\Big)\\
&\quad\times\Big(1-\big(q(x+1,0)(1-\varphi(x+1,1))\cdots(1-\varphi(x+1,t+1))\big)\Big)\\
&\quad\vdots\\
&\quad\times\Big(1-\big(q(x+r-1,0)(1-\varphi(x+r-1,1))\cdots(1-\varphi(x+r-1,\\
&\qquad\qquad t+r-1))\big)\Big).
\end{aligned}$$

Example 3.16 You are given mortality rates for lives aged 50 to 60 applying in 2010 in Table 3.9, and improvement factors for 2011 to 2020 in Table 3.10.

Calculate the difference between the expected number of deaths between ages 50 and 55 from 100 000 independent lives, assuming (a) mortality follows

Table 3.9 *Mortality rates for Example 3.16.*

x	$q(x,2010)$
50	0.002768
51	0.002905
52	0.003057
53	0.003225
54	0.003412
55	0.003622
56	0.003858
57	0.004128
58	0.004436
59	0.004789
60	0.005191

Table 3.10 *Improvement factors for Example 3.16.*

	$\varphi(x, 2010+t)$									
x	$t=1$	2	3	4	5	6	7	8	9	10
50	0.0206	0.0227	0.0238	0.0243	0.0241	0.0233	0.0221	0.0205	0.0188	0.0170
51	0.0180	0.0205	0.0221	0.0229	0.0230	0.0226	0.0216	0.0203	0.0188	0.0171
52	0.0156	0.0181	0.0201	0.0213	0.0218	0.0217	0.0210	0.0200	0.0186	0.0171
53	0.0124	0.0148	0.0168	0.0184	0.0193	0.0195	0.0192	0.0185	0.0175	0.0162
54	0.0093	0.0115	0.0134	0.0150	0.0164	0.0170	0.0171	0.0167	0.0160	0.0151
55	0.0066	0.0085	0.0104	0.0120	0.0134	0.0145	0.0150	0.0150	0.0146	0.0140
56	0.0045	0.0061	0.0078	0.0094	0.0109	0.0121	0.0130	0.0134	0.0134	0.0131
57	0.0033	0.0045	0.0060	0.0075	0.0090	0.0103	0.0113	0.0121	0.0125	0.0124
58	0.0031	0.0037	0.0049	0.0063	0.0078	0.0091	0.0102	0.0111	0.0117	0.0120
59	0.0039	0.0039	0.0046	0.0057	0.0071	0.0084	0.0096	0.0105	0.0112	0.0117
60	0.0055	0.0049	0.0050	0.0058	0.0069	0.0082	0.0094	0.0103	0.0110	0.0115

the base table with no improvement, and (b) mortality improvement follows the age-year improvement factors in the table, and the lives are all aged 50 in 2015.

Solution 3.16 With no mortality improvement, we have ${}_5p_{50} = 0.98473$, which means that the expected number of deaths before age 55 from 100 000 lives aged 50 is $100\,000(1-0.98473) = 1527$.

With mortality improvement we have

$$\begin{aligned}
q(50,5) &= 0.002768(1-0.0206)(1-0.0227)(1-0.0238)\\
&\qquad \times (1-0.0243)(1-0.0241)\\
&= 0.002768(0.889710) = 0.002463,\\
q(51,6) &= 0.002905(1-0.0180)\cdots(1-0.0226) = 0.002905(0.877640)\\
&= 0.002550,\\
q(52,7) &= 0.003057(1-0.0156)\cdots(1-0.0210) = 0.003057(0.868466)\\
&= 0.002655,\\
q(53,8) &= 0.003225(1-0.0124)\cdots(1-0.0185) = 0.003225(0.869233)\\
&= 0.002803,\\
q(54,9) &= 0.003412(1-0.0093)\cdots(1-0.0160) = 0.003412(0.875102)\\
&= 0.002986.
\end{aligned}$$

The five-year survival probability is then

$$(1-q(50,5))(1-q(51,6))\cdots(1-q(54,9)) = 0.98662,$$

and so the expected number of deaths before age 55 from 100 000 lives aged 50 in 2015, allowing for mortality improvement, is 1338. The difference between the number of expected deaths with and without mortality improvement is therefore 189. □

3.12.3 Cubic spline mortality improvement scales

In this section we describe the method used to construct the age-year improvement factors for the US and Canadian tables, which was first proposed by the Continuous Mortality Investigation Bureau (CMIB), a standing committee of the Institute and Faculty of Actuaries in the UK.

The improvement scales are determined in three steps.

1. Determine **short-term improvement factors**, using regression or other smoothing techniques applied to recent experience.
2. Determine **long-term improvement factors**, and the time at which the long-term factors will be reached. After this time, the factors are assumed to be constant. This step is usually based on subjective judgment.
3. Determine **intermediate improvement factors** using smooth functions that will connect the short- and long-term factors.

For the MP2014 tables, the Society of Actuaries used the following three steps to generate past and future improvement factors, $\varphi(x, t)$, where x is the age (integer values from 15 to 95) and t is the calendar year from 1950 forwards.

1. Improvement factors for calendar years 1950–2007 are determined by taking the raw mortality experience from the US Social Security Administration (SSA) database. A two-dimensional smoothing method is applied to the logarithm of the raw mortality rates, generating smooth log-mortality rates denoted $s(x, t)$. The two-dimensional smoothing ensures that $s(x, t)$ is smooth across ages x and across calendar years t. The smoothed historical mortality rates up to 2007 are then

   $$\tilde{q}(x, t) = e^{s(x,t)}$$

   and the historical improvement factors for 1950–2007 are

   $$\varphi(x, t) = 1 - \frac{\tilde{q}(x, t)}{\tilde{q}(x, t-1)} = 1 - e^{s(x,t)-s(x,t-1)}.$$

   We remark that two-dimensional smoothing techniques are beyond the scope of this book; in what follows we assume that the required smoothed historical improvement factors based on smoothed mortality rates exist.
2. Long-term improvement factors were set at 1% at all ages up to age 85, decreasing linearly to 0% at age 115, for both males and females. These factors are assumed to apply from 2027.
3. Intermediate factors covering calendar years 2008–2026 are determined using **cubic splines**. Two distinct approaches could be taken to determining

these intermediate factors. The first approach is age-based, under which we use historical improvement factors and the assumed long-term improvement factors for a given age to determine the intermediate factors for that age. The second approach is cohort-based, under which we use historical improvement factors and the assumed long-term improvement factors for a cohort. If we apply each approach over all ages and cohorts, we obtain two sets of intermediate improvement factors by age and by calendar year, and these sets of factors are typically different. In practice, the approach adopted was to average the factors.

A spline is a smooth function that can be used to interpolate between two other functions. In our context, we have the historical improvement factors up to 2007, and we have the assumed long-term improvement factors applying from 2027, which are assumed to be constant for each age. A cubic spline is a cubic function of time (in years) measured from 2007 which matches the improvement function values at 2007 and 2027, and also matches the gradient of the improvement function at 2007 and at 2027. The two end points joined by the spline are called **knots**. Using the two knots, and the gradients at the two knots, we have four equations, which we can solve for the four parameters of the cubic function.

The **age-based cubic spline** uses a fixed age for the spline function. The four equations for the function for age x are derived as follows.

1. We set 2007 as our base year, when $t = 0$, as this is the last year of the historic data. From the historic data, we have $\varphi(x, 2007)$ for each age x, and this will be set to match the cubic function at $t = 0$.
2. The first year of the assumed long-term factors is 2027, so $\varphi(x, 2027)$ will be set to match the cubic function at $t = 20$.
3. The gradient at $t = 0$ will be estimated as $\varphi(x, 2007) - \varphi(x, 2006)$, and this will be matched to the first derivative of the cubic function at $t = 0$.
4. The gradient at $t = 20$ will be estimated as $\varphi(x, 2028) - \varphi(x, 2027)$, and this will be matched to the first derivative of the cubic spline at $t = 20$. Generally $\varphi(x, 2028) - \varphi(x, 2027)$ will be zero, as we assume constant long-term improvement factors.

So, letting $C_a(x, t) = at^3 + bt^2 + ct + d$ represent the age-based cubic spline, with derivative $C_a'(x, t) = 3at^2 + 2bt + c$, the four conditions described above give the following four equations:

Knot at $t = 0$: $C_a(x, 0) = d = \varphi(x, 2007)$

Knot at $t = 20$: $C_a(x, 20) = 20^3\, a + 20^2\, b + 20c + d = \varphi(x, 2027)$

Gradient at $t = 0$: $C_a'(x, 0) = c = \varphi(x, 2007) - \varphi(x, 2006)$

Gradient at $t = 20$: $C_a'(x, 20) = 3a\, 20^2 + 40b + c = \varphi(x, 2028) - \varphi(x, 2027)$

So we immediately have the values of c and d from the equations for $t = 0$, leaving two equations to be solved for a and b to give the polynomial C_a.

The **cohort-based spline** is similar, but it smooths the improvement factors for a cohort, starting at age x, say, in 2007, and adding one year to the age as we move across one year in time. This gives the following four conditions for the cohort-based cubic spline.

1. The left side knot for the cohort spline for a life aged x in 2007 is $\varphi(x,\ 2007)$, as for the age-based spline, and again we set $t = 0$ at 2007.
2. The right side knot for the cohort-based spline matches the cubic function at $t = 20$ with the improvement factor for a life aged $x + 20$ in 2027, which is $\varphi(x + 20,\ 2027)$.
3. The left side gradient for the cohort-based spline is the difference between the improvement factor for age x in 2007 and the factor for age $x - 1$ in 2006, that is $\varphi(x, 2007) - \varphi(x - 1, 2006)$. This is matched to the first derivative of the cohort spline at $t = 0$.
4. The right side gradient for the cohort-based spline is the difference between the long-term improvement factor for age $x + 21$ in 2028 and the long-term improvement factor for age $x + 20$ in 2027. That is, we use $\varphi(x + 21, 2028) - \varphi(x + 20, 2027)$, which is matched to the first derivative of the cohort spline at $t = 20$.

So, letting $C_c(x, t) = a^*t^3 + b^*t^2 + c^*t + d^*$ represent the cohort-based cubic spline, with derivative $C'_c(x, t) = 3a^*t^2 + 2b^*t + c^*$, the four conditions described above give the following four equations:

Knot at $t = 0$: $C_c(x, 0) = d^* = \varphi(x, 2007)$

Knot at $t = 20$: $C_c(x + 20, 20) = 20^3a^* + 20^2b^* + 20c^* + d^*$

$$= \varphi(x + 20, 2027)$$

Gradient at $t = 0$: $C'_c(x, 0) = c^* = \varphi(x, 2007) - \varphi(x - 1, 2006)$

Gradient at $t = 20$:

$$C'_c(x + 20, 20) = 3a^*20^2 + 40b^* + c^* = \varphi(x + 21,\ 2028) - \varphi(x + 20, 2027)$$

For the US tables, the improvement factor for age x in year t is then taken as the average of the two splines, namely

$$\varphi(x,\ 2007 + t) = 0.5C_a(x, t) + 0.5C_c(x, t) \quad \text{for } t = 1, 2, \ldots, 19.$$

Example 3.17 Calculate the MP2014 one-year improvement factor for a female life aged 40 in 2020, given the following values for short- and long-term improvement factors:

$$\varphi(40,2006) = 0.0162,\ \varphi(40,2007) = 0.0192,\ \varphi(40,2027) = 0.01,$$
$$\varphi(40,2028) = 0.01,$$
$$\varphi(26,2006) = -0.0088,\ \varphi(27,2007) = -0.0088,\ \varphi(47,2027) = 0.01,$$
$$\varphi(48,2028) = 0.01.$$

Solution 3.17 The four equations for the age-based cubic spline are

$$\begin{aligned}
&\text{Knot at } t=0\text{:}\ \ C_a(40,0) = d = \varphi(40,2007) = 0.0192\\
&\text{Knot at } t=20\text{:}\ \ C_a(40,20) = 8000a + 400b + 20c + d = 0.01\\
&\text{Gradient at } t=0\text{: } C_a'(40,0) = c = \varphi(40,2007) - \varphi(40,2006) = 0.003\\
&\text{Gradient at } t=20\text{: } C_a'(40,20) = 1200a + 40b + c\\
&\qquad\qquad\qquad\qquad\qquad\qquad = \varphi(40,2028) - \varphi(40,2027) = 0
\end{aligned}$$

The equations for $t = 0$ give $c = 0.003$ and $d = 0.0192$, and the equations for $t = 20$ then yield $a = 9.8 \times 10^{-6}$ and $b = -3.69 \times 10^{-4}$.

So for a life aged 40 in 2020 we have

$$C_a(40,13) = 13^3a + 13^2b + 13c + d = 0.01737.$$

The four equations for the cohort-based cubic spline for a life who is aged 40 in 2020 apply to the cohort who are aged 27 in 2007, so the four equations for the spline are

$$\begin{aligned}
&\text{Knot at } t=0\text{:}\ \ C_c(27,0) = d^* = \varphi(27,2007) = -0.0088\\
&\text{Knot at } t=20\text{:}\ \ C_c(47,20) = 8000a^* + 400b^* + 20c^* + d^* = 0.01\\
&\text{Gradient at } t=0\text{: } C_c'(27,0) = c^* = \varphi(27,2007) - \varphi(26,2006) = 0\\
&\text{Gradient at } t=20\text{: } C_c'(47,20) = 1200a^* + 40b^* + c^*\\
&\qquad\qquad\qquad\qquad\qquad\qquad = \varphi(48,2028) - \varphi(47,2027) = 0
\end{aligned}$$

Solving as for the age-based spline we obtain

$$a^* = -4.7 \times 10^{-6}, \quad b^* = 1.41 \times 10^{-4}, \quad c^* = 0, \quad d^* = -0.0088.$$

So for a life aged 40 in 2020 we have

$$C_c(40,13) = 13^3a^* + 13^2b^* + 13c^* + d^* = 0.00470,$$

and hence the improvement factor for age 40 in 2020 is

$$\varphi(40,2020) = 0.5C_a(40,13) + 0.5C_c(40,13) = 0.011035.$$

□

3.13 Notes and further reading

The mortality rates in Section 3.4 are drawn from the following sources:

- Australian Life Tables 2010–12 were produced by the Australian Government Actuary (2014).
- English Life Table 17 was prepared by the UK Government Actuary and published by the Office for National Statistics (2013).
- US Life Tables 2013 were prepared in the Division of Vital Statistics of the National Center for Health Statistics in the USA.

The Continuous Mortality Investigation in the UK has been ongoing for many years. Findings on mortality and morbidity experience of UK policyholders are published via a series of formal reports and working papers. In this chapter we have drawn on CMI (2006).

In Section 3.5 we noted that there can be considerable variability in the mortality experience of different groups in a national population. Coleman and Salt (1992) give a good account of this variability in the UK population.

The paper by Gompertz (1825), who was the Actuary of the Alliance Insurance Company of London, introduced the force of mortality concept.

In Section 3.11 the data used are from the National Center for Health Statistics, *Vital Statistics of the United States, Volume II: Mortality, Part A*. Washington, D.C.: Government Printing Office. These were obtained through the Human Mortality Database, www.mortality.org, which is a vast database of mortality data from a large number of countries.

3.14 Exercises

Shorter exercises

Exercise 3.1 Sketch the following as functions of age x for a typical (human) population, and comment on the major features: (a) μ_x, (b) l_x, (c) d_x.

Exercise 3.2 You are given the following life table extract.

Age, x	l_x
52	89 948
53	89 089
54	88 176
55	87 208
56	86 181
57	85 093
58	83 940
59	82 719
60	81 429

Calculate each of the following probabilities assuming (i) uniform distribution of deaths between integer ages, and (ii) a constant force of mortality between integer ages:

(a) ${}_{0.2}q_{52.4}$,
(b) ${}_{5.7}p_{52.4}$,
(c) ${}_{3.2|2.5}q_{52.4}$.

Exercise 3.3 Let S_0 denote the survival function from birth. A number of fractional age assumptions satisfy the condition

$$h(S_0(x+t)) = (1-t)\,h(S_0(x)) + t\,h(S_0(x+1))$$

for some function h, where x is an integer and $0 < t < 1$. On the assumption of a constant force of mortality between integer ages, find an expression for $S_0(x+t)$ where x is an integer and $0 < t < 1$, and hence show that $h(s) = \log s$.

Exercise 3.4 Table 3.11 is an extract from a (hypothetical) select life table with a select period of two years. Note carefully the layout – each row relates to a fixed age at selection.

Use this table to calculate

(a) the probability that a life currently aged 75 who has just been selected will survive to age 85,
(b) the probability that a life currently aged 76 who was selected one year ago will die between ages 85 and 87, and
(c) ${}_{4|2}q_{[77]+1}$.

Table 3.11 *Extract from a (hypothetical) select life table.*

x	$l_{[x]}$	$l_{[x]+1}$	l_{x+2}	$x+2$
75	15 930	15 668	15 286	77
76	15 508	15 224	14 816	78
77	15 050	14 744	14 310	79
⋮	⋮	⋮	⋮	⋮
80			12 576	82
81			11 928	83
82			11 250	84
83			10 542	85
84			9 812	86
85			9 064	87

Table 3.12 *Mortality rates for Exercise 3.5.*

Age, x	Duration 0 $q_{[x]}$	Duration 1 $q_{[x-1]+1}$	Duration 2 $q_{[x-2]+2}$	Duration 3 $q_{[x-3]+3}$	Duration 4 $q_{[x-4]+4}$	Duration 5+ q_x
69	0.003974	0.004979	0.005984	0.006989	0.007994	0.009458
70	0.004285	0.005411	0.006537	0.007663	0.008790	0.010599
71	0.004704	0.005967	0.007229	0.008491	0.009754	0.011880
72	0.005236	0.006651	0.008066	0.009481	0.010896	0.013318
73	0.005870	0.007456	0.009043	0.010629	0.012216	0.014931
74	0.006582	0.008361	0.010140	0.011919	0.013698	0.016742
75	0.007381	0.009376	0.011370	0.013365	0.015360	0.018774
76	0.008277	0.010514	0.012751	0.014988	0.017225	0.021053
77	0.009281	0.011790	0.014299	0.016807	0.019316	0.023609

Exercise 3.5 CMI Table A23 is based on UK data from 1999 to 2002 for female non-smokers who are term insurance policyholders. It has a select period of five years. An extract from this table, showing values of $q_{[x-t]+t}$, is given in Table 3.12.

Use this survival model to calculate

(*a*) ${}_2p_{[72]}$ (*b*) ${}_3q_{[73]+2}$ (*c*) ${}_{1|}q_{[65]+4}$ (*d*) ${}_7p_{[70]}$

(*e*) ${}_{3.8}q_{[70]+0.2}$ assuming UDD.

Exercise 3.6 A select survival model has a select period of three years. Calculate ${}_3p_{53}$, given

$$q_{[50]} = 0.01601, \ {}_2p_{[50]} = 0.96411, \ {}_{2|}q_{[50]} = 0.02410, \ {}_{2|3}q_{[50]+1} = 0.09272.$$

Exercise 3.7 An insurer is constructing a two-factor mortality improvement model. You are given the following information for the improvement factors applying at age x.

- The one-year improvement factor in years 2019 and 2020 for age x are both equal to 0.02.
- The long-term improvement factors apply from 2040, and are assumed to be 0% at all ages.
- Improvement factors for intermediate years are calculated using an age-based cubic spline.

Calculate the improvement factor for age x in 2024.

Longer exercises

Exercise 3.8 A group of 100 000 independent lives, each aged 65, purchases one-year term insurance. At the start, 20% of the group are preferred lives, with $q_{65} = 0.002$ and 80% of the group are normal lives, with $q_{65} = 0.009$.

(a) Calculate the standard deviation of the number of survivors at the end of the year.
(b) Calculate the proportion of preferred lives expected in the survivor group.
(c) Using a normal approximation, without continuity correction, calculate the 90th percentile of the number of survivors at the end of the year.

Exercise 3.9 You are given l_x for integer valued x. Assuming a constant force of mortality between integer ages y and $y+1$ for $y = 0, 1, 2, \ldots$, show that for integer x,

$$\mathring{e}_x = \sum_{j=0}^{\infty} \frac{{}_jp_x - {}_{j+1}p_x}{a_j}$$

where a_j is a term that you should identify.

Exercise 3.10 When posted overseas to country A at age x, the employees of a large company are subject to a force of mortality such that, at exact duration t years after arrival overseas $(t = 0, 1, 2, 3, 4)$,

$$q^A_{[x]+t} = (6 - t)q_{x+t}$$

where q_{x+t} is on the basis of US Life Tables, 2002, Females. For those who have lived in country A for at least five years the force of mortality at each age is 50% greater than that of US Life Tables, 2002, Females, at the same age. Some l_x values for this table are shown in Table 3.13.
Calculate the probability that an employee posted to country A at age 30 will survive to age 40 if she remains in that country.

Exercise 3.11 A special survival model has a select period of three years. Functions for this model are denoted by an asterisk, *. Functions without an

Table 3.13 *An extract from the United States Life Tables, 2002, Females.*

Age, x	l_x
30	98 424
31	98 362
32	98 296
33	98 225
34	98 148
35	98 064
⋮	⋮
40	97 500

Table 3.14 *An extract from the Canada Life Tables 2000–02, Males.*

Age, x	l_x
15	99 180
16	99 135
17	99 079
18	99 014
19	98 942
20	98 866
21	98 785
22	98 700
23	98 615
24	98 529
25	98 444
26	98 363
⋮	⋮
62	87 503
63	86 455
64	85 313
65	84 074

asterisk are taken from the Canada Life Tables 2000–02, Males; an excerpt from this table is given in Table 3.14. You are given that, for all values of x,

$$p^*_{[x]} = {}_4p_{x-5}; \quad p^*_{[x]+1} = {}_3p_{x-1}; \quad p^*_{[x]+2} = {}_2p_{x+2}; \quad p^*_x = p_{x+1}.$$

A life table, tabulated at integer ages, is constructed on the basis of the special survival model. The radix, l^*_{25} is set at 98 363 (i.e. l_{26} from the Table 3.14).

(a) Construct the $l^*_{[x]}$, $l^*_{[x]+1}$, $l^*_{[x]+2}$, and l^*_{x+3} columns for $x = 20, 21, 22$.

(b) Calculate ${}_{2|38}q^*_{[21]+1}$, ${}_{40}p^*_{[22]}$, ${}_{40}p^*_{[21]+1}$, ${}_{40}p^*_{[20]+2}$, and ${}_{40}p^*_{22}$.

Exercise 3.12 (a) Show that a constant force of mortality between integer ages implies that the distribution of R_x, the fractional part of the future life time, conditional on $K_x = k$, has the following truncated exponential distribution for integer x, for $0 \leq s < 1$ and for $k = 0, 1, \ldots$

$$\Pr[R_x \leq s \mid K_x = k] = \frac{1 - \exp\{-\mu^*_{x+k}s\}}{1 - \exp\{-\mu^*_{x+k}\}} \tag{3.17}$$

where $\mu^*_{x+k} = -\log p_{x+k}$.

(b) Show that if formula (3.17) holds for $k = 0, 1, 2, \ldots$ then the force of mortality is constant between integer ages.

Exercise 3.13 The model in Example 3.13 has a two-year select period. The ultimate survival model follows Makeham's law, and the select part follows

$$\mu_{[x]+s} = 0.9^{2-s}\mu_{x+s}.$$

Verify equation (3.15); that is, for $0 \le t \le 2$:

$${}_tp_{[x]} = \exp\left\{0.9^{2-t}\left(\frac{1-0.9^t}{\log(0.9)}A + \frac{c^t - 0.9^t}{\log(0.9/c)}Bc^x\right)\right\}.$$

Exercise 3.14 Improvement factors are being developed based on data up to 2015 for a survival model. Long-term improvement factors are assumed to apply from 2040, and are assumed to be 1% at all ages up to age 90.

The improvement factors for the years 2016 to 2039 are to be developed using the average of an age-based and a cohort-based cubic spline, following the same approach as in Example 3.17.

Calculate the one-year improvement factor for a life aged 50 in 2025, given the following values for 2014 and 2015:

$$\varphi(50, 2014) = 0.0134, \; \varphi(50, 2015) = 0.0145,$$

$$\varphi(39, 2014) = 0.0168, \; \varphi(40, 2015) = 0.0182.$$

Answers to selected exercises

3.2 (a)(i) 0.001917 (ii) 0.001917
(b)(i) 0.935422 (ii) 0.935423
(c)(i) 0.030957 (ii) 0.030950

3.4 (a) 0.66177 (b) 0.09433 (c) 0.08993

3.5 (a) 0.987347 (b) 0.044998 (c) 0.010514 (d) 0.920271 (e) 0.027812

3.6 0.902942

3.7 0.01792

3.8 (a) 27.45 (b) 0.201 (c) 99 275

3.10 0.977497

3.11 (a)

x	$l^*_{[x]}$	$l^*_{[x]+1}$	$l^*_{[x]+2}$	l_{x+3}
20	99 180	98 942	98 700	98 529
21	99 135	98 866	98 615	98 444
22	99 079	98 785	98 529	98 363

(b) 0.121265, 0.872587, 0.874466, 0.875937, 0.876692

3.14 0.018615

4
Insurance benefits

4.1 Summary

In this chapter we develop formulae for the valuation of traditional insurance benefits. In particular, we consider whole life, term and endowment insurance. For each of these benefits we identify the random variables representing the present values of the benefits and we derive expressions for moments of these random variables. The functions we develop for traditional benefits will also be useful when we move to modern variable contracts.

We develop valuation functions for benefits based on the continuous future lifetime random variable, T_x, and the curtate future lifetime random variable, K_x, from Chapter 2. We introduce a new random variable, $K_x^{(m)}$, which we use to value benefits which depend on the number of complete periods of length $1/m$ years lived by a life (x). We explore relationships between the expected present values of different insurance benefits.

We also introduce the actuarial notation for the expected values of the present value of insurance benefits.

4.2 Introduction

In the previous two chapters, we have looked at models for future lifetime. The main reason that we need these models is to apply them to the valuation of payments which are dependent on the death or survival of a policyholder or pension plan member. Because of the dependence on death or survival, the timing and possibly the amount of the benefit are uncertain, so the present value of the benefit can be modelled as a random variable. In this chapter we combine survival models with time value of money functions to derive the distribution of the present value of an uncertain, life contingent future benefit.

We generally assume that the interest rate is constant and fixed. This is appropriate, for example, if the premiums for an insurance policy are invested in risk-free bonds, all yielding the same interest rate, so that the term structure is flat. In Chapter 12 we introduce more realistic term structures, and consider some models of interest that allow for uncertainty.

For the development of present value functions, it is generally easier, mathematically, to work in continuous time. In the case of a death benefit, working in continuous time means that we assume that the death payment is paid at the exact time of death. In the case of an annuity, a continuous benefit of, say, \$1 per year would be paid in infinitesimal units of \$$dt$ in every interval $(t, t+dt)$. Clearly both assumptions are impractical; it will take time to process a payment after death, and annuities will be paid at most weekly, not every moment (though the valuation of weekly payments is usually treated as if the payments were continuous, as the difference is very small). In practice, insurers and pension plan actuaries work in discrete time, often with cash flow projections that are, perhaps, monthly or quarterly. In addition, when the survival model being used is in the form of a life table with annual increments (that is, l_x for integer x), it is simplest to use annuity and insurance present value functions that assume payments are made at integer durations only. We work in continuous time in the first place because the mathematical development is more transparent, more complete and more flexible. It is then straightforward to adapt the results from continuous time analysis to discrete time problems.

4.3 Assumptions

To perform calculations in this chapter, we require assumptions about mortality and interest. We use the term **basis** to denote a set of assumptions used in life insurance or pension calculations, and we will meet further examples of bases when we discuss premium calculation in Chapter 6, policy values in Chapter 7 and pension liability valuation in Chapter 11.

In many of the examples and exercises of this and subsequent chapters, we use the following survival model which was introduced in Example 3.13.

The Standard Ultimate Survival Model:

$$\text{Makeham's Law,} \quad \mu_x = A + Bc^x,$$

$$A = 0.00022, \quad B = 2.7 \times 10^{-6}, \quad c = 1.124.$$

We also assume that interest rates are constant. As discussed above, this interest assumption can be criticized as unrealistic. However, it is a convenient assumption from a pedagogical point of view, is often accurate enough for practical purposes (but not always) and we relax the assumption in later chapters.

It is convenient to work with interest theory functions that are in common actuarial and financial use. We review some of these here.

Given an effective annual rate of interest $i > 0$, we use $v = 1/(1+i)$, so that the present value of a payment of S which is to be paid in t years' time is Sv^t. The force of interest per year is denoted δ where

$$\delta = \log(1+i), \quad 1+i = e^{\delta}, \quad \text{and} \quad v = e^{-\delta};$$

δ is also known as the continuously compounded rate of interest. In financial mathematics and corporate finance contexts, and in particular if the rate of interest is assumed risk free, the common notation for the continuously compounded rate of interest is r.

The nominal rate of interest compounded p times per year is denoted $i^{(p)}$ where

$$i^{(p)} = p\big((1+i)^{1/p} - 1\big) \Leftrightarrow 1+i = \big(1 + i^{(p)}/p\big)^p.$$

The effective rate of discount per year is d where

$$d = 1 - v = iv = 1 - e^{-\delta},$$

and the nominal rate of discount compounded p times per year is $d^{(p)}$ where

$$d^{(p)} = p\big(1 - v^{1/p}\big) \Leftrightarrow (1 - d^{(p)}/p)^p = v.$$

4.4 Valuation of insurance benefits

4.4.1 Whole life insurance: the continuous case, $\bar{A}_x$

For a whole life insurance policy, the time at which the benefit will be paid is unknown until the policyholder actually dies and the policy becomes a claim. Since the present value of a future payment depends on the payment date, the present value of the benefit payment is a function of the time of death, and is therefore modelled as a random variable. Given a survival model and an interest rate we can derive the distribution of the present value random variable for a life contingent benefit, and can therefore compute quantities such as the mean and variance of the present value.

We start by considering the value of a benefit of amount \$1 payable following the death of a life currently aged x. Using a benefit of \$1 allows us to develop valuation functions per unit of sum insured, then we can multiply these by the actual sum insured for different benefit amounts.

We first assume that the benefit is payable immediately on the death of (x). This is known as the continuous case since we work with the continuous future lifetime random variable T_x. Although in practice there would normally be a short delay between the date of a person's death and the time at which an insurance company would actually pay a death benefit (due to notification of death to the insurance company and legal formalities) the effect is slight and we will ignore that delay here.

For our life (x), the present value of a benefit of \$1 payable immediately on death is a random variable, Z, say, where

$$Z = v^{T_x} = e^{-\delta\, T_x}.$$

We are generally most interested in the expected value of the present value random variable for some future payment. We refer to this as the **Expected Present Value** or EPV. It is also commonly referred to as the **Actuarial Value** or the **Actuarial Present Value**.

The EPV of the whole life insurance benefit payment with sum insured \$1 is $\mathrm{E}[e^{-\delta\, T_x}]$. In actuarial notation, we denote this expected value by $\bar{A}_x$, where the bar above A denotes that the benefit is payable immediately on death.

As T_x has probability density function $f_x(t) = {}_tp_x\, \mu_{x+t}$, we have

$$\boxed{\bar{A}_x = \mathrm{E}[e^{-\delta\, T_x}] = \int_0^\infty e^{-\delta\, t}\ {}_tp_x\, \mu_{x+t}\, dt.} \tag{4.1}$$

It is worth looking at the intuition behind this formula. In Figure 4.1 we use the time-line format that was introduced in Section 2.4.

Consider time t, where $x \le x + t < \omega$. The probability that (x) is alive at time t is ${}_tp_x$, and the probability that (x) dies between ages $x+t$ and $x+t+dt$, having survived to age $x+t$, is, loosely, $\mu_{x+t}\, dt$, provided that dt is very small. In this case, the present value of the death benefit of \$1 is $e^{-\delta t}$. Note that we regard the period from t to $t + dt$ as so short that any payment in the interval can effectively be treated as occurring at time t for discounting.

Now we can integrate (that is, sum the infinitesimal components of) the product of present value and probability over all the possible death intervals t to $t + dt$ to obtain the EPV of the death benefit that will be paid in exactly

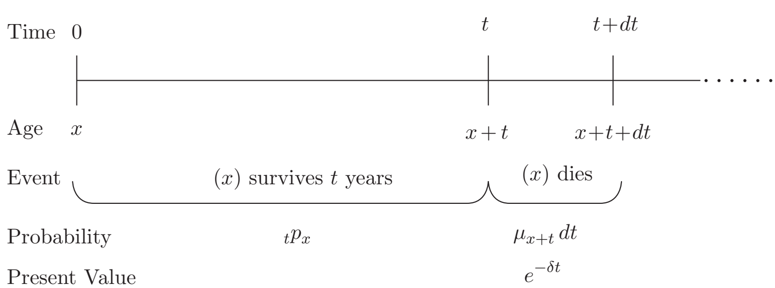


Figure 4.1 Time-line diagram for continuous whole life insurance.

one of these intervals. The result is the integral in (4.1). Similarly, the second moment (about zero) of the present value of the death benefit is

$$\begin{aligned} \mathrm{E}[Z^2] = \mathrm{E}[(e^{-\delta\, T_x})^2] = \mathrm{E}[e^{-2\delta\, T_x}] \\ &= \int_0^\infty e^{-2\delta t}\ {}_tp_x\, \mu_{x+t}\, dt \\ &= {}^2\bar{A}_x, \end{aligned} \tag{4.2}$$

where the superscript 2 indicates that calculation is at force of interest 2δ, or, equivalently, at rate of interest j, where $1 + j = e^{2\delta} = (1 + i)^2$.

The variance of the present value of a unit benefit payable immediately on death is

$$\mathrm{V}[Z] = \mathrm{V}[e^{-\delta\, T_x}] = \mathrm{E}[Z^2] - \mathrm{E}[Z]^2 = {}^2\bar{A}_x - \left(\bar{A}_x\right)^2. \tag{4.3}$$

Now, if we introduce a more general sum insured, S, say, then the EPV of the death benefit is

$$E[SZ] = E[Se^{-\delta\, T_x}] = S\bar{A}_x$$

and the variance is

$$\mathrm{V}[SZ] = \mathrm{V}[Se^{-\delta\, T_x}] = S^2\left({}^2\bar{A}_x - \bar{A}_x^2\right).$$

In fact we can calculate any probabilities associated with the random variable Z from the probabilities associated with T_x. Suppose we are interested in the probability $\Pr[Z \le 0.5]$, for example. We can rearrange this into a probability for T_x:

$$\begin{aligned} \Pr[Z \le 0.5] &= \Pr[e^{-\delta\, T_x} \le 0.5] \\ &= \Pr[-\delta\, T_x \le \log(0.5)] \\ &= \Pr[\delta\, T_x > -\log(0.5)] \\ &= \Pr[\delta\, T_x > \log(2)] \\ &= \Pr[T_x > \log(2)/\delta] \\ &= {}_up_x \end{aligned}$$

where $u = \log(2)/\delta$. We note that low values of Z are associated with high values of T_x. This makes sense because the benefit is more expensive to the insurer if it is paid early, as there has been little opportunity to earn interest. It is less expensive if it is paid later.

4.4.2 Whole life insurance: the annual case, A_x

Suppose now that the benefit of \$1 is payable at the end of the year of death of (x), rather than immediately on death. To value this we use the curtate future lifetime random variable, K_x, introduced in Chapter 2. Recall that K_x measures the number of complete years of future life of (x). The time to the end of the year of death of (x) is then $K_x + 1$. For example, if (x) lived for 25.6 years from the issue of the insurance policy, the observed value of K_x would be 25, and the death benefit payable at the end of the year of death would be payable 26 years from the policy's issue.

We again use Z to denote the present value of the whole life insurance benefit of \$1, so that Z is the random variable

$$Z = v^{K_x+1}.$$

The EPV of the benefit, E[Z], is denoted by A_x in actuarial notation.

In Chapter 2 we derived the probability function for K_x, $\Pr[K_x = k] = {}_{k|}q_x$, so the EPV of the benefit is

$$\boxed{A_x = \mathrm{E}\left[v^{K_x+1}\right] = \sum_{k=0}^{\infty} v^{k+1}\ {}_{k|}q_x = vq_x + v^2\,{}_{1|}q_x + v^3\,{}_{2|}q_x + \cdots .} \tag{4.4}$$

Each term on the right-hand side of this equation represents the EPV of a death benefit of \$1, payable at time k conditional on the death of (x) in $(k-1, k]$.

In fact, we can always express the EPV of a life-contingent benefit by considering each time point at which the benefit could be paid, and summing over all possible payment times the product of

(1) the amount of the benefit,
(2) the appropriate discount factor, and
(3) the probability that the benefit will be paid at that time.

We will justify this more rigorously in Section 4.6. We illustrate the process for the whole life insurance example in Figure 4.2.

The second moment of the present value is

$$\sum_{k=0}^{\infty} v^{2(k+1)}\ {}_{k|}q_x = \sum_{k=0}^{\infty} (v^2)^{(k+1)}\ {}_{k|}q_x = (v^2)q_x + (v^2)^2\,{}_{1|}q_x + (v^2)^3\,{}_{2|}q_x + \cdots .$$

Just as in the continuous case, we can calculate the second moment about zero of the present value by an adjustment in the rate of interest from i to $(1+i)^2 - 1$. We define

$${}^2A_x = \sum_{k=0}^{\infty} v^{2(k+1)}\ {}_{k|}q_x, \tag{4.5}$$

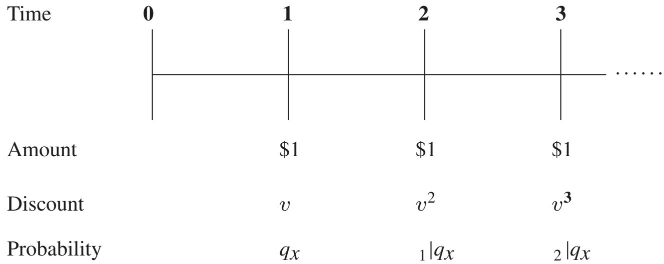


Figure 4.2 Time-line diagram for discrete whole life insurance.

and so the variance of the present value of a benefit of S payable at the end of the year of death is

$$\mathrm{V}\left[v^{K_x+1}\right] = S^2\left({}^2A_x - \left(A_x\right)^2\right). \tag{4.6}$$

4.4.3 *Whole life insurance: the* 1/m*thly case,* $A_x^{(m)}$

In Chapter 2 we introduced the random variable K_x, representing the curtate future lifetime of (x), and we saw in Section 4.4.2 that the present value of an insurance benefit payable at the end of the year of death can be expressed in terms of K_x.

We now define the **1/*m*thly curtate future lifetime random variable**, $K_x^{(m)}$, where $m > 1$ is an integer, to be the future lifetime of (x) in years rounded to the lower $\frac{1}{m}$th of a year. The most common values of m are 2, 4 and 12, corresponding to half years, quarter years and months. Thus, for example, $K_x^{(4)}$ represents the future lifetime of (x), rounded down to the lower 1/4.

Symbolically, if we let $\lfloor\ \rfloor$ denote the integer part (or floor) function, then

$$K_x^{(m)} = \frac{1}{m}\left\lfloor mT_x \right\rfloor. \tag{4.7}$$

For example, suppose (x) lives exactly 23.675 years. Then

$$K_x = 23, \quad K_x^{(2)} = 23.5, \quad K_x^{(4)} = 23.5, \quad \text{and} \quad K_x^{(12)} = 23\tfrac{8}{12} = 23.6667.$$

Note that $K_x^{(m)}$ is a discrete random variable. $K_x^{(m)} = k$ indicates that the life (x) dies in the interval $\left[k,\ k + \frac{1}{m}\right)$, for $k = 0, \frac{1}{m}, \frac{2}{m}, \ldots$.

The probability function for $K_x^{(m)}$ can be derived from the associated probabilities for T_x. For $k = 0, \frac{1}{m}, \frac{2}{m}, \ldots$,

$$\Pr[K_x^{(m)} = k] = \Pr\left[k \le T_x < k + \tfrac{1}{m}\right] = {}_{k|\frac{1}{m}}q_x = {}_kp_x - {}_{k+\frac{1}{m}}p_x.$$

In Figure 4.3 we show the time-line for the $1/m$thly benefit. At the end of each $1/m$ year period, we show the amount of benefit due, conditional on the death of the insured life in the previous $1/m$ year interval, the probability that the insured life dies in the relevant interval, and the appropriate discount factor.

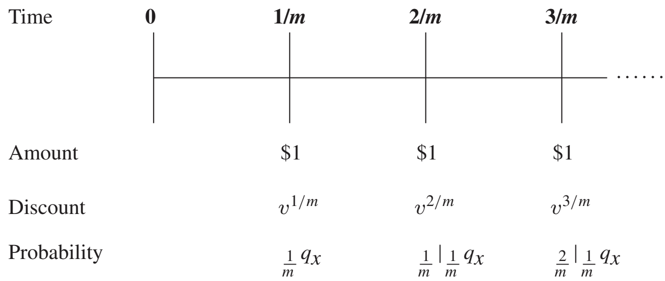


Figure 4.3 Time-line diagram for $1/m$thly whole life insurance.

Suppose, for example, that $m = 12$. A whole life insurance benefit payable at the end of the month of death has present value random variable Z where

$$Z = v^{K_x^{(12)}+1/12}.$$

We let $A_x^{(12)}$ denote the EPV of this benefit, so that

$$E[Z] = A_x^{(12)} = v^{\frac{1}{12}}\ {}_{\frac{1}{12}}q_x + v^{\frac{2}{12}}\ {}_{\frac{1}{12}|\frac{1}{12}}q_x + v^{\frac{3}{12}}\ {}_{\frac{2}{12}|\frac{1}{12}}q_x + v^{\frac{4}{12}}\ {}_{\frac{3}{12}|\frac{1}{12}}q_x + \cdots .$$

Similarly, for any m,

$$\begin{aligned} A_x^{(m)} &= v^{\frac{1}{m}}\ {}_{\frac{1}{m}}q_x + v^{\frac{2}{m}}\ {}_{\frac{1}{m}|\frac{1}{m}}q_x + v^{\frac{3}{m}}\ {}_{\frac{2}{m}|\frac{1}{m}}q_x + v^{\frac{4}{m}}\ {}_{\frac{3}{m}|\frac{1}{m}}q_x + \cdots \\ &= \sum_{k=0}^{\infty} v^{\frac{k+1}{m}}\ {}_{\frac{k}{m}|\frac{1}{m}}q_x. \end{aligned}$$

As for the continuous and annual cases, we can derive the variance of the present value of the $1/m$thly whole life benefit by adjusting the interest rate for the first term in the variance. We have

$$\mathrm{E}[Z^2] = \mathrm{E}[v^{2(K_x^{(12)}+1/12)}] = \mathrm{E}[(v^2)^{K_x^{(12)}+1/12}] = {}^2A_x^{(12)},$$

so the variance is

$$\mathrm{V}\left[v^{K^{(12)}+1/12}\right] = {}^2A_x^{(12)} - \left(A_x^{(12)}\right)^2.$$

4.4.4 Recursions

In practice, it would be very unusual for an insurance policy to provide the death benefit at the end of the year of death. Nevertheless, the annual insurance function A_x is still useful. We are often required to work with annual life tables, such as those in Chapter 3, in which case we would start by calculating the annual function A_x, then adjust for a more appropriate frequency using the relationships and assumptions we develop later in this chapter.

Using the annual life table in a spreadsheet, we can calculate the values of A_x using **backward recursion**. To do this, we start from the highest age in the table, ω. We assume all lives expire by age ω, so that $q_{\omega-1} = 1$. If the life table does not have a limiting age, we choose a suitably high value for ω so that $q_{\omega-1}$ is as close to 1 as we like. This means that any life attaining age $\omega - 1$ may be treated as certain to die before age ω, in which case we know that $K_{\omega-1} = 0$ and so

$$A_{\omega-1} = \mathrm{E}[v^{K_{\omega-1}+1}] = v.$$

Now, working from the summation formula for A_x, we have

$$\begin{aligned} A_x &= \sum_{k=0}^{\omega-x-1} v^{k+1}\, {}_kp_x\, q_{x+k} \\ &= v\, q_x + v^2\, p_x\, q_{x+1} + v^3\, {}_2p_x\, q_{x+2} + \cdots \\ &= v\, q_x + v\, p_x \big(v\, q_{x+1} + v^2\, p_{x+1}\, q_{x+2} + v^3\, {}_2p_{x+1}\, q_{x+3} + \cdots\big), \end{aligned}$$

giving the important recursion formula

$$\boxed{A_x = v\, q_x + v\, p_x\, A_{x+1}.} \tag{4.8}$$

This formula can be used in spreadsheet format to calculate A_x backwards from $A_{\omega-1}$ back to A_{x_0}, where x_0 is the minimum age in the table.

The intuition for equation (4.8) is that we separate the EPV of the whole life insurance into the value of the benefit due in the first year, followed by the value at age $x + 1$ of all subsequent benefits, multiplied by p_x to allow for the probability of surviving to age $x + 1$, and by v to discount the value back from age $x + 1$ to age x.

We can use the same approach for $1/m$thly benefits; now the recursion will give $A_x^{(m)}$ in terms of $A^{(m)}_{x+\frac{1}{m}}$. Again, we split the benefit into the part payable in the first period – now of length $1/m$ years – followed by the EPV of the insurance beginning after $1/m$ years. We have

$$\begin{aligned} A_x^{(m)} &= v^{\frac{1}{m}}\, {}_{\frac{1}{m}}q_x + v^{\frac{2}{m}}\, {}_{\frac{1}{m}}p_x\, {}_{\frac{1}{m}}q_{x+\frac{1}{m}} + v^{\frac{3}{m}}\, {}_{\frac{2}{m}}p_x\, {}_{\frac{1}{m}}q_{x+\frac{2}{m}} + \cdots \\ &= v^{\frac{1}{m}}\, {}_{\frac{1}{m}}q_x + v^{\frac{1}{m}}\, {}_{\frac{1}{m}}p_x \left(v^{\frac{1}{m}}\, {}_{\frac{1}{m}}q_{x+\frac{1}{m}} + v^{\frac{2}{m}}\, {}_{\frac{1}{m}}p_{x+\frac{1}{m}}\, {}_{\frac{1}{m}}q_{x+\frac{2}{m}} + \cdots\right), \end{aligned}$$

giving the recursion formula

$$A_x^{(m)} = v^{\frac{1}{m}}\, {}_{\frac{1}{m}}q_x + v^{\frac{1}{m}}\, {}_{\frac{1}{m}}p_x\, A^{(m)}_{x+\frac{1}{m}}.$$

Example 4.1 Using the Standard Ultimate Survival Model from Section 4.3, and an interest rate of 5% per year effective, construct a spreadsheet of values of A_x for $x = 20, 21, \ldots, 100$. Assume that $A_{129} = v$.

Table 4.1 *Sample values of A_x using the Standard Ultimate Survival Model, Example 4.1.*

x	A_x	x	A_x	x	A_x
30	0.07698	50	0.18931	98	0.85177
31	0.08054	51	0.19780	99	0.86153
32	0.08427	52	0.20664	100	0.87068
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Solution 4.1 The survival model for the Standard Ultimate Survival Model is the ultimate part of the model used in Example 3.13 and so values of ${}_tp_x$ can be calculated as explained in the solution to that example. The value of q_{129} is 0.99996, which is indeed close to 1. We can use the formula

$$A_x = vq_x + vp_xA_{x+1}$$

to calculate recursively $A_{128}, A_{127}, \ldots, A_{20}$, starting from $A_{129} = v$. Values for $x = 20, 21, \ldots, 80$, are given in Appendix D Table D.3. Some excerpts are shown in Table 4.1. □

Example 4.2 Using the Standard Ultimate Survival Model from Section 4.3, and an interest rate of 5% per year effective, develop a spreadsheet of values of $A_x^{(12)}$ for x starting at age 20, in steps of 1/12.

Solution 4.2 For this example, we follow exactly the same process as for the previous example, except that we let the ages increase by 1/12 years in each row. We construct a column of values of ${}_{\frac{1}{12}}p_x$ using

$${}_{\frac{1}{12}}p_x = \exp\left\{ -A/12 - Bc^x(c^{1/12} - 1)/\log c\right\}.$$

We again use 130 as the limiting age of the table. Then set $A^{(12)}_{129\frac{11}{12}} = v^{1/12}$, and for all the other values of $A_x^{(12)}$ use the recursion

$$A_x^{(12)} = v^{1/12}\ {}_{\frac{1}{12}}q_x + v^{1/12}\ {}_{\frac{1}{12}}p_x\, A^{(12)}_{x+\frac{1}{12}}.$$

The first and last few lines of the spreadsheet are reproduced in Table 4.2. □

It is worth making a remark about the calculations in Examples 4.1 and 4.2. In Example 4.1 we saw that $q_{129} = 0.99996$, which is sufficiently close to 1 to justify us starting our recursive calculation by setting $A_{129} = v$. In Example 4.2, our recursive calculation started from $A_{129\frac{11}{12}} = v^{1/12}$. If we calculate ${}_{\frac{1}{12}}q_{129\frac{11}{12}}$ we find its value is 0.58960, which is certainly not close to 1.

Table 4.2 *Sample values of $A_x^{(12)}$ using the Standard Ultimate Survival Model, Example 4.2.*

x	${}_{\frac{1}{12}}p_x$	${}_{\frac{1}{12}}q_x$	$A_x^{(12)}$
20	0.999979	0.000021	0.05033
$20\frac{1}{12}$	0.999979	0.000021	0.05051
$20\frac{2}{12}$	0.999979	0.000021	0.05070
$20\frac{3}{12}$	0.999979	0.000021	0.05089
$\vdots$	$\vdots$	$\vdots$	$\vdots$
50	0.999904	0.000096	0.19357
$50\frac{1}{12}$	0.999903	0.000097	0.19429
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$129\frac{10}{12}$	0.413955	0.586045	0.99427
$129\frac{11}{12}$			0.99594

What is happening in these calculations is that, for Example 4.1, we are replacing the exact calculation

$$A_{129} = v\,(q_{129} + p_{129}\,A_{130})$$

by $A_{129} = v$, which is justifiable because A_{130} is close to 1, meaning that $v(q_{129} + p_{129}\,A_{130})$ is very close to v. Similarly, for Example 4.2, we replace the exact calculation

$$A^{(12)}_{129\frac{11}{12}} = v^{1/12}\left({}_{\frac{1}{12}}q_{129\frac{11}{12}} + {}_{\frac{1}{12}}p_{129\frac{11}{12}}\,A^{(12)}_{130}\right)$$

by $A^{(12)}_{129\frac{11}{12}} = v^{1/12}$. As the value of $A^{(12)}_{130}$ is very close to 1, it follows that

$$v^{1/12}\left({}_{\frac{1}{12}}q_{129\frac{11}{12}} + {}_{\frac{1}{12}}p_{129\frac{11}{12}}\,A^{(12)}_{130}\right)$$

can by approximated by $v^{1/12}$.

Example 4.3 Using the Standard Ultimate Survival Model, and an interest rate of 5% per year effective, calculate the mean and standard deviation of the present value of a benefit of \$100 000 payable (a) immediately on death, (b) at the end of the month of death, and (c) at the end of the year of death for lives aged 20, 40, 60, 80 and 100, and comment on the results.

Solution 4.3 For part (a), we must calculate $100\,000\bar{A}_x$ and

$$100\,000\sqrt{{}^2\bar{A}_x - (\bar{A}_x)^2}$$

Table 4.3 *Mean and standard deviation of the present value of a whole life insurance benefit of \$100 000, for Example 4.3.*

	Continuous (a)		Monthly (b)		Annual (c)	
Age, x	Mean	St. Dev.	Mean	St. Dev.	Mean	St. Dev.
20	5 043	5 954	5 033	5 942	4 922	5 810
40	12 404	9 619	12 379	9 600	12 106	9 389
60	29 743	15 897	29 683	15 865	29 028	15 517
80	60 764	17 685	60 641	17 649	59 293	17 255
100	89 341	8 127	89 158	8 110	87 068	7 860

for $x = 20$, 40, 60 and 80, where ${}^2\bar{A}_x$ is calculated at effective rate of interest $j = 10.25\%$. For parts (b) and (c) we replace each $\bar{A}_x$ by $A_x^{(12)}$ and A_x, respectively. The values are shown in Table 4.3. The continuous benefit values in the first column are calculated by numerical integration, and the annual and monthly benefit values are calculated using the spreadsheets from Examples 4.1 and 4.2.

We can make the following observations about these values. First, values for the continuous benefit are larger than the monthly benefit, which are larger than the annual benefit. This is because the death benefit is payable soonest under (a) and latest under (c). Second, as x increases the mean increases for all three cases. This occurs because the smaller the value of x, the longer the expected time until payment of the death benefit. Third, as x increases, the standard deviation decreases relative to the mean, in all three cases. And further, as we get to very old ages, the standard deviation decreases in absolute terms, as the possible range of payout dates is reduced significantly.

It is also interesting to note that the continuous and monthly versions of the whole life benefit are very close. That is to be expected, as the difference arises from the change in the value of money in the period between the moment of death and the end of the month of death, a relatively short period. □

4.4.5 Term insurance

The continuous case, $\bar{A}^{\,1}_{x:\overline{n}|}$

Under a term insurance policy, the death benefit is payable only if the policyholder dies within a fixed term of, say, n years.

In the continuous case, the benefit is payable immediately on death. The present value of a benefit of \$1, which we again denote by Z, is

$$Z = \begin{cases} v^{T_x} = e^{-\delta\, T_x} & \text{if } T_x \le n, \\ 0 & \text{if } T_x > n. \end{cases}$$

The EPV of this benefit is denoted $\bar{A}^{\,1}_{x:\overline{n}|}$ in actuarial notation. The bar above A again denotes that the benefit is payable immediately on death, and the 1 above x indicates that the life (x) must die before the term of n years expires in order for the benefit to be payable.

Then

$$\boxed{\bar{A}^{\,1}_{x:\overline{n}|} = \int_0^n e^{-\delta t}\ {}_tp_x\,\mu_{x+t}\,dt} \tag{4.9}$$

and, similarly, the expected value of the square of the present value is

$$ {}^2\bar{A}^{\,1}_{x:\overline{n}|} = \int_0^n e^{-2\delta t}\ {}_tp_x\,\mu_{x+t}\,dt$$

which, as with the whole life case, is calculated by a change in the rate of interest used.

The annual case, $A^{\,1}_{x:\overline{n}|}$

Next, we consider the situation when a death benefit of 1 is payable at the end of the year of death, provided this occurs within n years. The present value random variable for the benefit is now

$$Z = \begin{cases} v^{K_x+1} & \text{if } K_x \le n-1, \\ 0 & \text{if } K_x \ge n. \end{cases}$$

The EPV of the benefit is denoted $A^{\,1}_{x:\overline{n}|}$ so that

$$\boxed{A^{\,1}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1}\ {}_{k|}q_x.} \tag{4.10}$$

The 1/$m$thly case, $A^{(m)\,1}_{\ x:\overline{n}|}$

We now consider the situation when a death benefit of 1 is payable at the end of the $1/m$th year of death, provided this occurs within n years. The present value random variable for the benefit is

$$Z = \begin{cases} v^{K_x^{(m)}+\frac{1}{m}} & \text{if } K_x^{(m)} \le n-\frac{1}{m}, \\ 0 & \text{if } K_x^{(m)} \ge n. \end{cases}$$

Table 4.4 *EPVs of term insurance benefits.*

x	$\bar{A}^{\ 1}_{x:\overline{10}\rceil}$	$A^{(4)\ 1}_{x:\overline{10}\rceil}$	$A^{\ 1}_{x:\overline{10}\rceil}$
20	0.00214	0.00213	0.00209
40	0.00587	0.00584	0.00573
60	0.04356	0.04329	0.04252
80	0.34550	0.34341	0.33722

The EPV of the benefit is denoted $A^{(m)\ 1}_{x:\overline{n}\rceil}$ so that

$$\boxed{A^{(m)\ 1}_{x:\overline{n}\rceil} = \sum_{k=0}^{mn-1} v^{(k+1)/m} \ {}_{\frac{k}{m}|\frac{1}{m}}q_x.} \tag{4.11}$$

Example 4.4 Using the Standard Ultimate Survival Model as specified in Section 4.3, with interest at 5% per year effective, calculate $\bar{A}^{\ 1}_{x:\overline{10}\rceil}$, $A^{(4)\ 1}_{x:\overline{10}\rceil}$ and $A^{\ 1}_{x:\overline{10}\rceil}$ for $x = 20$, 40, 60 and 80 and comment on the values.

Solution 4.4 We use formula (4.9) with $n = 10$ to calculate $\bar{A}^{\ 1}_{x:\overline{10}\rceil}$ (using numerical integration), and formulae (4.11) and (4.10), with $m = 4$ and $n = 10$ to calculate $A^{(4)\ 1}_{x:\overline{10}\rceil}$ and $A^{\ 1}_{x:\overline{10}\rceil}$.

The values are shown in Table 4.4, and we observe that values in each case increase as x increases, reflecting the fact that the probability of death in a 10-year period increases with age for the survival model we are using. The ordering of values at each age is the same as in Example 4.3, for the same reason – the ordering reflects the fact that any payment under the continuous benefit will be paid earlier than a payment under the quarterly benefit. The end year benefit is paid later than the quarterly benefit, except when the death occurs in the final quarter of the year, in which case the benefit is paid at the same time. □

4.4.6 Pure endowment

Pure endowment benefits are conditional on the survival of the policyholder at a policy maturity date. For example, a 10-year pure endowment with sum insured \$10 000, issued to (x), will pay \$10 000 in 10 years if (x) is still alive at that time, and will pay nothing if (x) dies before age $x + 10$. Pure endowment benefits are not sold as stand-alone policies, but may be sold in conjunction with term insurance benefits to create an endowment insurance, which is covered in the next section. However, pure endowment valuation functions turn out to be very useful when we value other benefits.

The pure endowment benefit of \$1, issued to a life aged x, with a term of n years has present value Z, say, where:

$$Z = \begin{cases} 0 & \text{if } T_x < n, \\ v^n & \text{if } T_x \geq n. \end{cases}$$

There are two ways to denote the EPV of the pure endowment benefit using actuarial notation. It may be denoted $A_{x:\overline{n|}}^{\;\;\,1}$. The '1' over the term subscript indicates that the term must expire before the life does for the benefit to be paid. This notation is consistent with the term insurance notation, but it can be cumbersome, considering that this is a function which is used very often in actuarial calculations. A more convenient standard actuarial notation for the EPV of the pure endowment is ${}_nE_x$.

If we rewrite the definition of Z above, we have

$$Z = \begin{cases} 0 & \text{with probability } 1 - {}_np_x, \\ v^n & \text{with probability } {}_np_x. \end{cases} \tag{4.12}$$

Then we can see that the EPV is

$$\boxed{A_{x:\overline{n|}}^{\;\;\,1} = {}_nE_x = v^n\,{}_np_x\,.} \tag{4.13}$$

Note that because the only possible payment date for the pure endowment is at time n, there is no need to specify continuous and discrete time versions; there is only a discrete time version.

It is sometimes useful to have an expression for $\mathrm{E}[Z^2]$, which we denote by ${}_n^2E_x$. As

$$Z^2 = \begin{cases} 0 & \text{with probability } 1 - {}_np_x, \\ v^{2n} & \text{with probability } {}_np_x, \end{cases} \tag{4.14}$$

we have

$${}_n^2E_x = v^{2n}\,{}_np_x = v^n\,{}_nE_x. \tag{4.15}$$

So the variance of the present value is

$$\mathrm{V}[Z] = {}_n^2E_x - {}_nE_x^2 = {}_nE_x\,v^n\,(1 - {}_np_x) = {}_nE_x\,v^n\,{}_nq_x. \tag{4.16}$$

We will generally use $v^n\,{}_np_x$ or ${}_nE_x$ for the pure endowment function, rather than the $A_{x:\overline{n|}}^{\;\;\,1}$ notation.

4.4.7 Endowment insurance

An endowment insurance provides a combination of a term insurance and a pure endowment. The sum insured is payable on the death of (x) if (x) dies within a fixed term, say n years, but if (x) survives for n years, the sum insured is payable at the end of the nth year.

Traditional endowment insurance policies were popular in Australia, North America and the UK up to the 1990s, but are rarely sold these days in these markets. However, as with the pure endowment, the valuation function turns out to be quite useful in other contexts. Also, companies operating in these territories will be managing the ongoing liabilities under the policies already written for some time to come. Furthermore, traditional endowment insurance is still relevant and popular in some emerging insurance markets, for example in the context of microinsurance.

We first consider the case when the death benefit (of amount 1) is payable immediately on death. The present value of the benefit is Z, say, where

$$
\begin{aligned}
Z &= \begin{cases} v^{T_x} = e^{-\delta T_x} & \text{if } T_x < n, \\ v^n & \text{if } T_x \geq n \end{cases} \\
&= v^{\min(T_x, n)} = e^{-\delta \min(T_x, n)}.
\end{aligned}
$$

Thus, the EPV of the benefit is

$$
\begin{aligned}
\mathrm{E}[Z] &= \int_0^n e^{-\delta t} \, {}_tp_x \, \mu_{x+t} \, dt + \int_n^\infty e^{-\delta n} \, {}_tp_x \, \mu_{x+t} \, dt \\
&= \int_0^n e^{-\delta t} \, {}_tp_x \, \mu_{x+t} \, dt + e^{-\delta n} \, {}_np_x \\
&= \bar{A}^{\ 1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\ \ 1}
\end{aligned}
$$

and in actuarial notation we write

$$
\boxed{\bar{A}_{x:\overline{n}|} = \bar{A}^{\ 1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\ \ 1} = \bar{A}^{\ 1}_{x:\overline{n}|} + {}_nE_x.} \tag{4.17}
$$

Similarly, the expected value of the squared present value of the benefit is

$$
\int_0^n e^{-2\delta t} \, {}_tp_x \, \mu_{x+t} \, dt + e^{-2\delta n} \, {}_np_x,
$$

which we denote ${}^2\bar{A}_{x:\overline{n}|}$.

In the situation when the death benefit is payable at the end of the year of death, the present value of the benefit is

$$
\begin{aligned}
Z &= \begin{cases} v^{K_x+1} & \text{if } K_x \leq n-1, \\ v^n & \text{if } K_x \geq n \end{cases} \\
&= v^{\min(K_x+1, n)}.
\end{aligned}
$$

The EPV of the benefit is then

$$\sum_{k=0}^{n-1} v^{k+1}\ {}_{k|}q_x + v^n \Pr[K_x \geq n] = A^{\ \,1}_{x:\overline{n}|} + v^n\ {}_n p_x, \tag{4.18}$$

and in actuarial notation we write

$$\boxed{A_{x:\overline{n}|} = A^{\ \,1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\ \ \ \ \,1} = A^{\ \,1}_{x:\overline{n}|} + {}_nE_x.} \tag{4.19}$$

Similarly, the expected value of the squared present value of the benefit is

$$\begin{aligned} {}^2A_{x:\overline{n}|} &= \sum_{k=0}^{n-1} v^{2(k+1)}\ {}_{k|}q_x + v^{2n}\,{}_np_x \\ &= {}^2A^{\ \,1}_{x:\overline{n}|} + {}^2_nE_x. \end{aligned}$$

Finally, when the death benefit is payable at the end of the $1/m$th year of death, the present value of the benefit is

$$\begin{aligned} Z &= \begin{cases} v^{K_x^{(m)}+\frac{1}{m}} & \text{if } K_x^{(m)} \leq n - \frac{1}{m}, \\ v^n & \text{if } K_x^{(m)} \geq n \end{cases} \\ &= v^{\min(K_x^{(m)}+\frac{1}{m},n)}. \end{aligned}$$

The EPV of the benefit is

$$\sum_{k=0}^{mn-1} v^{(k+1)/m}\ {}_{\frac{k}{m}|\frac{1}{m}}q_x + v^n \Pr[K_x^{(m)} \geq n] = A^{(m)\,1}_{\ \ x:\overline{n}|} + v^n\,{}_np_x,$$

and in actuarial notation we write

$$\boxed{A^{(m)}_{x:\overline{n}|} = A^{(m)\,1}_{\ \ x:\overline{n}|} + A_{x:\overline{n}|}^{\ \ \ \ \,1} = A^{(m)\,1}_{\ \ x:\overline{n}|} + {}_nE_x\,.} \tag{4.20}$$

Example 4.5 Using the Standard Ultimate Survival Model with interest at 5% per year effective, calculate $\bar{A}_{x:\overline{10}|}$, $A^{(4)}_{x:\overline{10}|}$, and $A_{x:\overline{10}|}$ for $x = 20$, 40, 60 and 80 and comment on the values.

Solution 4.5 We can obtain values of $\bar{A}_{x:\overline{10}|}$, $A^{(4)}_{x:\overline{10}|}$, and $A_{x:\overline{10}|}$ by adding ${}_{10}E_x = v^{10}\,{}_{10}p_x$ to the values of $\bar{A}^{\ \,1}_{x:\overline{10}|}$, $A^{(4)\,1}_{\ \ x:\overline{10}|}$ and $A^{\ \,1}_{x:\overline{10}|}$ in Example 4.4. The values are shown in Table 4.5.

The actuarial values of the 10-year endowment insurance functions do not vary greatly with x, unlike the values of the 10-year term insurance functions. The reason for this is that the probability of surviving 10 years is large (${}_{10}p_{20} = 0.9973$, ${}_{10}p_{60} = 0.9425$) and so for each value of x, the benefit is payable after 10 years with a high probability. Note that $v^{10} = 0.6139$, and because $t = 10$ is the latest possible payment date for the benefit, the values of $\bar{A}_{x:\overline{10}|}$, $A^{(4)}_{x:\overline{10}|}$ and $A_{x:\overline{10}|}$ must be greater than this for any age x. □

Table 4.5 *EPVs of endowment insurance benefits.*

x	$\bar{A}_{x:\overline{10}\rceil}$	$A^{(4)}_{x:\overline{10}\rceil}$	$A_{x:\overline{10}\rceil}$
20	0.61438	0.61437	0.61433
40	0.61508	0.61504	0.61494
60	0.62220	0.62194	0.62116
80	0.68502	0.68292	0.67674

4.4.8 Deferred insurance

Deferred insurance refers to insurance which does not begin to offer death benefit cover until the end of a deferred period. Suppose a benefit of \$1 is payable immediately on the death of (x) provided that (x) dies between ages $x+u$ and $x+u+n$. The present value random variable is

$$Z = \begin{cases} 0 & \text{if } T_x < u \text{ or } T_x \geq u+n, \\ e^{-\delta T_x} & \text{if } u \leq T_x < u+n. \end{cases}$$

This random variable describes the present value of a deferred term insurance. We can, similarly, develop random variables to value deferred whole life or endowment insurance.

The actuarial notation for the EPV of the deferred term insurance benefit is ${}_{u|}\bar{A}^{\;1}_{x:\overline{n}\rceil}$. Thus

$${}_{u|}\bar{A}^{\;1}_{x:\overline{n}\rceil} = \int_u^{u+n} e^{-\delta t}\; {}_tp_x\, \mu_{x+t}\, dt. \tag{4.21}$$

Changing the integration variable to $s = t - u$ gives

$$\begin{aligned} {}_{u|}\bar{A}^{\;1}_{x:\overline{n}\rceil} &= \int_0^n e^{-\delta(s+u)}\; {}_{s+u}p_x\, \mu_{x+s+u}\, ds \\ &= e^{-\delta u}\; {}_up_x \int_0^n e^{-\delta s}\; {}_sp_{x+u}\, \mu_{x+s+u}\, ds \\ &= e^{-\delta u}\; {}_up_x\, \bar{A}^{\;1}_{x+u:\overline{n}\rceil} = v^u\; {}_up_x\, \bar{A}^{\;1}_{x+u:\overline{n}\rceil} = {}_uE_x\, \bar{A}^{\;1}_{x+u:\overline{n}\rceil}. \end{aligned} \tag{4.22}$$

A further expression for ${}_{u|}\bar{A}^{\;1}_{x:\overline{n}\rceil}$ is

$$\boxed{{}_{u|}\bar{A}^{\;1}_{x:\overline{n}\rceil} = \bar{A}^{\;1}_{x:\overline{u+n}\rceil} - \bar{A}^{\;1}_{x:\overline{u}\rceil}.} \tag{4.23}$$

which follows from formula (4.21) since

$$\int_u^{u+n} e^{-\delta t}\,{}_tp_x\,\mu_{x+t}\,dt = \int_0^{u+n} e^{-\delta t}\,{}_tp_x\,\mu_{x+t}\,dt - \int_0^{u} e^{-\delta t}\,{}_tp_x\,\mu_{x+t}\,dt.$$

Thus, the EPV of a deferred term insurance benefit can be found by differencing the EPVs of term insurance benefits for terms $u+n$ and u.

Note the role of the pure endowment term ${}_uE_x = v^u\,{}_up_x$ in equation (4.22). This acts similarly to a discount function. If the life survives u years, to the end of the deferred period, then the EPV at that time of the term insurance is $\bar{A}^{\,1}_{x+u:\overline{n|}}$. Multiplying by $v^u\,{}_up_x$ converts this to the EPV at the start of the deferred period.

Our main interest in this EPV is as a building block. We observe, for example, that an n-year term insurance can be decomposed as the sum of n deferred term insurance policies, each with a term of one year, and we can write

$$\begin{aligned}\bar{A}^{\,1}_{x:\overline{n|}} &= \int_0^{n} e^{-\delta t}\,{}_tp_x\,\mu_{x+t}\,dt \\ &= \sum_{r=0}^{n-1}\int_r^{r+1} e^{-\delta t}\,{}_tp_x\,\mu_{x+t}\,dt \\ &= \sum_{r=0}^{n-1} {}_{r|}\bar{A}^{\,1}_{x:\overline{1|}}\,. \end{aligned} \tag{4.24}$$

A similar decomposition applies to a whole life insurance policy and we can write

$$\bar{A}_x = \sum_{r=0}^{\infty} {}_{r|}\bar{A}^{\,1}_{x:\overline{1|}}\,.$$

We can derive similar results for the deferred benefit payable at the end of the year of death, with EPV denoted ${}_{u|}A^{\,1}_{x:\overline{n|}}$.

In particular, it is useful to note that

$$\boxed{A_x = A^{\,1}_{x:\overline{n|}} + {}_{n|}A_x,}$$

where ${}_{n|}A_x$ is the EPV of a benefit of 1 payable at the end of the year of death of (x) if death occurs after time n, so that

$$\begin{aligned}A^{\,1}_{x:\overline{n|}} &= A_x - {}_{n|}A_x \\ &= A_x - v^n\,{}_np_x\,A_{x+n} = A_x - {}_nE_x\,A_{x+n}. \end{aligned} \tag{4.25}$$

This relationship can be used to calculate $A^{\ 1}_{x:\overline{n}|}$ for integer x and n given a table of values of A_x and l_x.

4.5 Relating $\bar{A}_x$, A_x and $A_x^{(m)}$

We mentioned in the introduction to this chapter that, even though insurance contracts with death benefits payable at the end of the year of death are very unusual, functions like A_x are still useful. The reason for this is that we can approximate $\bar{A}_x$ or $A_x^{(m)}$ from A_x, and we might wish to do this if the only information available is a life table, with integer age functions only, rather than a formula for the force of mortality that could be applied for all ages.

In Table 4.6 we show values of the ratios of $A_x^{(4)}$ to A_x and $\bar{A}_x$ to A_x, using the Standard Ultimate Survival Model from Section 4.3, with interest at 5% per year effective.

We see from Table 4.6 that, over a very wide range of ages, the ratios of $A_x^{(4)}$ to A_x and $\bar{A}_x$ to A_x are remarkably stable, giving the appearance of being independent of x up to around age 80. In the following section we show how we can approximate values of $A_x^{(m)}$ and $\bar{A}_x$ using values of A_x.

4.5.1 Using the uniform distribution of deaths assumption

The difference between $\bar{A}_x$ and A_x depends on the lifetime distribution between ages y and $y + 1$ for all $y \geq x$. If we do not have information about this, for example, because we have mortality information only at integer ages, we can approximate the relationship between the continuous function $\bar{A}_x$ and the discrete function A_x using one of the fractional age assumptions that we introduced in Section 3.3. The most convenient fractional age assumption for this purpose is the uniform distribution of deaths assumption, or UDD.

Table 4.6 *Ratios of $A_x^{(4)}$ to A_x and $\bar{A}_x$ to A_x, Standard Ultimate Survival Model.*

x	$A_x^{(4)}/A_x$	$\bar{A}_x/A_x$
20	1.0184	1.0246
40	1.0184	1.0246
60	1.0184	1.0246
80	1.0186	1.0248
100	1.0198	1.0261
120	1.0296	1.0368

Recall, from equation (3.11), that under UDD, for $0 \le s < 1$, and for integer y, we have ${}_s p_y\, \mu_{y+s} = q_y$. Using this assumption we can derive a relationship between A_x and $\bar{A}_x$, under the UDD assumption:

$$\begin{aligned}
\bar{A}_x &= \int_0^\infty e^{-\delta t}\, {}_t p_x\, \mu_{x+t}\, dt \\
&= \sum_{k=0}^\infty \int_k^{k+1} e^{-\delta t}\, {}_t p_x\, \mu_{x+t}\, dt \\
&= \sum_{k=0}^\infty {}_k p_x v^{k+1} \int_0^1 e^{(1-s)\delta}\, {}_s p_{x+k}\, \mu_{x+k+s}\, ds \\
&= \sum_{k=0}^\infty {}_k p_x v^{k+1}\, q_{x+k} \int_0^1 e^{(1-s)\delta}\, ds \quad \text{using UDD} \\
&= A_x\, \frac{e^\delta - 1}{\delta}.
\end{aligned}$$

Because $e^\delta = 1 + i$, under the assumption of UDD we have

$$\boxed{\bar{A}_x = \frac{i}{\delta} A_x.} \tag{4.26}$$

The same approximation applies to term insurance and deferred insurance, which we can show by changing the limits of integration in the proof above.

We may also want to derive a $1/m$thly death benefit EPV, such as $A_x^{(m)}$, from the annual function A_x.

Under the UDD assumption we also find that

$$\boxed{A_x^{(m)} = \frac{i}{i^{(m)}} A_x,} \tag{4.27}$$

and the proof of formula (4.27) is left as an exercise for the reader.

These results are exact where the UDD assumption applies, and are commonly used to find approximate values for continuous and $1/m$thly functions from the annual function, even if UDD is not an exact model for fractional ages.

We stress that these approximations apply only to death benefits. The endowment insurance combines the death and survival benefits, so we need to split off the death benefit before applying one of the approximations. That is, under the UDD approach

$$\boxed{\bar{A}_{x:\overline{n}|} \approx \frac{i}{\delta} A^{\ 1}_{x:\overline{n}|} + {}_nE_x.} \tag{4.28}$$

4.5.2 Using the claims acceleration approach

The claims acceleration approach is a more heuristic way of deriving an approximate relationship between the annual death benefit EPV, A_x, and the $1/m$thly or continuous EPVs, $A_x^{(m)}$ and $\bar{A}_x$. The only difference between these benefits is the timing of the payment. Consider, for example, A_x and $A_x^{(4)}$. The insured life, (x), dies in the year of age $x + K_x$ to $x + K_x + 1$. Under the end year of death benefit (valued by A_x), the sum insured is paid at time $K_x + 1$. Under the end of quarter-year of death benefit (valued by $A_x^{(4)}$), the benefit will be paid either at $K_x + \frac{1}{4}$, $K_x + \frac{2}{4}$, $K_x + \frac{3}{4}$ or $K_x + 1$ depending on the quarter year in which the death occurred. If the deaths occur evenly over the year (the same assumption as we use in the UDD approach), then, on average, the benefit is paid at time $K_x + \frac{5}{8}$, which is $3/8$ years earlier than the end of year of death benefit.

Similarly, suppose the benefit is paid at the end of the month of death. Assuming deaths occur uniformly over the year, then on average the benefit is paid at $K_x + \frac{13}{24}$, which is $11/24$ years earlier than the end year of death benefit.

In general, for a $1/m$thly death benefit, assuming deaths are uniformly distributed over the year of age, the average time of payment of the death benefit is $(m + 1)/2m$ in the year of death, which is $1 - \frac{m+1}{2m} = \frac{m-1}{2m}$ years earlier than the end year of death benefit would be paid.

So we have the resulting approximation

$$\begin{aligned} A_x^{(m)} &\approx q_x\, v^{\frac{m+1}{2m}} + {}_1|q_x\, v^{1+\frac{m+1}{2m}} + {}_2|q_x\, v^{2+\frac{m+1}{2m}} + \cdots \\ &= \sum_{k=0}^{\infty} {}_k|q_x\, v^{k+\frac{m+1}{2m}} \\ &= (1+i)^{\frac{m-1}{2m}} \sum_{k=0}^{\infty} {}_k|q_x\, v^{k+1}. \end{aligned}$$

That is

$$\boxed{A_x^{(m)} \approx (1+i)^{\frac{m-1}{2m}} A_x.} \tag{4.29}$$

For the continuous benefit EPV, $\bar{A}_x$, we let $m \to \infty$ in equation (4.29), to give the approximation

$$\boxed{\bar{A}_x \approx (1+i)^{\frac{1}{2}} A_x \,.} \tag{4.30}$$

This is explained by the fact that, if the benefit is paid immediately on death, and lives die uniformly through the year, then, on average, the benefit is paid half-way through the year of death, which is half a year earlier than the benefit valued by A_x.

As with the UDD approach, these approximations apply only to death benefits. Hence, for an endowment insurance using the claims acceleration approach we have

$$\boxed{\bar{A}_{x:\overline{n}|} \approx (1+i)^{\frac{1}{2}} A^{\,1}_{x:\overline{n}|} + {}_nE_x.} \tag{4.31}$$

Note that both the UDD and the claims acceleration approaches give values for $A_x^{(m)}$ or $\bar{A}_x$ such that the ratios $A_x^{(m)}/A_x$ and $\bar{A}_x/A_x$ are independent of x. Note also that for $i = 5\%$, $i/i^{(4)} = 1.0186$ and $i/\delta = 1.0248$, whilst $(1+i)^{3/8} = 1.0185$ and $(1+i)^{1/2} = 1.0247$. The values in Table 4.6 show that both approaches give good approximations in these cases.

4.6 Variable insurance benefits

For all the insurance benefits studied in this chapter, the EPV of the benefit can be expressed as the sum over all the possible payment dates of the product of three terms:

- the amount of benefit paid,
- the appropriate discount factor for the payment date, and
- the probability that the benefit will be paid at that payment date.

This approach works for the EPV of any traditional benefit – that is, where the future lifetime is the sole source of uncertainty. It will not generate higher moments or probability distributions.

The approach can be justified technically using **indicator random variables**. Consider a life contingent event E – for example, E could be the event that a life aged x dies in the interval $(k, k+1]$. The indicator random variable is

$$I(E) = \begin{cases} 1 & \text{if } E \text{ is true,} \\ 0 & \text{if } E \text{ is false.} \end{cases}$$

In this example, $\Pr[E \text{ is True}] = {}_{k|}q_x$, so the expected value of the indicator random variable is

$$\mathrm{E}[I(E)] = 1({}_{k|}q_x) + 0(1 - {}_{k|}q_x) = {}_{k|}q_x,$$

and, in general, the expected value of an indicator random variable is the probability of the indicator event.

Consider, for example, an insurance that pays \$1 000 after 10 years if (x) has died by that time, and \$2 000 after 20 years if (x) dies in the second 10-year period, with no benefit otherwise.

We can write the present value random variable as

$$1\,000\, I(T_x \leq 10)v^{10} + 2\,000\, I(10 < T_x \leq 20)v^{20}$$

and the EPV is then

$$1\,000\; {}_{10}q_x\, v^{10} + 2\,000\; {}_{10|10}q_x\, v^{20}.$$

Indicator random variables can also be used for continuous benefits. Here we consider indicators of the form

$$I(t < T_x \leq t + dt)$$

for infinitesimal dt, with associated probability

$$\begin{aligned} \mathrm{E}[I(t < T_x \leq t + dt)] &= \Pr[t < T_x \leq t + dt] \\ &= \Pr[T_x > t]\,\Pr[T_x < t + dt | T_x > t] \\ &\approx {}_t p_x\, \mu_{x+t}\, dt. \end{aligned}$$

Consider, for example, an increasing insurance policy with a death benefit of T_x payable at the moment of death. That is, the benefit is exactly equal to the number of years lived by an insured life from age x to his or her death. This is a continuous whole life insurance under which the benefit is a linearly increasing function.

To find the EPV of this benefit, we note that the payment may be made at any time, so we consider all the infinitesimal intervals $(t, t{+}dt)$, and we sum over all these intervals by integrating from $t = 0$ to $t = \infty$.

First, we identify the amount, discount factor and probability for a benefit payable in the interval $(t, t + dt)$. The amount is t, the discount factor is $e^{-\delta t}$; the probability that the benefit is paid in the interval $(t, t + dt)$ is the probability that the life survives from x to $x + t$, and then dies in the infinitesimal interval $(t, t + dt)$, which for the purpose of constructing the integral is ${}_t p_x\, \mu_{x+t}\, dt$.

So, we can write the EPV of this benefit as

$$\int_0^\infty t\, e^{-\delta t}\, {}_t p_x\, \mu_{x+t}\, dt. \tag{4.32}$$

In actuarial notation we write this as $(\bar{I}\bar{A})_x$. The I here stands for 'increasing' and the bar over the I denotes that the increases are continuous.

An alternative approach to deriving equation (4.32) is to identify the present value random variable for the benefit, denoted by Z, say, in terms of the future lifetime random variable,

$$Z = T_x \, e^{-\delta T_x}.$$

Then any moment of Z can be found from

$$\mathrm{E}[Z^k] = \int_0^\infty \left(t\, e^{-\delta t}\right)^k {}_tp_x \, \mu_{x+t} \, dt.$$

The advantage of the first approach is that it is very flexible and generally quick, even for very complex benefits.

If the policy term ceases after a fixed term of n years, the EPV of the death benefit is

$$(\bar{I}\bar{A})^{\,1}_{x:\overline{n|}} = \int_0^n t\, e^{-\delta t} \, {}_tp_x \, \mu_{x+t} \, dt.$$

There are a number of other increasing or decreasing benefit patterns that are fairly common. We present several in the following examples.

Example 4.6 Consider an n-year term insurance policy issued to (x) under which the death benefit is $k+1$, payable at the end of the year of death, if death occurs between ages $x+k$ and $x+k+1$, for $k = 0, 1, 2, \ldots, n-1$.

(a) Derive a formula for the EPV of the benefit using the first approach described, that is multiplying together the amount, the discount factor and the probability of payment, and summing for each possible payment date.
(b) Derive a formula for the variance of the present value of the benefit.

Solution 4.6 (a) If the benefit is paid at time $k+1$, then the benefit amount is \$$(k+1)$, the discount factor is v^{k+1}. The probability that the benefit is paid at $k+1$ is the probability that the policyholder died in the year $(k, k+1]$, which is ${}_{k|}q_x$. Summing over all the possible years of death, to the end of the policy term, we have the EPV of the death benefit is

$$\sum_{k=0}^{n-1} v^{k+1} (k+1) \; {}_{k|}q_x.$$

In actuarial notation the above EPV is denoted $(IA)^{\,1}_{x:\overline{n|}}$.

If the term n is infinite this is a whole life version of the increasing annual policy, with benefit $K_x{+}1$ paid at the end of the year of death. The EPV of the death benefit is denoted $(IA)_x$ where

$$(IA)_x = \sum_{k=0}^{\infty} v^{k+1}(k+1)\ {}_{k|}q_x.$$

(b) We must go back to first principles. First, we identify the random variable as

$$Z = \begin{cases} (K_x+1)v^{K_x+1} & \text{if } K_x < n, \\ 0 & \text{if } K_x \ge n. \end{cases}$$

So

$$\mathrm{E}[Z^2] = \sum_{k=0}^{n-1} v^{2(k+1)}(k+1)^2\ {}_{k|}q_x,$$

and the variance is

$$\mathrm{V}[Z] = \sum_{k=0}^{n-1} v^{2(k+1)}(k+1)^2\ {}_{k|}q_x - \left((IA)^{\ 1}_{x:\overline{n}|}\right)^2.$$

□

Example 4.7 A whole life insurance policy offers an increasing death benefit payable at the end of the quarter year of death. If (x) dies in the first year of the contract, then the benefit is 1, in the second year it is 2, and so on. Derive an expression for the EPV of the death benefit.

Solution 4.7 First, we note that the possible payment dates are $\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \ldots$. Next, if (x) dies in the first year, then the benefit payable is 1, if death occurs in the second year the benefit payable is 2, and so on. Third, corresponding to the possible payment dates, the discount factors are $v^{1/4}, v^{2/4}, \ldots$.

The probabilities associated with the payment dates are ${}_{\frac{1}{4}}q_x$, ${}_{\frac{1}{4}|\frac{1}{4}}q_x$, ${}_{\frac{2}{4}|\frac{1}{4}}q_x$, ${}_{\frac{3}{4}|\frac{1}{4}}q_x, \ldots$.

Hence, the EPV, which is denoted $(IA^{(4)})_x$, can be calculated as

$$\begin{aligned}
&{}_{\frac{1}{4}}q_x\, v^{\frac{1}{4}} + {}_{\frac{1}{4}|\frac{1}{4}}q_x\, v^{\frac{2}{4}} + {}_{\frac{2}{4}|\frac{1}{4}}q_x\, v^{\frac{3}{4}} + {}_{\frac{3}{4}|\frac{1}{4}}q_x\, v^{1} \\
&\quad + 2\left({}_{1|\frac{1}{4}}q_x\, v^{1\frac{1}{4}} + {}_{1\frac{1}{4}|\frac{1}{4}}q_x\, v^{1\frac{2}{4}} + {}_{1\frac{2}{4}|\frac{1}{4}}q_x\, v^{1\frac{3}{4}} + {}_{1\frac{3}{4}|\frac{1}{4}}q_x\, v^{2}\right) \\
&\quad + 3\left({}_{2|\frac{1}{4}}q_x\, v^{2\frac{1}{4}} + {}_{2\frac{1}{4}|\frac{1}{4}}q_x\, v^{2\frac{2}{4}} + {}_{2\frac{2}{4}|\frac{1}{4}}q_x\, v^{2\frac{3}{4}} + {}_{2\frac{3}{4}|\frac{1}{4}}q_x\, v^{3}\right) + \cdots \\
&\quad\ = A^{(4)\,1}_{\ x:\overline{1}|} + 2\ {}_{1|}A^{(4)\,1}_{\ x:\overline{1}|} + 3\ {}_{2|}A^{(4)\,1}_{\ x:\overline{1}|} + \cdots .
\end{aligned}$$

□

We now consider the case when the amount of the death benefit increases in geometric progression. This is important in practice because compound reversionary bonuses will increase the sum insured as a geometric progression.

Example 4.8 Consider an n-year term insurance issued to (x) under which the death benefit is paid at the end of the year of death. The benefit is 1 if death occurs between ages x and $x+1$, $1+j$ if death occurs between ages $x+1$ and $x+2$, $(1+j)^2$ if death occurs between ages $x+2$ and $x+3$, and so on. Thus, if death occurs between ages $x+k$ and $x+k+1$, the death benefit is $(1+j)^k$ for $k = 0, 1, 2, \dots, n-1$. Derive a formula for the EPV of this death benefit.

Solution 4.8 Using the technique of multiplying together the amount, the discount factor and the probability of payment, and summing for each possible payment date, the EPV can be constructed as

$$
\begin{aligned}
v\,q_x &+ (1+j)v^2\,{}_{1|}q_x + (1+j)^2\,v^3\,{}_{2|}q_x + \cdots + (1+j)^{n-1}\,v^n\,{}_{n-1|}q_x \\
&= \sum_{k=0}^{n-1} v^{k+1}(1+j)^k\,{}_{k|}q_x \\
&= \frac{1}{1+j}\sum_{k=0}^{n-1} v^{k+1}(1+j)^{k+1}\,{}_{k|}q_x \\
&= \frac{1}{1+j}A^{\,1}_{x:\overline{n}|\,i^*} \qquad \text{where } i^* = \frac{1+i}{1+j} - 1 = \frac{i-j}{1+j}. \qquad (4.33)
\end{aligned}
$$

□

The notation $A^{\,1}_{x:\overline{n}|\,i^*}$ indicates that the EPV is calculated using the rate of interest i^*, rather than i. In most practical situations, $i > j$ so that $i^* > 0$.

Example 4.9 Consider an insurance policy issued to (x) under which the death benefit is $(1+j)^t$ where t is the time of death (from the inception of the policy). The death benefit is payable immediately on death.

(a) Derive an expression for the EPV of the death benefit if the policy is an n-year term insurance.
(b) Derive an expression for the EPV of the death benefit if the policy is a whole life insurance.

Solution 4.9 (a) The present value of the death benefit is $(1+j)^{T_x}v^{T_x}$ if $T_x < n$, and is zero otherwise, so that the EPV of the death benefit is

$$
\int_0^n (1+j)^t\,v^t\,{}_tp_x\,\mu_{x+t}\,dt = \bar{A}^{\,1}_{x:\overline{n}|\,i^*} \quad \text{where } i^* = \frac{1+i}{1+j} - 1.
$$

(b) Similarly, if the policy is a whole life insurance rather than a term insurance, then the EPV of the death benefit would be

$$\int_0^\infty (1+j)^t\, v^t\, {}_tp_x\, \mu_{x+t}\, dt = (\bar{A}_x)_{i^*} \quad \text{where } i^* = \frac{1+i}{1+j} - 1.$$

□

4.7 Functions for select lives

Throughout this chapter we have developed results in terms of lives subject to ultimate mortality. We have taken this approach simply for ease of presentation. All of the above development equally applies to lives subject to select mortality.

For example, $\bar{A}_{[x]}$ denotes the EPV of a benefit of 1 payable immediately on the death of a select life (x). Similarly, $A_{[x]:\overline{n}|}$ denotes the EPV of a benefit of 1 payable at the end of the year of death within n years, of a newly selected life aged x, or at age $x + n$ if (x) survives.

4.8 Notes and further reading

The Standard Ultimate Survival Model incorporates Makeham's law as its survival model. A feature of Makeham's law is that we can integrate the force of mortality analytically and hence we can evaluate, for example, ${}_tp_x$ analytically, as in Chapter 2. This in turn means that the EPV of an insurance benefit payable immediately on death, for example $\bar{A}_x$, can be written as an integral where the integrand can be evaluated directly, as follows

$$\bar{A}_x = \int_0^\infty e^{-\delta t}\, {}_tp_x\, \mu_{x+t}\, dt.$$

This integral cannot be evaluated analytically but can be evaluated numerically. In many practical situations, the force of mortality cannot be integrated analytically, for example if μ_x is a GM(r,s) function with $s \geq 2$, from Section 2.7. In such cases, ${}_tp_x$ can be evaluated numerically but not analytically. Functions such as $\bar{A}_x$ can still be evaluated numerically but, since the integrand has to be evaluated numerically, the procedure may be a little more complicated. See Exercise 4.34 for an example. The survival model in Exercise 4.34 has been derived from data for UK whole life and endowment insurance policyholders (non-smokers), 1999–2002. See CMI (2006, Table 1).

4.9 Exercises

Some exercises in this and subsequent chapters are based on the Standard Select and Standard Ultimate Life Tables, which are given in Appendix D. Although the full model underlying these tables was described in Chapter 3, you are expected to use these life tables as if only the integer age information given in the tables is available. When you need to use a fractional age assumption, we specify which assumption to use in the exercise. We assume that an Excel version of the Standard Ultimate Life Table is available that can calculate annual functions at different rates of interest.

When we specify the use of the Standard Select Survival Model or Standard Ultimate Survival Model, then we assume knowledge of the full underlying distribution.

Shorter exercises

Exercise 4.1 You are given the following table of values for l_x and A_x, assuming an effective interest rate of 6% per year.

x	l_x	A_x
35	100 000.00	0.151375
36	99 737.15	0.158245
37	99 455.91	0.165386
38	99 154.72	0.172804
39	98 831.91	0.180505
40	98 485.68	0.188492

Calculate the following, assuming UDD between integer ages where necessary.

(a) ${}_5E_{35}$ (b) $A^{\ 1}_{35:\overline{5|}}$ (c) ${}_{5|}A_{35}$ (d) $\bar{A}_{35:\overline{5|}}$

Exercise 4.2 Using the Standard Ultimate Life Table, with interest at 5% per year effective, calculate the following, assuming UDD between integer ages where necessary.

(a) $A^{\ 1}_{30:\overline{20|}}$ (b) $\bar{A}_{40:\overline{20|}}$ (c) ${}_{10|}A_{25}$

Exercise 4.3 (a) Describe in words the insurance benefit with present value given by

$$Z = \begin{cases} T_{30}\, v^{T_{30}} & \text{if } T_{30} \le 25, \\ 25\, v^{T_{30}} & \text{if } T_{30} > 25. \end{cases}$$

(b) Write down an expression in terms of standard actuarial functions for $\mathrm{E}[Z]$.

Exercise 4.4 Put the following functions in order, from smallest to largest, assuming $i > 0$ and $\mu_{x+t} > 0$ for all $t > 0$. Explain your answer from general reasoning (i.e., from general principles, not from calculating the values).

$$A_x \qquad \bar{A}_x \qquad A^{(4)}_{x:\overline{10}|} \qquad \bar{A}_{x:\overline{10}|} \qquad A^{(4)\,1}_{x:\overline{10}|} \qquad A^{(12)}_{x:\overline{10}|}$$

Exercise 4.5 An insurer issues a five-year term insurance to (65), with a benefit of \$100 000 payable at the end of the year of death.

Calculate the probability that the present value of the benefit is greater than \$90 000, using the Standard Ultimate Life Table, with interest of 5% per year effective.

Exercise 4.6 Under an endowment insurance issued to a life aged x, let X denote the present value of a unit sum insured, payable at the moment of death or at the end of the n-year term.

Under a term insurance issued to a life aged x, let Y denote the present value of a unit sum insured, payable at the moment of death within the n-year term.

Given that

$$\mathrm{V}[X] = 0.0052, \quad v^n = 0.3, \quad {}_np_x = 0.8, \quad \mathrm{E}[Y] = 0.04,$$

calculate $\mathrm{V}[Y]$.

Exercise 4.7 A whole life insurance with sum insured \$50 000 is issued to (50). The benefit is payable immediately on death. Calculate the probability that the present value of the benefit is less than \$20 000. Use the Standard Ultimate Survival Model with interest at 4% per year effective.

Exercise 4.8 Assuming a uniform distribution of deaths over each year of age, show that

$$A^{(m)}_x = \frac{i}{i^{(m)}} A_x.$$

Exercise 4.9 Calculate A_{70} given that

$$A_{50:\overline{20}|} = 0.42247, \quad A^{\;1}_{50:\overline{20}|} = 0.14996, \quad A_{50} = 0.31266.$$

Exercise 4.10 Using the Standard Ultimate Life Table, with interest at 5% per year effective, calculate the standard deviation of the present value of the following benefits:

(a) \$100 000 payable at the end of the year of death of (30), and
(b) \$100 000 payable at the end of the year of death of (30), provided death occurs before age 50.

Exercise 4.11 You are given that $A_x = 0.25$, $A_{x+20} = 0.40$, $A_{x:\overline{20}|} = 0.55$ and $i = 0.03$. Calculate $10\,000\bar{A}_{x:\overline{20}|}$ using

(a) claims acceleration, and
(b) UDD.

Exercise 4.12 Show that $\bar{A}_x$ is a decreasing function of i, and explain this result by general reasoning.

Exercise 4.13 An insurer issues a whole life policy to a life aged 80, under which a benefit of \$10 000 is payable immediately on death during the first year, and \$100 000 is payable immediately on death during any subsequent year. Because the underwriting on the insurance is light, the insurer expects the mortality rate during the first year to be double the rate from the Standard Ultimate Life Table. After the first year, mortality is assumed to follow the Standard Ultimate Life Table with no adjustment.

Calculate the EPV of the benefit, using an interest rate of 5% per year effective. Assume UDD between integer ages.

Exercise 4.14 $(\bar{I}\bar{A})_{x:\overline{n}|}$ denotes the EPV of an increasing endowment insurance, where, if $T_x < n$ a benefit of T_x is payable immediately on death, and if $T_x \geq n$ a benefit of n is payable at time n.

Using the Standard Ultimate Survival Model, with 5% per year interest, calculate

$$\frac{d}{dt}(\bar{I}\bar{A})_{40:\overline{t}|} \qquad \text{at } t = 10.$$

Exercise 4.15 A whole life insurance policy issued to a life aged exactly 30 has an increasing sum insured. In the tth policy year, $t = 1, 2, 3, \ldots$, the sum insured is \$100 000 (1.03^{t-1}), payable at the end of the year of death. Using the Standard Ultimate Survival Model with interest at 5% per year, calculate the EPV of this benefit.

Exercise 4.16 Show that

$$(IA)^{\ 1}_{x:\overline{n}|} = (n+1)A^{\ 1}_{x:\overline{n}|} - \sum_{k=1}^{n} A^{\ 1}_{x:\overline{k}|}$$

and explain this result intuitively.

Exercise 4.17 You are given that the time to first failure, denoted T, of an industrial robot, has a probability density function for the first 10 years of operations of

$$f_T(t) = \begin{cases} 0.1 & \text{for } 0 \leq t < 2, \\ 0.4t^{-2} & \text{for } 2 \leq t < 10. \end{cases}$$

Consider a supplemental warranty on this robot which pays \$100 000 at time T, if $2 \leq T < 10$, with no benefits payable otherwise.

Calculate the 90th percentile of the present value of the benefit under the warranty, assuming a force of interest of $\delta = 0.05$.

(*Copyright 2017. The Society of Actuaries, Schaumburg, Illinois. Reproduced with permission.*)

Longer exercises

Exercise 4.18 You are given the following excerpt from a select life table.

$[x]$	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$	l_{x+4}	$x+4$
[40]	100 000	99 899	99 724	99 520	99 288	44
[41]	99 802	99 689	99 502	99 283	99 033	45
[42]	99 597	99 471	99 268	99 030	98 752	46
[43]	99 365	99 225	99 007	98 747	98 435	47
[44]	99 120	98 964	98 726	98 429	98 067	48

Assuming an interest rate of 6% per year, calculate

(a) $A_{[40]+1:\overline{4}|}$,
(b) the standard deviation of the present value of a four-year term insurance, deferred one year, issued to a newly selected life aged 40, with sum insured \$100 000, payable at the end of the year of death, and
(c) the probability that the present value of the benefit described in part (b) is less than or equal to \$85 000.

Exercise 4.19 A select life aged 50 purchases a whole life insurance policy, with sum insured \$1 000 000. The benefit is valued using the Standard Select Life Table, with interest of 5% per year. Assume UDD between integer ages where necessary.

(a) Assume the claim is payable four months after the death of the policyholder. Calculate the mean and standard deviation of the present value of the benefit.
(b) Now assume that there are two types of claims. A claim is either 'straightforward' or 'complex'. The straightforward claims are settled two months after the death of the policyholder. The complex claims are settled one year after the death of the policyholder. The probability that an individual claim is straightforward is 80% and the probability that it is complex is 20%. Calculate the mean and standard deviation of the present value of the benefit.

Exercise 4.20 (a) An insurer issues a 20-year term insurance with sum insured \$100, payable immediately on death, to a life currently aged 50. Calculate the 95th percentile of the present value of the benefit, assuming interest of 5% per year, and that mortality follows the Standard Ultimate Life Table, with UDD between integer ages.

(b) The insurer issues 10 000 identical policies to independent lives. Using a normal approximation, estimate the 95th percentile of the present value of the aggregate payment over all the policies.

(c) Explain why the normal approximation is reasonable for part (b) but would not be reasonable for part (a)

Exercise 4.21 (a) Describe **in words** the insurance benefits with the present values given below.

$$\text{(i)} \qquad Z_1 = \begin{cases} 20\, v^{T_x} & \text{if } T_x \le 15, \\ 10\, v^{T_x} & \text{if } T_x > 15. \end{cases}$$

$$\text{(ii)} \qquad Z_2 = \begin{cases} 0 & \text{if } T_x \le 5, \\ 10\, v^{T_x} & \text{if } 5 < T_x \le 15, \\ 10\, v^{15} & \text{if } T_x > 15. \end{cases}$$

(b) Write down in integral form the formula for the expected value for (i) Z_1 and (ii) Z_2.

(c) Derive expressions in terms of standard actuarial functions for the expected values of Z_1 and Z_2.

(d) Derive expressions in terms of standard actuarial functions for the variance of both Z_1 and Z_2.

(e) Derive an expression in terms of standard actuarial functions for the covariance of Z_1 and Z_2.

(f) Assume now that $x = 40$. Calculate the standard deviation of both Z_1 and Z_2, using the Standard Ultimate Life Table, with UDD between integer ages, and interest at 5% per year.

Exercise 4.22 Consider a five-year deferred, 20-year endowment insurance policy issued to (40). The policy pays no benefit on death before age 45. A death benefit of \$100 000 is payable immediately on death between ages 45 and 65. On survival to age 65 a benefit of \$50 000 is payable.

Assume mortality follows the Standard Ultimate Life Table, with UDD between integer ages, and with interest at 5% per year.

(a) Write down an expression for the present value of the benefits in terms of T_{40}.

(b) Sketch a graph of the present value of the benefit as a function of the time of death. Clearly label the axes and show all key values.
(c) Calculate the EPV of the benefit.
(d) Calculate the probability that the present value of the benefit is more than \$16 000.
(e) Calculate the 60% quantile of the present value of the benefit.

Exercise 4.23 Assume Gompertz' law, $\mu_x = Bc^x$, with $B = 2.5 \times 10^{-5}$, and $c = 1.1$, with interest at 4% per year effective.

(a) Evaluate

$$\frac{d}{dt}\bar{A}^{\,1}_{50:\overline{t}|} \qquad \text{at } t = 10.$$

(b) Hence, or otherwise, evaluate

$$\frac{d}{dt}\,{}_{t|}\bar{A}_{50} \qquad \text{at } t = 10.$$

Exercise 4.24 Show that if $\nu_y = -\log p_y$ for $y = x, x+1, x+2, \ldots$, then under the assumption of a constant force of mortality between integer ages,

$$\bar{A}_x = \sum_{t=0}^{\infty} v^t \,{}_tp_x \,\frac{\nu_{x+t}(1 - vp_{x+t})}{\delta + \nu_{x+t}}.$$

Exercise 4.25 For three insurance policies on the same life, you are given:

(i) Z_1 is the present value of a 20-year term insurance with sum insured \$200, payable at the end of the year of death, with $\mathrm{E}[Z_1] = 6.6$ and $\mathrm{V}[Z_1] = 748$.
(ii) Z_2 is the present value of a 20-year deferred whole life insurance with sum insured \$1000, payable at the end of the year of death, with $\mathrm{E}[Z_2] = 107.5$ and $\mathrm{V}[Z_2] = 5078$.
(iii) Z_3 is the present value of a whole life insurance with sum insured \$2000, payable at the end of the year of death.

Calculate the mean and standard deviation of Z_3.

Exercise 4.26 (a) Show that

$$A_{x:\overline{n}|} = \sum_{k=0}^{n-2} v^{k+1}\,{}_{k|}q_x + v^n\,{}_{n-1}p_x.$$

(b) Compare this formula with formula (4.18) and comment on the differences.

Exercise 4.27 A life insurance policy issued to a life aged 50 pays \$2000 at the end of the quarter year of death before age 65 and \$1000 at the end of the quarter year of death after age 65. Use the Standard Ultimate Life Table, with UDD between integer ages, and assuming interest at 5% per year, in the following.

(a) Calculate the EPV of the benefit.
(b) Calculate the standard deviation of the present value of the benefit.
(c) The insurer charges a single premium of \$500. Assuming that the insurer invests all funds at exactly 5% per year effective, what is the probability that the policy benefit has greater value than the accumulation of the single premium?

Exercise 4.28 Let Z_1 denote the present value of an n-year term insurance benefit, issued to (x). Let Z_2 denote the present value of a whole of life insurance benefit, issued to the same life.

Express the covariance of Z_1 and Z_2 in actuarial functions, simplified as far as possible.

Exercise 4.29 Show that

$$(IA^{(m)})_x = A_x^{(m)} + v p_x A_{x+1}^{(m)} + v^2 {}_2p_x A_{x+2}^{(m)} + \cdots$$

and explain this result intuitively.

Exercise 4.30 (a) Derive the following recursion formula for an n-year increasing term insurance:

$$(IA)_{x:\overline{n|}}^{\,1} = vq_x + vp_x \left((IA)_{x+1:\overline{n-1|}}^{\,1} + A_{x+1:\overline{n-1|}}^{\,1} \right).$$

(b) Give an intuitive explanation of the formula in part (a).
(c) You are given that $(IA)_{50} = 4.99675$, $A_{50:\overline{1|}}^{\,1} = 0.00558$, $A_{51} = 0.24905$ and $i = 0.06$. Calculate $(IA)_{51}$.

Exercise 4.31 Assuming a uniform distribution of deaths over each year of age, find an expression for $(\bar{I}\bar{A})_x$ in terms of A_x and $(IA)_x$.

Exercise 4.32 A two-year term insurance is issued to (70), and the sum insured is payable immediately on death. The amount payable on death at time t is \$100 000(1.05^t), for $0 < t \leq 2$.

(a) Calculate the EPV of the benefit, assuming mortality follows the Standard Ultimate Life Table with interest at 5% per year.
(b) Calculate the EPV of the benefit, assuming $\mu_{70+t} = 0.012$ for $0 \leq t \leq 2$, with interest at 4% per year.

Excel-based exercises

Exercise 4.33 Suppose that Makeham's law applies with $A = 0.0001$, $B = 0.00035$ and $c = 1.075$. Assume also that the effective rate of interest is 6% per year.

(a) Use Excel and backward recursion in parts (i) and (ii).
  (i) Construct a table of values of A_x for integer ages, starting at $x = 50$.
  (ii) Construct a table of values of $A_x^{(4)}$ for $x = 50, 50.25, 50.5, \ldots$. (Do not use UDD for this.)
  (iii) Hence, write down the values of A_{50}, A_{100}, $A_{50}^{(4)}$ and $A_{100}^{(4)}$.

(b) Use your values for A_{50} and A_{100} to estimate $A_{50}^{(4)}$ and $A_{100}^{(4)}$ using the UDD assumption.

(c) Compare your estimated values for the $A^{(4)}$ functions (from part (b)) with your accurate values (from part (a)). Comment on the differences.

Exercise 4.34 The force of mortality for a survival model is given by

$$\mu_x = A + BC^x D^{x^2},$$

where

$$A = 3.5 \times 10^{-4},\ B = 5.5 \times 10^{-4},\ C = 1.00085,\ D = 1.0005.$$

Use the repeated Simpson's rule to calculate

(a) ${}_tp_{60}$ for $t = 0, 1/40, 2/40, \ldots, 2$, and

(b) $\bar{A}^{\,1}_{60:\overline{2|}}$ using an effective rate of interest of 5% per year.

Answers to selected exercises

4.1 (a) 0.735942 (b) 0.012656 (c) 0.138719 (d) 0.748974
4.2 (a) 0.00645 (b) 0.38163 (c) 0.05907
4.5 0.012494
4.6 0.01
4.7 0.887277
4.9 0.59704
4.10 (a) 7 186 (b) 6 226
4.11 (a) 5 507.44 (b) 5 507.46
4.13 56 270
4.14 0.3120
4.15 33 569.47
4.17 81 873
4.18 (a) 0.79267 (b) \$7 519.71 (c) 0.99825
4.19 (a) EPV = 190 693, SD = 123 938 (b) EPV = 190 718, SD = 124 009

4.20 (a) 44.492 (b) 43 657

4.21 (f) $\mathrm{SD}[Z_1]=1.6414$, $\mathrm{SD}[Z_2]=0.3085$

4.22 (c) 15 972 (d) 0.0448 (e) 14 765

4.23 (a) 0.004896 (b) −0.004896

4.25 $\mathrm{E}[Z_3] = 281$ $\mathrm{SD}[Z_3] = 258.33$

4.27 (a) \$218.88 (b) \$239.88 (c) 0.04054

4.30 (c) 5.07307

4.32 (a) 2196.19 (b) 2394.18

4.33 (a)(iii) 0.33587, 0.87508, 0.34330, 0.89647

(b) 0.34333, 0.89453

4.34 (a) Selected values are ${}_{1/4}p_{60} = 0.999031$, $p_{60} = 0.996049$ and ${}_2p_{60} = 0.991885$

(b) 0.007725

5

Annuities

5.1 Summary

In this chapter we derive expressions for the valuation and analysis of life contingent annuities. We consider different payment frequencies, and we relate the valuation of annuity benefits to the valuation of the related insurance benefits.

If full survival model information is available, then the valuation of benefits payable at discrete time points can be determined exactly, regardless of the payment frequency. However, where we are using only an integer age life table, a very common situation in practice, then some approximation is required when valuing benefits paid more frequently than annually. We derive several commonly used approximations, using the UDD assumption and Woolhouse's formula, and explore their accuracy numerically.

5.2 Introduction

A **life annuity** is a series of payments which are contingent on a specified individual's survival to each potential payment date. The payments are normally made at regular intervals and the most common situation is that the payments are of the same amount. The valuation of annuities is important as they appear in the calculation of premiums (see Chapter 6), policy values (see Chapter 7) and pension benefits (see Chapter 11). The present value of a life annuity is a random variable, as it depends on the uncertain future lifetime of the individual; however, we will use some results and notation from the valuation of **annuities-certain**, for which the payments are certain, with no life contingency, so we start with a review of these.

Recall that an **annuity-due** is an annuity with each payment made at the start of a time interval (e.g at the beginning of each month), and an **immediate annuity** is one where each payment is made at the end of a time interval (e.g. at the end of each month).

5.3 Review of annuities-certain

For integer n and effective interest rate $i > 0$ per year, $\ddot{a}_{\overline{n}|}$ denotes the present value of an annuity-due of 1 per year payable annually (in advance) for n years. We have

$$\begin{aligned}\ddot{a}_{\overline{n}|} &= 1 + v + v^2 + \cdots + v^{n-1} \\ &= \frac{1 - v^n}{d}.\end{aligned} \tag{5.1}$$

Also, for integer n, $a_{\overline{n}|}$ denotes the present value of an immediate annuity of 1 per year payable annually (in arrear) for n years, where

$$\begin{aligned}a_{\overline{n}|} &= v + v^2 + v^3 + \cdots + v^n = \ddot{a}_{\overline{n}|} - 1 + v^n \\ &= \frac{1 - v^n}{i}.\end{aligned}$$

Thirdly, for any $n > 0$, $\bar{a}_{\overline{n}|}$ denotes the present value of an annuity-certain payable continuously at a rate of 1 per year for n years, where

$$\bar{a}_{\overline{n}|} = \int_0^n v^t \, dt = \frac{1 - v^n}{\delta}. \tag{5.2}$$

When payments of 1 per year are made every $1/m$ years in advance for n years, in instalments of $1/m$, the present value is

$$\begin{aligned}\ddot{a}^{(m)}_{\overline{n}|} &= \frac{1}{m}\left(1 + v^{\frac{1}{m}} + v^{\frac{2}{m}} + \cdots + v^{n-\frac{1}{m}}\right) \\ &= \frac{1 - v^n}{d^{(m)}}.\end{aligned}$$

For payments made in arrears,

$$\begin{aligned}a^{(m)}_{\overline{n}|} &= \tfrac{1}{m}\left(v^{\frac{1}{m}} + v^{\frac{2}{m}} + \cdots + v^n\right) = \ddot{a}^{(m)}_{\overline{n}|} - \tfrac{1}{m}\left(1 - v^n\right) \\ &= \frac{1 - v^n}{i^{(m)}}.\end{aligned}$$

In the equations for $1/m$thly annuities, we assume that n is an integer multiple of $1/m$.

5.4 Annual life annuities

The annual life annuity is paid once each year, conditional on the survival of a life (the **annuitant**) to the payment date. If the annuity is to be paid throughout the annuitant's life, it is called a **whole life annuity**. If there is to be a specified maximum term, it is called a **term annuity** or **temporary annuity**.

Annual annuities are quite rare. We would more commonly see annuities payable monthly or even weekly. However, the annual annuity is still important in the situation where we do not have full information about mortality between integer ages, for example because we are working with an integer age life table. Also, the development of the valuation functions for the annual annuity is a good starting point before considering more complex payment patterns.

As with the insurance functions, we are primarily interested in the EPV of a cash flow, and we identify present value random variables for the annuity payments in terms of the future lifetime random variables from Chapters 2 and 4, specifically, T_x, K_x and $K_x^{(m)}$.

5.4.1 Whole life annuity-due

Consider first a whole life annuity-due with annual payments of 1 per year, which depend on the survival of a life currently aged x. The first payment occurs immediately, the second payment is made in one year from now, provided that (x) is alive then, and payments follow at annual intervals with each payment conditional on the survival of (x) to the payment date. In Figure 5.1 we show the payments and associated probabilities and discount functions in a time-line diagram.

If (x) dies between ages $x + k$ and $x + k + 1$, for some integer $k \geq 0$, then annuity payments would be made at times $0, 1, 2, \ldots, k$, for a total of $k + 1$ payments. From the definition of the curtate future lifetime K_x, we know that (x) dies between ages $x + K_x$ and $x + K_x + 1$, so the number of annuity payments is $K_x + 1$, including the initial payment. This means that the random variable representing the present value of the whole life annuity-due for (x) is $Y = \ddot{a}_{\overline{K_x+1}|}$, and using equation (5.1), we have

$$Y = \ddot{a}_{\overline{K_x+1}|} = \frac{1 - v^{K_x+1}}{d}.$$

The expected value of Y is denoted $\ddot{a}_x$.

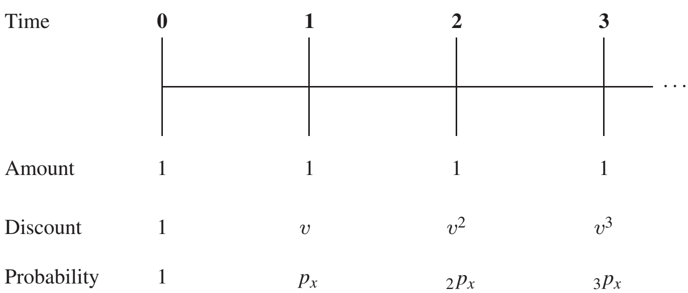


Figure 5.1 Time-line diagram for whole life annuity-due.

There are three useful ways to derive formulae for evaluating $\ddot{a}_x$ for a given survival model. We describe each of them below.

Using the insurance present value random variable, v^{K_x+1}

The expected value of Y is

$$\ddot{a}_x = \mathrm{E}\left[\frac{1-v^{K_x+1}}{d}\right] = \frac{1-\mathrm{E}[v^{K_x+1}]}{d}.$$

Using the mean of v^{K_x+1} which was derived in Section 4.4.2, we have

$$\boxed{\ddot{a}_x = \frac{1-A_x}{d}.} \tag{5.3}$$

Similarly, we can immediately obtain the variance of Y from the variance of v^{K_x+1} as

$$\begin{aligned} \mathrm{V}[Y] = \mathrm{V}\left[\frac{1-v^{K_x+1}}{d}\right] &= \frac{1}{d^2}\,\mathrm{V}\left[v^{K_x+1}\right] \\ &= \frac{{}^2A_x - A_x^2}{d^2}. \end{aligned} \tag{5.4}$$

Summing EPVs of individual payments

In Section 4.4.2 we stated that the EPV of any life contingent benefit can be found by considering each time point at which a benefit could be paid, and summing over all these time points the product of

(1) the amount of the benefit,
(2) the appropriate discount factor, and
(3) the probability that the benefit will be paid at that time.

For the annuity EPV, this approach is very helpful. Consider time t, where $t = 0, 1, 2, \ldots$. There will be an annuity payment at time t if (x) survives to time t. The amount of the annuity payment, in this case, is 1. The discount factor for a payment made at time t is v^t, and the probability that the payment is made is ${}_tp_x$, as it is contingent on the survival of (x) to age $x+t$. Note that $v^0 = 1$ and ${}_0p_x = 1$. Then, summing over all possible payment times, we have

$$\boxed{\ddot{a}_x = 1 + v\,p_x + v^2\,{}_2p_x + v^3\,{}_3p_x + \cdots = \sum_{t=0}^{\infty} v^t\,{}_tp_x.} \tag{5.5}$$

Equation (5.5) is the approach most commonly used in practice for evaluating $\ddot{a}_x$. However, it does not lead to useful expressions for higher moments of the present value random variable. Each term in equation (5.5) is the EPV of a single survival benefit of 1 at time t. That is, we can write the annuity present value random variable Y as the sum of pure endowment present random variables, Z_t, as

$$Y = Z_0 + Z_1 + Z_2 + \cdots ,$$

where

$$Z_t = \begin{cases} v^t & \text{if } K_x \geq t, \\ 0 & \text{if } K_x < t, \end{cases}$$

with

$$\mathrm{E}[Z_t] = v^t\,{}_tp_x \qquad \text{and} \qquad \mathrm{V}[Z_t] = v^{2t}\,{}_tp_x - \left(v^t\,{}_tp_x\right)^2 .$$

As the expected value of a sum of random variables is the sum of the expected values, we have

$$\mathrm{E}[Y] = \mathrm{E}[Z_1] + \mathrm{E}[Z_2] + \mathrm{E}[Z_3] + \cdots ,$$

which gives us (5.5). However, we cannot obtain the variance of a sum of random variables by summing the variances unless the random variables are independent, and in our case they are not. We can easily see this if we consider, say, Z_2 and Z_3. If we are given the information that $Z_2 = 0$, then we know that (x) died before time $t = 2$, which means we know that Z_3 (and all subsequent Z_t's) will also be zero, which means that the random variables are not independent.

Using the probability function for K_x

We know that $\Pr[K_x = k] = {}_{k|}q_x$, so that

$$\ddot{a}_x = \mathrm{E}\left[\ddot{a}_{\overline{K_x+1}|}\right] = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|}\,{}_{k|}q_x. \tag{5.6}$$

We can also use this approach to determine the variance of the annuity present value, although it does not reduce to a nice analytic form. We have

$$\mathrm{E}\left[(\ddot{a}_{\overline{K_x+1}|})^2\right] = \sum_{k=0}^{\infty} (\ddot{a}_{\overline{k+1}|})^2\,{}_{k|}q_x$$

and hence

$$\mathrm{V}\left[\ddot{a}_{\overline{K_x+1}|}\right] = \mathrm{E}\left[(\ddot{a}_{\overline{K_x+1}|})^2\right] - (\ddot{a}_x)^2.$$

Equation (5.6) is less often used in practice than equations (5.3) and (5.5). It is useful though to recognize the difference between the formulations for $\ddot{a}_x$ in equations (5.5) and (5.6). In equation (5.5) the summation is taken over all the possible payment times; in (5.6) the summation is taken over the possible years of death.

Example 5.1 Show that equations (5.5) and (5.6) are equivalent – that is, show that

$$\sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1|}}\ {}_{k|}q_x = \sum_{k=0}^{\infty} v^k\ {}_kp_x.$$

Solution 5.1 We can show this by using

$$\ddot{a}_{\overline{k+1|}} = \sum_{t=0}^{k} v^t$$

and

$$\sum_{k=t}^{\infty} {}_{k|}q_x = \sum_{k=t}^{\infty} ({}_kp_x - {}_{k+1}p_x) = {}_tp_x.$$

Then

$$\begin{aligned}\sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1|}}\ {}_{k|}q_x &= \sum_{k=0}^{\infty}\sum_{t=0}^{k} v^t\ {}_{k|}q_x \\ &= q_x + (1+v)\ {}_{1|1}q_x + (1+v+v^2)\ {}_{2|1}q_x \\ &\quad + (1+v+v^2+v^3)\ {}_{3|1}q_x + \cdots .\end{aligned}$$

Changing the order of summation on the right-hand side (that is, collecting together terms in powers of v) gives

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{t=0}^{k} v^t\ {}_{k|}q_x &= \sum_{t=0}^{\infty}\sum_{k=t}^{\infty} v^t\ {}_{k|}q_x \\ &= \sum_{t=0}^{\infty} v^t \sum_{k=t}^{\infty} {}_{k|}q_x \\ &= \sum_{t=0}^{\infty} v^t\ {}_tp_x\end{aligned}$$

as required. □

5.4.2 Term annuity-due

Now suppose we wish to value a term annuity-due of 1 per year. We assume the annuity is payable annually to a life now aged x for a maximum of n years.

Thus, payments are made at times $k = 0, 1, 2, \ldots, n-1$, provided that (x) has survived to age $x+k$. The present value of this annuity is Y, say, where

$$Y = \begin{cases} \ddot{a}_{\overline{K_x+1}|} & \text{if } K_x = 0, 1, \ldots, n-1, \\ \ddot{a}_{\overline{n}|} & \text{if } K_x \geq n. \end{cases}$$

That is

$$Y = \ddot{a}_{\overline{\min(K_x+1,\,n)}|} = \frac{1 - v^{\min(K_x+1,\,n)}}{d}.$$

The EPV of this annuity is denoted $\ddot{a}_{x:\overline{n}|}$.

We have seen the random variable $v^{\min(K_x+1,\,n)}$ before, in Section 4.4.7. It is the present value of a unit benefit under an n-year endowment insurance, with death benefit payable at the end of the year of death. Its EPV is $A_{x:\overline{n}|}$, so the EPV of the annuity is

$$\ddot{a}_{x:\overline{n}|} = \mathrm{E}[Y] = \frac{1 - \mathrm{E}[v^{\min(K_x+1,\,n)}]}{d},$$

that is,

$$\boxed{\ddot{a}_{x:\overline{n}|} = \frac{1 - A_{x:\overline{n}|}}{d}.} \tag{5.7}$$

The time-line for the term annuity-due cash flow is shown in Figure 5.2. Notice that, because the payments are made in advance, there is no payment due at time n, the end of the annuity term.

Using Figure 5.2, and summing the EPVs of the individual payments, we have

$$\boxed{\ddot{a}_{x:\overline{n}|} = 1 + v\,p_x + v^2\,{}_2p_x + v^3\,{}_3p_x + \cdots + v^{n-1}\,{}_{n-1}p_x = \sum_{t=0}^{n-1} v^t\,{}_tp_x.} \tag{5.8}$$

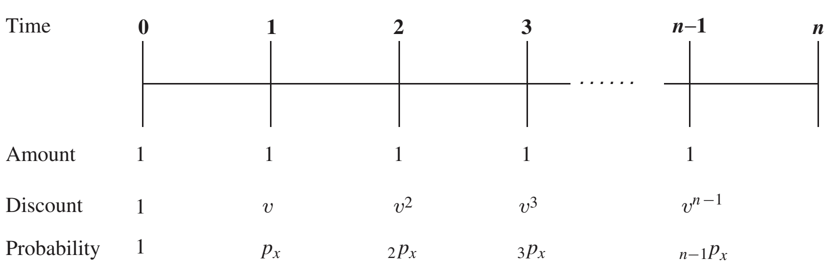


Figure 5.2 Time-line diagram for a term life annuity-due.

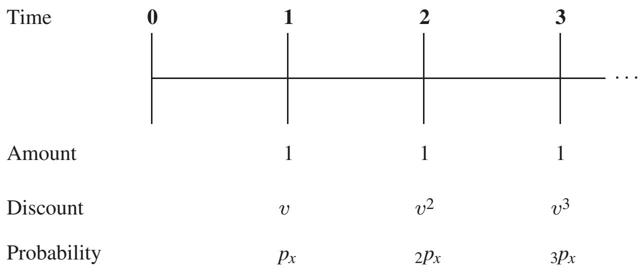


Figure 5.3 Time-line diagram for whole life immediate annuity.

Using the third approach from the previous section, we can adapt equation (5.6) to write the EPV as

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|}\ {}_{k|}q_x + {}_np_x\,\ddot{a}_{\overline{n}|}.$$

The second term here arises from the second term in the definition of Y – that is, if the annuitant survives for the full term, then the payments constitute an n-year annuity.

5.4.3 *Immediate life annuities*

A **whole life immediate annuity** of 1 per year, under which the payments are at the end of each year rather than the beginning, is illustrated in Figure 5.3. The actuarial notation for the EPV of this annuity is a_x.

We can see from the time-line that the difference in present value between the annuity-due and the immediate annuity is simply the first payment under the annuity-due, which is assumed to be paid at time $t = 0$, with certainty.

So, if Y is the random variable for the present value of the whole life annuity payable in advance, and Y^* is the random variable for the present value of the whole life annuity payable in arrear, we have $Y^* = Y - 1$, so that $\mathrm{E}[Y^*] = \mathrm{E}[Y] - 1$, and hence

$$a_x = \ddot{a}_x - 1. \tag{5.9}$$

Also, from equation (5.4) and the fact that $Y^* = Y - 1$, we have

$$\mathrm{V}[Y^*] = \mathrm{V}[Y] = \frac{{}^2A_x - A_x^2}{d^2}. \tag{5.10}$$

The EPV of an ***n*-year term immediate annuity** of 1 per year is denoted $a_{x:\overline{n}|}$. Under this annuity payments of 1 are made at times $k = 1, 2, \ldots, n$, conditional on the survival of the annuitant.

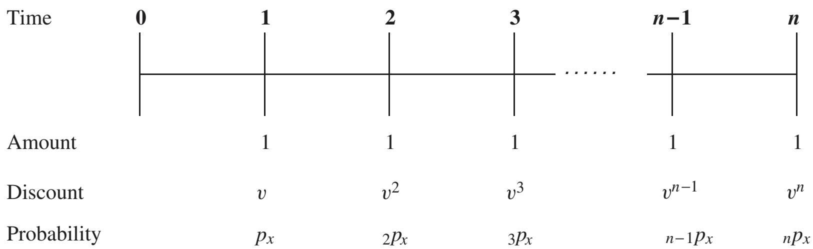


Figure 5.4 Time-line diagram for term life immediate annuity.

The random variable for the present value is

$$Y = a_{\overline{\min(K_x,n)}|},$$

and the time-line for the annuity cash flow is given in Figure 5.4.

Summing the EPVs of the individual payments, we have

$$\boxed{a_{x:\overline{n}|} = v\,p_x + v^2\,{}_2p_x + v^3\,{}_3p_x + \cdots + v^n\,{}_np_x = \sum_{t=1}^{n} v^t\,{}_tp_x.} \tag{5.11}$$

The difference between the annuity-due EPV, $\ddot{a}_{x:\overline{n}|}$, and the immediate annuity EPV, $a_{x:\overline{n}|}$, is found by differencing equations (5.8) and (5.11), to give

$$\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n}|} = 1 - v^n\,{}_np_x$$

so that

$$\boxed{a_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|} - 1 + v^n\,{}_np_x.} \tag{5.12}$$

The difference comes from the timing of the first payment under the annuity due and the last payment under the immediate annuity.

5.5 Annuities payable continuously

In practice annuities are payable at discrete time intervals, but if these intervals are short (say, weekly), it is convenient to treat payments as being made continuously. Consider an annuity which is payable at a rate of 1 per year as long as (x) survives. If the annuity is payable weekly (and we assume 52 weeks per year), then each week, the annuity payment is $1/52$. If payments were daily, for an annuity of 1 per year, the daily payment would be 1/365. Similarly, if the annuity is payable continuously, then for each infinitesimal interval $(t, t + dt)$ the payment under the annuity is dt.

A **continuous whole life annuity** of 1 per year payable to a life currently aged x, which ceases when (x) dies, has present value random variable

$$Y = \bar{a}_{\overline{T_x}|} = \frac{1 - v^{T_x}}{\delta}, \tag{5.13}$$

and the EPV of this annuity is denoted $\bar{a}_x$.

Analogous to the annual annuity-due, we can derive formulae for the EPV of the annuity in three different ways.

The first approach is to use the insurance present value random variable, v^{T_x}, which has expected value $\bar{A}_x$, so that

$$\boxed{\bar{a}_x = \frac{1 - \bar{A}_x}{\delta}.} \tag{5.14}$$

We can also derive the variance for the continuous annuity present value from the variance for the continuous insurance benefit

$$\mathrm{V}[Y] = \mathrm{V}\left[\frac{1 - v^{T_x}}{\delta}\right] = \frac{{}^2\bar{A}_x - \bar{A}_x^2}{\delta^2}.$$

The second approach is to integrate over all possible payment dates the product of the amount paid, the discount factor and the probability of payment. In the continuous case, the payment dates are infinitesimal intervals t to $t + dt$. For each interval, the amount paid is dt (as the total benefit over a whole year of payment is 1), the discount factor is $e^{-\delta t}$ and the probability of payment is ${}_tp_x$, giving

$$\boxed{\bar{a}_x = \int_0^\infty e^{-\delta t}\, {}_tp_x \, dt.} \tag{5.15}$$

The development of formula (5.15) is illustrated in Figure 5.5; we show the contribution to the integral from the contingent annuity payment made in an infinitesimal interval of time $(t, t + dt)$. The interval is so small that payments can be treated as being made exactly at time t.

Finally, we can directly write down the EPV from the distribution of T_x as

$$\bar{a}_x = \int_0^\infty \bar{a}_{\overline{t}|}\, {}_tp_x \, \mu_{x+t} \, dt.$$

We can evaluate this using integration by parts, noting that

$$\frac{d}{dt}\bar{a}_{\overline{t}|} = v^t = e^{-\delta t}.$$

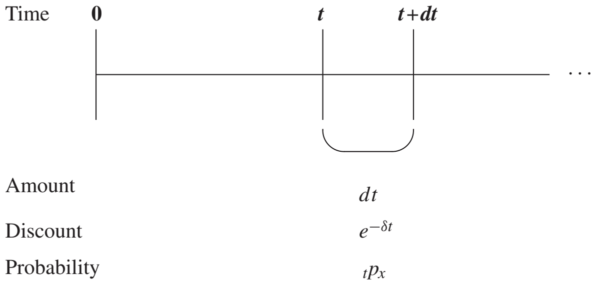


Figure 5.5 Time-line diagram for continuous whole life annuity.

So

$$\bar{a}_x = \int_0^\infty \bar{a}_{\overline{t}|} \frac{d}{dt}(-{}_tp_x)\,dt = -\left(\bar{a}_{\overline{t}|}\,{}_tp_x \Big|_0^\infty - \int_0^\infty {}_tp_x\, e^{-\delta t}\,dt \right)$$

$$= \int_0^\infty e^{-\delta t}\,{}_tp_x\,dt.$$

When $\delta = 0$, we see that $\bar{a}_x$ is equal to $\overset{\circ}{e}_x$, the complete expectation of life.

The **term continuous life annuity** present value random variable

$$\bar{a}_{\overline{\min(T_x,n)}|} = \frac{1 - v^{\min(T_x,n)}}{\delta}$$

has EPV denoted by $\bar{a}_{x:\overline{n}|}$. Analogous to the term annuity-due, we have three expressions for this EPV.

Using results for endowment insurance functions from Section 4.4.7, we have

$$\boxed{\bar{a}_{x:\overline{n}|} = \frac{1 - \bar{A}_{x:\overline{n}|}}{\delta}.} \tag{5.16}$$

Using the approach of integrating over all possible payment dates the product of the amount paid, the discount factor and the probability of payment, we have

$$\boxed{\bar{a}_{x:\overline{n}|} = \int_0^n e^{-\delta t}\,{}_tp_x\,dt,} \tag{5.17}$$

and taking the expected value of the present value random variable, using the probability density functon for T_x, we obtain

$$\boxed{\bar{a}_{x:\overline{n}|} = \int_0^n \bar{a}_{\overline{t}|}\ {}_tp_x\, \mu_{x+t}\, dt\ +\ \bar{a}_{\overline{n}|}\ {}_np_x.}$$

Similarly to the annuity-due case, the difference between the second and third approaches is that in the second approach we integrate over the possible payment dates, and in the third approach we integrate over the possible dates of death. The third approach is the least useful in practice.

5.6 Annuities payable 1/*m*thly

Premiums are commonly payable monthly, quarterly, or even weekly. Pension benefits and purchased annuities are payable with similar frequency to salary benefits, which means that weekly and monthly annuities are common. So it is useful to consider annuities which are payable in instalments every $1/m$ years, with each instalment contingent on the survival of the annuitant.

We define the present value of an annuity payable m times per year in terms of the $1/m$thly curtate future lifetime random variable, $K_x^{(m)}$. Recall that $K_x^{(m)}$ is the complete future lifetime rounded down to the lower $1/m$th of a year.

Recall also that $\ddot{a}^{(m)}_{\overline{n}|}$ is the present value of an annuity of 1 per year, payable each year in m instalments of $1/m$, for n years (certain), with first payment of $1/m$ at time $t = 0$ and final payment at time $n - \frac{1}{m}$. It is important to remember that $\ddot{a}^{(m)}_{\overline{n}|}$ is an **annual** factor, that is, it values a payment of 1 per year, and therefore for valuing annuities for other amounts, we need to multiply the $\ddot{a}^{(m)}_{\overline{n}|}$ factor by the **annual rate of annuity payment**, not the $1/m$thly rate.

Whole life 1/*m*thly annuity-due

Figure 5.6 shows the cash flow time-line diagram for an annuity of 1 per year, payable in advance m times per year throughout the lifetime of (x), paid in instalments of $1/m$. The present value random variable for this annuity is

$$Y = \ddot{a}^{(m)}_{\overline{K_x^{(m)}+\frac{1}{m}}|}.$$

It is probably easiest to understand this with an example. Suppose we are valuing an annuity-due of 1 per year, payable in quarterly instalments of 1/4 for the lifetime of (x).

Time	**0**	**1/m**	**2/m**	**3/m**	**4/m**	...
Amount	$1/m$	$1/m$	$1/m$	$1/m$	$1/m$	
Discount	1	$v^{1/m}$	$v^{2/m}$	$v^{3/m}$	$v^{4/m}$	
Probability	1	${}_{\frac{1}{m}}p_x$	${}_{\frac{2}{m}}p_x$	${}_{\frac{3}{m}}p_x$	${}_{\frac{4}{m}}p_x$	

Figure 5.6 Time-line diagram for whole life $1/m$thly annuity-due.

- If (x) dies in the first 1/4-year, then $K_x^{(4)} = 0$, and there will be a single annuity payment of 1/4 at time 0. The present value would be

$$\ddot{a}^{(4)}_{\overline{1/4}|} = \tfrac{1}{4}.$$

- If (x) dies in the second 1/4-year, then $K_x^{(4)} = 1/4$, and there will be two annuity payments, each of 1/4, made at times $t = 0$ and $t = 1/4$. The present value would be

$$\ddot{a}^{(4)}_{\overline{2/4}|} = \tfrac{1}{4} + \tfrac{1}{4}v^{\frac{1}{4}}.$$

- If (x) dies in the third 1/4-year, then $K_x^{(4)} = 2/4$, and there will be three annuity payments, each of 1/4, made at times $t = 0$, $t = 1/4$ and $t = 2/4$. The present value would be

$$\ddot{a}^{(4)}_{\overline{3/4}|} = \tfrac{1}{4} + \tfrac{1}{4}v^{\frac{1}{4}} + \tfrac{1}{4}v^{\frac{2}{4}}.$$

- Continuing this reasoning, if (x) dies in the kth 1/4-year (for $k = 1, 2, 3\ldots$), then $K_x^{(4)} = (k-1)/4$, and there will be k annuity payments, with present value $\ddot{a}^{(4)}_{\overline{k/4}|}$. So, whatever the value of $K_x^{(4)}$, the term of this annuity-due is $K_x^{(4)} + \frac{1}{4}$, and its present value is $\ddot{a}^{(4)}_{\overline{K_x^{(4)}+\frac{1}{4}}|}$.

The EPV of the $1/m$thly whole life annuity-due is denoted by $\ddot{a}_x^{(m)}$ and is given by

$$\mathrm{E}\left[\ddot{a}^{(m)}_{\overline{K_x^{(m)}+\frac{1}{m}}|}\right] = \frac{1 - \mathrm{E}\left[v^{K_x^{(m)}+\frac{1}{m}}\right]}{d^{(m)}},$$

and since $v^{K_x^{(m)}+\frac{1}{m}}$ is the present value of a death benefit of 1 payable at the end of the $1/m$th year of death, we have

$$\boxed{\ddot{a}_x^{(m)} = \frac{1 - A_x^{(m)}}{d^{(m)}}.} \tag{5.18}$$

Using the sum of individual EPVs approach, we have

$$\boxed{\ddot{a}_x^{(m)} = \sum_{r=0}^{\infty} \frac{1}{m}\, v^{\frac{r}{m}}\, {}_{\frac{r}{m}}p_x.} \tag{5.19}$$

Whole life $1/m$thly immediate annuity

We can derive the EPV of the $1/m$thly immediate annuity (i.e. payable at the end of each $1/m$ years) from the EPV of the annuity-due, as the only difference in the whole life case is the first payment of $1/m$. The EPV of the $1/m$thly immediate whole life annuity of 1 per year is denoted $a_x^{(m)}$, so

$$a_x^{(m)} = \ddot{a}_x^{(m)} - \tfrac{1}{m}. \tag{5.20}$$

Term life $1/m$thly annuity-due

For a $1/m$thly term life annuity-due, we have payments as illustrated in Figure 5.7.

The present value random variable for this annuity is

$$\ddot{a}^{(m)}_{\overline{H(x,n)}|} = \frac{1 - v^{H(x,n)}}{d^{(m)}} \qquad \text{where} \quad H(x,n) = \min\left(K_x^{(m)} + \tfrac{1}{m},\, n\right).$$

We know from the previous chapter that $v^{H(x,n)}$ is the present value of an n-year endowment insurance, with sum insured 1, with the death benefit payable at the end of the $1/m$th year of death.

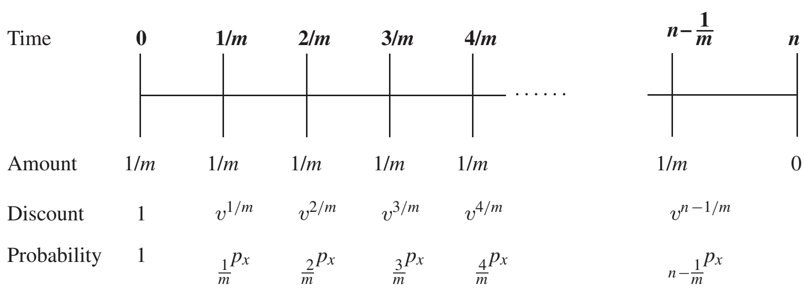


Figure 5.7 Time-line diagram for term life $1/m$thly annuity-due.

The EPV of the $1/m$thly n-year term annuity-due of 1 per year issued to (x) is denoted by $\ddot{a}^{(m)}_{x:\overline{n}|}$ and so

$$\boxed{\ddot{a}^{(m)}_{x:\overline{n}|} = \frac{1 - A^{(m)}_{x:\overline{n}|}}{d^{(m)}}.} \tag{5.21}$$

Using the sum of individual payment EPVs approach, we have

$$\boxed{\ddot{a}^{(m)}_{x:\overline{n}|} = \sum_{r=0}^{mn-1} \frac{1}{m} v^{r/m} \; {}_{\frac{r}{m}}p_x.} \tag{5.22}$$

Term life immediate annuity payable $1/m$thly

We can use the EPV of the $1/m$thly term annuity-due to determine the EPV for a $1/m$thly term immediate annuity; the difference is the first payment under the annuity-due, with EPV $1/m$, and the final payment under the immediate annuity, with EPV $\frac{1}{m} v^n {}_np_x$. So

$$a^{(m)}_{x:\overline{n}|} = \ddot{a}^{(m)}_{x:\overline{n}|} - \frac{1}{m}\left(1 - v^n \, {}_np_x\right). \tag{5.23}$$

Note that, by setting $m = 1$ in equations (5.19) and (5.22) we obtain equations (5.5) and (5.8) for $\ddot{a}_x$ and $\ddot{a}_{x:\overline{n}|}$. Also, by letting $m \to \infty$ in equations (5.19) and (5.22) we obtain equations (5.15) and (5.17) for continuous annuities, $\bar{a}_x$ and $\bar{a}_{x:\overline{n}|}$.

5.7 Comparison of annuities by payment frequency

In Table 5.1 we show values for a_x, $a^{(4)}_x$, $\ddot{a}_x$, $\ddot{a}^{(4)}_x$ and $\bar{a}_x$ for $x = 20$, 40, 60 and 80, using the Standard Ultimate Survival Model with interest of 5% per year.

We observe that each set of values decreases with age, reflecting the shorter expected life span as age increases. We also have, for each age, the ordering

$$a_x < a^{(4)}_x < \bar{a}_x < \ddot{a}^{(4)}_x < \ddot{a}_x.$$

There are two reasons for this ordering.

- While the life is alive, the payments in each year sum to 1 under each annuity, but on average, the payments under the annuity-due are paid earlier. The time value of money means that the value of an annuity with earlier payments will be higher than an annuity with later payment (provided the interest rate is greater than zero), so the annuity values are in increasing order from the latest average payment date (a_x payments are at each year end) to the earliest ($\ddot{a}_x$ payments are at the start of each year).

Table 5.1 *Values of* a_x , $a_x^{(4)}$, $\bar{a}_x$, $\ddot{a}_x^{(4)}$ *and* $\ddot{a}_x$.

x	a_x	$a_x^{(4)}$	$\bar{a}_x$	$\ddot{a}_x^{(4)}$	$\ddot{a}_x$
20	18.966	19.338	19.462	19.588	19.966
40	17.458	17.829	17.954	18.079	18.458
60	13.904	14.275	14.400	14.525	14.904
80	7.548	7.917	8.042	8.167	8.548

- In the year that (x) dies, the different annuities pay different amounts. Under the annual annuity-due the full year's payment of 1 is paid, as the life is alive at the payment date at the start of the year. Under the annual immediate annuity, in the year of death no payment is made as the life does not survive to the payment date at the year end. For the $1/m$thly and continuous annuities, less than the full year's annuity may be paid in the year of death.

  For example, suppose the life dies after seven months. Under the annual annuity-due, the full annuity payment is made for that year, at the start of the year. Under the quarterly annuity-due, three payments are made, each of 1/4 of the total annual amount, at times 0, 1/4 and 1/2. The first year's final payment, due at time 3/4, is not made, as the life does not survive to that date. Under the continuous annuity, the life collects 7/12ths of the annual amount. Under the quarterly immediate annuity, the life collects payments at times 1/4, 1/2, and misses the two payments due at times 3/4 and 1. Under the annual immediate annuity, the life collects no annuity payments at all, as the due date is the year end.

This second point explains why we cannot make a simple interest adjustment to relate the annuity-due and the continuous annuity. The situation here is different from the insurance benefits; A_x and $A_x^{(4)}$, for example, both value a payment of 1 in the year of death: A_x at the end of the year, and $A_x^{(4)}$ at the end of the quarter year of death. There is no difference in the amount of the payment, only in the timing. But for the annuities, the difference between $\ddot{a}_x$ and $\ddot{a}_x^{(4)}$ arises from differences in both cash flow timing and benefit amount in the year of death.

We also note from Table 5.1 that the $\bar{a}_x$ values are close to being half-way between a_x and $\ddot{a}_x$, suggesting the approximation $\bar{a}_x \approx a_x + \frac{1}{2}$. We will see in Section 5.11.3 that there is indeed a way of calculating an approximation to $\bar{a}_x$ from a_x, but it involves an extra adjustment term to a_x.

Example 5.2 Using the Standard Ultimate Survival Model, with 5% per year interest, calculate values of $a_{x:\overline{10}|}$, $a^{(4)}_{x:\overline{10}|}$, $\ddot{a}_{x:\overline{10}|}$, $\ddot{a}^{(4)}_{x:\overline{10}|}$ and $\bar{a}_{x:\overline{10}|}$ for $x = 20, 40, 60$ and 80, and comment.

Table 5.2 *Values of* $a_{x:\overline{10}|}$, $a^{(4)}_{x:\overline{10}|}$, $\bar{a}_{x:\overline{10}|}$, $\ddot{a}^{(4)}_{x:\overline{10}|}$ *and* $\ddot{a}_{x:\overline{10}|}$.

x	$a_{x:\overline{10}\|}$	$a^{(4)}_{x:\overline{10}\|}$	$\bar{a}_{x:\overline{10}\|}$	$\ddot{a}^{(4)}_{x:\overline{10}\|}$	$\ddot{a}_{x:\overline{10}\|}$
20	7.711	7.855	7.904	7.952	8.099
40	7.696	7.841	7.889	7.938	8.086
60	7.534	7.691	7.743	7.796	7.956
80	6.128	6.373	6.456	6.539	6.789

Solution 5.2 Using equations (5.8), (5.11), (5.17), (5.23) and (5.22), with $n = 10$ we obtain the values shown in Table 5.2.

We note that for a given annuity function, the values do not vary greatly with age, since the probability of death in a 10-year period is small. That means, for example, that the second term in equation (5.8) is much greater than the first term. The present value of an annuity certain provides an upper bound for each set of values. For example, for any age x, $a_{x:\overline{10}|} < a_{\overline{10}|} = 7.722$ and $\ddot{a}^{(4)}_{x:\overline{10}|} < \ddot{a}^{(4)}_{\overline{10}|} = 7.962$.

Due to the differences in timing of payments, and in amounts for lives who die during the 10-year annuity term, we have the same ordering of annuity values by payment frequency for any age x:

$$a_{x:\overline{10}|} < a^{(4)}_{x:\overline{10}|} < \bar{a}_{x:\overline{10}|} < \ddot{a}^{(4)}_{x:\overline{10}|} < \ddot{a}_{x:\overline{10}|}.$$

□

5.8 Deferred annuities

A deferred annuity is an annuity under which the first payment occurs at some specified future time. Consider an annuity payable to an individual now aged x under which annual payments of 1 will commence at age $x + u$, where u is an integer, and will continue until the death of (x). This is an annuity-due deferred u years. In standard actuarial notation, the EPV of this annuity is denoted by ${}_{u|}\ddot{a}_x$. Recall that we have used the format ${}_{u|}\ldots$ to indicate deferment before, both for mortality probabilities (${}_{u|t}q_x$) and for insurance benefits (${}_{u|}A_x$). Figure 5.8 shows the time-line for a u-year deferred annuity-due.

Combining Figure 5.8 with the time-line for a u-year term annuity, see Figure 5.2, we can see that the combination of the payments under a u-year term annuity-due and a u-year deferred annuity-due gives the same sequence of payments as under a lifetime annuity in advance, so we obtain

$$\ddot{a}_{x:\overline{u}|} + {}_{u|}\ddot{a}_x = \ddot{a}_x, \tag{5.24}$$

or, equivalently,

$$\boxed{{}_{u|}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{u}|}.} \tag{5.25}$$

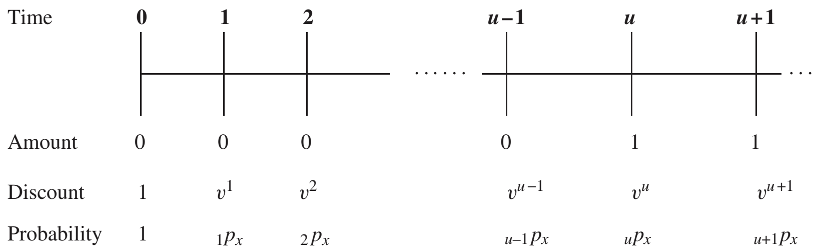


Figure 5.8 Time-line diagram for deferred annual annuity-due.

Similarly, the EPV of an annuity payable continuously at rate 1 per year to a life now aged x, commencing at age $x+u$, is denoted by ${}_{u|}\bar{a}_x$ and given by

$${}_{u|}\bar{a}_x = \bar{a}_x - \bar{a}_{x:\overline{u}|}.$$

Summing the EPVs of the individual payments for the deferred whole life annuity-due gives

$$\begin{aligned}{}_{u|}\ddot{a}_x &= v^u\,{}_up_x + v^{u+1}\,{}_{u+1}p_x + v^{u+2}\,{}_{u+2}p_x + \cdots \\ &= v^u\,{}_up_x\big(1 + v\,p_{x+u} + v^2\,{}_2p_{x+u} + \cdots\big)\end{aligned}$$

so that

$${}_{u|}\ddot{a}_x = v^u\,{}_up_x\,\ddot{a}_{x+u} = {}_uE_x\,\ddot{a}_{x+u}. \tag{5.26}$$

We see again that the pure endowment function acts like a discount function. In fact, we can use the ${}_uE_x$ function to find the EPV of any deferred benefit. For example, for a deferred term immediate annuity,

$${}_{u|}a_{x:\overline{n}|} = {}_uE_x\,a_{x+u:\overline{n}|},$$

and for an annuity-due payable $1/m$thly,

$${}_{u|}\ddot{a}^{(m)}_x = {}_uE_x\,\ddot{a}^{(m)}_{x+u}. \tag{5.27}$$

This result can be helpful when working with tables. Suppose we have available a table of whole life annuity-due values, say $\ddot{a}_x$, along with the life table function l_x, and we need the term annuity value $\ddot{a}_{x:\overline{n}|}$. Then, using equations (5.24) and (5.26), we have

$$\ddot{a}_{x:\overline{n}|} = \ddot{a}_x - {}_nE_x\,\ddot{a}_{x+n}. \tag{5.28}$$

For $1/m$thly payments, the corresponding formula is

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \ddot{a}^{(m)}_x - {}_nE_x\,\ddot{a}^{(m)}_{x+n}. \tag{5.29}$$

Example 5.3 Let Y_1, Y_2 and Y_3 denote present value random variables for a u-year deferred whole life annuity-due, a u-year term annuity-due and a whole life annuity-due, respectively. Show that $Y_3 = Y_1 + Y_2$. Assume annual payments.

Solution 5.3 The present value random variable for a u-year deferred whole life annuity-due, with annual payments is

$$Y_1 = \begin{cases} 0 & \text{if } K_x \leq u-1, \\ v^u \ddot{a}_{\overline{K_x+1-u}|} & \text{if } K_x \geq u, \end{cases}$$

$$= \begin{cases} 0 & \text{if } K_x \leq u-1, \\ \ddot{a}_{\overline{K_x+1}|} - \ddot{a}_{\overline{u}|} & \text{if } K_x \geq u. \end{cases} \tag{5.30}$$

From Section 5.4.2 we have

$$Y_2 = \begin{cases} \ddot{a}_{\overline{K_x+1}|} & \text{if } K_x \leq u-1, \\ \ddot{a}_{\overline{u}|} & \text{if } K_x \geq u. \end{cases}$$

Hence

$$Y_1 + Y_2 = \begin{cases} \ddot{a}_{\overline{K_x+1}|} & \text{if } K_x \leq u-1, \\ \ddot{a}_{\overline{K_x+1}|} & \text{if } K_x \geq u, \end{cases} \qquad = \ddot{a}_{\overline{K_x+1}|} \qquad = Y_3,$$

as required. □

We use deferred annuities as building blocks in later sections, noting that an n-year term annuity, with any payment frequency, can be decomposed as the sum of n deferred annuities, each with term 1 year. So, for example,

$$\bar{a}_{x:\overline{n}|} = \sum_{u=0}^{n-1} {}_{u|}\bar{a}_{x:\overline{1}|} \,. \tag{5.31}$$

5.9 Guaranteed annuities

A common feature of pension benefits is that the pension annuity is guaranteed to be paid for some period even if the life dies before the end of the period. For example, a pension benefit payable to a life aged 65, might be guaranteed for 5, 10 or even 15 years.

Suppose an annuity-due of 1 per year is payable annually to (x), and is guaranteed for a period of n years. Then the payment due at k years is paid whether or not (x) is then alive if $k = 0, 1, \ldots, n-1$, but is paid only if (x) is alive at age $x + k$ for $k = n, n+1, \ldots$. The present value random variable for this benefit is

$$
\begin{aligned}
Y &= \begin{cases} \ddot{a}_{\overline{n}|} & \text{if } K_x \le n-1, \\ \ddot{a}_{\overline{K_x+1}|} & \text{if } K_x \ge n, \end{cases} \\
&= \begin{cases} \ddot{a}_{\overline{n}|} & \text{if } K_x \le n-1, \\ \ddot{a}_{\overline{n}|} + \ddot{a}_{\overline{K_x+1}|} - \ddot{a}_{\overline{n}|} & \text{if } K_x \ge n, \end{cases} \\
&= \ddot{a}_{\overline{n}|} + \begin{cases} 0 & \text{if } K_x \le n-1, \\ \ddot{a}_{\overline{K_x+1}|} - \ddot{a}_{\overline{n}|} & \text{if } K_x \ge n, \end{cases} \\
&= \ddot{a}_{\overline{n}|} + Y_1\,,
\end{aligned}
$$

where Y_1 denotes the present value of an n-year deferred annuity-due of 1 per year, from equation (5.30), and

$$\mathrm{E}[Y_1] = {}_{n|}\ddot{a}_x = {}_nE_x\,\ddot{a}_{x+n}.$$

The EPV of the unit n-year guaranteed annuity-due is denoted $\ddot{a}_{\overline{\overline{x:\overline{n}|}}}$, so

$$\ddot{a}_{\overline{\overline{x:\overline{n}|}}} = \ddot{a}_{\overline{n}|} + {}_nE_x\,\ddot{a}_{x+n}. \tag{5.32}$$

Figure 5.9 shows the time-line for an n-year guaranteed unit whole life annuity-due. This time-line looks like the regular whole life annuity-due time-line, except that the first n payments, from time $t = 0$ to time $t = n-1$, are certain and not life contingent.

We can derive similar results for guaranteed benefits payable $1/m$thly; for example, a monthly whole life annuity-due guaranteed for n years has EPV

$$\ddot{a}^{(12)}_{\overline{\overline{x:\overline{n}|}}} = \ddot{a}^{(12)}_{\overline{n}|} + {}_nE_x\,\ddot{a}^{(12)}_{x+n}.$$

Example 5.4 A pension plan member is entitled to a benefit of \$1000 per month in advance, for life from age 65, with no guarantee. She can opt to take a lower benefit, with a 10-year guarantee. The revised benefit is calculated to have equal EPV at age 65 to the original benefit. Calculate the revised benefit using the Standard Ultimate Survival Model, with interest at 5% per year.

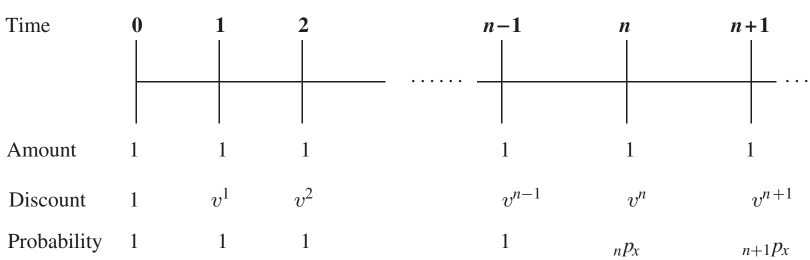


Figure 5.9 Time-line diagram for guaranteed annual annuity-due.

Solution 5.4 Let B denote the revised monthly benefit. To determine B we must equate the EPV of the original benefit with that of the revised benefit. The resulting equation of EPVs is usually called an **equation of value**. Our equation of value is

$$12\,000\,\ddot{a}_{65}^{(12)} = 12\,B\,\ddot{a}_{\overline{65:\overline{10}|}}^{(12)}$$

where $\ddot{a}_{65}^{(12)} = 13.0870$, and

$$\ddot{a}_{\overline{65:\overline{10}|}}^{(12)} = \ddot{a}_{\overline{10}|}^{(12)} + {}_{10}E_{65}\,\ddot{a}_{75}^{(12)} = 13.3791.$$

Thus, the revised monthly benefit is $B = \$978.17$. So the pension plan member can gain the security of the 10-year guarantee at a cost of a reduction of \$21.83 per month in her pension. □

5.10 Increasing annuities

In previous sections we have considered annuities with level payments. Some of the annuities which arise in actuarial work are not level. For example, annuity payments may increase over time. For these annuities, we are generally interested in determining the EPV, and are rarely concerned with higher moments. To calculate higher moments it is generally necessary to use first principles, and a computer.

The best approach for calculating the EPV of non-level annuities is to sum over all the payment dates the product of the amount of the payment, the probability of payment (that is, the probability that the life survives to the payment date) and the appropriate discount factor.

5.10.1 Arithmetically increasing annuities

We first consider annuities under which the amount of the annuity payment increases arithmetically with time. Consider an increasing annuity-due where the amount of the annuity is $t + 1$ at times $t = 0, 1, 2, \ldots$ provided that (x) is alive at time t. The time-line is shown in Figure 5.10.

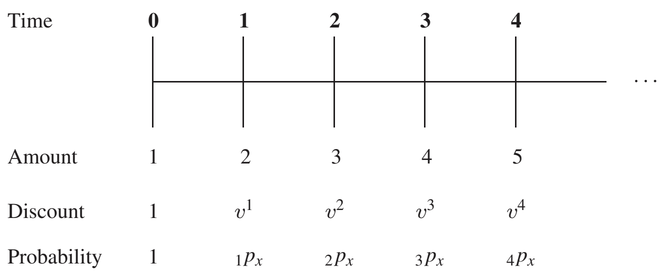


Figure 5.10 Time-line diagram for arithmetically increasing annual annuity-due.

The EPV of the annuity is denoted by $(I\ddot{a})_x$ in standard actuarial notation. From the diagram we see that

$$\boxed{(I\ddot{a})_x = \sum_{t=0}^{\infty} v^t (t+1)\ {}_tp_x\ .} \tag{5.33}$$

Similarly, if the annuity is payable for a maximum of n payments rather than for the whole life of (x), the EPV, denoted by $(I\ddot{a})_{x:\overline{n}|}$ in standard actuarial notation, is given by

$$\boxed{(I\ddot{a})_{x:\overline{n}|} = \sum_{t=0}^{n-1} v^t (t+1)\ {}_tp_x\ .} \tag{5.34}$$

If the annuity is payable continuously, with the payments increasing by 1 at each year end, so that the rate of payment in the tth year is constant and equal to t, for $t = 1, 2, \ldots, n$, then we may consider the n-year temporary annuity as a sum of one-year deferred annuities. By analogy with formula (5.31), the EPV of this annuity, denoted in standard actuarial notation by $(I\bar{a})_{x:\overline{n}|}$, is

$$(I\bar{a})_{x:\overline{n}|} = \sum_{m=0}^{n-1} (m+1)\ {}_{m|}\bar{a}_{x:\overline{1}|}\ .$$

We also have standard actuarial notation for the continuous annuity under which the rate of payment at time $t > 0$ is t; that is, the rate of payment is changing continuously. The notation for the EPV of this annuity is $(\bar{I}\bar{a})_x$ if it is a whole life annuity, and $(\bar{I}\bar{a})_{x:\overline{n}|}$ if it is a term annuity. For every infinitesimal interval, $(t, t+dt)$, the amount of annuity paid, if the life (x) is still alive, is $t\,dt$, the probability of payment is ${}_tp_x$ and the discount function is $e^{-\delta t} = v^t$. The time-line is shown in Figure 5.11.

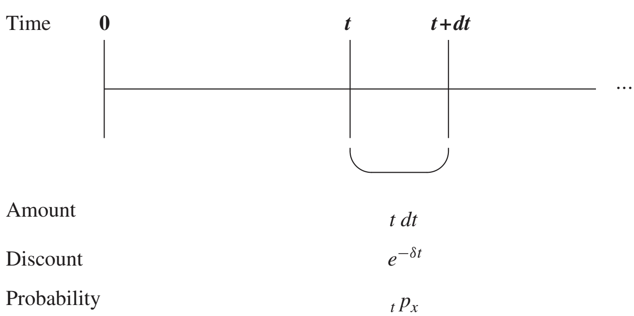


Figure 5.11 Time-line diagram for arithmetically increasing continuous whole life annuity.

To determine the EPV we integrate over all the possible intervals $(t, t+dt)$, so that

$$\boxed{(\bar{I}\bar{a})_{x:\overline{n}|} = \int_0^n t\, e^{-\delta t}\, {}_tp_x\, dt.} \tag{5.35}$$

5.10.2 Geometrically increasing annuities

An annuitant may be interested in purchasing an annuity that increases geometrically, to offset the effect of inflation on the purchasing power of the income. The approach is similar to the geometrically increasing insurance benefit which was considered in Examples 4.8 and 4.9.

Example 5.5 Consider an annuity-due with annual payments where the amount of the annuity is $(1+j)^t$ at times $t = 0, 1, 2, \ldots, n-1$ provided that (x) is alive at time t. Derive an expression for the EPV of this benefit, and simplify as far as possible.

Solution 5.5 First, consider the time-line diagram in Figure 5.12.

By summing the product of

- the amount of the payment at time t,
- the discount factor for time t, and
- the probability that the payment is made at time t,

over all possible values of t, we obtain the EPV as

$$\sum_{t=0}^{n-1} (1+j)^t\, v^t\, {}_tp_x = \ddot{a}_{x:\overline{n}|i^*},$$

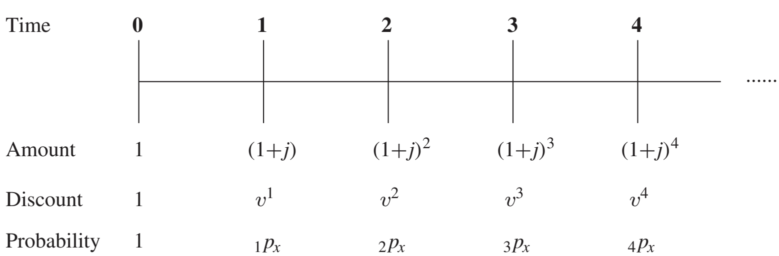


Figure 5.12 Time-line diagram for geometrically increasing annual annuity-due.

where $\ddot{a}_{x:\overline{n}|i^*}$ is the EPV of a term annuity-due evaluated at interest rate i^* where

$$i^* = \frac{1+i}{1+j} - 1 = \frac{i-j}{1+j}.$$

□

5.11 Evaluating annuity functions

If we have full information about the survival function for a life, then we can use summation or numerical integration to compute the EPV of any annuity. Often, though, we have only integer age information, for example when the survival function information is derived from a life table. In this section we consider how to evaluate the EPV of $1/m$thly and continuous annuities given only the EPVs of annuities at integer ages. Typically, we would have tables of $\ddot{a}_x$ values for integer x, as well as the life table function, l_x. We present two methods that are commonly used for estimating the EPV of $1/m$thly annuities from annual values, and we explore the accuracy of these methods for a fairly typical (Makeham) mortality model. First we consider recursive calculation of EPVs of annuities.

5.11.1 Recursions

In a spreadsheet, given values for ${}_tp_x$, we may calculate $\ddot{a}_x$ using a backward recursion. We assume that there is an integer limiting age, ω, so that $q_{\omega-1} = 1$. First, we set $\ddot{a}_{\omega-1} = 1$. The backward recursion for $x = \omega-2, \omega-3, \omega-4, \ldots$ is

$$\boxed{\ddot{a}_x = 1 + v\,p_x\,\ddot{a}_{x+1}} \tag{5.36}$$

since

$$\begin{aligned}\ddot{a}_x &= 1 + v\,p_x + v^2\,{}_2p_x + v^3\,{}_3p_x + \cdots \\ &= 1 + v\,p_x\big(1 + v\,p_{x+1} + v^2\,{}_2p_{x+1} + \cdots\big) \\ &= 1 + v\,p_x\,\ddot{a}_{x+1}.\end{aligned}$$

Similarly, for the $1/m$thly annuity-due,

$$\ddot{a}^{(m)}_{\omega-1/m} = \tfrac{1}{m},$$

and the backward recursion for $x = \omega - \frac{2}{m}, \omega - \frac{3}{m}, \omega - \frac{4}{m}, \ldots$ is

$$\ddot{a}^{(m)}_x = \frac{1}{m} + v^{\frac{1}{m}}\;{}_{\frac{1}{m}}p_x\;\ddot{a}^{(m)}_{x+\frac{1}{m}}. \tag{5.37}$$

We can calculate EPVs for term annuities and deferred annuities from the whole life annuity EPVs, using, for example, equations (5.24) and (5.26).

To find the EPV of an annuity payable continuously we can use numerical integration. Note, however, that Woolhouse's formula, which is described in Section 5.11.3, gives an excellent approximation to $1/m$thly and continuous annuity EPVs.

5.11.2 Applying the UDD assumption

We consider the evaluation of $\ddot{a}^{(m)}_{x:\overline{n}|}$ under the assumption of a uniform distribution of deaths (UDD). The indication from Table 4.6 is that, in terms of EPVs for insurance benefits, UDD offers a reasonable approximation at younger ages, but may not be sufficiently accurate at older ages.

From Section 4.5.1 recall the results from equations (4.27) and (4.26) that, under the UDD assumption,

$$A^{(m)}_x = \frac{i}{i^{(m)}} A_x \qquad \text{and} \qquad \bar{A}_x = \frac{i}{\delta} A_x.$$

We also know, from equations (5.3), (5.18) and (5.14) that for any survival model,

$$\ddot{a}_x = \frac{1 - A_x}{d}, \quad \ddot{a}^{(m)}_x = \frac{1 - A^{(m)}_x}{d^{(m)}} \quad \text{and} \quad \bar{a}_x = \frac{1 - \bar{A}_x}{\delta}.$$

Now, putting these equations together we have

$$\begin{aligned}
\ddot{a}^{(m)}_x &= \frac{1 - A^{(m)}_x}{d^{(m)}} \\
&= \frac{1 - \frac{i}{i^{(m)}} A_x}{d^{(m)}} \qquad \text{using UDD} \\
&= \frac{i^{(m)} - iA_x}{i^{(m)}\, d^{(m)}} \\
&= \frac{i^{(m)} - i(1 - d\ddot{a}_x)}{i^{(m)}\, d^{(m)}} \qquad \text{using (5.3)} \\
&= \frac{id}{i^{(m)}\, d^{(m)}}\, \ddot{a}_x - \frac{i - i^{(m)}}{i^{(m)}\, d^{(m)}} \\
&= \alpha(m)\, \ddot{a}_x - \beta(m)
\end{aligned}$$

where

$$\alpha(m) = \frac{id}{i^{(m)}\, d^{(m)}} \qquad \text{and} \qquad \beta(m) = \frac{i - i^{(m)}}{i^{(m)}\, d^{(m)}}. \tag{5.38}$$

For continuous annuities, we can let $m \rightarrow \infty$, noting that $\lim_{m\to\infty} i^{(m)} = \lim_{m\to\infty} d^{(m)} = \delta$, so that

$$\bar{a}_x = \frac{id}{\delta^2}\,\ddot{a}_x - \frac{i-\delta}{\delta^2}.$$

For term annuities, starting from equation (5.29) we have,

$$\begin{aligned}
\ddot{a}^{(m)}_{x:\overline{n}|} &= \ddot{a}^{(m)}_x - {}_nE_x\,\ddot{a}^{(m)}_{x+n} \\
&= \alpha(m)\ddot{a}_x - \beta(m) - {}_nE_x\left(\alpha(m)\ddot{a}_{x+n} - \beta(m)\right) \qquad \text{using UDD} \\
&= \alpha(m)\,(\ddot{a}_x - {}_nE_x\,\ddot{a}_{x+n}) - \beta(m)\,(1 - {}_nE_x) \\
&= \alpha(m)\,\ddot{a}_{x:\overline{n}|} - \beta(m)\,(1 - {}_nE_x)\,. \qquad (5.39)
\end{aligned}$$

Note that the functions $\alpha(m)$ and $\beta(m)$ depend only on the frequency of the payments, not on the underlying survival model. It can be shown (see Exercise 5.19) that $\alpha(m) \approx 1$ and $\beta(m) \approx \frac{m-1}{2m}$, leading to the approximation

$$\ddot{a}^{(m)}_{x:\overline{n}|} \approx \ddot{a}_{x:\overline{n}|} - \tfrac{m-1}{2m}\,(1 - {}_nE_x)\,. \qquad (5.40)$$

5.11.3 Woolhouse's formula

Woolhouse's formula is a method of calculating the EPV of annuities payable more frequently than annually that is not based on a fractional age assumption. It is based on the Euler–Maclaurin formula and expresses $\ddot{a}^{(m)}_x$ in terms of $\ddot{a}_x$. The Euler–Maclaurin formula is a numerical integration method. It gives a series expansion for the integral of a function, assuming that the function is differentiable a certain number of times. As discussed in Appendix B, in the case of a function g, where $\lim_{t\to\infty} g(t) = 0$, the formula can be written in terms of a constant $h > 0$ as

$$\int_0^\infty g(t)\,dt = h\sum_{k=0}^{\infty} g(kh) - \frac{h}{2}\,g(0) + \frac{h^2}{12}\,g'(0) - \frac{h^4}{720}\,g'''(0) + \cdots, \qquad (5.41)$$

where we have omitted terms on the right-hand side that involve higher derivatives of g.

We set $g(t) = {}_tp_x\,e^{-\delta t}$, so that the integral that we approximate is

$$\int_0^\infty g(t)\,dt = \int_0^\infty {}_tp_x\,e^{-\delta t}\,dt = \bar{a}_x.$$

We apply formula (5.41) twice, ignoring third and higher-order derivatives of g, which is reasonable as we are working with relatively smooth functions.

Note that $g(0) = 1$, $\lim_{t\to\infty} g(t) = 0$, and

$$\begin{aligned} g'(t) &= \frac{d}{dt}\,{}_tp_x\, e^{-\delta t} = {}_tp_x \frac{d}{dt} e^{-\delta t} + e^{-\delta t} \frac{d}{dt}\,{}_tp_x \\ &= -\,{}_tp_x\, \delta\, e^{-\delta t} - e^{-\delta t}\,{}_tp_x\, \mu_{x+t} \\ &= -{}_tp_x\, e^{-\delta t}\,(\delta + \mu_{x+t})\,, \end{aligned}$$

so $g'(0) = -\,(\delta + \mu_x)$.

Now, let $h = 1$. As we are ignoring third and higher-order derivatives, equation (5.41) gives

$$\bar{a}_x \approx \sum_{k=0}^{\infty} g(k) - \frac{1}{2} + \frac{1}{12} g'(0) = \sum_{k=0}^{\infty} v^k\, {}_kp_x - \frac{1}{2} - \frac{1}{12}\,(\delta + \mu_x)$$

giving us a formula for calculating $\bar{a}_x$ from $\ddot{a}_x$, together with the force of interest and the force of mortality at age x, namely

$$\bar{a}_x \approx \ddot{a}_x - \frac{1}{2} - \frac{1}{12}\,(\delta + \mu_x)\,. \tag{5.42}$$

Next, let $h = 1/m$. Again ignoring third and higher-order derivatives, equation (5.41) gives

$$\begin{aligned} \bar{a}_x &\approx \frac{1}{m} \sum_{k=0}^{\infty} g\left(\tfrac{k}{m}\right) - \frac{1}{2m} + \frac{1}{12m^2} g'(0) \\ &= \frac{1}{m} \sum_{k=0}^{\infty} v^{\frac{k}{m}}\, {}_{\frac{k}{m}}p_x - \frac{1}{2m} - \frac{1}{12m^2}\,(\delta + \mu_x) \\ &= \ddot{a}_x^{(m)} - \frac{1}{2m} - \frac{1}{12m^2}\,(\delta + \mu_x)\,. \end{aligned} \tag{5.43}$$

Since each of (5.42) and (5.43) approximates the same quantity, $\bar{a}_x$, we can obtain an approximate relationship between $\ddot{a}_x^{(m)}$ and $\ddot{a}_x$ by equating the right-hand sides, giving

$$\ddot{a}_x^{(m)} - \frac{1}{2m} - \frac{1}{12m^2}(\delta + \mu_x) \approx \ddot{a}_x - \frac{1}{2} - \frac{1}{12}(\delta + \mu_x).$$

Rearranging, we obtain the important formula

$$\boxed{\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}\,(\delta + \mu_x)\,.} \tag{5.44}$$

The right-hand side of equation (5.44) is called the **three-term Woolhouse formula**. It is an extremely accurate method of obtaining $\ddot{a}_x^{(m)}$ from $\ddot{a}_x$ across a wide range of ages, values for m and interest rates; it gives more accurate values than the UDD approach in all practical contexts. Where great accuracy is less important, we often omit the last term, giving the **two-term Woolhouse formula**. This is less accurate, in general, than the UDD approximation, but it is a short calculation, requires very little information, and is often adequate for practical purposes.

For term annuities, we start from equation (5.29), namely

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \ddot{a}^{(m)}_{x} - {}_{n}E_x\, \ddot{a}^{(m)}_{x+n},$$

and applying formula (5.44) to both $\ddot{a}^{(m)}_{x}$ and $\ddot{a}^{(m)}_{x+n}$, gives

$$\begin{aligned}\ddot{a}^{(m)}_{x:\overline{n}|} &\approx \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}\,(\delta + \mu_x) \\ &\quad - {}_{n}E_x \left(\ddot{a}_{x+n} - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}\,(\delta + \mu_{x+n})\right) \\ &= \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}\,(1 - {}_{n}E_x) - \frac{m^2-1}{12m^2}\big(\delta + \mu_x - {}_{n}E_x(\delta + \mu_{x+n})\big).\end{aligned} \tag{5.45}$$

Some notes on Woolhouse's approximation for $\ddot{a}^{(m)}_{x}$

1. An important difference between the approximation based on the three-term Woolhouse's formula and the UDD approximation is that we need extra information for the Woolhouse approach – specifically values for the force of mortality, and many mortality tables do not give values for μ_x. However, as long as we have values of l_x, we can still use the three-term Woolhouse's formula, using an approximate value for μ_x. If we assume that

$$\int_{x-1}^{x+1} \mu_s\, ds \approx 2\mu_x,$$

then

$$\frac{l_{x+1}}{l_{x-1}} = {}_{2}p_{x-1} = \exp\left\{-\int_{x-1}^{x+1} \mu_s\, ds\right\} \approx \exp\{-2\mu_x\}, \tag{5.46}$$

leading to the approximation

$$\mu_x \approx -\tfrac{1}{2}\log\left(\frac{l_{x+1}}{l_{x-1}}\right). \tag{5.47}$$

In the next section we compare approximation methods for annuities payable $1/m$thly, and we shall see that, for our survival model, there is almost no difference between using the exact force of mortality and using the approximation from (5.47).

2. It is worth noting that, although we do not apply Woolhouse's formula directly to insurance functions, we can obtain very accurate values for $1/m$thly and continuous life insurance functions, using the relationships between annuity and insurance functions. For example,

$$\begin{aligned} A_x^{(m)} &= 1 - d^{(m)}\ddot{a}_x^{(m)} \\ &\approx 1 - d^{(m)}\left(\ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}\left(\mu_x + \delta\right)\right). \end{aligned}$$

Values for insurance functions calculated using the three-term Woolhouse approximation for the associated annuity EPV are more accurate than those calculated using the UDD assumption, sometimes substantially so.

3. In Section 2.6.2 we saw an approximate relationship between the complete expectation of life and the curtate expectation of life, namely

$$\mathring{e}_x \approx e_x + \tfrac{1}{2}.$$

Setting the interest rate to 0 in equation (5.43) gives a refinement of this approximation, namely

$$\mathring{e}_x \approx e_x + \tfrac{1}{2} - \tfrac{1}{12}\mu_x.$$

5.12 Numerical illustrations

In this section we give some numerical illustrations of the different methods of computing $\ddot{a}_{x:\overline{n}|}^{(m)}$. Table 5.3 shows values of $\ddot{a}_{x:\overline{10}|}^{(12)}$ for $x = 20, 30, \ldots, 100$ when $i = 0.1$, while Table 5.4 shows values of $\ddot{a}_{x:\overline{25}|}^{(2)}$ when $i = 0.05$.

Table 5.3 *Values of* $\ddot{a}_{x:\overline{10}|}^{(12)}$ *for* $i = 0.1$.

x	Exact	UDD	W2	W3	W3*
20	6.4655	6.4655	6.4704	6.4655	6.4655
30	6.4630	6.4630	6.4679	6.4630	6.4630
40	6.4550	6.4550	6.4599	6.4550	6.4550
50	6.4295	6.4294	6.4344	6.4295	6.4295
60	6.3485	6.3482	6.3535	6.3485	6.3485
70	6.0991	6.0982	6.1044	6.0990	6.0990
80	5.4003	5.3989	5.4073	5.4003	5.4003
90	3.8975	3.8997	3.9117	3.8975	3.8975
100	2.0497	2.0699	2.0842	2.0497	2.0496

Table 5.4 *Values of* $\ddot{a}^{(2)}_{x:\overline{25}|}$ *for* $i = 0.05$.

x	Exact	UDD	W2	W3	W3*
20	14.5770	14.5770	14.5792	14.5770	14.5770
30	14.5506	14.5505	14.5527	14.5506	14.5506
40	14.4663	14.4662	14.4684	14.4663	14.4663
50	14.2028	14.2024	14.2048	14.2028	14.2028
60	13.4275	13.4265	13.4295	13.4275	13.4275
70	11.5117	11.5104	11.5144	11.5117	11.5117
80	8.2889	8.2889	8.2938	8.2889	8.2889
90	4.9242	4.9281	4.9335	4.9242	4.9242
100	2.4425	2.4599	2.4656	2.4424	2.4424

The mortality basis for the calculations is the Standard Ultimate Survival Model, from Section 4.3.

The legend for each table is as follows:

Exact denotes the true EPV, calculated from formula (5.37);

UDD denotes the approximation to the EPV based on the uniform distribution of deaths assumption;

W2 denotes the approximation to the EPV based on Woolhouse's formula, using the first two terms only;

W3 denotes the approximation to the EPV based on Woolhouse's formula, using all three terms, including the exact force of mortality;

W3* denotes the approximation to the EPV based on Woolhouse's formula, using all three terms, but using the approximate force of mortality estimated from integer age values of l_x.

From these tables we see that approximations based on Woolhouse's formula with all three terms yield excellent approximations, even where we have approximated the force of mortality from integer age l_x values. Also, note that the inclusion of the third term is important for accuracy; the two-term Woolhouse formula is the worst approximation. We also observe that the approximation based on the UDD assumption is good at younger ages, with some deterioration for older ages. In this case approximations based on Woolhouse's formula are superior, provided the three-term version is used.

It is also worth noting that calculating the exact value of, for example, $\ddot{a}^{(12)}_{20}$ using a spreadsheet approach takes around 1200 rows, one for each month from age 20 to the limiting age ω. Using Woolhouse's formula requires only the integer age table, of 100 rows, and the accuracy all the way up to age 100 is excellent, using the exact or approximate values for μ_x. Clearly, there can be significant efficiency gains using Woolhouse's formula.

5.13 Functions for select lives

Throughout this chapter we have assumed that lives are subject to an ultimate survival model, just as we did in deriving insurance functions in Chapter 4. Just as in that chapter, all the arguments in this chapter equally apply if we have a select survival model. Thus, for example, the EPV of an n-year term annuity payable continuously at rate 1 per year to a life who is aged $x+k$ and who was select at age x is $\bar{a}_{[x]+k:\overline{n}|}$, with

$$\bar{A}_{[x]+k:\overline{n}|} = 1 - \delta\, \bar{a}_{[x]+k:\overline{n}|}.$$

The approximations we have developed also hold for select survival models, so that, for example

$$\ddot{a}^{(m)}_{[x]+k} \approx \ddot{a}_{[x]+k} - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\delta + \mu_{[x]+k}),$$

where

$$\ddot{a}_{[x]+k} = \sum_{t=0}^{\infty} v^t \;{}_tp_{[x]+k}$$

and

$$\ddot{a}^{(m)}_{[x]+k} = \frac{1}{m}\sum_{t=0}^{\infty} v^{t/m} \;{}_{\frac{t}{m}}p_{[x]+k}.$$

Example 5.6 Use the Standard Select Survival Model described on page 82, with interest at 5% per year, to produce a table showing values of $\ddot{a}_{[x]}$, $\ddot{a}_{[x]+1}$ and $\ddot{a}_{x+2}$ for $x = 20, 21, \ldots, 80$. Assume that $q_{131} = 1$.

Solution 5.6 Since we are assuming that $q_{131} = 1$, we have $\ddot{a}_{131} = 1$. Annuity EPVs can then be calculated recursively using

$$\ddot{a}_x = 1 + v\,p_x\,\ddot{a}_{x+1},$$

$$\ddot{a}_{[x]+1} = 1 + v\,p_{[x]+1}\,\ddot{a}_{x+2},$$

$$\ddot{a}_{[x]} = 1 + v\,p_{[x]}\,\ddot{a}_{[x]+1}.$$

Values are shown in Appendix D, Table D.2. □

5.14 Notes and further reading

Woolhouse (1869) presented the formula that bears his name in a paper to the Institute of Actuaries in London. In this paper he also showed that his theory applied to joint-life annuities, a topic we discuss in Chapter 10. A derivation of Woolhouse's formula from the Euler–Maclaurin formula is given in Appendix B. The Euler–Maclaurin formula was derived independently (about

130 years before Woolhouse's paper) by the famous Swiss mathematician Leonhard Euler and by the Scottish mathematician Colin Maclaurin. A proof of the Euler–Maclaurin formula, and references to the original works, can be found in Graham *et al.* (1989).

The expression in part (b) of Exercise 5.15 is from Bowers *et al.* (1997).

5.15 Exercises

Shorter exercises

Exercise 5.1 Describe in words the benefits with the present values given and write down an expression in terms of actuarial functions for the expected present value.

$$\text{(a)} \qquad Y_1 = \begin{cases} \bar{a}_{\overline{T_x}|} & \text{if } T_x \le 15, \\ \bar{a}_{\overline{15}|} & \text{if } T_x > 15. \end{cases}$$

$$\text{(b)} \qquad Y_2 = \begin{cases} a_{\overline{15}|} & \text{if } 0 < K_x \le 15, \\ a_{\overline{K_x}|} & \text{if } K_x > 15. \end{cases}$$

Exercise 5.2 (a) Describe the annuity with the following present value random variable:

$$Y = \begin{cases} v^{T_x}\, \bar{a}_{\overline{n-T_x}|} & \text{if } T_x \le n, \\ 0 & \text{if } T_x \ge n. \end{cases}$$

(This random variable gives the present value of a Family Income Benefit of 1 issued to a life aged x.)

(b) Show that $\mathrm{E}[Y] = \bar{a}_{\overline{n}|} - \bar{a}_{x:\overline{n}|}$.

(c) Explain the answer in part (b) by general reasoning.

Exercise 5.3 Given that $\ddot{a}_{50:\overline{10}|} = 8.2066$, $a_{50:\overline{10}|} = 7.8277$, and ${}_{10}p_{50} = 0.9195$, what is the effective rate of interest per year?

Exercise 5.4 Given that ${}_{10|}\ddot{a}_x = 4$, $\ddot{a}_x = 10$, ${}_{10}E_x = 0.375$ and $v = 0.94$, calculate $A^{\,1}_{x:\overline{10}|}$.

Exercise 5.5 Given that $a_{60} = 10.996$, $a_{61} = 10.756$, $a_{62} = 10.509$ and $i = 0.06$, calculate ${}_{2}p_{60}$.

Exercise 5.6 Using the Standard Select Life Table at 5% per year interest, calculate the standard deviation of the present value of an annuity of \$50 000 per year payable annually in advance to a select life aged 60.

Exercise 5.7 You are given the following information for insurance functions at $i = 6\%$ per year interest:

$$ {}_{20}E_{50} = 0.26959, \qquad A_{50}^{(12)} = 0.18048, \qquad A_{70}^{(12)} = 0.41758. $$

Calculate $\ddot{a}^{(12)}_{50:\overline{20}|}$.

Exercise 5.8 Calculate the EPV of a five-year deferred whole-life annuity-due issued to (70), under which the first payment is \$10 000, and subsequent payments increase at 5% per year compound.

You are given that mortality follows the Standard Ultimate Life Table, $i = 5\%$ per year, and $e_{75} = 14.102$.

Exercise 5.9 Consider a whole life annuity-due of 1 per year issued to (50). Calculate the probability that the total payment made under the annuity (without discounting) is greater than the EPV of the annuity at issue.
Basis: Standard Ultimate Life Table, interest at 5% per year.

Exercise 5.10 Using the Standard Ultimate Life Table at 5% per year interest, calculate

(a) $\ddot{a}_{40:\overline{20}|}$,

(b) $\ddot{a}^{(4)}_{40:\overline{20}|}$ using Woolhouse's formula with two terms,

(c) $\bar{a}_{25:\overline{10}|}$ assuming UDD,

(d) $a^{(12)}_{50:\overline{20}|}$ using Woolhouse's formula with two terms,

(e) ${}_{20|}\ddot{a}_{45}^{(12)}$ assuming UDD,

(f) ${}_{20|}\ddot{a}_{45}^{(12)}$ using Woolhouse's formula with three terms.

Exercise 5.11 For a select life table with a two-year select period, you are given that

$$ \ddot{a}_{[x-1]+1:\overline{n}|} = 17.5 \quad \text{and} \quad \ddot{a}_{x:\overline{n}|} = 17. $$

Calculate $\dfrac{p_{[x-1]+1}}{p_x}$.

Exercise 5.12 Scott, who is aged 40, has just been involved in an accident. For the first year after the accident, his mortality follows the Standard Ultimate Life Table for a life five years older than Scott, with an addition of 0.05 to the force of mortality. After the first year, his mortality follows the Standard Ultimate Life Table for a life five years older than Scott, with no other adjustment. As compensation, Scott is awarded a whole life annuity-due of \$10 000 per year, payable annually.

Calculate the EPV of the annuity at an interest rate of 5% per year.

Longer Exercises

Exercise 5.13 Each member of a group of 200 independent lives aged x will receive a life annuity-due of \$100 per year paid from a trust fund. You are given that

$$\ddot{a}_x = \begin{cases} 13.01704 & \text{at } i = 4\%, \\ 9.59176 & \text{at } i = 8.16\%. \end{cases}$$

Let Y denote the present value random variable for the total annuity payments for all the group members.

Using a normal approximation to the distribution of Y, calculate the amount that should be deposited in the trust fund to be 90% certain that the funds will be sufficient to pay the benefits. Assume interest at 4% per year.

Exercise 5.14 You are given the following extract from a select life table.

$[x]$	$l_{[x]}$	$l_{[x]+1}$	l_{x+2}	$x+2$
40	33 519	33 485	33 440	42
41	33 467	33 428	33 378	43
42	33 407	33 365	33 309	44
43	33 340	33 294	33 231	45
44	33 265	33 213	33 143	46

Calculate the following, assuming an interest rate of 6% per year:

(a) $\ddot{a}_{[40]:\overline{4}|}$,

(b) $a_{[40]+1:\overline{4}|}$,

(c) $(Ia)_{[40]:\overline{4}|}$,

(d) $(IA)_{[40]:\overline{4}|}$,

(e) the standard deviation of the present value of a four-year term annuity-due, with annual payment \$1000, payable to a select life aged 41, and

(f) the probability that the present value of an annuity-due of 1 per year issued to a select life aged 40 is less than 3.

Exercise 5.15 Let $H = \min(K_x, n)$.

(a) Show that

$$\mathrm{V}[a_{\overline{H}|}] = \frac{{}^2A_{x:\overline{n+1}|} - \left(A_{x:\overline{n+1}|}\right)^2}{d^2}.$$

(b) An alternative form given for this variance is

$$\frac{(1+i)^2\,[{}^2A^{\,1}_{x:\overline{n}|} - (A^{\,1}_{x:\overline{n}|})^2] - 2(1+i)A^{\,1}_{x:\overline{n}|}\,v^n\,{}_np_x + v^{2n}\,{}_np_x(1-{}_np_x)}{i^2}.$$

Prove that this is equal to the expression in (a).

Exercise 5.16 The force of mortality for a certain population is exactly half the sum of the forces of mortality in two standard mortality tables, denoted A and B. Thus

$$\mu_x = (\mu_x^A + \mu_x^B)/2$$

for all x. A student has suggested the approximation

$$a_x = (a_x^A + a_x^B)/2.$$

Will this approximation overstate or understate the true value of a_x?

Exercise 5.17 Obtain the formula

$$(IA)_x = \ddot{a}_x - d(I\ddot{a})_x$$

by writing down the present value random variables for

(a) an increasing annuity-due to (x) with payments of $t+1$ at times $t = 0, 1, 2, \ldots$, and
(b) a whole life insurance benefit of amount t at time t, $t = 1, 2, 3 \ldots$, if the death of (x) occurs between ages $x+t-1$ and $x+t$.

Hint: Use the result

$$(I\ddot{a})_{\overline{n}|} = \sum_{t=1}^{n} t v^{t-1} = \frac{\ddot{a}_{\overline{n}|} - n v^n}{d}.$$

Exercise 5.18 Consider the random variables $Y = \bar{a}_{\overline{T_x}|}$ and $Z = v^{T_x}$.

(a) Derive an expression for the covariance in terms of standard actuarial functions.
(b) Show that the covariance is negative.
(c) Explain this result by general reasoning.

Exercise 5.19 Consider the quantities $\alpha(m)$ and $\beta(m)$ in formula (5.38). By expressing i, $i^{(m)}$, d and $d^{(m)}$ in terms of δ, show that

$$\alpha(m) \approx 1 \quad \text{and} \quad \beta(m) \approx \frac{m-1}{2m}.$$

Exercise 5.20 Consider the following portfolio of annuities-due currently being paid from the assets of a pension fund.

Age	Number of annuitants
60	40
70	30
80	10

Each annuity has an annual payment of \$10 000. The lives are assumed to be independent. Assume mortality follows the Standard Ultimate Life Table, with interest at 5% per year. Calculate

(a) the expected present value of the total outgo on annuities,
(b) the standard deviation of the present value of the total outgo on annuities, and
(c) the 95th percentile of the distribution of the present value of the total outgo on annuities, using a Normal approximation.

Exercise 5.21 Find, and simplify where possible:

(a) $\frac{d}{dx}\,\ddot{a}_x$, and
(b) $\frac{d}{dx}\,\ddot{a}_{x:\overline{n|}}$.

Exercise 5.22 You are given that

$$Y = \begin{cases} 0 & \text{if } T_x \le 5, \\ 1000v^5\,\bar{a}_{\overline{T_{x+5}|}} & \text{if } T_x > 5. \end{cases}$$

(a) Describe in words the benefit for which Y is the present value.
(b) Write down an expression for $\mathrm{E}[Y]$ in terms of standard actuarial symbols, simplified as far as possible.
(c) Let I be an indicator random variable, where

$$I = \begin{cases} 0 & \text{if } T_x \le 5, \\ 1 & \text{if } T_x > 5. \end{cases}$$

 (i) Show that

$$\mathrm{V}[Y|I=1] = 1000^2 v^{10}\,\frac{{}^2\bar{A}_{x+5} - \bar{A}_{x+5}^2}{\delta^2}.$$

 (ii) Show that

$$\mathrm{V}[Y] = 1000^2\,v^{10}\,{}_5p_x\left(\frac{{}^2\bar{A}_{x+5} - \bar{A}_{x+5}^2}{\delta^2} + (\bar{a}_{x+5})^2\,{}_5q_x\right).$$

(d) Assume now that $x = 65$, mortality follows the Standard Ultimate Life Table, with UDD between integer ages, and that $i = 5\%$ per year.
 (i) Calculate the mean and standard deviation of Y.
 (ii) Calculate $\Pr[Y > \mathrm{E}[Y]]$.
 (iii) Suppose an insurer sells 100 000 of these annuities to independent lives. Without further calculation, estimate the probability that the total present value of the payments exceeds the expected value of the total present value of the payments. Justify your answer.

(*Copyright 2018. The Society of Actuaries, Schaumburg, Illinois. Reproduced with permission.*)

Exercise 5.23 Jensen's inequality states that for a function f, whose first derivative is positive and whose second derivative is negative, and a random variable X,

$$\mathrm{E}[f(X)] \leq f(\mathrm{E}[X]).$$

Use Jensen's inequality to show that

$$\bar{a}_x \leq \bar{a}_{\overline{\mathrm{E}[T_x]}|}.$$

Excel-based exercises

Exercise 5.24 Using a spreadsheet, calculate the mean and standard deviation of the present value of the following, assuming mortality follows the Standard Ultimate Survival Model, and interest is 5% per year:

(a) an arithmetically increasing term annuity-due payable to a life aged 50 for at most 10 years under which the payment at time t is $t+1$ for $t = 0, 1, \ldots, 9$, and
(b) a geometrically increasing term annuity-due payable to a life aged 50 for at most 10 years under which the payment at time t is 1.03^t for $t = 0, 1, \ldots, 9$.

Exercise 5.25 Using a spreadsheet, calculate the mean and standard deviation of the present value of the following, assuming mortality follows the Standard Ultimate Survival Model, and interest is 5% per year:

(a) a whole life annuity-due to a life aged 65, with annual payments of 1, and
(b) a whole life annuity-due to a life aged 65, with annual payments of 1 and a guarantee period of 10 years.

Explain the ordering of the means and standard deviations.

Answers to selected exercises

5.3 4.0014%
5.4 0.265
5.5 0.98220
5.6 161 996
5.7 11.3974
5.8 110 690
5.9 0.94747
5.10 (a) 12.994 (b) 12.756 (c) 7.902 (d) 12.490 (e) 4.7102 (f) 4.7105
5.11 1.03125

5.12 169 960

5.13 268 086

5.14 (a) 3.6664 (b) 3.4506 (c) 8.3750 (d) 3.1630 (e) 119.14 (f) 0.00421

5.20 (a) 10 418 970 (b) 311 537 (c) 10 931 402

5.22 (d) (i) 8 679, 3 328 (ii) 0.6120

5.24 (a) 40.95, 3.325 (b) 9.121, 0.574

5.25 (a) 13.550, 3.535 (b) 13.814, 2.895

6
Premium calculation

6.1 Summary

In this chapter we discuss principles of premium calculation for insurance policies and annuities. We start by reviewing what we mean by the terms *premium*, *net premium* and *gross premium*. We next introduce the present value of loss at issue random variable, and show how this can be used in two ways to determine a premium.

The first approach is the equivalence premium principle. We show how this can be applied to calculate premiums for different types of policies, and we look at how to determine when an individual contract moves from loss to profit or vice versa.

The second approach is the portfolio percentile premium principle, and we show how, using the mean and variance of the loss at issue random variable, this principle can be used to determine a premium for a given initial portfolio size, and for a given probability of profit on the portfolio of policies.

The chapter concludes with a discussion of how a premium can be calculated when the insured life is subject to some extra level of risk.

6.2 Preliminaries

An insurance policy is a financial agreement between the insurance company and the policyholder. The insurance company agrees to pay some benefits, for example a sum insured on the death of the policyholder within a specified term, and the policyholder agrees to pay premiums to the insurance company to secure these benefits. The premiums also need to reimburse the insurance company for the expenses associated with the policy.

The calculation of the premium may not explicitly allow for the insurance company's expenses. In this case we refer to a **net premium** (also, sometimes, a risk premium or benefit premium). If the calculation does explicitly allow for expenses, the premium is called a **gross premium** or **office premium** or **expense-loaded premium**.

The premium may be a single payment by the policyholder – a **single premium** – or it may be a regular series of payments, possibly annually, quarterly, monthly or weekly. Monthly premiums are common since many employed people receive their salaries monthly and it is convenient to have payments made with the same frequency as income is received.

It is common for regular premiums to be a level amount, but they do not have to be.

A key feature of any life insurance policy is that premiums are payable in advance, with the first premium payable when the policy is purchased. To see why this is necessary, suppose it were possible to purchase a whole life insurance policy with annual premiums where the first premium is paid at the end of the year in which the policy was purchased. In this case, a person could purchase the policy and then withdraw from the contract at the end of the first year before paying the premium then due. This person would have had a year of insurance cover without paying anything for it.

Regular premiums for a policy on a single life cease to be payable on the death of the policyholder. The premium payment term for a policy is the maximum length of time for which premiums are payable. The premium payment term may be the same as the term of the policy, but it could be shorter. If we consider a whole life insurance policy, it would be usual for the death benefit to be secured by regular premiums and it would be common for premiums to cease at a certain age – perhaps at age 65 when the policyholder is assumed to retire, or at age 85 when the policyholder's real income may be diminishing, and their ability to manage their finances may be declining.

As we discussed in Chapter 1, premiums are payable to secure annuity benefits as well as life insurance benefits. Deferred annuities may be purchased using a single premium at the start of the deferred period, or by regular premiums payable throughout the deferred period. Immediate annuities are always purchased by a single premium. For example, a person aged 45 might secure a retirement income by paying regular premiums over a 20-year period to secure annuity payments from age 65. Or, a person aged 65 might secure a monthly annuity from an insurance company by payment of a single premium.

For traditional policies, the benchmark principle for calculating both gross and net premiums is called the equivalence principle, and we discuss its application in detail in this chapter. However, there are other methods of calculating premiums and we discuss one of these, the portfolio percentile principle.

A more contemporary approach, which is commonly used for non-traditional policies, is to consider the cash flows from the contract, and to set the premium to satisfy a specified profit criterion. This approach is discussed in Chapter 13.

6.2.1 Assumptions

For the numerical examples in this chapter we use the Standard Select Survival Model, described in Example 3.13. The ultimate force of mortality is $\mu_x = A + Bc^x$, where $A = 2.2 \times 10^{-4}$, $B = 2.7 \times 10^{-6}$ and $c = 1.124$. The select force of mortality, for $0 \leq s < 2$, is $\mu_{[x]+s} = 0.9^{2-s}\mu_{x+s}$. The table has a two-year select period.

Life table functions for this model are given in Appendix D, Table D.1 on page 737, and annuity and insurance functions, at $i = 5\%$ interest, are in Table D.2 on page 738.

6.3 The loss at issue random variable

The cash flows for a traditional life insurance contract consist of the insurance or annuity benefit outgo (and associated expenses) and the premium income. Both are generally life contingent, that is, the income and outgo cash flows depend on the future lifetime of the policyholder, unless the contract is purchased by a single premium, in which case there is no uncertainty regarding the premium income. We can model the present value at issue of the future loss (outgo minus income) on a policy with a random variable. When expenses are excluded we call this the **present value of net future loss at issue** random variable, or more commonly just the **net loss at issue**, which we denote by L_0^n. When expenses are included, then the premiums are the gross premiums, and the present value random variable is referred to as the **gross loss at issue**, denoted L_0^g. In other words,

$$L_0^n = \text{PV of benefit outgo} - \text{PV of net premium income},$$

$$L_0^g = \text{PV of benefit outgo} + \text{PV of expenses} - \text{PV of gross premium income}.$$

In cases where the meaning is obvious from the context, we drop the n or g superscript.

Example 6.1 An insurer issues a whole life insurance to a select life aged 60, with sum insured S payable immediately on death. Premiums are payable annually in advance, ceasing at age 80 or on earlier death. The net annual premium is P.

Write down the net loss at issue random variable, L_0^n, for this contract in terms of lifetime random variables for (60).

Solution 6.1 From Chapter 4, we know that the present value random variable for the benefit is $Sv^{T_{[60]}}$ and from Chapter 5 we know that the present value random variable for the premium income is $P\,\ddot{a}_{\overline{\min(K_{[60]}+1,20)}|}$, so

$$L_0^n = Sv^{T_{[60]}} - P\,\ddot{a}_{\overline{\min(K_{[60]}+1,20)}|}.$$

Since both terms of the random variable depend on the future lifetime of the same life they are clearly dependent.

Note that since premiums are payable in advance, premiums *payable annually in advance, ceasing at age 80 or on earlier death* means that the last possible premium is payable on the policyholder's 79th birthday. No premium is payable on reaching age 80. □

Given an appropriate survival model together with assumptions about future interest rates and, for gross premiums, expenses, the insurer can then determine a distribution for the present value of the future loss. This distribution can be used to find a suitable premium for a given benefit, or an appropriate benefit for a specified premium. To do this, the insurer needs to use a **premium principle**. This is a method of selecting an appropriate premium based on a given loss distribution. We discuss two premium principles in this chapter.

6.4 The equivalence principle premium

Under the **equivalence principle**, the premium is set such that the expected value of the loss at issue random variable is zero. The equivalence principle is the most common premium principle in traditional life insurance, and is our default principle – that is, if no other principle is specified, it is assumed that the equivalence principle is to be used.

6.4.1 Net premiums

The net premium for a contract is the equivalence principle premium using the net loss at issue random variable.

Expenses are never included in a net premium calculation, and a premium calculated using a different premium principle is not a net premium, even if expenses have been ignored.

So we have $\mathrm{E}[L_0^n] = 0$, which implies that

$$\mathrm{E}\big[\text{PV of benefit outgo} - \text{PV of net premium income}\big] = 0.$$

That is, the net premium can be defined as the premium that solves the equation of value:

$$\boxed{\textbf{EPV of benefit outgo} = \textbf{EPV of premium income}.} \qquad (6.1)$$

Example 6.2 Consider an endowment insurance with term n years and sum insured S payable at the earlier of the end of the year of death or at maturity, issued to a select life aged x. Premiums of amount P are payable annually throughout the term of the insurance.

(a) Derive expressions in terms of S, P and standard actuarial functions for
   (i) the net loss at issue, L_0^n,
   (ii) the mean of L_0^n,
   (iii) the variance of L_0^n, and
   (iv) the annual net premium for the contract.

(b) Assume now that the sum insured is \$100 000, mortality follows the Standard Select Life Table, $i = 5\%$, $x = 50$ and $n = 20$.
   (i) Calculate the net premium.
   (ii) Calculate the standard deviation of the net loss at issue.

Solution 6.2

(a) (i) The net loss at issue random variable is

$$L_0^n = S\,v^{\min(K_{[x]}+1,n)} - P\,\ddot{a}_{\overline{\min(K_{[x]}+1,n)}|}.$$

(ii) The mean of L_0^n is

$$\begin{aligned} \mathrm{E}[L_0^n] &= S\,\mathrm{E}\left[v^{\min(K_{[x]}+1,n)}\right] - P\,\mathrm{E}\left[\ddot{a}_{\overline{\min(K_{[x]}+1,n)}|}\right] \\ &= S\,A_{[x]:\overline{n}|} - P\,\ddot{a}_{[x]:\overline{n}|}. \end{aligned}$$

(iii) Expanding the expression above for L_0^n gives

$$\begin{aligned} L_0^n &= S\,v^{\min(K_{[x]}+1,n)} - P\,\frac{1 - v^{\min(K_{[x]}+1,n)}}{d} \\ &= \left(S + \frac{P}{d}\right) v^{\min(K_{[x]}+1,n)} - \frac{P}{d}, \end{aligned}$$

which isolates the random variable $v^{\min(K_{[x]}+1,n)}$. So the variance is

$$\begin{aligned} \mathrm{V}\left[L_0^n\right] &= \left(S + \frac{P}{d}\right)^2 \mathrm{V}\left[v^{\min(K_{[x]}+1,n)}\right] \\ &= \left(S + \frac{P}{d}\right)^2 \left({}^2A_{[x]:\overline{n}|} - (A_{[x]:\overline{n}|})^2\right). \end{aligned}$$

(iv) Setting the EPVs of the premiums and benefits to be equal gives the net premium as

$$P = S\,\frac{A_{[x]:\overline{n}|}}{\ddot{a}_{[x]:\overline{n}|}}. \tag{6.2}$$

(b) (i) The annual net premium is

$$P = 100\,000\,\frac{A_{[50]:\overline{20}|}}{\ddot{a}_{[50]:\overline{20}|}} = 100\,000\,\frac{0.38831}{12.8454} = \$3\,022.93.$$

(ii) The variance of L_0^n is

$$\mathrm{V}\left[L_0^n\right] = \left(100\,000 + \frac{P}{d}\right)^2 \left({}^2A_{[50]:\overline{20}|} - A^2_{[50]:\overline{20}|}\right)$$

where

$$\begin{aligned} {}^2A_{[50]:\overline{20}|} &= {}^2A_{[50]} - {}_{20}^{\;2}E_{[50]}\,{}^2A_{70} + {}_{20}^{\;2}E_{[50]} \\ &= {}^2A_{[50]} - v^{20}\,{}_{20}E_{[50]}\,{}^2A_{70} + v^{20}\,{}_{20}E_{[50]} \\ &= 0.153967, \end{aligned}$$

giving the standard deviation as

$$\begin{aligned} \mathrm{SD}\left[L_0^n\right] &= \left(100\,000 + \frac{3022.93}{0.04762}\right)\sqrt{0.153967 - 0.38831^2} \\ &= 9\,225.99. \end{aligned}$$

□

It is interesting to note in the above example that, using formula (6.2) and recalling that $\ddot{a}_{x:\overline{n}|} = (1 - A_{x:\overline{n}|})/d$, the premium equation can be written as

$$P = S\left(\frac{1}{\ddot{a}_{[x]:\overline{n}|}} - d\right)$$

so that the only actuarial function needed to calculate P in this case is $\ddot{a}_{[x]:\overline{n}|}$.

Example 6.3 An insurer issues a regular premium deferred annuity contract to a select life aged x. Premiums are payable monthly throughout the deferred period. The annuity benefit of X per year is payable monthly in advance from age $x + n$ for the remainder of the life of (x).

(a) Write down the net loss at issue random variable in terms of lifetime random variables for (x).
(b) Derive an expression for the monthly net premium.
(c) Assume now that, in addition, the contract offers a death benefit of S payable immediately on death during the deferred period. Write down the net loss at issue random variable for the contract, and derive an expression for the monthly net premium.

Solution 6.3 (a) Let P denote the monthly net premium, so that the total premium payable in a year is $12P$. Then

$$L_0^n = \begin{cases} 0 - 12P\,\ddot{a}^{(12)}_{\overline{K^{(12)}_{[x]} + \frac{1}{12}}|} & \text{if } T_{[x]} \le n, \\ X\,v^n\,\ddot{a}^{(12)}_{\overline{K^{(12)}_{[x]} + \frac{1}{12} - n}|} - 12P\,\ddot{a}^{(12)}_{\overline{n}|} & \text{if } T_{[x]} > n. \end{cases}$$

(b) The EPV of the annuity benefit is

$$X\,{}_nE_{[x]}\,\ddot{a}^{(12)}_{[x]+n},$$

and the EPV of the premium income is

$$12P\,\ddot{a}^{(12)}_{[x]:\overline{n|}}.$$

By equating these EPVs we obtain the premium equation which gives

$$P = \frac{{}_nE_{[x]}\,X\,\ddot{a}^{(12)}_{[x]+n}}{12\ddot{a}^{(12)}_{[x]:\overline{n|}}}.$$

(c) We now have

$$L_0^n = \begin{cases} S\,v^{T_{[x]}} - 12P\,\ddot{a}^{(12)}_{\overline{K^{(12)}_{[x]}+\frac{1}{12}|}} & \text{if } T_{[x]} \le n, \\ X\,v^n\,\ddot{a}^{(12)}_{\overline{K^{(12)}_{[x]}+\frac{1}{12}-n|}} - 12P\,\ddot{a}^{(12)}_{\overline{n|}} & \text{if } T_{[x]} > n. \end{cases}$$

The annuity benefit has the same EPV as in part (b); the death benefit during deferral is a term insurance benefit with EPV $S\bar{A}^{\;1}_{[x]:\overline{n|}}$, so the premium equation now becomes

$$P = \frac{S\bar{A}^{\;1}_{[x]:\overline{n|}} + X{}_nE_{[x]}\,\ddot{a}^{(12)}_{[x]+n}}{12\,\ddot{a}^{(12)}_{[x]:\overline{n|}}}.$$

□

Example 6.3 shows that the loss at issue random variable can be quite complicated to write down. Usually, the premium calculation does not require the identification of the random variable. Instead we can simply equate the EPV of income with the EPV of outgo.

Example 6.4 An insurer issues an endowment insurance with sum insured \$100 000 to a select life aged 45, with term 20 years, under which the death benefit is payable at the end of the year of death. Using the Standard Select Survival Model, with interest at 5% per year, calculate the total amount of net premium payable in a year if premiums are payable (a) annually, (b) quarterly, and (c) monthly, and comment on these values. Use Woolhouse's three-term formula for the quarterly and monthly functions.

Solution 6.4 Let P denote the total amount of premium payable in a year. Then the EPV of premium income is $P\,\ddot{a}^{(m)}_{[45]:\overline{20|}}$ (where $m = 1, 4$ or 12) and the EPV of benefit outgo is $100\,000A_{[45]:\overline{20|}}$, giving

$$P = \frac{100\,000A_{[45]:\overline{20|}}}{\ddot{a}^{(m)}_{[45]:\overline{20|}}}$$

where $\ddot{a}_{[45]:\overline{20|}} = 12.9409$ and $A_{[45]:\overline{20|}} = 0.383766$.

Hence, for $m = 1$ the net premium is $P = \$2\,965.52$. For the quarterly and monthly premiums we need the force of mortality for a select life aged 45 and an ultimate life aged 65 from the Standard Select Survival Model. Using the parameters from Section 6.2.1, we have $\mu_{[45]} = 0.000599$ and $\mu_{65} = 0.005605$, so

$$\begin{aligned}\ddot{a}^{(4)}_{[45]:\overline{20|}} &= \ddot{a}_{[45]} - \tfrac{3}{8} - \tfrac{15}{192}\left(\mu_{[45]} + \delta\right) - {}_{20}E_{[45]}\left(\ddot{a}_{65} - \tfrac{3}{8} - \tfrac{15}{192}\left(\mu_{65} + \delta\right)\right)\\ &= 12.6986\end{aligned}$$

and

$$\begin{aligned}\ddot{a}^{(12)}_{[45]:\overline{20|}} &= \ddot{a}_{[45]} - \tfrac{11}{24} - \tfrac{143}{1728}\left(\mu_{[45]} + \delta\right) - {}_{20}E_{[45]}\left(\ddot{a}_{65} - \tfrac{11}{24} - \tfrac{143}{1728}\left(\mu_{65} + \delta\right)\right)\\ &= 12.6451,\end{aligned}$$

which means that the annual premium, payable in quarterly instalments, is $P = \$3\,022.11$ and the annual premium payable in monthly instalments is $P = \$3\,034.89$. □

We note in the above example that the annually-paid premium is less than the quarterly-paid premium, which is less than the monthly-paid premium. This makes sense as the annually-paid premium is paid earliest in the year, and, on average, the quarterly premium is received earlier than the monthly premium. That means that the annually-paid premium has more time to earn interest than the quarterly, which has more time to earn interest than the monthly premium. Also, if the policyholder dies during the policy term, under the annually-paid premium, the insurer receives the entire year's premium in the year of death, which may not be true for the other cases; for the quarterly premium, it will only be true if the policyholder dies during the last quarter of the year, and for the monthly premium, the insurer will only receive the full year's premium in the year of death if the policyholder dies in the final month of the year.

So the annually-paid premium is smaller than the others, as the balance is made up from extra interest, and extra expected premium received in the year of death. Similarly, on average, the extra interest and extra expected premium paid in the year of death under the quarterly premium allow a slight smaller annual premium rate than under the monthly payment scheme.

6.4.2 Gross premiums

When we calculate a gross premium for an insurance policy or an annuity, we take account of the expenses the insurer incurs. There are three main types of expense associated with insurance policies and annuities – initial expenses, renewal expenses and termination or claim expenses.

Initial expenses, also (loosely) referred to as acquisition expenses, are incurred by the insurer around the time a policy is issued. When we calculate a gross premium, it is conventional to assume that the insurer incurs these expenses at exactly the same time as the first premium is payable, although in practice some of these expenses are usually incurred slightly ahead of this date. Later, in Chapter 13, we will separate the expenses incurred before the first premium from the expenses incurred with the first premium.

There are two major types of initial expenses – commission to agents for selling a policy and underwriting expenses. Commission is often paid to an agent in the form of a high percentage of the first year's premiums plus a much lower percentage of subsequent premiums, payable as the premiums are paid. Underwriting expenses may vary according to the amount of mortality risk involved. If there is significant risk of adverse selection, an insurer is likely to require much more stringent underwriting. For example, the adverse selection risk on a term life policy is much greater than a whole life policy, all else being equal. And the adverse selection risk associated with a \$10 million death benefit is rather greater than with a \$10 000 death benefit.

Renewal expenses, also called maintenance expenses, are normally incurred by the insurer each time a premium is payable, or when an annuity payment is made. Renewal expenses arise in a variety of ways, including renewal commissions (typically expressed as a percentage of the gross premium) and costs associated with processing premiums or annuity payments, or issuing annual statements. Renewal expenses should also cover a share of the ongoing fixed costs of the insurer such as staff salaries and rent for the insurer's premises.

Renewal expenses may be expressed as a percentage of premium, or as a **per policy** amount, meaning fixed for all policies, regardless of the size of the premium or sum insured, or as a combination of the two. Often, per policy costs are assumed to be increasing at a compound rate over the term of the policy, to approximate the effect of inflation.

Termination expenses, also called claim expenses, occur when a policy expires, typically on the death of a policyholder or annuitant, or on the maturity date of a term insurance or endowment insurance. Generally these expenses are small, and are associated with the paperwork required to finalize and pay a claim. In calculating gross premiums, specific allowance is often not made for termination expenses. Where it is made, it is usually a fixed sum, or is proportional to the benefit amount.

In practice, allocating the different expenses involved in running an insurance company is a complicated task, and in the examples in this chapter we simply assume that all expenses are known.

The equivalence principle using the gross loss at issue random variable gives us the equivalence principle gross premium. That is, $\mathrm{E}[L_0^g] = 0$ implies

$$\boxed{\textbf{EPV of gross premiums} = \textbf{EPV of benefits} + \textbf{EPV of expenses}.} \tag{6.3}$$

We illustrate the gross premium equation of value calculations with four examples.

Example 6.5 An insurer issues a 20-year annual premium endowment insurance with sum insured \$100 000 to a select life aged 30. The insurer incurs initial expenses of \$2000 plus 50% of the first premium, and renewal expenses of 2.5% of each subsequent premium. The death benefit is payable immediately on death.

(a) Write down the gross loss at issue random variable.
(b) Calculate the gross premium using the Standard Select Life Table with 5% per year interest. Assume UDD between integer ages.

Solution 6.5 (a) Let $S = 100\,000$, $x = 30$, $n = 20$ and let P denote the annual gross premium. Then

$$\begin{aligned} L_0^g &= S\,v^{\min(T_{[x]},n)} + 2000 + 0.475P + 0.025P\,\ddot{a}_{\overline{\min(K_{[x]}+1,n)}|} - P\,\ddot{a}_{\overline{\min(K_{[x]}+1,n)}|} \\ &= S\,v^{\min(T_{[x]},n)} + 2000 + 0.475P - 0.975P\,\ddot{a}_{\overline{\min(K_{[x]}+1,n)}|}. \end{aligned}$$

Note that the premium related expenses, of 50% of the first premium plus 2.5% of the second and subsequent premiums are more conveniently written as 2.5% of **all** premiums, plus an additional 47.5% of the first premium. By expressing the premium expenses this way, we can simplify the random variable, and the subsequent premium calculation.

(b) We may look separately at the three parts of the gross premium equation of value. The EPV of premium income is

$$P\,\ddot{a}_{[30]:\overline{20}|} = 13.0418\,P.$$

The EPV of all expenses is

$$\begin{aligned} 2000 + 0.475P + 0.025P\,\ddot{a}_{[30]:\overline{20}|} &= 2000 + 0.475P + 0.025 \times 13.0418P \\ &= 2000 + 0.801044P. \end{aligned}$$

The EPV of the death benefit, using the UDD assumption, is

$$\begin{aligned} 100\,000\,\bar{A}_{[30]:\overline{20}|} &= 100\,000 \left(\frac{i}{\delta} A^{\ 1}_{[30]:\overline{20}|} + {}_{20}E_{[30]} \right) \\ &= 100\,000 \left(\frac{i}{\delta} \left(A_{[30]:\overline{20}|} - {}_{20}E_{[30]} \right) + {}_{20}E_{[30]} \right) \end{aligned}$$

$$= 100\,000\left(\frac{i}{\delta}\left(1 - d\,\ddot{a}_{[30]:\overline{20}|} - {}_{20}E_{[30]}\right) + {}_{20}E_{[30]}\right)$$

$$= 100\,000(0.379122) = 37\,912.16.$$

Thus, the equivalence principle gives

$$P = \frac{37\,912.16 + 2\,000}{13.0418 - 0.801044} = \$3\,260.60.$$

□

Example 6.6 Calculate the monthly gross premium for a 10-year term insurance with sum insured \$50 000 payable immediately on death, issued to a select life aged 55, using the following basis:

Survival model:	Standard Select Life Table
	Assume UDD for fractional ages
Interest:	5% per year
Initial Expenses:	\$500 + 10% of each monthly premium in the first year
Renewal Expenses:	1% of each monthly premium in the second and subsequent policy years

Solution 6.6 Let P denote the monthly premium, so that the EPV of the premium income is $12P\,\ddot{a}^{(12)}_{[55]:\overline{10}|}$.

To find the EPV of the premium related expenses, we can apply the same idea as in the previous example, that is, apply the renewal expenses to all years including the first, and then separately allow for the additional initial expenses, in excess of the renewal expenses. Note in this case that the initial expenses apply to each monthly premium in the first year. Then we can write the EPV of the premium related expenses as

$$0.09 \times 12P\,\ddot{a}^{(12)}_{[55]:\overline{1}|} + 0.01 \times 12P\,\ddot{a}^{(12)}_{[55]:\overline{10}|}.$$

So the premium expenses for the first year have been split into the 1%, matching the renewal expenses, plus an additional 9% to meet the balance of the initial expenses.

The EPV of the insurance benefit is $50\,000\bar{A}^{\;1}_{[55]:\overline{10}|}$. So, setting the EPV of premiums less premium-related expenses equal to the EPV of benefits plus other expenses, the equation of value is

$$12P\left(\ddot{a}^{(12)}_{[55]:\overline{10}|}(1 - 0.01) - 0.09\ddot{a}^{(12)}_{[55]:\overline{1}|}\right) = 50\,000\,\bar{A}^{\;1}_{[55]:\overline{10}|} + 500.$$

We have $\ddot{a}_{[55]:\overline{10}|} = 8.0219$ and under UDD (using (5.38) and (5.39)),

$$\ddot{a}^{(12)}_{[55]:\overline{10}|} = \alpha(12)\,\ddot{a}_{[55]:\overline{10}|} - \beta(12)\left(1 - {}_{10}E_{[55]}\right) = 7.8339,$$

since

$$\alpha(12) = \frac{id}{i^{(12)}d^{(12)}} = 1.000197 \quad \text{and} \quad \beta(12) = \frac{i - i^{(12)}}{i^{(12)}d^{(12)}} = 0.466508.$$

Similarly, noting that $\ddot{a}_{[55]:\overline{1}|} = 1$, we have

$$\ddot{a}^{(12)}_{[55]:\overline{1}|} = \alpha(12) - \beta(12)\left(1 - {}_1E_{[55]}\right) = 0.9772.$$

Further,

$$\bar{A}^{\ \ 1}_{[55]:\overline{10}|} = \frac{i}{\delta} A^{\ \ 1}_{[55]:\overline{10}|} = \frac{i}{\delta}\left(1 - d\,\ddot{a}_{[55]:\overline{10}|} - {}_{10}E_{[55]}\right) = 0.024954,$$

giving $P = \$18.99$ per month. □

Example 6.7 A life insurance company issued a with-profit whole life policy to a select life aged 40. Under the policy, the basic sum insured of \$100 000 and attaching bonuses are payable at the end of the year of death. The company declares compound reversionary bonuses at the end of each year, and these bonuses do not apply to policies that became claims during the year. Level premiums are payable annually in advance under the policy.

Calculate the annual gross premium on the following basis:

Survival model:	Standard Select Survival Model
Interest:	5% per year
Bonus:	2.5% compound per year
Initial Expenses:	\$200
Renewal Expenses:	5% of each premium after the first

Solution 6.7 The EPV of premiums less premium related expenses is

$$0.95P\,\ddot{a}_{[40]} + 0.05P = (0.95 \times 18.4596 + 0.05)\,P = 17.5866P.$$

The death benefit increases by 2.5% each year, with the first bonus applying to deaths in the second year, so that the death benefit in the tth policy year, $t = 1, 2, 3, \ldots$, is $100\,000\,(1.025^{t-1})$.

The EPV of the death benefit is then

$$100\,000 \sum_{k=0}^{\infty} (1.025^k)\, v^{k+1}\ {}_{k|}q_{[40]} = \frac{100\,000}{1.025} A_{[40]j} = 32\,816.71$$

where $j = \frac{1.05}{1.025} - 1 = 0.0244$. Hence the equation of value is

$$17.5866P = 32816.71 + 200$$

giving a gross annual premium of $P = \$1877.38$. □

Example 6.8 Calculate the gross single premium for a deferred annuity of \$80 000 per year payable monthly in advance, issued to a select life now aged 50, with the first annuity payment due on the annuitant's 65th birthday. Allow for initial expenses of \$1 000, and renewal expenses on each anniversary of the issue date, provided that the policyholder is alive. Assume that the renewal expense will be \$20 on the first anniversary of the issue date, and that expenses will increase with inflation from that date at the compound rate of 1% per year. Assume the Standard Select Survival Model with interest at 5% per year.

Solution 6.8 The single premium is equal to the EPV of the deferred annuity plus the EPV of expenses.

The renewal expense on the tth policy anniversary is $20(1.01^{t-1})$ for $t = 1, 2, 3, \ldots$ so the EPV of the renewal expenses is

$$
\begin{aligned}
20 \sum_{t=1}^{\infty} 1.01^{t-1}\, v^t\, {}_tp_{[50]} &= \frac{20}{1.01} \sum_{t=1}^{\infty} 1.01^t\, v^t\, {}_tp_{[50]} \\
&= \frac{20}{1.01} \sum_{t=1}^{\infty} v_j^t\, {}_tp_{[50]} \\
&= \frac{20}{1.01} (\ddot{a}_{[50]_j} - 1) \\
&= \frac{20}{1.01} \times 18.4550 = 365.45
\end{aligned}
$$

where $1.01v = 1/(1 + j)$ so that $j = 0.0396$.

The EPV of the deferred annuity is

$$80\,000\ {}_{15|}\ddot{a}_{[50]}^{(12)} = 80\,000 \times 6.0413 = 483\,304,$$

and so the single premium is

$$P = 483\,304 + 1000 + 365.45 = \$484\,669.$$

□

We end this section with a comment on the premiums calculated in Examples 6.5 and 6.6. In Example 6.5, the annual premium is \$3 260.60 and the expenses at time 0 are \$2 000 plus 50% of the first premium, a total of \$3 630.30, which exceeds the first premium. Similarly, in Example 6.6 the total premium in the first year is \$227.88 and the total expenses in the first year are \$500 plus 10% of premiums in the first year. In each case, the premium income in the first year is insufficient to cover expenses in the first year. This situation is common in practice, especially when initial commission to agents is high, and is referred to as **new business strain**. A consequence of new business strain is that an insurer needs to have funds available in order to sell policies. From time to time insurers get into financial difficulties through pursuing an aggressive growth strategy without sufficient capital to support the new business strain. Essentially, the insurer borrows from shareholder (or participating policyholder) funds in order to write new business. These early expenses are gradually paid off by the expense loadings in future premiums.

6.5 Profit

The equivalence principle does not allow explicitly for a loading for profit. Since issuing new policies generally involves a loan from shareholder or participating policyholder funds to cover the new business strain, it is necessary for the business to generate sufficient income for the shareholders to make an adequate rate of return – in other words, the insurer needs to make a profit. In traditional insurance, we often load for profit implicitly, through margins in the valuation assumptions. For example, if we expect to earn an interest rate of 6% per year on assets, we might assume only 5% per year in the premium basis. The extra income from the invested premiums, known as the **interest spread**, will contribute to profit. In participating business, much of the profit will be distributed to the policyholders in the form of cash dividends or bonus. Some will be paid as dividends to shareholders. We may also use margins in the mortality assumptions. For a term insurance, we might use a slightly higher overall mortality rate than we expect. For an annuity, we might use a slightly lower rate.

Now, for each individual policy the experienced mortality rate in any year can take only the values 0 or 1. So, while the expected outcome under the equivalence principle is zero profit (assuming no margins), the actual outcome for each individual policy will either be a profit or a loss. For the actual profit from a group of policies to be reliably close to the expected profit, we need to sell contracts to a large number of individuals, whose future lifetimes can be regarded as statistically independent, so that the losses and profits from individual policies are combined.

As a simple illustration of this, consider a life who purchases a one-year term insurance with sum insured \$1000 payable at the end of the year of death. Let us suppose that the life is subject to a mortality of rate of 0.01 over the year, that the insurer can earn interest at 5% per year, and that there are no expenses. Then, using the equivalence principle, the premium is

$$P = 1\,000 \times 0.01/1.05 = 9.52.$$

The net loss at issue random variable is

$$L_0^n = \begin{cases} 1\,000v - P = 942.86 & \text{if } T_x \le 1, \quad \text{with probability } 0.01, \\ -P = -9.52 & \text{if } T_x > 1, \quad \text{with probability } 0.99. \end{cases}$$

The expected loss is $0.01 \times 942.86 + 0.99 \times (-9.52) = 0$, as required by the equivalence principle, but the probability of profit is 0.99, and the probability of loss is 0.01. The balance arises because the profit, if the policyholder survives the year, is small, and the loss, if the policyholder dies, is large. Using the equivalence principle, so that the expected loss at issue is zero, makes sense only if the insurer issues a large number of policies, so that the overall proportion of policies becoming claims will be close to the assumed proportion of 0.01.

Now suppose the insurer were to issue 100 such policies to independent lives. The insurer would expect to make a (small) profit on 99 of them. If the outcome from this portfolio is that all lives survive for the year, then the insurer makes a profit. If one life dies, there is no profit or loss. If more than one life dies, there will be a loss on the portfolio. Let D denote the number of deaths in the portfolio, so that $D \sim B(100, 0.01)$. The probability that the profit on the whole portfolio is greater than or equal to zero is

$$\Pr[D \leq 1] = 0.73576$$

compared with 99% for the individual contract. In fact, as the number of policies issued increases, the probability of profit will tend, monotonically, to 0.5. On the other hand, while the probability of loss is increasing with the portfolio size, the probability of very large aggregate losses (relative, say, to total premiums) is much smaller for a large portfolio, since there is a balancing effect from **diversification** of the risk amongst the large group of policies.

Let us now consider a whole life insurance policy with sum insured S payable at the end of the year of death, initial expenses of I, renewal expenses of e associated with each premium payment (including the first) issued to a select life aged x by annual premiums of P. For this policy,

$$L_0^g = S\, v^{K_{[x]}+1} + I + e\, \ddot{a}_{\overline{K_{[x]}+1}|} - P\, \ddot{a}_{\overline{K_{[x]}+1}|}.$$

If the policyholder dies shortly after the policy is issued, so that only a few premiums are paid, the insurer will make a loss. We can illustrate this by considering an example; suppose that $x = 30$, $S = \$100\,000$, $I = \$1\,000$, and $e = \$50$. Using the Standard Select Life Table at 5%, we find that $P = \$498.45$.

Suppose we know that $K_{[30]} = 5$. Then the value of L_0^g is

$$100\,000v^6 + 1000 + 50\ddot{a}_{\overline{6}|} - 498.45\ddot{a}_{\overline{6}|} = 73\,231.54,$$

which represents a substantial loss.

Conversely, if the policyholder lives to a ripe old age, we would expect that the insurer would make a profit as the policyholder will have paid a large number of premiums, and there will have been plenty of time for the premiums to accumulate interest. In the case of whole life policy sold to (30), suppose we know that $K_{[30]} = 60$. In this case the value of the gross loss at issue random variable is

$$100\,000v^{61} + 1000 + 50\ddot{a}_{\overline{61}|} - 498.45\ddot{a}_{\overline{61}|} = -2\,838.67,$$

which represents a decent profit to the insurer.

We can use the loss at issue random variable to find the minimum time that the policyholder needs to survive in order that the insurer makes a profit on an individual policy. The probability of a profit on the individual policy, $\Pr[L_0^g < 0]$, is given by

$$\Pr[L_0^g < 0] = \Pr\left[Sv^{K_{[x]}+1} + I + e\,\ddot{a}_{\overline{K_{[x]}+1}|} - P\,\ddot{a}_{\overline{K_{[x]}+1}|} < 0\right].$$

Rearranging, and replacing $\ddot{a}_{\overline{K_{[x]}+1}|}$ with $(1 - v^{K_{[x]}+1})/d$, gives

$$\begin{aligned}\Pr[L_0^g < 0] &= \Pr\left[v^{K_{[x]}+1} < \frac{\frac{P-e}{d} - I}{S + \frac{P-e}{d}}\right] \\ &= \Pr\left[K_{[x]} + 1 > \frac{1}{\delta}\log\left(\frac{P-e+Sd}{P-e-Id}\right)\right]. \qquad (6.4)\end{aligned}$$

Suppose we denote the right-hand side term of the inequality in (6.4) by τ. That means that the contract generates a profit for the insurer if $K_{[x]} + 1 > \tau$. Generally, τ is not an integer, and $K_{[x]}$ can only take integer values. So, if $\lfloor\tau\rfloor$ denotes the integer part of τ, then the insurer makes a profit on the policy if $K_{[x]} + 1 > \lfloor\tau\rfloor$, or, equivalently, if $K_{[x]} \geq \lfloor\tau\rfloor$. As

$$K_{[x]} \geq \lfloor\tau\rfloor \iff T_{[x]} > \lfloor\tau\rfloor,$$

the insurer makes a profit if $T_{[x]} > \lfloor\tau\rfloor$, the probability of which is ${}_{\lfloor\tau\rfloor}p_{[x]}$.

Continuing our example, with the whole life policy issued to [30], we have

$$\tau = \frac{1}{\delta}\log\left(\frac{498.45 - 50 + 100\,000\,d}{498.45 - 50 - 1000\,d}\right) = 52.57$$

so there is a profit if $K_{[30]} + 1 > 52.57$, which means $K_{[30]}$ must be greater than or equal to 52, so there is a profit if the life survives for 52 years, the probability of which is ${}_{52}p_{[30]} = 0.70704$.

We can check this by calculating the loss at issue random variable for $K_{[30]} = 51$, which gives $L_0^g = 237.07$ and for $K_{[30]} = 52$, which gives $L_0^g = -175.05$.

Figure 6.1 shows the value of the loss at issue against possible values for $K_{[30]}$. We see that large losses can occur in the early years of the policy, but the

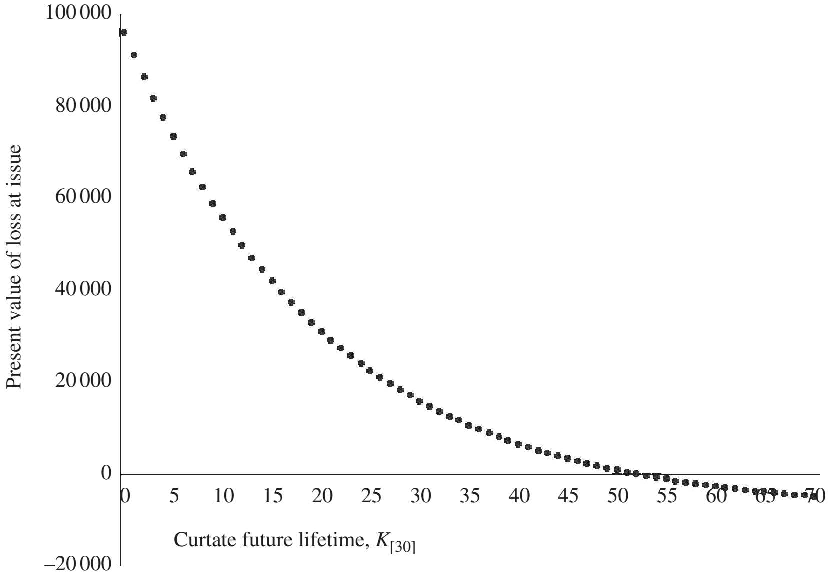


Figure 6.1 Present value of loss at issue by curtate future lifetime for the whole life insurance described in Section 6.5.

probability that the life dies in the first 20 years is small. The profits for long-lived policyholders are less substantial, but since the probabilities are greater, the net effect is an expected present value of zero.

Below, we present some different examples of the profit and loss calculation for an individual policy. The point at which a policy moves from loss to profit, or vice versa, will depend on the nature of the benefits and premiums, and it may not be as simple as the whole life example in this section. For example, a deferred insurance policy would have a period of profit, followed by a period of loss, followed by profit. It is important to consider each case from first principles.

Example 6.9 A life insurer is about to issue a 25-year endowment insurance with a basic sum insured of \$250 000 to a select life aged exactly 30. Premiums are payable annually throughout the term of the policy. Initial expenses are \$1200 plus 40% of the first premium and renewal expenses are 1% of the second and subsequent premiums. The insurer allows for a compound reversionary bonus of 2.5% of the basic sum insured, vesting on each policy anniversary (including the last). The death benefit is payable at the end of the year of death. Assume the Standard Select Survival Model with interest at 5% per year.

(a) Derive an expression for the loss at issue random variable, L_0^g, for this policy.

(b) Calculate the annual premium for this policy.

(c) Let $L_0(k)$ denote the present value of the loss on the policy given that $K_{[30]} = k$ for $k \le 24$ and let $L_0(25)$ denote the present value of the loss on the policy given that the policyholder survives to age 55. Calculate $L_0(k)$ for $k = 0, 1, \ldots, 25$.

(d) Calculate the probability that the insurer makes a profit on this policy.

(e) Calculate $\mathrm{V}[L_0^g]$.

Solution 6.9 (a) First we note that, if $K_{[30]} = k$ where $k = 0, 1, 2, \ldots, 24$, then the number of bonus additions is k, and the death benefit is payable $k + 1$ years from issue. The present value of the death benefit would be $250\,000(1.025^{K_{[30]}})v^{K_{[30]}+1}$. However, if the policyholder survives for 25 years, then 25 bonuses are applied. Thus, if P denotes the annual premium,

$$L_0^g = 250\,000(1.025^{\min(K_{[30]},\,25)})v^{\min(K_{[30]}+1,\,25)} \\ + 1200 + 0.39P - 0.99P\,\ddot{a}_{\overline{\min(K_{[30]}+1,\,25)|}}.$$

(b) The EPV of the premiums, less premium expenses, is

$$0.99P\,\ddot{a}_{[30]:\overline{25|}} = 14.5838P.$$

As the death benefit is \$250 000(1.025^t) if the policyholder dies in the tth policy year, the EPV of the death benefit is

$$250\,000\sum_{t=0}^{24} v^{t+1}\,{}_{t|}q_{[30]}\,(1.025^t) = 250\,000\left(\frac{1}{1.025}A^{\;1}_{[30]:\overline{25|}\,j}\right) = 3099.37$$

where $1 + j = (1 + i)/1.025$, so that $j = 0.0244$.

The EPV of the survival benefit is

$$250\,000v^{25}\,{}_{25}p_{[30]}\,1.025^{25} = 134\,295.43,$$

and the EPV of the remaining expenses is

$$1\,200 + 0.39P.$$

Hence, equating the EPV of premium income with the EPV of benefits plus expenses we find that $P = \$9\,764.44$.

(c) Given that $K_{[30]} = k$, where $k = 0, 1, \ldots, 24$, the present value of the loss is the present value of the death benefit payable at time $k + 1$ less the present value of $k + 1$ premiums plus the present value of expenses. Hence

$$L_0(k) = 250\,000(1.025^k)\,v^{k+1} + 1\,200 + 0.39P - 0.99P\,\ddot{a}_{\overline{k+1|}}.$$

Table 6.1 *Values of the loss at issue random variable for Example 6.9.*

Value of $K_{[30]}$, k	PV of loss, $L_0(k)$
0	233 437
1	218 561
⋮	⋮
23	1 737
24	−4 517
≥ 25	−1 179

If the policyholder survives to age 55, there is one extra bonus payment, and the present value of the loss at issue is

$$L_0(25) = 250\,000(1.025^{25})\,v^{25} + 1\,200 + 0.39P - 0.99P\,\ddot{a}_{\overline{25}|}.$$

Some values of the present value of the loss at issue are shown in Table 6.1.

(d) The full set of values for the present value of the loss at issue shows that there is a profit if and only if the policyholder survives 24 years and pays the premium at the start of the 25th policy year. Hence the probability of a profit is ${}_{24}p_{[30]} = 0.98297$.

Note that this probability is based on the assumption that the future expenses and interest rates used in the premium basis are fixed and known.

(e) From the full set of values for $L_0(k)$ we can calculate

$$\mathrm{E}[(L_0^g)^2] = \sum_{k=0}^{24} (L_0(k))^2\, {}_{k|}q_{[30]} + (L_0(25))^2\, {}_{25}p_{[30]} = 12\,115.55^2,$$

which is equal to the variance as $\mathrm{E}[L_0^g] = 0$. □

Generally speaking, for an insurance policy, the longer a life survives, the greater is the profit to the insurer, as illustrated in Figure 6.1. However, the converse is true for annuities, as the following example illustrates.

Example 6.10 An insurance company is about to issue a single premium deferred annuity to a select life aged 55. The first annuity payment is due 10 years from the issue date, and payments will be annual. The first annuity payment will be \$50 000, and each subsequent payment will be 3% greater than the previous payment. Assume that there are no expenses, that mortality follows the Standard Select Life Table, and that interest is $i = 5\%$ per year. You are given that, on this basis, the single premium is $P = 546\,812$.

Calculate the probability that

(a) the insurance company makes a profit from this policy, and
(b) the present value of the loss exceeds \$100 000.

Solution 6.10 (a) Let $L_0(k)$ denote the present value of the loss given that $K_{[55]} = k$, for $k = 0, 1, 2, \ldots$. Note first that if the life dies during the deferred period, the insurer receives the single premium and makes no payments. Next, if the life dies in the year k to $k+1$, where $k \geq 10$, then the present value of the loss is

$$50\,000\left(v^{10} + (1.03)v^{11} + (1.03^2)v^{12} + \cdots + (1.03^{k-10})v^k\right) - P$$

$$= 50\,000\,v^{10}\left(1 + (1.03)v + (1.03^2)v^2 + \cdots + (1.03^{k-10})v^{k-10}\right) - P$$

$$= 50\,000\,v^{10}\,\ddot{a}_{\overline{k+1-10}|j} - P$$

where $1 + j = 1.05/1.03$ giving $j = 0.0194$. So

$$L_0(k) = \begin{cases} -P & \text{for } k = 0, 1, \ldots, 9, \\ 50\,000\,v^{10}\,\ddot{a}_{\overline{k-9}|j} - P & \text{for } k = 10, 11, \ldots. \end{cases} \tag{6.5}$$

Since $\ddot{a}_{\overline{k-9}|j}$ is an increasing function of k, formula (6.5) shows that $L_0(k)$ is an increasing function of k for $k \geq 10$, which means that the insurer's loss increases with the survival of the annuitant, after 10 years. Clearly, the insurer makes a profit of P if the life dies during the deferred period, and makes a smaller profit if the life dies in the first few years of the payment period. Using formula (6.5), the condition for the individual contract to be profitable to the insurer is

$$50\,000\,v^{10}\,\ddot{a}_{\overline{k-9}|j} - P < 0,$$

or, equivalently,

$$\ddot{a}_{\overline{k-9}|j} < \frac{(1.05^{10})P}{50\,000}.$$

Writing $\ddot{a}_{\overline{k-9}|j} = \dfrac{1 - v^{k-9}}{d_j}$ and $d_j = j/(1+j)$, this condition becomes

$$v_j^{k-9} > 1 - \frac{d_j\,(1.05^{10})P}{50\,000}.$$

As $v_j = \exp\{-\delta_j\}$ where $\delta_j = \log(1+j)$ this gives

$$k < 9 - \frac{1}{\delta_j}\log\left(1 - \frac{d_j\,(1.05^{10})P}{50\,000}\right) = 30.55.$$

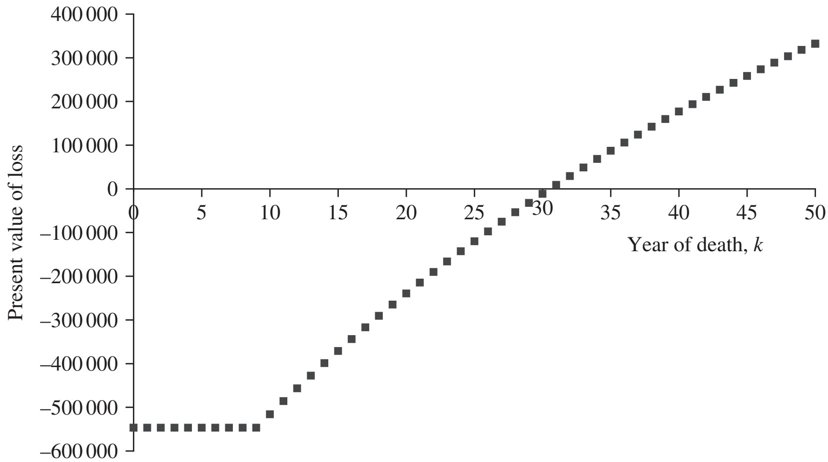


Figure 6.2 Present value of loss from Example 6.10.

So we need $K_{[55]} \leq 30$, which means we need $T_{[55]} < 31$ for the contract to make a profit. The probability of this is ${}_{31}q_{[55]} = 0.41051$.

(b) The present value of the loss will exceed 100 000 if

$$50\,000\, v^{10}\, \ddot{a}_{\overline{k-9}|j} - P > 100\,000,$$

and following through exactly the same arguments as in part (a) we find that $L_0(k) > 100\,000$ if $k > 35.68$, which requires $K_{[55]} \geq 36$, or equivalently $T_{[55]} > 36$, and the probability of this is ${}_{36}p_{[55]} = 0.38462$.

Figure 6.2 shows $L_0(k)$ for $k = 1, 2, \ldots, 50$. We can see that the loss is constant for the first 10 years at $-P$ and then increases due to annuity payments. In contrast to Figure 6.1, longevity results in large losses to the insurer. We can also clearly see from this figure that the loss is negative if k takes a value less than 31, confirming our answer to part (a). □

6.6 The portfolio percentile premium principle

The portfolio percentile premium principle is an alternative to the equivalence premium principle. We assume a large portfolio of identical and independent policies. By 'identical' we mean that the policies have the same premium, benefits, term, and so on, and that the policyholders are all subject to the same survival model. By 'independent' we mean that the policyholders are independent of each other with respect to mortality.

Suppose we know the sum insured for these policies, and wish to find an appropriate premium. As the policies are identical, each policy has the same loss at issue random variable. Let N denote the number of policies in the portfolio and let $L_{0,i}$ represent the loss at issue random variable for the ith policy in the portfolio, $i = 1, 2, 3, \ldots, N$. The total loss at issue in the portfolio is L, say, where $L = \sum_{i=1}^{N} L_{0,i}$,

$$\mathrm{E}[L] = \sum_{i=1}^{N} \mathrm{E}[L_{0,i}] = N\mathrm{E}[L_{0,1}],$$

$$\mathrm{V}[L] = \sum_{i=1}^{N} \mathrm{V}[L_{0,i}] = N\mathrm{V}[L_{0,1}].$$

(Note that as $\{L_{0,i}\}_{i=1}^{N}$ are identically distributed, the mean and variance of each $L_{0,i}$ are equal to the mean and variance of $L_{0,1}$.)

The portfolio percentile premium principle sets a premium to give a specified probability, say α, that the total loss at issue is negative. That is, P is set such that

$$\Pr[L < 0] = \alpha.$$

Now, if N is sufficiently large (say, greater than around 30), the central limit theorem tells us that L is approximately normally distributed, with mean $\mathrm{E}[L] = N\,\mathrm{E}[L_{0,1}]$ and variance $\mathrm{V}[L] = N\,\mathrm{V}[L_{0,1}]$. In this case, the portfolio percentile principle premium can be calculated from

$$\Pr[L < 0] = \Pr\left(\frac{L - \mathrm{E}[L]}{\sqrt{\mathrm{V}[L]}} < \frac{-\mathrm{E}[L]}{\sqrt{\mathrm{V}[L]}}\right) = \Phi\left(\frac{-\mathrm{E}[L]}{\sqrt{\mathrm{V}[L]}}\right) = \alpha,$$

which implies that

$$\frac{\mathrm{E}[L]}{\sqrt{\mathrm{V}[L]}} = -\Phi^{-1}(\alpha)$$

where Φ is the cumulative distribution function of the standard normal distribution.

Our aim is to calculate P, but P does not appear explicitly in either of the last two equations. However, as illustrated in the next example, both the mean and the variance of L are functions of P.

Example 6.11 An insurer issues whole life insurance policies to select lives aged 30. The sum insured of \$100 000 is paid at the end of the month of death and level monthly premiums are payable throughout the term of the policy. Initial expenses, incurred at the issue of the policy, are 15% of the total of the first year's premiums. Renewal expenses are 4% of every premium, including those in the first year.

Assume the Standard Select Survival Model with interest at 5% per year.

(a) Calculate the monthly premium using the equivalence principle.
(b) Calculate the monthly premium using the portfolio percentile principle, such that the probability that the loss at issue on the portfolio is negative is 95%. Assume a portfolio of 10 000 identical, independent policies.

Solution 6.11 (a) Let P be the monthly premium. Then the EPV of premiums is

$$12P\,\ddot{a}_{[30]}^{(12)} = 227.065P.$$

The EPV of benefits is

$$100\,000\,A_{[30]}^{(12)} = 7\,866.18,$$

and the EPV of expenses is

$$0.15 \times 12P + 0.04 \times 12P\,\ddot{a}_{[30]}^{(12)} = 10.8826P.$$

Equating the EPV of premiums with the EPVs of benefits and expenses gives the equivalence principle premium as \$36.39 per month.

(b) We need the mean and standard deviation of the total loss at issue random variable, which we calculate from the mean and standard deviation of $L_{0,i}$, the loss at issue for the ith policy, which is

$$L_{0,i} = 100\,000\,v^{K_{[30]}^{(12)}+\frac{1}{12}} + 0.15 \times 12P - 0.96 \times 12P\,\ddot{a}^{(12)}_{\overline{K_{[30]}^{(12)}+\frac{1}{12}}|}.$$

The expected value can be calculated using the solution to part (a) as

$$\mathrm{E}[L_{0,i}] = 7\,866.18 - 216.18P.$$

To find $\mathrm{SD}[L_{0,i}]$, we rewrite $L_{0,i}$ as

$$L_{0,i} = \left(100\,000 + \frac{0.96 \times 12P}{d^{(12)}}\right) v^{K_{[30]}^{(12)}+\frac{1}{12}} + 0.15 \times 12P - \frac{0.96 \times 12P}{d^{(12)}}$$

so that

$$\begin{aligned}\mathrm{V}[L_{0,i}] &= \left(100\,000 + \frac{0.96 \times 12P}{d^{(12)}}\right)^2 \left({}^2A_{[30]}^{(12)} - (A_{[30]}^{(12)})^2\right)\\ &= (100\,000 + 236.59P)^2\,(0.0053515)\end{aligned}$$

and $\mathrm{SD}[L_{0,i}] = (100\,000 + 236.59P)\,(0.073154)$. The loss at issue random variable for the whole portfolio of policies is $L = \sum_{i=1}^{10\,000} L_{0,i}$, so

$$\mathrm{E}[L] = 10\,000(7866.18 - 216.18P)$$

and, because the lives are independent, $\mathrm{V}[L] = \sum_{i=1}^{10\,000} \mathrm{V}[L_{0,i}]$, giving

$$\mathrm{SD}[L] = \sqrt{10\,000}\,(100\,000 + 236.59P)\,(0.073154).$$

Using the normal approximation to the distribution of L, we set P as follows, noting that for the standard normal distribution, $\Phi(1.645) = 0.95$:

$$\begin{aligned}
&\Pr[L < 0] = 0.95 \\
&\Rightarrow \Phi\left(\frac{-\mathrm{E}[L]}{\sqrt{\mathrm{V}[L]}}\right) = 0.95 \\
&\Rightarrow \Phi\left(\frac{10\,000(216.18P - 7\,866.18)}{100\,(100\,000 + 236.59P)\,(0.073154)}\right) = 0.95 \\
&\Rightarrow \frac{100(216.18P - 7\,866.18)}{(100\,000 + 236.59P)\,(0.073154)} = 1.645,
\end{aligned}$$

which gives $P = \$36.99$. □

Note that the solution to part (b) above depends on the number of policies in the portfolio (10 000) and the level of probability we set for the loss at issue being negative (0.95). If the portfolio had n policies instead of 10 000, then the equation we would have to solve for the premium, P, is

$$\frac{\sqrt{n}(216.18P - 7\,866.18)}{(100\,000 + 236.59P)\,(0.073154)} = 1.645. \tag{6.6}$$

Table 6.2 shows some values of P for different values of n. We note that P decreases as n increases. In fact, as $n \to \infty$, $P \to \$36.39$, which is the equivalence principle premium. The reason for this is that as $n \to \infty$ the insurer diversifies the mortality risk. We discuss diversification of risk further in Chapter 12.

Table 6.2 *Premiums according to portfolio size.*

n	P
1 000	38.31
2 000	37.74
5 000	37.24
10 000	36.99
20 000	36.81

6.7 Extra risks

As we discussed in Section 1.5.2, when an individual wishes to purchase a life insurance policy, underwriting takes place. If underwriting determines that an individual should not be offered insurance at standard rates, the individual might still be offered insurance, but above standard rates.

Another context for adjusting the mortality assumptions for an individual is in structured settlements, described in Chapter 1. The injuries involved in structured settlement compensation cases are often complex, and assessing appropriate mortality and morbidity rates is challenging. A serious injury would increase an individual's mortality, which would make an annuity cheaper compared with a life subject to a standard annuitants' mortality table. However, there is not much data on the relationship between different types of injury and mortality, so the adjustments to standard annuitant mortality rates may be quite arbitrary.

In this section we discuss some of the common ways in which extra mortality risk is modelled in EPV calculations.

6.7.1 Age rating

One reason why an individual might not be offered insurance at standard rates is that the individual suffers from a medical condition. In such circumstances we refer to the individual as an impaired life, and the insurer may compensate for this extra risk by treating the individual as being older. For example, an impaired life aged 40 might be asked to pay the same premium paid by a non-impaired life aged 45. This approach to modelling extra risk involves no new ideas in premium calculation – for example, we could apply the equivalence principle in our calculation, and we would simply change the policyholder's age. This is referred to as **age rating**, and this approach is commonly used for insurance premiums and in structured settlement calculations.

6.7.2 Constant addition to μ_x

Individuals can also be deemed to be ineligible for standard rates if they regularly participate in hazardous pursuits, for example parachuting. For such individuals the extra risk is largely independent of age, and so we could model this extra risk by adding a constant to the force of mortality – just as Makeham extended Gompertz' law of mortality. The application of this approach leads to some computational shortcuts. We are modelling the force of mortality as

$$\mu'_{[x]+s} = \mu_{[x]+s} + \phi$$

where functions with the superscript $'$ relate to the impaired life, functions without this superscript relate to a standard survival model and ϕ is the constant addition to the force of mortality. Then

$$ {}_tp'_{[x]} = \exp\left\{-\int_0^t \mu'_{[x]+s}\,ds\right\} = \exp\left\{-\int_0^t \left(\mu_{[x]+s} + \phi\right) ds\right\} = e^{-\phi t}\ {}_tp_{[x]}. $$

This formula is useful for computing the EPV of a survival benefit since

$$ e^{-\delta t}\ {}_tp'_{[x]} = e^{-(\delta+\phi)t}\ {}_tp_{[x]}, $$

so that, for example,

$$ {}_tE'_x = v^t\ {}_tp'_x = v^t e^{-\phi\,t}\,{}_tp_x = ({}_tE_x)_j $$

where j denotes calculation at interest rate $j = e^{\phi+\delta} - 1$. Further,

$$ \ddot{a}'_{[x]:\overline{n}|} = \sum_{t=0}^{n-1} e^{-\delta t}\ {}_tp'_{[x]} = \sum_{t=0}^{n-1} e^{-(\delta+\phi)t}\ {}_tp_{[x]} = \ddot{a}_{[x]:\overline{n}|\,j}, \tag{6.7} $$

where $\ddot{a}_{[x]:\overline{n}|\,j}$ is calculated using rate of interest j and the standard survival model. That means, for calculating the EPV of an annuity, the extra mortality can be modelled with an adjustment to the interest rate, which is simple to apply. However, this *only* applies to annuity and pure endowment functions, not to insurance functions involving mortality rates.

To understand why, consider the impaired life's curtate future lifetime $K'_{[x]}$. We know that

$$ \ddot{a}'_{[x]:\overline{n}|} = \mathrm{E}\left[\ddot{a}_{\overline{\min(K'_{[x]}+1,n)}|}\right] = \frac{1 - \mathrm{E}[v^{\min(K'_{[x]}+1,\,n)}]}{d} = \frac{1 - A'_{[x]:\overline{n}|}}{d}, $$

and hence

$$ A'_{[x]:\overline{n}|} = 1 - d\,\ddot{a}'_{[x]:\overline{n}|} = 1 - d\,\ddot{a}_{[x]:\overline{n}|\,j}. \tag{6.8} $$

So we see that for the insurance benefit we cannot just change the interest rate; in formula (6.8), the annuity function is evaluated at rate j, but d is calculated using the original rate of interest, that is $d = i/(1+i)$. Generally, when using the constant addition to the force of mortality, it is simplest to calculate the annuity function first, using a simple adjustment to the interest rate, then use formula (6.8) for any insurance functions.

Example 6.12 Calculate the annual premium for a 20-year endowment insurance with sum insured \$200 000 issued to a life aged 30 whose force of mortality at age $30 + s$ is given by $\mu_{[30]+s} + 0.01$. Allow for initial expenses of \$2000 plus 40% of the first premium, and renewal expenses of 2% of the second and subsequent premiums. Use the Standard Select Survival Model with interest at 5% per year.

Solution 6.12 Let P denote the annual premium. Then by applying formula (6.7), the EPV of premium income is

$$P\sum_{t=0}^{19} v^t\ {}_tp'_{[30]} = P\,\ddot{a}_{[30]:\overline{20}|j}$$

where $j = 1.05e^{0.01} - 1 = 0.0606$. Similarly, the EPV of expenses is

$$2000 + 0.38P + 0.02P\,\ddot{a}_{[30]:\overline{20}|j}.$$

The EPV of the benefit is $200\,000A'_{[30]:\overline{20}|}$, where the dash denotes extra mortality and the interest rate is $i = 0.05$. Using formula (6.8)

$$A'_{[30]:\overline{20}|} = 1 - d\,\ddot{a}_{[30]:\overline{20}|j}.$$

As $\ddot{a}_{[30]:\overline{20}|j} = 12.072$ and $d = 0.05/1.05$, we find that $A'_{[30]:\overline{20}|} = 0.425158$ and hence we find that $P = \$7\,600.84$. □

6.7.3 Constant multiple of mortality rates

A third method of allowing for extra mortality is to assume that lives are subject to mortality rates that are higher than the standard lives' mortality rates. For example, we might set $q'_{[x]+t} = 1.1q_{[x]+t}$ where the superscript $'$ again denotes extra mortality risk. With such an approach we can calculate the probability of surviving one year from any integer age, and hence we can calculate the probability of surviving an integer number of years. A computational disadvantage of this approach is that we have to apply approximations in calculating EPVs if payments are at other than annual intervals. Generally, this form of extra risk would be handled by recalculating the required functions in a spreadsheet.

Example 6.13 Calculate the monthly premium for a 10-year term insurance with sum insured \$100 000 payable immediately on death, issued to a life aged 50. Assume that each year throughout the 10-year term the life is subject to mortality rates that are 10% higher than for a standard life of the same age. Allow for initial expenses of \$1000 plus 50% of the first monthly premium and renewal expenses of 3% of the second and subsequent monthly premiums. Use the UDD assumption where appropriate, and use the Standard Select Survival Model with interest at 5% per year.

Solution 6.13 Let P denote the total premium per year. Then the EPV of premium income is $P\ddot{a}^{(12)\,\prime}_{50:\overline{10}|}$ and, assuming UDD, we compute $\ddot{a}^{(12)\,\prime}_{50:\overline{10}|}$ as

$$\ddot{a}^{(12)\,\prime}_{50:\overline{10}|} = \alpha(12)\,\ddot{a}'_{50:\overline{10}|} - \beta(12)\big(1 - v^{10}\ {}_{10}p'_{50}\big),$$

where $\alpha(12) = 1.000197$ and $\beta(12) = 0.466508$. As the initial expenses are \$1000 plus 50% of the first premium, which is $\frac{1}{12}P$, we can write the EPV of expenses as

$$1000 + \frac{0.47P}{12} + 0.03P\,\ddot{a}^{(12)\,\prime}_{50:\overline{10}|}.$$

Finally, the EPV of the death benefit is $100\,000(\bar{A}^{\;1}_{50:\overline{10}|})'$ and, using UDD, we can compute this as

$$\begin{aligned}(\bar{A}^{\;1}_{50:\overline{10}|})' &= \frac{i}{\delta}(A^{\;1}_{50:\overline{10}|})' \\ &= \frac{i}{\delta}\left((A_{50:\overline{10}|})' - v^{10}\ {}_{10}p'_{50}\right) \\ &= \frac{i}{\delta}\left(1 - d\ddot{a}'_{50:\overline{10}|} - v^{10}\ {}_{10}p'_{50}\right).\end{aligned}$$

The formula for $\ddot{a}'_{50:\overline{10}|}$ is

$$\ddot{a}'_{50:\overline{10}|} = \sum_{t=0}^{9} v^t\ {}_tp'_{50}$$

where

$${}_tp'_{50} = \prod_{r=0}^{t-1}(1 - 1.1q_{[50]+r}). \tag{6.9}$$

(We have written $q_{[50]+r}$ in formula (6.9) as standard lives are subject to select mortality.) Hence $\ddot{a}'_{50:\overline{10}|} = 8.0516$, $\ddot{a}^{(12)\,\prime}_{50:\overline{10}|} = 7.8669$ and $(\bar{A}^{\;1}_{50:\overline{10}|})' = 0.01621$, which give $P = \$345.18$ and so the monthly premium is \$28.76.

Table 6.3 *Spreadsheet calculations for Example 6.13.*

(1) t	(2) ${}_tp'_{50}$	(3) ${}_{t\mid}q'_{50}$	(4) v^t	(5) v^{t+1}	(6) (2) × (4)	(7) (3) × (5)
0	1.0000	0.0011	1.0000	0.9524	1.0000	0.0011
1	0.9989	0.0014	0.9524	0.9070	0.9513	0.0013
2	0.9975	0.0016	0.9070	0.8638	0.9047	0.0014
3	0.9959	0.0018	0.8638	0.8227	0.8603	0.0015
4	0.9941	0.0020	0.8227	0.7835	0.8178	0.0015
5	0.9921	0.0022	0.7835	0.7462	0.7774	0.0016
6	0.9899	0.0024	0.7462	0.7107	0.7387	0.0017
7	0.9875	0.0027	0.7107	0.6768	0.7018	0.0018
8	0.9849	0.0030	0.6768	0.6446	0.6666	0.0019
9	0.9819	0.0033	0.6446	0.6139	0.6329	0.0020
				Total	8.0516	0.0158

Table 6.3 shows how we could set out a spreadsheet to perform calculations. Column (2) was created from the original mortality rates using formula (6.9), with column (3) being calculated as

$$_{t|}q'_{50} = \, _tp'_{50} \, (1 - 1.1\, q_{50+t}).$$

The total in column (6) gives $\ddot{a}'_{50:\overline{10|}}$ while the total in column (7) gives the value for $\left(A^{\,1}_{50:\overline{10|}}\right)'$. Note that this must then by multiplied by i/δ to get $\left(\bar{A}^{\,1}_{50:\overline{10|}}\right)'$. □

6.8 Notes and further reading

The equivalence principle is the traditional approach to premium calculation, and we apply it again in Section 7.5, when we consider the possibility that a policy may terminate for reasons other than death.

A modification of the equivalence principle which builds an element of profit into a premium calculation is to select a profit target amount for each policy, Π, say, and set the premium to be the smallest possible such that $\mathrm{E}[L_0] \le \Pi$. Under this method of calculation we effectively set a level for the expected present value of future profit from the policy and calculate the premium by treating this amount as an additional cost at the issue date which will be met by future premium income.

Besides the premium principles discussed in this chapter, there is one further important method of calculating premiums. This is profit testing, which is the subject of Chapter 13.

The International Actuarial Notation for premiums may be found in Bowers *et al.* (1997). We have omitted it in this book because we find it has no particular benefit in practice.

6.9 Exercises

In the following questions we use the term **fully discrete** as a shorthand to mean that premiums are payable annually in advance and death benefits are payable at the end of the year of death, and **fully continuous** to mean that premiums are payable continuously, and death benefits are payable immediately on death. In all other cases the payment frequency is specified explicitly.

Shorter exercises

Exercise 6.1 Consider a fully discrete whole life insurance with sum insured \$200 000 issued to a select life aged 30. The premium payment term is

20 years. Assume that mortality follows the Standard Select Life Table, and assume an interest rate of 5% per year.

(a) Write down an expression for the net loss at issue random variable.
(b) Calculate the net annual premium.
(c) Calculate the probability that the contract makes a profit.

Exercise 6.2 Consider a five-year term insurance issued to a select life aged 40 by a single premium, with sum insured \$1 million payable immediately on death. Assume that mortality follows the Standard Select Life Table, with UDD between integer ages, and assume an interest rate of 5% per year.

(a) Write down an expression for the net loss at issue random variable.
(b) Calculate the net single premium.
(c) Calculate the probability that the contract makes a profit.

Exercise 6.3 You are given the following extract from a select life table with a four-year select period. A select individual aged 41 purchased a fully discrete three-year term insurance with a net annual premium of \$350.

$[x]$	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$	l_{x+4}	$x+4$
[40]	100 000	99 899	99 724	99 520	99 288	44
[41]	99 802	99 689	99 502	99 283	99 033	45
[42]	99 597	99 471	99 268	99 030	98 752	46

Use an effective rate of interest of 6% per year to calculate

(a) the sum insured,
(b) the standard deviation of L_0,
(c) $\Pr[L_0 > 0]$.

Exercise 6.4 An insurer issues a whole life insurance with sum insured \$100 000 to a select life aged 65. Premiums are payable quarterly, and the death benefit is payable at the end of the quarter-year of death. The premium basis is as follows.

- Mortality follows the Standard Select Life Table.
- Woolhouse's three-term formula is used to calculate quarterly annuity and insurance functions. You are given that $\mu_{[65]} = 0.00454$.
- $i = 5\%$.
- Maintenance expenses are 5% of gross premiums, including the first.
- Initial expenses are 1% of the sum insured, incurred at the inception of the policy.

(a) Calculate the gross annual premium using the equivalence principle.
(b) Calculate the gross annual premium which would give an expected profit of 110% of the gross annual premium at the issue date.

Exercise 6.5 Consider a fully discrete 10-year term insurance issued to a select life aged 50, with sum insured \$100 000. Assume that mortality follows the Standard Select Life Table, and assume an interest rate of 5% per year.

(a) Write down an expression for the net loss at issue random variable.
(b) Calculate the net annual premium.

Exercise 6.6 A select life aged 45 purchases a fully discrete 20-year endowment insurance with sum insured \$100 000. Calculate the annual premium using the following assumptions.

(i) Commission is 10% of the first premium and 2% of each subsequent premium.
(ii) Other expenses are \$50 at issue, and \$8 at each subsequent premium date.
(iii) Mortality follows the Standard Select Life Table.
(iv) Interest is 5% per year.

Exercise 6.7 Determine the annual premium for a 20-year term insurance with sum insured \$100 000 payable at the end of the year of death, issued to a select life aged 40 with premiums payable for at most 10 years, with expenses, which are incurred at the beginning of each policy year, as follows:

	Year 1		Years 2+	
	% of premium	Constant	% of premium	Constant
Taxes	4%	0	4%	0
Sales commission	25%	0	5%	0
Policy maintenance	0%	10	0%	5

Assume that mortality follows the Standard Select Life Table and use an interest rate of 5% per year.

Exercise 6.8 A fully discrete whole life insurance with unit sum insured is issued to (x).

- Let L_0 denote the net loss at issue random variable with the premium determined by the equivalence principle. You are given that $\mathrm{V}[L_0] = 0.75$
- Let L_0^* denote the net loss at issue random variable if the premium is determined such that $\mathrm{E}[L_0^*] = -0.5$.

Calculate $\mathrm{V}[L_0^*]$.

Exercise 6.9 A life insurance company issues a 10-year term insurance policy to a life aged 50, with sum insured \$100 000. Level premiums are payable monthly in advance throughout the term. You are given the following premium assumptions.

(i) Commission is 20% of each premium payment in the first year (incurred at the premium payment times) and 5% of all premiums after the first year.
(ii) Additional initial expenses are \$100.
(iii) Claim expenses are \$250.
(iv) The sum insured and claim expenses are payable one month after the date of death.
(v) Mortality follows the Standard Select Life Table, with UDD between integer ages.
(vi) $i = 5\%$.

Calculate the gross monthly premium.

Exercise 6.10 For a whole life insurance with sum insured \$150 000 paid at the end of the year of death, issued to (x), you are given:

(i) ${}^2A_x = 0.0143$,
(ii) $A_x = 0.0653$, and
(iii) the annual premium is determined using the equivalence principle.

Calculate the standard deviation of L_0^n.

Exercise 6.11 For a whole life insurance issued to (55), you are given:

- initial annual premiums are level for 10 years; thereafter annual premiums equal one-half of initial annual premiums,
- the death benefit is \$100 000 during the first 10 years of the contract, is \$50 000 thereafter, and is payable at the end of the year of death, and
- expenses are 25% of the first year's premium plus 3% of all subsequent premiums.

Calculate the initial annual gross premium, assuming that mortality follows the Standard Select Life Table, and that the interest rate is 5% per year.

Longer exercises

Exercise 6.12 Consider a fully discrete 20-year endowment insurance with sum insured \$100 000, issued to a select life aged 35. Initial expenses are 3% of the sum insured and 20% of the first premium, and renewal expenses are 3% of the second and subsequent premiums. Assume that mortality follows the Standard Select Life Table with interest at 5% per year.

(a) Write down an expression for the gross loss at issue random variable.
(b) Calculate the gross annual premium.
(c) Calculate the standard deviation of the gross loss at issue random variable.
(d) Calculate the probability that the contract makes a profit.

Exercise 6.13 A select life aged 45 purchases a single premium deferred annuity which provides an annuity of \$40 000 per year, payable annually in advance from age 65. In the event of death before age 65, the premium is returned at the end of the year of death. Assume that mortality follows the Standard Select Life Table with interest at 5% per year.

(a) Write down an expression for the net loss at issue random variable.
(b) Calculate the single premium.
(c) Now suppose that the annuity is guaranteed to be paid for at least five years if the life survives to age 65. Calculate the revised single premium.

Exercise 6.14 Consider a fully discrete with-profit whole life insurance issued to a select life aged 40. The basic sum insured is \$200 000 payable at the end of the year of death, and the premium term is 25 years. You are given the following premium assumptions:

- compound reversionary bonuses of 2.2% per year will be awarded to policies in force at the start of each year, from the second policy year,
- initial expenses are 60% of the annual premium,
- renewal commissions are 2.5% of all premiums after the first,
- other maintenance expenses, payable annually throughout the term of the contract, are \$5 at the beginning of the first year, increasing by 2% per year compound at the beginning of each subsequent year,
- mortality follows the Standard Select Life Table,
- $i = 5\%$.

You are also given the following table of values for $A_{[40]}$ at different interest rates:

j	2.64%	2.74%	2.84%	2.94%	3.04%
$A_{[40]\,j}$	0.30897	0.29627	0.28414	0.27256	0.26150

Calculate the annual premium.

Exercise 6.15 A life insurer is about to issue a 30-year deferred annuity-due with annual payments of \$20 000 to a select life aged 35. The policy has a single premium which is refunded without interest at the end of the year of death if death occurs during the deferred period. Assume that mortality follows the Standard Select Life Table, and that the interest rate is 5% per year.

(a) Calculate the single premium for this annuity.
(b) The insurer offers an option that if the policyholder dies before the total annuity payments exceed the single premium, then the balance will be

paid as a death benefit, at the end of the year of death. Calculate the revised premium.

This is called a **cash refund payout option**.

Exercise 6.16 A life insurance company sells annuities to men aged exactly 60. Each policyholder pays a single net premium, P, and then receives an annuity of \$30 000 a year in arrear (so that the first annuity payment is on the 61st birthday). Assume that mortality follows the Standard Select Life Table, and that the interest rate is 5% per year.

(a) Calculate P.
(b) Calculate the probability that the present value of profit on a single policy is positive.
(c) Calculate the standard deviation of the present value of profit on a single policy.
(d) Now suppose that the office sells 1 000 such annuities simultaneously to independent lives. Calculate the value of P such that the probability that the present value of the profit to the insurance company is positive is 95%.

Exercise 6.17 A life insurance company issues a single premium whole life immediate annuity with annual payments. The annuity is issued to (65) and payments are \$50 000 per year.

You are given the following information.

- Mortality follows the Standard Ultimate Life Table.
- Interest is 5% per year.
- Issue expenses are \$3 000 per policy, plus commission of 10% of the single premium.
- The single premium, G, is set at 110% of the expected present value of the benefits and expenses.

(a) Calculate G.
(b) Calculate the probability that the contract is profitable to the insurer.
(c) The insurer decides to change the pricing structure for the contract. The new price, G^*, is set using the portfolio percentile premium principle, assuming 8000 independent and identical contracts are sold, and using a profit percentile requirement of 90%.
Calculate G^*.

Exercise 6.18 A life is subject to extra risk that is modelled by a constant addition to the force of mortality, so that, if the extra risk functions are denoted by $'$, $\mu'_x = \mu_x + \phi$. Show that at rate of interest i,

$$\bar{A}'_x = (\bar{A}_x)_j + \phi\,(\bar{a}_x)_j,$$

where j is a rate of interest that you should specify.

Exercise 6.19 A life insurer is about to issue a 25-year annual premium endowment insurance with a basic sum insured of \$250 000 to a life aged exactly 30. Initial expenses are \$1200 plus 40% of the first premium and renewal expenses are 1% of the second and subsequent premiums. The office allows for a compound reversionary bonus of 2.5% of the basic sum insured, vesting on each policy anniversary (including the last). The death benefit is payable at the end of the year of death. Assume that mortality follows the Standard Select Survival Model, and that the interest rate is 5% per year.

(a) Let L_0 denote the gross loss at issue random variable for this policy. Show that

$$L_0 = 250\,000 Z_1 + \frac{0.99P}{d} Z_2 + 1\,200 + 0.39P - 0.99\frac{P}{d}$$

where P is the gross annual premium,

$$Z_1 = \begin{cases} v^{K_{[30]}+1}(1.025^{K_{[30]}}) & \text{if } K_{[30]} \leq 24, \\ v^{25}(1.025^{25}) & \text{if } K_{[30]} \geq 25, \end{cases}$$

and

$$Z_2 = \begin{cases} v^{K_{[30]}+1} & \text{if } K_{[30]} \leq 24, \\ v^{25} & \text{if } K_{[30]} \geq 25. \end{cases}$$

(b) Using the equivalence principle, calculate P.
(c) Calculate $\mathrm{E}[Z_1]$, $\mathrm{E}[Z_1^2]$, $\mathrm{E}[Z_2]$, $\mathrm{E}[Z_2^2]$ and $\mathrm{Cov}[Z_1, Z_2]$. Hence calculate $\mathrm{V}[L_0]$ using the value of P from part (b).
(d) Find the probability that the insurer makes a profit on this policy.

Hint: Recall the standard results from probability theory, that for random variables X and Y and constants a, b and c, $\mathrm{V}[X + c] = \mathrm{V}[X]$, and

$$\mathrm{V}[aX + bY] = a^2\mathrm{V}[X] + b^2\mathrm{V}[Y] + 2ab\,\mathrm{Cov}[X, Y],$$

with $\mathrm{Cov}[X, Y] = \mathrm{E}[XY] - \mathrm{E}[X]\mathrm{E}[Y]$.

Exercise 6.20 An insurer issues a 20-year endowment insurance policy to (40) with a sum insured of \$250 000, payable at the end of the year of death. Premiums are payable annually in advance throughout the term of the contract. Assume that mortality follows the Standard Select Life Table, and that the interest rate is 5% per year.

(a) Calculate the premium using the equivalence principle.
(b) Find the mean and standard deviation of the net loss at issue random variable using the premium in part (a).
(c) Assuming 10 000 identical, independent contracts, estimate the 99th percentile of the loss-at-issue random variable, using the premium in part (a).

(d) Assuming 10 000 identical, independent contracts, calculate the portfolio percentile premium such that the probability of profit on the portfolio is 99%.

Exercise 6.21 Matt, who is currently aged 35, takes out a mortgage of \$250 000 to purchase a home, at a loan interest rate of 5% per year. The loan is to be repaid with annual payments of X at the end of each year for 20 years.

NED Life offers Matt a mortgage term life insurance policy, which will pay off the outstanding mortgage at the end of the year of Matt's death, including the annual payment then due. The policy will have level annual premiums payable through the term of the contract.

Mortality is assumed to follow the Standard Ultimate Life Table, with interest at 5% per year.

(a) Write down an expression for the amount of death benefit payable at the end on the kth year of the policy, if Matt dies between times $k-1$ and k, in terms of X, k and interest rate functions.
(b) Let Z denote the present value of the death benefit at the policy issue date. Write down an expression for Z in terms of K_{35}, X, and interest rate functions.
(c) Show that

$$\mathrm{E}[Z] = \frac{X}{d}\left(A^{\ 1}_{35:\overline{20|}} - v^{21}\,{}_{20}q_{35}\right).$$

(d) Calculate X.
(e) Calculate the annual net premium.
(f) Explain why the insurer should expect a high lapse rate at later durations of the policy.

(*Copyright 2018. The Society of Actuaries, Schaumburg, Illinois. Reproduced with permission.*)

Exercise 6.22 An insurer issues a fully discrete three-year term insurance to a select life aged 80. The death benefit is \$1 000 in the first year, \$600 in the second year and \$200 in the third year. Assume that mortality follows the Standard Select Life Table with interest at 5%. There are no expenses.

(a) Calculate the annual premium.
(b) Calculate $\Pr[L_0 > 0]$.
(c) Calculate the standard deviation of L_0.
(d) The insurer sells 6 000 of these policies to identical, independent lives, all aged 80. Let L denote the aggregate loss at issue for these policies.
Using a normal approximation, calculate the 95th percentile of L.
(e) The insurer discovers that only 5 100 individuals purchased the 6 000 policies; 300 policyholders purchased 4 policies each, while the remaining

4 800 policyholders each purchased a single policy. Let L^* denote the aggregate loss at issue present value for these policies.

Using a normal approximation, calculate the 95th percentile of L^*.

(f) Give a general reasoning explanation of the difference between the 95th percentiles of L and L^*.

Excel-based exercises

Exercise 6.23 A select life aged exactly 40 has purchased a deferred annuity policy. Under the terms of the policy, the annuity payments will commence 20 years from the issue date and will be payable at annual intervals thereafter. The initial annuity payment will be \$50 000, and each subsequent payment will be 2% greater than the previous one.

The policy has monthly premiums, payable for at most 20 years.

Initial expenses are 2.5% of the first annuity payment and 20% of the first premium.

Renewal expenses are 5% of the second and subsequent premiums.

Terminal expenses, incurred at the end of the year of death, are \$20 inflated from the issue date, assuming an inflation rate of 3% per year.

Use the Standard Select Survival Model with interest at 5% per year to calculate the gross monthly premium.

Exercise 6.24 Calculate both the net and gross premiums for a whole life insurance issued to a select life aged 40. The sum insured is \$100 000 on death during the first 20 years, and \$20 000 thereafter, and is payable immediately on death. Premiums are payable annually in advance for a maximum of 20 years.

Use the following basis:

Survival model:	
ultimate rates	Makeham's law with $A = 0.0001, B = 0.00035,$ $c = 1.075$
select rates	2 year select period, $q_{[x]} = 0.75q_x$, $q_{[x]+1} = 0.9q_{x+1}$
Interest:	6% per year effective
Premium expenses:	30% of the first year's premium plus 3% of all premiums after the first year
Other expenses:	On each premium date an additional expense starting at \$10 and increasing at a compound rate of 3% per year

Exercise 6.25 An insurer calculates premiums for 20-year fully discrete term insurance for smokers assuming their mortality is 250% of the mortality of the Standard Select Survival Model. Non-smoker mortality is assumed to follow the Standard Select Survival Model without adjustment.

(a) Calculate the probability that a select life aged 50 dies during the 20-year term of the policy, assuming the life is (i) a non-smoker, and (ii) a smoker.
(b) Calculate the difference between the smoker and non-smoker net premiums for a life aged 50, with sum insured \$1 000 000. Assume interest at 4.5% per year.
(c) The insurer discovers some smokers are wrongly classified as non-smokers. Assume that 85% of policies are issued to non-smokers, correctly classified; 10% of policies are issued to smokers, correctly classified, and 5% of policies are issued to smokers wrongly classified as non-smokers. Calculate the expected value of the loss at issue, per policy issued, assuming that the misclassified policies are never identified.
(d) When a death benefit is claimed following the death of a policyholder, the insurer will find out if the policy has been misclassified. In this case, the insurer deducts from the sum insured the difference between the non-smoker premiums paid and the smoker premiums that should have been paid, with interest at 4.5%. Recalculate the expected value of the loss at issue, per policy issued.
(e) Explain why the adjustment in part (d) is not sufficient to eliminate the loss from misclassification.

Answers to selected exercises

6.1 (b) \$1179.74 (c) 0.70704
6.2 (b) \$2594 (c) 0.99704
6.3 (a) \$216 326.38 (b) \$13 731.03 (c) 0.0052
6.4 (a) \$2 958.63 (b) \$3 243.47
6.5 (b) \$178.50
6.6 \$3056.80
6.7 \$212.81
6.8 1.6875
6.9 \$17.87
6.10 \$16 076.72
6.11 \$1 131.11
6.12 (b) \$3 287.55 (c) \$4 964 (d) 0.98466
6.13 (b) \$199 850 (c) \$200 705
6.14 \$4 238.72
6.15 (a) \$60 691 (b) \$60 770
6.16 (a) \$417 402 (b) 0.36641 (c) \$97 197 (d) \$412 346
6.17 (a) 841 055 (b) 0.55761 (c) 758 914
6.19 (b) \$9 764.44
(c) \$0.54958, 0.30251, \$0.29852, 0.09020, 0.00071, 146 786 651
(d) 0.98297

6.20 (a) \$7 333.81 (b) 0, \$14 490 (c) \$3 370 929 (d) \$7 351

6.21 (d) \$20 060.65 (e) \$104.40

6.22 (a) \$18.63 (b) 0.10032 (c) 190.357 (d) 24 253.36 (e) 30 678.35

6.23 \$2 377.75

6.24 \$1 341.40 (net), \$1 431.08 (gross)

6.25 (a) (i) 0.0758 (ii) 0.1795 (b) \$4 624.70 (c) \$2 999.27 (d) \$2 578.95

7

Policy values

7.1 Summary

In this chapter we introduce the concept of a policy value for a life insurance policy. Policy values are a fundamental tool in insurance risk management, as they are used to determine the economic or regulatory capital needed to remain solvent, and also to determine the profit or loss for the company over any time period.

We start by considering the case where all cash flows take place at the start or end of a year. We define the policy value as the expected value of future net cash flows for a policy in force, and distinguish gross premium policy values, which explicitly allow for expenses and for the full gross premium, from net premium policy values, where expenses are excluded from the outgoing cash flows, and only the net premium is counted as income.

We show how to calculate policy values recursively from year to year. We also show how to calculate the profit from a policy in each policy year and we introduce the asset share for a policy.

We extend the analysis to policies where the cash flows are continuous and we derive Thiele's differential equation for policy values – the continuous time equivalent of the recursions for policies with annual cash flows.

We consider how policy values can be used to evaluate policy alterations, where a policyholder chooses to withdraw, or stop paying premiums, for example.

We show how a retrospective valuation has connections both with asset shares and with the policy values determined looking at future cash flows. Finally, we consider how a simple adjustment to the net premium policy value, through modifying the net premium assumed, can be used to approximate the gross premium policy value, and we discuss why this might be useful when acquisition expenses are high.

7.2 Policies with annual cash flows

7.2.1 The future loss random variable

In Chapter 6 we introduced the loss at issue random variable, L_0. In this chapter we are concerned with the estimation of future losses at intermediate times during the term of a policy, not just at inception. We therefore extend the concept of the loss at issue random variable to the time t future loss random variable, which we present in net and gross versions. For a policy in force at t, the net future loss random variable is denoted L_t^n, and the gross future loss random variable is denoted L_t^g. *For a policy in force at time t*, these are defined as follows:

$$\boxed{\begin{aligned} L_t^n \;&= \text{Present value at time } t \text{ of future benefits} \\ &\quad - \text{Present value at time } t \text{ of future net premiums} \end{aligned}}$$

and

$$\boxed{\begin{aligned} L_t^g \;&= \text{Present value at time } t \text{ of future benefits} \\ &\quad + \text{Present value at time } t \text{ of future expenses} \\ &\quad - \text{Present value at time } t \text{ of future gross premiums.} \end{aligned}}$$

We drop the n or g superscript where it is clear from the context which is meant. Note that the future loss random variable L_t is defined only if the contract is still in force t years after issue.

The example below will help establish some ideas.

Example 7.1 Consider a fully discrete 20-year endowment policy purchased by a life aged 50. Level premiums are payable annually throughout the term of the policy and the sum insured of \$500 000, is payable at the end of the year of death or at the end of the term, whichever is sooner.

The basis used by the insurance company for all calculations is the Standard Select Survival Model, 5% per year interest and no allowance for expenses.

(a) Show that the annual net premium, P, calculated using the equivalence principle, is \$15 114.33.
(b) Calculate $\mathrm{E}[L_t^n]$ for $t = 10$ and $t = 11$, in both cases just before the premium due at time t is paid.

Solution 7.1 (a) The equation of value for P is

$$P\,\ddot{a}_{[50]:\,\overline{20|}} - 500\,000\,A_{[50]:\,\overline{20|}} = 0, \tag{7.1}$$

giving

$$P = \frac{500\,000\,A_{[50]:\,\overline{20|}}}{\ddot{a}_{[50]:\,\overline{20|}}} = \$15\,114.33.$$

(b) L_{10}^n is the present value of the future net loss 10 years after the policy was purchased, assuming the policyholder is still alive at that time. The policyholder will then be aged 60 and the select period for the survival model, two years, will have expired eight years ago. The present value at that time of the future benefits is $500\,000\, v^{\min(K_{60}+1,10)}$ and the present value of the future premiums is $P\,\ddot{a}_{\overline{\min(K_{60}+1,10)}|}$, so we have

$$L_{10}^n = 500\,000\, v^{\min(K_{60}+1,10)} - P\,\ddot{a}_{\overline{\min(K_{60}+1,10)}|}$$

and similarly

$$L_{11}^n = 500\,000\, v^{\min(K_{61}+1,9)} - P\,\ddot{a}_{\overline{\min(K_{61}+1,9)}|}.$$

Taking expectations gives us

$$\mathrm{E}[L_{10}^n] = 500\,000 A_{60:\overline{10}|} - P\,\ddot{a}_{60:\overline{10}|} = \$190\,339 \tag{7.2}$$

and

$$E[L_{11}^n] = 500\,000 A_{61:\overline{9}|} - P\,\ddot{a}_{61:\overline{9}|} = \$214\,757.$$

□

We are now going to look at Example 7.1 in a little more detail. At the time when the policy is issued, at $t = 0$, the future loss random variable, L_0^n, is given by

$$L_0^n = 500\,000\, v^{\min(K_{[50]}+1,20)} - P\,\ddot{a}_{\overline{\min(K_{[50]}+1,20)}|}.$$

Since the premium is calculated using the equivalence principle, we know that $\mathrm{E}[L_0^n] = 0$, which is equivalent to equation (7.1). That is, at the time the policy is issued, the expected value of the present value of the loss on the contract is zero, so that, in expectation, the future premiums (from time 0) are exactly sufficient to provide the future benefits.

Consider the financial position of the insurer at time 10 with respect to this policy. The policyholder may have died before time 10. If so, the sum insured will have been paid and no more premiums will be received. In this case the insurer no longer has any liability with respect to this policy. Now suppose the policyholder is still alive at time 10. In this case the calculation in part (b) shows that the future loss random variable, L_{10}^n, has a positive expected value ($190 339) so that future premiums (from time 10) are **not** expected to be sufficient to provide the future benefits. For the insurer to be in a financially sound position at time 10, it should hold an amount of at least $190 339 in its assets so that, together with future premiums from time 10, it can expect to provide the future benefits.

Speaking generally, when a policy is issued the future premiums should be expected to be sufficient to pay for the future benefits and expenses. (If not, the premium should be increased!) However, it is usually the case that for a policy which is still in force t years after being issued, the future premiums (from time t) are not expected to be sufficient to pay for the future benefits and expenses. The amount needed to cover this shortfall is called the **policy value** for the policy at time t.

The insurer should be able to build up assets during the course of the policy because, with a regular level premium and an increasing level of risk, the premium in each of the early years is more than sufficient to pay the expected benefits in that year, given that the life has survived to the start of the year. For example, in the first year in the example policy, the premium is \$15 114.33 and the EPV of the first year's benefit is, $500\,000\, v\, q_{[50]} = \$492.04$. In fact, for the example policy, the premium exceeds the EPV of the benefits, in every year except the last; that is

$$P > 500\,000\, v\, q_{[50]+t} \qquad \text{for } t = 0, 1, \ldots, 18.$$

The final year is different because

$$P = 15\,114.33 < 500\,000\, v = 476\,190.$$

Note that if the policyholder is alive at the start of the final year, the sum insured will be paid at the end of the year whether or not the policyholder survives the year.

Figure 7.1 shows the excess of the premium over the EPV of the benefit payable at the end of the year for each year of this policy.

Figure 7.2 shows the corresponding values for a 20-year term insurance issued to (50). The sum insured is \$500 000, level annual premiums are payable throughout the term and all calculations use the same basis as in Example 7.1. The pattern is similar in that there is a positive surplus in the early years which can be used to build up the insurer's assets. These assets are needed in the later years when the premium is not sufficient to pay for the expected benefits.

The insurer will then, for a large portfolio, hold back some of the excess cash flow from the early years of the contract in order to meet the shortfall in the later years. This explains the concept of a policy value – we need to hold capital during the term of a policy to meet the liabilities in the period when outgo on benefits exceeds income from premiums. We give a formal definition of a policy value later in this section.

Before doing so, we return to Example 7.1. Suppose the insurer issues a large number, say N, of policies identical to the one in Example 7.1, to independent lives all aged 50. Suppose also that the experience of this group of policyholders is precisely as assumed in the basis used by the insurer in

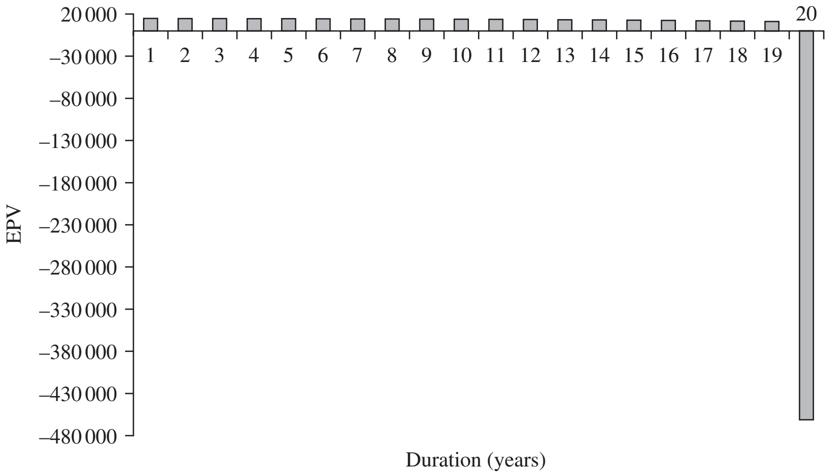


Figure 7.1 EPV at the start of each year of premiums minus claims in that year, for the 20-year endowment insurance in Example 7.1.

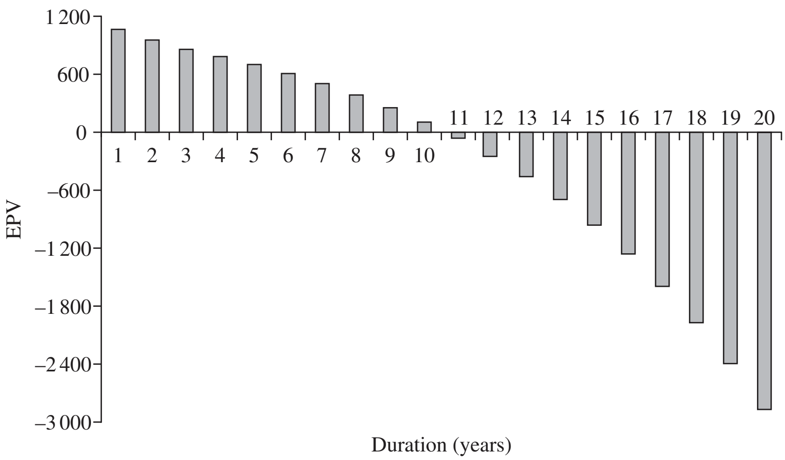


Figure 7.2 EPV at the start of each year of premiums minus claims in that year for a 20-year term insurance, sum insured \$500 000, issued to (50).

its calculations. In other words, interest is earned on investments at 5% every year, the mortality of the group of policyholders follows precisely the Standard Select Survival Model and there are no expenses.

Consider the financial situation of the insurer after these policies have been in force for 10 years. Some policyholders will have died, so that their sum insured of \$500 000 will have been paid at the end of the year in which they died, and some policyholders will still be alive. With our assumptions about the experience, precisely ${}_{10}p_{[50]}N$ policyholders will still be alive, $q_{[50]}N$ will have died in the first year, ${}_{1|}q_{[50]}N$ will have died in the second year, and so on, until the 10th year, when ${}_{9|}q_{[50]}N$ policyholders will have died. The accumulation to time 10 at 5% interest of all premiums received within the first 10 years, minus death benefits paid within the first 10 years, is

$$\begin{aligned}
& NP\left(1.05^{10} + p_{[50]}1.05^9 + \cdots + {}_9p_{[50]}1.05\right) \\
&\quad - 500\,000N\left(q_{[50]}1.05^9 + {}_{1|}q_{[50]}1.05^8 + \cdots + {}_{9|}q_{[50]}\right) \\
&= 1.05^{10}\,NP\left(1 + p_{[50]}1.05^{-1} + \cdots + {}_9p_{[50]}1.05^{-9}\right) \\
&\quad - 1.05^{10}\,500\,000N\left(q_{[50]}1.05^{-1} + {}_{1|}q_{[50]}1.05^{-2} + \cdots + {}_{9|}q_{[50]}1.05^{-10}\right) \\
&= 1.05^{10}\,N\left(P\,\ddot{a}_{[50]:\overline{10|}} - 500\,000\,A^{\,1}_{[50]:\overline{10|}}\right) \\
&= 186\,634N.
\end{aligned}$$

So, *if* the experience over the first 10 years were to follow precisely the assumptions set out in Example 7.1, the insurer would have built up a fund of \$186 634N after 10 years. The number of policyholders still alive at that time will be ${}_{10}p_{[50]}\,N$ and so the share of this fund for each surviving policyholder is

$$\frac{186\,634N}{{}_{10}p_{[50]}N} = \$190\,339.$$

This is precisely the amount the insurer needs, based on the calculation in equation (7.2). This is not a coincidence! In this example the premium was calculated using the equivalence principle, so that the EPV of the profit was zero when the policies were issued, and we have assumed the experience up to time 10 was exactly as we assumed in the calculation of the premium. Given these assumptions, it should not be surprising that the insurer is in a 'break even' position at time 10 – that is, that the excess premium received in the first 10 years, accumulated with interest, is exactly sufficient to meet the excess of the EPV of future outgo over future income at time 10 for each policy in force.

We can prove that this relationship must be true in this case by manipulating the equation of value, equation (7.1), as follows, where S denotes the sum insured:

$$\begin{aligned}
& P\,\ddot{a}_{[50]:\overline{20|}} = S\,A_{50:\overline{20|}} \\
\Rightarrow\; & P(\ddot{a}_{[50]:\overline{10|}} + v^{10}\,{}_{10}p_{[50]}\,\ddot{a}_{60:\overline{10|}}) = S\left(A^{\,1}_{[50]:\overline{10|}} + v^{10}\,{}_{10}p_{[50]}\,A_{60:\overline{10|}}\right) \\
\Rightarrow\; & P\,\ddot{a}_{[50]:\overline{10|}} - SA^{\,1}_{[50]:\overline{10|}} = v^{10}\,{}_{10}p_{[50]}\,\left(S\,A_{60:\overline{10|}} - P\,\ddot{a}_{60:\overline{10|}}\right) \\
\Rightarrow\; & \frac{1.05^{10}}{{}_{10}p_{[50]}}\left(P\,\ddot{a}_{[50]:\overline{10|}} - SA^{1}_{[50]:\overline{10|}}\right) = S\,A_{60:\overline{10|}} - P\,\ddot{a}_{60:\overline{10|}}. \qquad (7.3)
\end{aligned}$$

The left-hand side of equation (7.3) is the share of the fund built up at time 10 for each surviving policyholder; the right-hand side is the expected value of the future net loss random variable at time 10, $\mathrm{E}[L^n_{10}]$, and so is the amount needed by the insurer at time 10 for each policy still in force.

For this example, the proof that the total amount needed by the insurer at time 10 for all policies still in force is precisely equal to the amount of the fund built up, works because

(a) the premium was calculated using the equivalence principle,
(b) the expected value of the future loss random variable was calculated using the premium basis, and
(c) we assumed the experience followed precisely the assumptions in the premium basis.

In practice, (a) and (b) may or may not apply and assumption (c) is extremely unlikely to hold.

7.2.2 Policy values for policies with annual cash flows

In general terms, the policy value for a policy in force at duration t (≥ 0) years after it was purchased is the expected value at that time of the future loss random variable. At this stage we do not need to specify whether this is the gross or net future loss random variable – we will be more precise later in this section.

The general notation for a policy value t years after a policy was issued is ${}_tV$ (the V comes from 'Policy **V**alue') and we use this notation in this book. There is a standard actuarial notation associated with policy values for certain traditional contracts. This notation is not particularly useful, and so we do not use it. (Interested readers can consult the references in Section 7.10.)

Intuitively, for a policy in force at time t, the policy value at t represents the amount of assets the insurer should have set aside at that time, such that the policy value together with the expected present value of future premiums will exactly match the expected present value of the future benefits and expenses. In general terms, we have the equation, for a policy in force at t:

$$\boxed{{}_tV + \text{EPV at } t \text{ of future premiums} = \text{EPV at } t \text{ of future benefits} + \text{expenses.}}$$

An important element in the financial control of an insurance company is the calculation at regular intervals, usually at least annually, of the sum of the policy values for all policies in force at that time and also the value of all the company's investments. For the company to be financially sound, the investments should have a greater value than the total policy value. This process is called a **valuation** of the company. In most countries, valuations are required annually by the insurance supervisory authority.

In the literature, the terms **reserve**, **prospective reserve** and **prospective policy value** are sometimes used in place of *policy value*. We use *policy value* to mean the expected value of the future loss random variable, and restrict *reserve* to mean the actual provision made in respect of future policy obligations, which may be greater than or less than the policy value. We return to this topic in Chapter 13.

The precise definitions of policy value are as follows.

Definition 7.1 *The **gross premium policy value** for a policy in force at duration t (≥ 0) years after it was purchased is the expected value at that time of the gross future loss random variable on a specified basis. The premiums used in the calculation are the actual premiums payable under the contract.*

Definition 7.2 *The **net premium policy value** for a policy in force at duration t (≥ 0) years after it was purchased is the expected value at that time of the net future loss random variable on a specified basis (which makes no allowance for expenses). **The premiums used in the calculation are the net premiums calculated on the policy value basis using the equivalence principle, not the actual premiums payable.***

We make the following comments about these definitions.

(1) Throughout Section 7.2 we restrict ourselves to policies where the cash flows occur only at the start or end of a year since these policies have some simplifying features in relation to policy values. However, Definitions 7.1 and 7.2 apply to more general types of policy, as we show in later sections.

(2) The numerical value of a gross or net premium policy value depends on the assumptions – survival model, interest, expenses, future bonuses – used in its calculation. These assumptions, called the **policy value basis**, may differ from the assumptions used to calculate the premium, that is, the premium basis.

(3) A net premium policy value can be regarded as a special case of a gross premium policy value. Suppose the gross premium is calculated assuming no expenses. Then at issue, the gross and net premiums are the same. However, t years later, the gross premium policy value and the net premium policy value will be the same only if the policy value basis is the same as the original premium basis, which is unlikely to be the case. It is important to understand that the net premium policy value will use a net premium based on the policy value basis, not the original premium basis. The gross premium policy value uses the actual contract premium, with no recalculation.

    So, when the policy value basis changes, the net premium policy value requires the recalculation of the premium. You will see how this works in Example 7.2 below. The net premium policy value is a vestige of a time before modern computers, when easy calculation was a key issue – setting reserves based on a net premium policy value allowed the use of computational shortcuts. Although the need for easy computation is less critical than it was in the past, the net premium policy value is still widely used in the USA, so it is helpful to understand the concept. Where it is clear from the context whether the policy value is gross or net we refer simply to a *policy value*.

(4) If we are calculating a policy value at an integer duration, that is at the start/end of a year, there may be premiums and/or expenses and/or benefits payable at precisely that time and we need to be careful about which cash flows are included in our future loss random variable. It is the usual practice to regard a premium and any premium-related expenses due at that time as *future* payments and any insurance benefits (i.e. death or maturity claims) and related expenses as *past* payments. Under annuity contracts, the annuity payments and related expenses may be treated either as *future* payments or as *past* payments, so we need to be particularly careful to specify which it is in such cases.

(5) If an insurance policy has a finite term, n years, for example for an endowment insurance or a term insurance, then at the maturity date the policy value is technically undefined, as the policy is no longer in force. However, it is common to use ${}_nV$ to refer to the policy value immediately before the expiry of the policy, that is, ${}_nV = \lim_{t\to n^-} {}_tV$.

For a term insurance, immediately before the end of the term there are no more premiums or claims due, so that ${}_nV = 0$.

For an endowment insurance, immediately before the end of the term the present value of benefits is the maturity benefit, S, say, so ${}_nV = S$.

(6) If the premium is calculated using the equivalence principle *and* the policy value basis is the same as the premium basis, then ${}_0V = \mathrm{E}[L_0] = 0$.

In other cases we may have ${}_0V > 0$ or ${}_0V < 0$.

(7) In the discussion following Example 7.1 in Section 7.2.1 we saw how the insurer can build up a reserve for policies still in force by accumulating past premiums minus claims for a group of similar policies. Broadly speaking, this is what would happen in practice, though not with the artificial precision we saw in Section 7.2.1, where the accumulated funds were precisely the amount required by the insurer. In practice, the amount of reserve required for a policy is usually set by calculating a policy value on a specified basis; the funds for the reserve will come from accumulating past premiums, net of past benefit and expense costs. If the past experience of a portfolio of policies is better than the basis assumptions (from the insurer's perspective), for example, earning higher interest rates and (for

death benefits) experiencing lower mortality, then the accumulated cash flows will be more than needed to fund the reserve, and some profit can be taken. When the experience is worse than the basis, then the insurer will need to find extra funds to create adequate reserves.

Example 7.2 An insurer issues a whole life insurance policy to a life aged 50. The sum insured of \$100 000 is payable at the end of the year of death. Level premiums of \$1370 are payable annually in advance throughout the term of the contract.

(a) Write down an expression for the gross future loss random variable five years after issue, in terms of the curtate future lifetime random variable K_{55}, and interest rate functions.

(b) Calculate the gross premium policy value five years after the inception of the contract, assuming that the policy is still in force, using the following basis:

Survival model: Standard Select Survival Model
Interest: 5% per year effective
Expenses: 12.5% of each premium

(c) Calculate the net premium policy value five years after the issue of the contract, assuming that the policy is still in force, using the following basis:

Survival model: Standard Select Survival Model
Interest: 4% per year

You are given that at $i = 0.04$, $A_{[50]} = 0.25570$ and $A_{55} = 0.30560$.

Solution 7.2 We assume that the life is select at age 50, when the policy is purchased. At duration 5, the life is aged 55 and is no longer select since the select period for the Standard Select Survival Model is only two years. Note that a premium due at age 55 is regarded as a future premium in the calculation of a policy value.

(a) The gross future loss random variable at time 5 is

$$L_5^g = 100\,000 v^{K_{55}+1} - 0.875 \times 1\,370\, \ddot{a}_{\overline{K_{55}+1|}}.$$

(b) The policy value is the EPV of future outgo minus the EPV of future income, that is

$$\mathrm{E}[L_5^g] = {}_5V^g = 100\,000 A_{55} - 0.875 \times 1\,370\, \ddot{a}_{55} = \$4\,272.68.$$

(c) For net premium policy values we always re-calculate the (hypothetical) net premium for the contract on the policy value basis.

Let P denote the net premium for the policy value. At 4% per year,

$$\ddot{a}_{[50]} = \frac{1 - A_{[50]}}{d} = 19.3518 \quad \text{and} \quad \ddot{a}_{55} = \frac{1 - A_{55}}{d} = 18.0544,$$

so that

$$P = 100\,000\,\frac{A_{[50]}}{\ddot{a}_{[50]}} = \$1321.32,$$

giving

$${}_5V^n = 100\,000\,A_{55} - 1321.32\,\ddot{a}_{55} = \$6704.29.$$

Notice in this example that the net premium calculation ignores expenses, but uses a lower interest rate and a lower premium than the gross premium policy value, both of which provide a margin, implicitly allowing for expenses and other contingencies. The result is that the policy value under the net premium method indicates that the insurer needs to hold more funds for the policy than the policy value under the gross premium calculation. □

Example 7.3 A woman aged 60 purchases a 20-year endowment insurance with a sum insured of \$100 000 payable at the end of the year of death or on survival to age 80. An annual premium of \$5200 is payable for at most 10 years. The insurer uses the following basis for the calculation of policy values:

Survival model: Standard Select Survival Model
Interest: 5% per year effective
Expenses: 10% of the first premium, 5% of subsequent premiums, plus \$200 on payment of the sum insured

For each of $t = 0, 5, 6$, and 10, write down the gross future loss random variable, and calculate the gross premium policy values.

Solution 7.3 At time 0, when the policy is issued, the future loss random variable is

$$L_0 = 100\,200 v^{\min(K_{[60]}+1,20)} + 0.05\,P - 0.95\,P\,\ddot{a}_{\overline{\min(K_{[60]}+1,10)}|}$$

where $P = \$5\,200$. Hence

$${}_0V = \mathrm{E}[L_0] = 100\,200 A_{[60]:\,\overline{20}|} - (0.95\,\ddot{a}_{[60]:\,\overline{10}|} - 0.05)P = \$2023.$$

Similarly,

$$L_5 = 100\,200 v^{\min(K_{65}+1,15)} - 0.95\,P\,\ddot{a}_{\overline{\min(K_{65}+1,5)}|}$$

$$\Rightarrow {}_5V = \mathrm{E}[L_5] = 100\,200 A_{65:\,\overline{15}|} - 0.95\,P\,\ddot{a}_{65:\,\overline{5}|} = \$29\,068,$$

and

$$L_6 = 100\,200 v^{\min(K_{66}+1,14)} - 0.95\,P\,\ddot{a}_{\overline{\min(K_{66}+1,4)}|}$$

$$\Rightarrow {}_6V = \mathrm{E}[L_6] = 100\,200 A_{66:\,\overline{14}|} - 0.95\,P\,\ddot{a}_{66:\,\overline{4}|} = \$35\,324.$$

Finally, as no premiums are payable after time 9,

$$L_{10} = 100\,200 v^{\min(K_{70}+1,10)}$$

$$\Rightarrow {}_{10}V = \mathrm{E}[L_{10}] = 100\,200 A_{70:\,\overline{10}|} = \$63\,703.$$

□

In Example 7.3, the initial policy value, ${}_0V$, is greater than zero. This means that from the outset the insurer expects to make a loss on this policy. This sounds uncomfortable but is not uncommon in practice. The explanation is that the policy value basis may be more conservative than the premium basis. For example, the insurer may assume an interest rate of 6% in the premium calculation, but, for policy value calculations, assumes investments will earn only 5%. At 6% per year interest, and with a premium of \$5200, this policy generates an EPV of *profit* at issue of \$2869.

Example 7.4 A man aged 50 purchases a deferred annuity policy. The annuity will be paid annually for life, with the first payment on his 60th birthday. Each annuity payment will be \$10 000. Level premiums of \$11 900 are payable annually for at most 10 years. On death before age 60, all premiums paid will be returned, without interest, at the end of the year of death. The insurer uses the following basis for the calculation of policy values:

Survival model: Standard Select Survival Model
Interest: 5% per year
Expenses: 10% of the first premium, 5% of subsequent premiums, \$25 each time an annuity payment is paid, and \$100 when a death claim is paid.

Calculate the gross premium policy values for this policy at the start of the policy, at the end of the fifth year, and at the end of the 15th year, just *before* and just *after* the annuity payment and expense due at that time.

Solution 7.4 We can calculate the policy value at any time t as

EPV at t of future benefits + expenses − EPV at t of future premiums.

At the inception of the contract, the EPV of the death benefit is

$$P\,(IA)^{\ 1}_{[50]:\,\overline{10}|},$$

the EPV of the death claim expenses is

$$100A^{\ \ 1}_{[50]:\,\overline{10}|},$$

the EPV of the annuity benefit and associated expenses is

$$10\,025\ {}_{10}E_{[50]}\,\ddot{a}_{60},$$

and the EPV of future premiums less associated expenses is

$$0.95P\ddot{a}_{[50]:\,\overline{10}|} - 0.05P,$$

so that

$$\begin{aligned} {}_0V &= P(IA)^{\ \ 1}_{[50]:\,\overline{10}|} + 100A^{\ \ 1}_{[50]:\,\overline{10}|} + 10\,025\ {}_{10}E_{[50]}\,\ddot{a}_{60} - (0.95\,\ddot{a}_{[50]:\,\overline{10}|} - 0.05)P \\ &= \$485. \end{aligned}$$

At the fifth anniversary of the inception of the contract, assuming it is still in force, the future death benefit is $6P, 7P, \ldots, 10P$ depending on whether the life dies in the 6th, 7th, ..., 10th years, respectively. We can write this benefit as a level benefit of $5P$ plus an increasing benefit of $P, 2P, \ldots, 5P$.

So at time 5, the EPV of the future death benefit is

$$P\left((IA)^{\ \ 1}_{55:\,\overline{5}|} + 5A^{\ \ 1}_{55:\,\overline{5}|}\right),$$

the EPV of the death claim expenses is $100A^{\ \ 1}_{55:\,\overline{5}|}$,

the EPV of the annuity benefit and associated expenses is $10\,025\ {}_5E_{55}\,\ddot{a}_{60}$,

and the EPV of future premiums less associated expenses is $0.95P\ddot{a}_{55:\,\overline{10}|}$, so that

$$\begin{aligned} {}_5V &= P(IA)^{\ \ 1}_{55:\,\overline{5}|} + 5\,PA^{\ \ 1}_{55:\,\overline{5}|} + 100A^{\ \ 1}_{55:\,\overline{5}|} + 10\,025\ {}_5E_{55}\,\ddot{a}_{60} - 0.95P\,\ddot{a}_{55:\,\overline{5}|} \\ &= \$65\,470. \end{aligned}$$

Once the premium payment period of 10 years is completed there are no future premiums to value, so the policy value is the EPV of the future annuity payments and associated expenses.

Using the notation ${}_{15^-}V$ and ${}_{15^+}V$ to denote the policy values at duration 15 years just before and just after the annuity payment and expense due at that time, respectively, we have

$${}_{15^-}V = 10\,025\,\ddot{a}_{65} = \$135\,837,$$

and

$${}_{15^+}V = 10\,025\,a_{65} = {}_{15^-}V - 10\,025 = \$125\,812.$$

□

We make two comments about Example 7.4.

(1) As in Example 7.3, ${}_0V > 0$, which implies that the valuation basis is more conservative than the premium basis.
(2) In Example 7.4 we saw that ${}_{15^+}V = {}_{15^-}V - 10\,025$. This makes sense if we regard the policy value at any time as the amount of assets being held at that time in respect of a policy still in force. The policy value ${}_{15^-}V$ represents the assets required at time 15 just before the payment of the annuity, \$10 000, and the associated expense, \$25. Immediately after making these payments, the insurer's required assets will have reduced by \$10 025, and the new policy value is ${}_{15^+}V$.

We conclude this section by plotting policy values for the endowment insurance discussed in Example 7.1, in Figure 7.3 and for the term insurance with the same sum insured and term, in Figure 7.4. In Figure 7.3 we see that the policy values build up over time to provide the sum insured on maturity. By contrast, in Figure 7.4 the policy values increase then decrease. Towards the end of the term insurance it becomes increasingly likely the policyholder will survive the term, and that therefore no claim will be made. Towards the end of the endowment insurance, it becomes increasingly likely that the policyholder will survive the term, and the insurer will pay the sum insured at the maturity date.

A further contrast between these figures is the level of the policy values. In Figure 7.4 the largest policy value occurs at time 13, with ${}_{13}V = \$9563.00$,

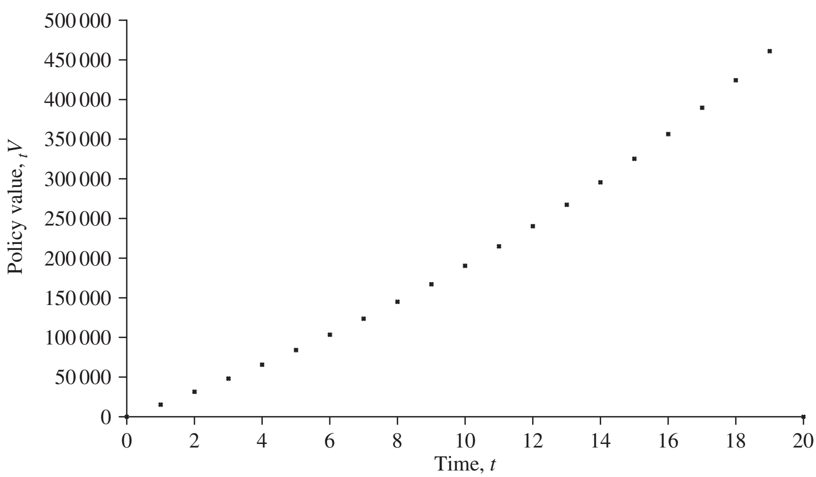


Figure 7.3 Policy values for each year of a 20-year endowment insurance, sum insured \$500 000, issued to (50).

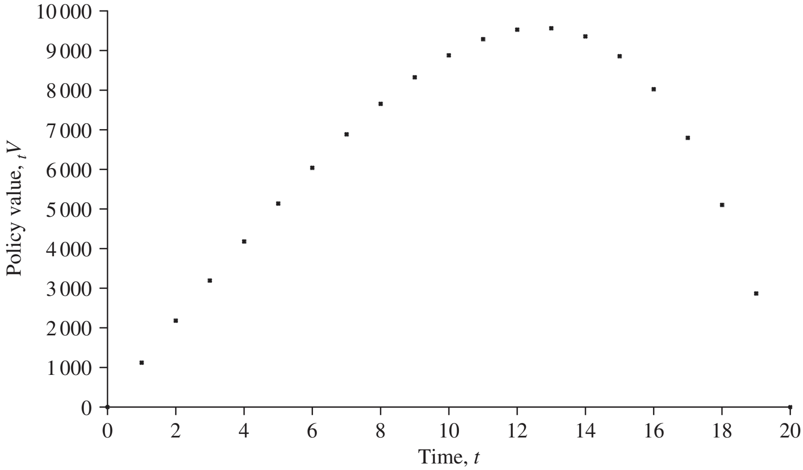


Figure 7.4 Policy values for each year of a 20-year term insurance, sum insured \$500 000, issued to (50).

which is a small amount compared with the sum insured of \$500 000. The reason why small policy values occur for the term insurance policy is simply that there is a small probability of the death benefit being paid.

7.2.3 Recursive formulae for policy values

In this section we show how to derive recursions for policy values for policies with discrete cash flows. These can be useful in the calculation of policy values in some cases – we give an example at the end of this section to illustrate this point – and they also provide an understanding of how the policy value builds up and how profit emerges while the policy is in force. We use Examples 7.1 and 7.4 to demonstrate the principles involved.

Example 7.5 In Example 7.1 we considered a fully discrete 20-year endowment insurance sold to a select life aged 50. There are no expenses.

Let P denote the net premium, and let S denote the sum insured. Show that, for $t = 0, 1, \ldots, 19$,

$$({}_tV + P)(1 + i) = S\, q_{[50]+t} + p_{[50]+t}\ {}_{t+1}V \tag{7.4}$$

where the interest and mortality/survival rates in (7.4) are from the policy value basis.

Solution 7.5 From the solution to Example 7.1 we know that for $t = 0, 1, \ldots, 19$,

$${}_tV = S\,A_{[50]+t:\,\overline{20-t}|} - P\,\ddot{a}_{[50]+t:\,\overline{20-t}|}.$$

Splitting off the terms for the first year for both the endowment and the annuity functions, we have

$$
\begin{aligned}
{}_tV &= S\left(vq_{[50]+t} + vp_{[50]+t}\, A_{[50]+t+1:\,\overline{19-t}|}\right) - P\left(1 + vp_{[50]+t}\, \ddot{a}_{[50]+t+1:\,\overline{19-t}|}\right) \\
&= v\left(Sq_{[50]+t} + p_{[50]+t}\big(SA_{[50]+t+1:\,\overline{19-t}|} - P\ddot{a}_{[50]+t+1:\,\overline{19-t}|}\big)\right) - P.
\end{aligned}
$$

Rearranging, multiplying both sides by $(1+i)$ and recognizing that

$$
{}_{t+1}V = S\,A_{[50]+t+1:\,\overline{19-t}|} - P\,\ddot{a}_{[50]+t+1:\,\overline{19-t}|}
$$

gives equation (7.4). □

We comment on Example 7.5 after the next example.

Example 7.6 In Example 7.4 we considered a deferred annuity policy issued to a life aged 50, with annual premiums during deferment, annuity payments of \$10 000 per year starting at age 60, and where any premiums paid are returned on death during the deferred period.

Expenses are 10% of the first premium and 5% of subsequent premiums, plus \$25 with each annuity payment, and \$100 on payment of a death benefit.

For this policy, show that for $t = 1, 2, \ldots, 9$,

$$
({}_tV + 0.95P)(1+i) = ((t+1)P + 100)\, q_{[50]+t} + p_{[50]+t}\ {}_{t+1}V \tag{7.5}
$$

where P is the premium, and the interest and mortality rates are from the policy value basis.

Solution 7.6 For $t = 1, 2, \ldots, 9$, ${}_tV$ has the same form as ${}_5V$, that is

$$
\begin{aligned}
{}_tV = {} & P(IA)^{\ \,1}_{[50]+t:\,\overline{10-t}|} + (tP + 100)A^{\ \,1}_{[50]+t:\overline{10-t}|} \\
& + 10\,025v^{10-t}\ {}_{10-t}p_{[50]+t}\, \ddot{a}_{60} - 0.95P\ddot{a}_{[50]+t:\overline{10-t}|}.
\end{aligned}
$$

We use recurrence relations for insurance and annuity functions to separate out the EPV of the first year's payments from subsequent years, for each of the EPV functions:

$$
\begin{aligned}
\ddot{a}_{[x]+t:\,\overline{n-t}|} &= 1 + vp_{[x]+t}\ \ddot{a}_{[x]+t+1:\,\overline{n-t-1}|}, \\
A^{\ \,1}_{[x]+t:\,\overline{n-t}|} &= vq_{[x]+t} + vp_{[x]+t}\, A^{\ \,1}_{[x]+t+1:\,\overline{n-t-1}|}, \\
(IA)^{\ \,1}_{[x]+t:\,\overline{n-t}|} &= vq_{[x]+t} + vp_{[x]+t}\left((IA)^{\ \,1}_{[x]+t+1:\,\overline{n-t-1}|} + A^{\ \,1}_{[x]+t+1:\,\overline{n-t-1}|}\right).
\end{aligned}
$$

Substituting these expressions into the policy value equation, we have, for $t = 1, 2, \ldots, 9$,

$$
\begin{aligned}
{}_tV = {} & P\left(vq_{[50]+t} + vp_{[50]+t}\left((IA)^{\;1}_{[50]+t+1:\,\overline{10-t-1}|} + A^{\;1}_{[50]+t+1:\,\overline{10-t-1}|}\right)\right) \\
& + (tP + 100)\left(vq_{[50]+t} + vp_{[50]+t}\, A^{\;1}_{[50]+t+1:\,\overline{10-t-1}|}\right) \\
& + 10\,025\; vp_{[50]+t}\left(v^{10-t-1}\;{}_{10-t-1}p_{[50]+t+1}\, \ddot{a}_{60}\right) \\
& - 0.95P\left(1 + vp_{[50]+t}\, \ddot{a}_{[50]+t+1:\,\overline{10-t-1}|}\right)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow {}_tV = {} & vq_{[50]+t}\,((t+1)P + 100) - 0.95P \\
& + vp_{[50]+t}\Big\{P(IA)^{\;1}_{[50]+t+1:\,\overline{10-t-1}|} + ((t+1)P + 100)\, A^{\;1}_{[50]+t+1:\,\overline{10-t+1}|} \\
& + 10\,025\;{}_{10-t-1}p_{[50]+t+1}\, v^{10-t-1}\, \ddot{a}_{60} - 0.95P\ddot{a}_{[50]+t+1:\overline{10-t-1}|}\Big\}.
\end{aligned}
$$

Notice that the expression in curly braces, { }, is ${}_{t+1}V$, so, substituting and rearranging,

$$
({}_tV + 0.95P)\,(1 + i) = ((t+1)P + 100)\, q_{[50]+t} + p_{[50]+t}\;{}_{t+1}V, \tag{7.6}
$$

as required. $\square$

Equations (7.4) and (7.5) are recursive formulae for policy values since they express ${}_tV$ in terms of ${}_{t+1}V$. Such formulae always exist but the precise form they take depends on the details of the policy being considered. The method we used to derive formulae (7.4) and (7.5) can be used for other policies with annual cash flows: first write down a formula for ${}_tV$ and then break up the EPVs into EPVs of payments in the coming year, t to $t + 1$, and EPVs of payments from $t + 1$ onwards. We can demonstrate this in a more general setting as follows.

Consider a policy issued to a life (x) where cash flows – that is, premiums, expenses and claims – can occur only at the start or end of a year. Suppose this policy has been in force for t years, where t is a non-negative integer. Consider the $(t + 1)$th year, and let

P_t denote the premium payable at time t,
e_t denote the renewal expenses payable at time t,
S_{t+1} denote the sum insured payable at time $t + 1$ if the policyholder dies in the year,
E_{t+1} denote the expense of paying the sum insured at time $t + 1$,

${}_tV$ denote the gross premium policy value for a policy in force at time t, and

${}_{t+1}V$ denote the gross premium policy value for a policy in force at time $t+1$.

Let $q_{[x]+t}$ denote the probability that the policyholder, alive at time t, dies in the year and let i_t denote the rate of interest assumed earned in the year. The quantities e_t, E_t, $q_{[x]+t}$ and i_t are all as assumed in the policy value basis.

Let L_t and L_{t+1} denote the gross future loss random variables at times t and $t + 1$, respectively, in both cases assuming the policyholder is alive at that time. Note that L_t involves present values at time t whereas L_{t+1} involves present values at time $t+1$. Then, by considering what can happen in the year, we have

$$L_t = \begin{cases} (1+i_t)^{-1}(S_{t+1}+E_{t+1}) - P_t + e_t & \text{if } K_{[x]+t} = 0, \text{ with probability } q_{[x]+t}, \\ (1+i_t)^{-1}L_{t+1} - P_t + e_t & \text{if } K_{[x]+t} \geq 1, \text{ with probability } p_{[x]+t}. \end{cases}$$

Taking expected values:

$$\begin{aligned} {}_tV = \mathrm{E}[L_t] &= q_{[x]+t}(1+i_t)^{-1}(S_{t+1}+E_{t+1}) - (q_{[x]+t} + p_{[x]+t})(P_t - e_t) \\ &\quad + p_{[x]+t}(1+i_t)^{-1}\mathrm{E}[L_{t+1}], \end{aligned}$$

which, after a little rearranging, and recognizing that ${}_{t+1}V = \mathrm{E}[L_{t+1}]$, gives the important equation

$$\boxed{({}_tV + P_t - e_t)(1+i_t) = q_{[x]+t}(S_{t+1}+E_{t+1}) + p_{[x]+t}\,{}_{t+1}V.} \qquad (7.7)$$

Equation (7.7) includes equations (7.4) and (7.5) as special cases and it is a little more general than either of them since it allows the premium, the sum insured, the expenses and the rate of interest all to be functions of t or of $t+1$, so that they can vary from year to year.

The intuition for equation (7.7) is explained below.

◇ Assume that at time t the insurer has assets of amount ${}_tV$ in respect of this policy. Recall that ${}_tV$ is the expected value on the policy value basis of the future loss random variable, assuming the policyholder is alive at time t. Hence we can interpret ${}_tV$ as the value of the assets the insurer should have at time t (in respect of a policy still in force) in order to expect to break even over the future course of the policy.

◇ Now add to ${}_tV$ the net cash flow received by the insurer at time t as assumed in the policy value basis. In equation (7.7) this is $P_t - e_t$; in Example 7.5 this was just the premium, P; in Example 7.6 this was the premium less the expense assumed in the policy value basis. The new amount is the amount of the insurer's assets at time t just after these cash flows. There are no further cash flows until the end of the year.

- ◇ These assets are rolled up to the end of the year with interest at the rate assumed in the policy value basis, i_t (= 5% in the two examples). This gives the amount of the insurer's assets at the end of the year before any further cash flows (assuming everything is as specified in the policy value basis). This gives the left-hand sides of equations (7.7), (7.4) and (7.5).
- ◇ We assumed the policyholder was alive at the start of the year, time t; we do not know whether the policyholder will be alive at the end of the year. With probability $p_{[x]+t}$ the policyholder will be alive, and with probability $q_{[x]+t}$ the policyholder will die in the year (where these probabilities are calculated on the policy value basis).
- ◇ If the policyholder is alive at time $t+1$ the insurer needs to have assets of amount $_{t+1}V$ at that time; if the policyholder has died during the year, the insurer must pay any death benefit and related expenses. The expected amount the insurer needs for the policy is given by the right-hand side of equation (7.7) (and of equations (7.4) and (7.5) for the two examples). In each recursion, the expected amount needed (right-hand side) is precisely the amount the insurer will have, based on experience matching assumptions. This happens because the policy value is defined as the expected value of the future loss random variable *and* because we assume cash flows from t to $t+1$ are exactly as specified in the policy value basis. We assumed that at time t the insurer had sufficient assets to *expect (on the policy value basis)* to break even over the future course of the policy. Since we have assumed that from t to $t+1$ all cash flows are as specified in the policy value basis, it is not surprising that at time $t+1$ the insurer still has sufficient assets to expect to break even.

One further point needs to be made about equations (7.7), (7.4) and (7.5). We can rewrite these three formulae as follows:

$$({}_tV + P_t - e_t)(1+i_t) = {}_{t+1}V + q_{[x]+t}(S_{t+1} + E_{t+1} - {}_{t+1}V), \tag{7.8}$$

$$({}_tV + P)(1+i) = {}_{t+1}V + q_{[x]+t}(500\,000 - {}_{t+1}V),$$

$$({}_tV + 0.95P)(1+i) = {}_{t+1}V + q_{[x]+t}((t+1)P\,q_{50+t} - {}_{t+1}V).$$

The left-hand side of each of these formulae is unchanged – in each case it still represents the amount of assets the insurer is assumed to have at time $t+1$ in respect of a policy which was in force at time t. The right-hand sides can now be interpreted slightly differently.

- For each policy in force at time t, the insurer needs to provide the policy value, $_{t+1}V$, at time $t+1$, whether the life died during the year or not.
- In addition, if the policyholder has died in the year (the probability of which is $q_{[x]+t}$), the insurer must also provide the extra amount to

increase the policy value to the death benefit payable plus any related expense, that is:

$S_{t+1} + E_{t+1} - {}_{t+1}V$ for the general policy,
$S - {}_{t+1}V$ in Example 7.5, and
$(t+1)P - {}_{t+1}V$ in Example 7.6.

The extra amount required to increase the policy value to the death benefit is called the **Sum at Risk** (SAR), or the **Death Strain At Risk** or the **Net Amount at Risk**, at time $t+1$. Generally (ignoring claim expenses) if the death benefit payable in the tth year is S_t, then the tth year SAR is $S_t - {}_tV$. This is an important measure of the insurer's risk if mortality exceeds the basis assumption, and is useful in determining risk management strategy, including reinsurance – which is the insurance that an insurer buys to protect itself against adverse experience.

In all the examples so far in this section it has been possible to calculate the policy value directly, as the EPV on the given basis of future benefits plus future expenses minus future premiums. In more complicated examples, in particular where the benefits are defined in terms of the policy value, this may not be possible. In these cases the recursive formula for policy values, equation (7.7), can be very useful, as the following example shows.

Example 7.7 Consider a 20-year endowment policy purchased by a life aged 50. Level premiums of \$23 500 per year are payable annually throughout the term of the policy. A sum insured of \$700 000 is payable at the end of the term if the life survives to age 70. On death before age 70 a sum insured is payable at the end of the year of death equal to the policy value at the start of the year in which the policyholder dies.

The policy value basis used by the insurance company is as follows:

Survival model: Standard Select Survival Model
Interest: 3.5% per year
Expenses: nil

Calculate ${}_{15}V$, the policy value for a policy in force at the start of the 16th year.

Solution 7.7 For this example, formula (7.7) becomes

$$({}_tV + P) \times 1.035 = q_{[50]+t}\, S_{t+1} + p_{[50]+t}\ {}_{t+1}V \quad \text{for } t = 0, 1, \ldots, 19,$$

where $P = \$23\,500$. For the final year of this policy, the death benefit payable at the end of the year is ${}_{19}V$ and the survival benefit is the sum insured, \$700 000. Putting $t = 19$ in the above equation gives:

$$({}_{19}V + P) \times 1.035 = q_{69}\ {}_{19}V + p_{69} \times 700\,000.$$

Tidying this up and noting that $S_{t+1} = {}_tV$, we can work backwards as follows:

$$
\begin{aligned}
{}_{19}V &= (p_{69} \times 700\,000 - 1.035P)/(1.035 - q_{69}) = 652\,401,\\
{}_{18}V &= (p_{68} \times {}_{19}V - 1.035P)/(1.035 - q_{68}) = 606\,471,\\
{}_{17}V &= (p_{67} \times {}_{18}V - 1.035P)/(1.035 - q_{67}) = 562\,145,\\
{}_{16}V &= (p_{66} \times {}_{17}V - 1.035P)/(1.035 - q_{66}) = 519\,362,\\
{}_{15}V &= (p_{65} \times {}_{16}V - 1.035P)/(1.035 - q_{65}) = 478\,063.
\end{aligned}
$$

Hence, the answer is \$478 063. □

7.2.4 Analysis of surplus

Analysis of surplus, or analysis of profit by source, is the process of identifying sources of profit or loss from a portfolio over an accounting period. Consider a group of identical policies issued at the same time. The recursive formulae for policy values show that *if* all cash flows between t and $t + 1$ are as specified in the policy value basis, then the insurer will be in a break-even position at time $t + 1$, given that it was in a break-even position at time t. These cash flows depend on mortality, interest, expenses and, for participating policies, bonus rates. In practice, it is very unlikely that all the assumptions will be met in any one year. If the assumptions are not met, then the value of the insurer's assets at time $t + 1$ may be more than sufficient to pay any benefits due at that time and to provide a policy value of ${}_{t+1}V$ for the policies still in force. In this case, the insurer will have made a profit in the year. If the insurer's assets at time $t + 1$ are less than the amount required to pay the benefits due at that time and to provide a policy value of ${}_{t+1}V$ for the policies still in force, the insurer will have made a loss in the year.

In general terms:

- Actual expenses less than the expenses assumed in the policy value basis will be a source of profit.
- Actual interest earned on investments less than the interest assumed in the policy value basis will be a source of loss.
- Actual mortality less than the mortality assumed in the policy value basis can be a source of either profit or loss. For most whole life, term and endowment policies it will be a source of profit; for annuity policies it will be a source of loss.
- Actual bonus or dividend rates less than the rates assumed in the policy value basis will be a source of profit.

The following example demonstrates how to calculate annual profit by source from a non-participating life insurance policy.

Example 7.8 An insurer issued a large number of policies identical to the policy in Example 7.3 to women aged 60. Each policy is a 20-year endowment insurance with a sum insured of \$100 000 and an annual premium of \$5200 payable for at most 10 years.

The insurer's policy value basis is:

Survival model: Standard Select Survival Model
Interest: 5% per year effective
Expenses: 10% of the first premium, 5% of subsequent premiums, plus \$200 on payment of the sum insured

Five years after they were issued, a total of 100 of these policies were still in force. In the following year,

- expenses of 6% of each premium paid were incurred,
- interest was earned at 6.5% on all assets,
- one policyholder died, and
- expenses of \$250 were incurred on the payment of the sum insured for the policyholder who died.

(a) Calculate the profit or loss on this group of policies for this year.
(b) Determine how much of this profit/loss is attributable to profit/loss from mortality, from interest and from expenses.

Solution 7.8 (a) At duration $t = 5$ we assume the insurer held assets for the portfolio with value exactly equal to the total of the policy values at that time for all the policies still in force. From Example 7.3 we know the value of ${}_5V$ and so we assume the insurer's assets at time 5, in respect of these policies, amounted to $100\,{}_5V$. If the insurer's assets were worth less (*resp.* more) than this, then losses (*resp.* profits) have been made in previous years. These do not concern us – we are concerned only with what happens in the 6th year.

Now consider the cash flows in the 6th year. For each of the 100 policies still in force at time 5 the insurer received a premium P (= \$5200) and paid an expense of $0.06P$ at time 5. Hence, the total assets at time 5 after receiving premiums and paying premium-related expenses were

$$100\,{}_5V + 100 \times 0.94\,P = \$3\,395\,551.$$

There were no further cash flows until the end of the year, so this amount grew for one year at the rate of interest actually earned, 6.5%, giving the value of the insurer's assets at time 6, before paying any death claims and expenses and setting up policy values, as

$$(100\,{}_5V + 100 \times 0.94\,P) \times 1.065 = \$3\,616\,262.$$

The death claim plus related expenses at the end of the year was 100 250. A policy value equal to ${}_6V$ (calculated in Example 7.3) is required at the end of the year for each of the 99 policies still in force. Hence, the total amount the insurer requires at the end of the year is

$$100\,250 + 99\,{}_6V = \$3\,597\,342.$$

Hence the insurer has made a *profit* in the sixth year of

$$(100\,{}_5V + 100 \times 0.94\,P) \times 1.065 - (100\,250 + 99\,{}_6V) = \$18\,919.$$

(b) In this example the sources of profit and loss in the sixth year are as follows.

(i) Interest: This is a source of profit since the actual rate of interest earned, 6.5%, is higher than the rate assumed in the policy value basis.

(ii) Expenses: These are a source of loss since the actual expenses, both premium related (6% of premiums) and claim related (\$250), are higher than assumed in the policy value basis (5% of premiums and \$200).

(iii) Mortality: The probability of dying in the year for any of these policyholders is q_{65} $(= 0.0059)$. Hence, out of 100 policyholders alive at the start of the year, the insurer expects $100\,q_{65}$ $(= 0.59)$ to die. In fact, one died. Each death reduces the profit since the amount required for a death, \$100 250, is greater than the amount required on survival, ${}_6V$ $(= \$35\,324)$, that is, the Sum At Risk, $(100\,250 - {}_6V) = 64\,926 > 0$, and so having more than the expected number of deaths generates a loss. If the Sum At Risk is negative, excess deaths generate profit.

Since the overall profit is positive, (i) has had a greater effect than (ii) and (iii) combined in this year.

We can attribute the total profit to the three sources as follows.

Interest: If expenses at the start of the year had been as assumed in the policy value basis, $0.05\,P$ per policy still in force, *and* interest had been earned at 5%, the total interest received in the year would have been

$$0.05 \times (100\,{}_5V + 100 \times 0.95\,P) = \$170\,038.$$

The actual interest earned, before allowing for actual expenses, was

$$0.065 \times (100\,{}_5V + 100 \times 0.95\,P) = \$221\,049.$$

Hence, there was a profit of \$51 011 attributable to interest.

Expenses: Now, we allow for the actual interest rate earned during the year (because the difference between actual and expected interest has already been accounted for in the interest profit above) but use the expected mortality. That is, we look at the loss arising from the expense experience given that the interest rate earned is 6.5%, but on the hypothesis that the number of deaths is $100\,q_{65}$.

The expected expenses on this basis, valued at the year end, are

$$100 \times 0.05P \times 1.065 + 100\,q_{65} \times 200 = \$27\,808.$$

The actual expenses, if deaths were as expected, are

$$100 \times 0.06P \times 1.065 + 100\,q_{65} \times 250 = \$33\,376.$$

The loss from expenses, allowing for the actual interest rate earned in the year but allowing for the expected, rather than actual, mortality, was

$$33\,376 - 27\,808 = \$5568.$$

Mortality: Now, we use actual interest (6.5%) and actual expenses, and look at the difference between the expected cost from mortality and the actual cost. For each death, the cost to the insurer is the death strain at risk, in this case $100\,000+250-{}_6V$, so the mortality profit is

$$(100\,q_{65} - 1) \times (100\,000 + 250 - {}_6V) = -\$26\,524.$$

This gives a total profit of

$$51\,011 - 5568 - 26\,524 = \$18\,919,$$

which is the amount calculated earlier.

□

We have calculated the profit from the three sources in the order: interest, expenses, mortality. At each step we assume that factors not yet considered are as specified in the policy value basis, whereas factors already considered are as actually occurred. This avoids 'double counting' and gives the correct total.

However, we could follow the same principle, building from expected to actual, one basis element at a time, but change the order of the calculation as follows.

Expenses: The loss from expenses, allowing for the assumed interest rate earned in the year and allowing for the expected mortality, was

$$100 \times (0.06 - 0.05)\,P \times 1.05 + 100\,q_{65} \times (250 - 200) = \$5490.$$

Interest: Allowing for the actual expenses at the start of the year, the profit from interest was

$$(0.065 - 0.05) \times (100\,{}_5V + 100 \times 0.94\,P) = \$50\,933.$$

Mortality: The profit from mortality, allowing for the actual expenses, was

$$(100q_{65} - 1) \times (100\,000 + 250 - {}_6V) = -\$26\,524.$$

This gives a total profit of

$$-5490 + 50\,933 - 26\,524 = \$18\,919$$

which is the same total as before, but with (slightly) different amounts of profit attributable to interest and to expenses.

The analysis of surplus is an important component of any valuation. It will indicate if any parts of the valuation basis are too conservative or too weak; it will assist in assessing the performance of the various managers involved in the business, and in determining the allocation of resources, and, for participating business it will help to determine how much surplus should be distributed.

7.2.5 Asset shares

In Section 7.2.1 we showed, using Example 7.1, that if the three conditions at the end of the section were fulfilled, then the accumulation of the premiums received minus the claims paid for a group of identical policies issued simultaneously would be precisely sufficient to provide the policy value required for the surviving policyholders at each future duration. We noted that condition (c) in particular would be extremely unlikely to hold in practice; that is, it is virtually impossible for the experience of a policy or a portfolio of policies to follow exactly the assumptions in the premium basis. In practice, the invested premiums may have earned a greater or smaller rate of return than that used in the premium basis, the expenses and mortality experience will differ from the premium basis. Each policy contributes to the total assets of the insurer through the actual investment, expense and mortality experience.

It is of practical importance to calculate the share of the insurer's assets attributable to each policy in force at any given time. This amount is known as the **asset share** of the policy at that time and it is calculated by assuming the policy being considered is one of a large group of identical policies issued simultaneously. The premiums minus claims and expenses for this notional group of policies are then accumulated using values for expenses, interest, mortality and bonus rates based on the insurer's experience for similar policies over the period. At any given time, the accumulated fund divided by the (notional) number of survivors gives the asset share at that time for each surviving policyholder. If the insurer's experience is close to the assumptions in the policy value basis, then we would expect the asset share to be close to the policy value.

The policy value at duration t represents the amount the insurer *needs to have* at that time in respect of each surviving policyholder; the asset share represents (an estimate of) the amount the insurer *actually does have*.

Example 7.9 Consider a policy identical to the policy studied in Example 7.4 – that is, a 10-year deferred annuity, with a return of premium death benefit. Suppose that this policy has now been in force for five years. Suppose that over the past five years the insurer's experience in respect of similar policies has been as follows.

- Annual interest earned on investments has been as shown in the following table.

Year	1	2	3	4	5
Interest %	4.8	5.6	5.2	4.9	4.7

- Expenses at the start of the year in which a policy was issued were 15% of the premium.
- Expenses at the start of each year after the year in which a policy was issued were 6% of the premium.
- The expense of paying a death claim was, on average, \$120.
- The mortality rate, $q_{[50]+t}$, for $t = 0, 1, \ldots, 4$, has been approximately 0.0015.

Calculate the asset share for the policy at the start of each of the first six years.

Solution 7.9 We assume that the policy we are considering is one of a large number, N, of identical policies issued simultaneously. As we will see, the value of N does not affect our final answers.

Let AS_t denote the asset share per policy surviving at time $t = 0, 1, \ldots, 5$. We calculate AS_t by accumulating to time t the premiums received minus the claims and expenses paid in respect of this notional group of policies using our estimates of the insurer's actual experience over this period and then dividing by the number of surviving policies. We adopt the convention that AS_t does not include the premium and related expense due at time t. With this convention, AS_0 is always 0 for any policy since no premiums will have been received and no claims and expenses will have been paid before time 0. Note that for our policy, using the policy value basis specified in Example 7.4, ${}_0V = \$490$.

The premiums minus expenses received at time 0 are

$$0.85 \times 11\,900\,N = 10\,115\,N.$$

This amount accumulates to the end of the year with interest at 4.8%, giving

$$10\,601\,N.$$

A notional $0.0015\,N$ policyholders die in the first year so that death claims plus expenses at the end of the year are

$$0.0015 \times (11\,900 + 120)\,N = 18\,N,$$

which leaves

$$10\,601\,N - 18\,N = 10\,582\,N$$

at the end of the year. Since $0.9985\,N$ policyholders are still surviving at the start of the second year, AS_1, the asset share for a policy surviving at the start of the second year, is given by

$$\mathrm{AS}_1 = 10\,582N/(0.9985\,N) = 10\,598.$$

These calculations, and the calculations for the next four years, are summarized in Table 7.1. You should check all the entries in this table. For example, the death claims and expenses in year 5 are calculated as

$$0.9985^4 \times 0.0015 \times (5 \times 11\,900 + 120)\,N = 89\,N$$

since $0.9985^4 N$ policyholders are alive at the start of the fifth year, a fraction 0.0015 of these die in the coming year, the death benefit is a return of the five premiums paid and the expense is \$120.

Note that the figures in Table 7.1, except the Survivors' column, have been rounded to the nearest integer for presentation; the underlying calculations have been carried out using far greater accuracy. □

We make the following comments about Example 7.9.

(1) As predicted, the value of N does not affect the values of the asset shares, AS_t. The only purpose of this notional group of N identical policies issued simultaneously is to simplify the presentation.

Table 7.1 *Asset share calculation for Example 7.9.*

Year, t	Fund at start of year	Cash flow at start of year	Fund at end of year before death claims	Death claims and expenses	Fund at end of year	Survivors	AS_t
1	0	$10\,115\,N$	$10\,601\,N$	$18\,N$	$10\,582\,N$	$0.9985\,N$	10 598
2	$10\,582\,N$	$11\,169\,N$	$22\,970\,N$	$36\,N$	$22\,934\,N$	$0.9985^2\,N$	23 003
3	$22\,934\,N$	$11\,152\,N$	$35\,859\,N$	$54\,N$	$35\,805\,N$	$0.9985^3\,N$	35 967
4	$35\,805\,N$	$11\,136\,N$	$49\,241\,N$	$71\,N$	$49\,170\,N$	$0.9985^4\,N$	49 466
5	$49\,170\,N$	$11\,119\,N$	$63\,123\,N$	$89\,N$	$63\,034\,N$	$0.9985^5\,N$	63 509

(2) The experience of the insurer over the five years has been close to the assumptions in the policy value basis specified in Example 7.4. The actual interest rate has been between 4.7% and 5.6%; the rate assumed in the policy value basis is 5%. The actual expenses, both premium-related (15% initially and 6% thereafter) and claim-related (\$120), are a little higher than the expenses assumed in the policy value basis (10%, 5% and \$100, respectively). The actual mortality rate is comparable to the rate in the policy value basis, e.g. $0.9985^5 = 0.99252$ is close to ${}_5p_{[50]} = 0.99283$.

As a result of this, the asset share, AS_5 (= \$63 509), is reasonably close to the policy value, ${}_5V$ (= \$65 470) in this example.

7.3 Policy values for policies with cash flows at 1/*m*thly intervals

Throughout Section 7.2 we assumed all cash flows for a policy occurred at the start or end of each year. This simplified the presentation and the calculations in the examples. In practice, this assumption does not often hold; for example, premiums are often payable monthly and death benefits are usually payable soon after death. Definitions 7.1 and 7.2 for policy values can be directly applied to policies with more frequent cash flows, and our interpretation of a policy value is unchanged – it is still the amount the insurer needs such that, with future premiums, it can expect (on the policy value basis) to pay future benefits and expenses.

The following example illustrates these points.

Example 7.10 A life aged 50 purchases a 10-year term insurance with sum insured \$500 000 payable at the end of the month of death. Level quarterly premiums, each of amount $P = \$460$, are payable for at most five years.

Calculate the (gross premium) policy values at durations 2.75, 3 and 6.5 years using the following basis.

Survival model: Standard Select Survival Model
Interest: 5% per year
Expenses: 10% of each gross premium

Solution 7.10 To calculate ${}_{2.75}V$ we need the EPV of future benefits and the EPV of premiums less expenses at that time, assuming the policyholder is still alive. Note that the premium and related expense due at time $t = 2.75$ are regarded as future cash flows. Note also that from duration 2.75 years the policyholder will be subject to the ultimate part of the survival model since the select period is only two years.

Hence

$${}_{2.75}V = 500\,000 A^{(12)\,1}_{52.75:\overline{7.25}|} - 0.9 \times 4 \times P\,\ddot{a}^{(4)}_{52.75:\overline{2.25}|} = \$3\,091.02.$$

Similarly

$$_{3}V = 500\,000 A^{(12)\,1}_{53:\overline{7}|} - 0.9 \times 4 \times P\,\ddot{a}^{(4)}_{53:\overline{2}|} = \$3\,357.94,$$

and

$$_{6.5}V = 500\,000 A^{(12)\,1}_{56.5:\overline{3.5}|} = \$4\,265.63.$$

□

7.3.1 Recursions with 1/mthly cash flows

We can derive recursive formulae for policy values for policies with cash flows at discrete times other than annually. For a policy with premiums of P every $1/m$ years, premium expenses of e_t at time t, and sum insured S payable at the end of the $1/m$th year of death, then following exactly the reasoning for the annual case, for $t = 0, \frac{1}{m}, \frac{2}{m}, \frac{3}{m}, \ldots$, we have

$$\begin{aligned}({}_tV + P - e_t)\,(1+i)^{\frac{1}{m}} &= {}_{\frac{1}{m}}q_{[x]+t}\,S + {}_{\frac{1}{m}}p_{[x]+t}\left({}_{t+\frac{1}{m}}V\right)\\ &= {}_{t+\frac{1}{m}}V + {}_{\frac{1}{m}}q_{[x]+t}\left(S - {}_{t+\frac{1}{m}}V\right).\end{aligned}$$

On the left-hand side we have the policy value brought forward, with the addition of the premium less premium expenses, and with interest added at the end of the $1/m$ years. On the right hand side we have the expected cost of death claims arising during the period, plus the expected cost of the policy value to be carried forward to the next period. In expectation, everything is in balance.

When premiums and benefits are payable with different frequencies, we can use the same principles, but we have slightly different recursions depending on whether the premium is payable or not in the interval. Consider, for example, ${}_{2.75}V$ and ${}_{3}V$ in the Example 7.10. In that case, benefits are monthly, but premiums are quarterly. So we can use a recurrence relationship to generate the policy value at each month end, but allowing for premiums only every third month. For the recursion starting at time 2.75, there is a premium payment and related expenses to account for, so

$$({}_{2.75}V + 460 - 0.1 \times 460)\,(1.05^{\frac{1}{12}}) = 500\,000\,{}_{\frac{1}{12}}q_{52.75} + {}_{\frac{1}{12}}p_{52.75}\,({}_{2.833}V). \tag{7.9}$$

For the following two months there are no premiums or premium expenses, so

$${}_{2.833}V\,(1.05^{\frac{1}{12}}) = 500\,000\,{}_{\frac{1}{12}}q_{52.833} + {}_{\frac{1}{12}}p_{52.833}\,({}_{2.917}V),$$

$${}_{2.917}V\,(1.05^{\frac{1}{12}}) = 500\,000\,{}_{\frac{1}{12}}q_{52.917} + {}_{\frac{1}{12}}p_{52.917}\,({}_{3}V).$$

7.3.2 Valuation between premium dates

All of the calculations in the sections above considered policy values at cash flow dates. We often need to calculate policy values between premium dates; typically, we value all policies on the same calendar dates each year as part of the insurer's liability valuation process. The principle when valuing between premium dates is the same as when valuing on premium dates; the policy value is the EPV of future benefits plus expenses minus premiums. The calculation may be a little more awkward though, as we demonstrate in the following examples.

Example 7.11 Consider a fully discrete whole life insurance issued to a select life aged 50. The sum insured is $S = \$200\,000$, and the annual premium is $P = \$2\,375$. Expenses are 30% of the first year's premium and 5% of all subsequent premiums. Policy values are calculated on a gross premium basis using the Standard Select Life Table, with interest at 5% per year.

(a) Calculate the policy value at time $t = 30$.
(b) Calculate the policy value at time $t = 31$ by recursion.
(b) Calculate the policy value at time $t = 30.8$, assuming a constant force of mortality between integer ages.

Solution 7.11 (a) The policy value at time 30 is

$$_{30}V = S A_{80} - 0.95P\,\ddot{a}_{80} = 99\,299.$$

(b) The recursive relationship between the policy values at times 30 and 31 is

$$(_{30}V + 0.95P) \times 1.05 = q_{80}\,S + p_{80} \times {}_{31}V,$$

which gives

$$_{31}V = \frac{(_{30}V + 0.95P) \times 1.05 - q_{80}\,S}{p_{80}} = 103\,480.$$

(c) For the policy value at time $t = 30.8$, we split the future income and outgo into the costs arising before and after time $t = 31$. That is

$$\begin{aligned} _{30.8}V &= {}_{0.2}q_{80.8}\,S\,v^{0.2} + {}_{0.2}p_{80.8}\,v^{0.2}\,S A_{81} - {}_{0.2}p_{80.8}\,v^{0.2}\,0.95P\,\ddot{a}_{81} \\ &= {}_{0.2}q_{80.8}\,S\,v^{0.2} + {}_{0.2}p_{80.8}\,v^{0.2}\,(S A_{81} - 0.95P\ddot{a}_{81}) \\ &= {}_{0.2}q_{80.8}\,S\,v^{0.2} + {}_{0.2}p_{80.8}\,v^{0.2}\,({}_{31}V)\,. \end{aligned}$$

Under the constant force fractional age assumption, ${}_{0.2}p_{80.8} = (p_{80})^{0.2} = 0.99338$, so that ${}_{0.2}q_{80.8} = 0.00662$. So we have

$$_{30.8}V = 0.00662(200\,000)v^{0.2} + 0.99338\,v^{0.2}(103\,480) = 103\,108.$$

□

Example 7.12 For the contract described in Example 7.10, (i.e., a 10-year term insurance with sum insured $S = \$500\,000$ payable at the end of the month of death, and with level quarterly premiums of $P = \$460$ payable for at most five years), calculate the policy value ${}_{2.8}V$.

Solution 7.12 As in the previous example, we split the EPV of future cash flows into the part up to the next premium date or benefit date and the parts after that date, noting that $A^{(m)\,1}_{x:\overline{n}|}$ and $\ddot{a}^{(m)}_{x:\overline{n}|}$ are only defined if n is an integer multiple of $1/m$.

The EPV of future benefits is

$$S\,v^{0.033}\,{}_{0.033}q_{52.8} + S\,v^{0.033}\,{}_{0.033}p_{52.8}\,A^{(12)\,1}_{52.833:\overline{7.167}|} = 6614.75,$$

and the EPV of future premiums less expenses is

$$0.9 \times 4P\,v^{0.2}\,{}_{0.2}p_{52.8}\,\ddot{a}^{(4)}_{53:\overline{2}|} = 3138.59.$$

In each of the above, we have used an exact calculation. Combining these,

$${}_{2.8}V = 6614.75 - 3138.59 = 3476.16.$$

An alternative approach is to use a form of recursion from time $t = 2.8$ up to the next benefit date, time $t = 2.833$, where we we can use formula (7.9) to calculate ${}_{2.833}V = 3456.72$. Accumulating ${}_{2.8}V$ for 0.033 of a year to provide S if death occurs before time 2.833 or ${}_{2.833}V$ if the policyholder survives to time 2.833, we have

$${}_{2.8}V(1+i)^{0.033} = {}_{0.033}q_{52.8}S + {}_{0.033}p_{52.8}\,({}_{2.833}V)$$

giving ${}_{2.8}V = 3476.16$, as before. □

For the general case, let h denote the time to the next benefit date, which is assumed to be at or before the next premium date; assume that we know ${}_{t+h}V$, and that we are calculating ${}_tV$ where t is not a benefit or premium payment date. Then for an insurance with death benefit S, we have

$${}_tV(1+i)^h = {}_hq_{[x]+t}S + {}_hp_{[x]+t}\,{}_{t+h}V$$

and this can be solved for ${}_tV$.

It is interesting to note here that it would **not** be appropriate to apply simple interpolation to the two policy values corresponding to the premium dates before and after the valuation date, as we have, from Example 7.12,

$${}_{2.75}V = \$3091.02, \quad {}_{2.8}V = \$3476.17 \quad \text{and} \quad {}_{2.833}V = \$34556.72,$$

so ${}_{2.8}V$ does not lie between ${}_{2.75}V$ and ${}_{2.833}V$.

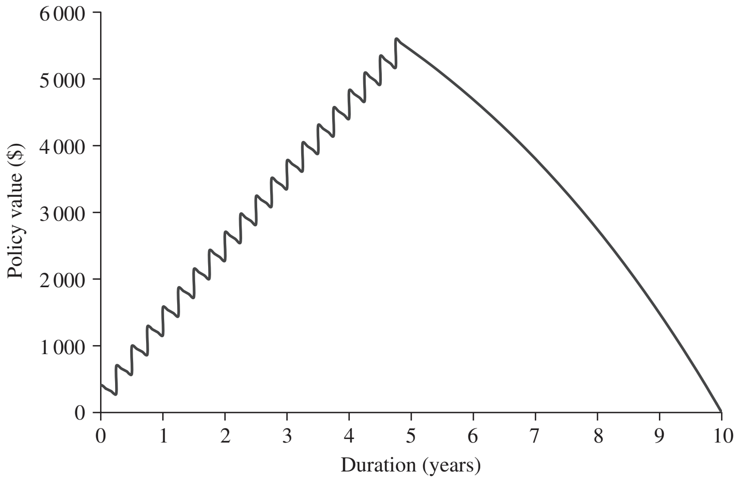


Figure 7.5 Policy values for the limited premium term insurance contract, Example 7.12.

The reason is that the function ${}_tV$ is not smooth. The policy value will jump immediately after each premium payment by the amount of that payment, less associated expenses. Before the premium payment, the premium immediately due is included in the EPV of future premiums, which is deducted from the EPV of future benefits to give the policy value. Immediately after the premium payment, it is no longer included, so the policy value increases by the amount of the premium.

In Figure 7.5 we show the policy values at all durations for the policy in Examples 7.10 and 7.12. The curve jumps at each premium date, and has an increasing trend until the premiums cease. In the second half of the contract, after the premium payment term, the policy value is run down. Other types of policy will have different patterns for policy values as we have seen in Figures 7.1 and 7.2.

A reasonable approximation to the policy value between premium dates can usually be achieved by interpolating between the policy value **just after the previous premium** and the policy value just before the next premium. That is, suppose the premium dates are $1/m$ years apart, and that t is a premium date. Then for $s < 1/m$, we approximate ${}_{t+s}V$ by interpolating between ${}_tV + P_t - e_t$ and ${}_{t+\frac{1}{m}}V$; more specifically,

$$ {}_{t+s}V \approx \frac{(\frac{1}{m} - s)({}_tV + P_t - e_t) + s\left({}_{t+\frac{1}{m}}V\right)}{1/m}. $$

In Example 7.12 above, where $s = 0.05$ and $m = 12$, this would give an approximate value for ${}_{2.8}V$ of

$$\frac{0.033(3091.02 + 0.9(460)) + 0.05(3456.72)}{0.0833} = 3476.04\,,$$

which is very close to the exact value.

In Example 7.11, we have

$${}_{30.8}V \approx 0.2(99\,299 + 0.95(2375)) + 0.8(103\,480) = 103\,095\,,$$

which is 99.99% of the accurate value of 103 108.

7.4 Policy values with continuous cash flows

7.4.1 Thiele's differential equation

In the previous sections, we have defined policy values for policies with cash flows at discrete intervals, and derived recursive formulae linking policy values at successive cash flow time points for these policies. These ideas extend to contracts where regular payments – premiums and/or annuities – are payable continuously and sums insured are payable immediately on death. In this case we can derive a differential equation, known as **Thiele's differential equation**. This is a continuous time version of the recursion equation which we derived in Section 7.2.3. Recall that for the discrete life insurance case

$$({}_tV + P_t - e_t)(1 + i_t) = {}_{t+1}V + q_{[x]+t}(S_{t+1} + E_{t+1} - {}_{t+1}V). \qquad (7.10)$$

Our derivation of Thiele's differential equation is somewhat different to the derivation of equation (7.10). However, once we have completed the derivation, we explain the link with this equation.

Consider a policy issued to a select life aged x under which premiums and premium-related expenses are payable continuously and the sum insured, together with any related expenses, is payable immediately on death. Suppose this policy has been in force for t years, where $t \geq 0$, and let

P_t = the annual rate of premium payable at time t,
e_t = the annual rate of premium-related expense payable at time t,
S_t = the sum insured payable immediately on death at time t,
E_t = the expense of paying the sum insured at time t,
$\mu_{[x]+t}$ = the force of mortality at age $[x] + t$,
δ_t = the force of interest per year assumed earned at time t, and,
${}_tV$ = the policy value for a policy in force at time t.

We assume that $P_t,\ e_t,\ S_t,\ \mu_{[x]+t}$ and δ_t are all continuous functions of t and that $e_t,\ E_t,\ \mu_{[x]+t}$ and δ_t are all as assumed in the policy value basis.

Just as we allowed the rate of interest to vary from year to year in Section 7.2.3, we are here letting the force of interest be a continuous function of time. Thus, if $v(t)$ denotes the present value of a payment of 1 at time t, we have

$$v(t) = \exp\left\{ -\int_0^t \delta_s \, ds \right\}. \tag{7.11}$$

Now ${}_tV$ represents the EPV of future benefits plus benefit-related expenses minus the EPV of future premiums less premium-related expenses, so we have

$$\begin{aligned} {}_tV = & \int_0^\infty \frac{v(t+s)}{v(t)} \left(S_{t+s} + E_{t+s}\right) \, {}_sp_{[x]+t} \, \mu_{[x]+t+s} \, ds \\ & - \int_0^\infty \frac{v(t+s)}{v(t)} \left(P_{t+s} - e_{t+s}\right) \, {}_sp_{[x]+t} \, ds. \end{aligned}$$

Note that we are measuring time, represented by s in the integrals, from time t, so that if, for example, the sum insured is payable at time s, the amount of the sum insured is S_{t+s} and as we are discounting back to time t, the discount factor is $v(t+s)/v(t)$. Changing the variable of integration to $r = t + s$ gives

$${}_tV = \int_t^\infty \frac{v(r)}{v(t)} \left(S_r + E_r\right) \, {}_{r-t}p_{[x]+t} \, \mu_{[x]+r} \, dr - \int_t^\infty \frac{v(r)}{v(t)} \left(P_r - e_r\right) \, {}_{r-t}p_{[x]+t} \, dr. \tag{7.12}$$

We could use formula (7.12) to calculate ${}_tV$ by numerical integration. However, we are instead going to turn this identity into a differential equation. There are two main reasons why we do this:

(1) There exist numerical techniques to solve differential equations, one of which is discussed in the next section. As we will see, an advantage of such an approach over numerical integration is that we can easily calculate policy values at multiple durations.
(2) In Chapter 8 we consider more general types of insurance policy than we have so far. For such policies it is usually the case that we are unable to calculate policy values using numerical integration, and we must calculate policy values using a set of differential equations. The following development of Thiele's differential equation sets the scene for the next chapter.

In order to turn equation (7.12) into a differential equation, we note that

$$ {}_{r-t}p_{[x]+t} = \frac{{}_r p_{[x]}}{{}_t p_{[x]}} $$

so that

$$ {}_tV = \frac{1}{v(t)\ {}_t p_{[x]}} \left(\int_t^{\infty} v(r)\,(S_r + E_r)\ {}_r p_{[x]}\ \mu_{[x]+r}\,dr - \int_t^{\infty} v(r)\,(P_r - e_r)\ {}_r p_{[x]}\,dr \right), $$

which we can write as

$$ v(t)\ {}_t p_{[x]}\ {}_tV = \int_t^{\infty} v(r)\,(S_r + E_r)\ {}_r p_{[x]}\ \mu_{[x]+r}\,dr - \int_t^{\infty} v(r)\,(P_r - e_r)\ {}_r p_{[x]}\,dr. \tag{7.13} $$

Differentiation of equation (7.13) with respect to t leads to Thiele's differential equation. First, differentiation of the right-hand side yields

$$ \begin{aligned} &- v(t)\,(S_t + E_t)\ {}_t p_{[x]}\,\mu_{[x]+t} + v(t)\,(P_t - e_t)\ {}_t p_{[x]} \\ &\quad = v(t)\ {}_t p_{[x]}\ \big(P_t - e_t - (S_t + E_t)\,\mu_{[x]+t}\big)\,. \end{aligned} \tag{7.14} $$

Differentiation of the left-hand side is most easily done in two stages, applying the product rule for differentiation at each stage. Treating $v(t)\ {}_t p_{[x]}$ as a single function of t we obtain

$$ \frac{d}{dt}\big(v(t)\,{}_t p_{[x]}\,{}_tV\big) = v(t)\ {}_t p_{[x]}\ \frac{d}{dt}\,{}_tV + \ {}_tV\ \frac{d}{dt}\big(v(t)\ {}_t p_{[x]}\big)\,. $$

Next,

$$ \frac{d}{dt}\big(v(t)\ {}_t p_{[x]}\big) = v(t)\frac{d}{dt}\,{}_t p_{[x]} + \ {}_t p_{[x]}\ \frac{d}{dt}\ v(t). $$

From Chapter 2 we know that

$$ \frac{d}{dt}\,{}_t p_{[x]} = -{}_t p_{[x]}\mu_{[x]+t} $$

and from formula (7.11)

$$ \frac{d}{dt}\ v(t) = -\delta_t\ \exp\left\{ -\int_0^t \delta_s\,ds \right\} = -\delta_t\ v(t). $$

Thus, the derivative of the left-hand side of equation (7.13) is

$$ \begin{aligned} \frac{d}{dt}\big(v(t)\ {}_t p_{[x]}\ {}_tV\big) &= v(t)\ {}_t p_{[x]}\ \frac{d}{dt}\,{}_tV - \ {}_tV\,\big(v(t)\ {}_t p_{[x]}\mu_{[x]+t} + \ {}_t p_{[x]}\ \delta_t\ v(t)\big) \\ &= v(t)\ {}_t p_{[x]}\ \left(\frac{d}{dt}\,{}_tV - \ {}_tV\,\big(\mu_{[x]+t} + \delta_t\big)\right). \end{aligned} $$

Equating this to (7.14) yields **Thiele's differential equation**, namely

$$\boxed{\frac{d}{dt}\,{}_tV = \delta_t\,{}_tV + P_t - e_t - (S_t + E_t - {}_tV)\,\mu_{[x]+t}.} \tag{7.15}$$

Formula (7.15) can be interpreted as follows. The left-hand side of the formula, $\frac{d}{dt}{}_tV$, is the rate of increase in the policy value at time t. We can derive a formula for this rate of increase by considering the individual factors affecting the value of ${}_tV$:

- Interest is being earned on the current amount of the policy value. The amount of interest earned in the time interval t to $t+h$ is $\delta_t\,{}_tV\,h\ (+o(h))$, so that the rate of increase at time t is $\delta_t\,{}_tV$.
- Premium income, minus premium-related expenses, is increasing the policy value at rate $P_t - e_t$. If there were annuity payments at time t, this would decrease the policy value at the rate of the annuity payment (plus any annuity-related expenses).
- Claims, plus claim-related expenses, decrease the amount of the policy value. The expected extra amount payable in the time interval t to $t+h$ is $\mu_{[x]+t}\,h\,(S_t + E_t - {}_tV)$ and so the rate of decrease at time t is $\mu_{[x]+t}$ $(S_t + E_t - {}_tV)$.

Hence the total rate of increase of the policy value at time t is

$$\delta_t\,{}_tV + P_t - e_t - \mu_{[x]+t}(S_t + E_t - {}_tV).$$

We can also relate formula (7.15) to equation (7.10) assuming that for some very small value h,

$$\frac{d}{dt}\,{}_tV \approx \frac{1}{h}\left({}_{t+h}V - {}_tV\right), \tag{7.16}$$

leading to the relationship

$$(1+\delta_t h)\ {}_tV + (P_t - e_t)h \approx {}_{t+h}V + h\mu_{[x]+t}(S_t + E_t - {}_tV).$$

Remembering that h is very small, the interpretation of the left-hand side is that it is the accumulation from time t to time $t+h$ of the policy value at time t plus the accumulation at time $t+h$ of the premium income less premium-related expenses over the interval $(t, t+h)$. (Note that for very small h, $\bar{s}_{\overline{h|}} \approx h$.) This total accumulation must provide the policy value at time $t+h$, and, if death occurs in the interval $(t, t+h)$, it must also provide the excess $S_t + E_t - {}_tV$ over the policy value. The probability of death in the interval $(t, t+h)$ is approximately $h\mu_{[x]+t}$.

7.4.2 Numerical solution of Thiele's differential equation

In this section we show how we can evaluate policy values by solving Thiele's differential equation numerically. The key to this is to apply equation (7.16) as an identity rather than an approximation, assuming that h is very small, that is

$$ {}_{t+h}V - {}_tV = h(\delta_t\, {}_tV + P_t - e_t - \mu_{[x]+t}(S_t + E_t - {}_tV)). \tag{7.17}$$

The smaller the value of h, the better this approximation is likely to be. The values of δ_t, P_t, e_t, $\mu_{[x]+t}$, S_t and E_t are assumed to be known, so this equation allows us to calculate ${}_tV$ provided we know the value of ${}_{t+h}V$, or ${}_{t+h}V$ if we know the value of ${}_tV$. But we always know the value of ${}_tV$ as t approaches the end of the policy term since, in the limit, it is the amount that should be held in respect of a policyholder who is still alive. For an endowment policy with term n years and sum insured S, the policy value builds up so that just before the maturity date it is exactly sufficient to pay the maturity benefit, that is

$$\lim_{t\to n-} {}_tV = S.$$

For a term insurance, with term n years and sum insured S, we have

$$\lim_{t\to n-} {}_tV = 0.$$

For a whole life insurance with sum insured S, if the policyholder is still alive as she approaches age ω (the upper limit of the life table), we assume the policyholder, and the policy expire at that time, so that

$$\lim_{t\to \omega-} {}_tV = S.$$

Using the endowment policy with term n years and sum insured S as an example, formula (7.17) with $t = n - h$ gives us

$$S - {}_{n-h}V = h\Big(\delta_{n-h}\, {}_{n-h}V + P_{n-h} - e_{n-h} - \mu_{[x]+n-h}(S_{n-h} + E_{n-h} - {}_{n-h}V)\Big), \tag{7.18}$$

from which we can calculate ${}_{n-h}V$. Another application of formula (7.17) with $t = n - 2h$ gives the value of ${}_{n-2h}V$, and so on.

This method for the numerical solution of a differential equation is known as **Euler's method**. It is also the continuous time version of the discrete time recursive method for calculating policy values illustrated in Example 7.7.

In some cases we also have a boundary condition at $t = 0$. For example, if the premium is calculated using the equivalence principle, we know that ${}_0V = 0$. However, if there is a single premium, or if there is a lump sum initial expense, there will be an immediate jump in the policy value, so a different boundary condition applies. In general, given a single equivalence principle premium G_0 and initial expenses e_0, the time $t = 0$ boundary condition is

$$\lim_{t \to 0+} {}_tV = G_0 - e_0.$$

Example 7.13 Consider a 20-year endowment insurance issued to a life aged 30. The sum insured, \$100 000, is payable immediately on death, or on survival to the end of the term, whichever occurs sooner. Premiums are payable continuously at a constant rate of \$2500 per year throughout the term of the policy. The policy value basis uses a constant force of interest, δ, and makes no allowance for expenses.

(a) Evaluate ${}_{10}V$.
(b) Use Euler's method with $h = 0.05$ years to calculate ${}_{10}V$.

Perform the calculations on the following basis:

Survival model: Standard Select Survival Model
Interest: $\delta = 0.04$ per year

Solution 7.13 (a) We have

$${}_{10}V = 100\,000\bar{A}_{40:\overline{10|}} - 2500\bar{a}_{40:\overline{10|}},$$

and as $\bar{A}_{40:\overline{10|}} = 1 - \delta\,\bar{a}_{40:\overline{10|}}$, we can calculate ${}_{10}V$ as

$${}_{10}V = 100\,000 - (100\,000\delta + 2500)\bar{a}_{40:\overline{10|}}.$$

Using numerical integration or the three-term Woolhouse formula, we get $\bar{a}_{40:\overline{10|}} = 8.2167$, and hence ${}_{10}V = 46\,591$.

(b) For this example, we use formula (7.18) with $h = 0.05$, $\delta = 0.04$, $P = 2500$, and $S = 100\,000$. Using Euler's method, working backwards from $n = 20$, and noting that $\mu_{49.95} = 0.001147$, we have

$$S - {}_{19.95}V = 0.05(0.04 \times {}_{19.95}V + 2\,500 - 0.001147(100\,000 - {}_{19.95}V))$$

which gives ${}_{19.95}V = 99\,676$. Calculating recursively ${}_{19.9}V$, ${}_{19.85}V, \ldots$, we arrive at ${}_{10}V = 46\,635$.

We note that the answer here is close to \$46 591, the value calculated in part (a). Using a value of $h = 0.01$ gives the closer answer of \$46 600. □

We remarked earlier that a useful feature of setting up and numerically solving a differential equation for policy values is that the numerical solution gives policy values at a variety of durations. We can see this in the above example. In part (a) we wrote down an expression for ${}_{10}V$ and evaluated it using numerical integration. By contrast, in part (b) with $h = 0.05$, as a by-product of our backwards recursive calculation of ${}_{10}V$ we also obtained values of ${}_{10+h}V, {}_{10+2h}V \dots, {}_{20-h}V$.

Other major advantages of Thiele's equation arise from its versatility and flexibility. We can easily accommodate variable premiums, benefits and interest rates. We can also use the equation to solve numerically for the premium given the benefits, interest model and boundary values for the policy values.

Some notes on Euler's method

There are two versions of Euler's method, the forward method and the backward method. In both methods, we approximate a function by using its derivative at intervals, of length h, say, to construct a piecewise linear function where each segment matches the derivative of the function at the start of the interval (under the forward method) or at the end of the interval (under the backward method). That is, suppose we have a differential equation $\frac{d}{dt}g(t) = d(t)$ for some functions g and d, and assume a step size h. Then we have the **forward Euler approximation**

$$\frac{g(t+h)-g(t)}{h} \approx d(t) \Rightarrow g(t+h) \approx g(t) + h\,d(t), \tag{7.19}$$

and the **backward Euler approximation**,

$$\frac{g(t+h)-g(t)}{h} \approx d(t+h) \Rightarrow g(t+h) \approx g(t) + h\,d(t+h). \tag{7.20}$$

As $h \to 0$ the forward and backward methods converge.

The examples in this section use the forward method, even though we iterated backwards from the boundary condition at the maximum policy term. In the next chapter we use both methods.

In practice, there are more efficient algorithms for solving differential equations; some references are given at the end of this chapter. However, Euler's method is intuitive, and provides a clear connection between the policy value recursions in the discrete case, and the differential equation in the continuous case.

7.5 Policy alterations

A life insurance policy is a contract between an individual, the policyholder, and the insurance company. This contract places obligations on both parties; for example, the policyholder agrees to pay regular premiums while he or she remains alive and the insurance company agrees to pay a sum insured, plus bonuses for a participating policy, on the death of the policyholder. So far in this book we have assumed that the terms of the contract are never broken or altered in any way. In practice, it is not uncommon, after the policy has been in force for some time, for the policyholder to request a change in the terms of the policy. Typical changes might be:

(1) The policyholder wishes to cancel the policy with immediate effect. In this case, it may be appropriate for the insurance company to pay a lump sum, called the **cash value** or **surrender value**, to the policyholder. This will be the case if the policy has a significant investment component – such as an endowment insurance, or whole life insurance. Term insurance and life annuity contracts (after any deferred period) do not have cash values. A policy which is cancelled at the request of the policyholder before the end of its originally agreed term, is said to **lapse** or to be **surrendered**.

We tend to use the term **lapse** to indicate a voluntary cessation when no surrender value is paid, and **surrender** when there is a return of assets of some amount to the policyholder, but the words may be used interchangeably.

In the USA and some other countries, insurers are required to offer cash surrender values on certain contract types once they have been in force for one or two years. The stipulation is known as the **non-forfeiture law**. Allowing zero cash values for early surrenders reflects the need of the insurers to recover the new business strain associated with issuing the policy.

(2) The policyholder wishes to pay no more premiums but does not want to cancel the policy, so that, in the case of an endowment insurance for example, a (reduced) sum insured is still payable on death or on survival to the end of the original term, whichever occurs sooner. Any policy for which no further premiums are payable is said to be **paid-up**, and the reduced sum insured for a policy which becomes paid-up before the end of its original premium paying term is called a **paid-up sum insured**.

(3) A whole life policy may be converted to a paid-up term insurance policy for the original sum insured.

(4) Some term insurance policies carry an option to convert to a whole life policy at certain times.

(5) Many other types of alteration can be requested: reducing or increasing premiums; changing the amount of the benefits; converting a whole life insurance to an endowment insurance; converting a non-participating policy to a with-profit policy; and so on. The common feature of these changes is that they are requested by the policyholder and were not part of the original terms of the policy.

If the change was not part of the original terms of the policy, and if it has been requested by the policyholder, it could be argued that the insurance company is under no obligation to agree to it. However, when the insurer has issued a contract with a substantive investment objective, rather than solely offering protection against untimely death, then at least part of the funds should be considered to be the policyholder's, under the stewardship of the insurer. In the USA the non-forfeiture law states that, for investment-type policies, each of (1), (2) and (3) would generally be available on pre-specified minimum terms. In particular, fixed or minimum cash surrender values, as a percentage of the sum insured, are specified in advance in the contract terms for such policies.

For policies with pre-specified cash surrender values, let CV_t denote the cash surrender value at duration t. Where surrender values are not set in advance, the actuary would determine an appropriate value for CV_t at the time of alteration.

Starting points for the calculation of CV_t could be the policy value at t, ${}_tV$, if it is to be calculated in advance, or the policy's asset share, AS_t, when the surrender value is not pre-specified. Recall that AS_t represents (approximately) the cash the insurer actually has and ${}_tV$ represents the amount the insurer should have at time t in respect of the original policy. Recall also that if the policy value basis is close to the actual experience, then ${}_tV$ will be numerically close to AS_t.

Setting CV_t equal to either AS_t or ${}_tV$ could be regarded as over-generous to the policyholder for several reasons, including:

(1) It is the policyholder who has requested that the contract be changed. The insurer will be concerned to ensure that surrendering policyholders do not benefit at the expense of the continuing policyholders – most insurers prefer the balance to go the other way, so that policyholders who maintain their contracts through to maturity achieve greater value than those who surrender early or change the contract. Another implication of the fact that the policyholder has called for alteration is that the policyholder may be acting on knowledge that is not available to the insurer. For example, a policyholder may alter a whole life policy to a term insurance (with lower premiums or a higher sum insured) if he or she becomes aware that their health is failing. This is an example of **adverse selection** or **selection against the insurer**.

(2) The insurance company will incur some expenses in making alterations to the policy, and even in calculating and informing the policyholder of the revised values, which the policyholder may not agree to accept.

(3) The alteration may, at least in principle, cause the insurance company to realize assets it would otherwise have held, especially if the alteration is a surrender. This **liquidity risk** may lead to reduced investment returns for the company. Under non-forfeiture law in the USA, the insurer has six months to pay the cash surrender value, so that it is not forced to sell assets at short notice.

For these reasons, CV_t is usually less than 100% of either AS_t or ${}_tV$ and may include an explicit allowance for the expense of making the alteration.

For alterations other than cash surrenders, we can apply CV_t as if it were a single premium, or an extra preliminary premium, for the future benefits. That is, we construct the equation of value for the altered benefits,

$$\begin{aligned} &CV_t + \text{EPV at } t \text{ of future premiums, altered contract} \\ &\quad = \text{EPV at } t \text{ of future benefits plus expenses, altered contract.} \end{aligned} \tag{7.21}$$

The numerical value of the revised benefits and/or premiums calculated using equation (7.21) depends on the basis used for the calculation, that is, the assumptions concerning the survival model, interest rate, expenses and future bonuses (for a participating policy). This basis may be the same as the premium basis, or the same as the policy value basis, but in practice usually differs from both of them.

The rationale behind equation (7.21) is the same as that which leads to the equivalence principle for calculating premiums: together with the cash currently available (CV_t), the future premiums are expected to provide the future benefits and pay for the future expenses.

Example 7.14 Consider the 10-year, regular premium deferred annuity with a return of premium death benefit discussed in Example 7.4.

You are given that the insurer's experience in the five years following the issue of this policy is as in Example 7.9. At the start of the sixth year, before paying the premium then due, the policyholder requests that the policy be altered in one of the following three ways.

(a) The policy is surrendered immediately.
(b) No more premiums are paid and a reduced annuity is payable from age 60. In this case, all premiums paid are refunded at the end of the year of death if the policyholder dies before age 60.
(c) Premiums continue to be paid, but the benefit is altered from an annuity to a lump sum (pure endowment) payable on reaching age 60. Expenses and benefits on death before age 60 follow the original policy terms. There is an expense of \$100 associated with paying the sum insured at the new maturity date.

Calculate (a) the surrender value, (b) the reduced annuity, and (c) the sum insured, assuming the insurer uses

(i) 90% of the asset share less a charge of \$200, or
(ii) 90% of the policy value less a charge of \$200

together with the assumptions in the policy value basis when calculating revised benefits and premiums.

Solution 7.14 We already know from Examples 7.4 and 7.9 that

$$ {}_5V = 65\,470 \quad \text{and} \quad AS_5 = 63\,509. $$

Hence, the amount CV_5 to be used in equation (7.21) is

(i) $0.9 \times AS_5 - 200 = 56\,958$,
(ii) $0.9 \times {}_5V - 200 = 58\,723$.

(a) The surrender values are the cash values CV_5, so we have
  (i) \$56 958,
  (ii) \$58 723.
(b) Let X denote the revised annuity amount. In this case, equation (7.21) gives

$$ CV_5 = 5 \times 11\,900 A^{\;1}_{55:\overline{5}|} + 100 A^{\;1}_{55:\overline{5}|} + (X + 25)\, {}_5E_{55}\, \ddot{a}_{60}. $$

  Using values calculated for the solution to Example 7.4, we can solve this equation for the two different values for CV_5 to give
  (i) $X = \$4859$,
  (ii) $X = \$5012$.
(c) Let S denote the new sum insured. Equation (7.21) now gives

$$ CV_5 + 0.95 \times 11\,900 \ddot{a}_{55:\overline{5}|} = 11\,900 \left((IA)^{1}_{55:\overline{5}|} + 5A^{\;1}_{55:\overline{5}|} \right) + 100 A^{\;1}_{55:\overline{5}|} + {}_5E_{55}\,(S + 100), $$

  which we solve using the two different values for CV_5 to give
  (i) $S = \$138\,314$,
  (ii) $S = \$140\,594$. □

Example 7.15 Ten years ago a man now aged 40 purchased a with-profit whole life insurance. The basic sum insured, payable at the end of the year of death, was \$200 000. Premiums of \$1500 were payable annually for life.

The policyholder now requests that the policy be changed to a with-profit endowment insurance with a remaining term of 20 years, with the same premium payable annually, but now for a maximum of 20 further years.

The insurer uses the following basis for the calculation of policy values and policy alterations.

Survival model: Standard Select Survival Model
Interest: 5% per year
Expenses: none
Bonuses: compound reversionary bonuses at rate 1.2% per year at the start of each policy year, including the first.

The insurer uses the full policy value less an expense of \$1000 when calculating revised benefits. You are given that the actual bonus rate declared in each of the past 10 years has been 1.6%.

(a) Calculate the revised sum insured, to which future bonuses will be added, assuming the premium now due has not been paid and the bonus now due has not been declared.
(b) Calculate the revised sum insured, to which future bonuses will be added, assuming the premium now due has been paid and the bonus now due has been declared to be 1.6%.

Solution 7.15 (a) Before the declaration of the bonus now due, the sum insured for the original policy is

$$200\,000 \times 1.016^{10} = 234\,405.$$

Hence, the policy value for the original policy, ${}_{10}V$, is given by

$${}_{10}V = 234\,405 A_{40\,j} - P\ddot{a}_{40}$$

where $P = 1500$ and the subscript j indicates that the rate of interest to be used is $j = 0.0375$ since $1.05/1.012 = 1.0375$. Let S denote the revised sum insured. Then, using equation (7.21)

$${}_{10}V - 1000 = S A_{40:\overline{20}|\,j} - P\ddot{a}_{40:\,\overline{20}|} \Rightarrow S = \$76\,039.$$

A point to note here is that the life was select at the time the policy was purchased, ten years ago. No further health checks are carried out at the time of a policy alteration and so the policyholder is now assumed to be subject to the ultimate part of the survival model.

(b) Let ${}_{10+}V$ denote the policy value just after the premium has been paid and the bonus has been declared at time 10. The term $A_{40\,j}$ used in the calculation of ${}_{10}V$ assumed the bonus to be declared at time 10 would be 1.2%, so that the sum insured in the 11th year would be $234\,405 \times 1.012$, in the 12th year would be $234\,405 \times 1.012^2$, and so on. Given that the bonus declared at time 10 is 1.6%, these sums insured are now $234\,405 \times 1.016$ (this value is known) and $234\,405 \times 1.016 \times 1.012$ (this is an assumed value since it assumes the bonus declared at the start of the 12th year will be 1.2%). Hence

$$\begin{aligned}{}_{10+}V &= (1.016/1.012) \times 234\,405 A_{40\,j} - P a_{40} \\ &= (1.016/1.012) \times 234\,405 A_{40\,j} - P\ddot{a}_{40} + P.\end{aligned}$$

Let S' denote the revised sum insured for the endowment policy in this case. Equation (7.21) now gives

$$
\begin{aligned}
{}_{10+}V - 1000 &= (S'/1.012)\, A_{40:\overline{20}|j} - P a_{40:\overline{19}|} \\
&= (S'/1.012)\, A_{40:\overline{20}|j} - P(\ddot{a}_{40:\overline{20}|} - 1),
\end{aligned}
$$

and hence the revised sum insured is $S' = \$77\,331$. □

Note that, in Example 7.15, the sum insured payable in the 11th year is $S \times 1.016 = \$149\,295$ in part (a) and \$149 381 in part (b). The difference between these values is not due to rounding – the timing of the request for the alteration has made a (small) difference to the sum insured offered by the insurer for the endowment insurance. This is caused partly by the charge of \$1000 for making the alteration and partly by the fact that the bonus rate in the 11th year is not as assumed in the policy value basis. In Example 7.15 we would have $S' = S \times 1.012$ if there were no charge for making the alteration and the bonus rate declared in the 11th year were the same as the rate assumed in the reserve basis (and the full policy value is still used in the calculation of the revised benefit).

7.6 Retrospective policy values

7.6.1 Prospective and retrospective valuation

Our definition of a policy value is based on the future loss random variable. As noted in Section 7.2.2, what we have called a policy value is called by some authors a *prospective policy value*. Since *prospective* means looking to the future, this name has some merit. We may also define the *retrospective policy value* at duration t, which is, loosely, the accumulated value of past premiums received, less the value of the past insurance, for a large group of identical policies, assuming the experience follows precisely the assumptions in the policy value basis, divided by the expected number of survivors. This is precisely the calculation detailed in the final part of Section 7.2.1 in respect of the policy studied in Example 7.1, so that the left-hand side of formula (7.3) is a formula for the retrospective policy value (at duration 10) for this particular policy.

The main purpose of policy values is to determine the liability value for policies that are in force. That is, the policy value is used to determine the capital that the insurer needs to hold such that, together with the expected future premiums, the insurer will have sufficient assets to meet the expected future liabilities. This fund is what actuaries call the reserve at time t for the policy. Since the purpose of the policy value is to assess future needs, it is natural to take the prospective approach.

The retrospective policy value will be defined more formally below; loosely, it measures the value at time t of all the cash flows from time 0 to time t, expressed per surviving policyholder. It is connected to the asset share, which tracks the accumulated contribution of each surviving policy to the insurer's funds. The difference between the retrospective policy value and the asset share is that, by definition, the asset share at time t uses the actual experience up to time t. The asset share at time t cannot be calculated until time t. The retrospective policy value can use any basis, and can be calculated at any time. If the retrospective policy value basis exactly matches the experience, then it will be equal to the asset share.

Intuitively, we interpret the prospective policy value as a measure of the funds needed at time t, and the retrospective policy value as a measure of the funds expected to be acquired at time t. The reserve must be prospective to meet natural requirements that assets should be sufficient to meet future liabilities. At time t, also, we have an exact measure of the asset share at that time. It is not clear why the retrospective reserve is necessary, and it is not commonly used in any country which uses gross premium policy values for setting reserves.

However, there is one way in which retrospective policy values may be useful, and it arises from the fact that, under very specific conditions, the prospective and retrospective policy values are equal. That is, let ${}_tV^R$ denote the retrospective policy value and let ${}_tV^P$ denote the prospective policy value of an n-year insurance policy. There are two conditions for ${}_tV^P$ to be equal to ${}_tV^R$, in general, namely:

(1) the premiums for the contract are determined using the equivalence principle, and
(2) the same basis is used for ${}_tV^R$, ${}_tV^P$ and the equivalence principle premium.

Now, in most cases, these conditions are very unlikely to be satisfied. Policies are very long term, and the basis used to determine the premiums will be updated regularly to reflect more up to date information about interest rates, expenses and mortality. The valuation assumptions might be quite different to the premium basis, as the former are likely to be more regulated to manage solvency risk.

However, there is one circumstance when the conditions may be satisfied, and the equality of the prospective and retrospective policy values may be useful. This is the case when the insurer uses the net premium policy value for determination of the reserves. Recall from Definition 7.2 that under the net premium policy value calculation, the premium used is *always* calculated using the valuation basis (regardless of the true or original premium). As the net premium is calculated using the equivalence principle, then the retrospective and prospective net premium policy values will be the same. This can be useful if the premium or benefit structure is complicated, so that it may be simpler to take the accumulated value of past premiums less accumulated value of benefits, per surviving policyholder (the retrospective policy value), than to use the prospective policy value. It is worth noting that many policies in the USA are still valued using net premium policy values, often using a retrospective formula. In all other major developed insurance markets, regulators require some form of gross premium policy value calculation, and in these countries the retrospective approach would be inappropriate.

7.6.2 *Defining the retrospective net premium policy value*

Consider an insurance sold to (x) at time $t = 0$ with term n (which may be ∞ for a whole life contract). For a policy in force at time t, let L_t denote the present value at time t of all the future benefits less net premiums, under the terms of the contract. The prospective policy value, ${}_tV^P$, was defined for policies in force at time $t < n$ as

$${}_tV^P = E[L_t].$$

If (x) does not survive to time t then L_t is undefined.

The value at issue of all future benefits less premiums payable from time $t < n$ onwards is the random variable

$$I(T_x > t)\, v^t\, L_t$$

where I is the indicator random variable.

Further, we define $L_{0,t}$, for $t \leq n$, as

$$\begin{aligned} L_{0,t} = {} & \text{PV at issue of future benefits payable up to time } t \\ & - \text{PV at issue, of future net premiums payable up to time } t. \end{aligned}$$

If premiums and benefits are paid at discrete intervals, and t is a premium or benefit payment date, then the convention is that $L_{0,t}$ would include benefits payable at time t, but not premiums. At issue (time 0) the loss at issue random variable, L_0, comprises the value of benefits less premiums up to time t, $L_{0,t}$, plus the value of benefits less premiums payable after time t, that is

$$L_0 = L_{0,t} + I(T_x > t) v^t L_t \,.$$

We now define the retrospective net premium policy value as

$${}_tV^R = \frac{-\mathrm{E}[L_{0,t}](1+i)^t}{{}_tp_x} = \frac{-\mathrm{E}[L_{0,t}]}{{}_tE_x}$$

and this formula corresponds to the calculation in Section 7.2 for the policy from Example 7.1. The term $-\mathrm{E}[L_{0,t}](1+i)^t$ is the expected value of premiums less benefits in the first t years, accumulated to time t. Dividing by ${}_tp_x$ expresses the expected accumulation per expected surviving policyholder.

Recall the conditions listed for equality of the retrospective and prospective values:

(1) the premium is calculated using the equivalence principle, and
(2) the same basis is used for prospective policy values, retrospective policy values and the equivalence principle premium.

By the equivalence principle,

$$\begin{aligned}
& \mathrm{E}[L_0] = \mathrm{E}\left[L_{0,t} + I(T_x > t)\, v^t\, L_t\right] = 0, \\
\Rightarrow\ & -\mathrm{E}[L_{0,t}] = \mathrm{E}\left[I(T_x > t)\, v^t\, L_t\right] \\
\Rightarrow\ & -\mathrm{E}[L_{0,t}] = {}_tp_x\, v^t\, {}_tV^P \\
\Rightarrow\ & {}_tV^R = {}_tV^P.
\end{aligned}$$

The same result could easily be derived for gross premium policy values, but the assumptions listed are very unlikely to hold when expenses and gross premiums are taken into consideration.

Example 7.16 An insurer issues a whole life insurance policy to a life aged 40. The death benefit in the first five years of the contract is \$5 000. In subsequent years, the death benefit is \$100 000. The death benefit is payable at the end of the year of death. Premiums are paid annually for a maximum of 20 years. Premiums are level for the first five years, then increase by 50%.

(a) Write down the equation of value for calculating the net premium, using standard actuarial functions.
(b) Write down equations for the net premium policy value at time $t = 4$ using (i) the retrospective policy value approach, and (ii) the prospective policy value approach.
(c) Write down equations for the net premium policy value at time $t = 20$ using (i) the retrospective policy value approach, and (ii) the prospective policy value approach.

Solution 7.16 For convenience, we work in units of \$1000.

(a) The equivalence principle premium is P for the first five years, and $1.5\,P$ thereafter, where

$$P = \frac{5\,A^{\;1}_{40:\overline{5|}} + 100\,{}_5E_{40}\,A_{45}}{\ddot{a}_{40:\overline{5|}} + 1.5\,{}_5E_{40}\,\ddot{a}_{45:\overline{15|}}}. \tag{7.22}$$

(b) The retrospective and prospective policy value equations at time $t = 4$ are

$$ {}_4V^R = \frac{P\,\ddot{a}_{40:\overline{4}|} - 5A^{\;1}_{40:\overline{4}|}}{{}_4E_{40}} \tag{7.23}$$

and

$$ {}_4V^P = 5\,A^{\;1}_{44:\overline{1}|} + 100\,{}_1E_{44}\,A_{45} - P\big(\ddot{a}_{44:\overline{1}|} + 1.5\,{}_1E_{44}\,\ddot{a}_{45:\overline{15}|}\big). \tag{7.24}$$

(c) The retrospective and prospective policy value equations at time $t = 20$ are

$$ {}_{20}V^R = \frac{P\big(\ddot{a}_{40:\overline{5}|} + 1.5\,{}_5E_{40}\,\ddot{a}_{45:\overline{15}|}\big) - 5\,A^{\;1}_{40:\overline{5}|} - 100\,{}_5E_{40}\,A_{45:\overline{15}|}}{{}_{20}E_{40}}$$

and

$$ {}_{20}V^P = 100\,A_{60}. $$

□

From these equations, we see that for this contract, the retrospective policy value offers an efficient calculation method at the start of the contract, when the premium and benefit changes are ahead, and the prospective approach is more efficient at later durations, when the changes are in the past.

Example 7.17 For Example 7.16 above, show that the prospective and retrospective policy values at time $t = 4$, given in equations (7.23) and (7.24), are equal under the standard assumptions (premium and policy values all use the same basis, and the equivalence principle premium).

Solution 7.17 Note that, assuming all calculations use the same basis,

$$\begin{aligned}
A^{\;1}_{40:\overline{5}|} &= A^{\;1}_{40:\overline{4}|} + {}_4E_{40}\,A^{\;1}_{44:\overline{1}|},\\
\ddot{a}_{40:\overline{5}|} &= \ddot{a}_{40:\overline{4}|} + {}_4E_{40}\,\ddot{a}_{44:\overline{1}|},\\
{}_5E_{40} &= {}_4E_{40}\;{}_1E_{44}.
\end{aligned}$$

Now we use these to re-write the equivalence principle premium equation (7.22),

$$\begin{aligned}
&P\left(\ddot{a}_{40:\overline{5}|} + 1.5\,{}_5E_{40}\,\ddot{a}_{45:\overline{15}|}\right) = 5A^{\;1}_{40:\overline{5}|} + 100\,{}_5E_{40}\,A_{45}\\
&\Rightarrow P\left(\ddot{a}_{40:\overline{4}|} + {}_4E_{40}\,\ddot{a}_{44:\overline{1}|} + 1.5\,{}_4E_{40}\,{}_1E_{44}\,\ddot{a}_{45:\overline{15}|}\right)\\
&\qquad\qquad = 5\left(A^{\;1}_{40:\overline{4}|} + {}_4E_{40}\,A^{\;1}_{44:\overline{1}|}\right) + 100\,{}_4E_{40}\,{}_1E_{44}\,A_{45}.
\end{aligned}$$

Rearranging gives

$$P\ddot{a}_{40:\overline{4}|} - 5A^{\,1}_{40:\overline{4}|} = {}_4E_{40}\left(5\,A^{\,1}_{44:\overline{1}|} + 100\,{}_1E_{44}\,A_{45} - P\left(\ddot{a}_{44:\overline{1}|} + 1.5\,{}_1E_{44}\,\ddot{a}_{45:\overline{15}|}\right)\right).$$

Dividing both sides by ${}_4E_{40}$ gives ${}_4V^R = {}_4V^P$ as required. □

7.7 Negative policy values

In all our examples in this chapter, the policy value was either zero or positive. It can happen that a policy value is negative. In fact, negative gross premium policy values are not unusual in the first few months of a contract, after the initial expenses have been incurred, and before sufficient premium is collected to defray these expenses. However, it would be unusual for policy values to be negative after the early period of the contract. If we consider the policy value equation

$${}_tV = \text{EPV at } t \text{ of Future Benefits} + \text{Expenses} - \text{EPV at } t \text{ of Future Premiums},$$

then we can see that, since the future benefits and premiums must both have non-negative EPVs, the only way for a negative policy value to arise is if the future benefits are worth less than the future premiums.

In practice, negative policy values would generally be set to zero when carrying out a valuation of the insurance company. Allowing them to be entered as assets (negative liabilities) ignores the policyholder's option to lapse the contract, in which case the excess premium will not be received.

Negative policy values arise when a contract is poorly designed, so that the value of benefits in early years exceeds the value of premiums, followed by a period when the order is reversed. If the policyholder lapses then the policyholder will have benefitted from the higher benefits in the early years without waiting around to pay for the benefit in the later years. In fact, the policyholder may be able to achieve the same benefit at a cheaper price by lapsing and buying a new policy. This is called the **lapse and re-entry option**.

7.8 Deferred acquisition expenses and modified net premium reserves

The principles of reserve calculation, such as whether to use a gross or net premium policy value, and how to determine the appropriate basis, are established by insurance regulators. While most jurisdictions use a gross premium policy value approach, as mentioned above, the net premium policy value is still used in the USA.

The use of the net premium approach can offer some advantages, in computation, and perhaps in smoothing results, but it can be quite a severe standard when there are large initial expenses (called **acquisition expenses**) incurred by the insurer. To reduce the impact, the reserve is not calculated directly as the net premium policy value, but can be modified, to approximate a gross premium policy value approach, whilst maintaining the advantages of the net premium approach. In this section we explain why this approach is used, by considering the impact of acquisition expenses on the policy value calculations.

Let ${}_tV^n$ denote the net premium policy value for a contract which is in force t years after issue, and let ${}_tV^g$ denote the gross premium policy value for the same contract, using the equivalence premium principle and using the original premium interest and mortality basis. Then we have

$$\begin{aligned}{}_tV^n &= \text{EPV future benefits} - \text{EPV future net premiums},\\ {}_tV^g &= \text{EPV future benefits} + \text{EPV future expenses}\\ &\quad - \text{EPV future gross premiums},\end{aligned}$$

and ${}_0V^n = {}_0V^g = 0$.

Since all the premium and policy valuations are on the same basis, the gross premium can be expressed as the net premium plus an amount to cover expenses. That is, if the gross premium for a level premium contract is P^g, and the net premium is P^n, the difference, P^e, say, is the **expense loading** (or **expense premium**) for the contract. This is the level annual amount paid by the policyholder to cover the policy expenses. So

$$\begin{aligned}{}_tV^g &= \text{EPV future benefits} + \text{EPV future expenses}\\ &\quad - (\text{EPV future net premiums} + \text{EPV future expense loadings})\\ &= {}_tV^n + \text{EPV future expenses} - \text{EPV future expense loadings}.\end{aligned}$$

That is

$${}_tV^g = {}_tV^n + {}_tV^e$$

where ${}_tV^e$ is the expense policy value, defined as

$${}_tV^e = \text{EPV future expenses} - \text{EPV future expense loadings}.$$

What is important about this relationship is that generally, for $t > 0$, ${}_tV^e$ is negative, meaning that the net premium policy value is greater than the gross premium policy value, assuming the same interest and mortality assumptions for both. This may appear counterintuitive – the reserve which takes expenses into consideration is smaller than the reserve which does not – but remember that the gross premium approach offsets the higher future outgo with higher future premiums. If expenses were incurred as a level annual amount, and assuming premiums are level and payable throughout the policy term, then the net premium and gross premium policy values would be the same, as the extra expenses valued in the gross premium case would be exactly offset by the extra premium. In practice though, expenses are not incurred at a flat rate. The acquisition expenses (commission, underwriting and administrative) are large relative to the renewal and claims expenses. This results in negative values for ${}_tV^e$, in general.

If expenses were incurred as a level sum at each premium date, then P^e would equal those incurred expenses (assuming premiums are paid throughout the policy term). If expenses are weighted to the start of the contract, as is normally the case, then P^e will be greater than the renewal expense as it must fund both the renewal and initial expenses. We illustrate these ideas with an example.

Example 7.18 An insurer issues a whole life insurance policy to a life aged 50. The sum insured of \$100 000 is payable at the end of the year of death. Level premiums are payable annually in advance throughout the term of the contract. All premiums and policy values are calculated using the Standard Select Survival Model, and an interest rate of 5% per year effective. Initial expenses are 50% of the gross premium plus \$250. Renewal expenses are 3% of the gross premium plus \$25 at each premium date after the first.

Calculate

(a) the expense loading, P^e, and
(b) ${}_{10}V^e$, ${}_{10}V^n$ and ${}_{10}V^g$.

Solution 7.18 (a) The expense loading, P^e, depends on the gross premium, P^g, which we calculate first as

$$P^g = \frac{100\,000\,A_{[50]} + 25\,\ddot{a}_{[50]} + 225}{0.97\,\ddot{a}_{[50]} - 0.47} = \$1219.09.$$

Now P^e can be calculated by finding the EPV of future expenses, and calculating the level premium to fund those expenses – that is

$$P^e\,\ddot{a}_{[50]} = 25\,\ddot{a}_{[50]} + 225 + 0.03P^g\,\ddot{a}_{[50]} + 0.47\,P^g.$$

Alternatively, we can calculate the net premium,

$$P^n = 100\,000A_{[50]}/\ddot{a}_{[50]} = \$1110.65,$$

and use $P^e = P^g - P^n$. Either method gives $P^e = \$108.43$.

Compare the expense premium with the incurred expenses. The annual renewal expenses, payable at each premium date after the first, are \$61.57. The rest of the expense loading, \$46.86 at each premium date, reimburses the acquisition expenses, which total \$859.54 at inception. Thus, at any premium date after the first, the value of the future expenses will be smaller than the value of the future expense loadings.

(b) The expense reserve at time $t = 10$ for an in force contract is

$$_{10}V^e = 25\,\ddot{a}_{60} + 0.03P^g\,\ddot{a}_{60} - P^e\,\ddot{a}_{60} = -46.86\,\ddot{a}_{60} = -698.42,$$

the net premium policy value is

$$_{10}V^n = 100\,000\,A_{60} - P^n\,\ddot{a}_{60} = 12\,474.94,$$

and the gross premium policy value is

$$_{10}V^g = 100\,000\,A_{60} + 25\,\ddot{a}_{60} - 0.97P^g\,\ddot{a}_{60} = 11\,776.52.$$

We note that, as expected, the expense reserve is negative, and that

$$_{10}V^g = {}_{10}V^n + {}_{10}V^e.$$

□

The negative expense reserve is referred to as the **deferred acquisition cost**, or DAC. The use of the net premium reserve can be viewed as being overly conservative, as it does not allow for the DAC reimbursement. The idea is that an insurer should not be required to hold the full net premium policy value as a reserve, when the true future liability value is smaller because of the DAC. One solution would be to use a gross premium reserve, but to do so would lose some of the numerical advantage offered by the net premium approach, including simple formulae for standard contracts, and the ability to use either a retrospective or prospective formula to perform the valuation. An alternative method, which maintains most of the numerical simplicity of the net premium approach, is to modify the net premium method to allow for the DAC, in a way that is at least approximately correct. **Modified net premium reserves** use a net premium policy value approach to reserve calculation, but instead of assuming a level annual premium, we assume a lower initial premium to allow implicitly for the DAC. The most common method of adjusting the net premium policy value is the **Full Preliminary Term** approach.

7.8.1 Full Preliminary Term reserve

We describe the full preliminary term (FPT) reserve for a simple contract, to illustrate the principles, but the method can be quite easily generalized to more complex products.

Consider a whole life insurance with level annual premiums payable throughout the term. Let $P_{[x]+s}$ denote the net premium for a contract issued to a life aged $[x]+s$, who was select at age x. Let $_1P_{[x]}$ denote the single premium required to fund the benefits payable during the first year of the contract (this is called the first year Cost of Insurance). Then the FPT reserve for a contract issued to a select life aged x is the net premium policy value assuming that the net premium in the first year is ${}_1P_{[x]}$ and in all subsequent years is $P_{[x]+1}$. This is equivalent to considering the policy as two policies, a one-year term insurance, and a separate contract issued to the same life one year later, if the life survives.

Example 7.19 (a) Calculate the premiums ${}_1P_{[50]}$ and $P_{[50]+1}$ for the policy in Example 7.18.

(b) Compare the net premium policy value, the gross premium policy value and the FPT reserve for the policy in Example 7.18 at durations 0, 1, 2 and 10.

Solution 7.19 (a) The modified net premium assumed at time $t = 0$ is

$$ {}_1P_{[50]} = 100\,000\,A^{\,1}_{[50]:\overline{1}|} = 100\,000\,v\,q_{[50]} = 98.41. $$

The modified net premium assumed to be paid at all subsequent premium dates is

$$ P_{[50]+1} = \frac{100\,000\,A_{[50]+1}}{\ddot{a}_{[50]+1}} = 1173.81. $$

(b) Let $P^g = 1219.09$ denote the gross premium for the policy. This is the gross premium calculated in Example 7.18, payable annually throughout the term. At time 0,

$$ \begin{aligned} {}_0V^n &= 100\,000\,A_{[50]} - P_{[50]}\,\ddot{a}_{[50]} = 0, \\ {}_0V^g &= 100\,000\,A_{[50]} + 225 + 25\,\ddot{a}_{[50]} + 0.47P^g - 0.97P^g\,\ddot{a}_{[50]} = 0, \\ {}_0V^{FPT} &= 100\,000\,A_{[50]} - {}_1P_{[50]} - P_{[50]+1}\,v\,p_{[50]}\,\ddot{a}_{[50]+1} \\ &= 100\,000\left(A^{\,1}_{[50]:\overline{1}|} + v\,p_{[50]}\,A_{[50]+1}\right) - 100\,000A^{\,1}_{[50]:\overline{1}|} \\ &\quad - \left(\frac{100\,000A_{[50]+1}}{\ddot{a}_{[50]+1}}\right) v\,p_{[50]}\,\ddot{a}_{[50]+1} \\ &= 0. \end{aligned} $$

At time 1,

$$ \begin{aligned} {}_1V^n &= 100\,000\,A_{[50]+1} - P_{[50]}\,\ddot{a}_{[50]+1} = 1063.96, \\ {}_1V^g &= 100\,000\,A_{[50]+1} + 25\,\ddot{a}_{[50]+1} - 0.97P^g\,\ddot{a}_{[50]+1} = 274.48, \\ {}_1V^{FPT} &= 100\,000\,A_{[50]+1} - P_{[50]+1}\,\ddot{a}_{[50]+1} = 0. \end{aligned} $$

At time 2,

$$
\begin{aligned}
{}_2V^n &= 100\,000\,A_{52} - P_{[50]}\,\ddot{a}_{52} = 2159.63,\\
{}_2V^g &= 100\,000\,A_{52} + 25\,\ddot{a}_{52} - 0.97P^g\,\ddot{a}_{52} = 1378.89,\\
{}_2V^{FPT} &= 100\,000\,A_{52} - P_{[50]+1}\,\ddot{a}_{52} = 1107.45.
\end{aligned}
$$

At time 10, we know from Example 7.18 that

$$ {}_{10}V^n = 12\,474.94 \qquad \text{and} \quad {}_{10}V^g = 11\,776.52, $$

and we have

$$ {}_{10}V^{FPT} = 100\,000\,A_{60} - P_{[50]+1}\,\ddot{a}_{60} = 11\,533.70. $$

□

Comments on full preliminary term reserves

1. The FPT reserve at time $t = 0$ is always equal to zero, because we are using the equivalence principle to determine the modified premium.
2. The FPT reserve at time $t = 1$ will also always be equal to zero. This is because we effectively start over at time 1, with a new equivalence principle premium, with an EPV that is exactly sufficient to meet the EPV of future benefits.
3. We emphasize that the FPT approach, and all other modified net premium approaches, use **net premium policy values**. That means that there is no explicit allowance for expenses in the modified premium calculation or in the policy value calculation. It also means that premiums must be calculated on the policy value basis using the equivalence principle. The only difference between the unmodified net premium policy value and the modified net premium policy value is that the unmodified one assumes level net premiums throughout, while the modified one assumes non-level net premiums.
4. The FPT reserve is intended, very roughly, to approximate the gross premium policy value, particularly in the first few years of the contract. As the policy matures, the net, gross and modified net premium policy values converge (though perhaps not until extremely advanced ages).
5. The FPT method implicitly assumes that the whole of the first year's gross premium is spent on the cost of insurance and the acquisition expenses. In the above example, that assumption overstates the acquisition expenses, with the result that the FPT reserve is actually a little lower than the gross premium policy value after the first year.
6. Modifications of the method may be used. For example, a partial preliminary term method would assume the first year's modified premium

is greater than the first year's cost of insurance, resulting in a slightly lower modified premium from the second year.

7. When the premium payment term is limited, the modified net premium would assume the same premium payment term as the original policy. So the FPT modified policy value would assume, as above, that the first year's premium covers the cost of insurance, and subsequent net premiums would be the net premium payable for a policy issued one year later, with a premium term ending at the same age as under the original policy.

7.9 Other reserves

At the start of this chapter, we discussed policy values, noting that these are often the basis for determining the insurer's reserves, by which we mean the funds held by the insurer to meet the costs arising from the current policies. In statutory reporting, these reserves are called **technical provisions**.

It is worth noting that the use of policy values to determine reserves only includes policies that are still in force. In practice there may be other reserves included in the technical provisions, and we discuss two of them briefly here.

The **Reported But Not Settled** (RBNS) reserve represents the value of outstanding claims for policies where the policyholder has died, but the claim has not yet been paid. For example, consider a whole life insurance with annual premiums, under which the death benefit is payable at the end of the year of death. Once the policyholder dies, the policy expires and there is no policy value. However, the sum insured will not be paid until the end of the policy year. So for each death reported an amount should be transferred into the RBNS reserve, ready to be paid out at the appropriate time.

In practice, while annual premiums are not uncommon, having a death benefit payable at the end of the year of death is not realistic. More commonly, the death benefit would be paid as soon as possible after the death is reported. There will be some checks made on the coverage; for example, were there any policy exclusions; were there any outstanding premiums or policy loans; is there any evidence that the policyholder gave misleading information – for example, with respect to smoking status or hazardous sports; is there any uncertainty about who should receive the payment. The checks will be more rigorous if the policy has been in force for less than about 3 years. So, as soon as a death is reported, a claims manager will assess how detailed the review process is likely to be, and the RBNS reserve would be suitably adjusted. Normally, the payment of the death benefit would occur, on average, around 2-4 months after the death is reported, for straightforward cases, and the RBNS would then be reduced by the claim amount at that time.

The **Incurred But Not Reported** (IBNR) reserve, as the name implies, allows for potential claims arising in respect of policyholders who have died, but whose deaths have not yet been recorded by the insurer. IBNR reserves will be more significant in countries where infrastructure is less reliable or accessible, for example, in microinsurance in developing nations.

Techniques for RBNS and IBNR reserves have been developed more extensively for Property and Casualty insurance, where they represent a much more substantial part of the full technical provision.

7.10 Notes and further reading

Thiele's differential equation is named after the Danish actuary Thorvald N. Thiele (1838–1910). For information about Thiele, see Hoem (1983) or Lauritzen *et al.* (2002).

For more information on more efficient methods for solving differential equations, see for example Burden *et al.* (2015) or Shampine (2018).

Texts such as Neill (1977) and Bowers *et al.* (1997) contain standard actuarial notation for policy values. We do not find the notation useful in practice, so it is not covered in this text.

7.11 Exercises

Shorter exercises

Exercise 7.1 For a whole life insurance issued to (40), you are given:

(i) The death benefit, which is payable at the end of the year of death, is \$50 000 in the first 20 years, and \$100 000 thereafter.
(ii) Level annual premiums are payable for 20 years or until earlier death.
(iii) The mortality basis for policy values is the Standard Ultimate Survival Model.
(iv) The interest basis for policy values is 5% per year.

Calculate the net premium policy value, ${}_{10}V$.

Exercise 7.2 A fully discrete whole life insurance with sum insured \$10 000 is issued to a select life aged x. The net premium is \$134. You are given that $q_{[x]} = 0.00106$ and $i = 0.045$. Calculate ${}_1V^n$.

Exercise 7.3 An insurer issues a whole life insurance with sum insured \$500 000, with premiums payable quarterly, to (70).

(a) Calculate the net premium policy value at time 10 assuming that the death benefit is payable at the end of the quarter year of death.
(b) Calculate the net premium policy value at time 10 assuming that the death benefit is payable 3 months after the death of the policyholder.

Basis:

- Mortality follows the Standard Ultimate Life Table.
- $i = 5\%$.
- Uniform distribution of deaths between integer ages.

Exercise 7.4 An insurer issues a whole life insurance to (70) with sum insured \$500 000 payable at the end of the year of death. Premiums are payable annually for a maximum of 15 years. Calculate the Full Preliminary Term reserve at the end of the fifth year, assuming mortality follows the Standard Ultimate Life Table, with interest at 5% per year.

Exercise 7.5 A special deferred annuity issued to (30) provides the following benefits:

A whole life annuity of \$10 000 per year, deferred for 30 years, payable monthly in advance.

The return of all premiums paid, without interest, at the moment of death, in the event of death within the first 30 years.

Premiums are payable continuously for a maximum of 10 years.

(a) Write down expressions for
  (i) the present value random variable for the benefits, and
  (ii) L_0, the future loss random variable for the contract.

(b) Write down an expression in terms of annuity and insurance functions for the net annual premium rate, P, for this contract.

(c) Write down an expression for L_5, the net present value of future loss random variable for a policy in force at duration 5.

(d) Write down an expression for ${}_5V$, the net premium policy value at time 5 for the contract, in terms of annuity and insurance functions, and the net annual premium rate, P.

Exercise 7.6 For a fully continuous whole life insurance with sum insured \$100 000 issued to (70) you are given

- Mortality follows the Standard Ultimate Survival Model.
- Renewal expenses of 5% of the premium are payable continuously.
- $i = 5\%$.
- The gross premium rate is \$4 000 per year.

(a) Calculate ${}_{10}V$, using Woolhouse's three-term formula for calculating the annuity and insurance functions.

(b) Calculate $\frac{d}{dt}\,{}_tV$ at $t = 10$.

Exercise 7.7 Recalculate the analysis of surplus in Example 7.8 in the order: mortality, interest, expenses. Check that the total profit is as before and note the small differences from each source.

Longer exercises

Exercise 7.8 You are given the following extract from a select life table with four-year select period. A select individual aged 41 purchased a three-year term insurance with a sum insured of $200 000, with premiums payable annually throughout the term.

$[x]$	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$	l_{x+4}	
[40]	100 000	99 899	99 724	99 520	99 288	44
[41]	99 802	99 689	99 502	99 283	99 033	45
[42]	99 597	99 471	99 268	99 030	98 752	46

Assume an effective rate of interest of 6% per year, and no expenses.

(a) Show that the premium for the term insurance is $P = \$323.59$.
(b) Calculate the mean and standard deviation of the present value of future loss random variable, L_1, for the term insurance.
(c) Calculate the sum insured for a three-year endowment insurance for a select life age 41, with the same premium as for the term insurance, $P = \$323.59$.
(d) Calculate the mean and standard deviation of the present value of future loss random variable, L_1, for the endowment insurance.
(e) Comment on the differences between the values for the term insurance and the endowment insurance.

Exercise 7.9 A whole life insurance with sum insured $100 000 is issued to a select life aged 35. Premiums are payable annually in advance and the death benefit is payable at the end of the year of death.

The premium is calculated using the Standard Select Survival Model, and assuming

Interest: 6% per year effective
Initial Expenses: 40% of the gross premium plus $125
Renewal expenses: 5% of gross premiums plus $40, due at the start of each policy year from the second onwards

(a) Calculate the gross premium.
(b) Calculate the net premium policy value at time $t = 1$ using the premium basis.
(c) Calculate the gross premium policy value at time $t = 1$ using the premium basis.
(d) Explain why the gross premium policy value is less than the net premium policy value.

(e) Calculate the asset share per policy at the end of the first year of the contract if the experienced mortality rate is given by $q_{[35]} = 0.0012$, the interest rate earned on assets was 10%, and expenses followed the premium basis, except that there was an additional initial expense of \$25 per policy.

(f) Calculate the surplus at the end of the first year per policy issued given that the experience follows (e) and assuming the policy value used is as calculated in (c) above.

(g) Analyse the surplus in (f) into components for interest, mortality and expenses.

Exercise 7.10 Consider a whole life insurance policy issued to a select life aged x. Premiums of \$$P$ per year are payable continuously throughout the policy term, and the sum insured of \$$S$ is paid immediately on death.

(a) Show that

$$\mathrm{V}[L_t] = \left(S + \frac{P}{\delta}\right)^2 \left({}^2\bar{A}_{[x]+t} - \left(\bar{A}_{[x]+t}\right)^2\right).$$

(b) Assume the life is aged 55 at issue, and that premiums are \$1200 per year. Show that the sum insured on the basis below is \$77 566.44.

Mortality:	Standard Select Survival Model
Interest:	5% per year effective
Expenses:	None

(c) Calculate the standard deviation of L_0^n, L_5^n and L_{10}^n. Comment briefly on the results.

Exercise 7.11 An insurer issues a fully discrete whole life insurance with sum insured \$100 000 to (60), under which premiums cease when the policyholder reaches age 80.

You are given the following assumptions for pricing and reserving.

- Mortality follows the Standard Ultimate Life Table.
- Commissions are 40% of the first premium, and 10% thereafter.
- Maintenance expenses are \$500 at issue and \$50 at the beginning of each subsequent year, while the policy is in force.
- $i = 0.05$.

(a) (i) Calculate the gross premium using the equivalence principle.
    (ii) Calculate the gross premium policy value at the end of the second policy year.

(b) The insurer holds reserves on a modified net premium policy value basis defined by these rules:

- The net premium payment term is the same as the gross premium payment term.
- The first year modified net premium is 50% of the renewal modified net premium.
- Renewal modified net premiums are level.

  (i) Calculate the first year modified net premium using this method.
  (ii) Calculate the modified net premium policy value at the end of the second policy year.

(c) (i) Calculate the first year modified net premium using the Full Preliminary Term (FPT) method.
  (ii) Without further calculation, state with reasons whether the FPT reserve at the end of the second policy year will be higher, lower, or the same as the modified net premium policy value in part (b).

(*Copyright 2018. The Society of Actuaries, Schaumburg, Illinois. Reproduced with permission.*)

Exercise 7.12 For an n-year endowment policy, level monthly premiums are payable throughout the term of the contract, and the sum insured is payable at the end of the month of death.

Derive the following formula for the net premium policy value at time t years, where t is a premium date:

$$ {}_tV = S\left(1 - \frac{\ddot{a}^{(12)}_{[x]+t:\overline{n-t}|}}{\ddot{a}^{(12)}_{[x]:\overline{n}|}}\right). $$

Exercise 7.13 A 10-year endowment insurance is issued to a life aged 40. The sum insured is payable at the end of the year of death or on survival to the maturity date. The sum insured is \$20 000 on death, \$10 000 on survival to age 50. Premiums are payable annually in advance.

(a) The premium basis is:

Expenses:	5% of each gross premium including the first
Interest:	5% per year effective
Survival model:	Standard Select Survival Model

Show that the gross premium is \$807.71.

(b) Calculate the policy value on the premium basis just before the fifth premium is due.
(c) Just before the fifth premium is due the policyholder requests that all future premiums, including the fifth, be reduced to one half their original amount. The insurer calculates the revised sum insured – the maturity benefit still

being half of the death benefit – using the policy value in part (b) with no extra charge for making the change.
Calculate the revised death benefit.

Exercise 7.14 An insurer issues a whole life insurance policy to a life aged 40. The death benefit in the first three years of the contract is \$1000. In subsequent years the death benefit is \$50 000. The death benefit is payable at the end of the year of death and level premiums are payable annually throughout the term of the contract.

Basis for premiums and policy values:

Survival model:	Standard Select Life Table
Interest:	5% per year effective
Expenses:	None

(a) Calculate the premium for the contract.
(b) Write down the policy value formula for any integer duration $t \geq 3$.
(c) Calculate the policy value at duration $t = 3$.
(d) Use the recurrence relation to determine the policy value after two years.
(e) The insurer issued 1000 of these contracts to identical, independent lives aged 40. After two years there are 985 still in force. In the following year there were four further deaths in the cohort, and the rate of interest earned on assets was 5.5%. Calculate the profit or loss from mortality and interest in the year.

Exercise 7.15 An insurer issued a single premium, 20-year level term insurance to a select life then aged exactly 60. The benefit is \$1.

Let ${}_tV$ denote the policy value after t years.

(a) Suppose the death benefit is payable at the year end. Write down and explain a recurrence relation between ${}_tV$ and ${}_{t+1}V$ for $t = 0, 1, \ldots, 19$.
(b) Suppose the benefit is payable at the end of every h years, where $h < 1$. Write down a recurrence relation between ${}_tV$ and ${}_{t+h}V$ for $t = 0, h, 2h, \ldots, 20 - h$.
(c) By considering the limit as $h \to 0$, show that Thiele's differential equation for the policy value for a benefit payable continuously is

$$\frac{d\,{}_tV}{dt} = (\mu_{[60]+t} + \delta)\,{}_tV - \mu_{[60]+t} \quad \text{for } 0 < t < 20$$

where δ is the force of interest, and state any boundary conditions.
(d) Show that

$${}_tV = \bar{A}^{\;\;1}_{[60]+t:\,\overline{20-t|}}$$

is the solution to the differential equation in part (c).

Exercise 7.16 An insurer issues identical deferred annuity policies to 100 independent lives aged 60 at issue. The deferred period is 10 years, after which the annuity of \$10 000 per year is paid annually in advance. Level premiums are payable annually throughout the deferred period. The death benefit during deferment is \$50 000, payable at the end of the year of death.

The basis for premiums and policy values is:

Survival model:	Standard Ultimate Life Table
Interest:	5% per year effective
Expenses:	None

(a) Calculate the premium for each contract.
(b) Write down the recursive relationship for the policy values, during and after the deferred period.
(c) Calculate the death strain at risk in the second year of the contract, for each contract still in force at the start of the year.
(d) Calculate the death strain at risk in the 13th year of the contract, per contract in force at the start of the year.
(e) One year after the issue date, 98 policies remain in force. In the second year, two lives die. Calculate the total mortality profit in the second year, assuming all other experience follows the assumptions in the premium basis.
(f) Twelve years after the issue date 80 lives survive; in the 13th year there are four deaths. Calculate the total mortality profit in the 13th year.

Exercise 7.17 An insurer issues a deferred annuity with a single premium to (x). The annuity is payable continuously at a level rate of \$50 000 per year after the 20-year deferred period, if the policyholder survives. On death during the deferred period, the single premium is returned at the time of death with interest at rate i per year effective.

(a) Write down an equation for the prospective net premium policy value (i) during the deferred period and (ii) after the deferred period, using standard actuarial functions. Assume an interest rate of i per year effective, the same as the accumulation rate for the return of premium benefit.
(b) Repeat part (a) for the retrospective net premium policy value.
(c) Show that the retrospective and prospective policy values are equal.

Exercise 7.18 An insurer issues a 20-year term insurance policy to (40). The sum insured is \$600 000 for the first five years, and \$300 000 for the remainder of the term. The gross premium is level for five years, and then reduces to 50% of the original value for the remainder of the term. Premiums are payable annually in advance and death benefits are payable at the end of the year of death.

The insurer calculates premiums and policy values using the following basis for mortality and expenses:

Initial Expenses:	50% of the first premium plus \$200
Renewal Expenses:	10% of all premiums after the first
Mortality:	Standard Select Survival Model

Gross premiums are calculated using an interest rate of 5% per year.

All policy values are calculated using an interest rate of 4.5% per year. You are given that, at 4.5% per year interest:

$$\ddot{a}_{[40]:\overline{5}|} = 4.5829, \quad \ddot{a}_{[40]:\overline{20}|} = 13.4968,$$

$$A^{\;1}_{[40]:\overline{5}|} = 0.002573, \quad A^{\;1}_{[40]:\overline{20}|} = 0.015400.$$

(a) Show that the gross premium in the first year is \$710 to the nearest 10.
(b) Calculate the gross premium policy values at times $t = 0, 1, 2$ and 19.
(c) Calculate the net premium policy values at times $t = 0, 1, 2$ and 19.
(d) (i) Calculate $\ddot{a}_{[40]+1:\overline{4}|}$ and $\ddot{a}_{[40]+1:\overline{19}|}$ at $i = 4.5\%$.
(ii) Calculate the Full Preliminary Term (FPT) modified premium in the second year. Assume that the modified net premium for the final 15 years is 50% of the value for the second to fifth years.
(iii) Calculate the FPT policy values at times $t = 0, 1, 2$ and 19.
(e) Explain briefly the rationale for the FPT approach. Does this example support the use of the FPT policy value? Justify your answer.

Exercise 7.19 Consider a whole life insurance policy issued to (40), with premiums of \$70 per month, and sum insured \$100 000 payable at the end of the month of death. At the end of the 15th policy year the policyholder is considering surrendering her policy.

The cash value for the policy is the policy value on the following basis:

Interest:	5% per year effective
Renewal expenses:	5% of premiums, and
	\$5 per month at the start of each month
Surrender charge:	\$500 (deducted from the policy value)
Mortality:	Standard Ultimate Life Table

(a) Calculate the cash value at time 15 years.
(b) Calculate the probability that L_{15} is less than the cash value, assuming the policy is not altered or surrendered. Assume UDD between integer ages in the life table.

(c) Calculate the paid-up sum insured based on the cash value. Assume that the \$5 per month maintenance expense continues under the paid-up policy.
(d) The policyholder proposes converting the policy to a term insurance, maturing at age 65. Assume the revised benefit would be payable at the end of the month of death and that the policyholder continues paying premiums until the end of the contract. Calculate the revised sum insured based on the cash value.
(e) Explain briefly why the insurer might not offer the term insurance conversion.
(f) The insurer values the liabilities for the whole life portfolio using the same assumptions, except with an interest rate of 4%. Explain briefly why the cash value interest rate is significantly higher than the valuation interest rate.

Excel-based exercises

Exercise 7.20 An insurer issues a 20-year term insurance policy to (35). The sum insured of \$100 000 is payable at the end of the year of death, and premiums are payable annually throughout the term of the contract. The basis for calculating premiums and policy values is:

Survival model:	Standard Select Survival Model	
Interest:	5% per year effective	
Expenses:	Initial:	\$200 plus 15% of the first premium
	Renewal:	4% of each premium after the first

(a) Show that the premium is \$91.37 per year.
(b) Show that the policy value immediately after the first premium payment is

$$_{0+}V = -\$122.33.$$

(c) Explain briefly why the policy value in part (b) is negative.
(d) Calculate the policy values at each year end for the contract, just before and just after the premium and related expenses incurred at that time, and plot them on a graph. At what duration does the policy value first become strictly positive?
(e) Suppose now that the insurer issues a large number, N say, of identical contracts to independent lives, all aged 35 and all with sum insured \$100 000. Show that if the experience exactly matches the premium/policy value basis, then the accumulated value at (integer) time k of all premiums less claims and expenses paid out up to time k, expressed per surviving policyholder, is exactly equal to the policy value at time k.

Exercise 7.21 Consider a 20-year endowment policy issued to (40), with premiums, P per year payable continuously, and sum insured of \$200 000 payable immediately on death. Premiums and policy values are calculated assuming:

Survival model:	Standard Select Survival Model
Interest:	5% per year effective
Expenses:	None

(a) Show that the premium, P, is \$6020.40 per year.
(b) Show that the policy value at duration $t = 4$, ${}_4V$, is \$26 131.42.
(c) Assume that the insurer decides to change the valuation basis at $t = 4$ to Makeham's mortality with $A = 0.0004$, with $B = 2.7 \times 10^{-6}$ and $c = 1.124$ as before. Calculate the revised policy value at $t = 4$ (using the premium calculated in part (a)).
(d) Explain why the policy value does not change very much.
(e) Now assume again that $A = 0.00022$ but that the interest assumption changes from 5% per year to 4% per year. Calculate the revised value of ${}_4V$.
(f) Explain why the policy value has changed considerably.
(g) A colleague has proposed that policyholders wishing to alter their contracts to paid-up status should be offered a sum insured reduced in proportion to the number of premiums paid. That is, the paid up sum insured after k years of premiums have been paid, out of the original total of 20 years, should be $S \times k/20$, where S is the original sum insured. This is called the **proportionate paid-up sum insured.**

Calculate the EPV of the proportionate paid-up sum insured at each year end, and compare these graphically with the policy values at each year end, assuming the original basis above is used for each. Explain briefly whether you would recommend the proportionate paid-up sum insured for this contract.

Exercise 7.22 A life aged 50 buys a participating whole life insurance policy with sum insured \$10 000. The sum insured is payable at the end of the year of death. The premium is payable annually in advance. Profits are distributed through cash dividends paid at each year end to policies in force at that time.

The premium basis is:

Initial expenses:	22% of the annual gross premium plus \$100
Renewal expenses:	5% of the gross premium plus \$10
Interest:	4.5%
Survival model:	Standard Select Survival Model

(a) Show that the annual premium, calculated with no allowance for future bonuses, is \$144.63 per year.
(b) Calculate the policy value at each year end for this contract using the premium basis.
(c) Assume the insurer earns interest of 5.5% each year. Calculate the dividend payable each year assuming
   (i) the policy is still in force at the end of the year,
   (ii) experience other than interest exactly follows the premium basis, and
   (iii) that 90% of the profit is distributed as cash dividends to policyholders.
(d) Calculate the expected present value of the profit to the insurer per policy issued, using the same assumptions as in (c).
(e) What would be a reasonable surrender benefit for lives surrendering their contracts at the end of the first year?

Exercise 7.23 A 20-year endowment insurance issued to a life aged 40 has level premiums payable continuously throughout the term. The sum insured on survival is \$60 000. The sum insured payable immediately on death within the term is \$20 000 if death occurs within the first 10 years and ${}_tV$ if death occurs after t years, $10 \le t < 20$, where ${}_tV$ is the policy value calculated on the premium basis.
Premium basis:

Survival model:	Standard Select Survival Model
Interest:	$\delta_t = 0.06 - 0.001t$ per year
Expenses:	None

(a) Write down Thiele's differential equation for ${}_tV$, separately for $0 < t < 10$ and $10 < t < 20$, and give any relevant boundary conditions.
(b) Determine the equivalence principle premium rate P by solving Thiele's differential equation using Euler's method, with a time step $h = 0.05$.
(c) Plot the graph of ${}_tV$ for $0 < t < 20$.

Exercise 7.24 Consider Example 7.1. Calculate the policy values at intervals of $h = 0.1$ years from $t = 0$ to $t = 2$.

Answers to selected exercises

7.1 \$11 149.02
7.2 129.57
7.3 (a) \$148 799 (b) \$147 895
7.4 \$80598.09
7.6 (a) 30 205.74 (b) 3 088.25
7.7 −\$26 504.04, \$51 011.26, −\$5 588.00

7.8 (b) \$116.68, \$11 663.78 (c) \$1 090.26 (d) \$342.15, \$15.73

7.9 (a) \$469.81 (b) \$381.39 (c) \$132.91 (e) \$25.10 (f) −\$107.67
(g) \$6.28, −\$86.45, −\$27.50

7.10 (c) \$14 540.32, \$16 240.72, \$17 619.98

7.11 (a)(i) \$2787.23 (ii) \$3153.24 (b)(i) \$1221.55 (ii) \$3191.66
(c)(i) \$323.65

7.13 (b) \$3 429.68 (c) \$14 565.95

7.14 (a) 323.83 (c) 1071.41 (d) 696.52
(e) 5268.06 (profit), 5025.21 (interest), 242.85 (mortality)

7.16 (a) \$9001 (c) 30 887 (d) −110 081 (e) −50297 (f) 325 130

7.18 (b) \$96.93, −\$6.95, \$338.85, \$555.40 (c) \$0, \$353.08, \$670.50
\$576.81 (d)(i) 3.7458, 13.0650 (ii) \$638.47 (iii) \$0, \$0, \$345.25,
\$555.81

7.19 (a) \$12 048 (b) 0.642291 (c) \$46 189 (d) \$705 374

7.20 (d) Selected values:
${}_4V = -\$32.53$, ${}_{4^+}V = \$55.18$; ${}_{13}V = \$238.95$, ${}_{13^+}V = \$326.67$
The policy value first becomes positive at duration 3+.

7.21 (c) \$26 348.41 (e) \$36 575.95

7.22 (b) Selected values: ${}_5V = \$509.93$, ${}_{10}V = \$1\,241.77$
(c) Selected values: Dividend at $t = 5$: \$4.55, Dividend at $t = 10$: \$10.96 (d) \$29.26 (e) \$0

7.23 (b) \$1 810.73
(c) Selected values: ${}_5V = \$10\,400.92$, ${}_{10}V = \$23\,821.21$, ${}_{15}V = \$40\,387.35$

7.24 Selected values: ${}_{0.5}V = \$15\,256$, ${}_1V = \$15\,369$, ${}_{1.5}V = \$30\,962$, ${}_2V = \$31\,415$

8
Multiple state models

8.1 Summary

In this chapter we reformulate the survival model introduced in Chapter 2 as an example of a multiple state model. We then introduce several other multiple state models which are useful for different types of life insurance policies, for example when benefits depend on the policyholder's health as well as survival, or when extra benefits are payable where death is accidental.

A general definition of a multiple state model, together with assumptions and notation, is given in Section 8.3. In Section 8.4 we discuss the derivation of formulae for probabilities and in Section 8.5 we consider the numerical evaluation of these probabilities. We then extend ideas from earlier chapters, first by introducing functions to value annuity and insurance benefits in Section 8.6, then we consider premium calculation in Section 8.7 and policy values in Section 8.8.

The general results in this chapter are derived for continuous-time transitions, for example, from healthy to disabled. In Section 8.10 we consider the discrete time model, where the states are observed at discrete intervals.

In Section 8.9 we show how multiple state models can be applied to specific long-term health contracts, including disability income insurance, critical illness insurance and funding of continuing care retirement communities.

8.2 Examples of multiple state models

Multiple state models are one of the most exciting developments in actuarial science in recent years. They are a natural tool for many important areas of practical interest to actuaries. They are intuitive, and easy to work with using some straightforward but powerful numerical techniques. They also simplify and provide a sound foundation for pricing and valuing complex insurance contracts. In this section we illustrate some of the applications of multiple state models, using four examples which are common in current actuarial practice,

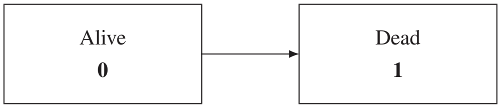


Figure 8.1 The alive–dead model.

and which offer some insight into the different levels of complexity that can be incorporated. In Section 8.9 we consider some further, more complex examples used in long-term health insurance and related contracts.

8.2.1 The alive–dead model

So far, we have modelled the uncertainty over the duration of an individual's future lifetime by regarding the future lifetime as a random variable, T_x, for an individual currently aged x. The cumulative distribution function of T_x is $\Pr[T_x \le t] = F_x(t) = {}_tq_x$, and the survival function is $\Pr[T_x > t] = S_x(t) = {}_tp_x$. This is a **probabilistic model** in the sense that for an individual aged x we have a single random variable, T_x, whose distribution is assumed to be known, and for which all associated probabilities can be determined.

We can represent this model diagrammatically as shown in Figure 8.1. Our individual is, at any time, in one of two **states**, 'Alive' and 'Dead'. For convenience we label these states '0' and '1', respectively. Transition from State 0 to State 1 is allowed, as indicated by the direction of the arrow, but transitions in the opposite direction cannot occur. This is an example of a **multiple state model** with two states.

We can use this multiple state model to reformulate our survival model as follows. Suppose we have a life aged $x \ge 0$ at time $t = 0$. For each $t \ge 0$, we define a random variable $Y(t)$ which takes one of the two values 0 and 1. The event '$Y(t) = 0$' means that our individual is alive at age $x + t$, and '$Y(t) = 1$' means that our individual died before age $x + t$. The set of random variables $\{Y(t)\}_{t\ge 0}$ is an example of a **continuous time stochastic process**. A continuous time stochastic process is a collection of random variables indexed by a continuous time variable. For all t, $Y(t)$ is either 0 or 1, and T_x is connected to this model as the time at which $Y(t)$ jumps from 0 to 1, that is

$$T_x = \max\{t : Y(t) = 0\}.$$

The alive–dead model represented by Figure 8.1 captures all the life contingent information that is necessary for calculating insurance premiums and policy values for policies where payments (premiums, benefits and expenses) depend only on whether the individual is alive or dead at any given age. We have seen examples of these contracts, such as term insurance or deferred annuities, in previous chapters. But there are more complicated forms of insurance which require more sophisticated models. We introduce some examples in this section, which we will use to illustrate the development of functions for calculating premiums and policy values. Further examples are presented in Section 8.9. Each of these models consists of a finite set of states with arrows indicating possible movements between some, but not necessarily all, pairs of states. Each state represents the status of an individual policy. Loosely speaking, each model is appropriate for a given insurance policy, in the sense that the condition for a payment relating to the policy – for example a premium, an annuity payment or a sum insured – is either that the individual is in a specified state at that time or that the individual makes an instantaneous transfer between a specified pair of states at that time.

8.2.2 *Term insurance with increased benefit on accidental death*

Suppose we are interested in a term insurance policy under which the death benefit is \$100 000 if death is due to an accident during the policy term and \$50 000 if it is due to any other cause. The alive–dead model in Figure 8.1 is not sufficient for this policy since, when the individual dies – that is, transfers from State 0 to State 1 – we do not know whether death was due to an accident, and so we do not know the amount of the death benefit to be paid.

An appropriate model for this policy is shown in Figure 8.2. This model has three states, and we can define a continuous time stochastic process, $\{Y(t)\}_{t\geq 0}$, where each random variable $Y(t)$ takes one of the values 0, 1 and 2. Hence, for example, the event '$Y(t) = 1$' indicates that the individual, who was aged x at time $t = 0$, has died from an accident before age $x + t$.

The model in Figure 8.2 is an extension of the model in Figure 8.1. In both cases an individual starts by being alive, that is, starts in State 0, and, at some future time, dies. The difference is that we now distinguish between deaths due to accident and deaths due to other causes. Notice that it is the benefits provided

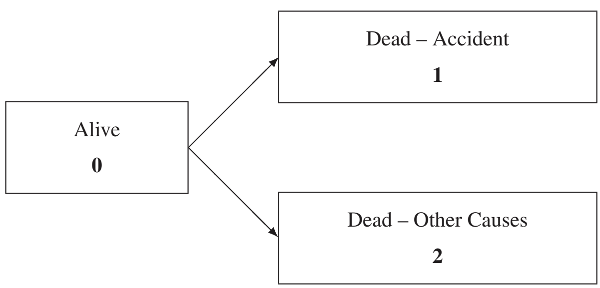


Figure 8.2 The accidental death model.

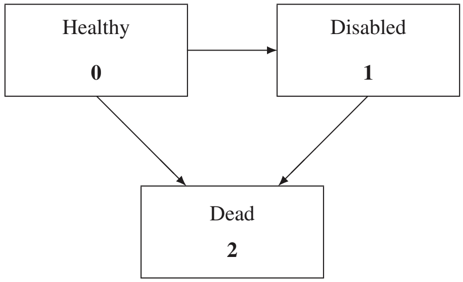


Figure 8.3 The permanent disability model.

by the insurance policy that determine the design of the model. Because the benefit is contingent on the cause of death, the model must specify the cause of death appropriately.

8.2.3 The permanent disability model

Figure 8.3 shows a model appropriate for a policy which provides some or all of the following payments:

(i) an annuity while permanently disabled,
(ii) a lump sum on becoming permanently disabled,
(iii) a lump sum on death, and
(iv) premiums payable while healthy.

An important feature of this model is that disability is permanent – there is no arrow from State 1 back to State 0.

8.2.4 The sickness–death model

In Chapter 1 we described disability income insurance, which pays an income benefit during periods of sickness. Figure 8.4 shows the sickness–death model which can be used for a policy which provides an annuity during periods of sickness, with premiums payable while the person is healthy. It could also be used for valuing lump sum payments contingent on becoming sick or on dying. The model represented by Figure 8.4 differs from that in Figure 8.3 in only one respect: it is possible to transfer from State 1 to State 0, that is, to recover from an illness.

This model illustrates an important general feature of multiple state models which was not present for the models in Figures 8.1, 8.2 and 8.3. This feature is the possibility of entering one or more states many times. In terms of our interpretation of the model, this means that several periods of sickness could occur before death, with healthy (premium paying) periods in between.

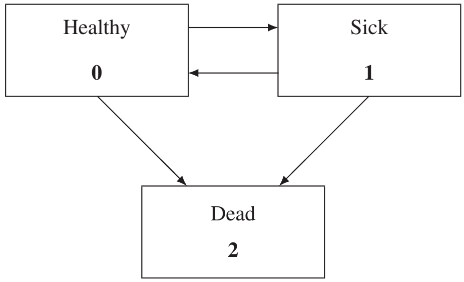


Figure 8.4 The sickness–death model for disability income insurance.

Tables of probabilities and actuarial functions for a parameterization of this model that we call the Standard Sickness–Death Model are given in Appendix D.

8.3 Assumptions and notation

The multiple state models introduced above are all extremely useful in an insurance context. We study several of these models in detail later in this chapter. Before doing so, we need to introduce some assumptions and some notation.

In this section we consider a general multiple state model. We have a finite set of $m + 1$ states labelled $0, 1, \ldots, m$, with instantaneous transitions being possible between selected pairs of states. These states represent different conditions for an individual. For each $t \geq 0$, the random variable $Y(t)$ takes one of the values $0, 1, \ldots, m$, and we interpret the event $Y(t) = i$ to mean that the individual is in State i at age $x + t$. The set of random variables $\{Y(t)\}_{t\geq 0}$ is then a continuous time stochastic process.

The multiple state model will be an appropriate model for an insurance policy if the payment of benefits or premiums is dependent on the life being in a given state or moving between a given pair of states, as illustrated in the examples in the previous section. Note that in these examples there is a natural starting state for the policy, which we always label State 0. This is the case for all insurance examples based on multiple state models. For example, a policy providing an annuity during periods of sickness in return for premiums payable while healthy, as described in Section 8.2.4 and illustrated in Figure 8.4, would be issued to a person who was healthy at the start of the policy.

Assumption 8.1 *We assume that for any states i and j and any times t and $t + s$, where $s \geq 0$, the conditional probability* $\Pr[Y(t + s) = j \mid Y(t) = i]$ *is well defined, in the sense that its value does not depend on any information about the process before time t.*

Intuitively, this means that the probabilities of future events for the process are completely determined by knowing the current state of the process. In particular, these probabilities do not depend on how the process arrived at the current state or how long it has been in the current state. This property, that probabilities of future events depend on the present but not on the past, is known as the **Markov property**. Using the language of probability theory, we are assuming that $\{Y(t)\}_{t \geq 0}$ is a Markov process.

Assumption 8.1 was not made explicitly for the models represented by Figures 8.1 and 8.2 since it was unnecessary, given our interpretation of these models. In each of these two cases, if we know that the process is in State 0 at time x (so that the individual is alive at age x) then we know the past of the process (the individual was alive at all ages before x). Assumption 8.1 is more interesting in relation to the models in Figures 8.3 and 8.4. Suppose, for example, in the sickness–death model (Figure 8.4) we know that $Y(t) = 1$, which means that the individual is sick at time t. Then Assumption 8.1 says that the probability of any future move after time t, either recovery or death, does not depend on any further information, such as how long the life has been sick up to time t, or how many different periods of sickness the life has experienced up to time t. In practice, we might believe that the probability of recovery in, say, the next week would depend on how long the current sickness has already lasted. If the current sickness has already lasted for, say, six months then it is likely to be a very serious illness and recovery within the next week is possible but not likely; if the current sickness has lasted only one day so far, then it may well be a trivial illness and recovery within a week could be very likely. It is important to understand the limitations of any model and also to bear in mind that no model is a perfect representation of reality. Assumption 8.1 can be relaxed to allow for some dependency on the process history, but these more general (non-Markov) models are beyond the scope of this book.

Assumption 8.2 *We assume that for any positive interval of time h,*

$$\Pr[2 \textit{ or more transitions within a time period of length } h] = o(h).$$

Recall that any function of h, say $g(h)$, is said to be $o(h)$ if

$$\lim_{h \to 0} \frac{g(h)}{h} = 0.$$

Intuitively, a function is $o(h)$ if, as h converges to 0, the function converges to zero faster than h. So, for example, h^k is $o(h)$ if and only if $k > 1$.

Assumption 8.2 tells us that for a very small time interval t to $t + h$, the probability of two or more transitions is vanishingly small. This assumption is unnecessary for the models in Figures 8.1 and 8.2 since in both cases only one transition can ever take place. However, it is an assumption we need to make for technical reasons for the models in Figures 8.3 and 8.4. In these cases, given our interpretation of the models, it is not unreasonable.

In Chapter 2 we introduced the standard actuarial notation for what we are now calling the alive–dead model, as shown in Figure 8.1; specifically, ${}_tp_x$, ${}_tq_x$, and μ_x. For multiple state models with more than two states, or more than one possible transition, we need a more flexible notation. We introduce the following notation for a general multiple state model.

Notation For states i and j in a multiple state model and for $x, t \geq 0$, we define

$$ {}_tp_x^{ij} = \Pr[Y(x+t) = j \mid Y(x) = i], \tag{8.1} $$

$$ {}_tp_x^{\overline{ii}} = \Pr\left[\, Y(x+s) = i \text{ for all } s \in [0,t] \mid Y(x) = i \right], \tag{8.2} $$

so that ${}_tp_x^{ij}$ is the probability that a life currently aged x and currently in State i is in State j at age $x+t$, where j may be equal to i, while ${}_tp_x^{\overline{ii}}$ is the probability that a life currently aged x and currently in State i stays in State i continuously throughout the period from age x to age $x+t$.

We define

$$ \mu_x^{ij} = \lim_{h \to 0^+} \frac{{}_hp_x^{ij}}{h} \quad \text{for } i \neq j. \tag{8.3} $$

Assumption 8.3 *For all states i and j and all ages $x \geq 0$, we assume that ${}_tp_x^{ij}$ is a differentiable function of t.*

Assumption 8.3 is a technical assumption needed to ensure that the mathematics proceeds smoothly. The first consequence of this assumption is that the limit in (8.3) always exists, which means that μ_x^{ij} always exists (though it may be zero for impossible transitions). Also, the probability of a transition taking place in a time interval of length h converges to 0 as h converges to 0. We also assume that μ_x^{ij} is a bounded and integrable function of x. These assumptions are not too restrictive in practice. However, there are some circumstances where we need to put aside Assumption 8.3, for example, when transitions between states occur at specific ages. This is discussed further in Chapter 9. To help interpret the multiple state model notation, we start by comparing it with standard actuarial notation, in terms of the alive–dead model (Figure 8.1) where we may use either notation. We make the following observations:

- For the alive–dead model, ${}_tp_x^{00}$ is the same as ${}_tp_x$, and ${}_tp_x^{01}$ is the same as ${}_tq_x$.
- ${}_tp_x^{10} = 0$ since backward transitions, 'Dead' to 'Alive', are not permitted in this model.
- μ_x^{01} is the same as μ_x, the force of mortality at age x.

In the general case, with states $0, 1, 2, \ldots, m$, we refer to μ_x^{ij} as the **force of transition** or **transition intensity** between states i and j at age x.

An **absorbing state** is a state from which no exit is possible. For example, in the sickness–death model, State 2 (dead) is an absorbing state, but the other two states are not. If State i is an absorbing state, then $\mu_x^{ij} = {}_tp_x^{ij} = 0$ for all $j \neq i$.

Another way of expressing formula (8.3) is to write for $h > 0$

$$ {}_hp_x^{ij} = h\,\mu_x^{ij} + o(h) \qquad \text{for } i \neq j. \tag{8.4}$$

From this formulation we can say that, for small positive values of h,

$$ {}_hp_x^{ij} \approx h\,\mu_x^{ij} \qquad \text{for } i \neq j. \tag{8.5}$$

This is equivalent to formula (2.8) in Chapter 2 for the alive–dead model and will be very useful to us.

Example 8.1 Explain why, for a general multiple state model, ${}_tp_x^{\overline{ii}}$ is not equivalent to ${}_tp_x^{ii}$. Write down an inequality linking these two probabilities and explain why

$$ {}_tp_x^{ii} = {}_tp_x^{\overline{ii}} + o(t). \tag{8.6}$$

Solution 8.1 From formulae (8.1) and (8.2) we can see that ${}_tp_x^{\overline{ii}}$ is the probability that the process/individual does not leave State i between ages x and $x + t$, whereas ${}_tp_x^{ii}$ is the probability that the process/individual is in State i at age $x + t$, in both cases given that the process was in State i at age x. The important distinction is that ${}_tp_x^{ii}$ includes the possibility that the process leaves State i between ages x and $x + t$, provided it is back in State i at age $x + t$. For any individual state which either (a) can never be left or (b) can never be re-entered once it has been left, these two probabilities *are* equivalent. This applies to all the states in the models illustrated in Figures 8.1, 8.2, 8.3, and 8.4 *except* States 0 and 1 in Figure 8.4.

The following inequality is always true since the left-hand side is the probability of an event which is included in the set of events whose probability is on the right-hand side

$$ {}_tp_x^{\overline{ii}} \leq {}_tp_x^{ii}. $$

The difference between these two probabilities is the probability of those paths where the process makes two or more transitions between ages x and $x + t$ so that it is back in State i at age $x + t$. From Assumption 8.2 we know that this probability is $o(t)$. This gives us formula (8.6). $\square$

Example 8.2 Show that, for a general multiple state model and for $h > 0$,

$$_hp_x^{\overline{ii}} = 1 - h \sum_{j=0, j\neq i}^{m} \mu_x^{ij} + o(h). \tag{8.7}$$

Solution 8.2 First note that $1 - {}_hp_x^{\overline{ii}}$ is the probability that the process *does* leave State i at some time between ages x and $x+h$, possibly returning to State i before age $x + h$. If the process leaves State i between ages x and $x + h$ then at age $x + h$ it must be in some State $j\,(\neq i)$ or be in State i having made at least two transitions in the time interval of length h. Using formula (8.4) and Assumption 8.2, the sum of these probabilities is

$$h \sum_{j=0, j\neq i}^{m} \mu_x^{ij} + o(h),$$

which proves (8.7). □

Note that, combining (8.6) and (8.7) we have

$$_hp_x^{ii} = 1 - h \sum_{j=0, j\neq i}^{m} \mu_x^{ij} + o(h). \tag{8.8}$$

Example 8.3 For the sickness–death model in Figure 8.4, you are given

x	$_{10}p_x^{00}$	$_{10}p_x^{01}$	$_{10}p_x^{11}$	$_{10}p_x^{10}$
40	0.93705	0.01953	0.70349	0.23942
50	0.83930	0.06557	0.81211	0.06057

(a) Calculate (i) $_{20}p_{40}^{00}$, (ii) $_{20}p_{40}^{10}$, (iii) $_{20}p_{40}^{02}$.
(b) Given that a healthy life aged 40 dies before age 60, what is the probability that she was healthy at age 50?

Solution 8.3

(a) (i) $_{20}p_{40}^{00}$ is the probability that (40) is healthy in 20 years, given that she is healthy now. We need to calculate this from the information on transition probabilities from age 40 to 50 and from age 50 to 60. Given that (40) is currently healthy, to be healthy at time 20, she must either be healthy at time 10, or sick at time 10, and then healthy again at time 20. So the required probability is

$$_{20}p_{40}^{00} = {}_{10}p_{40}^{00}\, {}_{10}p_{50}^{00} + {}_{10}p_{40}^{01}\, {}_{10}p_{50}^{10} = 0.78765.$$

(ii) $_{20}p_{40}^{10}$ is the probability that (40) is healthy in 20 years, given that she is sick now. The two possibilities for (40), based on the state at time

$t = 10$ are that she was sick at time 10, and healthy at time 20, or she was healthy at time 10, and healthy at time 20. So the required probability is

$$ {}_{20}p_{40}^{10} = {}_{10}p_{40}^{11}\,{}_{10}p_{50}^{10} + {}_{10}p_{40}^{10}\,{}_{10}p_{50}^{00} = 0.24356. $$

(iii) ${}_{20}p_{40}^{02}$ is the probability that a healthy life aged 40 dies within 20 years. There are three paths to consider.

Path 1: (40) dies before time 10, given that she is healthy currently. The probability of this is

$$ {}_{10}p_{40}^{02} = 1 - {}_{10}p_{40}^{00} - {}_{10}p_{40}^{01} = 0.04342. $$

Path 2: (40) is healthy at time 10, but dies between time 10 and time 20. The probability of this is

$$ {}_{10}p_{40}^{00}\,{}_{10}p_{50}^{02} = {}_{10}p_{40}^{00}\left(1 - {}_{10}p_{50}^{00} - {}_{10}p_{50}^{01}\right) = 0.08914. $$

Path 3: (40) is sick at time 10, and dies between time 10 and time 20. The probability of this is

$$ {}_{10}p_{40}^{01}\,{}_{10}p_{50}^{12} = {}_{10}p_{40}^{01}\left(1 - {}_{10}p_{50}^{10} - {}_{10}p_{50}^{11}\right) = 0.00249. $$

Hence

$$ {}_{20}p_{40}^{02} = 0.04341 + 0.08912 + 0.00248 = 0.13505. $$

(b) We want to find $\Pr\left[(40) \text{ is Healthy at age 50} \middle| \text{Dead before age 60}\right]$. Using Bayes' theorem for the conditional probability,

$$ \begin{aligned} &\Pr\left[(40) \text{ is Healthy at age 50} \middle| \text{Dead before age 60}\right] \\ &= \frac{\Pr\left[(40) \text{ is Healthy at age 50 and Dead before age 60}\right]}{\Pr\left[(40) \text{ is Dead before age 60}\right]} \\ &= \frac{{}_{10}p_{40}^{00}\,{}_{10}p_{50}^{02}}{{}_{20}p_{40}^{02}} = \frac{0.08914}{0.13505} = 0.66007. \end{aligned} $$

□

8.4 Formulae for probabilities

In this section we show how to derive formulae for all probabilities in terms of the transition intensities, which we assume to be known. This is the same approach as we adopted in Chapter 2, where we assumed the force of mortality, μ_x, was known and derived formula (2.20) for ${}_tp_x$ in terms of μ_{x+t}.

The fact that all probabilities can be expressed in terms of the transition intensities is important. It tells us that the transition intensities

$$\mu_x^{ij};\ x \geq 0;\ i,j = 0,\ 1, \ldots, m, i \neq j$$

are fundamental quantities which determine everything we need to know about a multiple state model.

The first result generalizes formula (2.20) from Chapter 2, and is valid for any multiple state model. It gives a formula for ${}_tp_x^{\overline{ii}}$ in terms of all the transition intensities out of State i, μ_x^{ij}.

For any State i in a multiple state model with $m+1$ states, satisfying Assumptions 8.1 to 8.3,

$$\boxed{{}_tp_x^{\overline{ii}} = \exp\left\{ -\int_0^t \sum_{j=0, j\neq i}^{m} \mu_{x+s}^{ij}\, ds \right\}.} \tag{8.9}$$

We can derive this as follows. For any $h > 0$, the probability ${}_{t+h}p_x^{\overline{ii}}$ is the probability that the life stays in State i throughout the time period $[0, t+h]$, given that the life was in State i at age x. We can split this event into two sub-events:

- the life stays in State i from age x until (at least) age $x+t$, given that it was in State i at age x, and then
- the life stays in State i from age $x+t$ until (at least) age $x+t+h$, given that it was in State i at age $x+t$ (note the different conditioning).

The probabilities of these two sub-events are ${}_tp_x^{\overline{ii}}$ and ${}_hp_{x+t}^{\overline{ii}}$, respectively, and, using the rules for conditional probabilities, we have

$${}_{t+h}p_x^{\overline{ii}} = {}_tp_x^{\overline{ii}}\ {}_hp_{x+t}^{\overline{ii}}.$$

Using equation (8.7), this can be rewritten as

$${}_{t+h}p_x^{\overline{ii}} = {}_tp_x^{\overline{ii}} \left(1 - h \sum_{j=0, j\neq i}^{m} \mu_{x+t}^{ij} + o(h) \right).$$

Rearranging this equation, we get

$$\frac{{}_{t+h}p_x^{\overline{ii}} - {}_tp_x^{\overline{ii}}}{h} = -\,{}_tp_x^{\overline{ii}} \sum_{j=0, j\neq i}^{m} \mu_{x+t}^{ij} + \frac{o(h)}{h}.$$

Dividing by ${}_tp_x^{\overline{ii}}$, and letting $h \to 0$ we have

$$\frac{1}{{}_tp_x^{\overline{ii}}}\left(\frac{d}{dt}\,{}_tp_x^{\overline{ii}}\right) = -\sum_{j=0,j\neq i}^{m} \mu_{x+t}^{ij}$$

$$\Rightarrow \frac{d}{dt}\log\,{}_tp_x^{\overline{ii}} = -\sum_{j=0,j\neq i}^{m} \mu_{x+t}^{ij}.$$

Integrating over $(0,t)$ gives

$$\log\,{}_tp_x^{\overline{ii}} - \log\,{}_0p_x^{\overline{ii}} = -\int_0^t \sum_{j=0,j\neq i}^{m} \mu_{x+s}^{ij}\,ds.$$

Note that ${}_0p_x^{\overline{ii}} = 1$, so we have

$$\log\,{}_tp_x^{\overline{ii}} = -\int_0^t \sum_{j=0,j\neq i}^{m} \mu_{x+s}^{ij}\,ds$$

$$\Rightarrow {}_tp_x^{\overline{ii}} = \exp\left\{-\int_0^t \sum_{j=0,j\neq i}^{m} \mu_{x+s}^{ij}\,ds\right\}$$

which proves (8.9).

We comment on this result after the next example.

Example 8.4 Consider the model for permanent disability illustrated in Figure 8.3.

(a) Explain why, for $x \geq 0$ and $t, h > 0$,

$$ {}_{t+h}p_x^{01} = {}_tp_x^{01}\,{}_hp_{x+t}^{\overline{11}} + {}_tp_x^{\overline{00}}\,h\,\mu_{x+t}^{01} + o(h). \tag{8.10}$$

(b) Hence show that

$$\frac{d}{dt}\left({}_tp_x^{01}\exp\left\{\int_0^t \mu_{x+s}^{12}\,ds\right\}\right) = {}_tp_x^{\overline{00}}\,\mu_{x+t}^{01}\exp\left\{\int_0^t \mu_{x+s}^{12}\,ds\right\}, \tag{8.11}$$

and hence that for $u > 0$,

$$ {}_up_x^{01} = \int_0^u {}_tp_x^{\overline{00}}\,\mu_{x+t}^{01}\,{}_{u-t}p_{x+t}^{\overline{11}}\,dt. \tag{8.12}$$

(c) Give a direct intuitive derivation of formula (8.12).

Solution 8.4 (a) To derive (8.10), consider a life who is healthy at age x. The left-hand side of (8.10) is the probability that this life is alive and disabled at age $x + t + h$. We can write down a formula for this probability by conditioning on which state the life was in at age $x + t$. Either:

- the life was disabled at age $x+t$ (probability ${}_tp_x^{01}$) and remained disabled between ages $x + t$ and $x + t + h$ (probability ${}_hp_{x+t}^{\overline{11}}$), or
- the life was healthy at age $x + t$ (probability ${}_tp_x^{\overline{00}}$) and then became disabled between ages $x + t$ and $x + t + h$ (probability $h\,\mu_{x+t}^{01} + o(h)$).

Combining the probabilities of these events gives (8.10). Note that the probability of the life being healthy at age $x + t$, becoming disabled before age $x + t + h$ and then dying before age $x + t + h$ is $o(h)$ since this involves two transitions in a time interval of length h.

(b) Using equation (8.7), we can rewrite equation (8.10) as

$$ {}_{t+h}p_x^{01} = {}_tp_x^{01}(1 - h\,\mu_{x+t}^{12}) + {}_tp_x^{\overline{00}}\,h\,\mu_{x+t}^{01} + o(h). \tag{8.13} $$

Rearranging, dividing by h and letting $h \to 0$ gives

$$ \frac{d}{dt}\,{}_tp_x^{01} + {}_tp_x^{01}\,\mu_{x+t}^{12} = {}_tp_x^{\overline{00}}\,\mu_{x+t}^{01}. $$

Multiplying all terms in this equation by $\exp\left\{\int_0^t \mu_{x+s}^{12}\,ds\right\}$, we have

$$ \frac{d}{dt}\left({}_tp_x^{01}\exp\left\{\int_0^t \mu_{x+s}^{12}\,ds\right\}\right) = {}_tp_x^{\overline{00}}\,\mu_{x+t}^{01}\exp\left\{\int_0^t \mu_{x+s}^{12}\,ds\right\}. $$

Integrating both sides of this equation from $t = 0$ to $t = u$, and noting that ${}_0p_x^{01} = 0$, we have

$$ {}_up_x^{01}\exp\left\{\int_0^u \mu_{x+s}^{12}\,ds\right\} = \int_0^u {}_tp_x^{\overline{00}}\mu_{x+t}^{01}\exp\left\{\int_0^t \mu_{x+s}^{12}\,ds\right\}dt. $$

Finally, dividing both sides by $\exp\left\{\int_0^u \mu_{x+s}^{12}\,ds\right\}$ and noting that, using formula (8.9),

$$ {}_{u-t}p_{x+t}^{\overline{11}} = \exp\left\{-\int_t^u \mu_{x+s}^{12}\,ds\right\}, $$

we have formula (8.12).

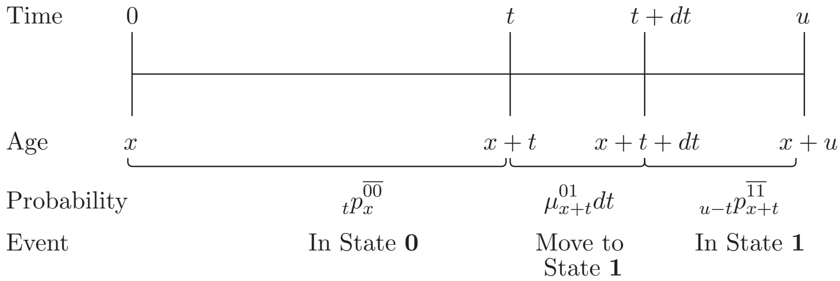


Figure 8.5 Time-line diagram for formula (8.12).

(c) The intuitive derivation of (8.12) is as follows: for the life to move from State 0 to State 1 between ages x and $x+u$, the life must stay in State 0 until some age $x+t$, transfer to State 1 between ages $x+t$ and $x+t+dt$, where dt is small, and then stay in State 1 from age $x+t+dt$ to age $x+u$. We illustrate this event sequence using the time-line in Figure 8.5.
The infinitesimal probability of this path is

$$ {}_tp_x^{\overline{00}}\ \mu_{x+t}^{01}\ {}_{u-t}p_{x+t}^{\overline{11}}\,dt, $$

where we have written ${}_{u-t}p_{x+t}^{\overline{11}}$ instead of ${}_{u-t-dt}p_{x+t}^{\overline{11}}$ since the two are equivalent for infinitesimal dt. Since the age at transfer, $x+t$, can be anywhere between x and $x+u$, the total probability, ${}_up_x^{01}$, is the sum (i.e. integral) of these probabilities from $t=0$ to $t=u$. □

We make the following comments about formula (8.9) and Example 8.4.

(1) As we have already noted, formula (8.9) is an extension of formula (2.20), which gives the relationship between the survival probability and the force of mortality as

$$ {}_tp_x = \exp\left\{-\int_0^t \mu_{x+s}\,ds\right\}. $$

(2) Throughout Example 8.4 we could have replaced ${}_tp_x^{\overline{ii}}$ by ${}_tp_x^{ii}$ for $i=0,1$, since for the disability insurance model, neither State 0 nor State 1 can be re-entered once it has been left.

(3) The intuition behind formula (8.12) can be applied to other models. Although the resulting formulae are often not directly used for calculating probabilities, they can have other uses as illustrated in Exercise 8.7. For example, consider ${}_tp_x^{01}$ for the sickness–death model. We can write

$$ {}_tp_x^{01} = \int_0^t {}_rp_x^{00}\, \mu_{x+r}^{01}\, {}_{t-r}p_{x+r}^{\overline{11}}\, dr. \tag{8.14}$$

This follows because the life must spend a continuous period in State 1 immediately prior to time t if the life is in State 1 at time t; we are integrating over all possible times of final transition to State 1 in equation (8.14).

(4) Perhaps the most important point to note about formula (8.9) and Example 8.4 is how similar the derivations are in their basic approach. In both cases we wrote down an expression for the probability of being in the required state at age $x + t + h$ by conditioning on the state occupied at age $x + t$. This led to a formula for the derivative of the required probability which we were then able to solve. An obvious question for us is, ‘Can this method be applied to a general multiple state model to derive formulae for probabilities?’ The answer is, ‘Yes’, as we demonstrate in the following section.

8.4.1 Kolmogorov’s forward equations

Let i and j be any two, not necessarily distinct, states in a multiple state model which has a total of $m + 1$ states, numbered $0, 1, \ldots, m$. For $x, t, h \geq 0$, we derive the formula

$$ {}_{t+h}p_x^{ij} = {}_tp_x^{ij} - h \sum_{k=0,\, k\neq j}^{m} \left({}_tp_x^{ij} \mu_{x+t}^{jk} - {}_tp_x^{ik}\, \mu_{x+t}^{kj} \right) + o(h), \tag{8.15}$$

and hence show the main result, that

$$ \boxed{\frac{d}{dt}\, {}_tp_x^{ij} = \sum_{k=0,\, k\neq j}^{m} \left({}_tp_x^{ik}\, \mu_{x+t}^{kj} - {}_tp_x^{ij} \mu_{x+t}^{jk} \right).} \tag{8.16}$$

Formula (8.16) gives a set of equations for a Markov process known as Kolmogorov’s forward equations.

To derive Kolmogorov’s forward equations, we proceed as we did in formula (8.9) and in Example 8.4. We consider the probability of being in the required state, j, at age $x + t + h$, and condition on the state at age $x + t$: either it is already in State j, in which case it needs to stay in j, or possibly leave and come back, between ages $x + t$ and $x + t + h$, or it is in some other state, say k, and must transition to State j before age $x + t + h$. Thus, we have

$$ {}_{t+h}p_x^{ij} = {}_tp_x^{ij}\, {}_hp_{x+t}^{jj} + \sum_{k=0,\, k\neq j}^{m} {}_tp_x^{ik}\, {}_hp_{x+t}^{kj}.$$

Using formula (8.8), we have

$$ {}_hp_{x+t}^{jj} = 1 - h \sum_{k=0,\, k\neq j}^{m} \mu_{x+t}^{jk} + o(h) $$

and using (8.4), we have

$$ {}_hp_{x+t}^{kj} = h\,\mu_{x+t}^{kj} + o(h). $$

So, substituting, and collecting all the $o(h)$ terms together, we have

$$ {}_{t+h}p_x^{ij} = {}_tp_x^{ij} \left(1 - h \sum_{k=0,\, k\neq j}^{m} \mu_{x+t}^{jk} \right) + \sum_{k=0, k\neq j}^{m} {}_tp_x^{ik}\, h\, \mu_{x+t}^{kj} + o(h) $$

$$ \Rightarrow \frac{{}_{t+h}p_x^{ij} - {}_tp_x^{ij}}{h} = \sum_{k=0,\, k\neq j}^{m} {}_tp_x^{ik}\, \mu_{x+t}^{kj} - {}_tp_x^{ij} \sum_{k=0, k\neq j}^{m} \mu_{x+t}^{jk} + \frac{o(h)}{h} $$

and letting $h \to 0$ gives

$$ \frac{d}{dt}\, {}_tp_x^{ij} = \sum_{k=0,\, k\neq j}^{m} \left({}_tp_x^{ik}\, \mu_{x+t}^{kj} - {}_tp_x^{ij}\, \mu_{x+t}^{jk} \right) $$

as required.

We can solve the Kolmogorov forward equations numerically, but we need boundary conditions. For *any* model, we use the facts that, for any state i,

$$ {}_0p_x^{ii} = 1 $$

and for any two states i and j where $i \neq j$,

$$ {}_0p_x^{ij} = 0. $$

8.5 Numerical evaluation of probabilities

In this section we discuss methods for the numerical evaluation of probabilities for a multiple state model given that all the transition intensities are known. In some cases, the probabilities can be calculated directly from formulae in terms of integrals, as the following example shows.

Example 8.5 Consider the permanent disability model illustrated in Figure 8.3.

(a) Suppose the transition intensities for this model are all constants, as follows

$$ \mu_x^{01} = 0.0279, \quad \mu_x^{02} = 0.0229, \quad \mu_x^{12} = \mu_x^{02}. $$

Calculate ${}_{10}p_{60}^{00}$ and ${}_{10}p_{60}^{01}$.

(b) Now suppose the transition intensities for this model are as follows

$$\mu_x^{01} = a_1 + b_1 \exp\{c_1 x\}, \quad \mu_x^{02} = a_2 + b_2 \exp\{c_2 x\}, \quad \mu_x^{12} = \mu_x^{02},$$

where

$$a_1 = 4 \times 10^{-4}, \ \ b_1 = 3.4674 \times 10^{-6}, \ \ c_1 = 0.138155,$$
$$a_2 = 5 \times 10^{-4}, \ \ b_2 = 7.5858 \times 10^{-5}, \ \ c_2 = 0.087498.$$

Calculate ${}_{10}p_{60}^{00}$ and ${}_{10}p_{60}^{01}$.

Solution 8.5 For this model, neither State 0 nor State 1 can be re-entered once it has been left, so that ${}_tp_x^{ii} \equiv {}_tp_x^{\overline{ii}}$ for $i = 0, 1$ and for any x, $t \geq 0$.

(a) Using formula (8.9), we have

$$ {}_tp_{60}^{00} \equiv {}_tp_{60}^{\overline{00}} = \exp\left\{ -\int_0^t (0.0279 + 0.0229)\, ds \right\} = \exp\{-0.0508t\}, \tag{8.17}$$

giving

$${}_{10}p_{60}^{00} = \exp\{-0.508\} = 0.60170.$$

Similarly

$${}_{10-t}p_{60+t}^{11} = \exp\{-0.0229(10-t)\},$$

and we can calculate ${}_{10}p_{60}^{01}$ using formula (8.12), integrating over the time of transition from State 0 to State 1, as

$$\begin{aligned}
{}_{10}p_{60}^{01} &= \int_0^{10} {}_tp_{60}^{00}\, \mu_{60+t}^{01}\, {}_{10-t}p_{60+t}^{11}\, dt \\
&= \int_0^{10} \exp\{-0.0508t\} \times 0.0279 \times \exp\{-0.0229(10-t)\}\, dt \\
&= 0.0279\, \exp\{-0.229\} \int_0^{10} \exp\{-0.0279t\}\, dt \\
&= \exp\{-0.229\}(1 - \exp\{-0.279\}) \\
&= 0.19363.
\end{aligned}$$

(b) In this case

$$
\begin{aligned}
{}_tp_{60}^{00} &= \exp\left\{-\int_0^t (\mu_{60+r}^{01} + \mu_{60+r}^{02})\,dr\right\} \\
&= \exp\left\{-\left((a_1+a_2)t + \frac{b_1}{c_1}e^{60\,c_1}(e^{c_1 t}-1) + \frac{b_2}{c_2}e^{60\,c_2}(e^{c_2 t}-1)\right)\right\}
\end{aligned}
$$

and

$$
{}_tp_{60}^{11} = \exp\left\{-\int_0^t \mu_{60+r}^{12}\,dr\right\} = \exp\left\{-\left(a_2\,t + \frac{b_2}{c_2}e^{60\,c_2}(e^{c_2 t}-1)\right)\right\}.
$$

Hence ${}_{10}p_{60}^{00} = 0.58395$.

Substituting the expressions for ${}_tp_{60}^{00}$ and ${}_{10-t}p_{60+t}^{11}$ and the formula for μ_{60+t}^{01} into formula (8.12) and integrating numerically, we obtain

$$
{}_{10}p_{60}^{01} = 0.20577.
$$

□

Probabilities of the form ${}_tp_x^{\overline{ii}}$ can be evaluated analytically provided the sum of the relevant intensities can be integrated analytically. In other cases numerical integration can be used. However, the approach used in Example 8.5 part (b) to calculate a more complicated probability, ${}_{10}p_{60}^{01}$, – that is, deriving an integral formula for the probability which can then be integrated numerically – is not tractable except in the simplest cases. Broadly speaking, this approach works if the model has relatively few states and if none of these states can be re-entered once it has been left. These conditions are met by the permanent disability model, illustrated in Figure 8.3 and used in Example 8.5, but are not met, for example, by the sickness–death model illustrated in Figure 8.4 since States 0 and 1 can both be re-entered. This means, for example, that ${}_tp_x^{01}$ is the sum of the probabilities of exactly one transition (0 to 1), plus three transitions (0 to 1, then 1 to 0, then 0 to 1 again), plus five transitions, and so on. A probability involving k transitions involves multiple integration with k nested integrals.

However, Euler's method, introduced in Chapter 7, can be used to evaluate probabilities for all models in which we are interested. The key to using this method is formula (8.15) and we illustrate it by applying it in the following example.

Example 8.6 Consider the sickness–death model illustrated in Figure 8.4. Suppose the transition intensities for this model are as follows

$$\mu_x^{01} = a_1 + b_1 \exp\{c_1 x\}, \qquad \mu_x^{10} = 0.1\,\mu_x^{01},$$
$$\mu_x^{02} = a_2 + b_2 \exp\{c_2 x\}, \qquad \mu_x^{12} = \mu_x^{02},$$

where $a_1,\ b_1,\ c_1,\ a_2,\ b_2$ and c_2 are as in Example 8.5, part (b) (though this is a different model).

Calculate ${}_{10}p_{60}^{00}$ and ${}_{10}p_{60}^{01}$ using formula (8.15) with a step size of $h = 1/12$ years.

Solution 8.6 For this particular model, formula (8.15) gives us the two formulae

$${}_{t+h}p_{60}^{00} = {}_tp_{60}^{00} - h\,{}_tp_{60}^{00}\left(\mu_{60+t}^{01} + \mu_{60+t}^{02}\right) + h\,{}_tp_{60}^{01}\,\mu_{60+t}^{10} + o(h)$$

and

$${}_{t+h}p_{60}^{01} = {}_tp_{60}^{01} - h\ {}_tp_{60}^{01}\ \left(\mu_{60+t}^{12} + \mu_{60+t}^{10}\right) + h\ {}_tp_{60}^{00}\,\mu_{60+t}^{01} + o(h).$$

As in Chapter 7, we choose a small step size h, ignore the $o(h)$ terms and regard the resulting approximations as exact formulae. This procedure changes the above formulae into

$${}_{t+h}p_{60}^{00} = {}_tp_{60}^{00} - h\,{}_tp_{60}^{00}\left(\mu_{60+t}^{01} + \mu_{60+t}^{02}\right) + h\,{}_tp_{60}^{01}\,\mu_{60+t}^{10}$$

and

$${}_{t+h}p_{60}^{01} = {}_tp_{60}^{01} - h\,{}_tp_{60}^{01}\ \left(\mu_{60+t}^{12} + \mu_{60+t}^{10}\right) + h\ {}_tp_{60}^{00}\,\mu_{60+t}^{01}.$$

By choosing successively $t = 0,\ h,\ 2h,\ \ldots,\ 10 - h$, we can use these formulae, together with the initial values ${}_0p_{60}^{00} = 1$ and ${}_0p_{60}^{01} = 0$, to calculate ${}_hp_{60}^{00},\ {}_hp_{60}^{01},\ {}_{2h}p_{60}^{00},\ {}_{2h}p_{60}^{01}$, and so on until we have a value for ${}_{10}p_{60}^{00}$, as required. These calculations are very well suited to a spreadsheet. For a step size of $h = 1/12$ years, selected values are shown in Table 8.1. Note that the calculations have been carried out using more significant figures than are shown in this table. □

The implementation of Euler's method in this example differs in two respects from the implementation in Chapter 7.

(1) We work forward recursively from initial values for the probabilities rather than backwards from the final value of the policy value. This is determined by the boundary conditions for the differential equations.
(2) We have two equations to solve simultaneously rather than a single equation. This is a typical feature of applying Euler's method to the calculation of probabilities for multiple state models. In general, the number of equations increases with the number of states in the model.

Table 8.1 *Calculation of* ${}_{10}p_{60}^{00}$ *and* ${}_{10}p_{60}^{01}$ *using a step size* $h = 1/12$ *years.*

t	μ_{60+t}^{01}	μ_{60+t}^{02}	μ_{60+t}^{10}	μ_{60+t}^{12}	${}_tp_{60}^{00}$	${}_tp_{60}^{01}$
0	0.01420	0.01495	0.00142	0.01495	1.00000	0.00000
$\frac{1}{12}$	0.01436	0.01506	0.00144	0.01506	0.99757	0.00118
$\frac{2}{12}$	0.01453	0.01517	0.00145	0.01517	0.99512	0.00238
$\frac{3}{12}$	0.01469	0.01527	0.00147	0.01527	0.99266	0.00358
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	0.01625	0.01628	0.00162	0.01628	0.96977	0.01479
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$9\frac{11}{12}$	0.05473	0.03492	0.00547	0.03492	0.59189	0.20061
10	0.05535	0.03517	0.00554	0.03517	0.58756	0.20263

8.6 State-dependent insurance and annuity functions

So far we have demonstrated that multiple state models are a natural way of modelling cash flows that are state-dependent, and we have shown how to evaluate probabilities for such models given only the transition intensities between pairs of states. The next stage in our study of multiple state models is to calculate premiums and policy values for a policy represented by such a model and for these we develop actuarial functions analogous to the $\bar{A}_x$ and $\bar{a}_x$ functions of Chapters 4 and 5.

We implicitly use the indicator function approach from Section 4.6 – that is, we develop expected present values by summing the product of the probabilities, amounts and discount functions over all possible payment dates. This leads directly to intuitive formulae for the EPVs, but does not give higher moments, which are outside the scope of this book.

There is no standard actuarial notation for insurance and annuity functions in the multiple state model framework; the notation used in this chapter generalizes International Actuarial Notation, and has become widespread in actuarial practice.

State-dependent insurance benefits

Insurance benefits are conditional on making a transition into a specified state. For example, a death benefit is payable on transition into the dead state; a permanent disability insurance policy might pay a sum insured on becoming disabled.

Suppose a payment of 1 is made immediately on each future transfer into State j within n years, given that the life is currently in State i (which may be equal to j). Then the EPV of the benefit is

$$\bar{A}^{ij}_{x:\overline{n}|} = \int_0^n \sum_{k=0,\,k\neq j}^{m} e^{-\delta t}\, {}_tp_x^{ik}\, \mu_{x+t}^{kj}\, dt. \tag{8.18}$$

Note that this benefit does not require the transition to be directly from State i to State j, and if there is a possibility of multiple transitions into State j, then it values a benefit of 1 paid *each time* the life transitions into State j. If there is no definite term to the policy, then we have the whole life function, $\bar{A}^{ij}_x$.

To derive (8.18), we consider a possible payment in the infinitesimal interval $(t, t+dt)$.

- If there is a payment, the amount is 1.
- The discount factor (for sufficiently small dt) is $e^{-\delta t}$.
- The probability that the payment is made is the probability that the life transfers into State j in $(t, t+dt)$, given that the life is in State i at time 0.

  In order to transfer into State j in $(t, t+dt)$, the life must be in some State k that is not j immediately before (the probability of two transitions in infinitesimal time being negligible), with probability ${}_tp_x^{ik}$, then transfer from State k to State j during the interval $(t, t+dt)$, with probability (loosely) $\mu_{x+t}^{kj}\, dt$.

Summing (that is, integrating) over all the possible time intervals gives equation (8.18).

If we need to calculate $\bar{A}^{ij}_{x:\overline{n}|}$ from a set of tables of $\bar{A}^{ij}_x$, we can do so provided that we know probabilities ${}_np_x^{ij}$. The relationship is

$$\bar{A}^{ij}_{x:\overline{n}|} = \bar{A}^{ij}_x - e^{-\delta n} \sum_{k=0}^{m} {}_np_x^{ik}\, \bar{A}^{kj}_{x+n}. \tag{8.19}$$

We can show this by starting with an integral expression for $\bar{A}^{ij}_x$, as follows. Letting $n \to \infty$ in (8.18) we have

$$\begin{aligned}
\bar{A}^{ij}_x &= \int_0^\infty \sum_{k=0,\,k\neq j}^{m} e^{-\delta t}\, {}_tp_x^{ik}\, \mu_{x+t}^{kj}\, dt \\
&= \int_0^n \sum_{k=0,\,k\neq j}^{m} e^{-\delta t}\, {}_tp_x^{ik}\, \mu_{x+t}^{kj}\, dt + \int_n^\infty \sum_{k=0,\,k\neq j}^{m} e^{-\delta t}\, {}_tp_x^{ik}\, \mu_{x+t}^{kj}\, dt \\
&= \bar{A}^{ij}_{x:\overline{n}|} + \int_0^\infty \sum_{k=0,\,k\neq j}^{m} e^{-\delta (n+r)}\, {}_{n+r}p_x^{ik}\, \mu_{x+n+r}^{kj}\, dr.
\end{aligned}$$

Considering the possible states at time n we can write

$$ {}_{n+r}p_x^{ik} = \sum_{z=0}^{m} {}_np_x^{iz}\, {}_rp_{x+n}^{zk} $$

giving

$$ \begin{aligned} \bar{A}_x^{ij} &= \bar{A}_{x:\overline{n|}}^{ij} + e^{-\delta n} \int_0^\infty \sum_{k=0,\, k\neq j}^{m} e^{-\delta r} \sum_{z=0}^{m} {}_np_x^{iz}\, {}_rp_{x+n}^{zk}\, \mu_{x+n+r}^{kj}\, dr \\ &= \bar{A}_{x:\overline{n|}}^{ij} + e^{-\delta n} \sum_{z=0}^{m} {}_np_x^{iz} \int_0^\infty \sum_{k=0,\, k\neq j}^{m} e^{-\delta r}\, {}_rp_{x+n}^{zk}\, \mu_{x+n+r}^{kj}\, dr \\ &= \bar{A}_{x:\overline{n|}}^{ij} + e^{-\delta n} \sum_{z=0}^{m} {}_np_x^{iz} \bar{A}_{x+n}^{zj}. \end{aligned} $$

A simple rearrangement and change in the variable of summation yield formula (8.19).

In the case of the alive–dead model, formula (8.19) gives the familiar expression

$$ \bar{A}_{x:\overline{n|}}^{00} = \bar{A}_x^{00} - e^{-\delta n}\, {}_np_x^{00}\, \bar{A}_{x+n}^{00}, $$

which is formula (4.25) written in the notation of multiple state models.

In the case of the sickness–death model, for the case $i = 0$ and $j = 1$, formula (8.19) gives

$$ \bar{A}_{x:\overline{n|}}^{01} = \bar{A}_x^{01} - e^{-\delta n}\, {}_np_x^{00}\, \bar{A}_{x+n}^{01} - e^{-\delta n}\, {}_np_x^{01}\, \bar{A}_{x+n}^{11}. \tag{8.20} $$

The final term on the left-hand side of (8.20) is there because even if (x) is in State 1 at time n, payments can still be made after time n if (x) moves to State 1 from State 0 after time n and then moves back to State 0 from State 1. (Note that multiple such moves are possible.)

We have assumed above that the benefit payable on transition from State i to State j is one. Suppose instead that the benefit on any transition to State j is $B_t^{(j)}$ in the next n years. Then the argument we used to construct the identity (8.18) still holds and the EPV of this new benefit would be

$$ \int_0^n \sum_{k=0,\, k\neq j}^{m} B_t^{(j)}\, e^{-\delta t}\, {}_tp_x^{ik}\, \mu_{x+t}^{kj}\, dt. \tag{8.21} $$

We make use of this expression in the next section when we discuss continuous sojourn annuities.

If we need a different function, for example, one that only values a payment on the first transition into State j, or that only values a benefit if the transition is direct from State i to State j, we can derive new functions as we need them,

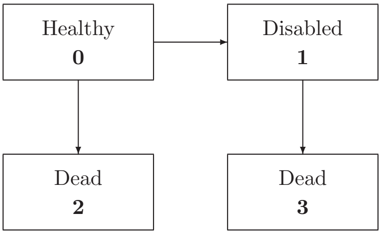


Figure 8.6 Four-state permanent disability model.

but we can often solve the problem by re-designing the model. For example, consider the permanent disability model, but with a death benefit that is only paid if the life does not become disabled. Using the three state model of Figure 8.3, we cannot, with our notation, separate the value of a benefit on transition from State 0 to State 2 directly, from a benefit on transition through State 1 to State 2. However, if we reframe the model as a four-state model, as in Figure 8.6, then $\bar{A}_x^{02}$ values the death benefit for a currently healthy life, conditional on the life not becoming disabled, as required.

8.6.1 State-dependent annuities

Suppose we have a life aged x currently in State i of a multiple state model. We wish to value an annuity of 1 per year payable continuously throughout (x)'s lifetime, but only while (x) is in some specified State j (which may be equal to i).

The EPV of the annuity, at force of interest δ per year, is denoted $\bar{a}_x^{ij}$, and is given by

$$\bar{a}_x^{ij} = \int_0^\infty e^{-\delta t}\, {}_tp_x^{ij}\, dt. \tag{8.22}$$

To derive this, we consider payment in the infinitesimal interval $(t, t+dt)$.

- If there is a payment, the amount paid in a small interval of length dt is dt, based on the payment rate of 1 per year.
- The discount factor is $e^{-\delta t}$ (or equivalently v^t).
- The probability that the benefit is paid in this interval is the probability that (x) is in State j at time t, given that (x) is in State i at time 0. This probability is ${}_tp_x^{ij}$.

Summing (that is, integrating) over all the possible time intervals gives equation (8.22).

State-dependent annuities are often payable for a fixed term. We use $\bar{a}^{ij}_{x:\overline{n}|}$ to denote the EPV, and a formula is derived similarly to (8.22), but with the upper limit of integration set at n years.

For annuities paid at discrete times, we can use the same principles. For a life (x), currently in State i, the EPV of an annuity of 1 per year, payable at the start of each year for a maximum of n years, conditional on the life being in State j at the payment date, is

$$\ddot{a}^{ij}_{x:\overline{n}|} = {}_0p^{ij}_x + v\,{}_1p^{ij}_x + v^2{}_2p^{ij}_x + \cdots + v^{n-1}{}_{n-1}p^{ij}_x = \sum_{k=0}^{n-1} v^k\,{}_kp^{ij}_x.$$

Note that if $i \neq j$ then ${}_0p^{ij}_x = 0$.

The EPVs of annuities payable $\frac{1}{m}$thly, denoted $\ddot{a}^{(m)ij}_x$ if payments are throughout life, and the EPVs of annuities paid at the end of each time interval, denoted $a^{ij}_{x:\overline{n}|}$ if payments are for at most n years, can be defined and valued similarly, adapting the notation of previous chapters.

In situations where we have a set of tables giving the EPV of a lifetime state-dependent annuity, we can use these tables to obtain the EPV of a state-dependent n-year term annuity provided that we know probabilities ${}_np^{ij}_x$. Using the same ideas that were used to derive formula (8.19) we can show that

$$\bar{a}^{ij}_{x:\overline{n}|} = \bar{a}^{ij}_x - e^{-\delta n} \sum_{k=0}^{m} {}_np^{ik}_x\,\bar{a}^{kj}_{x+n}. \tag{8.23}$$

For example, in the case of the alive–dead model, formula (8.23) gives the familiar expression

$$\bar{a}^{00}_{x:\overline{n}|} = \bar{a}^{00}_x - e^{-\delta n}\,{}_np^{00}_x\,\bar{a}^{00}_{x+n}.$$

In the case of the sickness–death model, for the case $i = j = 0$ formula (8.23) gives

$$\bar{a}^{00}_{x:\overline{n}|} = \bar{a}^{00}_x - e^{-\delta n}{}_np^{00}_x\,\bar{a}^{00}_{x+n} - e^{-\delta n}{}_np^{01}_x\,\bar{a}^{10}_{x+n}. \tag{8.24}$$

Note that the final term on the left-hand side of (8.24) is there because even if (x) is in State 1 at time n, payments can still made after time n if (x) moves from State 1 to State 0 after time n.

Woolhouse's formula and state-dependent annuities

The methods of the previous section have been used to produce the continuous annuity values given in Table D.8 in Appendix D. In this section we discuss how to adapt Woolhouse's formula to estimate the EPVs of discretely paid state-dependent annuities from the EPVs of continuously paid state-dependent annuities, or vice versa.

Recall from Section 5.11.3 the Euler–Maclaurin numerical integration formula

$$\int_0^\infty g(t)\,dt = h\sum_{k=0}^{\infty} g(kh) - \frac{h}{2}g(0) + \frac{h^2}{12}g'(0) - \frac{h^4}{720}g'''(0) + \cdots . \qquad (8.25)$$

In Chapter 5, we applied this formula to the function $g(t) = {}_tp_x\, e^{-\delta t}$, so that the left-hand side of (8.25) is $\bar{a}_x$. In this chapter, we apply the formula to the function $g(t) = {}_tp_x^{ij}\, e^{-\delta t}$, so that the left-hand side of (8.25) is $\bar{a}_x^{ij}$. For the right-hand side, we ignore terms after the first three (the third derivative of $g(t)$ will be very small in our applications), and consider the cases $i \neq j$ and $i = j$.

First, let $h = 1$, and consider the case when $i \neq j$. The first term on the right-hand side of the Euler–Maclaurin formula is

$$\sum_{t=0}^{\infty} {}_tp_x^{ij}\, e^{-\delta t} = \ddot{a}_x^{ij},$$

and (minus) the second term is

$$\frac{1}{2}\, {}_0p_x^{ij}\, e^{-0\delta} = 0 \quad \text{as } i \neq j.$$

For the third term we need the derivative of ${}_tp_x^{ij} e^{-\delta t}$:

$$\begin{aligned}\frac{d}{dt}\left({}_tp_x^{ij} e^{-\delta t}\right) &= e^{-\delta t}\frac{d}{dt}{}_tp_x^{ij} - \delta\, {}_tp_x^{ij}\, e^{-\delta t}\\ &= e^{-\delta t}\left(\sum_{k\neq j} {}_tp_x^{ik}\,\mu_{x+t}^{kj} - {}_tp_x^{ij}\sum_{k\neq j}\mu_{x+t}^{jk}\right) - \delta\, {}_tp_x^{ij}\, e^{-\delta t}\end{aligned}$$

using the Kolmogorov forward equation for $\frac{d}{dt}{}_tp_x^{ij}$. Setting $t = 0$, and noting that for $i \neq j, {}_0p_x^{ij} = 0$, we see that there is only one non-zero term in the derivative, that is

$$\frac{d}{dt}\left({}_tp_x^{ij} e^{-\delta t}\right)\Big|_{t=0} = \mu_x^{ij}.$$

So, for $i \neq j$

$$\bar{a}_x^{ij} \approx \ddot{a}_x^{ij} + \frac{\mu_x^{ij}}{12}.$$

Now let $i = j$, and $h = 1$. The first term on the left-hand side of (8.25) is $\ddot{a}_x^{ij}$, as before, and (minus) the second term is

$$\frac{1}{2}\, {}_0p_x^{ii}\, e^{-0\delta} = \frac{1}{2}.$$

For the third term, we need the derivative of ${}_tp_x^{ii}e^{-\delta t}$:

$$\frac{d}{dt}\left({}_tp_x^{ii}e^{-\delta t}\right) = e^{-\delta t}\frac{d}{dt}{}_tp_x^{ii} - \delta\,{}_tp_x^{ii}\,e^{-\delta t}$$
$$= e^{-\delta t}\left(\sum_{k\neq i}{}_tp_x^{ik}\mu_{x+t}^{ki} - {}_tp_x^{ii}\sum_{k\neq i}\mu_{x+t}^{ik}\right) - \delta\,{}_tp_x^{ii}\,e^{-\delta t}.$$

Setting $t = 0$ makes the first sum equal to 0 as ${}_tp_x^{ik} = 0$ for $i \neq k$, and hence

$$\frac{d}{dt}\left({}_tp_x^{ii}e^{-\delta t}\right)\bigg|_{t=0} = -\left(\sum_{k\neq i}\mu_x^{ik} + \delta\right),$$

giving

$$\bar{a}_x^{ii} \approx \ddot{a}_x^{ii} - \frac{1}{2} - \frac{1}{12}\left(\sum_{k\neq i}\mu_x^{ik} + \delta\right).$$

For discrete annuities payable $1/m$thly, the same method, but with $h = \frac{1}{m}$ gives

$$\bar{a}_x^{ij} \approx \ddot{a}_x^{(m)ij} + \frac{\mu_x^{ij}}{12m^2} \quad \text{for } i \neq j,$$
$$\bar{a}_x^{ii} \approx \ddot{a}_x^{(m)ii} - \frac{1}{2m} - \frac{1}{12m^2}\left(\sum_{k\neq i}\mu_x^{ik} + \delta\right).$$

Let $\mu_x^{i\bullet} = \sum_{k\neq i}\mu_x^{ik}$ denote the total transition intensity out of State i at age x. Then, rearranging the formulae above gives the following Woolhouse approximations for the discrete annuity EPVs, in terms of the continuous annuity EPVs.

Two-term Woolhouse approximations

$$\ddot{a}_x^{(m)ij} \approx \bar{a}_x^{ij} \qquad i \neq j, \tag{8.26}$$

$$\ddot{a}_x^{(m)ii} \approx \bar{a}_x^{ii} + \frac{1}{2m}. \tag{8.27}$$

Three-term Woolhouse approximations

$$\ddot{a}_x^{(m)ij} \approx \bar{a}_x^{ij} - \frac{\mu_x^{ij}}{12m^2} \qquad i \neq j, \tag{8.28}$$

$$\ddot{a}_x^{(m)ii} \approx \bar{a}_x^{ii} + \frac{1}{2m} + \frac{1}{12m^2}\left(\mu_x^{i\bullet} + \delta\right). \tag{8.29}$$

Table 8.2 *Standard Sickness–Death Model, EPVs of annual, quarterly and monthly annuities using Woolhouse's two-term and three-term formulae.*

	$m=1$			$m=4$			$m=12$		
	Exact	2-term	3-term	Exact	2-term	3-term	Exact	2-term	3-term
$\ddot{a}_{50}^{(m)00}$	12.2496	12.2446	12.2495	11.8700	11.8696	11.8699	11.7863	11.7863	11.7863
$\ddot{a}_{50}^{(m)01}$	1.9619	1.9622	1.9619	1.9622	1.9622	1.9622	1.9622	1.9622	1.9622
$\ddot{a}_{50}^{(m)11}$	12.8978	12.8918	12.8977	12.5172	12.5168	12.5172	12.4335	12.4335	12.4335
$\ddot{a}_{50}^{(m)10}$	0.6657	0.6668	0.6657	0.6668	0.6668	0.6667	0.6668	0.6668	0.6668

The formulae for approximating $1/m$thly annuity EPVs from annual annuity EPVs are

$$\ddot{a}_x^{(m)ij} \approx \ddot{a}_x^{ij} + \frac{m^2-1}{12m^2}\,\mu_x^{ij} \quad \text{for } i \neq j,$$

$$\ddot{a}_x^{(m)ii} \approx \ddot{a}_x^{ii} - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}\left(\mu_x^{i\bullet} + \delta\right).$$

In Table 8.2 we show annuity EPVs for the Standard Sickness–Death Model when $i = 0.05$, where the approximations are derived from the continuously paid annuity EPVs given in the tables in Appendix D. For the 'exact' columns we have used a more advanced technique than Euler's method to calculate probabilities ${}_tp_{50}^{ij}$ used to calculate $\ddot{a}_{50}^{(m)ij}$. We note that the three-term formula is very accurate, but the two-term version is sufficient for most purposes.

Continuous sojourn annuites

A **continuous sojourn annuity** is an annuity payable for the duration of a single stay in a given state. We define $\bar{a}_{x:\overline{n}|}^{\overline{ii}}$ to be the EPV of a continuous payment of 1 per year paid to a life currently aged x and currently in State i, where the payment continues as long as the life remains in State i, or until the expiry of the n year term if earlier. The annuity ceases if the life leaves State i, even if the life subsequently returns to it. We have

$$\bar{a}_{x:\overline{n}|}^{\overline{ii}} = \int_0^n {}_tp_x^{\overline{ii}}\, e^{-\delta t}\,dt. \tag{8.30}$$

The continuous sojourn annuity can be used for an alternative formulation of the EPV of the state-dependent annuity, $\bar{a}_x^{ij}$. Consider, for example, $\bar{a}_x^{01}$ for the sickness–death model. Instead of integrating over all the possible payment times, we can integrate over each possible transition time into State 1 (which in this model can only occur from State 0). Suppose the life transitions into State 1 in the infinitesimal interval $(t, t + dt)$, for which the probability is (loosely) ${}_tp_x^{00}\,\mu_{x+t}^{01}\,dt$. Then the EPV at time t of a payment of 1 per year, payable continuously until the life transitions out of State 1 is $\bar{a}_{x+t}^{\overline{11}}$; this is discounted back to time 0 with the discount factor $e^{-\delta t}$. Integrating over all the infinitesimal intervals, we have

$$\bar{a}_x^{01} = \int_0^\infty e^{-\delta t}\,{}_tp_x^{00}\,\mu_{x+t}^{01}\,\bar{a}_{x+t}^{\overline{11}}\,dt. \tag{8.31}$$

An alternative way of thinking about formula (8.31) is by considering formula (8.21) which gives the EPV of a benefit of $B_t^{(j)}$, payable at time t on any transition to State j for a life in State i at time 0. In the context of our sickness–death model, and with an infinite term, formula (8.21) would give the EPV of a benefit of $B_t^{(1)}$ payable on any transition to State 1 for a life in State 0 at time 0 as

$$\int_0^\infty B_t^{(1)}\,e^{-\delta t}\,{}_tp_x^{00}\,\mu_{x+t}^{01}\,dt. \tag{8.32}$$

In (8.32), the benefit on transition to State 1 is a lump sum, but if we replace the lump sum benefit with a continuous sojourn annuity at rate 1 per year, then that annuity has EPV $\bar{a}_{x+t}^{\overline{11}}$, and replacing $B_t^{(1)}$ by this in (8.32) yields (8.31). So, although (8.31) gives a state-dependent annuity EPV, we can also think of it as giving the EPV of state-dependent insurance benefits, where the benefits are in the form of continuous sojourn annuities. This is an important idea which we meet again in Chapter 10 under reversionary annuities.

When $i \neq j$, the general form for $\bar{a}_x^{ij}$ using continuous sojourn annuity EPVs is

$$\bar{a}_x^{ij} = \int_0^\infty \sum_{k\neq j,\,k=0}^{m} {}_tp_x^{ik}\,\mu_{x+t}^{kj}\,\bar{a}_{x+t}^{\overline{jj}}\,e^{-\delta t}\,dt. \tag{8.33}$$

For the case $i = j$, we have

$$\bar{a}_x^{ii} = \bar{a}_x^{\overline{ii}} + \int_0^\infty \sum_{k\neq i,\,k=0}^{m} {}_tp_x^{ik}\,\mu_{x+t}^{ki}\,\bar{a}_{x+t}^{\overline{ii}}\,e^{-\delta t}\,dt.$$

As we show in Section 8.9.1, this formulation can prove useful in practice.

8.7 Premiums

We assume that premiums are calculated using the equivalence principle and that lives are in State 0 at the policy inception date. Then premiums are calculated by solving the equation of value, using the appropriate annuity and insurance functions.

Example 8.7 An insurer issues a 10-year disability income insurance policy to a healthy life aged 60. Calculate the annual premium for the following two policy designs.

(a) Premiums are payable continuously while in the healthy state. A benefit of \$20 000 per year is payable continuously while in the disabled state. A death benefit of \$50 000 is payable immediately on death.
(b) Premiums are payable monthly in advance, conditional on the life being in the healthy state at the premium date. The sickness benefit of \$20 000 per year is payable monthly in arrear, if the life is in the sick state at the payment date. A death benefit of \$50 000 is payable immediately on death.

Basis: Use the Standard Sickness–Death Model in Appendix D.3, and Woolhouse's two-term formula. Assume interest is at 5% per year, and that there are no expenses.

Solution 8.7 (a) Letting P denote the total premium per year, the premium equation of value is

$$P\bar{a}^{00}_{60:\overline{10}|} = 20\,000\,\bar{a}^{01}_{60:\overline{10}|} + 50\,000\,\bar{A}^{02}_{60:\overline{10}|}.$$

We can find $\bar{a}^{00}_{60:\overline{10}|}$ from equation (8.24) as

$$\bar{a}^{00}_{60:\overline{10}|} = \bar{a}^{00}_{60} - v^{10}\,{}_{10}p^{00}_{60}\,\bar{a}^{00}_{70} - v^{10}\,{}_{10}p^{01}_{60}\,\bar{a}^{10}_{70} = 6.5885.$$

Similarly, using equation (8.23) we have

$$\bar{a}^{01}_{60:\overline{10}|} = \bar{a}^{01}_{60} - v^{10}\,{}_{10}p^{00}_{60}\,\bar{a}^{01}_{70} - v^{10}\,{}_{10}p^{01}_{60}\,\bar{a}^{11}_{70} = 0.6476.$$

Finally, we value the term death benefit using equation (8.19) as

$$\bar{A}^{02}_{60:\overline{10}|} = \bar{A}^{02}_{60} - v^{10}\,{}_{10}p^{00}_{60}\,\bar{A}^{02}_{70} - v^{10}\,{}_{10}p^{01}_{60}\,\bar{A}^{12}_{70} = 0.16382.$$

Substituting into the premium equation of value gives $P = \$3\,209$ per year.

(b) As premiums and sickness benefits are payable monthly, the premium equation of value is

$$P\ddot{a}^{(12)00}_{60:\overline{10}|} = 20\,000\,a^{(12)01}_{60:\overline{10}|} + 50\,000\,\bar{A}^{02}_{60:\overline{10}|},$$

where P is the total premium in the year. We find $\ddot{a}^{(12)00}_{60:\overline{10}|}$ from

$$\ddot{a}^{(12)00}_{60:\overline{10}|} = \ddot{a}^{(12)00}_{60} - v^{10}\,{}_{10}p^{00}_{60}\,\ddot{a}^{(12)00}_{70} - v^{10}\,{}_{10}p^{01}_{60}\,\ddot{a}^{(12)10}_{70},$$

and using the two-term Woolhouse formula, we have

$$\begin{aligned}
\ddot{a}^{(12)00}_{60:\overline{10}|} &\approx \left(\bar{a}^{00}_{60} + \tfrac{1}{24}\right) - v^{10}\,{}_{10}p^{00}_{60}\left(\bar{a}^{00}_{70} + \tfrac{1}{24}\right) - v^{10}\,{}_{10}p^{01}_{60}\,\bar{a}^{10}_{70} \\
&= \bar{a}^{00}_{60:\overline{10}|} + \tfrac{1}{24}\left(1 - v^{10}\,{}_{10}p^{00}_{60}\right) \\
&\approx 6.5885 + \tfrac{1}{24}(1 - 0.36286) \\
&= 6.6150.
\end{aligned}$$

To find $a^{(12)01}_{60:\overline{10}|}$, we first find $\ddot{a}^{(12)01}_{60:\overline{10}|}$ then make an adjustment to obtain $a^{(12)01}_{60:\overline{10}|}$. Proceeding as above, we have

$$\begin{aligned}
\ddot{a}^{(12)01}_{60:\overline{10}|} &= \ddot{a}^{(12)01}_{60} - v^{10}\,{}_{10}p^{00}_{60}\,\ddot{a}^{(12)01}_{70} - v^{10}\,{}_{10}p^{01}_{60}\,\ddot{a}^{(12)11}_{70} \\
&\approx \bar{a}^{01}_{60} - v^{10}\,{}_{10}p^{00}_{60}\,\bar{a}^{01}_{70} - v^{10}\,{}_{10}p^{01}_{60}\left(\bar{a}^{11}_{70} + \tfrac{1}{24}\right) \\
&= \bar{a}^{01}_{60:\overline{10}|} - \tfrac{1}{24}v^{10}\,{}_{10}p^{01}_{60} \\
&\approx 0.6476 - \tfrac{1}{24}(0.12027) \\
&= 0.6426.
\end{aligned}$$

Then, as ${}_{0}p^{01}_{60} = 0$,

$$a^{(12)01}_{60:\overline{10}|} = \ddot{a}^{(12)01}_{60:\overline{10}|} + \tfrac{1}{12}v^{10}\,{}_{10}p^{01}_{60} = 0.6426 + \tfrac{1}{12}(0.12027) = 0.6526.$$

As the EPV of the death benefit is unchanged from part (a), we find that $P = \$3\,211$.

□

8.8 Policy values

The definition of the time t policy value for a policy modelled using a multiple state model is exactly as in Chapter 7 – it is the expected value at time t of the future loss random variable – with one additional requirement; for a policy described by a multiple state model, the future loss random variable may depend on which state the policyholder is in at that time. For example, in disability income insurance, if a policy is still in force, the policyholder may be in State 0 (Healthy) or State 1 (Sick), and the policy value for a policyholder in State 0 is different from that for a policyholder in State 1. We therefore define state-contingent policy values as follows.

The **State *j* policy value** at time t, denoted ${}_tV^{(j)}$, is the expected value at time t of the future loss random variable for a policy which is in State j at time t.

In other words,

${}_tV^{(j)} =$ EPV at t of future benefits + expenses − EPV at t of future premiums, **given** that the insured is in State j at time t.

State-contingent policy values were not necessary in Chapter 7 since all the policies discussed in that chapter are based on the alive–dead model, for which the in-force policies are all in State 0.

As in Chapter 7, a policy value depends numerically on the basis used in its calculation, that is

(a) the model of transitions between states, which may be in terms of transition intensities or probabilities,
(b) the force of interest,
(c) the assumed expenses.

To establish some ideas we start by considering a particular example using the sickness–death model illustrated in Figure 8.4.

Example 8.8 Consider a 20-year disability income insurance policy issued to a healthy life aged 50. Benefits of \$60 000 per year are payable continuously while the life is sick; a death benefit of \$30 000 is payable immediately on death. Premiums of \$5 000 per year are payable continuously while the life is healthy.

Calculate ${}_{10}V^{(0)}$ and ${}_{10}V^{(1)}$, using the Standard Sickness-Death Table (Appendix D), with interest at 5% per year. Ignore expenses.

Solution 8.8 At time 10 the policyholder is aged 60 and the remaining term of the policy is 10 years. For a policyholder in State 0 at time 10, the EPV of future benefits minus the EPV of future premiums is

$$\begin{aligned}{}_{10}V^{(0)} &= 60\,000\,\bar{a}^{01}_{60:\overline{10}|} + 30\,000\bar{A}^{02}_{60:\overline{10}|} - 5\,000\,\bar{a}^{00}_{60:\overline{10}|} \\ &= 60\,000(0.6476) + 30\,000(0.16382) - 5000(6.5885) = 10\,829.\end{aligned}$$

For a policyholder in State 1 at time 10, we have

$$\begin{aligned}{}_{10}V^{(1)} &= 60\,000\,\bar{a}^{11}_{60:\overline{10}|} + 30\,000\,\bar{A}^{12}_{60:\overline{10}|} - 5\,000\,\bar{a}^{10}_{60:\overline{10}|} \\ &= 60\,000(6.9476) + 30\,000(0.21143) - 5000(0.0670) = 422\,862.\end{aligned}$$

Note that we must allow for the EPV of future premiums in ${}_{10}V^{(1)}$, as there is a possibility that the policyholder will move back into State 0 and resume paying premiums. □

The above example illustrates why the policy value at duration t depends on the state the policyholder is in at that time. If, in this example, the policyholder is in State 0 at time 10, then only a relatively modest policy value is required, as the life is not expected to spend much time in the Sick state over the next 10 years. On the other hand, if the policyholder is in State 1, it is very likely that benefits at the rate of \$60 000 per year will be paid for most of the next 10 years and no future premiums will be received. In this case, a substantial policy value is required.

In the traditional insurance policy values in Chapter 7, we showed that if experience exactly matches assumptions, and if the premium is calculated using the equivalence principle, then the expected value at time t of the premiums received, minus the cost of insurance in the first t years, is equal to the policy value at time t. In the example above, it is clear that this cannot be the case for the policy value in State 1, as it is much greater than the accumulated premiums, which are only \$5 000 per year. However, the premiums are rather more than sufficient to pay for the policy value in State 0. In fact, it can be shown that the EPV at time 0 of the premiums in the first 10 years, minus the EPV of the benefits in the first 10 years, is equal to the EPV at time 0 of the policy value at time 10, taking into consideration the probability of the policy being in State 0 or State 1 (or neither). See Exercise 8.19.

8.8.1 Recursions for state-dependent policy values

We start by recalling the policy value recursion for a traditional, two-state model, from Chapter 7. Consider a whole life policy issued to (x), with sum insured S payable at the end of the hth year of death, and with premium P per year paid in instalments of hP every h years. Ignoring expenses, we have policy values at times $t = 0, h, 2h, \ldots$ related recursively as follows:

$$({}_tV + hP)\,(1+i)^h = {}_hq_{x+t}\,S + {}_hp_{x+t}\,({}_{t+h}V).$$

The intuitive explanation of the recursion is that the left-hand side represents the available funds at the end of the period, if the policy value at the start of the period is held as a reserve, and if interest is earned at the assumed rate i per year.

The right-hand side represents the expected costs at the end of the period; if the policyholder dies, with probability ${}_hq_{x+t}$, then the funds must support the payment of the sum insured, S; on the other hand, if the policyholder survives, with probability ${}_hp_{x+t}$, then the new policy value (or reserve) must be carried forward to the next period. The policy value is determined such that the expected income and outgo in each period are balanced.

In this section, we use the sickness–death model from Figure 8.4 to illustrate the policy value recursion in a multiple state model setting. We construct the recursion using the principle of equating expected income and outgo each period, as we did in the previous section, noting that the policy value at the start of each period represents the funds available from the previous period, and is treated as income, while the policy value at the end of each period represents the cost of continuing the policy, and is treated as outgo.

Suppose an insurer issues a disability income insurance policy to (x), with term n years, and with level premiums of hP payable every h years, at the start of each interval t to $t+h$, provided the policyholder is in State 0 at the payment date. A benefit of hB is payable at the end of every h years provided the policyholder is in State 1 at the payment date. There are no other benefits, and we assume that n is an integer multiple of h.

The equation of value for the premium is

$$P\ddot{a}_{x:\overline{n}|}^{(\frac{1}{h})00} = Ba_{x:\overline{n}|}^{(\frac{1}{h})01}.$$

For cash flow dates, that is, for $t = 0, h, 2h, \ldots, n-h$, let ${}_tV^{(0)}$ denote the policy value given that the policyholder is in State 0 at time t, and let ${}_tV^{(1)}$ denote the policy value given that the policyholder is in State 1 at time t. The policy value at cash flow dates, considered as a prospective value of future outgo minus income, includes the premium payable at time t, but does not include the benefit payable at time t, if any. So we have

$${}_tV^{(0)} = Ba_{x+t:\overline{n-t}|}^{(\frac{1}{h})01} - P\ddot{a}_{x+t:\overline{n-t}|}^{(\frac{1}{h})00}, \tag{8.34}$$

$${}_tV^{(1)} = Ba_{x+t:\overline{n-t}|}^{(\frac{1}{h})11} - P\ddot{a}_{x+t:\overline{n-t}|}^{(\frac{1}{h})10}. \tag{8.35}$$

We now construct the recursions for ${}_tV^{(0)}$ and ${}_tV^{(1)}$ from first principles. For simplicity, we assume net premium policy values (so no expense allowance in premium or policy value), but it is straightforward to incorporate expenses in a gross premium approach.

Suppose the policyholder is in State 0 at time t. In this case, she pays her premium of hP at time t. This is added to the policy value brought forward, and accumulated to time $t+h$, giving the left-hand side of our recursion formula as

$$\left({}_tV^{(0)} + hP\right)(1+i)^h.$$

If the policyholder is in State 1 at time t, then there is no premium payable, and the left-hand side of our recursion formula is

$$\left({}_tV^{(1)}\right)(1+i)^h.$$

At time $t+h$, if the policyholder is in State 0 the insurer will need a policy value of ${}_{t+h}V^{(0)}$ to carry forward to the next time period; if she is in State 1, the insurer will need to pay the benefit, hB, and will also need a policy value of ${}_{t+h}V^{(1)}$ to carry forward. If the policyholder has moved to State 2, there is no payment, and no policy value is required. Applying the appropriate probabilities to the two relevant cases for the end of period states, we have the recursions

$$\left({}_tV^{(0)} + hP\right)(1+i)^h = {}_hp_{x+t}^{00}\left({}_{t+h}V^{(0)}\right) + {}_hp_{x+t}^{01}\left(hB + {}_{t+h}V^{(1)}\right) \tag{8.36}$$

and

$$\left({}_tV^{(1)}\right)(1+i)^h = {}_hp_{x+t}^{10}\left({}_{t+h}V^{(0)}\right) + {}_hp_{x+t}^{11}\left(hB + {}_{t+h}V^{(1)}\right). \tag{8.37}$$

We can prove the recursion formulae (8.36) and (8.37) more formally, starting from equations (8.34) and (8.35).

First, we note that for the sickness–death model, for any $k > h$,

$$ {}_kp_x^{00} = {}_hp_x^{00}\;{}_{k-h}p_{x+h}^{00} + {}_hp_x^{01}\;{}_{k-h}p_{x+h}^{10}. \tag{8.38}$$

Next, we decompose the state-dependent annuity EPVs as follows. First,

$$\begin{aligned}
\ddot{a}_{x:\overline{n}|}^{(\frac{1}{h})00} &= h\left(1 + v^h{}_hp_x^{00} + v^{2h}{}_{2h}p_x^{00} + \cdots + v^{n-h}{}_{n-h}p_x^{00}\right)\\
&= h + v^h{}_hp_x^{00}\,h\left(1 + v^h{}_hp_{x+h}^{00} + v^{2h}{}_{2h}p_{x+h}^{00} + \cdots + v^{n-2h}{}_{n-2h}p_{x+h}^{00}\right)\\
&\quad + v^h{}_hp_x^{01}\,h\left(v^h{}_hp_{x+h}^{10} + v^{2h}{}_{2h}p_{x+h}^{10} + \cdots + v^{n-2h}{}_{n-2h}p_{x+h}^{10}\right).
\end{aligned}$$

That is,

$$\ddot{a}_{x:\overline{n}|}^{(\frac{1}{h})00} = h + v^h{}_hp_x^{00}\,\ddot{a}_{x+h:\overline{n-h}|}^{(\frac{1}{h})00} + v^h{}_hp_x^{01}\,\ddot{a}_{x+h:\overline{n-h}|}^{(\frac{1}{h})10}.$$

Similarly,

$$\begin{aligned}
a_{x:\overline{n}|}^{(\frac{1}{h})01} &= hv^h{}_hp_x^{01} + v^h{}_hp_x^{01}\,a_{x+h:\overline{n-h}|}^{(\frac{1}{h})11} + v^h{}_hp_x^{00}\,a_{x+h:\overline{n-h}|}^{(\frac{1}{h})01},\\
\ddot{a}_{x:\overline{n}|}^{(\frac{1}{h})10} &= hv^h{}_hp_x^{10} + v^h{}_hp_x^{10}\,\ddot{a}_{x+h:\overline{n-h}|}^{(\frac{1}{h})00} + v^h{}_hp_x^{11}\,\ddot{a}_{x+h:\overline{n-h}|}^{(\frac{1}{h})10},\\
a_{x:\overline{n}|}^{(\frac{1}{h})11} &= hv^h{}_hp_x^{11} + v^h{}_hp_x^{11}\,a_{x+h:\overline{n-h}|}^{(\frac{1}{h})11} + v^h{}_hp_x^{10}\,a_{x+h:\overline{n-h}|}^{(\frac{1}{h})01}.
\end{aligned}$$

So we have

$$\begin{aligned}
{}_tV^{(0)} &= Ba_{x+t:\overline{n-t}|}^{(\frac{1}{h})01} - P\ddot{a}_{x+t:\overline{n-t}|}^{(\frac{1}{h})00}\\
&= B\left(hv^h{}_hp_{x+t}^{01} + v^h{}_hp_{x+t}^{01}\,a_{x+t+h:\overline{n-t-h}|}^{(\frac{1}{h})11} + v^h{}_hp_{x+t}^{00}\,a_{x+t+h:\overline{n-t-h}|}^{(\frac{1}{h})01}\right)\\
&\quad - P\left(h + v^h{}_hp_{x+t}^{00}\ddot{a}_{x+t+h:\overline{n-t-h}|}^{(\frac{1}{h})00} + v^h{}_hp_{x+t}^{01}\ddot{a}_{x+t+h:\overline{n-t-h}|}^{(\frac{1}{h})10}\right).
\end{aligned}$$

Multiply both sides by $(1+i)^h$, and collect together terms on the right-hand side in ${}_hp^{01}_{x+t}$ and ${}_hp^{00}_{x+t}$, to give the required result:

$$\begin{aligned}\left({}_tV^{(0)}+hP\right)(1+i)^h &= {}_hp^{01}_{x+t}\left(hB+B\,a^{(\frac{1}{h})11}_{x+t+h:\overline{n-t-h|}}-P\,\ddot{a}^{(\frac{1}{h})10}_{x+t+h:\overline{n-t-h|}}\right)\\ &\quad+{}_hp^{00}_{x+t}\left(B\,a^{(\frac{1}{h})01}_{x+t+h:\overline{n-t-h|}}-P\,\ddot{a}^{(\frac{1}{h})00}_{x+t+h:\overline{n-t-h|}}\right)\\ &={}_hp^{01}_{x+t}\left(hB+{}_{t+h}V^{(1)}\right)+{}_hp^{00}_{x+t}\left({}_{t+h}V^{(0)}\right).\end{aligned}$$

The recursion formula (8.37) can be derived similarly.

8.8.2 General recursion for h-yearly cash flows

Note that, in the recursions in the previous section, the cash flows depend only on the state at the payment dates. Discrete recursions for multiple state-dependent cash flows will not work if the payments at the end of the time period depend on intermediate transitions. For example, consider a model with a death benefit, payable at the end of the month of death, where the amount of benefit depends on whether the life became sick and then died, or died directly from the healthy state. In this model the cash flow at time $t+h$ depends on the intermediate states between times t and $t+h$, not solely on the starting and end states. The discrete recursion approach will not give accurate answers, as intermediate states are not accommodated. The inaccuracy will tend to zero as $h \to 0$, as the probability of intermediate transfers will also tend to zero.

For our general recursion, we assume that payments depend at most on the state at the start and end of the period between cash flows. (A stronger assumption, that is consistent with the recursions, but is stronger than we require, is that at most one transition may occur between cash flow dates.) We also assume, as in the previous section, that cash flows are h-yearly.

For the general recursion, we use the following notation:

- $hP^{(j)}_t$ denotes the amount of premium payable at the start of the interval $(t, t+h)$, conditional on the policyholder being in State j at time t.
- $hB^{(j)}_{t+h}$ denotes the amount of benefit payable at the end of the interval $(t, t+h)$, conditional on the policyholder being in State j at time $t+h$.
- $S^{(jk)}_{t+h}$ denotes a lump sum benefit payable at the end of the interval $(t, t+h)$, conditional on the policyholder being in State j at time t and in State k at time $t+h$.

Then the general net premium policy value recursion for a policy issued to (x), with h-yearly cash flows, and where the policy is in State j at time t, is

$$\left({}_tV^{(j)}+hP^{(j)}_t\right)(1+i)^h=\sum_{k=0}^{m}{}_hp^{jk}_{x+t}\left(hB^{(k)}_{t+h}+S^{(jk)}_{t+h}+{}_{t+h}V^{(k)}\right). \tag{8.39}$$

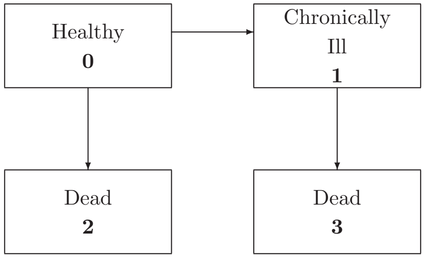


Figure 8.7 Life and chronic illness insurance model

Note that the application of this formula requires policy values at time t for different states, as illustrated in the next example.

Example 8.9 A whole life insurance policy with a **chronic illness rider** is sold to a healthy life aged 50. If the policyholder contracts a chronic illness, the policy pays a lump sum of \$10 000 at the end of the month of diagnosis (if the policyholder is still alive), plus an additional \$1 000 at the end of each subsequent month while the policyholder survives. A benefit of \$40 000 is payable at the end of the month of death if the policyholder dies after a chronic illness diagnosis. The policy also pays \$50 000 at the end of the month of death if the policyholder dies without a chronic illness.

Net monthly premiums of \$79.15 are payable while the policyholder is healthy.

The company uses the model in Figure 8.7 to evaluate premiums and policy values, with an interest rate of 5% per year effective.

You are given the following actuarial functions, at 5% per year interest.

x	$\ddot{a}_x^{(12)00}$	$a_x^{(12)01}$	$a_x^{(12)11}$
70	11.01946	0.60398	6.14524

x	$A_x^{(12)01}$	$A_x^{(12)02}$	$A_x^{(12)03}$	$A_x^{(12)13}$
70	0.246405	0.218444	0.215597	0.696724

(a) Calculate the net premium policy values ${}_{20}V^{(0)}$ and ${}_{20}V^{(1)}$ for this policy.

(b) You are given the following probabilities for a policyholder aged 70:

$$ {}_{\frac{1}{12}}p_{70}^{00} = 0.998866, \quad {}_{\frac{1}{12}}p_{70}^{01} = 0.000552, \quad {}_{\frac{1}{12}}p_{70}^{02} = 0.000582, \quad {}_{\frac{1}{12}}p_{70}^{03} = 0, $$

$$ {}_{\frac{1}{12}}p_{70}^{11} = 0.995489, \quad {}_{\frac{1}{12}}p_{70}^{13} = 0.004511. $$

Use equation (8.39) to calculate the policy values at time $t = 20\frac{1}{12}$.

Solution 8.9 (a) Before writing down the policy values, note that the benefits on transition from State 0 to State 1 are \$10 000 at the end of the month of transition provided the life is still in State 1 at the end of that month, plus \$1 000 at the end of each subsequent month (as long as the life remains in State 1). We can alternatively view these at \$9 000 at the end of the month of transition, plus \$1 000 at the end of each month (including the month of transition), so that the EPV of these benefits is $9\,000A_{70}^{(12)01} + 12\,000a_{70}^{(12)01}$. The policy values for States 0 and 1 at time 20 are

$$\begin{aligned}{}_{20}V^{(0)} &= 12\,000a_{70}^{(12)01} + 9\,000A_{70}^{(12)01} + 40\,000A_{70}^{(12)03} + 50\,000A_{70}^{(12)02} \\ &\quad - 12 \times 79.15\,\ddot{a}_{70}^{(12)00} \\ &= 18\,545,\end{aligned}$$

and ${}_{20}V^{(1)} = 12\,000\,a_{70}^{(12)11} + 40\,000A_{70}^{(12)13} = 101\,612$.

(b) With $h = \frac{1}{12}$ and $P = \$79.15$, equation (8.39) gives

$$\begin{aligned}\left({}_{20}V^{(0)} + P\right)(1+i)^h &= {}_hp_{70}^{00}\left({}_{20+h}V^{(0)}\right) + {}_hp_{70}^{01}\left(10\,000 + {}_{20+h}V^{(1)}\right) \\ &\quad + {}_hp_{70}^{02}\,(50\,000),\end{aligned} \tag{8.40}$$

$$\left({}_{20}V^{(1)}\right)(1+i)^h = {}_hp_{70}^{11}\left(1\,000 + {}_{20+h}V^{(1)}\right) + {}_hp_{70}^{13}\,(40\,000).$$

We solve for ${}_{20\frac{1}{12}}V^{(1)}$ first as

$${}_{20\frac{1}{12}}V^{(1)} = \frac{(101\,612)(1.05^{\frac{1}{12}}) - {}_{\frac{1}{12}}p_{70}^{13}(40\,000) - {}_{\frac{1}{12}}p_{70}^{11}(1\,000)}{{}_{\frac{1}{12}}p_{70}^{11}} = 101\,307$$

and then we use this in equation (8.40) for ${}_{20\frac{1}{12}}V^{(0)}$, giving

$$\begin{aligned}&{}_{20\frac{1}{12}}V^{(0)} \\ &= \frac{(18\,545 + 79.15)(1.05^{\frac{1}{12}}) - {}_{\frac{1}{12}}p_{70}^{01}(10\,000 + 101\,307) - {}_{\frac{1}{12}}p_{70}^{02}(50\,000)}{{}_{\frac{1}{12}}p_{70}^{00}} \\ &= 18\,631.\end{aligned}$$

□

8.8.3 Thiele's differential equation

If benefits and premiums are payable continuously, Thiele's differential equation provides a flexible and powerful tool for calculating policy values.

We first demonstrate how Thiele's differential equations can be developed for multiple state models using an example case, and then we present the more general formula.

Example 8.10 Consider a life insurance policy with a chronic illness rider, as in Example 8.9. Let P denote the total amount of premium per year, payable h-yearly in advance (where $h < 1$) as long as the life is healthy, let B denote the total amount of benefit per year payable h-yearly in arrear while the life is suffering a chronic illness, and let S denote the death benefit, payable at the end of the hth year in which death occurs.

(a) Write down expressions for ${}_tV^{(0)}$ in terms of ${}_{t+h}V^{(0)}$ and ${}_{t+h}V^{(1)}$, and for ${}_tV^{(1)}$ in terms of ${}_{t+h}V^{(1)}$.
(b) By considering the limit as $h \to 0$ of the expressions in part (a), show that

$$\frac{d}{dt}{}_tV^{(0)} = \delta\,{}_tV^{(0)} + P - \mu^{01}_{x+t}\big({}_tV^{(1)} - {}_tV^{(0)}\big) - \mu^{02}_{x+t}\big(S - {}_tV^{(0)}\big)$$

and

$$\frac{d}{dt}{}_tV^{(1)} = \delta\,{}_tV^{(1)} - B - \mu^{13}_{x+t}\big(S - {}_tV^{(1)}\big).$$

Solution 8.10 (a) We can write down an expression for ${}_tV^{(0)}$ for $t = 0, h, 2h, \ldots$ from first principles, as

$$\begin{aligned}({}_tV^{(0)} + hP)e^{\delta h} = {}&{}_hp^{00}_{x+t}\left({}_{t+h}V^{(0)}\right)\\ &+ {}_hp^{01}_{x+t}\left(hB + {}_{t+h}V^{(1)}\right) + ({}_hp^{02}_{x+t} + {}_hp^{03}_{x+t})S.\end{aligned}$$

Similarly, noting that ${}_hp^{12}_{x+t} = 0$, we have

$${}_tV^{(1)}\,e^{\delta h} = {}_hp^{11}_{x+t}\left(hB + {}_{t+h}V^{(1)}\right) + {}_hp^{13}_{x+t}\,S.$$

(b) Consider first ${}_tV^{(0)}$ and write ${}_hp^{00}_{x+t} = 1 - {}_hp^{01}_{x+t} - {}_hp^{02}_{x+t} - {}_hp^{03}_{x+t}$, giving

$$\begin{aligned}({}_tV^{(0)} + hP)e^{\delta h} &= {}_{t+h}V^{(0)}\left(1 - {}_hp^{01}_{x+t} - {}_hp^{02}_{x+t} - {}_hp^{03}_{x+t}\right)\\ &\quad + {}_hp^{01}_{x+t}\left(hB + {}_{t+h}V^{(1)}\right) + \left({}_hp^{02}_{x+t} + {}_hp^{03}_{x+t}\right)S\\ &= {}_{t+h}V^{(0)} + {}_hp^{01}_{x+t}(hB + {}_{t+h}V^{(1)} - {}_{t+h}V^{(0)})\\ &\quad + ({}_hp^{02}_{x+t} + {}_hp^{03}_{x+t})(S - {}_{t+h}V^{(0)}).\end{aligned}$$

Now we replace $e^{\delta h}$ with $1 + \delta h + o(h)$, so that the left-hand side becomes

$$ {}_tV^{(0)} + hP + \delta\, h\, {}_tV^{(0)} + o(h). $$

Note that $h^2 P\delta$ can be included in the $o(h)$ term. On the right-hand side, we replace ${}_hp^{0i}_{x+t}$ with $h\mu^{0i}_{x+t} + o(h)$ for $i = 1, 2$, and we replace ${}_hp^{03}_{x+t}$ with $o(h)$ as multiple transitions are required for a move from State 0 to State 3. Thus

$$ \begin{aligned} {}_tV^{(0)} + hP + \delta h\, {}_tV^{(0)} &= {}_{t+h}V^{(0)} + h\mu^{01}_{x+t}\big(hB + {}_{t+h}V^{(1)} - {}_{t+h}V^{(0)}\big) \\ &\quad + h\mu^{02}_{x+t}\big(S - {}_{t+h}V^{(0)}\big) + o(h). \end{aligned} $$

Rearranging, we obtain

$$ \begin{aligned} \frac{{}_{t+h}V^{(0)} - {}_tV^{(0)}}{h} &= \delta_t V^{(0)} + P - \mu^{01}_{x+t}\big(hB + {}_{t+h}V^{(1)} - {}_{t+h}V^{(0)}\big) \\ &\quad - \mu^{02}_{x+t}\big(S - {}_{t+h}V^{(0)}\big) + \frac{o(h)}{h}, \end{aligned} $$

and finally letting $h \to 0$ gives

$$ \frac{d}{dt}{}_tV^{(0)} = \delta_t V^{(0)} + P - \mu^{01}_{x+t}\big({}_tV^{(1)} - {}_tV^{(0)}\big) - \mu^{02}_{x+t}\big(S - {}_tV^{(0)}\big) $$

as required.

For ${}_tV^{(1)}$, we write ${}_hp^{11}_{x+t} = 1 - {}_hp^{13}_{x+t}$, and proceeding as above, we have

$$ \begin{aligned} {}_tV^{(1)} + \delta h\, {}_tV^{(1)} &= (1 - {}_hp^{13}_{x+t})(hB + {}_{t+h}V^{(1)}) + {}_hp^{13}_{x+t}\, S + o(h) \\ &= hB + {}_{t+h}V^{(1)} + {}_hp^{13}_{x+t}\,(S - hB - {}_{t+h}V^{(1)}) + o(h) \\ &= hB + {}_{t+h}V^{(1)} + h\,\mu^{13}_{x+t}(S - hB - {}_{t+h}V^{(1)}) + o(h). \end{aligned} $$

Rearranging, we obtain

$$ \frac{{}_{t+h}V^{(1)} - {}_tV^{(1)}}{h} = \delta\, {}_tV^{(1)} - B - \mu^{13}_{x+t}(S - hB - {}_{t+h}V^{(1)}) + \frac{o(h)}{h} $$

and letting $h \to 0$ gives

$$ \frac{d}{dt}{}_tV^{(1)} = \delta\, {}_tV^{(1)} - B - \mu^{13}_{x+t}(S - {}_tV^{(1)}). $$

□

The differential equations obtained in Example 8.10 are Thiele's differential equations for that particular insurance product. We next consider a general form of Thiele's differential equation.

Thiele's differential equation in the general multiple state case

Consider an insurance policy issued at age x and with term n years described by a multiple state model with $m+1$ states, labelled $0, 1, 2, \ldots, m$. Let

μ_y^{ij} denote the transition intensity between states i and j at age y, $i \neq j$;
δ_t denote the force of interest per year at time t;
$B_t^{(i)}$ denote the rate of payment of benefit at time t while the policyholder is in State i; and
$S_t^{(ij)}$ denote the lump sum benefit payable instantaneously at time t on transition from State i to State j, where $i \neq j$.

We assume that δ_t, $B_t^{(i)}$ and $S_t^{(ij)}$ are continuous functions of t. Note that premiums are included within this model as negative benefits and expenses can be included as additions to the benefits.

For this very general model, Thiele's differential equation is as follows.

For $i = 0, 1, \ldots, m$ and $0 < t < n$,

$$\boxed{\frac{d}{dt}\,{}_tV^{(i)} = \delta_t\,{}_tV^{(i)} - B_t^{(i)} - \sum_{j=0,\, j\neq i}^{m} \mu_{x+t}^{ij}\left(S_t^{(ij)} + {}_tV^{(j)} - {}_tV^{(i)}\right).} \tag{8.41}$$

Formula (8.41) can be interpreted in exactly the same way as formula (7.11). At time t the policy value for a policy in State i, ${}_tV^{(i)}$, is changing as a result of

- interest being earned at rate $\delta_t\,{}_tV^{(i)}$,
- benefits being paid at rate $B_t^{(i)}$,
- a jump from State i to another state, say j, which occurs with intensity μ_{x+t}^{ij}, causing the following additional changes to the policy value:
  - a decrease of $S_t^{(ij)}$ as the insurer has to pay any lump sum benefit contingent on jumping from State i to State j,
  - a decrease of ${}_tV^{(j)}$ as the insurer has to set up the appropriate policy value in State j, and
  - an increase of ${}_tV^{(i)}$ as this amount is no longer needed.

We can use formula (8.41) to calculate policy values using Euler's method, similarly to the examples in Chapter 7. Note that, for state-dependent policy values, it is often most convenient to use Euler's backward method, rather than the forward method we have used so far. To see why, consider Thiele's differential equations for an n-year disability income insurance policy, using the sickness–death model, with premium P per year payable continuously as long as the policyholder is in State 0, and benefit B per year payable continuously as long as the policyholder is in State 1, and with a benefit of

S payable immediately on death. Assume that the force of interest is δ per year and that there are no expenses. Thiele's differential equations are

$$\frac{d}{dt}{}_tV^{(0)} = \delta\ {}_tV^{(0)} + P - \mu^{01}_{x+t}\big({}_tV^{(1)} - {}_tV^{(0)}\big) - \mu^{02}_{x+t}\big(S - {}_tV^{(0)}\big)$$

and

$$\frac{d}{dt}{}_tV^{(1)} = \delta\ {}_tV^{(1)} - B - \mu^{10}_{x+t}\big({}_tV^{(0)} - {}_tV^{(1)}\big) - \mu^{12}_{x+t}\big(S - {}_tV^{(1)}\big)$$

with boundary conditions

$$\lim_{t\to n-} {}_tV^{(0)} = \lim_{t\to n-} {}_tV^{(1)} = 0.$$

We set ${}_nV^{(0)} = {}_nV^{(1)} = 0$, and work backwards with step size h, say. Using Euler's forward method, we need ${}_tV^{(1)}$ to calculate ${}_tV^{(0)}$, and we need ${}_tV^{(0)}$ to calculate ${}_tV^{(1)}$, and so we need to solve the equations simultaneously. However, if we use the backward method, then we have

$$\begin{aligned}{}_tV^{(0)} = {}_{t+h}V^{(0)} - h\Big(&\delta\ {}_{t+h}V^{(0)} + P - \mu^{01}_{x+t}\big({}_{t+h}V^{(1)} - {}_{t+h}V^{(0)}\big)\\ &- \mu^{02}_{x+t}\big(S - {}_{t+h}V^{(0)}\big)\Big)\end{aligned}$$

and

$$\begin{aligned}{}_tV^{(1)} = {}_{t+h}V^{(1)} - h\Big(&\delta\ {}_{t+h}V^{(1)} - B - \mu^{10}_{x+t}\big({}_{t+h}V^{(0)} - {}_{t+h}V^{(1)}\big)\\ &- \mu^{12}_{x+t}\big(S - {}_{t+h}V^{(1)}\big)\Big).\end{aligned}$$

Now, starting from ${}_nV^{(0)}$ and ${}_nV^{(1)}$, we can directly calculate ${}_{n-h}V^{(0)}$ and ${}_{n-h}V^{(1)}$, and then ${}_{n-2h}V^{(0)}$ and ${}_{n-2h}V^{(1)}$ and so on, much more efficiently than with the forward method.

8.9 Applications of multiple state models in long-term health and disability insurance

In this section we consider more explicitly how we can use multiple state models to perform calculations for some of the individual long-term health-related insurance contracts described in Chapter 1.

8.9.1 Disability income insurance

We can use the sickness–death model, illustrated in Figure 8.4, to calculate premiums, benefits and reserves under disability income insurance. As we have seen in earlier examples, an n-year disability income insurance policy written on a healthy life aged x, with premiums of P per year payable continuously while healthy, and with a benefit of B per year payable continuously while disabled, has equation of value at issue

$$P \int_0^n {}_tp_x^{00}\, e^{-\delta t}\, dt = B \int_0^n {}_tp_x^{01}\, e^{-\delta t}\, dt, \tag{8.42}$$

that is, $P\,\bar{a}^{00}_{x:\overline{n}|} = B\bar{a}^{01}_{x:\overline{n}|}$.

Unfortunately, this neat formulation does not include an allowance for the waiting period between the onset of disability and the payment of benefits, nor does it include an allowance for a maximum term for each period of disability income, both of which are common features, as described in Section 1.7.1. However, using the continuous sojourn annuity does make it quite straightforward to adjust the benefit value both for the waiting period and for the maximum period of benefits.

Recall equation (8.31),

$$\bar{a}_x^{01} = \int_0^\infty e^{-\delta t}\, {}_tp_x^{00}\, \mu_{x+t}^{01}\, \bar{a}_{x+t}^{\overline{11}}\, dt.$$

This shows that we can calculate $\bar{a}_x^{01}$ for the sickness–death model by integrating over each possible transition time into State 1, using the continuous sojourn annuity to value each period of benefit at the time of transition.

Now suppose that we are valuing a disability benefit with a waiting period of w years. That is, once the life becomes sick, she must wait for w years before receiving any benefit from that period of sickness. We can allow for this by subtracting the first w years of annuity from each period of sickness in equation (8.31).

That is, the EPV of a benefit of 1 payable continuously while disabled to a life currently aged x and healthy, with a waiting period of w years, and a policy term of $n > w$ years is

$$\int_0^{n-w} e^{-\delta t}\, {}_tp_x^{00}\, \mu_{x+t}^{01} \left(\bar{a}^{\overline{11}}_{x+t:\overline{n-t}|} - \bar{a}^{\overline{11}}_{x+t:\overline{w}|} \right) dt \tag{8.43}$$

$$= \int_0^n e^{-\delta t}\, {}_tp_x^{00}\, \mu_{x+t}^{01} \left(\bar{a}^{\overline{11}}_{x+t:\overline{n-t}|} - \bar{a}^{\overline{11}}_{x+t:\,\overline{\min(w,n-t)}|} \right) dt. \tag{8.44}$$

Note that the term in parentheses in both formulae is the EPV at time t of the w-year deferred, continuous sojourn sickness annuity starting at time t. The upper limit of integration in (8.43) is $n - w$ as we do not need to consider any sickness periods that start within w years of the end of the term of the contract, because the policy will expire before the waiting period ends. In the second expression, we allow for this in the term of the waiting period annuity. Note here that, for $t > n - w$, $\min(w, n-t) = n - t$ and so the terms in parentheses in (8.44) are identical, making their difference 0.

This approach can be adapted for discrete time payments. For example, suppose that the benefit payments are 1 per year, payable monthly, and that the waiting period, w years, is an integer number of months. (So, for example, in formula (8.45) below, if the waiting period is 3 months, then $w = \frac{3}{12}$ years.) We sum over each month of possible transition from healthy to sick, recalling that the final benefit payment date is time n, as the benefit is payable at the end of each month. The EPV of the benefit payment is then

$$\sum_{k=0}^{12(n-w-\frac{1}{12})} v^{\frac{k+1}{12}} \, {}_{\frac{k}{12}}p_x^{00} \, {}_{\frac{1}{12}}p_{x+\frac{k}{12}}^{01} \left(\ddot{a}^{(12)\,\overline{11}}_{x+\frac{k+1}{12}:\overline{n-\frac{k}{12}|}} - \ddot{a}^{(12)\,\overline{11}}_{x+\frac{k+1}{12}:\overline{w}|} \right). \tag{8.45}$$

Note that the term of the first annuity function in (8.45) is $n - \frac{k}{12}$ as the function is valuing payments made at times $\frac{k+1}{12}, \frac{k+2}{12}, \ldots, n$, that is an annuity payable monthly in advance for $n - \frac{k}{12}$ years from time $\frac{k+1}{12}$. Also, note that, if the life is in State 0 at time $\frac{k}{12}$ and in State 1 at time $\frac{k+1}{12}$, then no sickness benefit is payable at time $\frac{k+1}{12}$ as this time falls within the waiting period.

The rather messy form of (8.45) compared with the continuous version in (8.43) is one reason why we often work in continuous time and then adjust for discrete time payments.

We can adjust equation (8.43) to find the EPV of a unit of sickness benefit under a disability income insurance policy with a maximum payment term for each period of disability of, say, m years after the waiting period. We replace the term of the first continuous sojourn annuity in the equation with an $(m + w)$-year term annuity, unless the transition happens within $m + w$ years of the end of the contract, giving an EPV of

$$\int_0^{n-(m+w)} e^{-\delta t} \, {}_tp_x^{00} \, \mu_{x+t}^{01} \left(\bar{a}^{\overline{11}}_{x+t:\overline{m+w}|} - \bar{a}^{\overline{11}}_{x+t:\overline{w}|} \right) dt$$

$$+ \int_{n-(m+w)}^{n-w} e^{-\delta t} \, {}_tp_x^{00} \, \mu_{x+t}^{01} \left(\bar{a}^{\overline{11}}_{x+t:\overline{n-t}|} - \bar{a}^{\overline{11}}_{x+t:\overline{w}|} \right) dt.$$

The sums and integrals in this section can easily be evaluated numerically.

Example 8.11 A 10-year disability income insurance policy with a disability income benefit of \$40 000 per year payable continuously while the policyholder is disabled, is issued to a healthy life aged 50. Premiums are payable continuously at rate P per year while the policyholder is healthy. Using the Standard Sickness–Death Model, with interest at 5% per year, calculate

(a) the total premium per year if there is no waiting period, and
(b) the total premium per year if the waiting period is (i) one month, (ii) three months, (iii) six months, and (iv) one year.

Note that these calculations require nested numerical integration, for which standard Excel approaches are not very efficient. We have used more advanced programming in our solution.

Solution 8.11 (a) The premium equation of value is

$$P\,\bar{a}^{00}_{50:\overline{10}|} = 40\,000\,\bar{a}^{01}_{50:\overline{10}|},$$

which gives $P = \$1059.80$ as $\bar{a}^{00}_{50:\overline{10}|} = 7.4170$ and $\bar{a}^{01}_{50:\overline{10}|} = 0.1965$.

(b) We use equation (8.43) to value the disability income benefit as

$$40\,000 \int_0^{10-w} e^{-\delta t}\,{}_tp^{00}_{50}\,\mu^{01}_{50+t}\left(\bar{a}^{\overline{11}}_{50+t:\overline{10-t}|} - \bar{a}^{\overline{11}}_{50+t:\overline{w}|}\right) dt,$$

with $w = \frac{1}{12}, \frac{1}{4}, \frac{1}{2}$ and 1. Numerical integration is required to obtain the terms in parentheses, but note that for the Standard Sickness–Death Model an explicit expression for ${}_tp^{\overline{11}}_x$ can be found from

$${}_tp^{\overline{11}}_x = \exp\left\{-\int_0^t \left(\mu^{10}_{x+s} + \mu^{12}_{x+s}\right) ds\right\}.$$

The table below gives values for the EPV at issue of a disability benefit of 1 per year, payable continuously, along with the resulting premiums.

Waiting period (months)	EPV of benefit of 1 per year	Premium, P
1	0.1921	\$1035.76
3	0.1834	\$988.80
6	0.1708	\$921.24
12	0.1477	\$796.63

□

As expected, the numbers in the above exercise show that the premium reduces as the waiting period increases. In general, the amount of the decrease will depend on the benefit level and the age of the policyholder at issue.

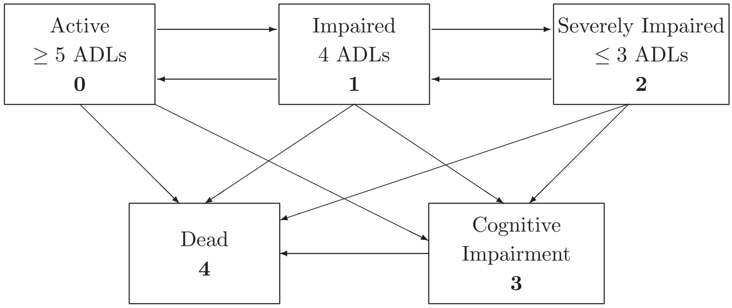


Figure 8.8 Example of an LTC insurance model.

8.9.2 Long-term care

In Section 1.7.2 we described how long-term care (LTC) insurance works in different countries. In this section we illustrate how multiple state models can be used for different types of LTC contracts. We would generally adapt the model depending on whether the policy reimburses the cost of care, or pays a predetermined annuity, possibly with inflation protection.

For a reimbursement policy, the severity of the disability will impact the level of benefit, so there is value in using different states to model different levels of disability. For example, we could use the model illustrated in Figure 8.8, where the number of Activities of Daily Living (ADLs) which a policyholder is able to manage acts as a marker for the expected amount of reimbursement, and we separately model the cognitive impairment state. The model is more complicated than previous models in this chapter, and could be more complicated still, for example, if we allow for recovery from cognitive impairment, or allow for simultaneous loss of more than one ADL. However, the principles presented in this chapter can be used to set out and evaluate all the required probabilities and actuarial functions.

Example 8.12 Write down the Kolmogorov forward equations for all the probabilities for a life aged x, currently in State 2, for the model in Figure 8.8, and give boundary conditions.

Solution 8.12 The equations are

$$\frac{d}{dt}\,{}_tp_x^{20} = {}_tp_x^{21}\,\mu_{x+t}^{10} - {}_tp_x^{20}\left(\mu_{x+t}^{01} + \mu_{x+t}^{03} + \mu_{x+t}^{04}\right),$$

$$\frac{d}{dt}\,{}_tp_x^{21} = {}_tp_x^{20}\,\mu_{x+t}^{01} + {}_tp_x^{22}\,\mu_{x+t}^{21} - {}_tp_x^{21}\left(\mu_{x+t}^{10} + \mu_{x+t}^{12} + \mu_{x+t}^{13} + \mu_{x+t}^{14}\right),$$

$$\frac{d}{dt}\,{}_tp_x^{22} = {}_tp_x^{21}\,\mu_{x+t}^{12} - {}_tp_x^{22}\left(\mu_{x+t}^{21} + \mu_{x+t}^{23} + \mu_{x+t}^{24}\right),$$

$$\frac{d}{dt}\,{}_tp_x^{23} = {}_tp_x^{20}\,\mu_{x+t}^{03} + {}_tp_x^{21}\,\mu_{x+t}^{13} + {}_tp_x^{22}\,\mu_{x+t}^{23} - {}_tp_x^{23}\,\mu_{x+t}^{34},$$

$$\frac{d}{dt}\,{}_tp_x^{24} = {}_tp_x^{20}\,\mu_{x+t}^{04} + {}_tp_x^{21}\,\mu_{x+t}^{14} + {}_tp_x^{22}\,\mu_{x+t}^{24} + {}_tp_x^{23}\,\mu_{x+t}^{34}.$$

For boundary conditions, we have ${}_0p_x^{22} = 1$ and ${}_0p_x^{2j} = 0$ for $j \neq 2$.

As a check, verify that the sum of all the terms on the right-hand sides of these equations is 0. □

Example 8.13 Write down Thiele's differential equation for $\bar{a}_{x+t}^{03}$ for the model in Figure 8.8, assuming a constant force of interest.

Solution 8.13 Thiele's differential equation applies to policy values, but we can use it here because we can view $\bar{a}_{x+t}^{03}$ as the policy value for a single premium annuity contract with benefit of 1 per year payable continuously in State 3, given that the life is currently in State 0. So, in terms of the general version of Thiele's differential equation, (8.41), we have

- ${}_tV^{(0)} = \bar{a}_{x+t}^{03}, \quad {}_tV^{(1)} = \bar{a}_{x+t}^{13}, \quad {}_tV^{(3)} = \bar{a}_{x+t}^{33}$;
- $\delta_t = \delta$, as the force of interest is constant;
- $B^{(3)} = 1$ and $B^{(j)} = 0$ otherwise;
- $S_t^{(ij)} = 0$ for all i and j.

The policy expires when the life moves into State 4, so we have ${}_tV^{(4)} = 0$; also, note that there are no direct transitions between State 0 and State 2. Using this information gives the Thiele differential equation

$$\frac{d}{dt}{}_tV^{(0)} = \delta\,{}_tV^{(0)} - \mu_{x+t}^{01}\left({}_tV^{(1)} - {}_tV^{(0)}\right) - \mu_{x+t}^{03}\left({}_tV^{(3)} - {}_tV^{(0)}\right) - \mu_{x+t}^{04}\left(-{}_tV^{(0)}\right).$$

Now ${}_tV^{(0)} = \bar{a}_{x+t}^{03}$, and similarly ${}_tV^{(1)} = \bar{a}_{x+t}^{13}$ and ${}_tV^{(3)} = \bar{a}_{x+t}^{33}$. We do not include State 2, since it is impossible to move directly into State 2 from State 0. Then

$$\begin{aligned}\frac{d}{dt}\bar{a}_{x+t}^{03} &= \delta\,\bar{a}_{x+t}^{03} - \mu_{x+t}^{01}\left(\bar{a}_{x+t}^{13} - \bar{a}_{x+t}^{03}\right) - \mu_{x+t}^{03}\left(\bar{a}_{x+t}^{33} - \bar{a}_{x+t}^{03}\right) - \mu_{x+t}^{04}\left(-\bar{a}_{x+t}^{03}\right)\\ &= \bar{a}_{x+t}^{03}\left(\delta + \mu_{x+t}^{01} + \mu_{x+t}^{03} + \mu_{x+t}^{04}\right) - \mu_{x+t}^{01}\,\bar{a}_{x+t}^{13} - \mu_{x+t}^{03}\,\bar{a}_{x+t}^{33}.\end{aligned}$$

Using simultaneous differential equations for the different state-dependent annuities, we can solve numerically to determine values for all relevant $\bar{a}_{x+t}^{ij}$. □

Example 8.14 Consider an LTC policy issued to (x). Premiums of P per year are payable continuously while in State 0; benefits payable continuously in States 1, 2 and 3 are assumed to increase geometrically at rate g, convertible continuously, with starting values at inception of $B^{(j)}$ in State $j = 1, 2, 3$.

Write down, and simplify as far as possible, the premium equation of value for the policy.

Solution 8.14 The equation of value is

$$P\bar{a}_x^{00} = \int_0^\infty B^{(1)} e^{gt} \, {}_tp_x^{01} \, e^{-\delta t} \, dt + \int_0^\infty B^{(2)} e^{gt} \, {}_tp_x^{02} \, e^{-\delta t} \, dt + \int_0^\infty B^{(3)} e^{gt} \, {}_tp_x^{03} \, e^{-\delta t} \, dt$$

$$\Rightarrow P\bar{a}_x^{00} = B^{(1)} \, \bar{a}_x^{01}|_{\delta^*} + B^{(2)} \, \bar{a}_x^{02}|_{\delta^*} + B^{(3)} \, \bar{a}_x^{03}|_{\delta^*},$$

where the annuity EPVs on the right-hand side are evaluated at a force of interest $\delta^* = \delta - g$.

8.9.3 Critical illness insurance

Critical illness insurance (CII) was discussed in Section 1.7.3. In Figure 8.9 we show some possible models for CII, where the most suitable version depends on the nature of the benefits. If the CII is a stand-alone policy, that is, with a benefit on CII diagnosis, but with no death benefit, then we could use the model illustrated in Figure 8.9(a). We can use the same model if the CII accelerates the death benefit in full. In both cases the policy expires on the earlier of the CII diagnosis or death. If there is an additional death benefit that is payable in the same amount, whether or not there is a CII diagnosis preceding, then we could use the model illustrated in Figure 8.9(b), because in this case the policy expires on the policyholder's death, but it does not make a difference to the death benefit whether the life dies from State 0 or from State 1. Finally, if the CII partially accelerates the death benefit, then we could use the model in Figure 8.9(c), which separates the case when death occurs without a preceding CII diagnosis and the case when death occurs after a CII diagnosis.

As Figure 8.9(c) is the most general form of the model, it could be used for any of the different CII forms described. The simpler models (a) and (b) in Figure 8.9 cannot be used for the partially accelerated benefit case.

Example 8.15 (a) Using Model 3 in Figure 8.9, write down the equations of value for the premiums for the following CII policies, in terms of the actuarial functions $\bar{A}^{ij}_{x:\overline{n}|}$ and $\bar{a}^{ij}_{x:\overline{n}|}$. Assume in each case that the policy is issued to a healthy life aged 50, that premiums are payable continuously while in State 0, and that all contracts are fully continuous, expiring on the policyholder's 70th birthday.

(i) A stand-alone CII policy with benefit \$20 000 payable immediately on CII diagnosis.
(ii) A combined CII and life insurance policy that pays \$20 000 on CII diagnosis and \$10 000 on death.
(iii) An accelerated death benefit CII policy that pays \$20 000 immediately on the earlier of CII diagnosis and death.

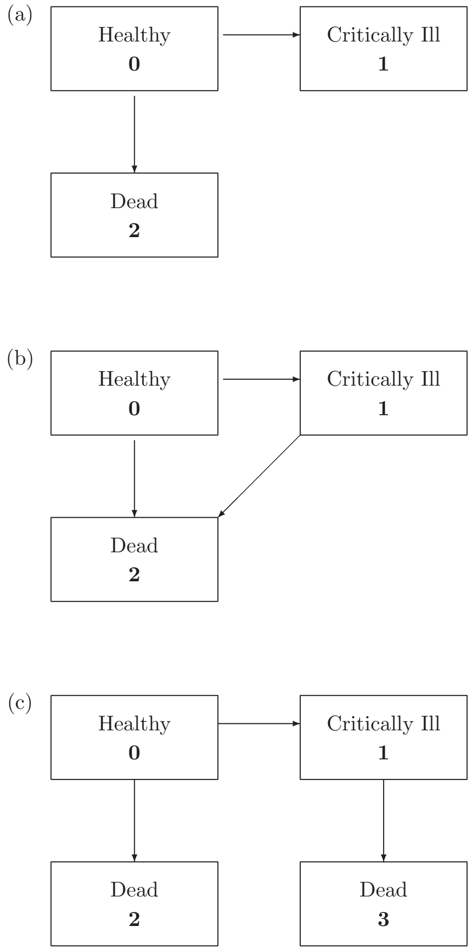


Figure 8.9 CII models: (a) Model 1, (b) Model 2 and (c) Model 3.

(iv) A partly accelerated death benefit policy, which pays \$20 000 on CII diagnosis, and pays \$30 000 if the policyholder dies without a CII claim, or \$10 000 if the policyholder dies after a CII claim.

(b) Use the functions given in the tables below to calculate the annual rate of premium for each of the policies described in part (a). The effective rate of interest is 5% per year.

(c) Use the functions in the tables below to calculate the policy value at time 10 for each of the policies described in part (a), assuming that (i) the life is in State 0 at time 10, and (ii) the life is in State 1 at time 10.

x	$\bar{a}_x^{00}$	$\bar{A}_x^{01}$	$\bar{A}_x^{02}$	$\bar{A}_x^{03}$	$\bar{A}_x^{13}$
50	13.31267	0.22409	0.12667	0.14176	0.34988
60	10.17289	0.34249	0.16140	0.22937	0.47904
70	6.56904	0.49594	0.18317	0.36019	0.62237

t	${}_{20-t}p_{50+t}^{00}$	${}_{20-t}p_{50+t}^{01}$	${}_{20-t}p_{50+t}^{02}$	${}_{20-t}p_{50+t}^{03}$	${}_{20-t}p_{50+t}^{11}$
0	0.68222	0.15034	0.13788	0.02956	0.66485
10	0.75055	0.13135	0.09943	0.01867	0.75283

Solution 8.15 (a) In each case, let P be the total premium per year. Then the equations of value are given below.

(i) $P\,\bar{a}^{00}_{50:\overline{20}|} = 20\,000\,\bar{A}^{01}_{50:\overline{20}|}$.

(ii) $P\,\bar{a}^{00}_{50:\overline{20}|} = 20\,000\,\bar{A}^{01}_{50:\overline{20}|} + 10\,000\left(\bar{A}^{02}_{50:\overline{20}|} + \bar{A}^{03}_{50:\overline{20}|}\right)$.

(iii) $P\,\bar{a}^{00}_{50:\overline{20}|} = 20\,000\left(\bar{A}^{01}_{50:\overline{20}|} + \bar{A}^{02}_{50:\overline{20}|}\right)$.

(iv) $P\,\bar{a}^{00}_{50:\overline{20}|} = 20\,000\,\bar{A}^{01}_{50:\overline{20}|} + 30\,000\,\bar{A}^{02}_{50:\overline{20}|} + 10\,000\,\bar{A}^{03}_{50:\overline{20}|}$.

(b) We need to calculate 20-year actuarial functions, as follows:

$$\bar{a}^{00}_{50:\overline{20}|} = \bar{a}^{00}_{50} - {}_{20}p^{00}_{50}\,v^{20}\,\bar{a}^{00}_{70} = 11.6236,$$
$$\bar{A}^{01}_{50:\overline{20}|} = \bar{A}^{01}_{50} - {}_{20}p^{00}_{50}\,v^{20}\,\bar{A}^{01}_{70} = 0.09657,$$
$$\bar{A}^{02}_{50:\overline{20}|} = \bar{A}^{02}_{50} - {}_{20}p^{00}_{50}\,v^{20}\,\bar{A}^{02}_{70} = 0.07957,$$
$$\bar{A}^{03}_{50:\overline{20}|} = \bar{A}^{03}_{50} - {}_{20}p^{00}_{50}\,v^{20}\,\bar{A}^{03}_{70} - {}_{20}p^{01}_{50}\,v^{20}\,\bar{A}^{13}_{70} = 0.01388.$$

Using the equations and functions above, we have premiums for each case as follows. Since we need the premiums for part (c) below, we use the superscripts to connect the premiums to the different contracts.

(i) $P^{(i)} = \$166.17$.

(ii) $P^{(ii)} = \$246.57$.

(iii) $P^{(iii)} = \$303.08$.

(iv) $P^{(iv)} = \$383.48$.

(c) We need the following actuarial functions:

$$\bar{a}^{00}_{60:\overline{10}|} = \bar{a}^{00}_{60} - {}_{10}p^{00}_{60}\,v^{10}\,\bar{a}^{00}_{70} = 7.14606,$$
$$\bar{A}^{01}_{60:\overline{10}|} = \bar{A}^{01}_{60} - {}_{10}p^{00}_{60}\,v^{10}\,\bar{A}^{01}_{70} = 0.11397,$$
$$\bar{A}^{02}_{60:\overline{10}|} = \bar{A}^{02}_{60} - {}_{10}p^{00}_{60}\,v^{10}\,\bar{A}^{02}_{70} = 0.07700,$$

$$\bar{A}^{03}_{60:\overline{10}|} = \bar{A}^{03}_{60} - {}_{10}p^{00}_{60}\, v^{10}\, \bar{A}^{03}_{70} - {}_{10}p^{01}_{60}\, v^{10}\, \bar{A}^{13}_{70} = 0.01322,$$
$$\bar{A}^{13}_{60:\overline{10}|} = \bar{A}^{13}_{60} - {}_{10}p^{11}_{60}\, v^{10}\, \bar{A}^{13}_{70} = 0.19140.$$

Using these, the policy values are

(i) $${}_{10}V^{(0)} = 20\,000\,\bar{A}^{01}_{60:\overline{10}|} - P^{(i)}\,\bar{a}^{00}_{60:\overline{10}|} = 1092.01,$$
$${}_{10}V^{(1)} = 0 \quad \text{(or undefined, as the policy has expired);}$$

(ii) $${}_{10}V^{(0)} = 20\,000\,\bar{A}^{01}_{60:\overline{10}|} + 10\,000\left(\bar{A}^{02}_{60:\overline{10}|} + \bar{A}^{03}_{60:\overline{10}|}\right) - P^{(ii)}\,\bar{a}^{00}_{60:\overline{10}|}$$
$$= 1419.67,$$
$${}_{10}V^{(1)} = 10\,000\,\bar{A}^{13}_{60:\overline{10}|} = 1914.00;$$

(iii) $${}_{10}V^{(0)} = 20\,000\left(\bar{A}^{01}_{60:\overline{10}|} + \bar{A}^{02}_{60:\overline{10}|}\right) - P^{(iii)}\,\bar{a}^{00}_{60:\overline{10}|}$$
$$= 1653.64,$$
$${}_{10}V^{(1)} = 0 \quad \text{(or undefined, as the policy has expired);}$$

(iv) $${}_{10}V^{(0)} = 20\,000\,\bar{A}^{01}_{60:\overline{10}|} + 30\,000\,\bar{A}^{02}_{60:\overline{10}|} + 10\,000\,\bar{A}^{03}_{60:\overline{10}|} - P^{(iv)}\,\bar{a}^{00}_{60:\overline{10}|}$$
$$= 1981.30,$$
$${}_{10}V^{(1)} = 10\,000\,\bar{A}^{13}_{60:\overline{10}|} = 1914.00.$$

8.9.4 Continuing care retirement communities

Continuing care retirement communities (CCRCs) are described in Section 1.9.1. The key features are that residents move between different parts of the facility, with each section providing different levels of care. The lowest level of care is provided in the Independent Living Units (ILUs); other levels typically include Assisted Living Units (ALUs), Specialized Nursing Facilities (SNFs) and, in some cases, a Memory Care Unit (MCU) that cares for residents with significant cognitive impairment. The model for a CCRC without a MCU would look something like the examples in Figure 8.10. In Figure 8.10(a), the model allows for a simple forward transition from the ILU through the ALU to the SNF. In Figure 8.10(b) the model allows explicitly for short-term stays in the skilled nursing facility (denoted STNF) while in the ILU. This would cover periods of temporary ill-health of residents who may recover sufficiently to return to independent living. Of course, the model could be made more complex by allowing periods of temporary disability from the ALU state, or by allowing for direct transitions from STNF to ALU or to SNF.

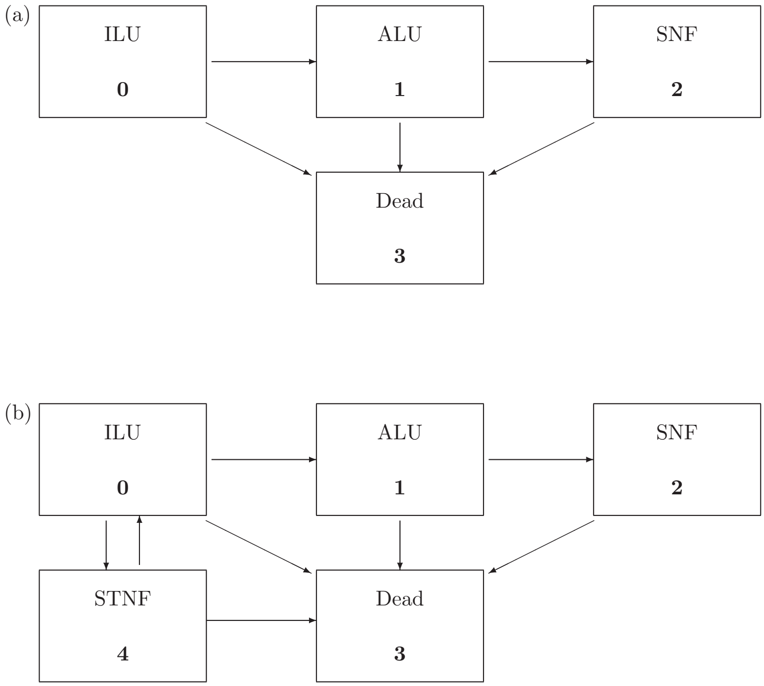


Figure 8.10 CCRC models. (a) The simplified CCRC model; ILU is Independent Living Unit; ALU is Assisted Living Unit; SNF is Specialized Nursing Facility. (b) The extended CCRC model, adding a Short-Term Nursing Facility (STNF) state.

Another possible complication is a joint life version of the model, which would allow for each partner of a couple moving separately through the stages. Multiple state models for joint life benefits are described in Chapter 10.

Some widely used CCRC contract types are described in Chapter 1. For the full life care (also called Type A) and modified life care (also called Type B) contracts, the price is expressed as a combination of entry fee and monthly fees; for Type A contracts the fees do not change when the resident moves between different residence categories. For Type B contracts, the fees increase as residents move through the different categories, but the increases are less than the actual differences in cost, so there is some pre-funding of the costs of the more expensive ALU and SNF facilities. Pay-as-you-go, also called Type C contracts, do not involve pre-funding, and therefore do not need actuarial modelling for costing purposes. Fees for each contract type will be adjusted periodically for inflation.

The allocation of costs between the entry fee and the monthly charge is mainly determined by market forces. Often, residents fund the entry fee by selling their home, and so the CCRC may set the entry fee to be close to the average home price in the area, and then allocate the remaining costs to the monthly fee.

Example 8.16 A CCRC wishes to charge an entry fee of \$200 000 under a full life care (Type A) contract, for lives entering the independent living unit. Subsequently, residents will pay a level monthly fee regardless of the level of care provided. Fees are payable at the start of each month.

The actual monthly costs incurred by the CCRC, including medical care, provision of services, maintenance of buildings and all other expenses and loadings, are as follows:

Independent Living Unit:	\$3 500
Assisted Living Unit:	\$6 000
Specialized Nursing Facility:	\$12 000

(a) The actuarial functions given in Table 8.3 have been calculated using the model in Figure 8.10(a), and an interest rate of 5%. Use these functions to calculate the level monthly fee for entrants age 65, 70, 80 and 90, assuming a \$200 000 entry fee, which is not refunded.
(b) Calculate the revised monthly fees from part (a), assuming 70% of the entry fee is refunded at the end of the month of death.
(c) The CCRC wants to charge a level monthly fee for all residents using the full life care contract, regardless of age at entry. Assume that all residents enter at one of the four ages in part (a), and the proportions of entrants at each age are as in the following table:

Entry age	Proportion
65	5%
70	30%
80	55%
90	10%

Table 8.3 *CCRC actuarial functions at 5% per year interest*

x	$\ddot{a}_x^{(12)00}$	$\ddot{a}_x^{(12)01}$	$\ddot{a}_x^{(12)02}$	$A_x^{(12)03}$
65	11.6416	0.75373	0.24118	0.38472
70	10.0554	0.74720	0.30944	0.45894
80	6.3846	0.69917	0.52113	0.62971
90	2.8414	0.44205	0.82295	0.80005

Calculate a suitable monthly fee which is not age-dependent, assuming (i) no refund of entry fee, and (ii) a 70% refund of entry fee on death.

(d) What are the advantages and disadvantages of offering the refund, compared with the no-refund contract?

Solution 8.16 (a) The EPV at entry of the future costs, for an entrant aged x, is

$$\mathrm{EPV}_x = 12\left(3500\ddot{a}_x^{(12)00} + 6000\ddot{a}_x^{(12)01} + 12000\ddot{a}_x^{(12)02}\right),$$

which gives the following values:

Entry age, x	EPV_x
65	\$577 946
70	\$520 685
80	\$393 536
90	\$269 671

To find the annual fee rate F_x, say, for an entrant aged x, we subtract the entry fee, and divide by the EPV of an annuity of 1 per year, payable monthly, while the life is in State 0, State 1 or State 2. To get the monthly fee, we divide this by 12, giving the fee equation

$$F_x = \frac{\mathrm{EPV}_x - 200\,000}{12\left(\ddot{a}_x^{(12)00} + \ddot{a}_x^{(12)01} + \ddot{a}_x^{(12)02}\right)},$$

resulting in the following monthly fees:

Entry age, x	F_x
65	\$2 492.42
70	\$2 404.93
80	\$2 120.74
90	\$1 413.87

(b) We need to add an extra term to the EPV in part (a) to allow for the refund. The revised EPV for entry age x is

$$\begin{aligned}\mathrm{EPV}_x^r = 12&\left(3\,500\,\ddot{a}_x^{(12)00} + 6\,000\,\ddot{a}_x^{(12)01} + 12\,000\ddot{a}_x^{(12)02}\right)\\ &+ 140\,000\,A_x^{(12)03}.\end{aligned}$$

To get the revised monthly fee, we proceed as in part (a), replacing EPV_x by EPV_x^r in the expression for F_x, giving

Entry age, x	F_x
65	\$2 847.61
70	\$2 886.78
80	\$3 086.78
90	\$3 686.89

(c) Let w_x denote the proportion of entrants aged x, and let F denote the level monthly fee, which is now not age-dependent. Then the EPV of the fee income, per individual entrant, is

$$F \sum_x w_x 12 \left(\ddot{a}_x^{(12)00} + \ddot{a}_x^{(12)01} + \ddot{a}_x^{(12)02} \right)$$

and the EPV of the costs, net of the entry fee, is

$$\sum_x w_x \mathrm{EPV}_x - 200\,000.$$

Equating these gives

$$F = \frac{\sum_x w_x \mathrm{EPV}_x - 200\,000}{\sum_x w_x 12 \left(\ddot{a}_x^{(12)00} + \ddot{a}_x^{(12)01} + \ddot{a}_x^{(12)02} \right)},$$

which results in a flat, non-age-dependent monthly fee of $F = \$2\,224.95$ with no refund, and $F = \$3\,020.01$ with the 70% entry fee refund.

(d) Adding the refund feature would be popular with residents who are concerned about losing much of their capital if the resident dies soon after entering the facility. It ensures a bequest is available for the resident's family.

A disadvantage of introducing the refund is that the monthly fees are higher, which might discourage potential residents who are more constrained by their ability to meet the monthly payments than they are concerned about their bequest. Those who have spare income could replace some or all of their bequest using separate life insurance.

□

8.9.5 Structured settlements

In Chapter 1 we described how Structured Settlements are used to settle legal liability cases between an injured party (IP) and a responsible person, or party (RP), where the settlement involves an annuity payable to the IP. In some cases, particularly for settlements under workers' compensation insurance, the payments may be reviewable. A reviewable structured settlement might cease when the IP recovers sufficiently to return to work, or it might revert

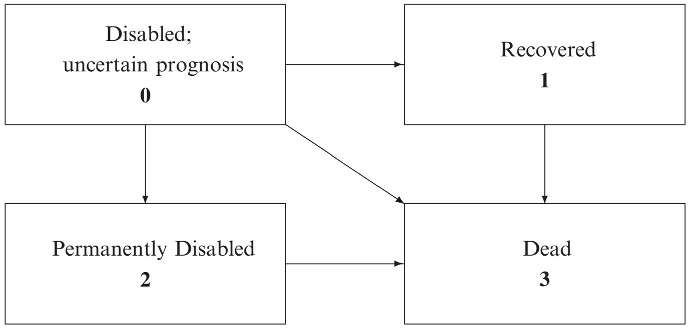


Figure 8.11 Model for workers' compensation benefits.

to a fixed structured settlement once the long-term prognosis for the IP is more certain, the time of 'maximum medical improvement'. This situation could be modelled using the multiple state model illustrated in Figure 8.11. An individual in State 0 has had an injury, but it is not yet known whether the disability will be permanent or temporary. If the individual recovers, they move to State 1. This may be associated with a lump sum back-to-work benefit. If the injury causes permanent impairment affecting income, then the individual moves to State 2, at which time the structured settlement would be implemented.

Example 8.17 A life aged 50 (the IP) has recently suffered a workplace injury which is covered by a workers' compensation insurance policy. The insurer uses the model illustrated in Figure 8.11 to value the benefits. You are given the following information.

- The IP is currently in State 0
- $i = 0.04$
- The insurer calculates the following probabilities and annuity EPVs (at 4%):

x	$\bar{a}_x^{00}$	$\bar{a}_x^{01}$	$\bar{a}_x^{02}$	$\bar{a}_x^{11}$	$\bar{a}_x^{22}$	${}_1p_x^{00}$	${}_1p_x^{01}$	${}_1p_x^{02}$
50	0.5585	5.2245	7.0033	18.8441	10.4798	0.17356	0.23578	0.54813
51	0.5585	5.1550	6.9646	18.6011	10.4250	0.17354	0.23575	0.54806

(a) The insurer considers a reviewable annuity of \$150 000 per year, payable continuously, which would cease when the IP dies or recovers. Alternatively, the insurer may pay a non-reviewable whole life annuity of \$100 000 per year, payable continuously.

Calculate the EPV of each of these two annuity options.

(b) (i) Assume the settlement uses the reviewable annuity option. Calculate the expected value at time 0 of the total policy value at time 1.
(ii) Assume the settlement uses the non-reviewable annuity option. Calculate the expected value at time 0 of the total policy value at time 1.

Solution 8.17 (a) The EPV of the non-reviewable annuity is

$$\begin{aligned}100\,000\left(\bar{a}_{50}^{00}+\bar{a}_{50}^{01}+\bar{a}_{50}^{02}\right) &= 100\,000\,(0.5585+5.2245+7.0033)\\ &= 1\,278\,630,\end{aligned}$$

and the EPV of the reviewable annuity is

$$\begin{aligned}150\,000\left(\bar{a}_{50}^{00}+\bar{a}_{50}^{02}\right) &= 150\,000.(0.5585+7.0033)\\ &= 1\,134\,270.\end{aligned}$$

(b) (i) First, we determine the state-dependent policy values as

$$\begin{aligned}{}_1V^{(0)} &= 150\,000\left(\bar{a}_{51}^{00}+\bar{a}_{51}^{02}\right) = 1\,128\,465,\\ {}_1V^{(1)} &= 0,\\ {}_1V^{(2)} &= 150\,000\,\bar{a}_{51}^{22} = 1\,563\,750.\end{aligned}$$

Then the EPV of the time-1 policy value at time 0 is

$$v\left({}_1p_{50}^{00}\;{}_1V^{(0)}+{}_1p_{50}^{02}\;{}_1V^{(2)}\right) = 1\,012\,495.$$

(ii) The policy values are now

$$\begin{aligned}{}_1V^{(0)} &= 100\,000\left(\bar{a}_{51}^{00}+\bar{a}_{51}^{01}+\bar{a}_{51}^{02}\right) = 1\,267\,810,\\ {}_1V^{(1)} &= 100\,000\,\bar{a}_{51}^{11} = 1\,860\,110,\\ {}_1V^{(2)} &= 100\,000\,\bar{a}_{51}^{22} = 1\,042\,500.\end{aligned}$$

Then the EPV of the time-1 policy value at time 0 is

$$v\left({}_1p_{50}^{00}\;{}_1V^{(0)}+{}_1p_{50}^{01}\;{}_1V^{(1)}+{}_1p_{50}^{02}\;{}_1V^{(2)}\right) = 1\,182\,734.$$

□

8.10 Markov multiple state models in discrete time

In this section, we derive some methods for working with Markov multiple state models in discrete time. Markov models in discrete time are called **Markov chains**. To be more precise, given a set of states labelled (say) $\{0, 1, \ldots, n\}$, a discrete time stochastic process $\{Y(t),\ t = 0, 1, 2, \ldots\}$ is a Markov chain if for any non-negative integers t and k, and for any State s,

$$\Pr\left[Y(t+k) = s | Y(0), Y(1), \ldots, Y(t)\right] = \Pr\left[Y(t+k) = s | Y(t)\right].$$

This means that (as in the continuous time case), the probability that the process is in any given state at some future date depends only on the current state, not on the history of the process before the current date.

Example 8.18 An insurer uses the Standard Sickness–Death Model for valuing sickness related benefits. One-year probabilities for this model are given in Table 8.4.

(a) Calculate the probability that a healthy life aged 60 is healthy at age 62.
(b) Calculate the probability that a healthy life aged 60 is sick at age 62.
(c) Calculate the probability that a healthy life aged 60 is healthy at age 63.

Solution 8.18 (a) The probability is ${}_2p_{60}^{00}$. We calculate this by considering the two possible states at age 61, that could lead to the event that (60) is healthy at age 62, given that she is healthy at age 60; the states at ages 60, 61 and 62 are either $\{0, 0, 0\}$ or $\{0, 1, 0\}$. So

$${}_2p_{60}^{00} = p_{60}^{00}\, p_{61}^{00} + p_{60}^{01}\, p_{61}^{10} = 0.93814.$$

It is worth noting here that we have no information from the one-year p_x^{00} probabilities about whether the life transitioned out and back into State 0 between integer ages. The table tells us only probabilities associated with end-year states, and should not be interpreted as offering any information about paths between states within each year.

(b) Similarly to part (a), given that the life is healthy at age 60, to be sick at age 62 the life must, either be healthy at age 61, and transition to sick by

Table 8.4 *One-year probabilities for the Standard Sickness–Death Model.*

x	p_x^{00}	p_x^{01}	p_x^{10}	p_x^{11}
60	0.97025	0.01467	0.00313	0.97590
61	0.96686	0.01674	0.00272	0.97449
62	0.96305	0.01911	0.00236	0.97286

age 62, or must be sick at age 61, and sick also at age 62. The probabilities combine for the two-year transition probability

$$_2p_{60}^{01} = p_{60}^{00}\,p_{61}^{01} + p_{60}^{01}\,p_{61}^{11} = 0.03054.$$

(c) The probability required is $_3p_{60}^{00}$. We can follow the argument from part (a), and work out all the possible states for the life at ages 61 and 62, but it is more convenient to work backwards from age 63; if she is to be healthy at age 63, then at age 62 she could be either healthy or sick. So, we can break down the three-year probability as

$$_3p_{60}^{00} = \,_2p_{60}^{00}\,p_{62}^{00} + \,_2p_{60}^{01}\,p_{62}^{10}$$

and since we calculated the two-year probabilities in parts (a) and (b), we have

$$_3p_{60}^{00} = 0.93814 \times 0.96305 + 0.03054 \times 0.00236 = 0.90354.$$

□

Generalizing the approach in part (c), we see that the values for $_tp_{60}^{00}$, $_tp_{60}^{01}$ and $_tp_{60}^{02}$, can be calculated recursively as

$$\begin{aligned}
_{t+1}p_{60}^{00} &= \,_tp_{60}^{00}\;p_{60+t}^{00} + \,_tp_{60}^{01}\;p_{60+t}^{10},\\
_{t+1}p_{60}^{01} &= \,_tp_{60}^{00}\;p_{60+t}^{01} + \,_tp_{60}^{01}\;p_{60+t}^{11},\\
_{t+1}p_{60}^{02} &= \,_tp_{60}^{00}\;p_{60+t}^{02} + \,_tp_{60}^{01}\;p_{60+t}^{12} + \,_tp_{60}^{02}\\
&= 1 - \,_{t+1}p_{60}^{00} - \,_{t+1}p_{60}^{01}.
\end{aligned}$$

Note that these recursions are specific to this model, as they depend on which transitions are possible.

8.10.1 The Chapman–Kolmogorov equations

In this section we generalize the recursions from the previous example for application to any Markov chain. The results are known as the Chapman–Kolmogorov equations for the Markov chain.

Consider a general Markov chain with $m+1$ states, that is, with state space $S = \{0, 1, \ldots, m\}$. For any non-negative integers t and r, and for any states $i, j \in S$ (where i and j could be the same state), we have

$$_{t+r}p_x^{ij} = \sum_{k=0}^{m} \,_tp_x^{ik}\;\,_rp_{x+t}^{kj}. \tag{8.46}$$

The intuition for these equations is exactly as explained in part (c) of Example 8.18. Any transition probability over $t + r$ years (and we use the term transition probability even when the starting and end state are the same) can be broken down into the probability for the first t years, followed by the remaining r years. That is, if $Y(t)$ denotes the state that (x) is in at time t, then the event

$$Y(t+r) = j|\,Y(0) = i$$

can be broken down into the $m+1$ mutually exclusive and exhaustive sub-events

$$[Y(t) = 0|Y(0) = i] \cap [Y(t+r) = j|Y(t) = 0]$$
$$[Y(t) = 1|Y(0) = i] \cap [Y(t+r) = j|Y(t) = 1]$$
$$\ldots$$
$$[Y(t) = m|Y(0) = i] \cap [Y(t+r) = j|Y(t) = m]\,,$$

which means that the event probability

$${}_{t+r}p_x^{ij} = \Pr[Y(t+r) = j|Y(0) = i]$$

can be similarly broken down as

$${}_{t+r}p_x^{ij} = {}_tp_x^{i0}\;{}_rp_{x+t}^{0j} + {}_tp_x^{i1}\;{}_rp_{x+t}^{1j} + \cdots + {}_tp_x^{im}\;{}_rp_{x+t}^{mj}.$$

In most examples of Markov chains in life contingent applications, many of the transitions will not be possible, so that the transition probabilities will be zero. Examples would include ${}_rp_{x+t}^{20}$ in Example 8.18.

8.10.2 Transition matrices

It is often convenient to express the transition probabilities ${}_tp_x^{ij}$, for integer $t \geq 1$ and for a given age x, in matrix form. Consider a multiple state model with $m+1$ states. The one-year transition matrix, P_x, say, is an $(m+1) \times (m+1)$ matrix

$$P_x = \begin{pmatrix} p_x^{00} & p_x^{01} & \cdots & p_x^{0m} \\ \vdots & & \ddots & \vdots \\ p_x^{m0} & p_x^{m1} & \cdots & p_x^{mm} \end{pmatrix}.$$

This is a **stochastic matrix**, meaning that all entries are non-negative, and the sum of entries in each row is 1.

In the three-state model of Example 8.18, the transition matrix of one-year probabilities for (x) (noting, again, that there are no transitions out of State 2) is

$$P_x = \begin{pmatrix} p_x^{00} & p_x^{01} & p_x^{02} \\ p_x^{10} & p_x^{11} & p_x^{12} \\ 0 & 0 & 1 \end{pmatrix}.$$

The reason why this is a useful representation comes from the fact that multiplying matrices generates Chapman–Kolmogorov equations for the system. For example, consider the three-state model from Example 8.18. We have

$$P_x \times P_{x+1}$$
$$= \begin{pmatrix} p_x^{00}\, p_{x+1}^{00} + p_x^{01}\, p_{x+1}^{10} & p_x^{00}\, p_{x+1}^{01} + p_x^{01}\, p_{x+1}^{11} & p_x^{00}\, p_{x+1}^{02} + p_x^{01}\, p_{x+1}^{12} + p_x^{02} \\ p_x^{10}\, p_{x+1}^{00} + p_x^{11}\, p_{x+1}^{10} & p_x^{10}\, p_{x+1}^{01} + p_x^{11}\, p_{x+1}^{11} & p_x^{10}\, p_{x+1}^{02} + p_x^{11}\, p_{x+1}^{12} + p_x^{12} \\ 0 & 0 & 1 \end{pmatrix}.$$

Comparing the individual terms with the Chapman–Kolmogorov equations, with $t = r = 1$, we see that each element in the matrix product is a two-year transition probability. That is, the product of one-year transition matrices at successive ages is a two-year transition matrix:

$$P_x \times P_{x+1} = {}_2P_x = \begin{pmatrix} {}_2p_x^{00} & {}_2p_x^{01} & {}_2p_x^{02} \\ {}_2p_x^{10} & {}_2p_x^{11} & {}_2p_x^{12} \\ {}_2p_x^{20} & {}_2p_x^{21} & {}_2p_x^{22} \end{pmatrix}.$$

Continuing the matrix multiplication through subsequent ages, we can determine the t-year transition matrix for any integer t, and this can be used for projecting expected numbers of claims or for valuing state contingent benefits.

A special case of the transition matrices and Chapman–Kolmogorov equations arises when probabilities for each successive time period are the same. In most examples for life insurance this will not be the case, as probabilities depend on age, which changes each time unit. There are cases however where the overall time scale is sufficiently short that it may be possible to simplify the process such that probabilities do not depend on age. In this case, the transition matrix does not change over successive time units, and the process is said to be **time-homogeneous**. Otherwise, the process is **time-inhomogeneous**.

Example 8.19 Employees in Company OHB transition at the end of every six months between the three states

State 0 – junior management
State 1 – senior management
State 2 – terminated

The probability of moving from junior to senior management is 0.1; the probability of being terminated from junior management is 0.04.

No senior managers are demoted to junior management. The probability of termination for senior managers is 0.08 every six months.

At time $t = 0$ Olivia is a senior manager and Harriet is a junior manager. Calculate the probability that both are terminated within two years. Assume transitions are independent, and ignore mortality or any other modes of exit.

Solution 8.19 We work in time units of six months, which means that there are four time periods.

The transition matrix for each time unit is

$$P = \begin{pmatrix} 0.86 & 0.10 & 0.04 \\ 0.00 & 0.92 & 0.08 \\ 0.00 & 0.00 & 1.00 \end{pmatrix}.$$

As this process is time-homogeneous, the four-period transition matrix is

$$P^4 = \begin{pmatrix} 0.5470 & 0.2823 & 0.1707 \\ 0.0000 & 0.7164 & 0.2836 \\ 0.0000 & 0.0000 & 1.0000 \end{pmatrix}.$$

The probability that Olivia is terminated within two years is ${}_4p^{12} = 0.2836$. The probability that Harriet is terminated within two years is ${}_4p^{02} = 0.1707$. So, the probability that both are terminated (given independence) is $0.2836 \times 0.1707 = 0.0484$. □

8.11 Notes and further reading

The general, continuous time multiple state models of this chapter are known to probabilists as Markov processes with discrete states in continuous time. The processes of interest to actuaries are usually time-inhomogeneous since the transition intensities are functions of time/age. Good references for such processes are Cox and Miller (1977) and Ross *et al.* (1996). Rolski *et al.* (2009) provide a brief treatment of such models within an insurance context.

Andrei Andreyevich Markov (1865–1922) was a Russian mathematician best known for his work in probability theory. Andrei Nikolaevich Kolmogorov (1903–1987) was also a Russian mathematician, who made many fundamental contributions to probability theory and is generally credited with putting probability theory on a sound mathematical basis.

The application of multiple state models to problems in actuarial science goes back at least to Sverdrup (1965). Hoem (1988) provides a very comprehensive treatment of the mathematics of such models. Multiple state models are not only a natural framework for modelling conventional life and health insurance policies, they are also a valuable research tool in actuarial science. See, for example, Macdonald *et al.* (2003) and Ji *et al.* (2012)

Norberg (1995) shows how to calculate the kth moment, $k = 1, 2, 3, \ldots$, for the present value of future cash flows from a very general multiple state model.

In Section 8.4 we remarked that the transition intensities are fundamental quantities which determine everything we need to know about a multiple state model. They are also often the natural quantities to estimate from data. We will discuss this further in Chapter 18. For more details, see, for example, Sverdrup (1965) or Waters (1984).

We can extend multiple state models in various ways. One way is to allow the transition intensities out of a state to depend not only on the individual's current age but also on how long they have been in the current state. This breaks the Markov assumption and the new process is known as a **semi-Markov process**. This could be appropriate for the disability income insurance process (Figure 8.4) where the intensities of recovery and death from the sick state could be assumed to depend on how long the individual had been sick, as well as on current age. Precisely this model has been applied to UK insured lives data. See CMI (1991).

As noted at the end of Chapter 7, there are more sophisticated ways of solving systems of differential equations than Euler's method. Waters and Wilkie (1987) present a method specifically designed for use with multiple state models. For a discussion on how to use mathematical software to tackle the problems discussed in this chapter, see Dickson (2006).

8.12 Exercises

Shorter exercises

Exercise 8.1 You are given the following transition intensities for the permanent disability model (Figure 8.3), for $0 \le t \le 5$:

$$\mu_{x+t}^{01} = 0.02, \quad \mu_{x+t}^{02} = 0.03, \quad \mu_{x+t}^{12} = 0.04.$$

(a) Calculate the probability that a healthy life aged x is still healthy at age $x+5$.
(b) Calculate the probability that a healthy life aged x is still healthy at age $x+5$, given that (x) survives to age $x+5$.

Exercise 8.2 Use the permanent disability model (Figure 8.3) with

$$\mu_{50+t}^{01} = 0.02, \quad \mu_{50+t}^{02} = 0.03, \quad \mu_{50+t}^{12} = 0.11, \qquad 0 \le t \le 15.$$

An insurance policy will pay a benefit only if the life, currently aged 50 and healthy, has been disabled for one full year, before age 65.

Calculate the probability that the benefit is paid.

Exercise 8.3 Use the sickness–death model of Figure 8.4 with

$$\mu_{x+t}^{01} = 0.08, \quad \mu_{x+t}^{02} = 0.04, \quad \mu_{x+t}^{10} = 0.1, \quad \mu_{x+t}^{12} = 0.05, \qquad 0 \le t \le 15.$$

(a) Calculate the probability that a life aged x who is currently in State 1 remains in State 1 for the next 15 years.

(b) Calculate the probability that a life aged x who is currently in State 1 makes exactly one transition to State 0, and then remains in State 0, over the next 15 years.

Exercise 8.4 Consider the permanent disability model in Figure 8.3, and suppose that $\mu_x^{02} = \mu_x^{12}$ for all x.

(a) Show that

$$ {}_tp_x^{00}\ {}_{n-t}p_{x+t}^{11} = \exp\left\{-\int_0^t \mu_{x+s}^{01}\, ds\right\} \exp\left\{-\int_0^n \mu_{x+s}^{02}\, ds\right\}. $$

(b) Hence show that

$$ {}_np_x^{01} = \exp\left\{-\int_0^n \mu_{x+s}^{02}\, ds\right\}\left(1 - \exp\left\{-\int_0^n \mu_{x+s}^{01}\, ds\right\}\right). $$

Exercise 8.5 For a multiple state model with $m+1$ states, labelled 0, 1, ..., m, show that for any non-absorbing State i,

$$ \sum_{j=0}^{m} \bar{a}_x^{ij} = \frac{1}{\delta}. $$

Exercise 8.6 Calculate $\bar{A}^{02}_{79:\overline{1}|}$ using the Standard Sickness–Death Model tables in Appendix D, with an interest rate of 5% per year.

Exercise 8.7 For the sickness–death model, use formula (8.14) for ${}_tp_x^{01}$ and formula (8.22) for $\bar{a}_x^{ij}$ to obtain formula (8.31); i.e. show that

$$ \bar{a}_x^{01} = \int_0^\infty e^{-\delta t}\ {}_tp_x^{00}\ \mu_{x+t}^{01}\ \bar{a}_{x+t}^{\overline{11}}\, dt. $$

Exercise 8.8 An insurer calculates premiums for permanent disability insurance using the model in Figure 8.3. A life aged 60 purchases a policy with a five-year term which provides a benefit of $100 000 on exit from the healthy state.

(a) Write down an expression in terms of transition intensities, probabilities and δ for the EPV of this benefit at force of interest δ per year.
(b) Calculate the EPV of the benefit when $\mu_x^{01} = 0.01$ and $\mu_x^{02} = 0.015$ for $60 \le x \le 65$ and $\delta = 0.05$.

Exercise 8.9 Consider the permanent disability model illustrated in Figure 8.3. An insurer uses this model to price an insurance policy with term two years issued to a life aged 58. The policy provides a benefit of \$100 000 if death occurs from the healthy state, \$75 000 if the policyholder becomes permanently disabled, and \$25 000 if death occurs after permanent disability. Benefits are payable at the end of the year in which a transition takes place, and premiums are payable at the start of each policy year.

Annual transition probabilities are as follows:

x	p_x^{00}	p_x^{01}	p_x^{02}	p_x^{11}	p_x^{12}
58	0.995	0.002	0.003	0.992	0.008
59	0.993	0.003	0.004	0.990	0.010

Calculate the annual premium assuming an effective rate of interest of 5% per year.

Exercise 8.10 An insurer issues a 10-year disability income insurance policy to a healthy life aged 50. Premiums of \$2 400 per year are payable monthly in advance if the life is healthy. Benefits of \$36 000 per year are payable at the end of each month if the life is sick at that time. A death benefit of \$150 000 is payable at the end of the month of death.

The policy value basis is the Standard Sickness–Death Model at 5% per year interest. You are given the following values:

x	${}_{\frac{1}{12}}p_x^{00}$	${}_{\frac{1}{12}}p_x^{01}$	${}_{\frac{1}{12}}p_x^{02}$	${}_{\frac{1}{12}}p_x^{10}$	${}_{\frac{1}{12}}p_x^{11}$	${}_{\frac{1}{12}}p_x^{12}$
$59\frac{10}{12}$	0.99765	0.00115	0.00120	0.00029	0.99803	0.00168
$59\frac{11}{12}$	0.99763	0.00116	0.00121	0.00029	0.99803	0.00168

Calculate ${}_tV^{(0)}$ and ${}_tV^{(1)}$ for $t = 9\frac{11}{12}$ and for $t = 9\frac{10}{12}$.

Exercise 8.11 In a homogeneous Markov model with three states, you are given the following matrix of annual transition probabilities for all x:

$$\begin{pmatrix} p_x^{00} & p_x^{01} & p_x^{02} \\ p_x^{10} & p_x^{11} & p_x^{12} \\ p_x^{20} & p_x^{21} & p_x^{22} \end{pmatrix} = \begin{pmatrix} 0.5 & 0.4 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.0 & 0.0 & 1.0 \end{pmatrix}.$$

At time 0 there are eight independent lives in State 0 and four independent lives in State 1.

(a) Calculate the three-year transition matrix, ${}_3P_x$.
(b) Calculate the expected number of lives in State 2 at the end of three years.
(c) Calculate the probability that exactly two lives, out of the original 12, are in State 2 at the end of three years.

Longer exercises

Exercise 8.12 A CCRC offers three levels of care. The lower two levels are provided to individuals in Independent Living Units (ILUs), and the highest level is provided in a Specialized Nursing Facility (SNF). Transitions are modelled as a Markov multiple state model with transitions and states as shown in Figure 8.12.

(a) Write down the Kolmogorov differential equations for ${}_tp_x^{0j}$ for $j = 0, 1, 2, 3$, for this model, and give the boundary conditions.

(b) Show that

$$\sum_{j=0}^{3} \frac{d}{dt}\, {}_tp_x^{0j} = 0.$$

(c) Explain the result in part (b).

Exercise 8.13 Use the same CCRC model as in Exercise 8.12.

You are given that, for this model,

$$\mu_y^{01} = 0.01y, \quad \mu_y^{03} = Be^{cy}, \quad \mu_y^{10} = 0.05,$$
$$\mu_y^{12} = 0.015, \quad \mu_y^{13} = 2\mu_y^{03}, \quad \mu_y^{23} = 8\mu_y^{03},$$

where $B = 10^{-5}$ and $c = 0.1$. You are also given the following information.

- The care costs are \$12 000 per year in Level 0, and \$25 000 per year in Level 1. Costs in Level 2 are \$40 000 per year up to age 85, and \$45 000 per year after age 85.
- Additional costs of \$10 000 are incurred on transfer between Level 1 and Level 2, and \$15 000 on the death of the resident.

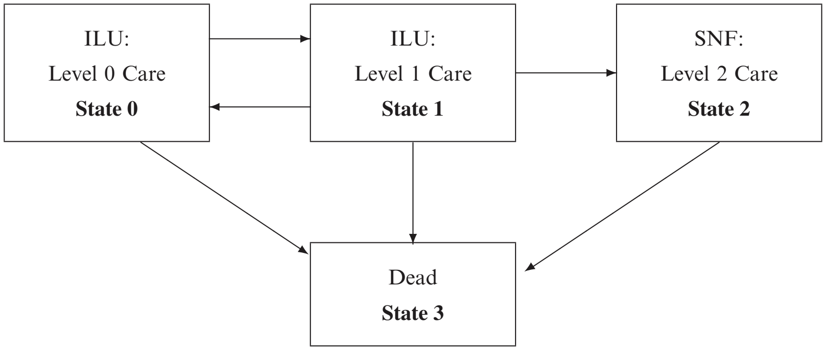


Figure 8.12 CCRC model for Exercises 8.12 and 8.13.

- Fees are set using the equivalence principle, assuming interest at 5% per year. Expenses and costs other than those listed are ignored.
- Service fees are payable continuously by residents under Full Life Care (Type A) contracts, and are \$$F$ per year for all levels.
- Fees payable by residents under Modified Life Care (Type B) contracts are \$$X$ per year in Level 0, \$$2X$ per year in Level 1, and \$$3X$ per year in Level 2.

The following actuarial functions have been calculated for the model, using an effective rate of interest of 5% per year:

x	$\bar{a}_x^{00}$	$\bar{a}_x^{01}$	$\bar{a}_x^{02}$	$\bar{a}_x^{11}$	$\bar{a}_x^{10}$	$\bar{a}_x^{12}$	$\bar{a}_x^{22}$
80	1.3691	4.9471	0.1406	5.7249	0.2982	0.1741	2.7899
85	1.2156	3.7976	0.0775	4.4893	0.2188	0.0981	1.9196

x	$\bar{A}_x^{02}$	$\bar{A}_x^{03}$	$\bar{A}_x^{12}$	$\bar{A}_x^{13}$	$\bar{A}_x^{23}$
80	0.07421	0.68497	0.08587	0.69764	0.86388
85	0.05696	0.75162	0.06734	0.76551	0.90634

x	${}_5p_x^{00}$	${}_5p_x^{01}$	${}_5p_x^{02}$
80	0.04994	0.62570	0.02273

(a) Calculate F for a new entrant aged 80, who is entering Level 0 care under a Type A contract.

(b) Calculate X for a new entrant aged 80 who is entering Level 0 care under a Type B contract.

(c) Assume now that (80) entered the facility under a Type A contract with a fee rate of \$24 350 per year.

   (i) Write down Thiele's differential equations for ${}_tV^{(j)}$, for $j = 0$, 1, and 2.

   (ii) Calculate ${}_5V^{(0)}$, ${}_5V^{(1)}$, and ${}_5V^{(2)}$.

   (iii) Calculate $\frac{d}{dt}{}_tV^{(j)}$ for $t = 5$, and for $j = 0, 1, 2$. Are the reserves increasing or decreasing? Can you explain the results?

Exercise 8.14 An insurer prices disability income insurance using the sickness–death model in Figure 8.4. A 20-year policy issued to (55) provides an income of \$30 000 payable continuously while sick, and a death benefit of \$100 000 immediately on death (from either live state). Premiums are payable continuously at a level rate while the policyholder is healthy. The insurer uses the Standard Sickness–Death Model, with interest at 5% per year.

(a) Calculate the annual net premium.
(b) Calculate the state-dependent net premium policy values at time 10.
(c) The insurer decides to add a return of premium feature to the policy. At the end of the policy term, if the policyholder has made no claims, they receive a 'refund' of one half of the total premium paid.

For valuing the policy with the return of premium feature, the insurer uses a 4-state model to separate healthy lives who have never claimed disability from healthy lives who have made at least one disability claim. It is assumed that the new benefit does not change any of the transition intensities.

You are given that, for the Standard Sickness–Death Model,

$$\mu_x^{01} = A_1 + B_1 e^{c_1 x} \quad \text{and} \quad \mu_x^{02} = A_2 + B_2 e^{c_2 x}$$

where

$$A_1 = 4 \times 10^{-4}, \quad B_1 = 3.47 \times 10^{-6}, \quad c_1 = 0.138,$$

and

$$A_2 = 5 \times 10^{-4}, \quad B_2 = 7.58 \times 10^{-5}, \quad c_2 = 0.087.$$

(i) Sketch the new model and identify the transition intensities, in terms of the original three-state model transitions.
(ii) Calculate the probability that the return of premium benefit is paid for the 20-year policy issued to (55).
(iii) Calculate the revised premium for the policy.
(iv) Calculate the state-dependent net premium policy values at time 10.

(d) Is it reasonable to assume that the return of premium benefit will not affect the transition intensities?

Exercise 8.15 Recall that an **absorbing state** is a state from which no exit is possible. Suppose that State k is an absorbing state in a model with $m+1$ states, labelled $i = 0, 1, \ldots, m$. Prove that for any non-absorbing State i where $i \neq k$,

$$\bar{A}_x^{ik} = 1 - \delta \sum_{j=0, j\neq k}^{m} \bar{a}_x^{ij}.$$

Hint: Use Exercise 8.5 and equation (8.33).

Exercise 8.16 An insurer prices disability income insurance using the sickness–death model in Figure 8.4. The following probabilities apply for a healthy life aged 50:

$$ {}_tp_{50}^{00} = \frac{2}{3} e^{-0.015t} + \frac{1}{3} e^{-0.01t}, $$
$$ {}_tp_{50}^{02} = 1 - e^{-0.01t}. $$

Policyholders pay premiums continuously throughout the policy term while they are healthy, and receive benefits while they are disabled. Policyholders are assumed to be healthy at the issue date.

Using a force of interest of 5% per year, calculate the annual premium rate for a policy with term two years for a life aged 50 that provides a disability income benefit at the rate of \$60 000 per year.

Exercise 8.17 A disability income insurance policy pays B per year, continuously, while the policyholder is sick, and pays a lump sum death benefit of S. The policyholder pays a premium of P per year, continuously, when in the healthy state. The policy is issued to a healthy life aged 40, and ceases at age 60. The insurer prices policies using the model in Figure 8.4.

(a) Write down the Kolmogorov forward equations for ${}_tp_x^{00}$, ${}_tp_x^{01}$, and ${}_tp_x^{02}$.
(b) Explain, in words, the meaning of $\bar{a}_{x:\overline{n}|}^{01}$ and $\bar{A}_{x:\overline{n}|}^{02}$, and write down the integral formulae to calculate them.
(c) Write down a formula in terms of $\bar{a}_{x:\overline{n}|}^{ij}$ and $\bar{A}_{x:\overline{n}|}^{ij}$ functions for the net premium policy value for this disability income insurance policy at duration $t < 20$, given that the life is sick at that time.
(d) You are given the following transition intensities for this model.

t	μ_{40+t}^{01}	μ_{40+t}^{02}	μ_{40+t}^{10}	μ_{40+t}^{12}
0	0.01074	0.00328	0.09012	0.00712
0.1	0.01094	0.00330	0.09003	0.00719
0.2	0.01116	0.00333	0.08994	0.00726
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
19.8	0.54261	0.01700	0.07663	0.05813
19.9	0.55356	0.01714	0.07658	0.05876
20.0	0.56474	0.01730	0.07653	0.05941

(i) Using a time step of $h = 0.1$, estimate ${}_{0.2}p_{40}^{00}$ and ${}_{0.2}p_{40}^{01}$.
(ii) Write down Thiele's differential equations for ${}_tV^{(0)}$ and ${}_tV^{(1)}$, $0 < t < 20$ for this policy.
(iii) You are given that $B = 20\,000$, $S = 100\,000$, $P = 6\,000$ and $\delta = 0.05$. Use the table above to estimate ${}_{19.9}V^{(1)}$ and ${}_{19.9}V^{(0)}$.

Exercise 8.18 Use the Long-Term Care (LTC) multiple state model illustrated in Figure 8.13 for this question.

An insurance policy pays a benefit of \$3000 per year, payable continuously, while the policyholder is in State 2. Policyholders may opt to have benefits start immediately on transition into State 2, or, for a lower premium, may select a six-month waiting period before benefits begin. Benefits are valued at a force

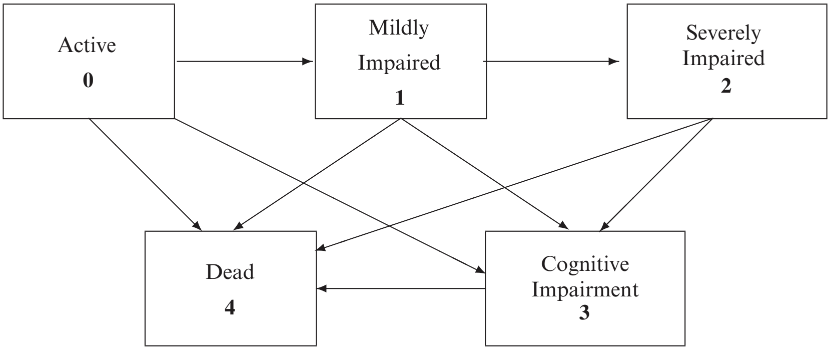


Figure 8.13 LTC insurance model for Exercise 8.18.

of interest of $\delta = 0.05$, and transition intensities for all ages $x \geq 90$ are

$$\mu_x^{12} = 0.10, \qquad \mu_x^{13} = 0.04, \qquad \mu_x^{14} = 0.10,$$
$$\mu_x^{23} = 0.04, \qquad \mu_x^{24} = 0.30, \qquad \mu_x^{34} = 0.20.$$

(a) Derive the Kolmogorov forward differential equation for ${}_tp_x^{12}$.
(b) Show that ${}_tp_{90}^{12} = e^{-0.24t} - e^{-0.34t}$.
(c) Calculate the EPV of the LTC benefit in State 2 for a life currently aged 90 and in State 1, assuming benefits begin immediately on transition (i.e. there is no waiting period).
(d) Show that $\bar{a}^{\overline{22}}_{y:\overline{0.5}|} = 0.45$ to the nearest 0.01, for any $y \geq 90$.
(e) Calculate the EPV of the LTC benefit in State 2 for a life currently aged 90 and in State 1, assuming a waiting period of six months between transition to State 2 and the start of benefit payments.
(f) Suggest two reasons why the insurer uses waiting periods in long-term health insurance products.

Exercise 8.19 Consider an n-year disability income insurance policy issued to (x), with a benefit of B per year payable continuously while the policyholder is sick, and premiums of P per year payable continuously while the policyholder is healthy.

Show that, for $0 < t < n$, the EPV at issue of the net premium policy value at time t is equal to the EPV at issue of the income minus the cost of insurance in the period $(0, t)$.

Exercise 8.20 In Section 8.8.3 Thiele's differential equation for a general multiple state model was stated as

$$\frac{d}{dt}\,{}_tV^{(i)} = \delta_t\,{}_tV^{(i)} - B_t^{(i)} - \sum_{j=0,\,j\neq i}^{m} \mu_{x+t}^{ij}\left(S_t^{(ij)} + {}_tV^{(j)} - {}_tV^{(i)}\right).$$

(a) Let $v(t) = \exp\{-\int_0^t \delta_s\, ds\}$. Explain why

$$
{}_tV^{(i)} = \sum_{j=0,\, j\neq i}^{m} \int_0^{\infty} \frac{v(t+s)}{v(t)} \left(S_{t+s}^{(ij)} + {}_{t+s}V^{(j)}\right) {}_sp_{x+t}^{\overline{ii}}\, \mu_{x+t+s}^{ij}\, ds
$$

$$
+ \int_0^{\infty} \frac{v(t+s)}{v(t)} B_{t+s}^{(i)}\, {}_sp_{x+t}^{\overline{ii}}\, ds.
$$

(b) Using the techniques introduced in Section 7.5.1, differentiate the above expression to obtain Thiele's differential equation.

Excel-based exercises

Exercise 8.21 Consider the accidental death model illustrated in Figure 8.2. For $x \geq 0$, let

$$
\mu_x^{01} = 10^{-5} \quad \text{and} \quad \mu_x^{02} = A + Bc^x
$$

where $A = 5 \times 10^{-4}, B = 7.6 \times 10^{-5}$ and $c = 1.09$.

(a) Calculate
  - (i) ${}_{10}p_{30}^{00}$,
  - (ii) ${}_{10}p_{30}^{01}$, and
  - (iii) ${}_{10}p_{30}^{02}$.

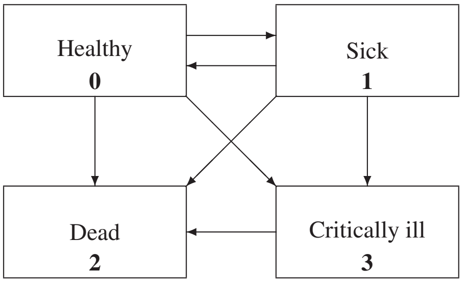
(b) An insurance company uses the model to calculate premiums for a special 10-year term life insurance policy. The basic sum insured is \$100 000, but the death benefit doubles to \$200 000 if death occurs as a result of an accident. The death benefit is payable immediately on death. Premiums are payable continuously throughout the term. For a policy issued to a life aged 30, using an effective rate of interest of 5% per year and ignoring expenses, calculate
  - (i) the annual premium for this policy, and
  - (ii) the policy value at time 5.

Exercise 8.22 An insurer prices critical illness insurance policies using Model 1 from Figure 8.9. For all $x \geq 0$,

$$
\mu_x^{01} = A + Bc^x \quad \text{and} \quad \mu_x^{02} = 0.05\mu_x^{01},
$$

where $A = 0.0001, B = 0.00035$ and $c = 1.075$. On the basis of interest at 4% per year effective, calculate the monthly premium, payable for at most 20 years, for a life aged exactly 30 at the issue date of a policy which provides \$50 000 immediately on death, provided that the critical illness benefit has not already been paid, and \$75 000 immediately on becoming critically ill, should either event occur within 20 years of the policy's issue date. Ignore expenses.

Exercise 8.23 Consider the following model for an insurance policy combining disability income insurance benefits and critical illness benefits.



The transition intensities are as follows:

$$\mu_x^{01} = a_1 + b_1 \exp\{c_1 x\}, \qquad \mu_x^{02} = a_2 + b_2 \exp\{c_2 x\},$$

$$\mu_x^{12} = \mu_x^{02}, \qquad \mu_x^{32} = 1.2\mu_x^{02},$$

$$\mu_x^{10} = 0.1\mu_x^{01}, \qquad \mu_x^{03} = 0.05\mu_x^{01}, \qquad \mu_x^{13} = \mu_x^{03}$$

where

$$a_1 = 4 \times 10^{-4}, \ \ b_1 = 3.5 \times 10^{-6}, \ \ c_1 = 0.14,$$

$$a_2 = 5 \times 10^{-4}, \ \ b_2 = 7.6 \times 10^{-5}, \ \ c_2 = 0.09.$$

(a) Using Euler's method with a step size of $\frac{1}{12}$, calculate values of ${}_tp_{30}^{00}$ for $t = 0, \frac{1}{12}, \frac{2}{12}, \ldots, 35$.

(b) An insurance company issues a policy with term 35 years to a life aged 30 which provides a death benefit, a disability income benefit, and a critical illness benefit as follows:

- a lump sum payment of \$100 000 is payable immediately on the life becoming critically ill,
- a lump sum payment of \$100 000 is payable immediately on death, provided that the life has not already been paid a critical illness benefit,
- a disability income annuity of \$75 000 per year payable whilst the life is disabled payable continuously.

Premiums are payable monthly in advance provided that the policyholder is healthy.

(i) Calculate the monthly premium for this policy on the following basis:
Transition intensities: as in (a)
Interest: 5% per year effective
Expenses: Nil
Use the repeated Simpson's rule with $h = \frac{1}{12}$.

(ii) Suppose that the premium is payable continuously rather than monthly. Use Thiele's differential equation to solve for the total premium per year, using Euler's method with a step size of $h = \frac{1}{12}$.

(iii) Using your answer to part (ii), find the policy value at time 10 for a healthy life.

Exercise 8.24 Parameter values and tables for the Standard Sickness–Death Model are given in Appendix D. Calculate values for $\bar{a}_x^{ij}$ and $\bar{A}_x^{ij}$ for ages x and values of i and j shown in the tables. Use Euler's method with a step size of $h = 0.01$ to solve appropriate versions of Thiele's differential equation; appropriate values of premiums and benefits are typically 0 and 1. (You should not expect to reproduce the values in Appendix D, but your answers should be close, and in some cases they will be identical.)

Answers to selected exercises

8.1 (a) 0.77880 (b) 0.90699

8.2 0.18039

8.3 (a) 0.10540 (b) 0.19967

8.6 0.0749

8.8 (b) \$10 423.69

8.9 \$509.90

8.10 ${}_{9\frac{11}{12}}V^{(0)} = -15.77$, ${}_{9\frac{11}{12}}V^{(1)} = 3232.92$, ${}_{9\frac{10}{12}}V^{(0)} = -29.26$, ${}_{9\frac{10}{12}}V^{(1)} = 6446.37$

8.11 (a) $\begin{pmatrix} 0.253 & 0.396 & 0.351 \\ 0.198 & 0.352 & 0.450 \\ 0.000 & 0.000 & 1.000 \end{pmatrix}$ (b) 4.608 (c) 0.07594

8.13 (a) 24 342.38 (b) 13 450.79
(c)(ii) ${}_5V^{(0)} = 900.05$, ${}_5V^{(1)} = 14\,397.68$, ${}_5V^{(2)} = 53\,234.84$.
(c)(iii) $\frac{d}{dt}\,{}_tV^{(0)}\Big|_{t=5} = 227.68$, $\frac{d}{dt}\,{}_tV^{(1)}\Big|_{t=5} = -64.42$, $\frac{d}{dt}\,{}_tV^{(2)}\Big|_{t=5} = -3\,019.43$

8.14 (a) \$6583.73 (b) ${}_{10}V^{(0)} = 19\,110$, ${}_{10}V^{(1)} = 227\,092$

(c)(ii) 0.288199 (iii) \$7410 (iv) 32 046, 14 320, 227 068

8.16 \$195.99

8.17 (d)(i) 0.99719, 0.00216 (iii) −\$427, \$2 594

8.18 (c) 2652.52 (e) 2182.59

8.21 (a)(i) 0.979122 (ii) 0.020779 (iii) 0.000099
(b)(i) \$206.28 (ii) \$167.15

8.22 \$28.01

8.23 (a) ${}_{35}p_{30}^{00} = 0.581884$
(b)(i) \$206.56 (ii) \$2498.07 (iii) \$16 925.88

9

Multiple decrement models

9.1 Summary

Multiple decrement models are a special class of multiple state models which occur frequently in actuarial applications. Their key feature is that at most one transition can take place from the initial state. Actuaries have a long history of utilizing multiple decrement models, and have developed some specific assumptions, techniques and notation, which are covered in this chapter. In Section 9.2 we give some examples of multiple decrement models, and in Section 9.3 we use results from the previous chapter to obtain probabilities and actuarial functions for such models. In Sections 9.4 and 9.5 we consider multiple decrement tables and their construction. We briefly discuss notation in Section 9.6, then in Section 9.7 we discuss techniques to allow multiple state methods and models to be used when there are exact age transitions, contravening Assumption 8.3 from the previous chapter.

9.2 Examples of multiple decrement models

A multiple decrement model with $m + 1$ states is characterized by having a single initial state, and m exit states. We generically refer to the initial state as the **active state** or State 0. The process makes at most one transition, into one of the exit states. There are no further transitions, so all the exit states are **absorbing states**, and all the probabilities in the model are of the form ${}_tp_x^{0j}$ for $j = 0, 1, \ldots, m$. Typically, we refer to modes of exit as **decrements**.

Figure 9.1 illustrates a general multiple decrement model. We have seen other examples in Chapter 8; the alive–dead model is sometimes called a single decrement model; the accidental death model in Figure 8.2 is an example of a double decrement model, that is, with two modes of exit. In the models proposed for critical illness insurance in Figure 8.9, the first version is a double decrement model, but the other two are not multiple decrement models, as both allow for more than one transition.

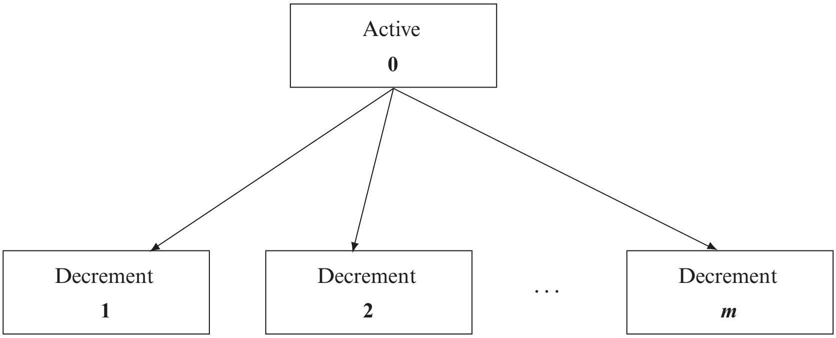


Figure 9.1 A general multiple decrement model.

As the multiple decrement model is a special case of the multiple state model, we have already developed the necessary techniques for developing probabilities and actuarial functions. In the following sections we apply the multiple state results to the multiple decrement framework.

9.3 Actuarial functions for multiple decrement models

We can write down expressions for probabilities in a multiple decrement model using results from the previous chapter. First we note that as no transition into State 0 is possible, ${}_tp_x^{00} = {}_tp_x^{\overline{00}}$, and as formula (8.9) applies to any multiple state model,

$$ {}_tp_x^{00} = {}_tp_x^{\overline{00}} = \exp\left\{ -\int_0^t \sum_{i=1}^m \mu_{x+s}^{0i}\, ds \right\}. $$

For the probability ${}_tp_x^{0j}$, $j = 1, 2, \ldots, m$, the Kolmogorov forward equations (8.16) have the simple form

$$ \frac{d}{dt}\, {}_tp_x^{0j} = {}_tp_x^{00}\, \mu_{x+t}^{0j}. $$

Integrating this gives the intuitively appealing formula

$$ {}_tp_x^{0j} = \int_0^t {}_sp_x^{00}\, \mu_{x+s}^{0j}\, ds. $$

Assuming we know the transition intensities as functions of age, we can calculate ${}_tp_x^{00}$ and ${}_tp_x^{0j}$ using numerical or, in some cases, analytic integration.

Since the only possible transitions are from State 0 to one of the exit states, the only annuity EPVs used are for annuities payable in State 0, (for example, $\bar{a}^{00}_{x:\overline{n}|}$), and the only insurance functions used are those which value a unit benefit payable on transition from State 0 into one of the exit states (for example, $\bar{A}^{01}_{x:\overline{n}|}$).

The following example illustrates a feature which commonly occurs when a multiple decrement model is used. We discuss the general point after completing the example.

Example 9.1 A 10-year term insurance policy is issued to a life aged 50. The sum insured, payable immediately on death, is \$200 000 and premiums are payable continuously at a constant rate throughout the term. No benefit is payable if the policyholder lapses.

Calculate the annual premium rate using the following two sets of assumptions.

(a) The force of interest is 2.5% per year.
The force of mortality is given by $\mu_x = 0.002 + 0.0005(x - 50)$.
No allowance is made for lapses.
No allowance is made for expenses.

(b) The force of interest is 2.5% per year.
The force of mortality is given by $\mu_x = 0.002 + 0.0005(x - 50)$.
The transition intensity for lapses is a constant equal to 0.05.
No allowance is made for expenses.

Solution 9.1 (a) Since lapses are being ignored, an appropriate model for this policy is the alive–dead model shown in Figure 8.1.

The annual premium rate, P, calculated using the equivalence principle, is given by

$$P\,\bar{a}^{00}_{50:\overline{10}|} = 200\,000\,\bar{A}^{01}_{50:\overline{10}|}$$

where

$$\bar{A}^{01}_{50:\overline{10}|} = \int_0^{10} e^{-\delta t}\,{}_tp^{00}_{50}\,\mu^{01}_{50+t}\,dt \quad \text{and} \quad \bar{a}^{00}_{50:\overline{10}|} = \int_0^{10} e^{-\delta t}\,{}_tp^{00}_{50}\,dt\,.$$

Further, ${}_tp^{00}_{50} = \exp\{-0.002t - 0.00025t^2\}$. So we can use numerical integration to calculate the integrals, and we find

$$P = 200\,000 \times 0.03807/8.6961 = \$875.49.$$

(b) To allow for lapses, the model should be as in Figure 9.2. Note that this has the same structure as the accidental death model in Figure 8.2 – a single starting state and two exit states – but with different labels for the states.

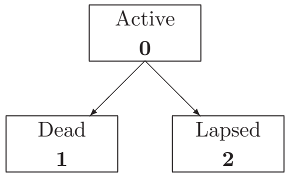


Figure 9.2 The insurance-with-lapses model.

Using this model, the formula for the premium, P, is, again,

$$P = 200\,000 \frac{\bar{A}^{01}_{50:\overline{10}|}}{\bar{a}^{00}_{50:\overline{10}|}}$$

but now ${}_tp^{00}_{50} = \exp\{-0.052t - 0.00025t^2\}$, which gives

$$P = 200\,000 \times 0.02890/6.9269 = \$834.54.$$

□

We make the following observations about Example 9.1.

(1) The premium allowing for lapses is lower than the premium which does not allow for lapses. This was to be expected. The insurer will make a profit from any lapses in this example because, without allowing for lapses, the policy value at any duration is positive and a lapse (with no benefit payable) releases this amount as profit to the insurer. If the insurer allows for lapses, these profits can be used to reduce the premium.

(2) In practice, the insurer may prefer not to allow for lapses when pricing policies if, as in this example, this leads to a higher premium. The decision to lapse is totally at the discretion of the policyholder and depends on many factors, both personal and economic, beyond the control of the insurer. Where lapses are used to reduce the premium, the business is called **lapse supported**. Because lapses are unpredictable, lapse supported pricing is considered somewhat risky and has proved in some cases to be a controversial technique.

(3) Note that two different models were used in the example to calculate a premium for the policy. The choice of model depends on the terms of the policy and on the assumptions made by the insurer.

(4) The two models used in this example are clearly different, but they are connected. The difference is that the model in Figure 9.2 has more exit states; the connections between the models are that the single exit state in Figure 8.1, 'Dead', is one of the exit states in Figure 9.2 *and* the transition intensity into this state, μ^{01}_x, is the same in the two models.

(5) The probability that the policyholder, starting at age 50, 'dies', that is enters State 1, before age $50 + t$ is different for the two models. For the model in Figure 8.1 this is

$$\int_0^t \exp\{-0.002r - 0.00025r^2\}\,(0.002 + 0.0005r)dr,$$

which, over the full 10-year term is ${}_{10}p_{50}^{01} = 0.044$, whereas for the model in Figure 9.2 it is

$$\int_0^t \exp\{-0.052r - 0.00025r^2\}\,(0.002 + 0.0005r)dr,$$

which over the full 10-year term is ${}_{10}p_{50}^{01} = 0.033$.

The explanation for this is that in the second case we are eliminating the policyholders who die after lapsing their policies – that is, we interpret the 'Dead' state here as having died whilst in the active state. If we increase the intensity of the lapse decrement, the probability of dying (from active) decreases, as more lives lapse before they die.

Points (4) and (5) illustrate common features in the application of multiple decrement models. When working with a multiple decrement model we are often interested in a hypothetical simpler model with only one of the exit states and with the same transition intensity into this state. For exit state j, the reduced model, represented in Figure 9.3 is called **the independent single decrement model for decrement** j.

For the multiple decrement model in Figure 9.1, starting in State 0 at age x, the probability of being in State $j \neq 0$ at age $x+t$ is

$${}_tp_x^{0j} = \int_0^t {}_sp_x^{00}\ \mu_{x+s}^{0j}\,ds$$

where (from equation (8.9))

$${}_tp_x^{00} = \exp\left\{ -\int_0^t \sum_{i=1}^m \mu_{x+u}^{0i}\,du \right\}.$$

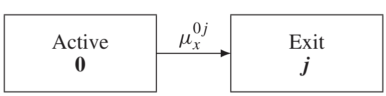


Figure 9.3 Independent single decrement model, decrement j.

For the independent single decrement model in Figure 9.3, we consider only two states (0 and j), but with transition intensity μ_{x+t}^{0j} taken from the full multiple decrement model. We can then use the methods and results of Chapter 2 to derive survival and transition probabilities, denoted by ${}_tp_x^{*(j)}$ and ${}_tq_x^{*(j)}$, respectively, for $j = 1, 2, \ldots, m$, giving

$$ {}_tp_x^{*(j)} = \exp\left\{ -\int_0^t \mu_{x+u}^{0j}\, du \right\} \quad \text{and} \quad {}_tq_x^{*(j)} = \int_0^t {}_sp_x^{*(j)}\, \mu_{x+s}^{0j}\, ds. $$

The probabilities ${}_tp_x^{00}$ and ${}_tp_x^{0j}$ are called the **dependent** probabilities or rates of survival or exit by decrement j, because the values depend on the values of the other transition intensities, μ_{x+s}^{0k} for $0 \leq s \leq t$ and $k \neq j$. The probabilities from the reduced, two state model, ${}_tp_x^{*(j)}$ and ${}_tq_x^{*(j)}$, are called the **independent** probabilities or rates of surviving and exiting for decrement j, because the values are independent of the effect of other transitions. The independent probabilities are the hypothetical probabilities applying if the only way to leave the 'Active' state is by the single decrement j.

In the remainder of this chapter we use terminology that has been used for many years by actuaries. We refer to one-year probabilities as *rates*, so that, for example, p_x^{0j} is the dependent rate of decrement by decrement j at age x. This is consistent with the terminology *mortality rate* which describes a probability rather than a rate. In the previous chapter we noted that the term *transition intensity* was interchangeable with *force of transition*, and the latter terminology is more commonly used for multiple decrement models. So, for example, we refer to μ_x^{0j} as the force of transition at age x.

In many cases, the independent rates are not real quantities – for example, modelling insurance lapses assuming that there are no deaths does not appear to make much sense. However, identifying the hypothetical independent rates can be useful when changing assumptions for other decrements, an issue that we explore further in Section 9.5.

9.4 Multiple decrement tables

It is often convenient to express a multiple decrement model in tabular form, similar to the use of the life table functions l_x and d_x for the alive–dead model. The multiple decrement table can be used to calculate survival probabilities and exit probabilities, by mode of exit, for integer ages and durations. With a fractional age assumption for decrements between integer ages, the multiple decrement table can be used to estimate all survival and exit probabilities for ages within the range of the table. We expand the life table notation of Section 3.2 as follows.

Let l_{x_o} be the radix of the table (an arbitrary positive number) at the initial age x_o. Define

$$l_{x+t} = l_{x_o}\ {}_tp_{x_o}^{00}$$

and for $j = 1, 2, \ldots, m$, and $x \geq x_o$,

$$d_x^{(j)} = l_x\ p_x^{0j}.$$

Given integer age values for l_x and for $d_x^{(j)}$, all integer age and duration probabilities can be calculated.

We can interpret l_x, $x > x_o$, as the expected number of active lives (i.e. in State 0) at age x out of l_{x_o} active lives at age x_o, although as l_{x_o} is an arbitrary starting value, it does not need to be an integer.

Similarly, $d_x^{(j)}$ may be interpreted as the expected number of lives exiting by mode of decrement j in the year of age x to $x+1$, out of l_{x_0} lives in the starting state at age x_o.

Example 9.2 An excerpt from a multiple decrement table for an insurance policy offering benefits on death or diagnosis of critical illness is given in Table 9.1. The insurance expires on the earliest event of death ($j = 1$), surrender ($j = 2$) and critical illness diagnosis ($j = 3$).

(a) Calculate (i) ${}_3p_{45}^{00}$, (ii) p_{40}^{01}, and (iii) ${}_5p_{41}^{03}$.
(b) Calculate the probability that a policy issued to a life aged 45 generates a claim for death or critical illness before age 47.
(c) Calculate the probability that a policy issued to a life aged 40 is surrendered between ages 45 and 47.

Table 9.1 *Excerpt from a critical illness multiple decrement table.*

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$	$d_x^{(3)}$
40	100000	51	4 784	44
41	95121	52	4 526	47
42	90496	53	4 268	50
43	86125	54	4 010	53
44	82008	55	3 753	56
45	78144	56	3 496	59
46	74533	57	3 239	62
47	71175	57	2 983	65
48	68070	58	2 729	67
49	65216	58	2 476	69
50	62613	58	2 226	70

Solution 9.2

(a) (i) ${}_3p_{45}^{00} = l_{48}/l_{45} = 0.87108.$

(ii) $p_{40}^{01} = d_{40}^{(1)}/l_{40} = 0.00051.$

(iii) ${}_5p_{41}^{03} = (d_{41}^{(3)} + d_{42}^{(3)} + \cdots + d_{45}^{(3)})/l_{41} = 0.00279.$

(b) ${}_2p_{45}^{01} + {}_2p_{45}^{03} = (d_{45}^{(1)} + d_{46}^{(1)} + d_{45}^{(3)} + d_{46}^{(3)})/l_{45} = 0.00299.$

(c) ${}_5p_{40}^{00}\,{}_2p_{45}^{02} = (d_{45}^{(2)} + d_{46}^{(2)})/l_{40} = 0.06735.$

□

9.4.1 Fractional age assumptions for decrements

Often, the only information that we have about a multiple decrement model are the integer age values of l_x and $d_x^{(j)}$. To calculate non-integer age or duration probabilities, we need to make an assumption about the decrement probabilities or forces between integer ages.

UDD in the multiple decrement table

Here UDD stands for uniform distribution of decrements. For $0 \le t \le 1$, and integer x, and for each exit mode j, assume that for $j \ne 0$,

$$ {}_tp_x^{0j} = t\,p_x^{0j}. \tag{9.1}$$

The assumption of UDD in the multiple decrement model can be interpreted as assuming that, for each decrement, the exits from the starting state are uniformly spread over each year.

Constant transition forces

For $0 \le t < 1$, and integer x, assume that for each exit mode j, μ_{x+t}^{0j} is a constant for each age x, equal to $\mu^{0j}(x)$, say. Let

$$\mu^{0\bullet}(x) = \sum_{k=1}^{m} \mu^{0k}(x)$$

so that $\mu^{0\bullet}(x)$ represents the total force of transition out of State 0 at age $x+t$ for $0 \le t < 1$, and ${}_tp_x^{00} = e^{-\mu^{0\bullet}(x)t}$. It is convenient also to denote the total exit probability from State 0 for the year of age x to $x+1$ as $p_x^{0\bullet}$. That is

$$p_x^{0\bullet} = 1 - p_x^{00} = \sum_{k=1}^{m} p_x^{0k} = 1 - e^{-\mu^{0\bullet}(x)}.$$

Then for integer x and for $0 \le t < 1$, we have

$$ {}_tp_x^{0j} = \frac{p_x^{0j}}{p_x^{0\bullet}}\left(1 - \left(p_x^{00}\right)^t\right). \tag{9.2}$$

We prove this as follows:

$$
\begin{aligned}
{}_tp_x^{0j} &= \int_0^t {}_rp_x^{00}\, \mu_{x+r}^{0j}\, dr && (9.3)\\
&= \int_0^t e^{-r\mu^{0\bullet}(x)}\, \mu^{0j}(x)\, dr \qquad \text{by the constant force assumption}\\
&= \frac{\mu^{0j}(x)}{\mu^{0\bullet}(x)} \left(1 - e^{-t\mu^{0\bullet}(x)}\right)\\
&= \frac{\mu^{0j}(x)}{\mu^{0\bullet}(x)} \left(1 - \left(p_x^{00}\right)^t\right). && (9.4)
\end{aligned}
$$

Now let $t \to 1$, and rearrange, giving

$$\frac{\mu^{0j}(x)}{\mu^{0\bullet}(x)} = \frac{p_x^{0j}}{p_x^{0\bullet}} \tag{9.5}$$

where the left-hand side is the ratio of the mode j force of exit to the total force of exit, and the right-hand side is the ratio of the mode j probability of exit to the total probability of exit. Substitute from equation (9.5) back into (9.4) to complete the proof.

The intuition here is that the term $1 - \left(p_x^{00}\right)^t$ represents the total probability of exit under the constant transition force assumption, and the term $p_x^{0j}/p_x^{0\bullet}$ divides this exit probability into the different decrements in proportion to the full one-year exit probabilities.

Example 9.3 Calculate ${}_{0.2}p_{50}^{0j}$ for $j = 1, 2, 3$ using the model summarized in Table 9.1, and assuming (a) UDD in all the decrements between integer ages, and (b) constant transition forces in all decrements between integer ages.

Solution 9.3 (a) Under UDD we have ${}_{0.2}p_{50}^{0j} = 0.2\, p_{50}^{0j}$, which gives

$${}_{0.2}p_{50}^{01} = 0.000185, \qquad {}_{0.2}p_{50}^{02} = 0.007110, \qquad {}_{0.2}p_{50}^{03} = 0.000224.$$

(b) Under constant transition forces,

$${}_{0.2}p_{50}^{0j} = \frac{p_{50}^{0j}}{p_{50}^{0\bullet}} \left(1 - (p_{50}^{00})^{0.2}\right),$$

which gives

$${}_{0.2}p_{50}^{01} = 0.000188, \qquad {}_{0.2}p_{50}^{02} = 0.007220, \qquad {}_{0.2}p_{50}^{03} = 0.000227.$$

□

9.5 Constructing a multiple decrement table

Suppose an insurer is using a double decrement table for deaths and lapses to model the liabilities for a lapse-supported product. When a new mortality table is issued, the insurer may want to adjust the dependent rates to allow for the more up-to-date mortality probabilities. However, the mortality table is an independent table – the probabilities are the pure mortality probabilities. In the double decrement table, what we are interested in is the dependent probability that death occurs from the ‘Active’ state – so deaths after lapsation do not count.

The relationship between dependent and independent rates depends on exit patterns between integer ages, for each decrement in the table. For example, suppose we have dependent rates of mortality and withdrawal for some age x in a double decrement table, of $p_x^{01} = 0.01$ and $p_x^{02} = 0.10$ respectively. This means that, given 100 active lives aged x, we expect one life to die from the active state, and 10 lives to withdraw, before age $x + 1$. What does this tell us about the independent rates of lapse or death? Consider the following two extreme cases.

- Suppose we know that withdrawals all happen right at the end of the year. Then the one expected death during the year is not affected by the lapses; if the lapses doubled, we would still have 1 expected death from 100 active lives aged x, so the independent mortality rate is $q_x^{*(1)} = 1/100$.
  Now the 10 expected withdrawals are all coming from the 99 lives that are expected to survive the year, so the independent withdrawal rate would be $q_x^{*(2)} = 10/99$.
- Suppose instead that all the withdrawals occur right at the beginning of the year. Then the independent rate of withdrawal is $q_x^{*(2)} = 10/100$, as there is not time for the mortality rate to affect the withdrawals.
  Following the initial withdrawals, we have 90 expected active lives at age x, of whom one is expected to die before age $x + 1$, so the independent mortality rate is $q_x^{*(1)} = 1/90$.

If we do not have specific information on the timing of exits we use the fractional age assumptions of the previous section to derive the relationships between the dependent and independent probabilities. Note that this methodology relates specifically to the situation where we have a table with probabilities at integer ages only, and so we need to use a fractional age assumption.

9.5.1 Deriving independent rates from dependent rates

UDD in the multiple decrement table

Assume that each decrement is uniformly distributed in the multiple decrement model. Then we know that for integer x, and for $0 \leq t < 1$,

$$ {}_tp_x^{0k} = t\,p_x^{0k}, \quad {}_tp_x^{00} = 1 - t\,p_x^{0\bullet} \quad \text{and} \quad {}_tp_x^{00}\,\mu_{x+t}^{0j} = p_x^{0j}, \tag{9.6} $$

where the last equation is derived exactly analogously to equation (3.9), where we showed that UDD in the alive–dead model implies that ${}_tp_x\,\mu_{x+t} = q_x$ for integer x and for $0 \le t < 1$. The key observation is that the right-hand side of the equation does not depend on t. Then from (9.6) above

$$ \mu_{x+t}^{0j} = \frac{p_x^{0j}}{1 - t\,p_x^{0\bullet}} \tag{9.7} $$

and integrating both sides gives

$$ \int_0^1 \mu_{x+t}^{0j}\,dt = \frac{p_x^{0j}}{p_x^{0\bullet}}\left(-\log(1 - p_x^{0\bullet})\right) = \frac{p_x^{0j}}{p_x^{0\bullet}}\left(-\log p_x^{00}\right) = -\log\left((p_x^{00})^{p_x^{0j}/p_x^{0\bullet}}\right) $$

Note that the decrement j independent survival probability is

$$ p_x^{*(j)} = \exp\left\{-\int_0^1 \mu_{x+t}^{0j}\,dt\right\} = \exp\left\{\log\left((p_x^{00})^{p_x^{0j}/p_x^{0\bullet}}\right)\right\} $$

$$ \Rightarrow p_x^{*(j)} = \left(p_x^{00}\right)^{p_x^{0j}/p_x^{0\bullet}}. \tag{9.8} $$

So, given the table of dependent rates of exit, p_x^{0j}, we can use equation (9.8) to calculate the associated independent rates, under the assumption of UDD in the multiple decrement table.

Example 9.4 Calculate the independent one-year exit probabilities for each decrement for ages 45 to 50, using Table 9.1 above, assuming a uniform distribution of decrements in the multiple decrement model.

Solution 9.4 The results are given in Table 9.2. □

Table 9.2 *Independent rates of exit for the multiple decrement model, Table 9.1, assuming UDD in the multiple decrement table.*

x	$q_x^{*(1)}$	$q_x^{*(2)}$	$q_x^{*(3)}$
45	0.000733	0.044771	0.000773
46	0.000782	0.043493	0.000851
47	0.000819	0.041947	0.000933
48	0.000870	0.040128	0.001005
49	0.000907	0.038004	0.001079
50	0.000944	0.035589	0.001139

You might notice in Table 9.2 that the independent rates are greater than the dependent rates. This is always true, as the effect of exposure to multiple forces of decrement must reduce the probability of exit by each individual mode, compared with the probability when only a single force of exit is present.

Constant forces of transition in the multiple decrement table

Interestingly, the relationship between dependent and independent rates under the constant force fractional age assumption is exactly that in equation (9.8). From Equation (9.5), we have

$$\mu^{0j}(x) = \mu^{0\bullet}(x)\,\frac{p_x^{0j}}{p_x^{0\bullet}}\,,$$

so

$$p_x^{*(j)} = e^{-\mu^{0j}(x)} = \left(e^{-\mu^{0\bullet}(x)}\right)^{p_x^{0j}/p_x^{0\bullet}} = \left(p_x^{00}\right)^{p_x^{0j}/p_x^{0\bullet}}.$$

However, the approximations are not the same for non-integer durations, as we demonstrate in the next example.

Example 9.5 Derive expressions for ${}_tp_x^{*(j)}$ for a double decrement table, for $0 < t < 1$, assuming (a) UDD in the multiple decrement table, and (b) constant forces of transition.

Solution 9.5 (a) We have, under the UDD assumption for ${}_tp_x^{0j}$,

$$\int_0^t \mu_{x+r}^{0j}\,dr = \int_0^t \frac{p_x^{0j}}{1 - r\,p_x^{0\bullet}}\,dr = \frac{p_x^{0j}}{p_x^{0\bullet}}\left(-\log(1 - t\,p_x^{0\bullet})\right)$$
$$= -\log(1 - t\,p_x^{0\bullet})^{p_x^{0j}/p_x^{0\bullet}}.$$

Then

$${}_tp_x^{*(j)} = \exp\left\{-\int_0^t \mu_{x+r}^{0j}\,dr\right\} = \left(1 - t\,p_x^{0\bullet}\right)^{p_x^{0j}/p_x^{0\bullet}} = \left({}_tp_x^{00}\right)^{p_x^{0j}/p_x^{0\bullet}}.$$

(b) Now we have

$${}_tp_x^{*(j)} = \exp\left\{-\int_0^t \mu_{x+r}^{0j}\,dr\right\} = e^{-t\,\mu_x^{0j}} = \left(\left(p_x^{00}\right)^t\right)^{p_x^{0j}/p_x^{0\bullet}} = \left({}_tp_x^{00}\right)^{p_x^{0j}/p_x^{0\bullet}}.$$

In both cases, the final expression is the same, but in calculating ${}_tp_x^{00}$ under the UDD assumption, we use ${}_tp_x^{00} = 1 - t\,p_x^{0\bullet}$ and under the constant force assumption, we use ${}_tp_x^{00} = \left(p_x^{00}\right)^t$. □

Next, we consider how to construct the table of dependent rates given the independent rates.

9.5.2 Deriving dependent rates from independent rates

UDD in the multiple decrement table or constant forces of transition

We can rearrange equation (9.8), which applies to both fractional age assumptions, to give

$$p_x^{0j} = \frac{\log p_x^{*(j)}}{\log p_x^{00}}\, p_x^{0\bullet}. \tag{9.9}$$

In order to apply this, we use the fact that the product of the independent survival probabilities gives the dependent survival probability, that is

$$\prod_{j=1}^{m} {}_tp_x^{*(j)} = \prod_{j=1}^{m} \exp\left\{-\int_0^t \mu_{x+r}^{0j}dr\right\} = \exp\left\{-\int_0^t \sum_{j=1}^{m} \mu_{x+r}^{0j}dr\right\} = {}_tp_x^{00}.$$

So, given the independent survival probabilities, we calculate

$$p_x^{00} = \prod_{j=1}^{m} p_x^{*(j)} \quad \text{and} \quad p_x^{0\bullet} = 1 - p_x^{00}\,,$$

and use (9.9), with the appropriate $p_x^{*(j)}$ to determine the dependent rate, p_x^{0j}.

UDD in the independent models

If we assume a uniform distribution of decrements in each of the independent models, the result is slightly different from the assumption of UDD in the multiple decrement table. That is, if we assume UDD in the independent models, then the transitions in the multiple decrement model are not uniformly distributed.

The UDD assumption in the independent models means that for each decrement j, and for integer x, $0 \le t < 1$,

$${}_tq_x^{*(j)} = t\,q_x^{*(j)} \Rightarrow {}_tp_x^{*(j)}\,\mu_{x+t}^{0j} = q_x^{*(j)}.$$

Then

$$p_x^{0j} = \int_0^1 {}_tp_x^{00}\mu_{x+t}^{0j}dt = \int_0^1 {}_tp_x^{*(1)}\,{}_tp_x^{*(2)}\,\ldots\,{}_tp_x^{*(m)}\,\mu_{x+t}^{0j}\,dt.$$

Extract ${}_tp_x^{*(j)}\,\mu_{x+t}^{0j} = q_x^{*(j)}$ to give

$$p_x^{0j} = q_x^{*(j)} \int_0^1 \prod_{k=1,k\neq j}^{m} {}_tp_x^{*(j)}\,dt$$

$$= q_x^{*(j)} \int_0^1 \prod_{k=1,k\neq j}^{m} \left(1 - t\,q_x^{*(k)})\right) dt.$$

The integrand here is just a polynomial in t, so for example, if there are two decrements, we have

$$p_x^{01} = q_x^{*(1)} \int_0^1 \left(1 - t\,q_x^{*(2)})\right) dt$$

$$= q_x^{*(1)} \left(1 - \frac{1}{2} q_x^{*(2)}\right)$$

and similarly for p_x^{02}.

It is simple to show that, with three decrements, under the assumption of UDD in each of the single decrement models, we have

$$p_x^{01} = q_x^{*(1)} \left(1 - \frac{1}{2}\left(q_x^{*(2)} + q_x^{*(3)}\right) + \frac{1}{3}\,q_x^{*(2)}\,q_x^{*(3)}\right),$$

and similarly for p_x^{02} and p_x^{03}. The proof is left as an exercise.

Generally it makes little difference whether the assumption used is UDD in the multiple decrement model or UDD in the single decrement models. The differences may be noticeable though when the transition forces are changing rapidly between integer ages.

Example 9.6 The insurer using Table 9.1 wishes to revise the underlying assumptions. The independent annual surrender probabilities are to be decreased by 10% and the independent annual critical illness diagnosis probabilities are to be increased by 30%. The independent mortality probabilities are unchanged.

Construct the revised multiple decrement table for ages 45 to 50 assuming UDD in the multiple decrement model and comment on the impact of the changes on the dependent mortality probabilities.

Solution 9.6 This is a straightforward application of equation (9.9), and the results are given in Table 9.3. We note the increase in the mortality ($j = 1$) probabilities, even though the underlying (independent) mortality rates were not changed. This arises because fewer lives are withdrawing, so more lives are expected to die before withdrawal. □

Table 9.3 *Revised multiple decrement table for Example 9.6.*

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$	$d_x^{(3)}$
45	80 021.90	57.47	3 221.64	78.73
46	76 664.06	58.75	2 998.07	83.09
47	73 524.15	59.00	2 772.92	87.48
48	70 604.74	60.28	2 547.17	90.53
49	67 906.76	60.50	2 319.97	93.58
50	65 432.71	60.71	2 093.26	95.27

Table 9.4 *Summary of multiple decrement model notation.*

	AMLCR	USA and Canada	UK and Australia
Dependent survival probability	${}_tp_x^{00}$	${}_tp_x^{(\tau)}$	${}_t(ap)_x$
Dependent transition probability	${}_tp_x^{0j}$	${}_tq_x^{(j)}$	${}_t(aq)_x^j$
Dependent total transition probability	${}_tp_x^{0\bullet}$	${}_tq_x^{(\tau)}$	${}_t(aq)_x$
Independent transition probability	${}_tq_x^{*(j)}$	${}_tq_x^{\prime(j)}$	${}_tq_x^j$
Independent survival probability	${}_tp_x^{*(j)}$	${}_tp_x^{\prime(j)}$	${}_tp_x^j$
Forces of transition	μ_{x+t}^{0j}	$\mu_x^{(j)}(t)$	μ_{x+t}^j
Total force of transition	$\mu_{x+t}^{0\bullet}$	$\mu_x^{(\tau)}(t)$	$(a\mu)_{x+t}$
Multiple Decrement Table:			
Active lives	l_x	$l_x^{(\tau)}$	$(al)_x$
Decrements	$d_x^{(j)}$	$d_x^{(j)}$	$(ad)_x^j$

9.6 Comments on multiple decrement notation

Multiple decrement models have been used by actuaries for many years, but the associated notation is not in the standardized international actuarial notation. We have retained the more general multiple state notation for multiple decrement (dependent) probabilities, although it is unnecessarily unwieldy in this context, since every probability is in the form p_x^{0j}.

The introduction of the reduced single decrement, or independent, models associated with the multiple decrement model is not easily incorporated into our multiple state model notation, which is why we have fallen back on the p and q notation from the alive–dead model.

In Table 9.4 we have summarized the multiple decrement notation that has evolved in North America and in the UK and Australia. In the first column we show the equivalent probabilities in the multiple state notation, which is mainly what we use in this text. We also use the North American notation where it is more convenient (in particular, in double decrement death and surrender models).

For convenience, we list some of the key results from this chapter in both the alternative notation sets:

USA and Canada	UK and Australia
$${}_tp_x^{(\tau)} = \exp\left\{-\int_0^t \mu_x^{(\tau)}(r)\,dr\right\}$$	$${}_t(ap)_x = \exp\left\{-\int_0^t (a\mu)_{x+r}\,dr\right\}$$
$${}_tq_x^{(j)} = \int_0^t {}_rp_x^{(\tau)}\,\mu_x^{(j)}(r)\,dr$$	$${}_t(aq)_x^j = \int_0^t {}_r(ap)_x\,\mu_{x+r}^j\,dr$$
$$q_x^{(\tau)} = \frac{d_x^{(\tau)}}{l_x^{(\tau)}}$$	$$(aq)_x = \frac{(ad)_x}{(al)_x}$$
$$p_x^{\prime(j)} = \left(p_x^{(\tau)}\right)^{q_x^{(j)}/q_x^{(\tau)}}$$	$$p_x^j = \left((ap)_x\right)^{(aq)_x^j/(aq)_x} \qquad (9.10)$$

Note that (9.10) is the relationship between dependent and independent rates assuming constant transition forces, or UDD in the dependent rates.

9.7 Transitions at exact ages

A feature of all the multiple state models considered so far in this chapter, and in Chapter 8, is that transitions take place in continuous time, and the probability of a transition taking place in a time interval of length h converges to 0 as h converges to 0. This follows from Assumption 8.3. In practice, there are situations where this assumption is not realistic. One common example arises in pension plans, when there is a minimum retirement age, leading to a mass of retirements at that age. For example, if employees can retire with an immediate pension at age 60 or above, there may be a significant probability that an individual retires at exact age 60, breaching Assumption 8.3, which requires (loosely) that the probability of exit at any specified instant must be infinitesimal. The situation also arises in relation to surrenders in life insurance, where policyholders might surrender immediately before a premium payment date.

Exact age or duration transitions can be managed by separating the analysis into periods of continuous transition, where Assumption 8.3 applies, and periods of discrete transition, where there are bulk exits from the starting state.

In this section, we demonstrate this with three examples.

Example 9.7 For a group of life insurance policies issued to lives currently aged x, the mortality rate is $q_x = 0.02$, and the insurer expects 5% of survivors at the year end to surrender their policies. Assume mortality is decrement 1 and surrender is decrement 2. Identify

(a) the independent and dependent mortality rates,
(b) the independent and dependent surrender rates, and
(c) p_x^{00}.

Solution 9.7 (a) During the course of the year, the only decrement applying to the group is mortality, so the independent rate of mortality is the same as the dependent rate, i.e. 0.02.

(b) At the year end, 5% of survivors surrender their policies, regardless of the mortality rate, so the independent rate of surrender is 0.05.
The dependent rate of surrender is the probability that a life aged x surrenders their policy before age $x + 1$. For this to happen, the life must survive the year, and then surrender, with probability

$$p_x^{02} = 0.98 \times 0.05 = 0.049.$$

(c) The probability that a life aged x has neither died nor surrendered by age $x + 1$ can be derived from the independent models, by multiplying the independent survival probabilities,

$$p_x^{00} = 0.98 \times 0.95 = 0.931,$$

or from the dependent model, by subtracting the dependent exit probabilities from 1:

$$p_x^{00} = 1 - 0.02 - 0.049 = 0.931.$$

□

Example 9.8 You are given the following excerpt from a multiple decrement table with two modes of decrement.

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$
60	10 000	400	3000
61	6 600	300	2000

Assume that decrement 1 is uniformly distributed over the age-year; for decrement 2 assume that one third of the exits occur three months into the year, and the rest occur at the end of the year.

Calculate the independent rates of decrement at age 60.

Solution 9.8 UDD in the multiple decrement table for decrement 1 implies that, out of l_x active lives age x, we expect $t\,d_x^{(1)}$ to exit by decrement 1 in any interval of t years within the period from age x to $x+1$.

Following the general principle for dealing with exact age exits, we split the year up into

(A) the first 0.25 years, up to just before the first tranche of exits by decrement 2; we indicate this age as 60.25^-,
(B) the exits (by decrement 2) that are assumed to occur at exact age 60.25,
(C) exits by decrement 1 from age 60.25^+, i.e. just after the decrement 2 exits, to age 61^-, just before the second tranche of decrement 2 exits, and
(D) exact age exits (by decrement 2) at the end of the year.

We deal with each portion of the year as follows.

(A) Out of 10 000 active lives aged 60, $0.25 \times 400 = 100$ are expected to leave by decrement 1 between age 60 and age 60.25^-. That means that $l_{60.25^-} = 9900$.
(B) At age 60.25 we expect $3000/3 = 1000$ lives to exit by decrement 2, leaving $l_{60.25^+} = 8900$.
(C) Between age 60.25 and 61^- we expect exits by decrement 1 to be $0.75 \times 400 = 300$. So, just before the second tranche of decrement 2 exits, we have $l_{61^-} = 8600$.
(D) At exact age 61, the remainder of the decrement 2 exits occur; we expect 2000 exits at this time, leaving $l_{61} = 6600$ as required.

Over the first 0.25 years, the only decrement applying is decrement 1, so the independent survival probability for decrement 1, up to the first tranche of decrement 2 exits, is

$$ {}_{0.25}p_{60}^{*(1)} = \frac{9900}{10000} = 0.99. $$

Similarly, once the decrement 2 exits at time 0.25 have left, only decrement 1 applies until just before the end of the year, so that

$$ {}_{0.75}p_{60.25}^{*(1)} = \frac{8600}{8900} = 0.96629. $$

So the independent rate of decrement 1 for the full year is

$$ 1 - {}_{0.25}p_{60}^{*(1)} \times {}_{0.75}p_{60.25}^{*(1)} = 1 - p_{60}^{*(1)} = 1 - 0.95663 = 0.04337. $$

The independent survival probability for decrement 2 over the first 0.25 years is the survival probability if there were no exits from decrement 1 in this time, which is

$$ {}_{0.25}p_{60}^{*(2)} = \frac{8900}{9900} = 0.89899 $$

and similarly, the independent survival probability for decrement 2 over the last 0.75 years is

$$ {}_{0.75}p_{60.25}^{*(2)} = \frac{6600}{8600} = 0.76744. $$

So the independent rate of decrement 2 is

$$ 1 - {}_{0.25}p_{60}^{*(2)} \times {}_{0.75}p_{60.25}^{*(2)} = 1 - p_{60}^{*(2)} = 1 - 0.68992 = 0.31008. $$

As a check, the product of the independent survival probabilities must give the dependent survival probability:

$$ p_{60}^{*(1)}\, p_{60}^{*(2)} = 0.95663 \times 0.68992 = 0.66 $$

as required. □

Example 9.9 Exits from employment are modelled using a three decrement model, as shown in Figure 9.4. You are given the following information on exits between integer ages:

$$ \mu_x^{01} = \begin{cases} 0.05 & x < 60, \\ 0.00 & x \geq 60, \end{cases} $$

$$ \mu_x^{02} = \begin{cases} 0.00 & x < 60, \\ 0.10 & 60 < x < 62, \end{cases} $$

$$ \mu_x^{03} = \begin{cases} 0.003 & 59 \leq x < 60, \\ 0.004 & 60 \leq x < 61, \\ 0.005 & 61 \leq x < 62. \end{cases} $$

In addition to the continuous time exits modelled using the transition intensities above, there are retirements at exact age 60, when 20% of all employees still working retire immediately, and at exact age 62, when all remaining employees retire immediately.

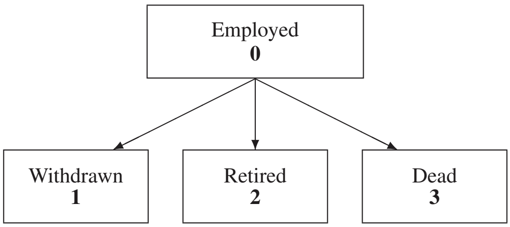


Figure 9.4 Multiple decrement model for Example 9.9.

(a) Construct the multiple decrement table for this model, for integer ages from 59 to 62, using a radix of $l_{59} = 100\,000$.
(b) Calculate the probability that an employee aged 59 will retire before age 62.
(c) Calculate the EPV of a death in service benefit of \$100 000, payable immediately on death, for an employee aged 59. Assume an interest rate of 4% per year, and use (i) exact calculation and (ii) the multiple decrement table from part (a), with claims acceleration – that is, assume all payments between integer durations occur exactly half-way through the year.
(d) Calculate the EPV of a lump sum benefit of \$100 000, payable immediately on retirement, for an employee aged 59. Assume an interest rate of 4% per year, and use claims acceleration as in part (c).

Solution 9.9 (a) We first calculate the transition probabilities, then build up the multiple decrement table, starting from age 59. The probability of survival to age $59 + t$, for $0 < t < 1$, is

$$ {}_tp_{59}^{00} = \exp\left\{ -\int_0^t 0.053\,dt \right\} = e^{-0.053t}. $$

So the probabilities of exit for the year are

$$ \begin{aligned} p_{59}^{01} &= \int_0^1 {}_tp_{59}^{00}\,\mu_{59+t}^{01}\,dt \\ &= \int_0^1 e^{-0.053t}\,0.05\,dt = \frac{0.05(1-e^{-0.053})}{0.053} = 0.04870, \\ p_{59}^{02} &= 0, \\ p_{59}^{03} &= \int_0^1 e^{-0.053t}\,0.003\,dt = 0.00292, \end{aligned} $$

which gives the first row of the multiple decrement table as:

Age	l_x	$d_x^{(1)}$	$d_x^{(2)}$	$d_x^{(3)}$
59	100 000	4 869.8	0	292.2

So, from $l_{59} = 100\,000$ employees aged 59, we expect $4869.8 + 292.2$ to exit before age 60, leaving 94 838 in employment at exact age 60. Of these, 20% are expected to retire immediately, leaving the remainder to continue working beyond age 60. We identify the exact age exits with a special row in the multiple decrement table. Exits between ages 60 and 61, after the exact age transitions, will be denoted as applying to age 60^+. The table becomes:

Age	l_x	$d_x^{(1)}$	$d_x^{(2)}$	$d_x^{(3)}$
59	100 000.0	4 869.8	0.0	292.2
60 exact	94 838.0	0.0	18 967.6	0.0
60^+	75 870.4			

Once the exact age transitions are accounted for, transitions between ages 60^+ and 61 follow using the standard multiple state model formulae. We have ${}_tp_{60^+}^{00} = e^{-0.104t}$ for $0 \le t \le 1$, $p_{60^+}^{01} = 0$,

$$p_{60^+}^{02} = \int_0^1 e^{-0.104t}\, 0.1\, dt = 0.09498,$$

$$p_{60^+}^{03} = \int_0^1 e^{-0.104t}\, 0.004\, dt = 0.00380,$$

and multiplying these by l_{60^+} gives the $d_{60^+}^{(j)}$ values required:

Age	l_x	$d_x^{(1)}$	$d_x^{(2)}$	$d_x^{(3)}$
59	100 000.0	4 869.8	0.0	292.2
60 exact	94 838.0	0.0	18 967.6	0.0
60^+	75 870.4	0.0	7205.8	288.2
61	68 376.3			

We then repeat the process for age 61:

$$p_{61}^{01} = 0,$$

$$p_{61}^{02} = \int_0^1 e^{-0.105t}\, 0.1\, dt = 0.09493,$$

$$p_{61}^{03} = \int_0^1 e^{-0.105t}\, 0.005\, dt = 0.00475,$$

which give the full table up to exact age 62, when all remaining employees are assumed to retire immediately:

Age	l_x	$d_x^{(1)}$	$d_x^{(2)}$	$d_x^{(3)}$
59	100 000.0	4 869.8	0.0	292.2
60 exact	94 838.0	0.0	18 967.6	0.0
60^+	75 870.4	0.0	7 205.8	288.2
61	68 376.3	0.0	6 490.9	324.5
62 exact	61 560.9	0.0	61 560.9	0.0

(b) To distinguish between probabilities before exact age exits and after exact age exits, we use superscripts t^- and t^+ , for the age or term, as appropriate. So, using the table, the probability that an employee aged 59 retires before age 62^- is

$$ {}_{3^-}p_{59}^{02} = \frac{18\,967.6 + 7\,205.8 + 6\,490.9}{100\,000} = 0.32664. $$

(c) (i) The death in service benefit may be valued in three parts, corresponding to the three possible years of service. We then have the EPV as $100\,000\bar{A}^{03}_{59:\overline{3}|}$ where $\delta = \log 1.04$ and

$$ \bar{A}^{03}_{59:\overline{3}|} = \int_0^1 e^{-0.053t}\, 0.003\, e^{-\delta t} dt + {}_{1^+}p_{59}^{00}\, e^{-\delta} \int_0^1 e^{-0.104t}\, 0.004\, e^{-\delta t} dt $$

$$ + {}_2p_{59}^{00}\, e^{-2\delta} \int_0^1 e^{-0.105t}\, 0.005\, e^{-\delta t} dt $$

$$ = 0.0085281 $$

so the EPV for the death in service benefit as \$852.81.

(ii) With the claims acceleration approach (which is very commonly used in pension calculations), we can use the table to determine the probability of death in service over each year, and assume that the benefits are paid, on average, midway through the year of age, giving the EPV as

$$ 100\,000\, \frac{292.2v^{0.5} + 288.2v^{1.5} + 324.5v^{2.5}}{100\,000} = \$852.51. $$

We notice that the claims acceleration approach is simpler to calculate and quite accurate, compared with the exact calculation, in this case.

(d) The continuous time retirement benefits are discounted from the mid-year of payment, and the exact age retirements at ages 60 and 62 are discounted from the exact dates, giving the EPV of the retirement lump sum as

$$ 100\,000 \left(\frac{18\,967.6\, v + 7\,205.8\, v^{1.5} + 6\,490.9\, v^{2.5} + 61\,560.9\, v^3}{100\,000} \right) = \$85\,644.28. $$

□

9.8 Exercises

Shorter exercises

Exercise 9.1 You are given the following excerpt from a double decrement table. In the table, $*$ indicates a missing value. Calculate ${}_2p_{50}^{01}$.

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$
50	$*$	80	140
51	980	$*$	200
52	710		

Exercise 9.2 The following table is an extract from a multiple decrement table modelling withdrawals from life insurance contracts. Decrement 1 represents withdrawal, and decrement 2 represents death.

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$
40	15 490	2 400	51
41	13 039	2 102	58
42	10 879	1 507	60

(a) Calculate $q_{40}^{*(2)}$ assuming UDD in both single decrement tables.
(b) Repeat the calculation in part (a), but assuming that all withdrawals occurred at the start of the year.

Exercise 9.3 In a double decrement model, the independent rates of decrement at age x are $q_x^{*(1)} = 0.05$ and $q_x^{*(2)} = 0.2$.

You are given that decrement 1 is uniformly distributed in the single decrement model, and decrement 2 occurs at exact age $x + 0.75$ for integer age x.

Calculate the dependent rate of exit by decrement 1.

Exercise 9.4 You are given the following excerpt from a three-decrement table:

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$	$d_x^{(3)}$
60	8000	400	800	640

You are also given that decrement 1 occurs exactly one-third of the way through the year, that decrement 2 is uniformly distributed in the multiple decrement model, and that decrement 3 is uniformly distributed in the multiple decrement model over the first half of the year, with no exits during the second half of the year.

(a) Calculate ${}_{0.8}p_{60}^{00}$.
(b) Calculate ${}_{0.4}p_{60}^{00}$.
(c) Calculate the probability that a life exits by decrement 3 given that the life exits during the first 0.4 years.

Longer exercises

Exercise 9.5 Consider the insurance-with-lapses model illustrated in Figure 9.2. Suppose that this model is adjusted to include death after withdrawal, i.e. the transition intensity μ_x^{21} is introduced into the model.

(a) Show that if withdrawal does not affect the transition intensity to State 1 (i.e. that $\mu_x^{21} = \mu_x^{01}$), then the probability that an individual aged x is dead by age $x+t$ is the same as that under the alive–dead model with the transition intensity μ_x^{01}.
(b) Why is this intuitively obvious?

Exercise 9.6 Assuming UDD in the single decrement models, derive an expression for ${}_tp_x^{01}$ for $0 < t < 1$, and for integer x, in a double decrement model.

Hence, show that UDD in the single decrement model implies that the decrement in the multiple decrement model is *not* uniformly distributed.

Exercise 9.7 Employees of a certain company enter service at exact age 20, and, after a period in Canada, may be transferred to an overseas office. While in Canada, the only causes of decrement, apart from transfer to the overseas office, are death and resignation from the company.

(a) Using a radix of 100 000 at exact age 39, construct a multiple decrement table covering service in Canada for ages 39, 40 and 41 last birthday, given the following information about independent probabilities:

Mortality ($j = 1$): Standard Ultimate Survival Model.
Transfer ($j = 2$): $q_{39}^{*(2)} = 0.09$, $q_{40}^{*(2)} = 0.10$, $q_{41}^{*(2)} = 0.11$.
Resignation ($j = 3$): 20% of those reaching age 40 resign on their 40th birthday. No other resignations take place.

Assume a uniform distribution of deaths and transfers between integer ages in the single decrement models.
(b) Calculate the probability that an employee in service in Canada at exact age 39 will still be in service in Canada at exact age 42.
(c) Calculate the probability that an employee in service in Canada at exact age 39 will transfer to the overseas office between exact ages 41 and 42.
(d) The company has decided to set up a scheme to give each employee transferring to the overseas office between exact ages 39 and 42 a grant

of \$10 000 at the date of transfer. To pay for these grants the company will deposit a fixed sum in a special account on the 39th, 40th and 41st birthday of each employee still working in Canada (excluding resignations on the 40th birthday). The special account is invested to produce interest of 8% per year.

Calculate the annual deposit required.

Exercise 9.8 You are given the following excerpt from a double-decrement table:

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$
60	10 000	1 000	600

You are also given that both decrements are uniformly distributed in the single decrement tables.

(a) Calculate the independent rates of decrement 1 and 2 for age 60.

(b) Calculate ${}_{0.8}p_{60}^{00}$.

(c) Write down, and simplify as far as possible, the Kolmogorov forward differential equation for ${}_tp_x^{02}$.

(d) Using part (c), or otherwise, calculate $\mu_{60.8}^{02}$.

Exercise 9.9 The employees of a large corporation can leave the corporation in three ways: they can withdraw from the corporation, they can retire or they can die while they are still employees. Figure 9.5 illustrates this model.

You are given:

- The force of mortality depends on the individual's age but not on whether the individual is an employee, has withdrawn or is retired, so that for all ages x

$$\mu_x^{03} \equiv \mu_x^{13} \equiv \mu_x^{23} = \mu_x, \text{ say.}$$

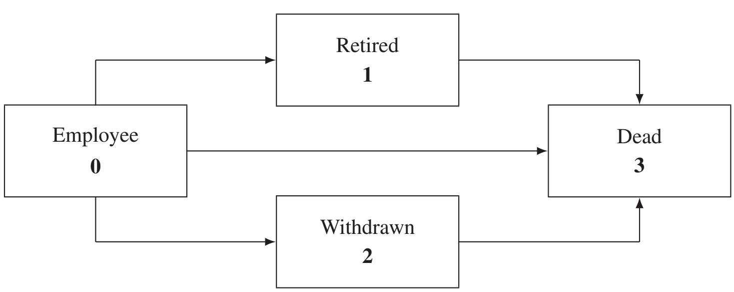


Figure 9.5 A withdrawal/retirement model.

- Withdrawal can take place at any age up to age 60 and the intensity of withdrawal is a constant denoted μ^{02}. Hence

$$\mu_x^{02} = \begin{cases} \mu^{02} & \text{for } x < 60, \\ 0 & \text{for } x \geq 60. \end{cases}$$

- Retirement can take place only on an employee's 60th, 61st, 62nd, 63rd, 64th or 65th birthday. It is assumed that 40% of employees reaching exact age 60, 61, 62, 63 and 64 retire at that age and 100% of employees who reach age 65 retire immediately.

The corporation offers the following benefits to the employees:

- For those employees who die while still employed, a lump sum of \$200 000 is payable immediately on death.
- For those employees who retire, a lump sum of \$150 000 is payable immediately on death after retirement.

Show that the EPVs of these benefits to an employee currently aged 40 can be written as follows, where $\bar{A}_{65}$ and ${}_{25}E_{40}$ are standard single life functions based on the force of mortality μ_x, and assuming a constant force of interest δ per year.

Death in service benefit

$$200\,000\left(\bar{A}^{\,1}_{40:\overline{20|}} + {}_{20}E_{40}\, e^{-20\mu^{02}}\left(\sum_{k=1}^{5} 0.6^k\, {}_{k-1|}\bar{A}^{\,1}_{60:\overline{1|}}\right)\right).$$

Death after retirement benefit

$$150\,000\, {}_{20}E_{40}\, e^{-20\mu^{02}}\left(0.4\sum_{k=0}^{4} 0.6^k\, {}_{k|}\bar{A}_{60} + 0.6^5\, {}_{5|}\bar{A}_{60}\right).$$

Excel-based exercises

Exercise 9.10 A university offers a four-year degree course. Semesters are half a year in length. The probability that a student progresses from one semester of study to the next is 0.85 in the first year of study, 0.9 in the second year, 0.95 in the third, and 0.98 in the fourth, conditional in each case on the student surviving to the end of the semester.

You are given the following information.

- Students pay tuition fees at the start of each semester.
- For the first semester the tuition fee is \$10 000.
- Fees increase by 2% each semester.
- Students who fail in any semester may not continue in the degree.
- Interest is 5% per year.

Calculate the EPV of fee income to the university for a new student aged 19. Assume that the student is subject to a constant force of mortality between integer ages x and $x+1$ of $5x \times 10^{-5}$ for $x = 19$, 20, 21 and 22, and that there are no means of leaving the course other than by death or failure.

Exercise 9.11 In a certain country, members of its regular armed forces can leave active service (State 0) by transfer (State 1), by resignation (State 2) or by death (State 3). The transition intensities are

$$\mu_x^{01} = 0.001x,$$
$$\mu_x^{02} = 0.01,$$
$$\mu_x^{03} = A + Bc^x,$$

where $A = 0.001$, $B = 0.0004$ and $c = 1.07$. New recruits join only at exact age 25.

(a) Calculate the probability that a new recruit
  (i) is transferred before age 27,
  (ii) dies aged 27 last birthday, and
  (iii) is in active service at age 28.
(b) New recruits who are transferred within three years of joining receive a lump sum payment of \$10 000 immediately on transfer. This sum is provided by a levy on all recruits in active service on the first and second anniversary of joining. On the basis of interest at 6% per year effective, calculate the levy payable by a new recruit.
(c) Those who are transferred enter an elite force. Members of this elite force are subject to a force of mortality at age x equal to $1.5\mu_x^{03}$, but are subject to no other decrements. Calculate the probability that a new recruit into the regular armed forces dies before age 28 as a member of the elite force.

Exercise 9.12 An insurance company sells 10-year term insurance policies with sum insured \$100 000 payable immediately on death to lives aged 50. Calculate the monthly premium for this policy on the following basis.

Survival: Makeham's law, with $A = 0.0001$, $B = 0.0004$ and $c = 1.075$
Lapses: 2% of policyholders lapse their policy on each of the first two policy anniversaries
Interest: 5% per year effective
Initial expenses: \$200
Renewal expenses: 2.5% of each premium (including the first)

Value the death benefit using the UDD assumption.

Exercise 9.13 You are given the following three-decrement service table for modelling employment.

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$	$d_x^{(3)}$
60	10 000	350	150	25
61	9 475	360	125	45
62	8 945	380	110	70

(a) Calculate ${}_3p_{60}^{01}$.

(b) Calculate ${}_2p_{61}^{00}$.

(c) Calculate the EPV of a benefit of \$10 000 payable at the end of the year of exit, if a life aged 60 leaves by decrement 3 before age 63. Use an effective rate of interest of 5% per year.

(d) Calculate the EPV of an annuity of \$1000 per year payable at the start of each of the next 3 years if a life currently aged 60 remains in service. Use an effective rate of interest of 5% per year.

(e) Show that $q_{62}^{*(1)} = 0.04292$, assuming a constant force of decrement for each decrement.

(f) Calculate the revised service table for age 62 if $q_{62}^{*(1)}$ is increased to 0.1, with the other independent rates remaining unchanged. Use (i) the constant force assumption, and (ii) UDD in the single decrement models assumption.

Exercise 9.14 Consider the model in Exercise 9.9, and suppose that

$$\mu_x = A + Bc^x \quad \text{and} \quad \mu^{02} = 0.02,$$

where $A = 0.0001$, $B = 0.0004$ and $c = 1.07$.

A corporation contributes \$10 000 to a pension fund when an employee joins the corporation and on each anniversary of that person joining the corporation, provided the person is still an employee. On the basis of interest at 5% per year effective, calculate the EPV of contributions to the pension fund in respect of a new employee aged 30.

Answers to selected exercises

9.1 0.125

9.2 (a) 0.00357 (b) 0.00390

9.3 0.0475

9.4 (a) 0.790 (b) 0.846 (c) 0.4156

9.7 (b) 0.58220 (c) 0.07198 (d) \$943.11

9.8 (a) 0.103267, 0.063267 (b) 0.870955 (d) 0.06664

9.10 \$53 285.18

9.11 (a)(i) 0.050002 (ii) 0.003234 (iii) 0.887168
(b) \$397.24 (c) 0.000586

9.12 \$181.27

9.13 (a) 0.109 (b) 0.8850 (c) \$125.09 (d) \$2 713.72
(f)(i) 885.4, 106.7, 67.9 (ii) 885.3, 106.8, 68.0

9.14 \$125 489.33

10
Joint life and last survivor benefits

10.1 Summary

Insurance benefits which are dependent on the joint mortality of two lives, typically a married couple, form an important part of many insurance portfolios. In this chapter we develop the concepts and models from previous chapters to examine **joint life** insurance policies. There are also important applications in pension design and valuation, as spousal benefits are a common part of a pension benefit package.

In Section 10.2 we describe the typical benefits offered and introduce standard notation for actuarial functions dependent on two lives. In Section 10.4 we develop an approach for pricing and valuing these policies, based on the future lifetime random variables, and making the strong assumption that the two lives are independent with respect to mortality.

In Section 10.5 we show how joint life mortality can be analysed using multiple state models. This creates a flexible framework to introduce dependence between lives, and we can apply the methods of Chapter 8 to calculate probabilities and value benefits.

10.2 Joint life and last survivor benefits

All the development in previous chapters relates to life insurance or annuity benefits based on a single insured life. In practice, policies based on two lives are common. The two lives are typically a couple who are jointly organizing their financial security, but other situations are also feasible, for example, annuities dependent on the lives of a parent and their child, or insurance on the lives of business partners.

In this chapter we consider policies based on two lives, whom we label, for convenience, (x) and (y). Throughout, we assume that at some inception time $t = 0$, (x) and (y) are alive, are then aged x and y, respectively, and are partners in some joint life contingent benefit context.

The most common types of benefit which are contingent on two lives are described briefly below.

A **joint life annuity** is an annuity payable until the first death of (x) and (y).

A **last survivor annuity** is an annuity payable until the second death of (x) and (y).

A common benefit design is an annuity payable at a higher rate while both partners are alive and at a lower rate following the first death. The annuity ceases on the second death. This could be viewed as a last survivor annuity for the lower amount, plus a joint life annuity for the difference.

A **reversionary annuity** is a life annuity that starts payment on the death of a specified life, say (x), provided that (y) is alive at that time, and continues through the lifetime of (y). A pension plan may offer a reversionary annuity benefit as part of the pension package, where (x) would be the pension plan member, and (y) would be a partner eligible for spousal benefits.

A **joint life insurance** pays a death benefit on the first death of (x) and (y).

A **last survivor insurance** pays a death benefit on the second death of (x) and (y).

A **contingent insurance** pays a death benefit contingent on multiple events. The most common is a benefit paid on the death of (x), say, but only if (y) is still living at that time.

10.3 Joint life notation

Following the same approach as in Chapters 4 and 5, we express the present values of the joint life benefits in terms of random variables, so that we can value them using the expected value of the present value.

Throughout this chapter, we assume that at the inception of a contract, the lives (x) and (y) are alive. From Chapter 2, we recall that the future lifetimes of (x) and (y) are represented by T_x and T_y. At this stage we make no assumption about the independence, or otherwise, of these two random variables.

Given T_x and T_y, we define two more random variables, representing the times until the first and second to die of (x) and (y).

Time to first death: $T_{xy} = \min(T_x, T_y)$.
Time to last death: $T_{\overline{xy}} = \max(T_x, T_y)$.

We refer to the subscript, xy or $\overline{xy}$, as a **status**; xy (also written as $\{xy\}$) is the **joint life** status and $\overline{xy}$ is the **last survivor** status. Hence, T_{xy} and $T_{\overline{xy}}$ are random variables representing the time until the failure of the joint life status and the last survivor status, respectively. It is implicit throughout that the definitions of joint life and last survivor random variables for the couple (x) and (y) are conditional on both lives being alive, and a couple (at least, in the sense of being connected for the purpose of the insurance contract) at some starting point.

It is very useful to observe that with only two lives involved, either

1. (x) dies first so that the realized values of T_x and T_{xy} are the same, and consequently the realized values of T_y and $T_{\overline{xy}}$ are the same, or
2. (y) dies first so that the realized values of T_y and T_{xy} are the same, and consequently the realized values of T_x and $T_{\overline{xy}}$ are the same.

Thus, regardless of the order of deaths, the realized value of T_x matches one of T_{xy} and $T_{\overline{xy}}$, and the realised value of T_y matches the other.

Important consequences of these observations include the following relationships:

$$T_x + T_y = T_{xy} + T_{\overline{xy}}, \tag{10.1}$$

$$v^{T_x} + v^{T_y} = v^{T_{xy}} + v^{T_{\overline{xy}}}, \tag{10.2}$$

$$\bar{a}_{\overline{T_x}|} + \bar{a}_{\overline{T_y}|} = \bar{a}_{\overline{T_{xy}}|} + \bar{a}_{\overline{T_{\overline{xy}}}|}. \tag{10.3}$$

International actuarial notation for probabilities, annuity and insurance functions on multiple lives extends the notation for single lives introduced in Chapters 2, 4 and 5. The list below shows the notation for probabilities based on two lives, in each case followed by the definition in words, and in terms of the T_{xy} or $T_{\overline{xy}}$ random variables.

$$
\begin{aligned}
{}_tp_{xy} &= \Pr[(x) \text{ and } (y) \text{ are both alive in } t \text{ years}] = \Pr[T_{xy} > t].\\
{}_tq_{xy} &= \Pr[(x) \text{ and } (y) \text{ are not both alive in } t \text{ years}] = \Pr[T_{xy} \le t].\\
{}_{u|t}q_{xy} &= \Pr[(x) \text{ and } (y) \text{ are both alive in } u \text{ years, but not in } u+t \text{ years}]\\
&= \Pr[u < T_{xy} \le u+t].\\
{}_tq^{\,1}_{xy} &= \Pr[(x) \text{ dies before } (y) \text{ and within } t \text{ years}] = \Pr[T_x \le t \text{ and } T_x < T_y].\\
{}_tq^{\,2}_{xy} &= \Pr[(x) \text{ dies after } (y) \text{ and within } t \text{ years}] = \Pr[T_y < T_x \le t].\\
{}_tp_{\overline{xy}} &= \Pr[\text{at least one of } (x) \text{ and } (y) \text{ is alive in } t \text{ years}] = \Pr[T_{\overline{xy}} > t].\\
{}_tq_{\overline{xy}} &= \Pr[(x) \text{ and } (y) \text{ are both dead in } t \text{ years}] = \Pr[T_{\overline{xy}} \le t].\\
{}_{u|t}q_{\overline{xy}} &= \Pr[\text{at least one of } (x) \text{ and } (y) \text{ is alive in } u \text{ years, but both have died}\\
&\qquad \text{within } u+t \text{ years}]\\
&= \Pr[u < T_{\overline{xy}} \le u+t].
\end{aligned}
$$

The '1' over x in ${}_tq^{\,1}_{xy}$ indicates that we are interested in the probability of (x) dying *first*. We have already used this notation, in $A^{\;1}_{x:\overline{n}|}$, where the benefit is paid only if x dies first, before the term, n years, expires.

In cases where it makes the notation clearer, we put a colon between the ages in the right subscript. For example, we write ${}_tp_{\overline{30:40}}$ rather than ${}_tp_{\overline{30\,40}}$.

For each of the common joint life benefits, we list here the notation for the EPV, and indicate the appropriate function of the joint life random variables T_{xy} or $T_{\overline{xy}}$. We assume a constant force of interest δ per year.

$\bar{a}_{xy}$ **Joint life annuity:** a continuous payment at unit rate per year while both (x) and (y) are still alive;

$$\bar{a}_{xy} = \mathrm{E}\left[\bar{a}_{\overline{T_{xy}}|}\right].$$

If there is a maximum period, n years, for the annuity, then we refer to a 'temporary' or 'term' joint life annuity. The notation for the EPV is $\bar{a}_{xy:\overline{n}|}$ and the formula for this is

$$\bar{a}_{xy:\overline{n}|} = \mathrm{E}\left[\bar{a}_{\overline{\min(T_{xy},n)}|}\right].$$

$\bar{A}_{xy}$ **Joint life insurance:** a unit payment immediately on the death of the first to die of (x) and (y);

$$\bar{A}_{xy} = \mathrm{E}\left[v^{T_{xy}}\right].$$

$\bar{a}_{\overline{xy}}$ **Last survivor annuity:** a continuous payment at unit rate per year while at least one of (x) and (y) is still alive;

$$\bar{a}_{\overline{xy}} = \mathrm{E}\left[\bar{a}_{\overline{T_{\overline{xy}}}|}\right].$$

$\bar{A}_{\overline{xy}}$ **Last survivor insurance:** a unit payment immediately on the death of the second to die of (x) and (y);

$$\bar{A}_{\overline{xy}} = \mathrm{E}\left[v^{T_{\overline{xy}}}\right].$$

$\bar{a}_{x|y}$ **Reversionary annuity:** a continuous payment at unit rate per year, starting on the death of (x) if (y) is still alive then, and continuing until the death of (y);

$$\bar{a}_{x|y} = \mathrm{E}\left[\left(v^{T_x}\,\bar{a}_{\overline{T_y-T_x}|}\right) I(T_x < T_y)\right].$$

$\bar{A}^{\,1}_{xy}$ **Contingent insurance:** a unit payment immediately on the death of (x) provided (x) dies before (y);

$$\bar{A}^{\,1}_{xy} = \mathrm{E}\left[v^{T_x}\, I(T_x < T_y)\right],$$

where I is the indicator function.

If there is a time limit on this payment, say n years, then it is called a 'temporary' or 'term' contingent insurance. The notation for the EPV is $\bar{A}^{\;1}_{x:\,y:\overline{n}|}$ and the formula is

$$\bar{A}^{\;1}_{x:\,y:\overline{n}|} = \mathrm{E}\left[v^{T_x}\, I(T_x < T_y)\, I(T_x < n)\right].$$

Although we have defined these functions in terms of continuous benefits, the annuity and insurance functions can easily be adapted for payments made at discrete points in time. For example, the EPV of a monthly joint life annuity-due would be denoted $\ddot{a}_{xy}^{(12)}$, and would represent the EPV of an annuity of $\frac{1}{12}$ payable at the start of each month, contingent on both (x) and (y) surviving to the payment date.

We can write down the following important relationships:

$$\boxed{{}_tp_{\overline{xy}} = {}_tp_x + {}_tp_y - {}_tp_{xy},} \tag{10.4}$$

$$\boxed{\bar{a}_{\overline{xy}} = \bar{a}_x + \bar{a}_y - \bar{a}_{xy},} \tag{10.5}$$

$$\boxed{\bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y - \bar{A}_{xy},} \tag{10.6}$$

$$\boxed{\bar{a}_{x|y} = \bar{a}_y - \bar{a}_{xy}.} \tag{10.7}$$

Formula (10.4) follows by noting that $T_{\overline{xy}} > t$ if $T_x > t$ or $T_y > t$, giving

$$\Pr[T_{\overline{xy}} > t] = \Pr[T_x > t] + \Pr[T_y > t] - \Pr[T_{xy} > t],$$
$$\text{i.e. } {}_tp_{\overline{xy}} = {}_tp_x + {}_tp_y - {}_tp_{xy}.$$

Formula (10.5) follows by taking expectations in formula (10.3), and formula (10.6) follows by taking expectations in formula (10.2).

Formula (10.7) for a reversionary annuity to (y) following the death of (x) is most easily derived by noting that $\bar{a}_{xy}$ is the EPV of an annuity of 1 per year payable continuously to (y) while (x) is alive, and $\bar{a}_{x|y}$ is the EPV of an annuity of 1 per year payable to (y) after (x) has died. The total is the EPV of an annuity payable to (y) whether (x) is alive or dead, so

$$\bar{a}_y = \bar{a}_{xy} + \bar{a}_{x|y}.$$

The notation for the EPV of the reversionary annuity uses the status $x|y$; the vertical line indicates deferral in standard actuarial notation. In this case, the status is deferred until the death of (x), and then continues as long as (y) is alive.

Equations (10.5), (10.6) and (10.7) can be derived, alternatively, by considering the cash flows involved. This is a useful trick for verifying joint life EPV formulae more generally. For example, in formula (10.7) for the reversionary annuity, the right-hand side values an annuity of 1 per year payable continuously while (y) is alive, minus an annuity of 1 per year payable continuously while both (x) and (y) are alive. What remains is an annuity of 1 per year payable continuously after the death of (x) while (y) is alive.

Similarly, in formula (10.6) for the last survivor insurance, the right-hand side values a payment of 1 when (x) dies, plus 1 when (y) dies, minus 1 on the first death; what remains is 1 paid on the second death, which is the last survivor insurance benefit.

Recall formula (5.14), linking whole life annuity and insurance functions for a single life:

$$\bar{a}_x = \frac{1 - \bar{A}_x}{\delta}.$$

The following corresponding relationships are derived in exactly the same way:

$$\bar{a}_{xy} = \frac{1 - \bar{A}_{xy}}{\delta} \quad \text{and} \quad \bar{a}_{\overline{xy}} = \frac{1 - \bar{A}_{\overline{xy}}}{\delta}.$$

All the expressions and relationships in this section are true for *any* model of dependence between T_x and T_y. Our objective now is to specify appropriate joint models for the future lifetimes of (x) and (y) and derive expressions for EPVs of future payments. We can then calculate premium rates and policy values for benefits and premiums which are dependent on two lives, analogously to the premiums and policy values derived for single lives in Chapters 6 and 7. In the remainder of this chapter we introduce and discuss different ways of modelling the future lifetimes of two possibly dependent lives.

10.4 Independent future lifetimes

Our first approach to modelling the survival of two lives assumes that the future lifetime for each individual is not affected in any way by the other life. This is a very strong assumption, and in later sections we relax it somewhat. However, it is a simple assumption which gives practical formulae that are easily implemented. It is still commonly used in practice, though models incorporating dependence are also becoming popular, especially if the dependence could have a material impact on the valuation. We often use the phrase **independent lives** as a short way of saying the lives have independent future lifetimes.

To be precise, throughout this section, including all the examples, we make the following very important assumption.

Independence Assumption 1. The random variables T_x and T_y are independent.

We also assume throughout this section that we know all about the survival models for these two random variables, so that we know the survival functions, ${}_tp_x$ and ${}_tp_y$, and the forces of mortality μ_{x+t} and μ_{y+t}. Note that we do not assume that these survival functions come from the same survival model. If the two lives are indeed husband and wife, then, since mortality rates for females are generally lower than those for males, the survival models are likely to be different.

From our assumption of independence, we can write for $t \geq 0$

$$ {}_tp_{xy} = \Pr[T_x > t \text{ and } T_y > t] = {}_tp_x \, {}_tp_y, \tag{10.8} $$

and

$$ \begin{aligned} {}_tp_{\overline{xy}} &= \Pr[T_x > t \text{ or } T_y > t] = 1 - {}_tq_x \, {}_tq_y \\ &= {}_tp_x + {}_tp_y - {}_tp_x \, {}_tp_y. \end{aligned} \tag{10.9} $$

The probability on the left-hand side of formula (10.8) relates to the survival model for the joint lives, whereas the probabilities on the right-hand side relate to two, possibly different, survival models for the individual lives. When we use labels, such as x and y, for the lives, we assume the context makes it clear what we mean by ${}_tp_x$ and ${}_tp_y$. We often have notational confusion though if we use numbers for x and y; for example, if (x) is a male aged 40, and (y) is a female aged 40, then we have two probabilities on the right-hand side, both labelled ${}_tp_{40}$, but which are actually based on different survival models. When we need to be more specific about the probabilities we might write the joint probability relationship more carefully as ${}_tp^{m\,f}_{x:y} = {}_tp^{m}_{x} \, {}_tp^{f}_{y}$, so, for example, ${}_tp^{m\;\;f}_{45:40} = {}_tp^{m}_{45} \, {}_tp^{f}_{40}$, which indicates that the two single life survival probabilities are from different distributions.

Using the joint life and last survivor probabilities, we can develop annuity and insurance functions for the EPV of benefits contingent on two lives, by summing over all the possible payment dates the EPVs of the individual payments. For example,

$$ \ddot{a}_{xy:\overline{n}|} = \sum_{k=0}^{n-1} v^k \, {}_kp_{xy}, $$

$$ \ddot{a}_{\overline{xy}} = \sum_{k=0}^{\infty} v^k \, {}_kp_{\overline{xy}} = \ddot{a}_x + \ddot{a}_y - \ddot{a}_{xy}, $$

$$ A_{xy} = \sum_{k=0}^{\infty} v^{k+1} \, {}_{k|}q_{xy}, $$

where ${}_{k|}q_{xy} = {}_kp_{xy}(1 - p_{x+k:\,y+k}) = {}_kp_{xy} - {}_{k+1}p_{xy}$, and

$$ A_{\overline{xy}} = \sum_{k=0}^{\infty} v^{k+1} \, {}_{k|}q_{\overline{xy}} = A_x + A_y - A_{xy}. $$

Example 10.1 The table below shows extracts from two life tables appropriate for a husband and wife, who are assumed independent with respect to mortality.

Husband		Wife	
x	l_x	y	l_y
65	43 302	60	47 260
66	42 854	61	47 040
67	42 081	62	46 755
68	41 351	63	46 500
69	40 050	64	46 227

(a) Calculate ${}_3p_{xy}$ for a husband aged $x = 66$ and a wife aged $y = 60$.
(b) Calculate ${}_2p_{\overline{xy}}$ for a husband aged $x = 65$ and a wife aged $y = 62$.
(c) Calculate the probability that a husband, currently aged 65, dies within two years and that his wife, currently aged 61, survives at least two years.
(d) Explain the meaning of the symbol $\ddot{a}_{xy:\overline{n}|}$.
(e) Explain the meaning of the symbol $\ddot{a}_{\overline{xy}:\overline{n}|}$.
(f) Calculate $\ddot{a}_{xy:\overline{5}|}$ and $\ddot{a}_{\overline{xy}:\overline{5}|}$ for a husband aged $x = 65$ and a wife aged $y = 60$ at a rate of interest 5% per year.

Solution 10.1 (a) Using formula (10.8)

$$ {}_3p_{xy} = {}_3p_x\, {}_3p_y = \frac{40\,050}{42\,854} \times \frac{46\,500}{47\,260} = 0.9195. $$

(b) Using formula (10.9)

$$ {}_2p_{\overline{xy}} = \frac{42\,081}{43\,302} + \frac{46\,227}{46\,755} - \frac{42\,081}{43\,302} \times \frac{46\,227}{46\,755} = 0.9997. $$

(c) Since the two lives are assumed to be independent with respect to mortality, the required probability is

$$ \left(1 - \frac{42\,081}{43\,302}\right) \times \frac{46\,500}{47\,040} = 0.0279. $$

(d) The symbol $\ddot{a}_{xy:\overline{n}|}$ represents the EPV, at a given constant rate of interest, of a series of at most n annual payments, each of unit amount with the first payment due now, with each payment being made only if the lives (x) and (y) are both alive at the time the payment is due.
(e) The symbol $\ddot{a}_{\overline{xy}:\overline{n}|}$ represents the EPV, at a given constant rate of interest, of a series of at most n annual payments, each of unit amount with the first payment due now, with each payment being made only if at least one of (x) and (y), is alive at the time the payment is due.
(f) From the definitions in parts (d) and (e), we can write down the following formulae

$$ \ddot{a}_{xy:\overline{5}|} = \sum_{t=0}^{4} v^t\, {}_tp_{xy}, \quad \ddot{a}_{\overline{xy}:\overline{5}|} \sum_{t=0}^{4} v^t\, {}_tp_{\overline{xy}}, $$

where $v = 1/1.05$, $x = 65$ and $y = 60$. These are derived in exactly the same way as formula (5.8). The numerical values of the annuities are

$$\ddot{a}_{xy:\overline{5}|} = 4.3661 \quad \text{and} \quad \ddot{a}_{\overline{xy}:\overline{5}|} = 4.5437.$$

Note that $\ddot{a}_{xy:\overline{5}|} \leq \ddot{a}_{\overline{xy}:\overline{5}|}$ since $T_{xy} \leq T_{\overline{xy}}$.

□

For the following example, we use the $1/m$thly joint life and last survivor annuities

$$\ddot{a}_{xy}^{(m)} = \frac{1}{m}\left(1 + {}_{\frac{1}{m}}p_{xy}\, v^{\frac{1}{m}} + {}_{\frac{2}{m}}p_{xy}\, v^{\frac{2}{m}} + \cdots\right)$$

and

$$\ddot{a}_{\overline{xy}}^{(m)} = \ddot{a}_{x}^{(m)} + \ddot{a}_{y}^{(m)} - \ddot{a}_{xy}^{(m)},$$

and the joint life term insurance

$$A_{\overset{1}{\lceil xy \rceil}:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1}\, {}_{k|}q_{xy}.$$

Note that we use $\lceil xy \rceil$ here to denote that it is the failure of the *joint life* status, before n years, that triggers the death benefit payment. Without the $\lceil\ \rceil$, the symbol could be confused with the EPV of a contingent term insurance payable on the death of (x) before (y), and before n years.

Example 10.2 A husband, currently aged 55, and his wife, currently aged 50, have just purchased a deferred annuity policy. Level premiums are payable monthly for at most 10 years but only if both are alive. If either dies within 10 years, a sum insured of \$200 000 is payable at the end of the year of death. If both lives survive 10 years, an annuity of \$50 000 per year is payable monthly in advance while both are alive, reducing to \$30 000 per year, still payable monthly, while only one is alive. The annuity ceases on the death of the last survivor.

Calculate the monthly premium on the following basis:

Survival model:

Standard Select Survival Model for both lives
(55) and (50) are independent with respect to mortality.
(55) and (50) are select at the time the policy is purchased.

Interest: 5% per year effective
Expenses: None

Solution 10.2 Since the two lives are independent with respect to mortality, we can write the probability that they both survive t years as ${}_tp_{[55]}\,{}_tp_{[50]}$, where each single life probability is calculated using the Standard Select Survival Model.

Let P denote the annual amount of the premium. Then the EPV of the premiums is

$$P\,\ddot{a}^{(12)}_{[55]:[50]:\overline{10|}} = 7.7786\,P.$$

The EPV of the death benefit is

$$200\,000\,A^{\;\;1}_{\overline{[55]:[50]}:\overline{10|}} = 7660.$$

To find the EPV of the annuities we note that if both lives are alive at time 10 years, the EPV of the payments at that time is

$$30\,000\,\ddot{a}^{(12)}_{\overline{65:60}} + 20\,000\,\ddot{a}^{(12)}_{65:60} = 30\,000\,\ddot{a}^{(12)}_{65} + 30\,000\,\ddot{a}^{(12)}_{60} - 10\,000\,\ddot{a}^{(12)}_{65:60}. \tag{10.10}$$

For the EPV at issue, we discount for survival for the 10 year deferred period, and for interest, giving

$$v^{10}\;{}_{10}p_{[55]}\;{}_{10}p_{[50]}\left(30\,000\,\ddot{a}^{(12)}_{65} + 30\,000\,\ddot{a}^{(12)}_{60} - 10\,000\,\ddot{a}^{(12)}_{65:60}\right) = 411\,396.$$

Hence the monthly premium, $P/12$, is

$$P/12 = (7\,660 + 411\,396)/(12 \times 7.7786) = \$4\,489.41.$$

□

In this solution we calculated the monthly premium values exactly, by summing the monthly terms. However, we have noted in earlier chapters that it is sometimes the case in practice that the only information available to us to calculate the EPV of an annuity payable more frequently than annually is a life table specified at integer ages only. In Section 5.11 we illustrated methods of approximating the EPV of an annuity payable m times per year, and these methods can also be applied to joint life annuities. To illustrate, consider the annuity EPVs in equation (10.10). These can be approximated from the corresponding annual values using UDD as

$$\begin{aligned}
\ddot{a}_{65}^{(12)} &\approx \alpha(12)\,\ddot{a}_{65} - \beta(12) \\
&= 1.000197 \times 13.5498 - 0.466508 \\
&= 13.0860, \\
\ddot{a}_{60}^{(12)} &\approx \alpha(12)\,\ddot{a}_{60} - \beta(12) \\
&= 1.000197 \times 14.9041 - 0.466508 \\
&= 14.4405, \\
\ddot{a}_{65:60}^{(12)} &\approx \alpha(12)\,\ddot{a}_{65:60} - \beta(12) \\
&= 1.000197 \times 12.3738 - 0.466508 \\
&= 11.9097, \\
\ddot{a}_{[55]:[50]:\overline{10}|}^{(12)} &\approx \alpha(12)\,\ddot{a}_{[55]:[50]:\overline{10}|} - \beta(12)(1 - {}_{10}p_{[55]}\,{}_{10}p_{[50]}\,v^{10}) \\
&= 1.000197 \times 7.9716 - 0.466508 \times 0.41790 \\
&= 7.7782.
\end{aligned}$$

The approximate value of the monthly premium is then

$$\begin{aligned}
P/12 &\approx \frac{32\,715 + 0.54923 \times (30\,000(13.0860 + 14.4405) - 10\,000 \times 11.9097)}{12 \times 7.7782} \\
&= \$4489.33.
\end{aligned}$$

An important point to appreciate here is that, under UDD for individual lives, we have

$$\ddot{a}_x^{(m)} = \alpha(m)\ddot{a}_x - \beta(m)$$

but under the assumption of UDD for each individual life, we cannot assume UDD for the joint life status; the assumptions are incompatible. That means, assuming UDD for the individual lives, we do not have simple exact relationships between $1/m$thly and annual functions for joint lives. It is, however, true that applying the UDD formulae will give close to exact values (assuming UDD for the individual lives) in most cases. So we may use the fact that, assuming UDD for the individual lives

$$\ddot{a}_{xy}^{(m)} \approx \alpha(m)\ddot{a}_{xy} - \beta(m). \tag{10.11}$$

Our calculations above illustrate the general point that this approximation is usually very accurate.

We can also derive a form of Woolhouse's formula for $\ddot{a}_{xy}^{(m)}$, for independent lives, as

$$\ddot{a}_{xy}^{(m)} \approx \ddot{a}_{xy} - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}\left(\delta + \mu_x + \mu_y\right). \tag{10.12}$$

The derivation of this formula (in a more general form) is the subject of Exercise 10.16.

Example 10.3 Derive integral expressions in terms of survival probabilities and a constant force of interest δ for $\bar{a}_{xy}$ and $\bar{a}_{\overline{xy}}$.

Solution 10.3 Integral expressions are as follows:

$$\bar{a}_{xy} = \int_0^\infty e^{-\delta t}\, {}_tp_{xy}\, dt, \tag{10.13}$$

$$\bar{a}_{\overline{xy}} = \int_0^\infty e^{-\delta t}\, {}_tp_{\overline{xy}}\, dt. \tag{10.14}$$

These expressions can be derived in the same way that Figure 5.5 was used to derive formula (5.15) for a single life annuity. Consider formula (10.13): the amount paid between times t and $t + dt$ is dt, provided both lives survive to that time, and its present value is $e^{-\delta t}\, dt$; the probability of this amount being paid is ${}_tp_{xy}$; hence the EPV of this possible payment is $e^{-\delta t}\, {}_tp_{xy}\, dt$ and the total EPV is the sum (integral) of this expression over all values of t. □

We can derive integral expressions for insurance functions using arguments similar to those used in Section 4.4.1 for a single life based on Figure 4.1. For example, $\bar{A}^{\;1}_{x\,y}$ can be written

$$\bar{A}^{\;1}_{x\,y} = \int_0^\infty e^{-\delta t}\, {}_tp_{xy}\, \mu_{x+t}\, dt \tag{10.15}$$

and this can be justified as follows. Consider the possible payment between times t and $t + dt$, where dt is small. If there is a payment, the amount will be 1, and the present value of the payment is $e^{-\delta t}$. For this payment to be made, both lives must be alive at time t (probability ${}_tp_{xy}$) and x must die before time $t + dt$ (probability $\mu_{x+t}\, dt$) – the probability that both die before time $t + dt$ is negligible if dt is small. Hence the EPV of this payment is $e^{-\delta t}\, {}_tp_{xy}\, \mu_{x+t}\, dt$ and the total EPV is the integral over all possible values of t. The following integral expressions can be justified in similar ways:

$$\bar{A}_{xy} = \int_0^\infty e^{-\delta t} \, {}_tp_{xy} \, (\mu_{x+t} + \mu_{y+t}) \, dt = \bar{A}_{\overset{1}{x}y} + \bar{A}_{x\overset{1}{y}},$$

$$\bar{A}_{\overline{xy}} = \int_0^\infty e^{-\delta t} \left({}_tp_x \, \mu_{x+t} \, (1 - {}_tp_y) + {}_tp_y \, \mu_{y+t} \, (1 - {}_tp_x) \right) dt.$$

Similar arguments can be used to construct integral expressions for many annuity and insurance functions based on two lives, but the approach has drawbacks. When benefits are complex, it is easy to mis-state probabilities, for example. In the following section we present another approach to deriving these equations, using a multiple state model, that gives the insurance and annuity EPV formulae more directly, and that also proves fruitful in terms of generalizing the model to incorporate dependence.

10.5 A multiple state model for independent future lifetimes

In Section 8.2, we described how the single life future lifetime random variable, T_x, is related to a two-state multiple state model, which we called the alive–dead model. Loosely, for a life aged x, T_x is the time to the transition from State 0 (alive) to State 1 (dead).

In our joint life case, we can, similarly, create a multiple state model that provides a different perspective on the future lifetime random variables, T_x and T_y, as well as the joint life and last survivor random variables, T_{xy} and $T_{\overline{xy}}$.

We use the multiple state model shown in Figure 10.1. For illustration, we assume (x) is male and (y) is female, and that the male and female transition intensities are labelled m and f respectively. The process starts in State 0 with both (x) and (y) alive. It moves to State 1 on the death of (y) if she dies

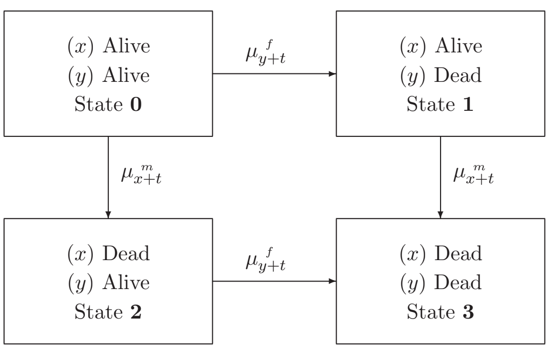


Figure 10.1 The independent joint life and last survivor model.

before (x), or to State 2 if (x) dies first. The process moves to State 3 on the death of the surviving partner. The model is specified in terms of the transition intensities between the states. We assume these are known.

In this and the following sections, we use the multiple state probability notation from Chapter 8, but with a right subscript that references both (x) and (y), for probabilities involving State 0, where both are alive. For transitions from State 1 and State 2, we use the subscript appropriate to the surviving life. So, for example,

$$
\begin{aligned}
{}_tp_{xy}^{00} &= \Pr[(x) \text{ and } (y) \text{ both alive at time } t \,|\, \text{both alive at time } 0], \\
{}_tp_{xy}^{01} &= \Pr[(x) \text{ alive and } (y) \text{ dead at time } t \,|\, \text{both alive at time } 0],
\end{aligned}
$$

and for $0 < s < t$,

$$
{}_tp_{y+s}^{22} = \Pr[(y) \text{ alive at time } t+s \,|\, (y) \text{ alive and } (x) \text{ dead at time } s].
$$

Our model, in this form, incorporates the following important assumption. We will show that this assumption is equivalent to assuming independence of T_x and T_y.

Independence Assumption 2. The transition intensities from State 0 to State 1, and from State 2 to State 3, are identical and depend on (y)'s age, but not on (x)'s age or survival. Similarly, the transition intensities from State 0 to State 2 and from State 1 to State 3, representing the death of (x), are identical functions of (x)'s age, with no dependence on (y)'s age or survival.

Under this assumption, the force of mortality for (x) at age $x+t$ is μ_{x+t}^{m}, whether (y) is then alive or not. Hence, T_x has a distribution determined by $\{\mu_{x+t}^{m}\}_{t\geq 0}$. Similarly, T_y has a distribution determined by $\{\mu_{y+t}^{f}\}_{t\geq 0}$. Since there is no connection between the mortality of (x) and (y), it is not surprising that the future lifetimes of (x) and (y) are independent random variables, and, hence, that the multiple state model in this section is equivalent to the model studied in Section 10.4. We show this more formally in Example 10.4.

Example 10.4 Show that, under the multiple state model in Figure 10.1, with the assumption labelled 'Independence Assumption 2', the future lifetimes of (x) and (y), both currently in State 0, are independent.

Solution 10.4 Let s and t be any positive numbers. It is sufficient to show that

$$
\Pr[T_x > s \;\text{ and }\; T_y > t] = \Pr[T_x > s]\Pr[T_y > t], \tag{10.16}
$$

and we can assume, without loss of generality, that $s \leq t$. Let

$$ {}_sp_x = \exp\left\{-\int_0^s \mu^m_{x+u}\,du\right\} \quad \text{and} \quad {}_tp_y = \exp\left\{-\int_0^t \mu^f_{y+u}\,du\right\}. $$

We know that these are the survival probabilities in the single life models. We need to show that they are also survival probabilities in the multiple state model in Figure 9.1. We have

$$ \Pr[T_y > t] = {}_tp^{00}_{xy} + {}_tp^{02}_{xy}. $$

Now

$$ {}_tp^{00}_{xy} = {}_tp_{xy} = \exp\left\{-\int_0^t (\mu^m_{x+u} + \mu^f_{y+u})\,du\right\} = {}_tp_x\ {}_tp_y \tag{10.17} $$

and

$$ \begin{aligned} {}_tp^{02}_{xy} &= \int_0^t {}_rp^{00}_{xy}\ \mu^m_{x+r}\ {}_{t-r}p^{22}_{y+r}\,dr \\ &= \int_0^t {}_rp_x\ {}_rp_y\ \mu^m_{x+r}\ {}_{t-r}p_{y+r}\,dr \\ &= {}_tp_y \int_0^t {}_rp_x\ \mu^m_{x+r}\,dr \\ &= {}_tp_y\ (1 - {}_tp_x)\,. \end{aligned} \tag{10.18} $$

So

$$ \begin{aligned} \Pr[T_y > t] &= {}_tp^{00}_{xy} + {}_tp^{02}_{xy} = {}_tp_x\ {}_tp_y + {}_tp_y\ (1 - {}_tp_x) \\ &= {}_tp_y. \end{aligned} $$

Similarly, $\Pr[T_x > s] = {}_sp_x$.

Now consider $\Pr[T_x > s \text{ and } T_y > t]$; this requires either (i) both (x) and (y) survive to time t, or (ii) (x) and (y) both survive to time $s < t$, and, subsequently, (x) dies and (y) survives in the interval from time s to time t. This gives the joint survival probability

$$ \Pr[T_x > s \text{ and } T_y > t] = {}_tp^{00}_{xy} + {}_sp^{00}_{xy}\ {}_{t-s}p^{02}_{x+s:y+s}, $$

where we have used the Markov property to write the probability for case (ii). Using (10.17) and (10.18), we have

$$ {}_tp^{00}_{xy} = {}_tp_x\,{}_tp_y \quad \text{and} \quad {}_{t-s}p^{02}_{x+s:\,y+s} = {}_{t-s}p_{y+s}\,(1 - {}_{t-s}p_{x+s})\,. $$

Hence

$$\begin{aligned}\Pr[T_x > s \text{ and } T_y > t] &= {}_tp_x\, {}_tp_y + {}_sp_x\, {}_sp_y \left({}_{t-s}p_{y+s}\,(1 - {}_{t-s}p_{x+s})\right)\\ &= {}_tp_x\, {}_tp_y + {}_tp_y\,({}_sp_x - {}_tp_x)\\ &= {}_sp_x\, {}_tp_y\\ &= \Pr[T_x > s]\Pr[T_y > t],\end{aligned}$$

which completes the proof. □

We emphasize here that throughout this proof we have used the assumption that forces of mortality for (x) and (y) are unaffected by the model state. More generally, this assumption will not be true, future lifetimes are not independent, and the breakdown of the joint life probabilities into single life probabilities will no longer be valid.

Example 10.5 Using the multiple state model, with the independence assumption, shown in Figure 10.1, write down integral equations for each of the following, and describe the benefit being valued:
(a) $\bar{A}^{01}_{xy}$, (b) $\bar{A}^{01}_{xy} + \bar{A}^{02}_{xy}$, (c) $\bar{a}^{01}_{xy}$.

Solution 10.5 (a) This is the EPV of a unit benefit payable on the death of (y) provided (x) is still alive at that time. As an integral,

$$\bar{A}^{01}_{xy} = \int_0^\infty e^{-\delta t}\, {}_tp^{00}_{xy}\, \mu^f_{y+t}\, dt = \bar{A}_{x:\overset{1}{y}}$$

noting that $\mu^{01}_{x+t:y+t} = \mu^f_{y+t}$.

(b) This is the EPV of a unit benefit payable on the first death of (x) and (y). As an integral,

$$\bar{A}^{01}_{xy} + \bar{A}^{02}_{xy} = \int_0^\infty e^{-\delta t}\, {}_tp^{00}_{xy}\,(\mu^f_{y+t} + \mu^m_{x+t})\, dt = \bar{A}_{xy}.$$

(c) This is the EPV of a unit reversionary annuity payable to (x) after the death of (y). As an integral,

$$\bar{a}^{01}_{xy} = \int_0^\infty e^{-\delta t}\, {}_tp^{01}_{xy}\, dt = \int_0^\infty e^{-\delta t}\, {}_tp_x(1 - {}_tp_y)\, dt = \bar{a}_x - \bar{a}_{xy} = \bar{a}_{y|x}.$$

□

Note here that, as expected, the independent multiple state model generates results consistent with the model of Section 10.4.

Example 10.6 Use the independent multiple state model to write down equations for the EPVs of the following benefits, and simplify as far as possible. Assume that at time 0 the lives (x) and (y) are in State 0.

(a) An insurance of 1 payable on the death of (y), conditional on (x) dying first.
(b) A joint life annuity of 1 per year, payable continuously, guaranteed for n years.
(c) A last survivor annuity of 1 per year, payable continuously, deferred for n years.

Solution 10.6 (a) The EPV is

$$\bar{A}_{x:\overset{2}{y}} = \int_0^\infty e^{-\delta t}\, {}_tp_{xy}^{02}\, \mu_{y+t}^{f}\, dt = \int_0^\infty e^{-\delta t}\, {}_tp_y (1 - {}_tp_x)\, \mu_{y+t}^{f}\, dt$$
$$= \bar{A}_y - \bar{A}_{x:\overset{1}{y}}.$$

(b) For guaranteed annuities, we separate the value of the first n years payments, which are certain, and the value of the annuity after n years, which depends on which state the process is in at that time. In this example, after n years, the annuity continues if the process is then in State 0, and ceases otherwise. Hence

$$\bar{a}_{\overline{xy:\overline{n|}}} = \bar{a}_{\overline{n|}} + e^{-\delta n}\, {}_np_{xy}^{00}\, \bar{a}_{x+n:y+n}^{00}.$$

(c) This follows similarly to (b), but we must now also take into consideration the possibility that exactly one life survives the guarantee period. We have

$$ {}_{n|}\bar{a}_{\overline{xy}} = e^{-\delta n}\, {}_np_{xy}^{00}\, \bar{a}_{\overline{x+n:y+n}} + e^{-\delta n}\, {}_np_{xy}^{01}\, \bar{a}_{x+n} + e^{-\delta n}\, {}_np_{xy}^{02}\, \bar{a}_{y+n}$$
$$= {}_{n|}\bar{a}_x + {}_{n|}\bar{a}_y - {}_{n|}\bar{a}_{xy}.$$

□

10.6 A model with dependent future lifetimes

The disadvantage of the approach in Sections 10.4 and 10.5 is that the assumption of independence may not be appropriate for couples who purchase joint life insurance or annuity products. The following three factors are often cited as sources of dependence between married partners.

- The death of the first to die could adversely affect the mortality of the survivor. This is sometimes called the ‘broken heart syndrome’.
- The two lives could die together in an accident. This is called the ‘common shock’ risk.

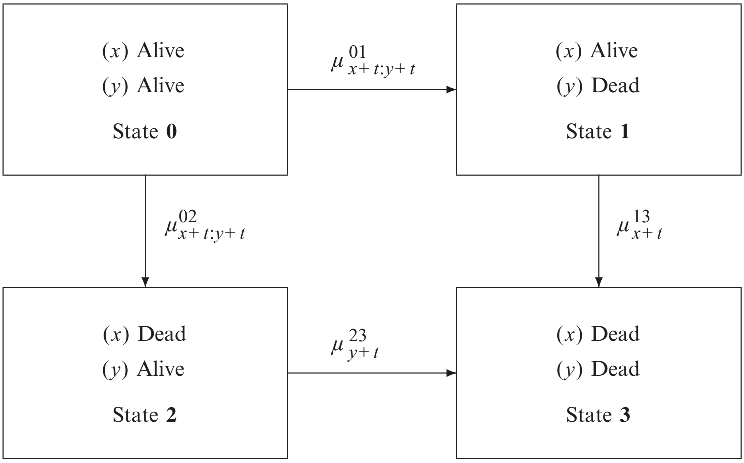


Figure 10.2 A joint life and last survivor model.

- The two lives are likely to share a common lifestyle. For example, mortality is related to wealth and levels of education. Married couples tend to have similar levels of wealth and similar levels of education. They may also share interests, for example, related to health and fitness.

In this section, we relax the assumption of independence, introducing dependence in a relatively intuitive way, allowing us to apply the methods and results of Chapter 8 to cash flows contingent on two dependent lives.

The modification is illustrated in Figure 10.2; we now allow for the force of mortality of (x) to depend on whether (y) is still alive, and what age (y) is, and *vice versa*.

More formally, we incorporate the following assumption.

State-Dependent Mortality Assumption. The force of mortality for each life depends on whether the other partner is still alive. If the partner is alive, the intensity may depend on the exact age of the partner, as well as the age of the life being considered. If one partner has died, the transition intensity for the survivor depends only on the survivor's age and state.

Our notation is adjusted appropriately for this assumption. For example, $\mu^{01}_{x+t:y+t}$ is the intensity of mortality for (y) at age $y+t$ given that (x) is still alive and aged $x+t$. However, if one partner, say (x), has died, the intensity of mortality for (y) depends on her then current age, and the fact that (x) has died, but not on how long he has been dead. Since the age at death of (x) is assumed not to affect the transition intensity from State 2 to State 3, this intensity is denoted μ^{23}_{y+t}, where $y+t$ is the current age of (y).

This model allows for some dependence between the lives; the death of, say, (x) affects the transition intensity of (y). It does not allow for both lives dying simultaneously; we discuss a way of incorporating this in the next section.

Since none of the states in the model can be re-entered once it has been left, we have

$$ {}_tp_{xy}^{ii} \equiv {}_tp_{xy}^{\overline{ii}} \quad \text{for } i = 0, 1, 2, 3 $$

so that using formula (8.9)

$$ {}_tp_{xy}^{00} = \exp\left\{ -\int_0^t (\mu_{x+s:y+s}^{01} + \mu_{x+s:y+s}^{02})\, ds \right\}, \tag{10.19} $$

$$ {}_tp_{x}^{11} = \exp\left\{ -\int_0^t \mu_{x+s}^{13}\, ds \right\}, $$

$$ {}_tp_{y}^{22} = \exp\left\{ -\int_0^t \mu_{y+s}^{23}\, ds \right\}, $$

and, for example,

$$ {}_tp_{xy}^{01} = \int_0^t {}_sp_{xy}^{00}\ \mu_{x+s:y+s}^{01}\ {}_{t-s}p_{x+s}^{11}\, ds. \tag{10.20} $$

Assuming, as usual, that we know the transition intensities, probabilities for the model can be evaluated either by starting with Kolmogorov's forward equations, (8.16), and then using Euler's method, or some more sophisticated, method, or, alternatively, by starting with formulae corresponding to (10.19) and (10.20) and integrating, probably numerically.

The probabilities listed in Section 10.2 do not all correspond to ${}_tp_{xy}^{ij}$ type probabilities. We examine two in more detail, in the context of the model discussed in this section, in the following example.

Example 10.7 (a) Explain why ${}_tq_{xy}^{1}$ is not the same as ${}_tp_{xy}^{02}$, and write down an integral equation for ${}_tq_{xy}^{1}$.

(b) Write down an integral equation for ${}_tq_{xy}^{2}$.

Solution 10.7 (a) The probability ${}_tq_{\ xy}^{1}$ is the probability that (x) dies within t years, and that (y) is alive at the time of (x)'s death.

The probability ${}_tp_{xy}^{02}$ is the probability that (x) dies within t years, and that (y) is alive at time t years. So the first probability allows for the possibility that (y) dies after (x), within t years, and the second does not.

The probability that (x) dies within t years, and that (y) is alive at the time of the death of (x) can be constructed by summing (integrating) over all the infinitesimal intervals in which (x) could die, conditioning on the survival of both (x) and (y) up to that time, so that

$$ {}_tq^{\,1}_{xy} = \int_0^t {}_rp^{00}_{xy}\,\mu^{02}_{x+r:y+r}\,dr. $$

(b) The probability ${}_tq^{\,2}_{xy}$ is the probability that (x) dies within t years and that (y) is already dead when (x) dies, conditional on (x) and (y) both being alive at time 0. In terms of the model in Figure 10.2, the process must move into State 1 and then into State 3 within t years, given that it starts in State 0 at time 0. Summing all the probabilities of such a move over infinitesimal intervals, we have

$$ {}_tq^{\,2}_{xy} = \int_0^t {}_rp^{01}_{xy}\,\mu^{13}_{x+r}\,dr. \qquad \square $$

Example 10.8 Derive the following expression for the probability that (x) has died before reaching age $x+t$, given that (x) and (y) are in State 0 at time 0;

$$ \int_0^t {}_sp^{00}_{xy}\,\mu^{02}_{x+s:y+s}\,ds + \int_0^t\int_0^s {}_up^{00}_{xy}\,\mu^{01}_{x+u:y+u}\,{}_{s-u}p^{11}_{x+u}\,du\,\mu^{13}_{x+s}\,ds. $$

Solution 10.8 For (x) to die before time t, we require the process either to

(1) enter State 2 from State 0 at some time s $(0 < s \le t)$, or
(2) enter State 1 ((y) dies while (x) is alive) at some time u $(0 < u \le t)$ and then enter State 3 at some time s $(u < s \le t)$.

The total probability of these events, integrating over the time of death of (x), is

$$ \int_0^t {}_sp^{00}_{xy}\,\mu^{02}_{x+s:y+s}\,ds + \int_0^t {}_sp^{01}_{xy}\,\mu^{13}_{x+s}\,ds $$

where

$$ {}_sp^{00}_{xy} = \exp\left\{-\int_0^s \left(\mu^{01}_{x+u:y+u} + \mu^{02}_{x+u:y+u}\right)du\right\}, $$

$$ {}_sp^{01}_{xy} = \int_0^s {}_up^{00}_{xy}\,\mu^{01}_{x+u:y+u}\,{}_{s-u}p^{11}_{x+u}du, $$

and

$$ {}_{s-u}p_{x+u}^{11} = \exp\left\{ -\int_{0}^{s-u} \mu_{x+u+r}^{13}\, dr \right\}. $$

This gives the required formula. □

We can write down formulae for the EPVs of annuities and insurances in terms of probabilities, the transition intensities and the interest rate. For annuities we have the following formulae, given (x) and (y) are alive at time $t = 0$:

$$ \bar{a}_y = \int_0^\infty e^{-\delta t} \left({}_tp_{xy}^{00} + {}_tp_{xy}^{02} \right) dt, $$

$$ \bar{a}_x = \int_0^\infty e^{-\delta t} \left({}_tp_{xy}^{00} + {}_tp_{xy}^{01} \right) dt, $$

$$ \bar{a}_{xy} = \int_0^\infty e^{-\delta t}\, {}_tp_{xy}^{00}\, dt, $$

$$ \bar{a}_{\overline{xy}} = \int_0^\infty e^{-\delta t} \left({}_tp_{xy}^{00} + {}_tp_{xy}^{01} + {}_tp_{xy}^{02} \right) dt, $$

$$ \bar{a}_{x|y} = \int_0^\infty e^{-\delta t}\, {}_tp_{xy}^{02}\, dt. $$

For the EPVs of the lump sum payments we have the following formulae:

$$ \bar{A}_y = \int_0^\infty e^{-\delta t} \left({}_tp_{xy}^{00}\, \mu_{x+t:y+t}^{01} + {}_tp_{xy}^{02}\, \mu_{y+t}^{23} \right) dt, $$

which values a unit benefit paid on transition from State 0 to State 1, or from State 2 to State 3;

$$ \bar{A}_{xy} = \int_0^\infty e^{-\delta t}\, {}_tp_{xy}^{00} \left(\mu_{x+t:y+t}^{01} + \mu_{x+t:y+t}^{02} \right) dt, $$

which values a unit benefit paid on transition out of State 0, with EPV $A_{xy}^{01} + A_{xy}^{02}$;

$$ \bar{A}_{\overline{xy}} = \int_0^\infty e^{-\delta t} \left({}_tp_{xy}^{01}\, \mu_{x+t}^{13} + {}_tp_{xy}^{02}\, \mu_{y+t}^{23} \right) dt, $$

which values a benefit paid on transition from State 1 to State 3, or from State 2 to State 3;

$$\bar{A}^{\,1}_{xy} = \int_0^\infty e^{-\delta t}\, {}_tp^{00}_{xy}\, \mu^{02}_{x+t:y+t}\, dt,$$

which values a benefit paid on transition from State 0 to State 2, that is A^{02}_{xy}; and

$$\bar{A}^{\,1}_{xy:\overline{n}|} = \int_0^n e^{-\delta t}\, {}_tp^{00}_{xy}\, \mu^{02}_{x+t:y+t}\, dt,$$

which values a benefit paid on transition from State 0 to State 2, provided it occurs within n years.

Example 10.9 For the model illustrated in Figure 10.2, you are given the following information:

$$\mu^{01}_{x+t:y+t} = B_f\, c_f^{y+t}, \qquad \mu^{02}_{x+t:y+t} = B_m\, c_m^{x+t},$$

$$\mu^{13}_{x+t} = C_m\, d_m^{x+t}, \qquad \mu^{23}_{y+t} = C_f\, d_f^{y+t},$$

where

$$B_f = 9.741 \times 10^{-7}, \quad c_f = 1.1331, \quad B_m = 2.622 \times 10^{-5}, \quad c_m = 1.0989,$$

$$C_m = 3.899 \times 10^{-4}, \quad d_m = 1.0725, \quad C_f = 2.638, \times 10^{-5}, \quad d_f = 1.1020.$$

For a couple (x, y), where $x = 65$ and $y = 62$,

(a) Calculate the probability that both (x) and (y) are alive in 15 years.
(b) Calculate the probability that (x) is alive in 15 years, using numerical integration.
(c) Calculate the probability that (y) is alive in 15 years, using numerical integration.
(d) Hence, calculate the probability that at least one is alive in 15 years.

Solution 10.9 Note first that each of the four forces of mortality has a Gompertz formula, so that each can be integrated analytically. Also, we see that the force of mortality for each partner depends on whether the spouse is still alive, though not on the age of the spouse. If (x) survives (y), his force of mortality increases from μ^{02}_{x+t} to μ^{13}_{x+t}. Similarly, (y)'s mortality increases if she is widowed. Thus, the two lives are not independent with respect to mortality.

(a) The probability that both are alive in 15 years is

$$
\begin{aligned}
{}_{15}p^{00}_{65:62} &= \exp\left\{-\int_0^{15} (\mu^{01}_{65+t:62+t} + \mu^{02}_{65+t:62+t})\, dt\right\} \\
&= g_f^{c_f^{62}(c_f^{15}-1)} \times g_m^{c_m^{65}(c_m^{15}-1)} \\
&= 0.905223 \times 0.671701 = 0.608039
\end{aligned}
$$

where

$$
g_f = \exp\left\{-\frac{B_f}{\log c_f}\right\} \quad \text{and} \quad g_m = \exp\left\{-\frac{B_m}{\log c_m}\right\}.
$$

(b) The probability that (x) is alive in 15 years is ${}_{15}p^{00}_{65:62} + {}_{15}p^{01}_{65:62}$. We already know from part (a) that ${}_{15}p^{00}_{65:62} = 0.608039$. By considering the time, t, at which (y) dies, we can write

$$
{}_{15}p^{01}_{65:62} = \int_0^{15} {}_tp^{00}_{65:62}\, \mu^{01}_{65+t:62+t}\, {}_{15-t}p^{11}_{65+t}\, dt, \tag{10.21}
$$

where, following the steps in part (a),

$$
{}_tp^{00}_{65:62} = g_f^{c_f^{62}(c_f^{t}-1)} \times g_m^{c_m^{65}(c_m^{t}-1)}
$$

and

$$
\begin{aligned}
{}_{15-t}p^{11}_{65+t} &= \exp\left\{-\int_0^{15-t} \mu^{13}_{65+t+s}\, ds\right\} \\
&= h_m^{d_m^{65+t}(d_m^{15-t}-1)}
\end{aligned}
$$

where $h_m = \exp\{-C_m/\log d_m\}$.
The integral in formula (10.21) can now be evaluated numerically, giving

$$
{}_{15}p^{01}_{65:62} = 0.050402
$$

so that

$$
{}_{15}p^{00}_{65:62} + {}_{15}p^{01}_{65:62} = 0.658442.
$$

(c) The probability that (y) is alive in 15 years is ${}_{15}p^{00}_{65:62} + {}_{15}p^{02}_{65:62}$, where

$$
{}_{15}p^{02}_{65:62} = \int_0^{15} {}_tp^{00}_{65:62}\, \mu^{02}_{65+t:62+t}\, {}_{15-t}p^{22}_{62+t}\, dt \tag{10.22}
$$

and

$$
{}_{15-t}p^{22}_{62+t} = \exp\left\{ - \int_0^{15-t} \mu^{23}_{62+t+s}\, ds \right\}
$$

$$
= h_f^{\,d_f^{62+t}\,(d_f^{15-t}-1)}
$$

where $h_f = \exp\{-C_f / \log d_f\}$.

We can evaluate numerically the integral in formula (10.22), giving

$$
{}_{15}p^{02}_{65:62} = 0.258823
$$

so that

$$
{}_{15}p^{00}_{65:62} + {}_{15}p^{02}_{65:62} = 0.866862.
$$

(d) The probability that at least one is alive in 15 years is

$$
\begin{aligned}
{}_{15}p^{00}_{65:62} + {}_{15}p^{01}_{65:62} + {}_{15}p^{02}_{65:62} &= 0.608039 + 0.050402 + 0.258823 \\
&= 0.917265.
\end{aligned}
$$

□

10.7 The common shock model

The model illustrated in Figure 10.2 incorporates dependence between (x) and (y) by allowing the transition intensity of each to depend on whether the other is still alive. We can extend this dependence by allowing for (x) and (y) to die at the same time, for example as the result of a car accident. This is illustrated in Figure 10.3, the so–called **common shock model**.

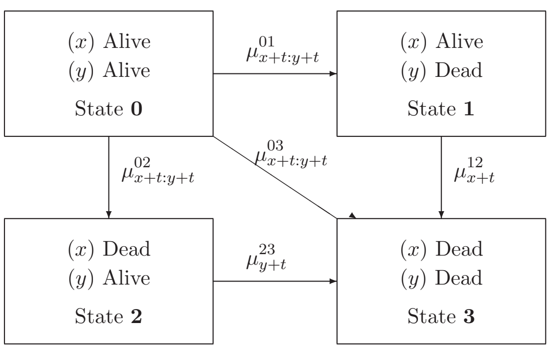


Figure 10.3 A common shock model.

Example 10.10 You are given the following transition intensities for the common shock model:

$$\mu^{01}_{x+t:y+t} = B_f\, c_f^{y+t}, \qquad \mu^{02}_{x+t:y+t} = B_m\, c_m^{x+t},$$
$$\mu^{13}_{x+t} = C_m\, d_m^{x+t}, \qquad \mu^{23}_{y+t} = C_f\, d_f^{y+t}, \qquad \mu^{03}_{x+t:y+t} = \lambda,$$

where

$$B_f = 9.741 \times 10^{-7}, \quad c_f = 1.1331, \quad B_m = 2.622 \times 10^{-5}, \quad c_m = 1.0989,$$
$$C_m = 3.899 \times 10^{-4}, \quad d_m = 1.0725, \quad C_f = 2.638 \times 10^{-5}, \quad d_f = 1.1020,$$
$$\lambda = 1.407 \times 10^{-3}.$$

Calculate ${}_{10}p^{00}_{70:65}$, ${}_{10}p^{01}_{70:65}$, and ${}_{10}p^{02}_{70:65}$.

Note that the transition intensities in this example are the same as in Example 10.9, except that we have added the common shock transition intensity, which is assumed to be constant.

Solution 10.10 We can evaluate ${}_{10}p^{00}_{70:65}$ directly from

$$\begin{aligned}
{}_{10}p^{00}_{70:65} &= \exp\left\{ -\int_0^{10} (\mu^{01}_{70+u:60+u} + \mu^{02}_{70+u:60+u} + \mu^{03}_{70+u:60+u})\, du \right\} \\
&= g_f^{c_f^{65}\,(c_f^{10}-1)} \times g_m^{c_m^{70}\,(c_m^{10}-1)} \times e^{-10\,\lambda} \\
&= 0.670051,
\end{aligned}$$

where g_m and g_f are as defined in the solution to Example 10.9.

For the probabilities of transition to State 1 or State 2, we have

$${}_{10}p^{01}_{70:65} = \int_0^{10} {}_tp^{00}_{70:65}\, \mu^{01}_{70+t:65+t}\, {}_{10-t}p^{11}_{70+t}\, dt, \tag{10.23}$$

$${}_{10}p^{02}_{70:65} = \int_0^{10} {}_tp^{00}_{70:65}\, \mu^{02}_{70+t:65+t}\, {}_{10-t}p^{22}_{65+t}\, dt \tag{10.24}$$

and

$$\begin{aligned}
{}_sp^{00}_{70:65} &= g_f^{c_f^{65}\,(c_f^{s}-1)}\; g_m^{c_m^{70}\,(c_m^{s}-1)}\; e^{-s\,\lambda}, \\
{}_sp^{11}_{70} &= h_m^{d_m^{70}\,(d_m^{s}-1)}, \\
{}_sp^{22}_{65} &= h_f^{d_f^{65}\,(d_f^{s}-1)},
\end{aligned}$$

where h_m and h_f are as defined in the solution to Example 10.9.

Numerical integration gives

$$ {}_{10}p^{01}_{70:65} = 0.03771 \quad \text{and} \quad {}_{10}p^{02}_{70:65} = 0.23255. $$

□

Note the similarity between formulae (10.23) and (10.24) and formulae (10.21) and (10.22). The difference is that the ${}_tp^{00}_{xy}$ values are calculated differently, with the extra term $e^{-\lambda t}$ allowing for the common shock risk.

Example 10.11 A husband and wife, aged 63 and 61 respectively, have just purchased a joint life 15–year term insurance with sum insured \$200 000 payable immediately on the death of the first to die, or on their simultaneous deaths, within 15 years. Level premiums are payable monthly for at most 15 years while both are still alive.

Calculate

(a) the monthly premium, and
(b) the gross premium policy value after 10 years, assuming both husband and wife are still alive.

Use the common shock model with the following basis.
Interest: 4% per year
Survival model: As in Example 10.10
Expenses:
Initial expenses of \$500
Renewal expenses of 10% of all premiums
\$200 on payment of the sum insured

Solution 10.11 (a) The EPV of a unit sum insured is

$$ \int_0^{15} v^t \, {}_tp^{00}_{63:61} \left(\mu^{01}_{63+t:61+t} + \mu^{02}_{63+t:61+t} + \mu^{03}_{63+t:61+t}\right) dt $$

and the EPV of a unit premium per month is

$$ 12\,\ddot{a}^{(12)}_{63:61:\,\overline{15|}} = \sum_{t=0}^{179} v^{t/12} \; {}_{t/12}p^{00}_{63:61}. $$

Let $\mu^{0\bullet}_{x+t:y+t} = \sum_{j=1}^{3} \mu^{0j}_{x+t:y+t}$. Then the formula for the monthly premium, P, is

$$ 12P\,\ddot{a}^{(12)}_{63:61:\overline{15|}} = 200\,200 \int_0^{15} v^t \, {}_tp^{00}_{63:61} \, \mu^{0\bullet}_{63+t:\,61+t} \, dt $$

$$ + \, 500 + 0.1 \times 12P\,\ddot{a}^{(12)}_{63:61:\overline{15|}}. $$

Using spreadsheet calculations (with numerical integration to value the death benefit), we find that

$$P = (200\,200 \times 0.25574 + 500)/(0.9 \times 12 \times 9.87144)$$
$$= \$484.94.$$

(b) The gross premium policy value at duration 10 years, just before the premium then due is paid, is

$$
\begin{aligned}
{}_{10}V &= 200\,200 \int_0^5 v^t \, {}_tp^{00}_{73:71} \, \mu^{0\bullet}_{73+t:71+t} \, dt - 0.9\,P \sum_{t=0}^{59} v^{t/12} \, {}_{t/12}p^{00}_{73:71} \\
&= 200\,200 \times 0.17776 - 0.9 \times 484.94 \times 12 \times 4.14277 \\
&= \$13\,890.
\end{aligned}
$$

□

10.8 Notes and further reading

The multiple state model approach is very flexible, allowing us to introduce dependence in various ways. In Section 10.6 we modelled 'broken heart syndrome' by allowing the mortality of each partner to depend on whether the other was still alive and in Section 10.7 we extended this by allowing for the possibility that the two lives die simultaneously. A more realistic model for broken heart syndrome would allow for the mortality of the surviving partner to depend not only on their current age and the fact that their spouse had died, but also on the time since the spouse died. This would make the model semi–Markov, rather than Markov, and is beyond the scope of this book.

The paper by Ji *et al.* (2012) discusses Markov, semi–Markov and other models for multiple life statuses. They use data from a large Canadian insurance company from 1988–93 to parameterize the common shock model in Section 10.7. Their parameterization is used in Example 10.10. Note that their definition of 'simultaneous death' is death within five days, though they do investigate the effects of altering this definition.

10.9 Exercises

Some actuarial functions for joint lives, both subject to the Standard Ultimate Survival Model, and assuming independent future lifetimes, are given in Appendix D.

Shorter exercises

Exercise 10.1 Two lives aged 60 and 70 are independent with respect to mortality. Given that ${}_{10}p_{60} = 0.94$, ${}_{10}p_{70} = 0.83$ and ${}_{10}p_{80} = 0.72$, calculate the probability that

(a) both lives are alive 10 years from now,
(b) at least one life is alive 10 years from now,
(c) exactly one of the lives is alive 10 years from now,
(d) the joint life status fails within the next 10 years,
(e) the last survivor status fails within the next 10 years,
(f) the joint life status holds 10 years from now, but not 20 years from now,
(g) the last survivor status holds 10 years from now, but not 20 years from now.

Exercise 10.2 A couple, currently aged x and y, have the following individual and joint one-year survival probabilities for $t = 0, 1, 2$:

$$p_{x+t} = 0.75, \quad p_{y+t} = 0.85, \quad p_{x+t:y+t} = 0.7.$$

Calculate the probability that (x) survives two years and dies in the third year, and (y) survives three years.

Exercise 10.3 Using the Joint Life Tables in Appendix D, calculate the following functions. Assume UDD (in single and joint life models) where necessary.

(a) $\ddot{a}_{50:60:\overline{10}|}$, (b) $\ddot{a}_{\overline{50:60}}$, (c) $\ddot{a}_{60|50}$, (d) $\bar{A}_{50:60}$, (e) $\bar{A}^{\;1}_{60:60:\overline{10}|}$, (f) $\bar{A}^{\;\;2}_{60:60}$.

Exercise 10.4 Two lives aged 60 and 70 are independent with respect to mortality, and the Standard Ultimate Survival Model is applicable for each. On the basis of an effective rate of interest of 5% per year, calculate the EPV of

(a) an annuity of \$20 000 a year, payable in arrear as long as at least one of the lives is alive,
(b) an annuity of \$30 000 a year, payable annually in advance for at most 10 years, provided that both lives are alive, and
(c) a reversionary annuity of \$25 000 a year, payable annually to (60) following the death of (70).

Exercise 10.5 For independent lives (x) and (y), show that

$$\mathrm{Cov}\left(v^{T_{\overline{xy}}}, v^{T_{xy}}\right) = \left(\bar{A}_x - \bar{A}_{xy}\right)\left(\bar{A}_y - \bar{A}_{xy}\right).$$

Exercise 10.6 An insurer sells an annuity policy to Shannon and Riley, who are both aged 60, with the following premium and benefit terms.

- Level annual premiums are payable for at most 10 years, while both Shannon and Riley are alive.

- There are no annuity payments during the first 10 years.
- After 10 years, at the start of each year the annuity pays:
  - $120 000 if both Shannon and Riley are alive at the payment date, and
  - $70 000 if only one of them is alive at the payment date.

Assume independent future lifetimes, and that mortality follows the Standard Ultimate Life Table with interest at 5%.

Calculate the annual net premium.

Exercise 10.7 By considering the cases $T_y > T_x$ and $T_y \leq T_x$, show that the present value random variable for a reversionary annuity payable continuously at rate 1 per year to (y) following the death of (x) is

$$Y = \bar{a}_{\overline{T_y}|} - \bar{a}_{\overline{T_{xy}}|}.$$

Exercise 10.8 Tom and John are both aged 75 with independent future lifetimes. They purchase an insurance policy which pays $100 000 immediately on the death of John provided he dies after Tom. Premiums are payable continuously at a rate P per year while both lives are alive. You are given that

(i) $\bar{A}_{75} = 0.46570$ and $\bar{A}_{75:75} = 0.57481$,
(ii) $i = 6\%$.

Calculate P.

Longer exercises

Exercise 10.9 Two independent lives, (x) and (y), experience mortality according to Gompertz' law, that is, $\mu_x = Bc^x$.

(a) Show that ${}_tp_{xy} = {}_tp_w$ for $w = \log(c^x + c^y)/\log c$. In this case w is called the **equivalent single age**.
(b) Show that

$$\bar{A}^{\,1}_{x:y} = \frac{c^x}{c^w}\,\bar{A}_w.$$

Exercise 10.10 Two independent lives, (x) and (y), experience mortality according to Makeham's law, that is, $\mu_x = A + Bc^x$.

(a) Show that ${}_tp_{xy} = {}_tp_{w:w}$ for $w = \log((c^x + c^y)/2)/\log c$. In this case w is called the **equivalent equal age**.
(b) Amanda is aged 48.24 and Zoe is aged 60. Both are subject to the Standard Ultimate Survival Model (which follows Makeham's model with $c = 1.124$). Assume their future lifetimes are independent.

Use the result in part (a), and the joint life tables in Appendix D, to determine the EPV of the following annuities written on the lives of Amanda and Zoe:

(i) a joint life annuity-due of \$10 000 per year, and
(ii) a last survivor annuity-due of \$10 000 per year.

Exercise 10.11 Assume that T_x and T_y are independent.

(a) Show that the probability density function of T_{xy} is

$$f_{T_{xy}}(t) = {}_tp_{xy}\,(\mu_{x+t} + \mu_{y+t}).$$

(b) What is the joint probability density function of (T_x, T_y)? Use this joint probability density function to obtain formula (10.15) for $\bar{A}^{\,1}_{xy}$.
(c) $\bar{A}^{\,2}_{xy}$ is the EPV of a benefit of 1 payable on the death of (x) provided that the death of (x) occurs after the death of (y). Using the same approach as in part (b), show that

$$\bar{A}^{\,2}_{xy} = \bar{A}_x - \bar{A}^{\,1}_{xy}.$$

Explain this result.

Exercise 10.12 Bob and Mike are independent lives, both aged 50. They purchase an insurance policy which provides \$100 000, payable at the end of the year of Bob's death, provided Bob dies after Mike. Annual premiums are payable in advance throughout Bob's lifetime. Calculate

(a) the net annual premium, and
(b) the net premium policy value after 10 years (before the premium then due is payable) if
  (i) only Bob is then alive, and
  (ii) both lives are then alive.

Basis:

Survival model: Standard Ultimate Life Table, independent lives
Interest: 5% per year effective
Expenses: None

Exercise 10.13 A husband and wife, aged 65 and 60 respectively, purchase an insurance policy, under which the benefits payable on first death are a lump sum of \$10 000, payable immediately on death, plus an annuity of \$5000 per year payable continuously throughout the lifetime of the surviving spouse. A benefit of \$1000 is paid immediately on the second death. Premiums are payable continuously until the first death.

You are given that $\bar{A}_{60} = 0.353789$, $\bar{A}_{65} = 0.473229$ and that $\bar{A}_{60:65} = 0.512589$ at 4% per year effective rate of interest. The lives are assumed to be independent.

(a) Calculate the EPV of the lump sum death benefits, at 4% per year interest.
(b) Calculate the EPV of the reversionary annuity benefit, at 4% per year interest.
(c) Calculate the annual rate of premium, at 4% per year interest.
(d) Ten years after the contract is issued the insurer is calculating the policy value.
  (i) Write down an expression for the policy value at that time assuming that both lives are still surviving.
  (ii) Write down an expression for the policy value assuming that (x) has died but (y) is still alive.
  (iii) Write down Thiele's differential equation for the policy value assuming (1) both lives are still alive, and (2) only (y) is alive.

Exercise 10.14 Use the multiple state model illustrated in Figure 10.4, and an interest rate of 5%, for this question.

You are given the following transition intensities, where μ_z^* is the force of mortality for a life aged z under the Standard Ultimate Survival Model:

$$\mu_{x:y}^{01} = \mu_y^* - 0.0005, \qquad \mu_{x:y}^{02} = \mu_x^* - 0.0005, \qquad \mu_{x:y}^{04} = 0.0005,$$
$$\mu_{x:y}^{13} = \mu_x^*, \qquad \mu_{x:y}^{23} = 1.2\mu_y^*.$$

Reminder: $\mu_z^* = A + Bc^z$ where $A = 0.00022$, $B = 2.7 \times 10^{-6}$ and $c = 1.124$.

(a) (i) Explain why the t-year survival probability for (x) does not depend on whether (y) is alive or dead.

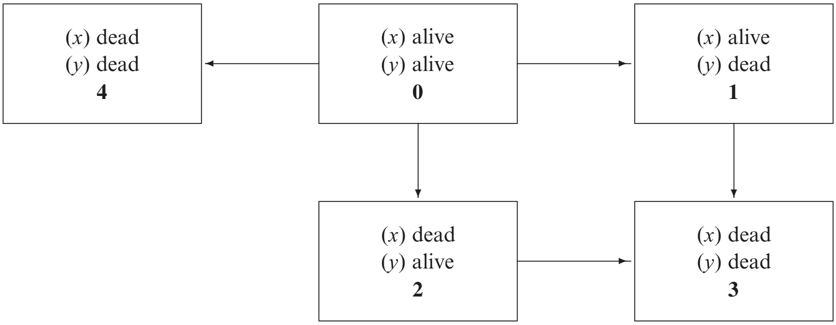


Figure 10.4 Model for Exercise 10.14

(ii) Explain why the future lifetimes of (x) and (y) are not independent (despite the result in part (i)).

(b) Calculate ${}_{20}p^{00}_{60:55}$.

A couple, (x), who is 60 and (y), who is 55, buys a policy combining a last survivor insurance and an annuity, with the following features.

- Level premiums are payable continuously in State 0 for a maximum of 20 years.
- A death benefit of \$200 000 is payable immediately on the second death, if deaths are not simultaneous.
- If the deaths are simultaneous, the death benefit is increased to \$400 000.
- A reversionary annuity of \$20 000 per year is payable continuously to (y) following the death of (x), continuing until the death of (y).

The following values have been calculated based on the model and parameters described above, with interest at 5% per year.

$x{:}y$	${}_{20}p^{01}_{x:y}$	${}_{20}p^{02}_{x:y}$	$\bar{a}^{00}_{x:y}$	$\bar{a}^{01}_{x:y}$	$\bar{a}^{02}_{x:y}$	$\bar{a}^{22}_{y}$
60:55	0.09570	0.18100	13.4155	0.9844	2.0300	15.2083
80:75	0.06189	0.18869	6.7127	1.3294	2.8729	9.2615

$x{:}y$	$\bar{A}^{03}_{x:y}$	$\bar{A}^{04}_{x:y}$	$\bar{A}^{13}_{x}$	$\bar{A}^{23}_{y}$
60:55	0.19307	0.00537	0.29743	0.25798
80:75	0.46488	0.00269	0.60764	0.54813

(c) Calculate the EPV of the non-simultaneous death benefit.

(d) Calculate the annual rate of premium.

(e) Calculate the policy values at time 20 for each of the in-force states.

(f) Write down Thiele's differential equations for ${}_tV^{(0)}$, ${}_tV^{(1)}$ and ${}_tV^{(2)}$, for $t < 20$.

(g) Using $h = 0.25$ estimate ${}_{19.5}V^{(0)}$, ${}_{19.5}V^{(1)}$ and ${}_{19.5}V^{(2)}$ using Euler's backward method.

(h) Explain why we use Figure 10.4 for this policy, rather than Figure 10.3.

Exercise 10.15 Let A_{xy} denote the EPV of a benefit of 1 payable at the end of the year in which the first death of (x) and (y) occurs, and let $A^{(m)}_{xy}$ denote the EPV of a benefit of 1 payable at the end of the $\frac{1}{m}$th of a year in which the first death of (x) and (y) occurs.

(a) As an EPV, what does the following expression represent?

$$\sum_{t=1}^{m} v^{\frac{t}{m}} \left({}_{\frac{t-1}{m}}p_{xy} - {}_{\frac{t}{m}}p_{xy} \right).$$

(b) Write down an expression for $A_{xy}^{(m)}$ in summation form by considering the insurance benefit as comprising a series of deferred one year term insurances with the benefit payable at the end of the $\frac{1}{m}$th of a year in which the first death of (x) and (y) occurs.

(c) Assume that two lives (x) and (y) are independent with respect to mortality. Show that under the UDD assumption,

$${}_{\frac{t-1}{m}}p_{xy} - {}_{\frac{t}{m}}p_{xy} = \frac{1}{m}(1 - p_{xy}) + \frac{m - 2t + 1}{m^2} q_x\, q_y$$

and that

$$v^{\frac{t}{m}} \left({}_{\frac{t-1}{m}}p_{xy} - {}_{\frac{t}{m}}p_{xy} \right)$$
$$= (1 - p_{xy}) \frac{iv}{i^{(m)}} + q_x\, q_y \sum_{t=1}^{m} v^{t/m} \frac{m - 2t + 1}{m^2}.$$

(d) Deduce that under the assumptions in part (c),

$$A_{xy}^{(m)} \approx \frac{i}{i^{(m)}} A_{xy}.$$

Exercise 10.16 (a) Show that

$$\frac{d}{dt}\, {}_tp_{x:y}^{00}\, v^t = -{}_tp_{x:y}^{00}\, v^t \left(\delta + \mu_{x+t:\,y+t}^{0\bullet} \right),$$

where $\mu_{x+t:\,y+t}^{0\bullet}$ is the total transition intensity out of State 0 for the joint life process.

(b) Use Woolhouse's formula to show that

$$\ddot{a}_{xy}^{(m)} \approx \ddot{a}_{xy} - \frac{m-1}{2m} - \frac{m^2 - 1}{12m^2} \left(\delta + \mu_{x:y}^{0\bullet} \right).$$

Excel-based exercises

Exercise 10.17 Two lives aged 30 and 40 are independent with respect to mortality, and each is subject to Makeham's law of mortality with $A = 0.0001$, $B = 0.0003$ and $c = 1.075$. Calculate

(a) ${}_{10}p_{30:40}$,
(b) ${}_{10}q_{\overset{1}{30}:40}$,

(c) ${}_{10}q^{2}_{30:40}$, and
(d) ${}_{10}p_{\overline{30:40}}$.

Exercise 10.18 Smith and Jones are both aged exactly 30. Smith is subject to Gompertz' law of mortality with $B = 0.0003$ and $c = 1.07$, and Jones is subject to a force of mortality at all ages x of $Bc^x + 0.039221$. Calculate the probability that Jones dies before reaching age 50 and before Smith dies. Assume that Smith and Jones are independent with respect to mortality.

Exercise 10.19 Ryan is entitled to an annuity of \$100 000 per year at retirement, paid monthly in advance, and the normal retirement age is 65. Ryan's wife, Lindsay, is two years younger than Ryan.

(a) Calculate the EPV of the annuity at Ryan's retirement date.
(b) Calculate the revised annual amount of the annuity (payable in the first year) if Ryan chooses to take a benefit which provides Lindsay with a monthly annuity following Ryan's death equal to 60% of the amount payable whilst both Ryan and Lindsay are alive.
(c) Calculate the revised annual amount of the annuity (payable in the first year) if Ryan chooses the benefit in part (b), with a 'pop-up' – that is, the annuity reverts to the full \$100 000 on the death of Lindsay if Ryan is still alive. (Note that under a **pop-up annuity**, the benefit reverts to the amount to which Ryan was originally entitled on the death of the reversionary beneficiary.)

Basis:

Male mortality before and after widowerhood:
Makeham's law, $A = 0.0001$, $B = 0.0004$ and $c = 1.075$
Female survival before widowhood:
Makeham's law, $A = 0.0001$, $B = 0.00025$ and $c = 1.07$
Female survival after widowhood:
Makeham's law, $A = 0.0001$, $B = 0.0003$ and $c = 1.072$
Interest: 5% per year effective

Exercise 10.20 A husband and wife are aged 28 and 24, respectively. They are about to effect an insurance policy that pays \$100 000 immediately on the first death. Calculate the premium for this policy, payable monthly in advance as long as both are alive and limited to 25 years, on the following basis:

- Male survival: Makeham's law, with $A = 0.0001$, $B = 0.0004$ and $c = 1.075$
- Female survival: Makeham's law, with $A = 0.0001$, $B = 0.0003$ and $c = 1.07$
- Interest: 5% per year effective

- Initial expenses: \$250
- Renewal expenses: 3% of each premium

Assume that this couple are independent with respect to mortality.

Exercise 10.21 An insurance company issues a joint life insurance policy to a married couple. The husband, (x), is aged 28 and his wife, (y), is aged 27. The policy provides a benefit of \$500 000 immediately on the death of (x) provided that he dies first. The policy terms stipulate that if the couple die at the same time, the elder life is deemed to have died first. Premiums are payable annually in advance while both lives are alive for at most 30 years.

Use the common shock model illustrated in Figure 10.3 to calculate the annual net premium using an effective rate of interest of 5% per year and transition intensities of

$$\mu_{xy}^{01} = A + Bc^y, \quad \mu_{xy}^{02} = A + Dc^x, \quad \mu_{xy}^{03} = 5 \times 10^{-5},$$

where $A = 0.0001$, $B = 0.0003$, $c = 1.075$ and $D = 0.00035$.

Answers to selected exercises

10.1 (a) 0.7802 (b) 0.9898 (c) 0.2096 (d) 0.2198 (e) 0.0102 (f) 0.3140 (g) 0.0782

10.2 0.0735

10.3 (a) 7.9044 (b) 17.6587 (c) 2.7546 (d) 0.32843 (e) 0.04242 (f) 0.10837

10.4 (a) \$293 808.37 (b) \$225 329.46 (c) \$92 052.87

10.6 \$110 650

10.8 \$2443.39

10.10 (b)(i) \$143 725 (ii) \$173 163

10.12 (a) \$387.47 (b)(i) 23 253 (ii) 4 800

10.13 (a) \$5 440.32 (b) \$25 262.16 (c) \$2470.55

10.14 (b) 0.67498 (c) 38 614 (d) \$6 949.38 (e) 151 510, 121 528, 294 856 (g) 146 391, 119 792, 296 663

10.17 (a) 0.886962 (b) 0.037257 (c) 0.001505 (d) 0.997005

10.18 0.567376

10.19 (a) \$802 639 (b) \$76 846 (c) \$73 942

10.20 \$161.78

10.21 \$4 948.24

11
Pension mathematics

11.1 Summary

In this chapter we introduce some of the notation and concepts of pension plan valuation and funding. We discuss the difference between defined benefit (DB) and defined contribution (DC) pension plans. We introduce the salary scale function, and show how to calculate an appropriate contribution rate in a DC plan to meet a target level of pension income.

We define the service table, which is a summary of the multiple state model appropriate for a pension plan. Using the service table and the salary scale, we can value the benefits and contributions of a pension plan, using the same principles as we have used for valuing benefits under an insurance policy.

We introduce benefit design, valuation and accruals based funding for final salary and career average earnings DB plans. In the final section, we apply the principles of pension funding and valuation to retiree health benefits.

11.2 Introduction

The pension plans we discuss in this chapter are typically employer sponsored plans, designed to provide employees with retirement income. Employers sponsor plans for a number of reasons, including

- competition for new employees;
- to facilitate turnover of older employees by ensuring that they can afford to retire;
- to provide incentive for employees to stay with the employer;
- pressure from trade unions;
- to provide a tax efficient method of remunerating employees;
- responsibility to employees who have contributed to the success of the company.

The plan design will depend on which of these motivations is most important to the sponsor. If competition for new employees is the most important factor, for example, then the employer's plan will closely resemble other employer sponsored plans within the same industry. Ensuring turnover of older employees, or rewarding longer service might lead to a different benefit design.

The two major categories of employer sponsored pension plans are **defined contribution** (DC) and **defined benefit** (DB).

The defined contribution pension plan specifies how much the employer will contribute, as a percentage of salary, into a plan. The employee may also contribute, and the employer's contribution may be related to the employee's contribution (for example, the employer may agree to match the employee's contribution up to some maximum). The contributions are invested, and the accumulated funds are available to the employee when he or she leaves the company. The contributions may be set to meet a target benefit level, but the actual retirement income may be well below or above the target, depending on the investment experience.

The defined benefit plan specifies a level of benefit, usually in relation to salary near retirement (final salary plans), or to salary throughout employment (career average salary plans). The contributions, from the employer and, possibly, employee are accumulated to meet the benefit. If the investment or demographic experience is adverse, the contributions can be increased; if experience is favourable, the contributions may be reduced. The pension plan actuary monitors the plan funding on a regular basis to assess whether the contributions need to be changed.

The benefit under a DB plan, and the target under a DC plan, are set by consideration of an appropriate **replacement ratio** or **replacement rate**. The pension plan replacement ratio is defined as

$$R = \frac{\text{pension income in the year after retirement}}{\text{salary in the year before retirement}}$$

where we assume the plan member survives the year following retirement. The target for the plan replacement ratio depends on other post-retirement income, such as government benefits. A total replacement ratio, including government benefits and personal savings, of around 70% is often assumed to allow retirees to maintain their pre-retirement lifestyle. Employer sponsored plans often target 50%–70% as the replacement ratio for an employee with a full career in the company.

11.3 The salary scale function

The contributions and the benefits for most employer sponsored pension plans are related to salaries, so we need to model the progression of salaries through an individual's employment. We use a deterministic model, even though future changes in salary cannot usually be predicted with the certainty a deterministic model implies. However, this model is almost universally used in practice and a more realistic model would complicate the presentation in this chapter.

We start by defining the **rate of salary function**, $\{\bar{s}_y\}_{y\ge x_o}$, where x_o is some suitable initial age. The value of $\bar{s}_{x_o}$ can be set arbitrarily as any positive number. For $y > x \ge x_o$, the value of $\bar{s}_y/\bar{s}_x$ is defined to be the ratio of the annual rate of salary at age y to the annual rate of salary at age x, where we assume the individual is employed from age x to age y.

Example 11.1 Consider an employee aged 30 whose current annual salary rate is \$30 000 and assume she will still be employed at exact age 41.

(a) Suppose the employee's rate of salary function $\{\bar{s}_y\}_{y\ge 20}$ is given by

$$\bar{s}_y = 1.04^{y-20}.$$

- (i) Calculate her annual rate of salary at exact age 30.5.
- (ii) Calculate her salary for the year of age 30 to 31.
- (iii) Calculate her annual rate of salary at exact age 40.5.
- (iv) Calculate her salary for the year of age 40 to 41.

(b) Now suppose that each year the rate of salary increases by 4%, three months after an employee's birthday and then remains constant for a year. Repeat parts (i) to (iv) of (a) above.

Solution 11.1

(a) (i) From the definition of the rate of salary function, the employee's annual rate of salary at age 30.5 will be

$$30\,000 \times 1.04^{0.5} = \$30\,594.$$

(ii) Consider a small interval from age $30 + t$ to $30 + t + dt$, where $0 \le t < t + dt \le 1$. The rate of salary in this age interval will be $30\,000\,\bar{s}_{30+t}/\bar{s}_{30}$ and the amount received by the employee will be $30\,000\,(\bar{s}_{30+t}/\bar{s}_{30})\,dt$.
Hence, her total income for the year of age 30 to 31 will be

$$\begin{aligned}\int_0^1 30\,000\,(\bar{s}_{30+t}/\bar{s}_{30})\,dt &= \int_0^1 30\,000 \times 1.04^t\,dt \\ &= 30\,000\,(1.04 - 1)/\log 1.04 \\ &= \$30\,596.\end{aligned}$$

(iii) Her salary rate at exact age 40.5 will be $30\,000 \times 1.04^{10.5} = \$45\,287$.

(iv) Using the same argument as in part (ii), her income for the year of age 40 to 41 will be

$$30\,000 \int_{10}^{11} 1.04^t \, dt = 30\,000\,(1.04^{11} - 1.04^{10}) / \log 1.04$$
$$= \$45\,290.$$

(b) (i) The employee will receive a salary increase of 4% at age 30.25 and her rate of salary will then remain constant until she reaches age 31.25, so her rate of salary at age 30.5 will be

$$30\,000 \times 1.04 = \$31\,200.$$

(ii) The income for the three months following her 30th birthday will be $30\,000\,(0.25) = \$7\,500$ and her income for the following nine months will be $30\,000\,(1.04)\,(0.75) = \$23\,400$. Hence, her income for the year of age 30 to 31 will be

$$7\,500 + 23\,400 = \$30\,900.$$

(iii) Her rate of salary income at age 40.5 will be

$$30\,000 \times 1.04^{10} = \$44\,407.$$

(iv) Her income for the year of age 40 to 41 will be

$$30\,000 \times (0.25 \times 1.04^9 + 0.75 \times 1.04^{10}) = \$43\,980.$$

□

In practice it is very common to model the progression of salaries using a **salary scale**, $\{s_y\}_{y \geq x_0}$, rather than a rate of salary function. The salary scale can be derived from the rate of salary function as follows. The value of s_{x_0} can be set arbitrarily as any positive number. For $y > x \geq x_0$, we define

$$\frac{s_y}{s_x} = \frac{\int_0^1 \bar{s}_{y+t} \, dt}{\int_0^1 \bar{s}_{x+t} \, dt}$$

so that, using the same argument as in Example 11.1 parts (ii) and (iv),

$$\frac{s_y}{s_x} = \frac{\text{salary received in year of age } y \text{ to } y+1}{\text{salary received in year of age } x \text{ to } x+1},$$

where we assume the individual remains in employment throughout the period from age x to age $y + 1$.

Compare this with the salary rate function,

$$\frac{\bar{s}_y}{\bar{s}_x} = \frac{\text{salary rate at exact age } y}{\text{salary rate at exact age } x},$$

where we assume the individual remains in employment throughout the period from age x to age y.

Salaries usually increase as a result of promotional increases and inflation adjustments. We assume in general that the salary scale allows for both forces, but it is straightforward to manage these separately.

Example 11.2 The final average salary for the pension benefit provided by a pension plan is defined as the average salary in the three years before retirement. Members' salaries are increased each year, six months before the valuation date.

(a) A member aged exactly 35 at the valuation date received \$75 000 in salary in the year to the valuation date. Calculate his predicted final average salary assuming retirement at age 65.
(b) A member aged exactly 55 at the valuation date was paid salary at a rate of \$100 000 per year at that time. Calculate her predicted final average salary assuming retirement at age 65.

Assume

(i) a salary scale where $s_y = 1.04^y$, and
(ii) the integer age example salary scale in Table 11.1, with linear interpolation between integer ages where necessary.

Solution 11.2 (a) The member is aged 35 at the valuation date, so that the salary in the previous year is the salary from age 34 to age 35. The predicted final average salary in the three years to age 65 is then

Table 11.1 *Example Salary Scale.*

x	s_x	x	s_x	x	s_x	x	s_x
30	1.000	40	2.005	50	2.970	60	3.484
31	1.082	41	2.115	51	3.035	61	3.536
32	1.169	42	2.225	52	3.091	62	3.589
33	1.260	43	2.333	53	3.139	63	3.643
34	1.359	44	2.438	54	3.186	64	3.698
35	1.461	45	2.539	55	3.234		
36	1.566	46	2.637	56	3.282		
37	1.674	47	2.730	57	3.332		
38	1.783	48	2.816	58	3.382		
39	1.894	49	2.897	59	3.432		

$$75\,000\,\frac{s_{62}+s_{63}+s_{64}}{3\,s_{34}},$$

which gives \$234 019 under assumption (i) and \$201 067 under assumption (ii).

(b) The current annual salary rate is \$100 000, so the final average salary is

$$100\,000\,\frac{s_{62}+s_{63}+s_{64}}{3\,\bar{s}_{55}} \approx 100\,000\,\frac{s_{62}+s_{63}+s_{64}}{3\,s_{54.5}}.$$

Under assumption (i) this is \$139 639. Under assumption (ii) we need to estimate $s_{54.5}$ as

$$s_{54.5} = (s_{54}+s_{55})/2 = 3.210,$$

giving a final average salary of \$113 499. □

Example 11.3 The current annual salary rate of an employee aged exactly 40 is \$50 000. Salaries are revised continuously. Using the salary scale function $s_y = 1.03^y$, estimate

(a) the employee's salary between ages 50 and 51, and
(b) the employee's annual rate of salary at age 51.

In both cases, you should assume the employee remains in employment until at least age 51.

Solution 11.3 (a) The estimated earnings between ages 50 and 51 are given by

$$50\,000\,\frac{s_{50}}{\bar{s}_{40}} \approx 50\,000\,\frac{s_{50}}{s_{39.5}} = 50\,000 \times 1.03^{10.5} = \$68\,196.$$

(b) The estimated salary rate at age 51 is given by

$$50\,000\,\frac{\bar{s}_{51}}{\bar{s}_{40}} \approx 50\,000\,\frac{s_{50.5}}{s_{39.5}} = 50\,000 \times 1.03^{11} = \$69\,212.$$

□

11.4 Setting the contribution for a DC plan

To set the contribution rate for a DC plan to aim to meet a target replacement ratio for a 'model' employee, we need

- the target replacement ratio and retirement age,
- assumptions on the rate of return on investments, interest rates at retirement, a salary scale and a model for post-retirement mortality, and
- the form the benefits should take.

With this information we can set a contribution rate that will be adequate if experience follows all the assumptions. We might also want to explore sensitivity to the assumptions, to assess a possible range of outcomes for the plan member's retirement income. The following example illustrates these points.

Example 11.4 An employer establishes a DC pension plan. On withdrawal from the plan before retirement age, 65, for any reason, the proceeds of the invested contributions are paid to the employee or the employee's survivors.

The contribution rate is set using the following assumptions.

- The employee will use the proceeds at retirement to purchase a pension for his lifetime, plus a reversionary annuity for his wife at 60% of the employee's pension.
- At age 65, the employee is married, and the age of his wife is 61.
- The target replacement ratio is 65%.
- The salary rate function is given by $s_y = 1.04^y$ and salaries are assumed to increase continuously.
- Contributions are payable monthly in arrear at a fixed percentage of the salary rate at that time.
- Contributions are assumed to earn investment returns of 10% per year.
- Annuities purchased at retirement are priced assuming an interest rate of 5.5% per year.
- Male survival: Makeham's law, with $A = 0.0004$, $B = 4 \times 10^{-6}$, $c = 1.13$.
- Female survival: Makeham's law, with $A = 0.0002$, $B = 10^{-6}$, $c = 1.135$.
- Members and their spouses are independent with respect to mortality.

Consider a male new entrant aged 25.

(a) Calculate the contribution rate required to meet the target replacement ratio for this member.

(b) Assume now that the contribution rate will be 5.5% of salary, and that over the member's career, his salary will actually increase by 5% per year, investment returns will be only 8% per year and the interest rate for calculating annuity values at retirement will be 4.5% per year. Calculate the actual replacement ratio for the member.

Solution 11.4 (a) First, we calculate the accumulated DC fund at retirement. Mortality is not relevant here, as in the event of the member's death, the fund is paid out anyway; the DC fund is more like a bank account than an insurance policy.

We then equate the accumulated fund with the EPV at retirement of the pension benefits.

Suppose the initial salary rate is \$$S$. As everything is described in proportion to salary, the value assumed for S does not matter. The annual salary rate at age $x > 25$ is $S\,(1.04^{x-25})$, which means that the contribution at time t, where $t = 1/12, 2/12, \ldots, 40$, is

$$\frac{c}{12}\,S\,(1.04^t)$$

where c is the contribution rate per year. Hence, the accumulated amount of contributions at retirement is

$$\frac{cS}{12}\sum_{k=1}^{480} 1.04^{\frac{k}{12}}\,1.1^{40-\frac{k}{12}} = cS\left(\frac{1.1^{40}-1.04^{40}}{12\left(\left(\frac{1.1}{1.04}\right)^{\frac{1}{12}}-1\right)}\right) = 719.6316\,cS.$$

The salary received in the year prior to retirement, under the assumptions, is

$$\frac{s_{64}}{s_{24.5}}\,S = 1.04^{39.5}\,S = 4.7078S.$$

Since the target replacement ratio is 65%, the target pension benefit per year is $0.65 \times 4.7078S = 3.0601S$.

The EPV at retirement of a benefit of $3.0601S$ per year to the member, plus a reversionary benefit of $0.6 \times 3.0601S$ per year to his wife, is

$$3.0601S\left(\ddot{a}^{(12)}_{\overset{m}{65}} + 0.6\,\ddot{a}^{(12)}_{\overset{m\ f}{65|61}}\right),$$

where the m and f scripts indicate male and female mortality, respectively. Using the given survival models and an interest rate of 5.5% per year, we have

$$\begin{aligned}
\ddot{a}^{(12)}_{\overset{m}{65}} &= 10.5222,\\
\ddot{a}^{(12)}_{\overset{m\ f}{65|61}} &= \ddot{a}^{(12)}_{\overset{f}{61}} - \ddot{a}^{(12)}_{\overset{m\ f}{65:61}},\\
\ddot{a}^{(12)}_{\overset{f}{61}} &= 13.9194,\\
\ddot{a}^{(12)}_{\overset{m\ f}{65:61}} &= \sum_{k=0}^{\infty}\frac{1}{12}v^{\frac{k}{12}}\;{}_{\frac{k}{12}}p_{\overset{m}{65+k}}\;{}_{\frac{k}{12}}p_{\overset{f}{61+k}}\\
&= 10.0066, \qquad (11.1)
\end{aligned}$$

giving

$$\ddot{a}^{(12)}_{\overset{m\ f}{65|61}} = 3.9128.$$

Note that we can write the joint life survival probability in formula (11.1) as the product of the single life survival probabilities using the independence assumption, as in Section 10.5

Hence, the EPV of the benefit at retirement is

$$3.0601S\,(10.5222 + 0.6 \times 3.9128) = 39.3826S.$$

Equating the accumulation of contributions to age 65 with the EPV of the benefits at age 65 gives

$$c = 5.4726\% \text{ per year.}$$

(b) We now repeat the calculation, using the actual experience rather than estimates. We use an annual contribution rate of 5.5%, and solve for the amount of benefit funded by the accumulated contributions, as a proportion of the final year's salary.

The accumulated contributions at age 65 are now $28.6360S$, and the annuity EPVs at 4.5% per year interest are

$$\ddot{a}^{(12)}_{\substack{m\\65}} = 11.3576, \qquad \ddot{a}^{(12)}_{\substack{f\\61}} = 15.4730, \qquad \ddot{a}^{(12)}_{\substack{m\ f\\65:61}} = 10.7579.$$

Thus, the EPV of a benefit of X per year to the member and of $0.6X$ reversionary benefit to his spouse is $14.1867X$. Equating the accumulation of contributions to age 65 with the EPV of benefits at age 65 gives $X = 2.0185S$.

The final year salary, with 5% per year increases, is $6.8703S$. Hence, the replacement ratio is

$$R = \frac{2.0185S}{6.8703S} = 29.38\%.$$

$\square$

We note that apparently quite small differences between the assumptions used to set the contribution and the experience can make a significant difference to the level of benefit, in terms of the pre-retirement income. This is true for both DC and DB benefits. In the DC case, the risk is taken by the member, who takes a lower benefit, relative to salary, than the target. In the DB case, the risk is usually taken by the employer, whose contributions are adjusted when the difference becomes apparent. If the differences are in the opposite direction, then the member benefits in the DC case, and the employer contributions may be reduced in the DB case.

11.5 The service table

The demographic elements of the basis for pension plan calculations include assumptions about survival models for members and their spouses, and about the exit patterns from employment. There are several reasons why a member might exit the plan. At early ages, the employee might withdraw to take another job with a different employer. At later ages, employees may be offered a range of ages at which they may retire with the pension that they have accumulated. A small proportion of employees will die while in employment, and another group may leave early through disability retirement.

In a DC plan the benefit on exit is the same, regardless of the reason for the exit, so there is no need to model the member employment patterns.

In a DB plan different benefits may be payable on the different forms of exit. In the UK it is common on the death in service of a member for the pension plan to offer both a lump sum and a pension benefit for the member's surviving spouse. In North America, any lump sum benefit is more commonly funded through separate group life insurance, and so the liability does not fall on the plan. There may be a contingent spouse's benefit.

The extent to which the DB plan actuary needs to model the different exits depends on how different the values of benefits are from the values of benefits for people who do not leave until the normal retirement age.

For example, if an employer offers a generous benefit on disability (or ill health) retirement, that is worth substantially more than the benefit that the employee would have been entitled to if they had remained in good health, then it is necessary to model that exit and to value that benefit explicitly. Otherwise, the liability will be understated. On the other hand, if there is no benefit on death in service (for example, because of a separate group life arrangement), then to ignore mortality before retirement would overstate the liabilities within the pension plan.

If all the exit benefits have roughly the same value as the normal age retirement benefit, the actuary may assume that all employees survive to retirement. It is not a realistic assumption, but it simplifies the calculation and is appropriate if it does not significantly over-estimate or under-estimate the liabilities.

It is relatively common to ignore withdrawals in the basis, even if a large proportion of employees do withdraw, especially at younger ages. By ignoring withdrawals, we are implicitly valuing age retirement benefits for lives who withdraw, instead of valuing the withdrawal benefits. This is a reasonable shortcut if the age retirement benefits have similar value to the withdrawal benefits, which is often the case. For example, in a final salary plan, if withdrawal benefits are increased in line with inflation, the value of withdrawal and age benefits will be similar. Even if the difference is relatively large,

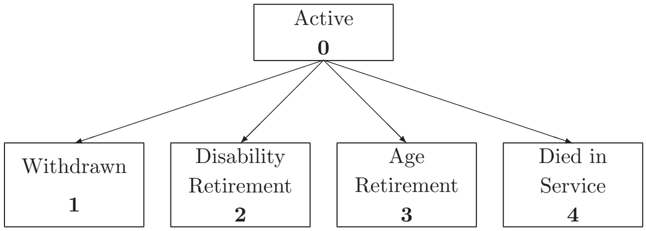


Figure 11.1 A multiple decrement model for a pension plan.

withdrawals may be ignored. This creates an implicit margin in the valuation if withdrawal benefits are less valuable than retirement benefits, which is often the case. An additional consideration is that withdrawals are notoriously unpredictable, as they are strongly affected by economic and social factors, so that historical trends may not provide a good indicator of future exit patterns.

When the actuary does model the exits from a plan, an appropriate multiple decrement model could be similar to the one shown in Figure 11.1. All the model assumptions of Chapters 8 and 9 apply to this model, except that, typically, some age retirements will be exact age retirements, as discussed in Section 9.7.

Example 11.5 A pension plan member is entitled to a lump sum benefit on death in service of four times the salary paid in the year up to death.

Assume the appropriate multiple decrement model is as in Figure 11.1, with

$$\mu_x^{01} \equiv \mu_x^w = \begin{cases} 0.1 & \text{for } x < 35, \\ 0.05 & \text{for } 35 \le x < 45, \\ 0.02 & \text{for } 45 \le x < 60, \\ 0 & \text{for } x \ge 60, \end{cases}$$

$$\mu_x^{02} \equiv \mu_x^i = 0.001,$$

$$\mu_x^{03} \equiv \mu_x^r = \begin{cases} 0 & \text{for } x \le 60, \\ 0.1 & \text{for } 60 < x < 65. \end{cases}$$

In addition, 30% of the members surviving in employment to age 60 retire at that time, and 100% of the lives surviving in employment to age 65 retire at that time. For transitions to State 4, we assume μ_x^{04} is equal to the force of mortality of the Standard Ultimate Survival Model at age x.

(a) Calculate the probability that an active member who is currently aged 35 retires at age 65.

(b) For each mode of exit, calculate the probability that an active member currently aged 35 exits employment by that mode.

Solution 11.5 (a) Since all surviving members retire at age 65, the probability can be written ${}_{30}p_{35}^{00}$. To calculate this, we need to consider separately the periods before and after the jump in the withdrawal transition intensity, and before and after the exact age retirements at age 60.

For $0 < t \leq 10$,

$$\begin{aligned} {}_{t}p_{35}^{00} &= \exp\left\{-\int_0^t \left(\mu_{35+s}^{w} + \mu_{35+s}^{i} + \mu_{35+s}^{d}\right) ds\right\} \\ &= \exp\left\{-\left((A+0.051)t + \frac{B}{\log c}c^{35}(c^t - 1)\right)\right\} \end{aligned}$$

giving

$${}_{10}p_{35}^{00} = 0.597342.$$

For $10 < t < 25$,

$$\begin{aligned} {}_{t}p_{35}^{00} &= {}_{10}p_{35}^{00} \exp\left\{-\int_0^{t-10} \left(\mu_{45+s}^{w} + \mu_{45+s}^{i} + \mu_{45+s}^{d}\right) ds\right\} \\ &= {}_{10}p_{35}^{00} \exp\left\{-\left((A+0.021)(t-10) + \frac{B}{\log c}c^{45}(c^{t-10} - 1)\right)\right\} \end{aligned}$$

giving

$${}_{25^-}p_{35}^{00} = 0.597342 \times 0.712105 = 0.425370.$$

At $t = 25$, 30% of the survivors retire, so at $t = 25^+$, we have

$${}_{25^+}p_{35}^{00} = 0.7 \times {}_{25^-}p_{35}^{00} = 0.297759.$$

For $25 < t < 30$,

$$\begin{aligned} {}_{t}p_{35}^{00} &= {}_{25^+}p_{35}^{00} \exp\left\{-\int_0^{t-25} \left(\mu_{60+s}^{r} + \mu_{60+s}^{i} + \mu_{60+s}^{d}\right) ds\right\} \\ &= 0.297759 \exp\left\{-\left((A+0.1+0.001)(t-25) + \frac{B}{\log c}c^{60}(c^{t-25} - 1)\right)\right\} \end{aligned}$$

giving

$${}_{30^-}p_{35}^{00} = 0.297759 \times 0.590675 = 0.175879.$$

Since all surviving active employees at age 65 retire immediately, the probability of retirement at exact age 65 is then 0.175879.

(b) We know that all members leave employment by or at age 65.
All withdrawals occur by age 60. To compute the probability of withdrawal, we split the period into before and after the change in the withdrawal force at age 45.
The probability of withdrawal by age 45 is

$$ {}_{10}p_{35}^{01} = \int_0^{10} {}_tp_{35}^{00}\, \mu_{35+t}^{w}\, dt = 0.05 \int_0^{10} {}_tp_{35}^{00}\, dt, $$

which we calculate using numerical integration to give

$$ {}_{10}p_{35}^{01} = 0.05 \times 7.8168 = 0.3908. $$

The probability of withdrawal between ages 45 and 60 is

$$ \begin{aligned} {}_{10}p_{35}^{00}\ {}_{15}p_{45}^{01} &= 0.597342 \int_0^{15} {}_tp_{45}^{00}\, \mu_{45+t}^{w}\, dt \\ &= 0.597342 \times 0.02 \int_0^{15} {}_tp_{45}^{00}\, dt, \end{aligned} $$

which, again using numerical integration, gives

$$ {}_{10}p_{35}^{00}\ {}_{15}p_{45}^{01} = 0.597342 \times 0.02 \times 12.7560 = 0.1524. $$

So, the total probability of withdrawal is 0.5432.
The probability of disability retirement by age 45 is

$$ \begin{aligned} {}_{10}p_{35}^{02} &= \int_0^{10} {}_tp_{35}^{00}\, \mu_{35+t}^{i}\, dt = 0.001 \int_0^{10} {}_tp_{35}^{00}\, dt \\ &= 0.001 \times 7.8168 = 0.0078, \end{aligned} $$

and the probability of disability retirement between ages 45 and 60 is

$$ \begin{aligned} {}_{10}p_{35}^{00}\, {}_{15}p_{45}^{02} &= 0.597342 \int_0^{15} {}_tp_{45}^{00}\, \mu_{45+t}^{i}\, dt \\ &= 0.597342 \times 0.001 \int_0^{15} {}_tp_{45}^{00}\, dt \\ &= 0.597342 \times 0.001 \times 12.7560 = 0.0076. \end{aligned} $$

The probability of disability retirement in the final five years is

$$
{}_{25^+}p_{35}^{00}\ {}_5p_{60}^{02} = 0.297759 \int_0^5 {}_tp_{60}^{00}\,\mu_{60+t}^{i}\,dt
$$

$$
= 0.297759 \times 0.001 \times 3.8911 = 0.0012.
$$

So, the total probability of disability retirement is 0.0166.
The probability of age retirement is the sum of the probabilities of exact age retirements at age 60 and 65, and the probability of retirement between ages 60^+ and 65^-.
The probability of exact age 60 retirement is

$$
0.3\ {}_{25^-}p_{35} = 0.1276,
$$

and the probability of exact age 65 retirement is

$$
{}_{30^-}p_{35} = 0.1759.
$$

The probability of retirement between exact ages 60 and 65 is

$$
{}_{25^+}p_{35}^{00}\ {}_5p_{60}^{03} = 0.297759 \int_0^5 {}_tp_{60}^{00}\,\mu_{60+t}^{r}\,dt
$$

$$
= 0.297759 \times 0.1 \times 3.8911 = 0.1159.
$$

So, the total age retirement probability is 0.4194.
We could infer the death in service probability, by the law of total probability, but we instead calculate it directly as a check on the other results. We use numerical integration for all these calculations.
The probability of death in the first 10 years is

$$
{}_{10}p_{35}^{04} = \int_0^{10} {}_tp_{35}^{00}\,\mu_{35+t}^{d}\,dt = 0.0040,
$$

and the probability of death in the next 15 years is

$$
{}_{10}p_{35}^{00}\ {}_{15}p_{45}^{04} = 0.59734 \int_0^{15} {}_tp_{45}^{00}\,\mu_{45+t}^{d}\,dt = 0.0120.
$$

The probability of death in the final five years is

$$
\begin{aligned}
{}_{25}p_{35}^{00}\ {}_5p_{60}^{04} &= 0.297759 \int_0^5 {}_tp_{60}^{00}\,\mu_{60+t}^{r}\,dt \\
&= 0.297759 \times 0.016323 = 0.0049.
\end{aligned}
$$

So the total death in service probability is 0.0208.

We can check our calculations by summing the probabilities of exiting by each mode. This gives a total of 1 $(= 0.5432+0.0166+0.4194+0.0208)$, as it should. □

Often the multiple decrement model is summarized in tabular form at integer ages, in the same way that a life table summarizes a survival model. Such a summary is called a pension plan **service table**. We start at some minimum integer entry age, x_0, by defining an arbitrary radix, for example, $l_{x_0} = 1\,000\,000$. Using the model of Figure 11.1, we then define for integer ages $x_0 + k$ $(k = 0, 1, \ldots)$

$$
\begin{aligned}
w_{x_0+k} &= l_{x_0}\ {}_kp_{x_0}^{00}\ p_{x_0+k}^{01}, \\
i_{x_0+k} &= l_{x_0}\ {}_kp_{x_0}^{00}\ p_{x_0+k}^{02}, \\
r_{x_0+k} &= l_{x_0}\ {}_kp_{x_0}^{00}\ p_{x_0+k}^{03}, \\
d_{x_0+k} &= l_{x_0}\ {}_kp_{x_0}^{00}\ p_{x_0+k}^{04}, \\
l_{x_0+k} &= l_{x_0}\ {}_kp_{x_0}^{00}.
\end{aligned}
$$

Since the probability that a member aged x_0 withdraws between ages $x_0 + k$ and $x_0 + k + 1$ is ${}_kp_{x_0}^{00}\,p_{x_0+k}^{01}$, we can interpret w_{x_0+k} as the number of members expected to withdraw between ages x_0+k and x_0+k+1 out of l_{x_0} members aged exactly x_0; i_{x_0+k}, r_{x_0+k} and d_{x_0+k} can be interpreted similarly. We can interpret l_{x_0+k} as the expected number of lives who are still active plan members at age $x_0 + k$ out of l_{x_0} active members aged exactly x_0. We can extend these interpretations to say that for any integer ages x and $y \geq x$, w_y is the number of members expected to withdraw between ages y and $y + 1$, out of l_x active members aged exactly x, and l_y is the expected number of active members at age y out of l_x active members aged exactly x. These interpretations are precisely in line with those for the life table in Chapter 3 and the multiple decrement tables in Chapter 9.

Note that, using the law of total probability, we have the following identity for any integer age $x > x_0$

$$
l_x = l_{x-1} - w_{x-1} - i_{x-1} - r_{x-1} - d_{x-1}. \tag{11.2}
$$

A service table summarizing the model in Example 11.5 is shown in Table D.9 in Appendix D, from age 20, with the radix $l_{20} = 1\,000\,000$. This service table has been constructed by calculating, for each integer age x (> 20), w_x, i_x, r_x and d_x as described above. The value of l_x shown in the table is then calculated recursively from age 20. The table is internally consistent in the sense that identity (11.2) holds for each row of the table. However, this does not appear to be the case in Table D.9 because all values have been rounded to the nearer integer. The exact age exits at ages 60 and 65 are shown in the rows labelled 60^- and 65^-. In all subsequent calculations based on Table D.9, we use the exact values rather than the rounded ones.

We use the model underlying this service table for several examples and exercises, where we refer to it (for convenience) as the Standard Service Table.

Having constructed a service table, the calculation of the probability of any event between integer ages can be performed relatively simply. To see this, consider the calculations required for Example 11.5. For part (a), the probability that a member aged 35 survives in service to age 65, calculated using Table D.9, is

$$\frac{l_{65}}{l_{35}} = \frac{38\,488}{218\,834} = 0.1759.$$

For part (b), the probability that a member aged 35 withdraws is

$$(w_{35} + w_{36} + \cdots + w_{59})/l_{35}$$
$$= \frac{10\,665 + 10\,131 + \cdots + 1\,930 + 1\,884}{218\,834} = 0.5432.$$

The probability that the member retires in ill health is

$$(i_{35} + i_{36} + \cdots + i_{64})/l_{35}$$
$$= \frac{213 + 203 + \cdots + 45 + 41}{218\,834} = 0.0166,$$

the probability that the member retires on age grounds is

$$(r_{35} + r_{36} + \cdots + r_{65})/l_{35}$$
$$= \frac{27\,926 + 6\,188 + 5\,573 + 5\,018 + 4\,515 + 4\,061 + 38\,488}{218\,834} = 0.4194,$$

and the probability that the member dies in service is

$$(d_{35} + d_{36} + \cdots + d_{64})/l_{35}$$
$$= \frac{83 + 84 + \cdots + 214 + 215}{218\,834} = 0.0208.$$

Example 11.6 Employees in a pension plan pay contributions of 6% of their previous month's salary at each month end. Calculate the EPV at entry of contributions for a new entrant aged 35, with a starting salary rate of \$100 000, using

(a) exact calculation using the multiple decrement model specified in Example 11.5, and
(b) the values in Table D.9, adjusting the EPV of an annuity payable annually in the same way as under the UDD assumption in Chapter 5.

Other assumptions:

- Salary rate: Salaries increase at 4% per year continuously.
- Interest: 6% per year effective.

Solution 11.6 (a) The EPV is

$$\frac{0.06 \times 100\,000}{12} \left(\sum_{k=1}^{299} {}_{\frac{k}{12}}p_{35}^{00}\,(1.04^{\frac{k}{12}})\,v^{\frac{k}{12}} + {}_{25^-}p_{35}^{00}\,(1.04^{25})\,v^{25} + \sum_{k=301}^{360} {}_{\frac{k}{12}}p_{35}^{00}\,(1.04^{\frac{k}{12}})\,v^{\frac{k}{12}} \right)$$

$$= \frac{0.06 \times 100\,000}{12} \left(\sum_{k=1}^{299} {}_{\frac{k}{12}}p_{35}^{00}\,v_j^{\frac{k}{12}} + {}_{25^-}p_{35}^{00}\,v_j^{25} + \sum_{k=301}^{360} {}_{\frac{k}{12}}p_{35}^{00}\,v_j^{\frac{k}{12}} \right)$$

$$= 6\,000 \times 13.3529 = \$80\,117,$$

where $j = 0.02/1.04 = 0.0192$, and where we have separated out the term relating to age 60 to emphasize the point that contributions would be paid by all employees reaching ages 60 and 65, even those who retire at those ages.

(b) Recall from Chapter 5 that the UDD approximation to the EPV of a term annuity payable monthly in arrear, $a_{x:\overline{n}|}^{(12)}$, in terms of the corresponding value for annual payments in advance, $\ddot{a}_{x:\overline{n}|}$, is

$$a_{x:\overline{n}|}^{(12)} \approx \alpha(12)\,\ddot{a}_{x:\overline{n}|} - \left(\beta(12) + \frac{1}{12}\right)(1 - v^n\,{}_np_x).$$

This approximation will work for the monthly multiple decrement annuity, provided that the decrements, in total, are approximately UDD. This is **not** the case for our service table, because between ages 60^- and 61, the vast majority of decrements occur at exact age 60. We can take account of this by splitting the annuity into two parts, up to age 60^- and from age 60^+, and applying a UDD-style adjustment to each part as follows:

$$
\begin{aligned}
a^{(12)00}_{35:\overline{30}|} &= a^{(12)00}_{35:\overline{25}|} + \frac{l_{60^+}}{l_{35}} v_j^{25}\, a^{(12)00}_{60^+:\overline{5}|} \\
&\approx \alpha(12)\, \ddot{a}^{00}_{35:\overline{25}|} - \left(\beta(12) + \frac{1}{12}\right)\left(1 - \frac{l_{60^-}}{l_{35}} v_j^{25}\right) \\
&\quad + \frac{l_{60^+}}{l_{35}} v_j^{25} \left(\alpha(12)\, \ddot{a}^{00}_{60^+:\overline{5}|} - \left(\beta(12) + \frac{1}{12}\right)\left(1 - \frac{l_{65^-}}{l_{60^+}} v_j^{5}\right)\right).
\end{aligned}
$$

As $\ddot{a}^{00}_{35:\overline{25}|} = 13.0693$ and $\ddot{a}^{00}_{60^+:\overline{5}|} = 3.9631$ we find that

$$
6\,000\, a^{00(12)}_{35:\overline{30}|} \approx \$80\,131.
$$

□

Using the service table and the UDD-based approximation has resulted in a relative error of the order of 0.03% in this example. This demonstrates again that the service table summarizes the underlying multiple decrement model sufficiently accurately for practical purposes.

In applying the UDD adjustment we are effectively saying that the arguments we applied to deaths in Chapter 5 can be applied to total decrements. However, just as in Section 9.4, if we were to assume a uniform distribution of decrements in each of the related single decrement models, we would find that there is not a uniform distribution of the overall decrements. Nevertheless, the assumption of a uniform distribution of total decrements provides a useful, and relatively accurate, means of calculating the EPV of an annuity payable m times a year from a service table.

It is very common in pension plan valuation to use approximations, primarily because of the long-term nature of the liabilities and the huge uncertainty in the parameters of the models used. To calculate values with great accuracy when there is so much uncertainty involved would be spurious. Nevertheless, it is important to ensure that the approximation methods do not introduce potentially significant biases in the final results, for example, by systematically underestimating the value of liabilities.

11.6 Valuation of final salary plans

11.6.1 Accrued benefits

In a DB final salary pension plan, the basic annual age retirement pension benefit for a life who entered the pension plan at age xe and retires at age xr is equal to $B = (xr - xe)\, S^{F}_{xr}\, \alpha$, where

$xr - xe$ is the total number of years of service within the pension plan.

S^{F}_{xr} is the average salary in a specified period before retirement; it is called the **final average salary**, for obvious reasons. Typical averaging periods for the final average salary range from three to five years.

α is the **accrual rate** of the plan, typically between 0.01 and 0.02. For an employee who has been a member of the plan all her working life, say 40 years, this gives a replacement ratio in the range 40%–80%.

The pension is paid in the form of a life annuity of B per year. The **normal form** of pension defines the terms of the annuity. For example, the normal form may be a monthly life annuity-due, or may include a guarantee period of, say, five or 10 years. Another common normal form includes a reversionary annuity of, say, 60% of the original pension, payable to the member's surviving spouse for their lifetime, following the death of the member. Members may have the option to take their benefit in a different form, with an adjustment to the pension amount, such that the benefits have the same actuarial value as under the normal form.

We interpret the benefit formula to mean that the employee earns (or *accrues*) a pension of $100\alpha\%$ of final average salary over each year of employment.

Consider a member who is currently aged y, who joined the pension plan at age x ($\leq y$) and for whom the normal retirement age is 65. Our estimate of her annual pension at retirement is

$$(65 - x)\,\hat{S}^{\mathrm{F}}_{65}\,\alpha$$

where $\hat{S}^{\mathrm{F}}_{65}$ is the current estimate of S^{F}_{65}, calculated using her current salary and an appropriate salary scale.

We can split this annual amount into two parts as

$$(65 - x)\,\hat{S}^{\mathrm{F}}_{65}\,\alpha = (y - x)\,\hat{S}^{\mathrm{F}}_{65}\,\alpha + (65 - y)\,\hat{S}^{\mathrm{F}}_{65}\,\alpha.$$

The first part is related to her past service, and is called the **projected accrued benefit**. The second part is related to future service. Note that both parts use an estimate of the final average salary at retirement.

In the expression 'projected accrued benefit', *projected* refers to the salary, where we take future increases into consideration, and *accrued* refers to the fact that we only take the past (or accrued) service into consideration.

The employer who sponsors the pension plan retains the right to stop offering pension benefits in the future. If this were to happen, the final benefit would be based on the member's past service at the wind-up of the pension plan; in this sense, the accrued benefits (also known as the past service benefits) are already secured (assuming the plan is adequately managed and funded). The future service benefits are more of a statement of intent, but do not have the contractual nature of the accrued benefits. Because of this, in valuing the plan liabilities, modern valuation approaches often consider only the accrued benefits, even when the plan is valued as a going concern.

A similar argument applies to future salary increases, which *are* included in the projected accrued benefit. The **current accrued benefit** (or just **accrued benefit**) is the retirement benefit for an active member based on their current salary and their past service. We discuss the current accrued benefit further later in this chapter.

In the following example, we illustrate the steps required to find the EPV of the projected accrued age retirement pension; after that, we present a more generalized valuation formula. In the solution to this example and elsewhere we use the notation S_x to denote salary aged x to $x+1$; do not confuse this with the salary scale $\{s_x\}$.

Example 11.7 A pension plan offers an age retirement pension of 1.5% of final average salary for each year of service, where final average salary is defined as the earnings in the three years before retirement. The pension benefit is payable monthly in advance for life, with no spouse's benefit.

Calculate the EPV of the projected accrued age retirement benefit for an active member, currently aged 55 with 20 years of service, whose salary in the year prior to the valuation date was \$50 000.

Basis:

- Standard Service Table (Table D.9)
- Post-retirement survival: Standard Ultimate Survival Model
- Interest: 5% per year effective

Solution 11.7 According to the service table, age retirement can take place at exact age 60, at exact age 65, or at any age in between. We assume that mid-year age retirements (the retirements that do not occur at exact age 60 or 65) that occur between ages $60+t$ and $60+t+1$ ($t=0, 1, \dots, 4$) take place at age $60+t+0.5$ exact. This is a common assumption in pensions calculations and is analogous to the claims acceleration approach for continuous benefits in Section 4.5. The assumption considerably simplifies calculations for complex benefits, as it converts a continuous model for exits into a discrete model, more suitable for efficient spreadsheet calculation, and the inaccuracy introduced is generally small.

Suppose retirement takes place at age y. Then the projected final average salary is

$$\hat{S}_y^F = 50\,000\,\frac{s_{y-1}+s_{y-2}+s_{y-3}}{3\,s_{54}} = 50\,000\,\frac{z_y}{s_{54}},$$

where $z_y = (s_{y-1}+s_{y-2}+s_{y-3})/3$.

The function z_y is the averaging function for the salary scale to give the final average salary, and would be adjusted for different averaging periods. The salary scale values are taken from Table 11.1, and we use linear interpolation to calculate values at non-integer ages.

For this example, the salary information is $S_{54} = 50\,000$, so if the member retires at exact age 60, the projected accrued benefit, based on 20 years of past service and an accrual rate of 1.5%, is a pension payable monthly in advance from age 60 of annual amount

$$S_{54} \frac{z_{60}}{s_{54}} \times 20 \times 0.015 = \$15\,922.79.$$

The total projected pension if the member works until age 60 and retires at that time is based on the full service, not just the past service, giving

$$S_{54} \frac{z_{60}}{s_{54}} \times 25 \times 0.015 = \$19\,903.49,$$

but while that number might be interesting to the member, in this example we are valuing only the accrued benefit, not the full service benefit.

The EPV of the age retirement pension is found by summing the EPVs of the benefits at each possible retirement age, i.e. for each age $y = 60, 60.5, 61.5, \ldots, 64.5, 65$. We multiply together the probability that the life retires at age y, the EPV of the benefit for a life retiring at age y, and the discount factor to bring the value back to the valuation date, when the life is aged 55.

For the two exact age retirement dates, the EPV of the projected accrued benefit is as follows, where $S_{54} = 50\,000$ is the salary earned in the year immediately before the valuation date:

$$\begin{aligned} S_{54} \times \alpha \times 20 \times & \left(\frac{r_{60^-}}{l_{55}} \frac{z_{60}}{s_{54}} \ddot{a}^{(12)}_{60}\, v^5 + \frac{r_{65^-}}{l_{55}} \frac{z_{65}}{s_{54}} \ddot{a}^{(12)}_{65}\, v^{10} \right) \\ &= S_{54} \times \alpha \times 20 \times (3.204 + 3.378) = 98\,727. \end{aligned}$$

For the mid-year retirements, we proceed similarly. In the calculations below, we have used exact values for the monthly annuities starting at the half-year, but interpolating the integer age values will give very similar answers. The EPV of the projected accrued benefits is

$$\begin{aligned} S_{54} \times \alpha \times 20 \times & \left(\frac{r_{60^+}}{l_{55}} \frac{z_{60.5}}{s_{54}} \ddot{a}^{(12)}_{60.5}\, v^{5.5} + \frac{r_{61}}{l_{55}} \frac{z_{61.5}}{s_{54}} \ddot{a}^{(12)}_{61.5}\, v^{6.5} + \frac{r_{62}}{l_{55}} \frac{z_{62.5}}{s_{54}} \ddot{a}^{(12)}_{62.5}\, v^{7.5} \right. \\ & \left. + \frac{r_{63}}{l_{55}} \frac{z_{63.5}}{s_{54}} \ddot{a}^{(12)}_{63.5}\, v^{8.5} + \frac{r_{64}}{l_{55}} \frac{z_{64.5}}{s_{54}} \ddot{a}^{(12)}_{64.5}\, v^{9.5} \right) \\ &= S_{54} \times \alpha \times 20 \times (0.692 + 0.591 + 0.505 + 0.431 + 0.366) = 38\,781, \end{aligned}$$

so that the total EPV is 137 508.

Note the differences between the exact age and the mid-year retirement calculations.

□

11.6.2 A general formula for the EPV of the projected accrued age retirement pension

Because pension plan retirement benefits and provisions vary, we present below a generalized approach to valuing the projected accrued benefits, which can be adapted for the specific plan benefits and structure.

In general, we define the **salary averaging function** z_y such that, given a salary scale function s_x, and a valuation salary in the year of age x to $x+1$ of S_x, then

$$\hat{S}_y^{\mathrm{F}} = S_x \, \frac{z_y}{s_x}.$$

We also use a generic function a_y^r to denote the EPV of an annual pension benefit of 1 per year payable at age y to a member who retires at that age. This function is adapted to the specific normal form of the individual pension plan. In the simplest case, if the pension is paid as a monthly life annuity-due, then $a_y^r = \ddot{a}_y^{(12)}$.

Consider an active plan member aged x at the valuation date. We assume the salary information provided gives us S_x, which is the salary expected to be earned in the year following the valuation, that the member has n years past service at the valuation date, that the accrual rate is α, and that all retirements occur at or before age xr.

Then the EPV of (x)'s projected accrued age retirement benefit is

$$\boxed{S_x \times \alpha \times n \times \left(\underbrace{\sum_{y=x}^{xr-1}}_{\text{mid-year exits}} \frac{r_y}{l_x} \, \frac{z_{y+\frac{1}{2}}}{s_x} \, a^r_{y+\frac{1}{2}} \, v^{y+\frac{1}{2}-x} + \underbrace{\sum_{y=x+1}^{xr}}_{\text{exact age exits}} \frac{r_{y^-}}{l_x} \, \frac{z_y}{s_x} \, a_y^r \, v^{y-x} \right)}. \tag{11.3}$$

In the first summation, we sum over all possible ages y where the retirement is assumed to occur at age $y+\frac{1}{2}$, and in the second we sum over all ages y at which exact age retirement is possible under the service table model. In the Standard Service Table in Appendix D, there are two terms in the second sum, corresponding to ages $y=60$ and $y=65$.

Often, as in Example 11.7, we have salary information at age $x-1$ rather than age x, in which case we replace S_x/s_x with S_{x-1}/s_{x-1} in (11.3).

In Example 11.7, the benefit annuity was a monthly life annuity-due. Below, we give some other common payment forms for the retirement benefit. Note that we often use continuous functions when payments are made at weekly intervals.

- Monthly life annuity-due with a 10-year guarantee: $a_y^r = \ddot{a}^{(12)}_{\overline{y:\overline{10|}}}$.
- Continuously paid annuity, increasing continuously at rate j per year:

$$a_y^r = \bar{a}_{y|i^*}, \quad \text{where } i^* = (1+i)/(1+j) - 1.$$

- Monthly life annuity-due to the member, plus 60% reversionary benefit to the member's spouse, who is assumed to be age $y+d$:

$$a_y^r = \ddot{a}_y^{(12)} + 0.6\,\ddot{a}^{(12)}_{y\,|y+d},$$

where the reversionary annuity would be calculated using suitable mortality models for the member and the spouse.

We assume in the valuation equation (11.3) that employees can take retirement at any age without a reduction in the benefit formula. In practice, the benefits may be reduced by a factor known as the **actuarial reduction factor**. The factor is a simple rate of reduction for each year (part years counting pro-rata) by which retirement precedes the minimum normal retirement age (typically expressed as a rate per month in pension documentation). That is, suppose that the actuarial reduction factor is k per year, and that the minimum retirement age without actuarial reduction is xr. A life retiring at age $y < xr$ with n years of past service would have an unreduced annual pension of $B = S_y^{\mathrm{F}}\alpha\, n$, but would receive a benefit of only $B^* = (1-(xr-y)k)S_y^{\mathrm{F}}\alpha\, n$. It is a relatively straightforward matter to adjust the valuation equation for actuarial reduction factors, potentially by redefining the a_y^r factors appropriately.

11.6.3 Withdrawal benefits

When an employee leaves employment before being eligible to take an immediate pension, the usual benefit (subject to some minimum period of employment) in a DB plan is a deferred pension. The benefit would be based on the same formula as the retirement pension, that is,

$$\text{Accrual Rate} \times \text{Service} \times \text{Final Average Salary},$$

but would not be paid until the member attains the specified **normal retirement age**. Note that the final average salary here is based on earnings in the years immediately preceding withdrawal.

The deferred period could be very long, perhaps 35 years for an employee who changes jobs at age 30. If the deferred benefit is not increased during the deferred period, then inflation, even at relatively low levels, will have a significant effect on the purchasing power of the pension. In some plans the withdrawal benefit is adjusted through the deferred period to make some, possibly partial, allowance for inflation. Such adjustments are called **cost of living adjustments**, or COLAs. In the UK, some inflation adjustment is mandatory. Some plans outside the UK do not guarantee any COLA but apply increases on a discretionary basis.

To value the withdrawal benefit we proceed similarly to the age retirement case; we assume all withdrawals occur half-way through the year of age. The probability that an active member aged x withdraws between ages $x+t$ and $x+t+1$ is w_{x+t}/l_x. For each potential time of withdrawal, we project the benefits, apply the benefit annuity, multiply by the probability of withdrawal, and discount to the valuation date. Let a_y^w denote the EPV at age y of a withdrawal pension of one per year deferred to the final possible retirement age, xr (which is usually the same as the normal retirement age); for example, if the withdrawal pension is payable as a monthly annuity-due from age 60, we would have $a_y^w = {}_{60-y}E_y\,\ddot{a}_{60}^{(12)}$.

As before, for a member aged x, let S_x, n and α denote the valuation salary from age x to $x+1$, the number of years of past service at the valuation date, and the accrual rate, respectively.

The EPV of the projected accrued withdrawal benefit is

$$\boxed{S_x \times \alpha \times n \times \sum_{y=x}^{xr-1} \frac{w_y}{l_x}\,\frac{z_{y+\frac{1}{2}}}{s_x}\,a^w_{y+\frac{1}{2}}\,v^{y+\frac{1}{2}-x}.} \tag{11.4}$$

Example 11.8 A final salary pension plan offers an accrual rate of 2%, and the normal retirement age is 65. The final average salary is the average salary in the three years before retirement or withdrawal. Pensions are paid monthly in advance for life from age 65, with no spouse's benefit, and are guaranteed for five years.

(a) Estimate the EPV of the projected accrued withdrawal pension for a life now aged 35 with 10 years of service whose salary in the past year was \$100 000
 (i) with no COLA, and
 (ii) with a COLA in deferment of 3% per year.
(b) On death during deferment, a lump sum benefit of five times the accrued annual pension, with a COLA of 3% per year, is payable immediately. Estimate the EPV of this benefit.

Basis:

- Standard Service Table (Table D.9)
- Salary scale: from Table 11.1
- Post-withdrawal survival: Standard Ultimate Survival Model
- Interest: 5% per year effective

Solution 11.8 According to the service table, the member can withdraw at any age up to 60. If the member withdraws between ages y and $y+1$ ($y = 35, 36, \ldots, 59$) we assume that withdrawal takes place at age $y+0.5$.

The averaging period for the final average salary is (again) three years, so

$$z_x = (s_{x-3} + s_{x-2} + s_{x-1})/3.$$

(a) (i) For a life withdrawing at age $y + \frac{1}{2}$, who will start receiving their pension at age 65, the annuity function in equation (11.4) is

$$a^w_{y+\frac{1}{2}} = {}_{65-(y+\frac{1}{2})}E_{y+\frac{1}{2}}\ \ddot{a}^{(12)}_{\overline{65:\overline{5}|}},$$

calculated using the Standard Ultimate Survival Model (once the life has withdrawn from the plan, the service table mortality is no longer relevant). We have

$$\ddot{a}^{(12)}_{\overline{65:\overline{5}|}} = \ddot{a}^{(12)}_{\overline{5}|} + {}_5E_{65}\,\ddot{a}^{(12)}_{70} = 13.1573.$$

To obtain the EPV of the withdrawal pension, we make a small adjustment to (11.4) to allow for the fact that the salary information here is for the year immediately prior to the valuation date, rather than the year following the valuation date; this is the situation we noted just after (11.3). So, using s_{34} rather than s_{35} in (11.4) gives the EPV as

$$100\,000 \times 0.02 \times 10 \times \sum_{y=35}^{59} \frac{w_y}{l_{35}}\,\frac{z_{y+\frac{1}{2}}}{s_{34}}\,a^w_{y+\frac{1}{2}}\,v^{y+\frac{1}{2}-35} = 48\,246.$$

(ii) The change here is that the withdrawal annuity must now include allowance for the 3% annual increase in the accrued benefit between withdrawal and the normal retirement age. That can be allowed for by setting

$$a^w_{y+\frac{1}{2}} = 1.03^{65-(y+\frac{1}{2})}\ {}_{65-(y+\frac{1}{2})}E_{y+\frac{1}{2}}\ \ddot{a}^{(12)}_{\overline{65:\overline{5}|}},$$

which gives a revised EPV of the withdrawal pension benefit of 88 853.

(b) Now we must adapt the withdrawal pension benefit valuation formula in (11.4). If the member withdraws at age $y + \frac{1}{2}$, the EPV of the death benefit at the time of withdrawal is

$$5\,\alpha\,n\,\hat{S}^{\mathrm{F}}_{y+0.5} \int_0^{65-(y+0.5)} v^t\,1.03^t\,{}_tp_{y+0.5}\,\mu_{y+0.5}\,dt$$
$$= 5\,\alpha\,n\,\hat{S}^{\mathrm{F}}_{y+0.5}\,\bar{A}^{\ \ 1}_{y+0.5\,:\,\overline{65-(y+0.5)}|\,j}$$

where the rate of interest for the insurance function is $j = 1.05/1.03-1 = 0.0194$. So we replace $S_x \times \alpha \times n \times a_{y+\frac{1}{2}}$ in (11.4) with this EPV giving the full EPV of the death in deferment benefit as

$$5 \times 100\,000 \times 0.02 \times 10 \times \sum_{y=35}^{59} \frac{w_y}{l_{35}} \frac{z_{y+0.5}}{s_{34}} \, v^{y+\frac{1}{2}-35} \bar{A}^{\;\;1}_{y+0.5:\overline{65-(y+0.5)}|j}$$

$$= 1\,813.$$

□

11.6.4 Valuing the current accrued benefit

As mentioned above, future salary increases are not guaranteed, and there is a case for omitting them from the valuation liabilities, so that salary increases would be brought into the liability valuation only once they are awarded. The accrued benefit based on current salary is the **current accrued benefit**.

There are (at least) two different ways to value the current accrued benefit.

1. Value the benefit to a life aged x at the valuation date using their 'final average salary' at the valuation date, as if they were exiting the pension plan at that time.
2. Value the benefit to a life aged x at the valuation date assuming that they are awarded no future salary increases between the valuation date and their exit date.

To understand the difference between these approaches, consider a hypothetical plan member called Bob, who is currently aged 50, and who belongs to a plan that uses a three-year averaging period for the final average salary. Bob's annual earnings in the past three years have been \$70 000, \$75 000 and \$80 000.

Under the first method, Bob's final average salary for valuation purposes would be \$75 000 (the average of the past three years' earnings), regardless of the projected exit date. Under the second method, we assume Bob's future earnings will be \$80 000 until retirement. For exits at age 52 or older, the final average salary would be \$80 000 (we assume the same earnings in each year from age 49 onwards). For exits before age 52, there would be some adjustment for the earnings before age 49.

Both methods are used in practice. In this chapter we use the first approach, which is the standard in North America.

The EPV of the current accrued age retirement benefit is

$$\boxed{S_x^{\mathrm{F}} \times \alpha \times n \times \left(\underbrace{\sum_{y=x}^{x_r-1}}_{\substack{\text{mid-year}\\ \text{exits}}} \frac{r_y}{l_x} \, a^r_{y+\frac{1}{2}} \, v^{y+\frac{1}{2}-x} + \underbrace{\sum_{y=x+1}^{x_r}}_{\substack{\text{exact age}\\ \text{exits}}} \frac{r_{y^-}}{l_x} \, a^r_y \, v^{y-x} \right)} \tag{11.5}$$

where, as in (11.3), the life is aged x at the valuation date, with n years of past service, and has a final average salary of S_x^{F}.

Example 11.9 A pension plan offers an age retirement pension of 1.5% of final average salary for each year of service, where the final average salary is defined as the earnings in the three years before retirement. The pension benefit is payable monthly in advance for life, with no spouse's benefit.

Calculate the EPV of the accrued age retirement benefit for an active member, currently aged 55 with 20 years of service, whose salaries in each of the three years prior to the valuation date were \$47 900, \$48 800 and \$50 000.

Basis:

- Standard Service Table (Table D.9)
- Post-retirement survival: Standard Ultimate Survival Model
- Interest: 5% per year effective

(This is the same as Example 11.7, except that we are using the current accrued benefit in place of the projected accrued benefit.)

Solution 11.9 We have $S_{55}^{\mathrm{F}} = (47\,900 + 48\,800 + 50\,000)/3 = 48\,900$ and $a_y^r = \ddot{a}_y^{(12)}$, and so the EPV of the current accrued retirement benefit is

$$\begin{aligned} S_{55}^{\mathrm{F}} \times \alpha \times 20 \times \Bigg(& \frac{r_{60^-}}{l_{55}}\, \ddot{a}_{60}^{(12)}\, v^5 + \frac{r_{60^+}}{l_{55}}\, \ddot{a}_{60.5}^{(12)}\, v^{5.5} + \frac{r_{61}}{l_{55}}\, \ddot{a}_{61.5}^{(12)}\, v^{6.5} \\ & + \cdots + \frac{r_{64}}{l_{55}}\, \ddot{a}_{64.5}^{(12)}\, v^{9.5} + \frac{r_{65^-}}{l_{55}}\, \ddot{a}_{65}^{(12)}\, v^{10} \Bigg) \\ = {} & 122\,204. \end{aligned}$$

□

As expected, this is smaller than the EPV in Example 11.7 as we are not allowing for any further salary increases after age 55, meaning that the amount (and hence the EPV) of the retirement benefit is less than in Example 11.7.

11.7 Valuing career average earnings plans

Under a career average earnings (CAE) defined benefit pension plan, the benefit formula is based on the average salary during the period of pension plan membership, rather than the final average salary. Suppose a plan member enters at age xe, retires at age xr with $xr-xe$ years of service, and has total pensionable earnings during their pensionable employment of $(\mathrm{TPE})_{xr}$. Then their career average earnings are $(\mathrm{TPE})_{xr}/(xr - xe)$. So a CAE plan with an accrual rate of α would provide a pension benefit on retirement at age xr of

$$\alpha\,(xr - xe)\frac{(\mathrm{TPE})_{xr}}{(xr - xe)} = \alpha\,(\mathrm{TPE})_{xr}.$$

Under a career average earnings plan, the accrued, or past service, benefit that we value at age x is $\alpha\,(\mathrm{TPE})_x$, where $(\mathrm{TPE})_x$ denotes the total pensionable earnings up to age x. The methods available for valuing such benefits are the same as for a final salary benefit. The EPV of the accrued age retirement benefit for an active member aged x with total past pensionable earnings of $(\mathrm{TPE})_x$ is

$$(\mathrm{TPE})_x \times \alpha \times \left(\underbrace{\sum_{y=x}^{xr-1}}_{\substack{\text{mid-year}\\ \text{exits}}} \frac{r_y}{l_x}\, \ddot{a}^r_{y+\frac{1}{2}}\, v^{y+\frac{1}{2}-x} + \underbrace{\sum_{y=x+1}^{xr}}_{\substack{\text{exact age}\\ \text{exits}}} \frac{r_{y^-}}{l_x}\, \ddot{a}^r_y\, v^{y-x} \right). \tag{11.6}$$

A popular variation of the career average earnings plan is the **career average revalued earnings** (CARE) plan, in which an inflation adjustment of the salary is made before averaging. The accrual principle is the same. The accrued benefit is based on the total past earnings after the revaluation calculation.

Let $(\mathrm{TPRE})_x$ denote the total past revalued earnings for an active member aged x at the valuation date, where past earnings have been adjusted for inflation up to the valuation date. For projecting the benefits, we assume a constant rate of future inflation of j per year.

The current accrued benefit is $(\mathrm{TPRE})_x\,\alpha$ and the projected accrued benefit under the CARE plan, assuming exit at age y, is $(\mathrm{TPRE})_x\,\alpha(1+j)^{y-x}$. The EPV of the projected accrued benefit is

$$(\mathrm{TPRE})_x \times \alpha$$
$$\times \left(\underbrace{\sum_{y=x}^{xr-1}}_{\substack{\text{mid-year}\\ \text{exits}}} \frac{r_y}{l_x}(1+j)^{y+\frac{1}{2}-x}\, \ddot{a}^r_{y+\frac{1}{2}}\, v^{y+\frac{1}{2}-x} + \underbrace{\sum_{y=x+1}^{xr}}_{\substack{\text{exact age}\\ \text{exits}}} \frac{r_{y^-}}{l_x}(1+j)^{y-x}\, \ddot{a}^r_y\, v^{y-x} \right). \tag{11.7}$$

Example 11.10 A pension plan offers a retirement benefit of 4% of career average earnings for each year of service. The pension benefit is payable monthly in advance for life, guaranteed for five years. On withdrawal, a deferred pension is payable from age 65.

Consider a member now aged 35 who has 10 years of service, with total past earnings of \$525 000. Calculate the EPV of the accrued age retirement and withdrawal benefits.

Basis:

- Standard Service Table (Table D.9)
- Post-retirement/withdrawal survival: Standard Ultimate Survival Model
- Interest: 5% per year effective

Solution 11.10 The annuity function in (11.6) for the age retirement benefit is $a_y^r = \ddot{a}^{(12)}_{\overline{y:\overline{5}|}}$, and the EPV is

$$\begin{aligned}(\mathrm{TPE})_{35} \times 0.04 \times 10 \times \Bigg(& \frac{r_{60^-}}{l_{35}}\, a^r_{60}\, v^{25} + \frac{r_{60^+}}{l_{35}}\, a^r_{60.5}\, v^{25.5} + \frac{r_{61}}{l_{35}}\, a^r_{61.5}\, v^{26.5} \\ & + \cdots + \frac{r_{64}}{l_{55}}\, a^r_{64.5}\, v^{29.5} + \frac{r_{65^-}}{l_{35}}\, a^r_{65}\, v^{30} \Bigg) \\ = 31\,666. &\end{aligned}$$

We value the withdrawal benefit using a formula similar to (11.6) but without any exact age exits, and with w_y replacing r_y. The annuity function for the withdrawal benefit is $a_y^w = {}_{65-y}E_y\ \ddot{a}^{(12)}_{\overline{65:\overline{5}|}}$, and the EPV of the accrued withdrawal pension is

$$\begin{aligned}&(\mathrm{TPE})_{35} \times 0.04 \times 10 \\ &\times \left(\frac{w_{35}}{l_{35}}\, a^w_{35.5}\, v^{0.5} + \frac{w_{36}}{l_{35}}\, a^w_{36.5}\, v^{1.5} + \cdots + \frac{w_{59}}{l_{35}}\, a^w_{64.5}\, v^{29.5} \right) = 33\,173. \ \square\end{aligned}$$

11.8 Funding the benefits

In a typical DB pension plan the employee pays a fixed contribution (or no contribution at all), and the balance of the cost of the employee benefits is funded by the employer. The employer's contribution is set at the regular actuarial valuations, and is expressed as a percentage of salary.

With an insurance policy, the policyholder pays for a contract, typically, with a level, regular premium. The nature of the pension plan is that there is no need for the cost to be constant, as contributions can be adjusted from time to time. The level of contribution from the employer is not usually a part of a contract, in the way that the premium is specified in the insurance contract. Nevertheless, because the employer will have an interest in smoothing its costs, there is some incentive for the funding to be reasonably smooth and predictable. Also, as a matter of principle, each individual's benefits should be expected to be funded over their working lifetime, so that on average, by the time each member leaves active employment, the contributions paid should be sufficient to fund the benefits earned.

We assume that the benefit valuation approach from the previous section is used to establish a **reserve** level at the start of the year. The reserve refers to the assets set aside to meet the accrued liabilities as they fall due in the future. So, the reserve at time t, say, is the sum of the EPVs of all the accrued benefits at that time, taking into consideration all the appropriate benefits, and allowing also for the accrued benefits of inactive plan members – that is, the pensions in payment and in deferment, as well as contingent benefits of members' spouses and dependents. This reserve is called the **actuarial liability**.

We then set the **normal contribution**, also called the **normal cost**, to be the amount required to be paid such that, together with the fund value at the start of the year, the pension plan assets are exactly sufficient to pay the expected cost of any benefits due during the year, and to pay the expected cost of establishing the new actuarial liability at the year end. The normal contribution is often expressed as a percentage of the salary in the year following the valuation, known as the **normal contribution rate**.

We assume that (i) all employer contributions are payable at the start of the year, (ii) there are no employee contributions, and (iii) any benefits payable during the year are paid exactly half-way through the year. These are simplifying assumptions that make the development of the principles and formulae clearer, but they can be relaxed quite easily.

Consider an individual active pension plan member, aged x at the valuation date. Let ${}_tV^{(0r)}$ denote the actuarial liability of the member's age retirement benefits at time t, given that the member is still in the active state at time t. Let ${}_tV^{(r)}$ denote the actuarial liability of the member's age retirement benefits at time t, given that the member has left the active state by decrement r (age retirement) at or before time t, and let C^r denote the normal contribution due at the start of the valuation year (which runs from time $t = 0$ to $t = 1$), in respect of the age retirement benefits for an active member aged x at time $t = 0$. Let ${}_{1^-}p_x^{0r}$ be the probability that (x) exits active service by age retirement before age $x + 1$. Then we can determine C^r from the **funding equation**

$$ {}_0V^{(0r)} + C^r = \underbrace{v^{1/2}\,{}_{1^-}p_x^{0r}\,{}_{1/2}V^{(r)}}_{\substack{\text{EPV of benefits}\\ \text{for mid-year exits}}} + \underbrace{v\,{}_{1^-}p_x^{00}\,{}_{1^-}V^{(0r)}}_{\substack{\text{EPV of AL at } t=1^-\\ \text{for active lives}}}. \tag{11.8} $$

The funding equation (11.8) is interpreted as follows: the start of year actuarial liability plus normal contributions must be sufficient, on average, to pay for the EPV of the retirement benefits payable if the member exits during the year, or to fund the EPV at time $t = 1$ of the actuarial liability at the year end if the member remains in employment for the full year. We assume that lives exiting during the year do so exactly half-way through the year, and we also note that ${}_{1^-}V^{(0r)}$ includes the cost of lives retiring at exact age $x + 1$, if there are any.

So the right-hand side of (11.8) is the EPV at the valuation date of all the benefits accrued by the end of the valuation year, including any payments made to members retiring during the year (at age $x+1/2$ or at exact age $x+1$). In terms of the service table functions, we can express the funding equation as

$$\boxed{{}_0V^{(0r)} + C^r = v^{1/2}\,\frac{r_x}{l_x}\,{}_{1/2}V^{(r)} + v\,\frac{l_{x+1^-}}{l_x}\,{}_{1^-}V^{(0r)}.} \tag{11.9}$$

Similarly, we can derive an expression for the normal contribution for withdrawal benefits for (x), replacing the age retirement decrement with the withdrawal decrement. In this case, we do not have to allow for exact age exits, so we have

$$\boxed{{}_0V^{(0w)} + C^w = v^{1/2}\,\frac{w_x}{l_x}\,{}_{1/2}V^{(w)} + v\,\frac{l_{x+1}}{l_x}\,{}_1V^{(0w)}.} \tag{11.10}$$

The total normal contribution for (x) is the sum of the contributions in respect of all the plan benefits, and the aggregate normal contribution for the plan is the sum of the normal contributions for each active plan member. We assume that no normal contributions are paid in respect of inactive members.

The ideas behind the contribution equations are similar to the policy value recursions developed in Chapter 8.

As we noted in previous sections, there is some flexibility about how we define the actuarial liability, particularly for final salary plans. We consider two commonly used methods in this chapter.

- The **Projected Unit Credit** (PUC) funding method sets ${}_tV^{(0j)}$ equal to the EPV at time t of the *projected* accrued benefits for decrement j.
- The **Traditional Unit Credit** (TUC) funding method (also known as **Current Unit Credit**) sets ${}_tV^{(0j)}$ equal to the EPV at time t of the *current* accrued benefits for decrement j.

Because the contributions are dependent on the start and end year actuarial liability values, the two methods generate different contribution rates. In both cases, the normal contribution rates are expected to increase with age, but the level and gradient of contributions can be very different.

11.9 Projected Unit Credit funding

Under this method, we find the actuarial liability, ${}_tV^{(0j)}$, as the EPV at time t of the projected accrued benefits for decrement j. To illustrate how to use this method to calculate normal contribution rates for the decrement j benefits, we begin with an example, and then present a more general result.

Example 11.11 Peter and Alison are members of a final salary pension plan offering age retirement benefits, with an accrual rate of 1.5%.

Alison is aged 50 at the valuation date, and has 20 years past service. Her salary in the year following the valuation is expected to be \$100 000 (assuming she remains in employment for the full year). Her earnings in the previous year were \$96 150.

Peter is aged 63 at the valuation date, and has 25 years past service. His salary in the year following the valuation is expected to be \$90 000 (assuming he remains in employment for the full year). His earnings in the previous year were \$88 000.

Assuming PUC funding, calculate both the actuarial liability and the normal contribution payable at the start of the year in respect of the age retirement benefits for Alison and Peter.

You are given the pension plan information and valuation assumptions below.

- Final average salary is the salary in the year before retirement.
- The pension benefit is a life annuity payable monthly in advance.

Assumptions:

- Standard Service Table (Table D.9).
- Interest rate: 5% per year effective.
- Salaries increase at 4% per year at the start of each year.
- Mortality before and after retirement follows the Standard Ultimate Survival Model.

Solution 11.11 For exact age y retirements, $z_y/s_{50} = 1.04^{y-51}$, since the final average salary is the salary in the year prior to retirement. For mid-year retirements, at age $y + 0.5$, say, the final average salary is the salary earned between ages $y - 0.5$ and y, plus the salary earned between ages y and $y + 0.5$. The first part is half of the salary earned between ages $y - 1$ and y, and the second part is half of the salary that would have been earned between ages y and $y + 1$. So for Alison, we have

$$\frac{z_{y+0.5}}{s_{50}} = \frac{1}{2}\left(1.04^{y-50} + 1.04^{y-51}\right).$$

The actuarial liability at the valuation date for Alison is

$$\begin{aligned} {}_0V^{(0r)} = S_{50}\,\alpha\,20 \Bigg(& \frac{r_{60^-}}{l_{50}}\,\frac{z_{60}}{s_{50}}\,\ddot{a}^{(12)}_{60}\,v^{10} + \frac{r_{60^+}}{l_{50}}\,\frac{z_{60.5}}{s_{50}}\,\ddot{a}^{(12)}_{60.5}\,v^{10.5} \\ & + \frac{r_{61}}{l_{50}}\,\frac{z_{61.5}}{s_{50}}\,\ddot{a}^{(12)}_{61.5}\,v^{11.5} + \cdots + \frac{r_{64}}{l_{50}}\,\frac{z_{64.5}}{s_{50}}\,\ddot{a}^{(12)}_{64.5}\,v^{14.5} \\ & + \frac{r_{65^-}}{l_{50}}\,\frac{z_{65}}{s_{50}}\,\ddot{a}^{(12)}_{65}\,v^{15} \Bigg) = 274\,546. \end{aligned}$$

Similarly, for Peter the actuarial liability at the valuation date is

$$
\begin{aligned}
{}_0V^{(0r)} &= S_{63}\,\alpha\,25\left(\frac{r_{63}}{l_{63}}\,\frac{z_{63.5}}{s_{63}}\,\ddot{a}^{(12)}_{63.5}v^{0.5} + \frac{r_{64}}{l_{63}}\,\frac{z_{64.5}}{s_{63}}\ddot{a}^{(12)}_{64.5}v^{1.5} + \frac{r_{65^-}}{l_{63}}\,\frac{z_{65}}{s_{63}}\,\ddot{a}^{(12)}_{65}\,v^2\right)\\
&= 414\,941. \qquad (11.11)
\end{aligned}
$$

Note that here we use the *actual* final average salary for the age 63.5 retirement (i.e. $(90\,000+88\,000)/2$), rather than the salary based on the value of $z_{63.5}/s_{63}$.

Recall the funding equation (11.9):

$$
{}_0V^{(0r)} + C^r = v^{1/2}\,\frac{r_x}{l_x}\,{}_{1/2}V^{(r)} + v\,\frac{l_{x+1^-}}{l_x}\,{}_{1^-}V^{(0r)}.
$$

In Alison's case, the first term on the right-hand side is zero, as there are no mid-year (or exact age) retirements during the valuation year. So we consider only the second term, which is the EPV at time 0 of the actuarial liability in the active state at the start of the next year, ${}_1V^{(0r)}$. The difference between ${}_0V^{(0r)}$ and ${}_1V^{(0r)}$ is that (i) Alison will have one extra year of service, (ii) Alison will be one year older, and (iii) her salary is projected to be $S_{51} = S_{50}\,(s_{51}/s_{50})$. Then

$$
\begin{aligned}
{}_1V^{(0r)} = S_{51}\alpha 21\Bigg(&\frac{r_{60^-}}{l_{51}}\,\frac{z_{60}}{s_{51}}\,\ddot{a}^{(12)}_{60}\,v^9 + \frac{r_{60^+}}{l_{51}}\,\frac{z_{60.5}}{s_{51}}\,\ddot{a}^{(12)}_{60.5}\,v^{9.5} + \frac{r_{61}}{l_{51}}\,\frac{z_{61.5}}{s_{51}}\,\ddot{a}^{(12)}_{61.5}\,v^{10.5}\\
&+\cdots+\frac{r_{64}}{l_{51}}\,\frac{z_{64.5}}{s_{51}}\,\ddot{a}^{(12)}_{64.5}\,v^{13.5} + \frac{r_{65^-}}{l_{51}}\,\frac{z_{65}}{s_{51}}\,\ddot{a}^{(12)}_{65}\,v^{14}\Bigg)
\end{aligned}
$$

and so (noting that $S_{51}/s_{51} = S_{50}/s_{50}$) the expected present value at time 0 of the actuarial liability at time 1 is

$$
\begin{aligned}
&v\,\frac{l_{51}}{l_{50}}\,{}_1V^{(0r)}\\
&= S_{50}\,\alpha\,21\Bigg(\frac{r_{60^-}}{l_{50}}\,\frac{z_{60}}{s_{50}}\,\ddot{a}^{(12)}_{60}\,v^{10} + \frac{r_{60^+}}{l_{50}}\,\frac{z_{60.5}}{s_{50}}\,\ddot{a}^{(12)}_{60.5}\,v^{10.5} + \frac{r_{61}}{l_{50}}\,\frac{z_{61.5}}{s_{50}}\,\ddot{a}^{(12)}_{61.5}\,v^{11.5}\\
&\qquad+\cdots+\frac{r_{64}}{l_{50}}\,\frac{z_{64.5}}{s_{50}}\,\ddot{a}^{(12)}_{64.5}\,v^{14.5} + \frac{r_{65^-}}{l_{50}}\,\frac{z_{65}}{s_{50}}\,\ddot{a}^{(12)}_{65}\,v^{15}\Bigg)\\
&= \frac{21}{20}\,{}_0V^{(0r)}.
\end{aligned}
$$

So for Alison we have a contribution equation under the PUC method of

$$
{}_0V^{(0r)} + C^r = \frac{21}{20}\,{}_0V^{(0r)} \Rightarrow C^r = \frac{1}{20}\,{}_0V^{(0r)} = 13\,727,
$$

which gives a normal contribution rate (that is, percentage of salary in the valuation year) of $C^r/S_{50} = 13.73\%$.

In fact, if there are no exits by decrement j in the valuation year, then the normal contribution for decrement j is always equal to the starting actuarial liability divided by the service, as long as the service is greater than 0 years. This makes sense intuitively; the purpose of the normal contribution is to pay for benefits accruing from the additional year of service earned in the year following the valuation. If n years of accrued benefit is valued at ${}_0V^{(0j)}$, then one year's additional accrual has an EPV of ${}_0V^{(0j)}/n$.

For Peter, we again start with the right-hand side of the funding equation (11.9). First, we have

$$ {}_{1/2}V^{(r)} = S^{\mathrm{F}}_{63.5} \times \alpha \times 25.5\, \ddot{a}^{(12)}_{63.5} = S_{63} \times \alpha \times 25.5\, \frac{z_{63.5}}{s_{63}}\, \ddot{a}^{(12)}_{63.5}, $$

which is the EPV at age 63.5 of Peter's benefits if he retires at that time.

Recall that in equation (11.9) ${}_{1^-}V^{(0r)}$ includes the cost of lives retiring at exact age $x+1$. As there are no exact age retirements at age 64, we need only consider the reserve at the year end assuming Peter is still active. Assuming that $S_{64} = S_{63}(s_{64}/s_{63})$, we have

$$ {}_1V^{(0r)} = S_{64} \times \alpha \times 26 \times \left(\frac{r_{64}}{l_{64}}\, \frac{z_{64.5}}{s_{64}}\, \ddot{a}^{(12)}_{64.5}\, v^{0.5} + \frac{r_{65^-}}{l_{64}}\, \frac{z_{65}}{s_{64}}\, \ddot{a}^{(12)}_{65}\, v \right), $$

and so

$$ v\,\frac{l_{64}}{l_{63}}\, {}_1V^{(0r)} = S_{63} \times \alpha \times 26 \times \left(\frac{r_{64}}{l_{63}}\, \frac{z_{64.5}}{s_{63}}\, \ddot{a}^{(12)}_{64.5}\, v^{1.5} + \frac{r_{65^-}}{l_{63}}\, \frac{z_{65}}{s_{63}}\, \ddot{a}^{(12)}_{65}\, v^{2} \right). \tag{11.12} $$

Applying the normal contribution equation (11.9) with expression (11.11), we find that

$$ \begin{aligned} C^r = S_{63} \times \alpha \times \Bigg(& \frac{1}{2}\left(\frac{r_{63}}{l_{63}}\, \frac{z_{63.5}}{s_{63}}\, \ddot{a}^{(12)}_{63.5}\, v^{0.5} \right) + \frac{r_{64}}{l_{63}}\, \frac{z_{64.5}}{s_{63}}\, \ddot{a}^{(12)}_{64.5}\, v^{1.5} \\ & + \frac{r_{65^-}}{l_{63}}\, \frac{z_{65}}{s_{63}}\, \ddot{a}^{(12)}_{65}\, v^{2} \Bigg) \\ = 15\,762, & \end{aligned} $$

which is 17.51% of Peter's salary in the valuation year.

We can explain this contribution by observing that it must fund the EPV of one half-year of accrual if Peter retires at age 63.5 (the first term in parentheses) and a full extra year of accrual if Peter is still active at age 64, and retires at age 64.5 or 65 (the second and third terms in parentheses).

□

11.9.1 The normal contribution formula using PUC funding

The explanation given at the end of the previous example about the normal contribution for Peter can be applied more generally to obtain an expression for the normal contribution for PUC funding.

Consider the age retirement normal contribution, due at the start of the year, under the PUC funding method for a life aged x with n years of past service. The contribution must fund an additional year of accrual for retirement at or after age $x+1$, but only half a year if retirement takes place at age $x+0.5$. Then the normal contribution, C^r, is

$$S_x\,\alpha\left(\frac{1}{2}\frac{r_x}{l_x}\frac{z_{x+\frac{1}{2}}}{s_x}\,a^r_{x+\frac{1}{2}}\,v^{\frac{1}{2}}+\underbrace{\sum_{y=x+1}^{xr-1}}_{\substack{\text{mid-year}\\ \text{exits}}}\frac{r_y}{l_x}\frac{z_{y+\frac{1}{2}}}{s_x}\,a^r_{y+\frac{1}{2}}\,v^{y+\frac{1}{2}-x}+\underbrace{\sum_{y=x+1}^{xr}}_{\substack{\text{exact age}\\ \text{exits}}}\frac{r_{y^-}}{l_x}\frac{z_y}{s_x}\,a^r_y\,v^{y-x}\right)$$

$$=S_x\,\alpha\left(\underbrace{\sum_{y=x}^{xr-1}}_{\substack{\text{mid-year}\\ \text{exits}}}\frac{r_y}{l_x}\frac{z_{y+\frac{1}{2}}}{s_x}\,a^r_{y+\frac{1}{2}}\,v^{y+\frac{1}{2}-x}+\underbrace{\sum_{y=x+1}^{xr}}_{\substack{\text{exact age}\\ \text{exits}}}\frac{r_{y^-}}{l_x}\frac{z_y}{s_x}\,a^r_y\,v^{y-x}\right.$$

$$\left.-\,\frac{1}{2}\frac{r_x}{l_x}\frac{z_{x+\frac{1}{2}}}{s_x}\,a^r_{x+\frac{1}{2}}\,v^{\frac{1}{2}}\right).$$

Further,

$${}_0V^{(0r)}=S_x\,\alpha\,n\left(\underbrace{\sum_{y=x}^{xr-1}}_{\substack{\text{mid-year}\\ \text{exits}}}\frac{r_y}{l_x}\frac{z_{y+\frac{1}{2}}}{s_x}\,a^r_{y+\frac{1}{2}}\,v^{y+\frac{1}{2}-x}+\underbrace{\sum_{y=x+1}^{xr}}_{\substack{\text{exact age}\\ \text{exits}}}\frac{r_{y^-}}{l_x}\frac{z_y}{s_x}\,a^r_y\,v^{y-x}\right).$$

So for $n>0$, we have

$$C^r=\frac{{}_0V^{(0r)}}{n}-S_x\,\alpha\left(\frac{1}{2}\frac{r_x}{l_x}\frac{z_{x+\frac{1}{2}}}{s_x}\,a^r_{x+\frac{1}{2}}\,v^{\frac{1}{2}}\right).$$

Note the similarities to the actuarial liability calculation in equation (11.3) (which gives ${}_0V^{(0r)}$ above); the summations are the same, but we are valuing only half a year of benefit for all exits between ages x and $x+1$, and only one year of benefit for all exits at or after age $x+1$, compared with the n years valued for ${}_0V^{(0r)}$. If there are no mid-year exits at age $x + 0.5$, and assuming $n > 0$, then this simplifies to

$$C^r = \frac{{}_0V^{(0r)}}{n}.$$

11.10 Traditional Unit Credit funding

Using the TUC approach, the valuation liability is based on the **current** final average salary, rather than on the projected final average salary used in the PUC method. That is, ${}_tV^{(0j)}$ is the EPV at time t of current accrued benefits for decrement j. As we assume that salaries increase with service, the TUC reserves are generally lower than the PUC reserves, except at two points; at the very start of the member's service, when $n = 0$ (i.e. when there is no past service) the reserve under both methods is zero, and, if the member is still active just before the final age at which all members are assumed to retire, then both the PUC reserves and the TUC reserves will be equal to the EPV at that time of the full retirement benefits. Again, we illustrate the principles through an example, and follow with a more general formula.

Example 11.12 Repeat the previous example, but using the Traditional Unit Credit funding method.

Solution 11.12 Now we value the age retirement benefits assuming a final average salary as at the valuation date, which gives $S^{\mathrm{F}}_{50} = S_{49} = \$96\,150$ for Alison and $S^{\mathrm{F}}_{63} = S_{62} = \$88\,000$ for Peter.

The actuarial liability at the valuation date for Alison is

$$\begin{aligned}{}_0V^{(0r)} &= S^{\mathrm{F}}_{50} \times \alpha \times 20 \times \left(\frac{r_{60^-}}{l_{50}}\, \ddot{a}^{(12)}_{60}\, v^{10} + \frac{r_{60^+}}{l_{50}}\, \ddot{a}^{(12)}_{60.5}\, v^{10.5} + \frac{r_{61}}{l_{50}}\, \ddot{a}^{(12)}_{61.5}\, v^{11.5} \right.\\ &\qquad\left. + \cdots + \frac{r_{64}}{l_{50}}\, \ddot{a}^{(12)}_{64.5}\, v^{14.5} + \frac{r_{65^-}}{l_{50}}\, \ddot{a}^{(12)}_{65}\, v^{15} \right)\\ &= 168\,249,\end{aligned}$$

and the actuarial liability at the valuation date for Peter is

$$\begin{aligned}{}_0V^{(0r)} &= S^{\mathrm{F}}_{63} \times \alpha \times 25 \times \left(\frac{r_{63}}{l_{63}}\, \ddot{a}^{(12)}_{63.5}\, v^{0.5} + \frac{r_{64}}{l_{63}}\, \ddot{a}^{(12)}_{64.5}\, v^{1.5} + \frac{r_{65^-}}{l_{63}}\, \ddot{a}^{(12)}_{65}\, v^{2} \right)\\ &= 392\,811.\end{aligned}$$

For the normal contributions, we once again start with the funding equation (11.9):

$${}_0V^{(0r)} + C^r = v^{1/2}\, \frac{r_x}{l_x}\, {}_{1/2}V^{(r)} + v\, \frac{l_{x+1^-}}{l_x}\, {}_{1^-}V^{(0r)}.$$

As in the previous example, in Alison's case the first term on the right-hand side is zero. However, the difference between ${}_0V^{(0r)}$ and ${}_1V^{(0r)}$ is now that (i) Alison will have one extra year of service, (ii) Alison will be one year older, and (iii) her final average salary at age 51 will be $S^{\mathrm{F}}_{51} = 100\,000$. So

$$ {}_1V^{(0r)} = S^{\mathrm{F}}_{51} \times \alpha \times 21 \times \left(\frac{r_{60^-}}{l_{51}}\, \ddot{a}^{(12)}_{60}\, v^{9} + \frac{r_{60^+}}{l_{51}}\, \ddot{a}^{(12)}_{60.5}\, v^{9.5} + \frac{r_{61}}{l_{51}}\, \ddot{a}^{(12)}_{61.5}\, v^{10.5} \right. $$
$$ \left. + \cdots + \frac{r_{64}}{l_{51}}\, \ddot{a}^{(12)}_{64.5}\, v^{13.5} + \frac{r_{65^-}}{l_{50}}\, \ddot{a}^{(12)}_{65}\, v^{14} \right) $$

and so the EPV at time 0 of the actuarial liability at time 1 is

$$ v\, \frac{l_{51}}{l_{50}}\, {}_1V^{(0r)} = S^{\mathrm{F}}_{50} \times \frac{S^{\mathrm{F}}_{51}}{S^{\mathrm{F}}_{50}} \times \alpha \times 21 \times \left(\frac{r_{60^-}}{l_{50}}\, \ddot{a}^{(12)}_{60}\, v^{10} + \frac{r_{60^+}}{l_{50}}\, \ddot{a}^{(12)}_{60.5}\, v^{10.5} \right. $$
$$ \left. + \cdots + \frac{r_{64}}{l_{50}}\, \ddot{a}^{(12)}_{64.5}\, v^{14.5} + \frac{r_{65^-}}{l_{50}}\, \ddot{a}^{(12)}_{65}\, v^{15} \right) $$
$$ = \frac{21}{20}\, \frac{S^{\mathrm{F}}_{51}}{S^{\mathrm{F}}_{50}}\, {}_0V^{(0r)}. $$

So for Alison, we have a contribution equation under the TUC method of

$$ {}_0V^{(0r)} + C^r = \frac{21}{20}\, \frac{S^{\mathrm{F}}_{51}}{S^{\mathrm{F}}_{50}}\, {}_0V^{(0r)} $$
$$ \Rightarrow C^r = {}_0V^{(0r)} \left(\frac{21}{20} \times \frac{100\,000}{96\,150} - 1 \right) = 15\,486, $$

which gives a normal contribution rate of 15.49%.

For Peter, the key expressions on the right-hand side of the funding equation are

$$ {}_{1/2}V^{(r)} = S^{\mathrm{F}}_{63.5} \times \alpha \times 25.5 \times \ddot{a}^{(12)}_{63.5} = 460\,048 $$

and

$$ {}_1V^{(0r)} = S^{\mathrm{F}}_{64} \times \alpha \times 26 \times \left(\frac{r_{64}}{l_{64}}\, \ddot{a}^{(12)}_{64.5}\, v^{0.5} + \frac{r_{65-}}{l_{64}}\, \ddot{a}^{(12)}_{65}\, v \right) = 436\,359 $$

and so the right-hand side of the funding equation is

$$ v^{1/2}\, \frac{r_{63}}{l_{63}}\, {}_{1/2}V^{(r)} + v\, \frac{l_{64}}{l_{63}}\, {}_1V^{(0r)} = 416\,485. $$

We subtract the reserve at the start of the year, to give

$$ C^r = 416\,485 - 392\,811 = 23\,674, $$

which is 26.30% of salary.

□

11.10.1 The normal contribution formula using TUC funding

We now generalize the results for the age retirement normal contribution under TUC funding; contributions for other decrements can be derived similarly.

Assume first that there are no exits during the valuation year. In this case, as we saw in Example 11.12 (for Alison), the normal contribution for a life aged x, with past service $n > 0$, can be written as

$$C^r = {}_0V^{(0r)} \left(\frac{n+1}{n} \frac{S^{\mathrm{F}}_{x+1}}{S^{\mathrm{F}}_x} - 1 \right),$$

where n is the past service at the valuation date, and $S^{\mathrm{F}}_{x+1}/S^{\mathrm{F}}_x$ is the ratio of the projected year end ‘final average salary’ to the starting ‘final average salary’.

We can rearrange this formula into two parts, as

$$C^r = {}_0V^{(0r)} \, \frac{1}{n} \frac{S^{\mathrm{F}}_{x+1}}{S^{\mathrm{F}}_x} + {}_0V^{(0r)} \left(\frac{S^{\mathrm{F}}_{x+1}}{S^{\mathrm{F}}_x} - 1 \right).$$

The first term on the right-hand side funds the cost of one additional year of accrued benefit, based on the final average salary at the year end. The second term funds the cost arising from one additional year of salary escalation applied to all the past service benefits at the valuation date. So we see clearly the major difference between the PUC and TUC methods; under the PUC method we pre-fund all the projected future salary increases on the accrued benefit. The normal contribution funds new accrued benefits, based on the same projected final average salary. Under the TUC method we still have to fund the newly accrued benefits, but only based on the one-year projected final average salary. In addition, we have to fund the impact of each year’s salary escalation on the past service benefit. In the case of Alison’s normal contribution in Example 11.12, the cost of the additional year of accrual was \$8 749, and the cost of funding the increase in past service costs caused by the salary escalation (from \$96 150 to \$100 000) is \$6 737. The total is \$15 486 as before.

In the general case, when there is a possibility of a mid-year exit at age $x + 0.5$, the same principles apply, but the mid-year benefits and reserve are based on the final average salary half-way through the year, which means that we need to treat them separately. That gives four parts to the normal contribution, as follows.

1. The cost of an additional 0.5 years of accrual on exit at age $x+0.5$, which is

$$0.5 \times S^{\mathrm{F}}_{x+0.5} \times \alpha \times \left(\frac{r_x}{l_x} \, v^{0.5} \, a^r_{x+0.5} \right).$$

2. The cost of an additional year's accrual for exits from age $x+1$, based on the final average salary at the next valuation date, which is

$$S^{\mathrm{F}}_{x+1} \times \alpha \times \left(\underset{\substack{\text{mid-year}\\ \text{exits}}}{\sum_{y=x+1}^{xr-1}} \frac{r_y}{l_x}\, a^r_{y+\frac{1}{2}}\, v^{y+\frac{1}{2}-x} + \underset{\substack{\text{exact age}\\ \text{exits}}}{\sum_{y=x+1}^{xr}} \frac{r_{y^-}}{l_x}\, a^r_y\, v^{y-x} \right).$$

3. The cost of uprating the accrued pension for exits at age $x+0.5$ to allow for salary escalation between ages x and $x+0.5$. The increase to be funded is

$$\left(\frac{S^{\mathrm{F}}_{x+0.5}}{S^{\mathrm{F}}_x} - 1 \right) \times S^{\mathrm{F}}_x \times n \times \alpha \times \left(\frac{r_x}{l_x}\, v^{0.5}\, a^r_{x+0.5} \right).$$

4. The cost of uprating the accrued pension for exits at or after age $x+1$ to allow for salary growth between ages x and $x+1$. The increase to be funded is

$$\left(\frac{S^{\mathrm{F}}_{x+1}}{S^{\mathrm{F}}_x} - 1 \right) \times \left({}_0V^{(0r)} - S^{\mathrm{F}}_x \times n \times \alpha \times \left(\frac{r_x}{l_x}\, v^{0.5}\, a^r_{x+0.5} \right) \right).$$

In Example 11.12, Peter's normal contribution can be broken down into these four parts as follows:

1.	Additional 1/2-year accrual for age 63.5 retirements:	835
2.	Additional full year accrual for retirements after age 64:	14 380
3.	Salary uprating of past accruals for age 63.5 retirements:	469
4.	Salary uprating of past accruals for retirements after age 64:	7 989
		23 674

11.11 Comparing PUC and TUC funding methods

As mentioned earlier, the PUC actuarial liability will be greater than the TUC actuarial liability at all points in the working lifetime of an active member, except at the very start and very end of the member's active service, when the PUC and TUC actuarial liabilities are equal. The larger reserve under PUC funding arises because the PUC method prepays for salary increases. Since the actuarial liability is made up from the normal contributions, plus the investment income earned on those contributions, the PUC normal contributions must start out greater than the TUC normal contributions, to build up the larger reserves. On the other hand, because the PUC and TUC reserves end up equal at the very end of the member's working lifetime, at some point the TUC contributions must catch up with and overtake the PUC contributions.

In the early years, the TUC contributions are considerably less than the PUC contributions; the TUC contribution is paying for accruals based on the then current salary, while the PUC contribution is paying for accruals based on the final salary. As there is not much reserve, the second part of the TUC contribution, that funds the impact of salary escalation on the past accruals, is not very significant. At older ages, the part of the TUC contribution funding new accruals is still less than the PUC contribution (until the final year when they are equal), but the cost of salary escalation becomes substantial, so the TUC contributions become much greater than the PUC contributions at older ages.

In Figure 11.2(b) we show typical patterns for the actuarial liability and the normal contribution rate over the working lifetime of an active member of a final salary pension plan, who enters at age 25 and retires at age 65. We see that the initial contribution rates are very low for the TUC method, but can increase to very high levels near to retirement. Because the employer has a mix of younger and older employees, the total normal contribution rate may not vary too much from year to year, but if the demographic mix of the active membership changes, the impact on the overall contribution rate is likely to be more significant under the TUC approach.

11.12 Retiree health benefits

11.12.1 Introduction

Retiree health benefits are provided by some employers through supplementary health insurance cover. This insurance reimburses the retiree for some portion of their health care expenses, supplementing the cover provided by relevant government funded systems. For example, seniors in the USA are eligible for socialized health cover through the Medicare programme, but there are significant costs that are not covered; most retirees have to pay a significant amount for each doctor's visit and hospital stay. Supplementary health insurance may also provide cover for care that is not included in the Medicare programme, such as dental treatment.

Retiree health benefits are typically offered only to those who retire from an organization, so that those who leave before they are eligible to retire would not receive retirement health benefits, even if they have a deferred pension benefit in the pension plan. In addition there are typically minimum service requirements to qualify for retiree health benefits, such as at least 10 years of service with the organization.

Retiree health benefits may be pre-funded, similarly to defined benefit pensions, or may be funded on a pay-as-you-go basis, meaning that the annual premiums are met by the organization from year-to-year earnings. However,

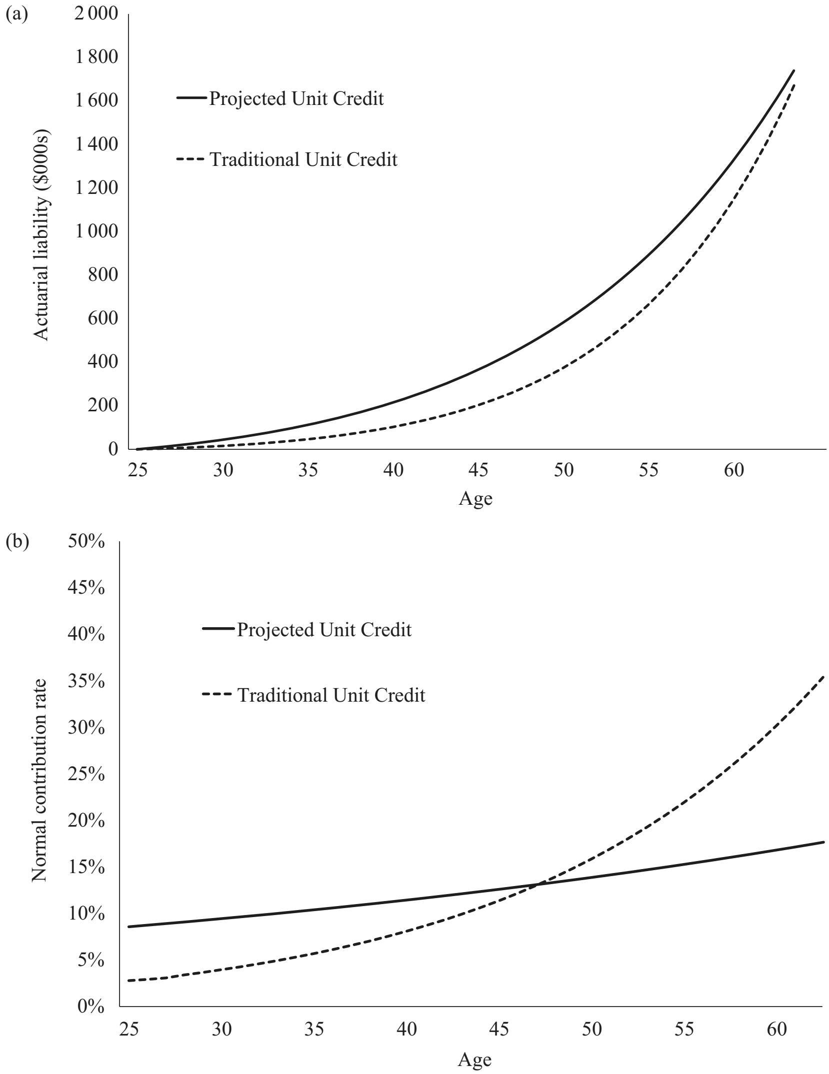


Figure 11.2 Actuarial Liability (a) and Normal Contribution Rates (b) for a final salary pension plan under PUC and TUC funding.

even when pay-as-you-go is used, there may be accounting regulations requiring benefits to be valued and declared in financial statements. Unlike accrued pension benefits, post-retirement health benefits do not represent a legal obligation for an employer. The benefits could be withdrawn at any point with no ongoing liability for the employer. This means that there is no need to hold separate funds to meet the future liability.

11.12.2 Valuing retiree health benefits

In this section we consider the valuation and funding of a simplified retiree health care benefit plan. We assume that the health care benefits are provided through a health insurance company. The employer pays premiums to the insurer to secure the post-retirement health cover, so the cost of the benefits to the sponsoring employer is the cost of the premiums that it pays to the health insurance company.

Premiums for health insurance increase with age and also with time. As health care inflation tends to exceed normal price inflation, it is an important factor in the valuation.

Let $B(x,t)$ denote the annual premium payable for health insurance under an employer-sponsored plan, for a life aged x at time t.

For a life retiring at age y at time t, the EPV at retirement of the supplementary health insurance is

$$\begin{aligned} &B(y,t) + v\,p_y\,B(y+1,t+1) + v^2\,{}_2p_y\,B(y+2,t+2) \\ &\qquad + v^3\,{}_3p_y\,B(y+3,t+3) + \cdots \\ &\qquad = B(y,t)\left(1 + v\,p_y\,\frac{B(y+1,t+1)}{B(y,t)} + v^2\,{}_2p_y\,\frac{B(y+2,t+2)}{B(y,t)} + \cdots\right) \\ &\qquad = B(y,t)\;a^B_{y,t}, \end{aligned}$$

where we define

$$a^B_{y,t} = 1 + v\,p_y\frac{B(y+1,t+1)}{B(y,t)} + v^2\,{}_2p_y\frac{B(y+2,t+2)}{B(y,t)} + \cdots .$$

Now, suppose that the annual rate of inflation for the health care premiums is j, and that premiums also increase exponentially with age, so that $B(y+1,t)/B(y,t) = c$, say. Then for any k_1 and k_2,

$$B(y+k_1, t+k_2) = c^{k_1}(1+j)^{k_2}B(y,t), \tag{11.13}$$

so that

$$\begin{aligned} a^B_{y,t} &= 1 + v\,p_y\,c\,(1+j) + v^2{}_2p_y\,c^2(1+j)^2 + \cdots \\ &= 1 + v_{i^*}\,p_y + v^2_{i^*}\,{}_2p_y + v^3_{i^*}\,{}_3p_y + \cdots \\ &= \ddot{a}_{y|i^*}, \end{aligned} \tag{11.14}$$

where $i^* = (1+i)/(c\,(1+j)) - 1$. In this case we can drop the subscript t, and we have

$$a^B_y = \ddot{a}_{y|i^*}.$$

We assume for convenience that the age and inflation increases apply continuously, so that (11.13) applies for all k_1 and k_2, not just integers. If the age or inflation increases are discrete, then some adjustment to the valuation formulae would be required.

Example 11.13 Calculate the EPV of the post-retirement health benefits for a retired life currently aged 65 who is subject to the Standard Ultimate Survival Model. You are given that $B(65,0) = \$4\,000$, and that $i = 5\%$, $c = 1.02$ and $j = 4\%$.

Solution 11.13 We have

$$i^* = \frac{1.05}{1.02 \times 1.04} - 1 = -1.018\%$$

and so $\ddot{a}_{65|i^*} = 26.6074$, giving an EPV of

$$4\,000 \times 26.6074 = 106\,430.$$

Note that the adjusted interest rate $i^* = -1.018\%$ is negative because the rate of increase of the premiums is greater than the interest rate which discounts them. □

To value the contingent post-retirement health care benefits for an active employee currently aged x, we use the same approach as for pension benefit valuation.

The actuarial value of the total post-retirement health care benefits, which we denote by AVTHB, is

$$\underbrace{\sum_{y=x}^{xr-1}}_{\substack{\text{mid-year}\\ \text{exits}}} \frac{r_y}{l_x} B(y+0.5, y+0.5-x) a^B_{y+\frac{1}{2}} \, v^{y+0.5-x} + \underbrace{\sum_{y=x+1}^{xr}}_{\substack{\text{exact age}\\ \text{exits}}} \frac{r_{y^-}}{l_x} B(y, y-x) a^B_y \, v^{y-x}.$$

Noting from equation (11.13) that

$$B(y+0.5, y+0.5-x) = B(x,0)\, c^{y-x+0.5}\, (1+j)^{y-x+0.5},$$

we can write the *AVTHB* as

$$B(x,0)\left(\sum_{\substack{y=x\\ \text{mid-year}\\ \text{exits}}}^{xr-1} \frac{r_y}{l_x}\,(c(1+j))^{y+0.5-x} a^B_{y+\frac{1}{2}}\; v^{y+0.5-x}\right.$$

$$\left. + \sum_{\substack{y=x+1\\ \text{exact age}\\ \text{exits}}}^{xr} \frac{r_{y^-}}{l_x}\,(c(1+j))^{y-x}\; a^B_y\, v^{y-x}\right)$$

$$= B(x,0)\left(\sum_{\substack{y=x\\ \text{mid-year}\\ \text{exits}}}^{xr-1} \frac{r_y}{l_x}\, a^B_{y+\frac{1}{2}}\; v_{i*}^{y+0.5-x} + \sum_{\substack{y=x+1\\ \text{exact age}\\ \text{exits}}}^{xr} \frac{r_{y^-}}{l_x}\, a^B_y\, v_{i^*}^{y-x}\right), \quad (11.15)$$

where $i^* = (1+i)/(c\,(1+j)) - 1$ and $a^B_y = \ddot{a}_{y|i^*}$.

Example 11.14 The annual benefit premium for retiree health care cover for a life aged 60 at the valuation date is $B(60,0) = \$5\,000$. You are given that

- $c = 1.02, j = 5\%$ and $i = 6\%$,
- retirements follow the Standard Service Table (Appendix D),
- mortality after age retirement follows the Standard Ultimate Survival Model.

Calculate the AVTHB for an active life

(a) aged 63 at the valuation date, and
(b) aged 50 at the valuation date.

Solution 11.14 (a) From (11.15), the AVTHB is

$$B(63,0)\left(\frac{r_{63}}{l_{63}}\,\ddot{a}_{63.5|\,i^*}\, v_{i^*}^{0.5} + \frac{r_{64}}{l_{63}}\,\ddot{a}_{64.5|\,i^*}\, v_{i^*}^{1.5} + \frac{r_{65-}}{l_{63}}\,\ddot{a}_{65|\,i^*}\, v_{i^*}^{2}\right)$$

$$= B(60,0)\,c^3\left(\frac{r_{63}}{l_{63}}\,\ddot{a}_{63.5|\,i^*}\, v_{i^*}^{0.5} + \frac{r_{64}}{l_{63}}\,\ddot{a}_{64.5|\,i^*}\, v_{i^*}^{1.5} + \frac{r_{65-}}{l_{63}}\,\ddot{a}_{65|\,i^*}\, v_{i^*}^{2}\right).$$

We have $i^* = (1+i)/(c(1+j)) - 1 = -1.027\%$, and we can calculate the required annuity functions as follows:

x	$\ddot{a}_{x\|\,i^*}$	x	$\ddot{a}_{x\|\,i^*}$
60.0	32.5209	63.5	28.3496
60.5	31.9097	64.5	27.2047
61.5	30.7024	65.0	26.6403
62.5	29.5156		

Using these values and the Standard Service Table, we find the AVTHB is \$143 593.

(b) Again using (11.15), the AVTHB is

$$
\begin{aligned}
& B(50,0)\left(\frac{r_{60^-}}{l_{50}}\ddot{a}_{60|\,i^*}\, v_{i^*}^{10} + \frac{r_{60}}{l_{50}}\,\ddot{a}_{60.5|\,i^*}\, v_{i^*}^{10.5}\right. \\
& \qquad \left. + \cdots + \frac{r_{64}}{l_{50}}\,\ddot{a}_{64.5|\,i^*}\, v_{i^*}^{14.5} + \frac{r_{65-}}{l_{50}}\,\ddot{a}_{65|\,i^*}\, v_{i^*}^{15}\right) \\
& = \frac{B(60,0)}{c^{10}}\left(\frac{r_{60^-}}{l_{50}}\ddot{a}_{60|\,i^*}\, v_{i^*}^{10} + \frac{r_{60}}{l_{50}}\,\ddot{a}_{60.5|\,i^*}\, v_{i^*}^{10.5}\right. \\
& \qquad \left. + \cdots + \frac{r_{64}}{l_{50}}\,\ddot{a}_{64.5|\,i^*}\, v_{i^*}^{14.5} + \frac{r_{65-}}{l_{50}}\,\ddot{a}_{65|\,i^*}\, v_{i^*}^{15}\right) \\
& = 107\,168.
\end{aligned}
$$

□

We see from this example that it is relatively straightforward to write down the AVTHB from first principles, and this is usually preferable to applying memorized valuation formulae, which may need adapting depending on the information given, and on the specific details of each case. Calculations generally need to be performed on a spreadsheet.

11.12.3 Funding retiree health benefits

Although employers may not be required to pre-fund retiree health benefits, they may choose to do so. Even if they choose the pay-as-you-go route, it may be necessary to determine the value of the benefits and a (nominal) normal cost for accounting purposes.

Both of the funding methods discussed earlier in this chapter are **accruals-based methods**, meaning that we assume the actuarial liability at each valuation date is based on the benefits earned from past service; we do not include future service benefits in the liability valuation, as these are assumed to be funded from future contributions.

The notion of accrual for retiree health benefits is less natural than for a final salary pension plan, as the health benefits do not depend on service, except as a threshold for qualifying for benefits. So, two employees retiring on the same day at the same age will both be entitled to the same benefits, costing the same amount, even if one has 20 years of service and the other has 40 years. The benefit is also independent of salary, so paying contributions as a percentage of payroll is less natural than for salary-related benefits.

We can adapt the accruals principle to retiree health benefits by assuming that the benefits accrue linearly over each employee's period of employment. This means that if the accruals period is set at $xr - xe$ (i.e. retirement age − entry age) then the actuarial liability for an active life aged x, with $n = x - xe$ years of past service, denoted ${}_0V^{(0B)}$, is

$$ {}_0V^{(0B)} = \frac{n}{xr - xe}\, AVTHB. $$

If there are multiple possible retirement dates, then we might use the same proportional approach, but separately for each retirement date. That is, adapting the AVTHB formula in (11.15), we have ${}_0V^{(0B)}$ equal to

$$ B(x,0)\Bigg(\underset{\substack{\text{mid-year}\\ \text{exits}}}{\sum_{y=x}^{xr-1}} \frac{r_y}{l_x}\, a^B_{y+0.5}\, v_{i*}^{y+0.5-x} \left(\frac{n}{y+0.5-xe}\right) + \underset{\substack{\text{exact age}\\ \text{exits}}}{\sum_{y=x+1}^{xr}} \frac{r_y^{-}}{l_x}\, a^B_y\, v_{i*}^{y-x} \left(\frac{n}{y-xe}\right)\Bigg) \tag{11.16} $$

so that each term is adjusted by a factor of (past service/total service), based on retirement at age $y + 0.5$ for mid-year retirements and age y for exact age retirements.

Once we have defined the actuarial liability, the normal cost is found using the same recursion as in previous sections. That is, if

- ${}_tV^{(0B)}$ represents the actuarial liability at time t for an active employee aged x at time $t = 0$ (the valuation date),
- C^B represents the normal cost for the year, payable at time 0, for the same employee, and
- ${}_tV^{(B)}$ represents the actuarial liability at time t for post-retirement health benefits given that (x) has retired at time t,

then the general funding valuation recursion is

$$ {}_0V^{(0B)} + C^B = v^{1/2}\, \frac{r_x}{l_x}\, {}_{1/2}V^{(B)} \; + v\, \frac{l_{x+1}^{-}}{l_x}\, {}_{1^-}V^{(0B)}, \tag{11.17} $$

where ${}_tV^{(B)} = B(x+t,t)\, a^B_{x+t}$.

Example 11.15

(a) Suppose the retiree health benefits in Example 11.14 are funded using a pro-rata accruals method, assuming retirement at age 60 for the accruals period. Calculate the accrued liability and normal cost for an employee

(i) aged 50 with 15 years service, and
(ii) aged 63 with 25 years service.

(b) Repeat part (a), but assume linear accrual to each retirement age.

Solution 11.15

(a) (i) We assume the benefits are accrued over 25 years, from entry until age 60. So the actuarial liability is

$$ {}_0V^{(0B)} = AVTHB\,\frac{15}{25} = 64\,301. $$

Since there are no mid-year exits by age retirement between ages 50 and 51, the normal cost is

$$ C^B = \frac{{}_0V^{(0B)}}{15} = \frac{AVTHB}{25} = 4\,287. $$

The normal cost is exactly sufficient to pay for one more year of accrual, and, under this approach, each year's accrual costs $(AVTHB/25)$, up to age 60.

(ii) Under this approach, by age 63 the benefit is fully funded, so the actuarial liability is ${}_0V^{(0B)} = AVTHB = 143\,593$, and the normal cost is 0. There is no additional accrual to fund from the normal cost.

(b) (i) We cannot use the AVTHB calculation here, because each term in the valuation is assumed to accrue at a different rate, depending on the retirement age.

Instead we consider each possible retirement date separately. For the employee aged 50 with 15 years past service, the benefit payable on retirement at age 60 is assumed to accrue over 25 years; the benefit payable if the employee retires at age 60.5 is assumed to accrue over 25.5 years, and so on to the age 65 retirement benefit, which accrues over the 30 years of total service. So, for each possible retirement age y, say, the accrued benefit at age 50 is $15/(y-35)$ years of the total cost given retirement at age y.

Using the approach from equation (11.16) with $xe = 35$, we have

$$ \begin{aligned} {}_0V^{(0B)} &= B(50,0)\left(\frac{r_{60^-}}{l_{50}}\,\ddot{a}_{60|\,i^*}\,v^{10.5}\left(\frac{15}{25}\right) + \frac{r_{60}}{l_{50}}\,\ddot{a}_{60.5|\,i^*}\,v_{i^*}^{11.5}\left(\frac{15}{25.5}\right)\right. \\ &\qquad \left. + \cdots + \frac{r_{64}}{l_{50}}\,\ddot{a}_{64.5|\,i^*}\,v_{i^*}^{4.5}\left(\frac{15}{29.5}\right) + v_{i^*}^{15}\,\frac{r_{65-}}{l_{50}}\,\ddot{a}_{65|\,i^*}\left(\frac{15}{30}\right)\right) \\ &= 58\,677, \end{aligned} $$

noting that $B(50,0) = B(60,0)/c^{10}$. The normal cost is the EPV of one additional year of accrual, as there are no mid-year retirements between ages 50 and 51, so there are no half-year complications. That means that

$$C^B = \frac{58\,677}{15} = \$3\,912.$$

Adding the normal cost to the actuarial liability gives a new actuarial liability based on 16 years of service, rather than 15, so the normal cost funds the additional year of (notional) accrual.

(ii) For the life aged 63, we note that $B(63,0) = B(60,0)\,c^3$. Then, from equation (11.16) with $xe = 38$, the actuarial liability is

$$\begin{aligned} {}_0V^{(0B)} &= B(63,0)\left(\frac{r_{63}}{l_{63}}\,\ddot{a}_{63.5|\,i^*}\,v_{i*}^{0.5}\left(\frac{25}{25.5}\right) + \frac{r_{64}}{l_{63}}\,\ddot{a}_{64.5|\,i^*}\,v_{i^*}^{1.5}\left(\frac{25}{26.5}\right)\right. \\ &\qquad \left. + \frac{r_{65-}}{l_{63}}\,\ddot{a}_{65|\,i^*}\,v_{i^*}^{2}\left(\frac{25}{27}\right)\right) \\ &= 133\,957. \end{aligned}$$

For the normal cost, consider each of the terms in the actuarial liability calculation.

- The age 63.5 term is $25/25.5 = 98.04\%$ funded in the actuarial liability, and by the end of the year must be fully funded, so the normal cost must fund the additional half-year of accrual, at a cost of
$$\frac{0.5}{25.5}\,B(63,0)\,\frac{r_{63}}{l_{63}}\,\ddot{a}_{63.5|i^*}\,v_{i^*}^{0.5} = 281.35.$$
- The age 64.5 term is $25/26.5 = 94.34\%$ funded in the actuarial liability and must be $26/26.5$ funded by the end of the year. So the normal cost for age 64.5 retirements is the cost of an additional year of accrual,
$$\frac{1}{26.5}\,B(63,0)\,\frac{r_{64}}{l_{63}}\,\ddot{a}_{64.5|i^*}\,v_{i^*}^{1.5} = 472.19.$$
- Similarly, for retirement age 65 the normal cost must fund an extra year of accrual at a cost of
$$\frac{1}{27}\,B(63,0)\,\frac{r_{65-}}{l_{63}}\,\ddot{a}_{65|i^*}\,v_{i^*}^{2} = 4323.39.$$

Combining these costs of accrual, we have $C^B = \$5\,077$.

Alternatively, using the funding recursion (11.17) with $x = 63$, we have

$$C^B = v^{1/2}\,\frac{r_{63}}{l_{63}}\,{}_{1/2}V^{(B)} + v\,\frac{l_{64}}{l_{63}}\,{}_1V^{(0B)} - {}_0V^{(0B)},$$

where

$$_{1/2}V^{(B)} = B(63.5, 0.5)\ \ddot{a}_{63.5|i^*} = 155\,672$$

and

$$\begin{aligned} {}_1V^{(0B)} &= B(64, 1)\left(\frac{r_{64}}{l_{64}}\ \ddot{a}_{64.5|i^*}\ v_{i^*}^{0.5}\ \left(\frac{26}{26.5}\right) + \frac{r_{65}}{l_{64}}\ \ddot{a}_{65|i^*}\ v_{i^*}\ \left(\frac{26}{27}\right)\right) \\ &= 146\,907, \end{aligned}$$

which again yields $C^B = 5\,077$. □

We see, in this example, that the funding recursion can be expressed in the form below, given our assumptions about benefit premium increases due to age and inflation. For an active member aged x who entered at age xe, the contribution must fund an additional factor of $0.5/(n + 0.5)$ of the costs for exits at age $x + 0.5$, and a factor of $1/(y - xe)$ of the costs for all retirements at ages $y \geq x + 1$. That is,

$$\begin{aligned} C^B = B(x, 0)\Bigg(&\frac{r_x}{l_x}\ \ddot{a}_{x+0.5|i^*}\ v_{i^*}^{0.5}\ \left(\frac{0.5}{n + 0.5}\right) \\ &+ \sum_{\substack{y=x+1 \\ \text{mid-year} \\ \text{exits}}}^{xr-1} \frac{r_y}{l_x}\ \ddot{a}_{y+0.5|i^*}\ v_{i*}^{y+0.5-x}\ \left(\frac{1}{y + 0.5 - xe}\right) \\ &+ \sum_{\substack{y=x+1 \\ \text{exact age} \\ \text{exits}}}^{xr} \frac{r_{y^-}}{l_x}\ \ddot{a}_{y|i^*}\ v_{i^*}^{y-x}\ \left(\frac{1}{y - xe}\right)\Bigg). \end{aligned}$$

11.13 Notes and further reading

In this chapter we have introduced some of the language and concepts of pension plan funding and valuation. The presentation has been relatively simplified to bring out some of the major concepts, in particular, accruals funding principles. The difference between the normal contribution and the actual contribution paid represents a paying down of surplus or deficit. Such practical considerations are beyond the scope of this book – we are considering pensions here in the specific context of the application of life contingent mathematics. For more information on pension plan design and related issues, texts such as McGill *et al.* (2005) and Blake (2006) are useful. For information on post-retirement health benefits, see Yamamoto (2015).

11.14 Exercises

When an exercise uses the Standard Service Table summarized in Appendix D, the calculations are based on the exact model underlying the table. Using the rounded values presented in the table may result in slight differences from the numerical answers at the end of this chapter.

Shorter exercises

Exercise 11.1 A pension plan member, whose date of birth is 1 April 1961, had earnings of \$75 000 during 2018. Using the salary scale in Table 11.1, estimate the member's expected earnings during 2019.

Exercise 11.2 Assume the salary scale in Table 11.1 and a valuation date of 1 January.

(a) A plan member aged 35 at valuation received \$75 000 in salary in the year prior to the valuation date. Given that final average salary is defined as the average salary in the four years before retirement, calculate the member's expected final average salary on retirement at age 60.

(b) A plan member aged 55.5 at valuation was paid salary at a rate of \$100 000 per year at the valuation date. Salaries are increased on average half-way through each calendar year.

Calculate the expected average salary earned in the four years before retirement at age 64.5.

Exercise 11.3 Xiaoxiao, who is 35, is a new employee of a firm offering a career average earnings pension plan with an accrual rate of 2.5%.

Calculate Xiaoxiao's projected replacement ratio assuming that salaries increase by 4% at the start of each year, and that Xiaoxiao will remain in the firm until she retires at age 65.

Exercise 11.4 DHW offers a pension plan with a death-in-service benefit of \$20 000 per year of service payable immediately on death. Calculate the actuarial liability and the normal contribution for this benefit, for an employee who is aged 62 at the valuation date, and has 30 years service.

Basis: Standard Service Table, with interest at 5% per year. Assume deaths occur half-way through the year of age.

Exercise 11.5 An employer offers post-retirement health care benefits for employees to bridge the two-year period between the retirement age, which is 63, and the age of eligibility for government funded health benefits, which is 65.

Calculate the EPV of this benefit for an active employee currently aged 53, given the following information:

- $B(63,0) = \$5000$.
- $c = 1.025$, $j = 4\%$ and $i = 5\%$.
- There are no exits from employment before age 63 other than by death, and all surviving employees retire at age 63.
- Mortality follows the Standard Ultimate Life Table.

Exercise 11.6 LVB offers its employees a final salary pension plan, with an accrual rate of 1.5%. Salaries are averaged over the final two years of service. The normal retirement age is 65, and the pension benefit is payable as a monthly life annuity-due.

Quinn is an employee of LVB. At the valuation date, she is 35 and has 10 years of service in the plan. Her earnings in the previous two years were \$100 000 and \$104 000.

The valuation basis for the plan is as follows.

Exits before age 65:	None
Mortality after age 65:	Standard Ultimate Life Table
Salary increases:	4% per year, at the start of each year
Interest:	5% per year

Calculate the actuarial liability and the normal contribution for Quinn's age retirement benefits, using the traditional unit credit approach and Woolhouse's two-term formula to value the pension benefit.

Longer exercises

Exercise 11.7 A pension plan member is aged 55. One of the plan benefits is a death in service benefit payable on death before age 60.

(a) Calculate the probability that the employee dies in service before age 60.
(b) Assuming that the death in service benefit is \$200 000, and assuming that the death benefit is payable immediately on death, calculate the EPV at age 55 of the death in service benefit.
(c) Now assume that the death in service benefit is twice the annual salary rate at death. At age 55 the member's salary rate is \$85 000 per year. Assuming that deaths occur evenly throughout the year, estimate the EPV of the death in service benefit.

Basis: Standard Service Table
Interest of 6% per year effective
Salary scale follows Table 11.1

Exercise 11.8 (a) A new employee aged 25 joins a DC pension plan. Her starting salary is \$40 000 per year. Her salary is assumed to increase continuously at a rate of 7% per year for the first 20 years of her career and 4% per year for the following 15 years.

At retirement she is to receive a pension payable monthly in advance, guaranteed for 10 years. She plans to retire at age 60, and she wishes to achieve a replacement ratio of 70% through the pension plan. Using the assumptions below, calculate the level annual contribution rate c (% of salary) that would be required to achieve this replacement ratio.

Assumptions:

- Interest rate of 7% per year effective before retirement, 5% per year effective after retirement.
- Survival after retirement follows the Standard Ultimate Survival Model.

(b) Now assume that this contribution rate is paid, but her salary increases at a rate of 5% throughout her career, and interest is earned at 6% on her contributions, rather than 7%. In addition, at retirement, interest rates have fallen to 4.5% per year. Calculate the replacement ratio achieved using the same mortality assumptions.

Exercise 11.9 An employer currently offers a final salary pension plan with an accrual rate of 1.2%. Salaries are averaged over the final three years of employment. The employer is planning to convert to a career average earnings plan.

(a) Calculate the accrual rate under the career average earnings plan that would give the same projected replacement ratio as the current plan for a new employee aged 30. Assume all retirements occur at age 65, and that salaries increase at a rate of 5% per year at the end of each of the first 15 years of employment, and at 4% at the end of each subsequent year.
(b) Repeat part (a), but assume now that the new plan is a career average revalued earnings plan, where earnings are revalued from the end of each year of employment to the retirement date. Inflation is expected to be 2.25% per year.

Exercise 11.10 A pension plan member aged 61 has 35 years of past service at the funding valuation date. His salary in the year to the valuation date was \$50 000.

The death in service benefit is 10% of salary at death for each year of service. Calculate the value of the accrued death in service benefit and the normal contribution rate for the death in service benefit.

Basis:

- Standard Service Table
- An interest rate of 6% per year effective

- The salary scale follows Table 11.1; all salary increases take place on the valuation date
- Projected unit credit funding method

Exercise 11.11 A new company employee is 25 years old. Her company offers a choice of a defined benefit or a defined contribution pension plan. All contributions are paid by the employer, none by the employee.

Her starting salary is \$50 000 per year. Salaries are assumed to increase at a rate of 5% per year, increasing at each year end.

Under the defined benefit plan her final pension is based on the salary received in the year to retirement, using an accrual rate of 1.6% for each year of service. The normal retirement age is 65. The pension is payable monthly in advance for life.

Under the defined contribution plan, contributions are deposited into the member's account at a rate of 12% of salary per year. The total accumulated contribution is applied at the normal retirement age to purchase a monthly life annuity-due.

(a) Assuming the employee chooses the defined benefit plan and that she stays in employment through to age 65, calculate her projected annual rate of pension.
(b) Calculate the contribution, as a percentage of her starting salary, for the retirement pension benefit for this life, for the year of age 25–26, using the projected unit credit method. Assume no exits except mortality, and that the survival probability is ${}_{40}p_{25} = 0.8$. The valuation interest rate is 6% per year effective. The annuity factor $\ddot{a}_{65}^{(12)}$ is expected to be 11.0.
(c) Now assume that the employee joins the defined contribution plan. Contributions are expected to earn a rate of return of 8% per year. The annuity factor $\ddot{a}_{65}^{(12)}$ is expected to be 11.0. Assuming the employee stays in employment through to age 65, calculate (i) the projected fund at retirement, and (ii) her projected annual rate of pension, payable from age 65.
(d) Explain briefly why the employee might choose the defined benefit plan even though the projected pension is smaller.
(e) Explain briefly why the employer might prefer the defined contribution plan even though the contribution rate is higher.

Exercise 11.12 A plan sponsor wishes to calculate the actuarial reduction factor such that the EPV at early retirement of the reduced pension benefit is the same as the EPV of the accrued benefit payable at age 65, assuming no exits from mortality or any other decrement before age 65, and ignoring pay increases up to age 65. The pension is assumed to be payable monthly in advance for the member's lifetime, and is valued using Woolhouse's two-term formula.

Calculate k, expressed as a simple rate of reduction per year of early retirement, for a person who entered the plan at age 25 and who wishes to retire at age (i) 55 and (ii) 60, using the following further assumptions:

Post-retirement mortality:	Standard Ultimate Life Table
Interest rate:	5% per year effective

Exercise 11.13 Allison is a member of a pension plan. At the valuation date, 31 December 2018, she is exactly 45, and her salary in the year before valuation is \$100 000. Final average salary is defined as the average salary in the two years before exit, and salaries are revised annually on 1 July each year in line with the salary scale in Table 11.1.

The pension plan provides a benefit of 1.5% of final average salary for each year of service. The benefits are valued using the Standard Ultimate Survival Model, using an interest rate of 5% per year effective. Allison has 15 years service at the valuation date. She is contemplating three possible retirement dates.

- She could retire at 60.5, with an actuarial reduction applied to her pension of 0.5% per month (simple) up to age 62.
- She could retire at age 62 with no actuarial reduction.
- She could retire at age 65 with no actuarial reduction.

(a) Calculate the replacement ratio provided by the pension for each of the retirement dates.
(b) Calculate the EPV of Allison's retirement pension for each of the possible retirement dates, assuming mortality is the only decrement. The basic pension benefit is a single life annuity, payable monthly in advance.
(c) Now assume Allison leaves the company and withdraws from active membership of the pension plan immediately after the valuation. Her total salary in the two years before exit is \$186 000. She is entitled to a deferred pension of 1.5% of her final average earnings in the two years before withdrawal for each year of service, payable at age 62. There is no COLA for the benefit. Calculate the EPV of the withdrawal benefit using the valuation assumptions.

Exercise 11.14 Using the current unit credit method, calculate the actuarial liability and the normal contribution for the following pension plan.

Benefit:	\$300 per year pension for each year of service
Normal retirement age:	60
Survival model:	Standard Ultimate Survival Model

Interest:	6% per year effective
Pension:	Payable weekly, guaranteed for five years
Pre-retirement exits:	Mortality only

Active membership data at valuation

Age of employee	Service (years)	Number of employees
25	0	3
35	10	3
45	15	1
55	25	1

Inactive membership data at valuation

Age	Service	Number of employees
35	7	1 (deferred pensioner)
75	25	1 (pension in payment)

Exercise 11.15 A defined benefit pension plan offers an annual pension of 2% of the final year's salary for each year of service, payable monthly in advance. You are given the following information.

Interest rate:	4% per year effective
Salary scale:	Table 11.1, and all increases occur on 31 December
Retirement age:	65
Pre-retirement exits:	None
Post-retirement mortality:	Standard Ultimate Survival Model

Membership

Name	Age at entry	Age at 1 January 2019	Salary at 1 January 2018	Salary at 1 January 2019
Giles	30	35	38 000	40 000
Faith	30	60	47 000	50 000

(a) (i) Calculate the actuarial liability at 1 January 2019 using the projected unit credit method.

(ii) Calculate the normal contribution rate in 2019 separately for Giles and Faith, as a proportion of their 2019 salary, using the projected unit credit funding method.

(b) (i) Calculate the actuarial liability at 1 January 2019 using the traditional unit credit method.
(ii) Calculate the normal contribution rate in 2019 separately for Giles and Faith, as a proportion of their 2019 salary, using the traditional unit credit funding method.
(c) Comment on your answers.

Exercise 11.16 Oscar is aged 63 and his employer offers post-retirement supplementary health insurance to all retirees. The premium at the valuation date for all retirees aged 65 last birthday or younger is \$1500.

For older ages, the premium increases by 2.5% on the retiree's birthday.

For each age, the rate of premium inflation is assumed to be 4% per year. Inflation increases apply continuously. Other assumptions are:

- Standard Service Table
- 5% per year interest
- Post-retirement mortality follows the Standard Ultimate Survival Model
- Uniform distribution of deaths

You are given that $a^B_{66} = 27.2166$ and $a^B_{65.5} = 27.8386$.

(a) Calculate a^B_y for $y = 65$, 64.5 and 63.5.
(b) Hence calculate the actuarial value at the valuation date of Oscar's post-retirement health insurance benefits.

Excel-based exercises

Exercise 11.17 A pension plan has only one member, who is aged 35 at the valuation date, with five years past service. The plan benefit is \$350 per year pension for each year of service, payable monthly in advance. There is no actuarial reduction for early retirement.

Calculate the actuarial liability and the normal contribution for the age retirement benefit for the member. Use the service table from Appendix D. Post-retirement mortality follows the Standard Ultimate Survival Model. Assume 5% per year interest and use the unit credit funding method.

Exercise 11.18 An employer offers a career average pension scheme, with accrual rate 2.5%. A plan member is aged 35 with five years past service, and total past salary \$175 000. His salary in the year following valuation is projected to be \$40 000.

Using the service table from Appendix D, calculate the actuarial liability and the normal contribution for the age retirement benefit for the member. There is no actuarial reduction for early retirement. Post-retirement mortality follows the Standard Ultimate Survival Model. Assume 5% per year interest and use the unit credit funding method.

Exercise 11.19 A pension plan offers an annual retirement pension of 2% of the member's career average earnings for each year of service. The pension is payable as a monthly life annuity-due. Salaries are reviewed annually on the valuation date. There are no other benefits. The plan is valued annually to set the contribution rate for the benefits using the traditional unit credit approach.

David is 64 years old, and has 30 years' service at the valuation date. His total past earnings are \$2 500 500. His salary in the year following the valuation date will be \$180 000 if he works for the full year.

Rui joined the company on the valuation date, at age 30, with a starting salary of \$120 000.

(a) Calculate the actuarial liability for David.
(b) Calculate the normal contribution for David.
(c) Calculate the normal contribution for Rui.
(d) Calculate the revised normal contribution for Rui if the plan includes revaluation of earnings, from the end of the age-year to the date of retirement. Assume an inflation rate of 2.5% per year.

Basis:	Standard Service Table
	Interest of 5% per year effective
	Post-retirement mortality follows the Standard Ultimate Life Table

Exercise 11.20 A new member aged exactly 35, expecting to earn \$40 000 in the next 12 months, has just joined a pension plan. The plan provides a pension on age retirement of 1/60th of final pensionable salary for each year of service, with fractions counting proportionately, payable monthly in advance for life. There are no spousal benefits.

Final pensionable salary is defined as the average salary over the three years prior to retirement. Members contribute a percentage of salary, the rate depending on age. Those under age 50 contribute 4% and those aged 50 and over contribute 5%.

The employer contributes a constant multiple of members' contributions to meet exactly the expected cost of pension benefits. Calculate the multiple needed to meet this new member's age retirement benefits. Assume all contributions are paid exactly half-way through the year of age in which they are paid.

Basis:

Service Table:	Standard Service Table
Post-retirement mortality:	Standard Ultimate Survival Model
Interest:	4% per year effective

Answers to selected exercises

11.1 \$76 130
11.2 (a) \$185 265 (b) \$111 000
11.3 44.96%
11.4 \$6 773.31, \$186.51
11.5 \$8 88.52
11.6 \$46 345, \$6 674
11.7 (a) 0.01171 (b) \$2011.21 (c) \$1776.02
11.8 (a) 20.3% (b) 56.1%
11.9 (a) 2.12% (b) 1.56%
11.10 \$2 351.48, \$58.31
11.11 (a) \$214 552 (b) 9.18% (c) (i) \$3 052 123 (ii) \$277 466
11.12 4.85%, 5.80%
11.13 (a) 41.3%, 47.6%, 52.1% (b) \$383 700, \$406 686, \$372 321 (c) \$123 143
11.14 Total actuarial liability: \$197 691, Total normal contribution: \$8619
11.15 (a) (i) \$422 201 (ii) Giles: 22.5%, Faith: 25.1%
(b) (i) \$350 945 (ii) Giles: 11.1%, Faith: 66.3%
11.16 (a) $a^B_{65} = 28.4679$, $a^B_{64.5} = 28.4193$, $a^B_{63.5} = 29.0078$ (b) \$41 584
11.17 AL: \$2628.39, NC: \$525.68
11.18 AL: \$6570.98, NC: \$1501.94
11.19 (a) \$621 718 (b) \$42 550 (c) \$1 702 (d) \$3 699
11.20 2.15

12

Yield curves and non-diversifiable risk

12.1 Summary

In this chapter we consider the effect on annuity and insurance valuation of interest rates varying with the duration of investment, as summarized by a yield curve, and of uncertainty over future interest rates, which we will model using stochastic interest rates. We introduce the concepts of diversifiable and non-diversifiable risk and give conditions under which mortality risk can be considered to be diversifiable. In the final section we demonstrate the use of Monte Carlo methods to explore distributions of uncertain cash flows and loss random variables through simulation of both future lifetimes and future interest rates.

12.2 The yield curve

In practice, at any given time interest rates vary with the duration of the investment; that is, a sum invested for a period of, say, five years, would typically earn a different rate of interest than a sum invested for a period of 15 years or a sum invested for a period of six months.

Let $v(t)$ denote the current market price of a t-year **zero-coupon bond** with face value 1; that is $v(t)$ is the current market price of an investment which pays 1, with certainty, t years from now. Note that, at least in principle, there is no uncertainty over the value of $v(t)$ although this value can change at any time as a result of trading in the market. The t-year **spot rate of interest**, denoted y_t, is the yield per year on this zero-coupon bond, so that

$$v(t)(1+y_t)^t = 1 \iff v(t) = (1+y_t)^{-t}. \qquad (12.1)$$

The **term structure of interest rates** describes the relationship between the term of the investment and the interest rate on the investment, and it is expressed graphically by the **yield curve**, which is a plot of $\{y_t\}_{t>0}$ against t. Figures 12.1–12.4 show different yield curves, derived using government

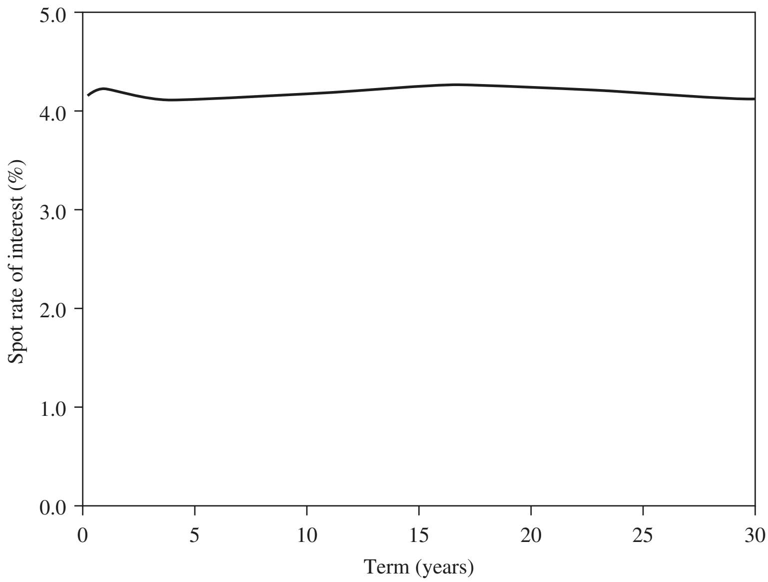


Figure 12.1 Canadian government bond yield curve (spot rates), May 2007.

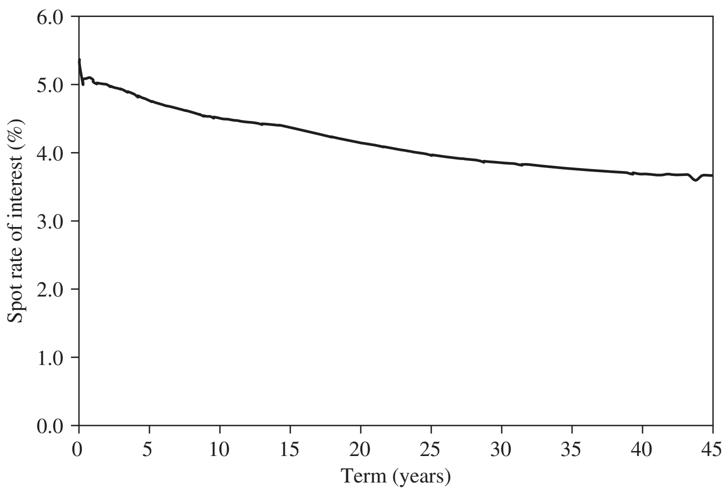


Figure 12.2 UK government bond yield curve (spot rates), November 2006.

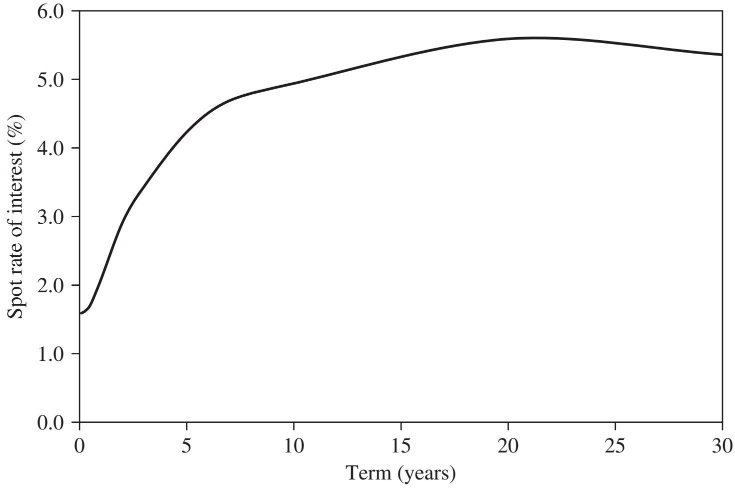


Figure 12.3 US government bond yield curve (spot rates), January 2002.

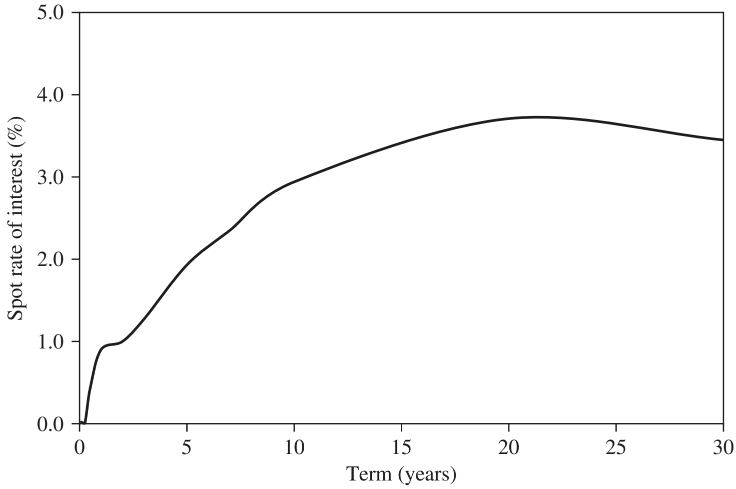


Figure 12.4 US government bond yield curve (spot rates), November 2008.

issued bonds from the UK, the USA and Canada at various dates. The UK issues longer term bonds than most other countries, so the UK yield curve is longer.

These figures illustrate some of the shapes a yield curve can have. Figure 12.1 shows a relatively flat curve, so that interest rates vary little with the term of the investment. Figure 12.2 shows a falling curve. Both of these shapes are relatively uncommon; the most common shape is that shown in Figures 12.3 and 12.4, a rising yield curve, flattening out after 10–15 years, with spot rates increasing at a decreasing rate.

Previously in this book we have assumed a flat term structure. This assumption has allowed us to use v^t or $e^{-\delta t}$ as discount functions for any term t, with v and δ as constants. When we relax this assumption, and allow interest rates to vary by term, the v^t discount function is no longer appropriate. Figure 12.3 shows that the rate of interest on a one-year US government bond in January 2002 was 1.6% per year and on a 20-year bond was 5.6%. The difference of 4% may have a significant effect on the valuation of an annuity or insurance benefit. The present value of a 20-year annuity-due of \$1 per year payable in advance, valued at 1.6%, is \$17.27; valued at 5.6% it is \$12.51.

The value of the annuity should be the amount required to be invested now to produce payments of 1 at the start of each of the next 20 years – this is how we have been implicitly valuing annuities when we discount at the rate of interest on assets. When we have a term structure this means we should discount each future payment using the spot interest rate appropriate to the term until that payment is due. This is a replication argument: the present value of any cash flow is the cost of purchasing a portfolio which exactly replicates the cash flow.

Since an investment now of amount $v(t)$ in a t-year zero-coupon bond will accumulate to 1 in t years, $v(t)$ can be interpreted as a discount function which generalizes v^t.

The price of the 20-year annuity-due with this discount function is $\sum_{t=0}^{19} v(t)$ which means that the price of the annuity-due is the cost of purchasing 20 zero-coupon bonds, each with \$1 face value, with maturity dates corresponding to the annuity payment dates. The spot rates underlying the yield curve in Figure 12.3 give a value of \$13.63 for the 20-year annuity-due, closer to, but significantly higher than the cost using the long-term rate of 5.6%.

At any given time the market will determine the price of zero-coupon bonds and this will determine the yield curve. These prices also determine **forward rates of interest** at that time. Let $f(t, t+k)$ denote the forward rate, contracted at time zero, effective from time t to $t+k$, expressed as an effective annual rate. This represents the interest rate contracted at time 0 earned on an investment made at t, maturing at $t+k$. To determine forward rates in terms of spot rates of interest, consider two different ways of investing 1 for $t+k$ years. Investing for the whole period, the $(t+k)$-year spot rate, y_{t+k}, gives the accumulation of this investment as $(1+y_{t+k})^{t+k}$. On the other hand, if the unit sum is invested first for t years at the t-year spot rate, then reinvested for k years at the k year forward rate starting at time t, the accumulation will be $(1+y_t)^t(1+f(t,t+k))^k$. Since there is no uncertainty involved in either of these schemes – note that y_{t+k}, y_t and $f(t, t+k)$ are all known now – the accumulation at $t+k$ under these two schemes must be the same. That is,

$$(1+f(t,t+k))^k = \frac{(1+y_{t+k})^{t+k}}{(1+y_t)^t} = \frac{v(t)}{v(t+k)}.$$

This is (implicitly) a *no arbitrage argument*, which, essentially, says in this situation that we should not be able to make money from nothing in risk free bonds by disinvesting and then reinvesting. The no arbitrage assumption is discussed further in Chapter 16.

12.3 Valuation of insurances and life annuities

The present value random variable for a life annuity-due with annual payments, issued to a life aged x, given a yield curve $\{y_t\}$, is

$$Y = \sum_{k=0}^{K_x} v(k) \tag{12.2}$$

where $v(k) = (1+y_k)^{-k}$. The EPV of the annuity, denoted $\ddot{a}(x)_y$, can be found using the payment-by-payment (or indicator function) approach, so that

$$\ddot{a}(x)_y = \sum_{k=0}^{\infty} {}_kp_x\, v(k). \tag{12.3}$$

Similarly, the present value random variable for a whole life insurance for (x), payable immediately on death, is

$$Z = v(T_x) \tag{12.4}$$

and the EPV is

$$\bar{A}(x)_y = \int_0^{\infty} v(t)\, {}_tp_x\, \mu_{x+t}\, dt. \tag{12.5}$$

Note that we have to depart from International Actuarial Notation here as it is defined in terms of interest rates that do not vary by term, though we retain the spirit of the notation.

By allowing for a non-flat yield curve we lose many of the relationships that we have developed for flat interest rates, such as the equation linking $\ddot{a}_x$ and A_x.

Example 12.1 You are given the following spot rates of interest per year.

y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}
0.032	0.035	0.038	0.041	0.043	0.045	0.046	0.047	0.048	0.048

Table 12.1 *Calculations for Example 12.1.*

t	$v(t)$	p_{80+t}	${}_tp_{80}$	t	$v(t)$	p_{80+t}	${}_tp_{80}$
0	1.0000	0.88845	1.00000	6	0.7679	0.83320	0.42456
1	0.9690	0.88061	0.88845	7	0.7299	0.82188	0.35374
2	0.9335	0.87226	0.78237	8	0.6925	0.80988	0.29073
3	0.8941	0.86337	0.68243	9	0.6558	0.79718	0.23546
4	0.8515	0.85391	0.58919	10	0.6257	0.78374	0.18770
5	0.8102	0.84387	0.50312				

(a) Calculate the discount function $v(t)$ for $t = 1, 2, \ldots, 10$.

(b) A survival model follows Makeham's law with $A = 0.0001$, $B = 0.00035$ and $c = 1.075$. Calculate the net level annual premium for a 10-year term insurance policy, with sum insured \$100 000 payable at the end of the year of death, issued to a life aged 80:
  (i) using the spot rates of interest in the table above, and
  (ii) using a level interest rate of 4.8% per year effective.

Solution 12.1 (a) Use equation (12.1) for the discount function values and (2.12) for the Makeham survival probabilities. Table 12.1 summarizes some of the calculations.

(b) (i) The EPV for the 10-year life annuity-due is

$$\ddot{a}(80 : \overline{10|}) = \sum_{k=0}^{9} v(k)\,{}_kp_{80} = 5.0507,$$

and the EPV of the term insurance benefit is

$$100\,000\,A\big(\overset{1}{80} : \overline{10|}\big) = \sum_{k=0}^{9} {}_kp_{80}(1 - p_{80+k})\ v(k+1) = 66\,739.$$

So the annual premium is \$13 213.72.

(ii) Assuming a 4.8% per year flat yield curve gives a premium of \$13 181.48.

□

In general, life insurance contracts are relatively long term. The influence of the yield curve on long-term contracts may not be very great since the yield curve tends to flatten out after around 15 years. It is common actuarial practice to use the long-term rate in traditional actuarial calculations, and in many cases, as in the example above, the answer will be close. However, using the long-term rate may overstate the interest income when the yield curve is rising, which is the most common shape. Overstating the interest results in a premium that is lower than the true premium. An insurer that systematically charges premiums less than the true price, even if each is only a little less, may face solvency problems in time. With a rising yield curve, if a level interest rate is assumed, it should be a little less than the long-term rate.

12.3.1 Replicating the cash flows of a traditional non-participating product

In this section we continue Example 12.1. Recall that the forward rate is contracted at the inception of the contract. This means that, assuming an equivalence principle premium, if we take the premium and any surplus cash flow brought forward each year and invest them at the forward rate, then there will be exactly enough to fund the sums insured, provided that the mortality and survival experience exactly follows the premium assumptions. That is, if the premiums and benefit cash flows are completely predictable, and are invested in the forward rates each year, the resulting cash flows exactly match the claims outgo, without risk.

This may be illustrated using the policy value recursion from Section 7.2.3, adjusted now to allow for the yield curve. Suppose we have a regular premium policy with term n years, issued to (x), with annual premium and benefit cash flows. Let P denote the premium and S denote the sum insured. We ignore expenses to keep things simple. As usual, the policy value at integer duration $t \geq 0$ is the EPV of the future loss random variable at time t, for a policy in force at that time. The recursive relationship between policy values can be written as

$$({}_tV + P)\,(1 + f(t, t+1)) = S\,q_{x+t} + {}_{t+1}V\,p_{x+t}.$$

The recursion shows that if we have N identical, independent policies at time t, and exactly $p_{x+t}\,N$ lives survive to time $t+1$, and exactly $q_{x+t}\,N$ lives do not, then the forward rate of interest is exactly enough to balance income and outgo, with no residual interest rate risk.

What this means is that if the cash flows are certain, and if the policy term is not so long that it extends beyond the scope of risk-free investments, then there is no need for the policy to involve interest rate uncertainty. At the inception of the contract, we can lock in forward rates that will exactly replicate the required cash flows.

This raises two interesting questions.

First, we know that mortality is uncertain, so that the mortality related cash flows are not certain. To what extent does this invalidate the replication argument? The answer is that, if the portfolio of life insurance policies is sufficiently large, and, crucially, if mortality can be treated as **diversifiable**, then it is reasonable to treat the life contingent cash flows as if they were certain. In Section 12.4.1 we discuss in detail what we mean by diversifiability, and under what conditions it might be a reasonable assumption for mortality.

The second question is, what risks are incurred by an insurer if it chooses not to replicate, or is unable to replicate for lack of appropriate risk-free investments? If the insurer does not replicate the cash flows, then interest rate risk is introduced, and must be modelled and managed. Interest rate risk is inherently non-diversifiable, as we shall discuss in Section 12.4.2.

12.4 Diversifiable and non-diversifiable risk

Consider a portfolio consisting of N life insurance policies. We can model as a random variable, X_i, $i = 1, \ldots, N$, many quantities of interest for the ith policy in this portfolio. For example, X_i could take the value 1 if the policyholder is still alive, say, 10 years after the policy was issued and the value zero otherwise. In this case, $\sum_{i=1}^{N} X_i$ represents the number of survivors after 10 years. Alternatively, X_i could represent the present value of the loss on the ith policy so that $\sum_{i=1}^{N} X_i$ represents the present value of the loss on the whole portfolio.

Suppose that the X_is are identically distributed with common mean μ and standard deviation σ. Let ρ denote the correlation coefficient for any pair X_i and X_j ($i \neq j$), and let $\bar{X} = \sum_{i=1}^{N} X_i/N$. Then

$$\mathrm{E}\left[\sum_{i=1}^{N} X_i\right] = N\mu \Rightarrow \mathrm{E}[\bar{X}] = \mu$$

and

$$\mathrm{V}\left[\sum_{i=1}^{N} X_i\right] = N\sigma^2 + N(N-1)\rho\,\sigma^2$$

$$\Rightarrow \mathrm{V}[\bar{X}] = \frac{N\sigma^2 + N(N-1)\rho\sigma^2}{N^2} = \frac{\sigma^2}{N} + \frac{N-1}{N}\rho\,\sigma^2.$$

Suppose now that the X_is are independent, so that ρ is zero. Then $\mathrm{V}[\bar{X}] = \sigma^2/N$ and the central limit theorem (which is described in Appendix A) tells us that, provided N is large, $\bar{X}$ will be approximately normally distributed, with mean μ and variance σ^2/N.

For any $k > 0$, the probability that $\bar{X}$ deviates from its mean, μ, by at least k is

$$\Pr\left[\,\left|\bar{X} - \mu\right| \geq k\right] = \Pr\left[\left|\frac{\bar{X} - \mu}{\sigma/\sqrt{N}}\right| \geq \frac{k\sqrt{N}}{\sigma}\right].$$

If we now let $N \to \infty$, then we can assume from the central limit theorem that $Z = (\bar{X} - \mu)/(\sigma/\sqrt{N})$ has a standard normal (i.e. $N(0, 1)$) distribution, and the probability can be written as

$$\lim_{N\to\infty} \Pr\left[|Z| \geq \frac{k\sqrt{N}}{\sigma} \right] = \lim_{N\to\infty} 2\,\Phi\left(-\frac{k\sqrt{N}}{\sigma} \right) = 0,$$

where $\Phi(z) = \Pr[Z \leq z]$.

So, if $\mathrm{V}\left[\sum_{i=1}^{N} X_i\right]$ is linear in N, then $\mathrm{V}[\bar{X}]$ is of order N^{-1}, and as N increases, the variation of the mean of the X_is from the expected value will tend to zero, which means that in this case we can reduce the risk measured by X_i, relative to its mean value, by increasing the size of the portfolio.

However, this result relies on the fact that we have assumed that the X_is are independent; it is not generally true if ρ is not equal to zero; in that case $\mathrm{V}\left[\sum_{i=1}^{N} X_i\right]$ is of order N^2, which means that increasing the number of policies increases the risk, relative to the mean value.

So, we say that the risk within our portfolio, as measured by the random variable X_i, is **diversifiable** if the following condition holds

$$\lim_{N\to\infty} \frac{\sqrt{\mathrm{V}\left[\sum_{i=1}^{N} X_i\right]}}{N} = 0,$$

or, equivalently,

$$\lim_{N\to\infty} \mathrm{V}[\bar{X}] = 0.$$

A risk is **non-diversifiable** if this condition does not hold. In simple terms, a risk is diversifiable if we can eliminate it (relative to its expectation) by increasing the number of policies in the portfolio. An important aspect of financial risk management is to identify those risks which can be regarded as diversifiable and those which cannot. Diversifiable risks are generally easier to deal with than those which are not.

12.4.1 Diversifiable mortality risk

In Section 12.2 we employed the no arbitrage principle to argue that the value of a deterministic payment stream should be the same as the price of the zero-coupon bonds that replicate that payment stream. In Section 12.3.1 we explored the replication idea further. To do this we need to assume that the mortality risk associated with a portfolio is diversifiable and we discuss conditions for this to be a reasonable assumption.

Consider a group of N lives all now aged x who have just purchased identical insurance or annuity policies. We will make the following two assumptions throughout the remainder of this chapter, except where otherwise stated.

(i) The N lives are independent with respect to their future mortality.
(ii) The survival model for each of the N lives is known.

We also assume, for convenience, that each of the N lives has the same survival model.

The cash flow at any future time t for this group of policyholders will depend on how many are still alive at time t and on the times of death for those not still alive. These quantities are uncertain. However, with the two assumptions above the mortality risk is diversifiable. This means that, provided N is large, the variability of, say, the number of survivors at any time, relative to the expected number, is small so that we can regard mortality, and hence the cash flows for the portfolio, as deterministic. This is demonstrated in the following example.

Example 12.2 For $0 \le t \le t+s$, let $N_{t,s}$ denote the number of deaths between ages $x+t$ and $x+t+s$ from N lives aged x. Show that

$$\lim_{N\to\infty} \frac{\sqrt{\mathrm{V}[N_{t,s}]}}{N} = 0.$$

Solution 12.2 The random variable $N_{t,s}$ has a binomial distribution with parameters N and ${}_tp_x\,(1 - {}_sp_{x+t})$. Hence

$$\begin{aligned}
\mathrm{V}[N_{t,s}] &= N\,{}_tp_x\,(1 - {}_sp_{x+t})\,(1 - {}_tp_x\,(1 - {}_sp_{x+t})) \\
\Rightarrow \frac{\sqrt{\mathrm{V}[N_{t,s}]}}{N} &= \sqrt{\frac{{}_tp_x(1 - {}_sp_{x+t})(1 - {}_tp_x(1 - {}_sp_{x+t}))}{N}} \\
\Rightarrow \lim_{N\to\infty} \frac{\sqrt{\mathrm{V}[N_{t,s}]}}{N} &= 0.
\end{aligned}$$

□

In practice most insurers sell so many contracts over all their life insurance or annuity portfolios that mortality risk can be treated in many situations as fully diversified away. There are exceptions; for example, for very old age mortality, where the number of policyholders tends to be small, or where an insurance has a very high sum at risk, in which case the outcome of that particular contract may have a significant effect on the portfolio as a whole, or where the survival model for the policyholders cannot be predicted with confidence.

If mortality risk can be treated as fully diversified then we can assume that the mortality experience is deterministic – that is, we may assume that the number of claims each year is equal to the expected number. In the following section we use this deterministic assumption for mortality to look at the replication of the term insurance cash flows in Example 12.1 above.

12.4.2 Non-diversifiable risk

In practice, many insurers do not replicate with forward rates or zero-coupon bonds either because they choose not to or because there are practical difficulties in trying to do so. By locking into forward rates at the start of a contract, the insurer can remove (much of) the investment risk, as noted in Section 12.3.1. However, while this removes the risk of losses, it also removes the possibility of profits. Also there may be practical constraints. For example, in some countries it may not be possible to find risk-free investments for terms longer than around 20 years, which is often not long enough. A whole life insurance contract issued to a life aged 40 may not expire for 50 years. The rate of interest that would be appropriate for an investment to be made over 20 years ahead could be very difficult to predict.

If an insurer does not lock into the forward rates at inception, there is a risk that interest rates will move, resulting in premiums that are either too low or too high. The risk that interest rates are lower than those expected in the premium calculation is an example of non-diversifiable risk. Suppose an insurer has a large portfolio of whole life insurance policies issued to lives aged 40, with level premiums payable throughout the term of the contract, and that mortality risk can be considered diversified away. The insurer decides to invest all premiums in 10-year bonds, reinvesting when the bonds mature. The price of 10-year bonds at each of the future premium dates is unknown now. If the insurer determines the premium assuming a fixed interest rate of 6% per year, and the actual interest rate earned is 5% per year, then the portfolio will make a substantial loss, and in fact each individual contract is expected to make a loss. Writing more contracts will only increase the loss, because each policy experiences the same interest rates. The key point here is that the policies are not independent of each other with respect to the interest rate risk.

Previous chapters in this book have focused on the mortality risk in insurance, which, under the conditions discussed in Section 12.4.1 can be considered to be diversifiable. However, non-diversifiable risk is, arguably, even more important. Most life insurance company failures occur because of problems with non-diversifiable risk related to assets. Note also that not all mortality risk is diversifiable. In Example 12.4 below, we look at a situation where the mortality risk is not fully diversifiable. First, in Example 12.3 we look at an example of non-diversifiable interest rate risk.

Example 12.3 An insurer issues a whole life insurance policy to (40), with level premiums payable continuously throughout the term of the policy, and with sum insured \$50 000 payable immediately on death. The insurer assumes that an appropriate survival model is given by Makeham's law with parameters $A = 0.0001$, $B = 0.00035$ and $c = 1.075$.

(a) Suppose the insurer prices the policy assuming an interest rate of 5% per year effective. Show that the annual premium rate is $P = \$1\,010.36$.

(b) Now suppose that the effective annual interest rate, $\mathbf{i}$, is modelled stochastically, with the following distribution:

$$\mathbf{i} = \begin{cases} 4\% \text{ with probability } 0.25, \\ 5\% \text{ with probability } 0.5, \\ 6\% \text{ with probability } 0.25. \end{cases}$$

Calculate the expected value and the standard deviation of the present value of the future loss on the contract. Assume that the future lifetime is independent of the interest rate.

Solution 12.3 (a) At 5% we have $\bar{a}_{40} = 14.49329$ and $\bar{A}_{40} = 0.29287$ giving a premium of

$$P = 50\,000\,\frac{\bar{A}_{40}}{\bar{a}_{40}} = \$1\,010.36.$$

(b) Let $S = 50\,000$, $P = 1\,010.36$ and $T = T_{40}$. The present value of the future loss on the policy, L_0, is given by

$$L_0 = S\,v_{\mathbf{i}}^{T} - P\,\bar{a}_{\overline{T|}\mathbf{i}}.$$

To calculate the moments of L_0, we condition on the value of $\mathbf{i}$ and then use iterated expectation (see Appendix A for a review of conditional expectation). As

$$L_0|\mathbf{i} = S\,v_{\mathbf{i}}^{T} - P\,\bar{a}_{\overline{T|}\mathbf{i}}, \tag{12.6}$$

$$\begin{aligned} \mathrm{E}[L_0|\mathbf{i}] &= (S\bar{A}_{40} - P\,\bar{a}_{40})|\mathbf{i} \\ &= \begin{cases} 1\,587.43 & \text{with probability } 0.25\ \ (\mathbf{i} = 4\%), \\ 0 & \text{with probability } 0.50\ \ (\mathbf{i} = 5\%), \\ -1\,071.49 & \text{with probability } 0.25\ \ (\mathbf{i} = 6\%), \end{cases} \end{aligned} \tag{12.7}$$

and so

$$\begin{aligned} \mathrm{E}[L_0] = \mathrm{E}\,[\mathrm{E}[L_0|\mathbf{i}]] &= 0.25\,(1\,587.43) + 0.5\,(0) + 0.25\,(-1\,071.49) \\ &= \$128.99. \end{aligned} \tag{12.8}$$

For the standard deviation, we use

$$\mathrm{V}[L_0] = \mathrm{E}[\mathrm{V}[L_0|\mathbf{i}]] + \mathrm{V}[\mathrm{E}[L_0|\mathbf{i}]]. \tag{12.9}$$

We can interpret the first term as the risk due to uncertainty over the future lifetime and the second term as the risk due to the uncertain interest rate.

Now

$$L_0|\mathbf{i} = S\,v_{\mathbf{i}}^T - P\,\bar{a}_{\overline{T}|\mathbf{i}} = \left(S + \frac{P}{\delta_{\mathbf{i}}}\right) v_{\mathbf{i}}^T - \frac{P}{\delta_{\mathbf{i}}}$$

so

$$\begin{aligned}\mathrm{V}[L_0|\mathbf{i}] &= \left(S + \frac{P}{\delta_{\mathbf{i}}}\right)^2 \left({}^2\bar{A}_{40} - \bar{A}_{40}^2\right)_{\mathbf{i}} \\ &= \begin{cases} 14\,675^2 & \text{with probability } 0.25 \quad (\mathbf{i} = 4\%), \\ 14\,014^2 & \text{with probability } 0.5 \quad\;\, (\mathbf{i} = 5\%), \\ 13\,316^2 & \text{with probability } 0.25 \quad (\mathbf{i} = 6\%). \end{cases}\end{aligned}$$

Hence

$$\mathrm{E}[\mathrm{V}[L_0|\mathbf{i}]] = \$196\,364\,762.$$

Also, from equation (12.7),

$$\begin{aligned} V[\mathrm{E}[L_0|\mathbf{i}]] &= \left((1\,587.43^2)\,0.25 + (0^2)\,0.5 + (-1\,071.49^2)\,0.25\right) - 128.99^2 \\ &= 900\,371 \\ &= \$948.88^2. \end{aligned}$$

So

$$\mathrm{V}[L_0] = 196\,364\,762 + 900\,371 = 197\,265\,133 = \$14\,045^2.$$

□

Comments

This example illustrates some important points.

(1) The fixed interest assumption, 5% in this example, is what is often called the '**best estimate**' assumption – it is the expected value, as well as the most likely value, of the future interest rate. It is tempting to calculate the premium using the best estimate assumption, but this example illustrates that doing so may lead to systematic losses. In this example, using a 5% per year interest assumption to price the policy leads to an expected loss of $128.99 on every policy issued.

(2) Breaking the variance down into two terms separates the diversifiable risk from the non-diversifiable risk. Consider a portfolio of, say, N contracts. Let $L_{0,j}$ denote the present value of the loss at inception on the jth policy and let

$$L = \sum_{j=1}^{N} L_{0,j}$$

so that L denotes the total future loss random variable.

Following formula (12.9), and noting that, given our assumptions at the start of this section, the random variables $\{L_{0,j}|\,\mathbf{i}\}_{j=1}^{N}$ are independent and identically distributed, we can write

$$\begin{aligned}
\mathrm{V}[L] &= \mathrm{E}[\mathrm{V}[L|\mathbf{i}]] + \mathrm{V}[\mathrm{E}[L|\mathbf{i}]] \\
&= \mathrm{E}\left[\mathrm{V}\left[\sum_{j=1}^{N} L_{0,j}|\mathbf{i}\right]\right] + \mathrm{V}\left[\mathrm{E}\left[\sum_{j=1}^{N} L_{0,j}|\mathbf{i}\right]\right] \\
&= \mathrm{E}[N\,\mathrm{V}[L_0|\mathbf{i}]] + \mathrm{V}[N\,\mathrm{E}[L_0|\mathbf{i}]] \\
&= 196\,364\,762\,N + 900\,371\,N^2.
\end{aligned}$$

Now consider separately each component of the variance of L. The first term represents diversifiable risk since it is a multiple of N and the second term represents non-diversifiable risk since it is a multiple of N^2. We can see that, for an individual policy ($N = 1$), the future lifetime uncertainty is very much more influential than the interest rate uncertainty, as the first term is much greater than the second term. But, for a large portfolio, the contribution of the interest uncertainty to the total standard deviation is far more important than the future lifetime uncertainty.

The conclusion above, that for large portfolios, interest rate uncertainty is more important than mortality uncertainty relies on the assumption that the future survival model is known and that the separate lives are independent with respect to mortality. The following example shows that if these conditions do not hold, mortality risk can be non-diversifiable.

Example 12.4 A portfolio consists of N identical one-year term insurance policies issued simultaneously. Each policy was issued to a life aged 70, has a sum insured of \$50 000 payable at the end of the year of death and was purchased with a single premium of \$1300. The insurer uses an effective interest rate of 5% for all calculations but is unsure about the mortality of this group of policyholders over the term of the policies. The probability of dying within the year, regarded as a random variable $\mathbf{q_{70}}$, is assumed to have the following distribution:

$$\mathbf{q_{70}} = \begin{cases} 0.022 & \text{with probability } 0.25, \\ 0.025 & \text{with probability } 0.5, \\ 0.028 & \text{with probability } 0.25. \end{cases}$$

The value of $\mathbf{q_{70}}$ is the same for all policies in the portfolio and, given this value, the policies are independent with respect to mortality.

(a) Let $D(N)$ denote the number of deaths during the one year term. Show that
$$\lim_{N\to\infty}\frac{\sqrt{\mathrm{V}[D(N)]}}{N}\neq 0.$$

(b) Let $L(N)$ denote the present value of the loss from the whole portfolio. Show that
$$\lim_{N\to\infty}\frac{\sqrt{\mathrm{V}[L(N)]}}{N}\neq 0.$$

Solution 12.4 (a) We have
$$\mathrm{V}[D(N)] = \mathrm{V}[\mathrm{E}[D(N)|\mathbf{q_{70}}]] + \mathrm{E}[\mathrm{V}[D(N)|\mathbf{q_{70}}]].$$

Now
$$\begin{aligned}\mathrm{V}[\mathrm{E}[D(N)|\mathbf{q_{70}}]] &= 0.25((0.022-0.025)N)^2 + 0 + 0.25((0.028-0.025)N)^2\\ &= 4.5\times 10^{-6}N^2\end{aligned}$$

and
$$\begin{aligned}\mathrm{E}[\mathrm{V}[D(N)|\mathbf{q_{70}}]] &= 0.25\times 0.022(1-0.022)N\\ &\quad + 0.5\times 0.025(1-0.025)N\\ &\quad + 0.25\times 0.028(1-0.028)N\\ &= 0.0243705N.\end{aligned}$$

Hence
$$\mathrm{V}[D(N)] = 4.5\times 10^{-6}N^2 + 0.0243705N$$

and so
$$\lim_{N\to\infty}\frac{\sqrt{\mathrm{V}[D(N)]}}{N} = 0.002121.$$

(b) The arguments are as in part (a). We have
$$\mathrm{V}[L(N)] = \mathrm{E}[\mathrm{V}[L(N)|\mathbf{q_{70}}]] + \mathrm{V}[\mathrm{E}[L(N)|\mathbf{q_{70}}]].$$

As
$$L(N) = 50\,000vD(N) - 1\,300N,$$

we have
$$\begin{aligned}\mathrm{V}[L(N)|\mathbf{q_{70}}] &= (50\,000v)^2\mathrm{V}[D(N)|\mathbf{q_{70}}]\\ &= (50\,000v)^2N\mathbf{q_{70}}(1-\mathbf{q_{70}})\end{aligned}$$

and

$$\mathrm{E}[L(N)|\mathbf{q_{70}}] = 50\,000vN\mathbf{q_{70}} - 1\,300N.$$

Thus

$$\begin{aligned}\mathrm{E}[\mathrm{V}[L(N)|\mathbf{q_{70}}]] &= (50\,000v)^2N\left(\mathrm{E}[\mathbf{q_{70}}] - \mathrm{E}[\mathbf{q_{70}}^2]\right)\\ &= (50\,000v)^2N(0.025 - 0.0006295)\end{aligned}$$

and

$$\begin{aligned}\mathrm{V}[\mathrm{E}[L(N)|\mathbf{q_{70}}]] &= (50\,000v)^2N^2\mathrm{V}[\mathbf{q_{70}}]\\ &= (50\,000v)^2N^2 \times 4.5 \times 10^{-6}.\end{aligned}$$

Hence

$$\lim_{N\to\infty}\frac{\sqrt{\mathrm{V}[L(N)]}}{N} = 50\,000v\sqrt{\mathrm{V}[\mathbf{q_{70}}]} = 101.02.$$

□

12.5 Monte Carlo simulation

Suppose we wish to explore a more complex example of interest rate variation than in Example 12.3. If the problem is too complicated, for example if we want to consider both lifetime variation and the interest rate uncertainty, then the numerical methods used in previous chapters may be too unwieldy. An alternative is Monte Carlo, or stochastic, simulation. Using Monte Carlo techniques allows us to explore the distributions of present values for highly complicated problems, by generating a random sample from the distribution. If the sample is large enough, we can get good estimates of the moments of the distribution, and, even more interesting, the full picture of a loss distribution. Appendix C gives a brief review of Monte Carlo simulation.

In this section we demonstrate the use of Monte Carlo methods to simulate future lifetimes and future rates of interest, using a series of examples based on the following deferred annuity policy issued to a life aged 50.

- Policy terms:
  - An annuity of \$10 000 per year is payable continuously from age 65 contingent on the survival of the policyholder.
  - Level premiums of amount $P =$ \$4 447 per year are payable continuously throughout the period of deferment.
  - If the policyholder dies during the deferred period, a death benefit equal to the total premiums paid (without interest) is due immediately on death.

- Basis for all calculations:
  - The survival model follows Gompertz' law with parameters $B = 0.0004$ and $c = 1.07$.
  - The force of interest during deferment is $\delta = 5\%$ per year.
  - The force of interest applying at age 65 is denoted $\mathbf{r}$.

In the next three examples we assume that $\mathbf{r}$ is fixed and known. In the final example we assume that $\mathbf{r}$ has a fixed but unknown value.

Example 12.5 Assume the force of interest from age 65 is 6% per year, so that $\mathbf{r} = 0.06$.

(a) Calculate the EPV of the loss on the contract.
(b) Calculate the probability that the present value of the loss on the policy will be positive.

Solution 12.5 (a) The EPV of the loss on this contract is

$$10\,000\,{}_{15}E_{50}\,\bar{a}^{*}_{65} + P\,(\bar{I}\bar{A})^{\;\;1}_{50:\,\overline{15}|} - P\,\bar{a}_{50:\,\overline{15}|},$$

where * denotes calculation using a force of interest 6% per year and all other functions are calculated using a force of interest 5% per year. This gives the EPV of the loss as

$$\begin{aligned}&10\,000 \times 0.34773 \times 8.51058 + 4\,447 \times 1.32405 - 4\,447 \times 9.49338\\ &\quad = -\$6735.38.\end{aligned}$$

(b) The present value of the loss, L, can be written in terms of the expected future lifetime, T_{50}, as follows

$$L = \begin{cases} P\,T_{50}\,v^{T_{50}} - P\,\bar{a}_{\overline{T_{50}}|} & \text{if } T_{50} \le 15,\\ 10\,000\,\bar{a}^{*}_{\overline{T_{50}-15}|}\,v^{15} - P\,\bar{a}_{\overline{15}|} & \text{if } T_{50} > 15.\end{cases}$$

By looking at the relationship between L and T_{50} we can see that the policy generates a profit if the life dies in the deferred period, or in the early years of the annuity payment period, and that

$$\begin{aligned}\Pr[L > 0] &= \Pr\left[10\,000\,e^{-15\delta}\bar{a}_{\overline{T_{50}-15}|6\%} - P\bar{a}_{\overline{15}|5\%} > 0\right]\\ &= \Pr\left[T_{50} > 15 - \frac{1}{0.06}\log\left(1 - \frac{P}{10^4}e^{15(0.05)}\bar{a}_{\overline{15}|5\%}(0.06)\right)\right]\\ &= \Pr\left[T_{50} > 30.109\right] = {}_{30.109}p_{50} = 0.3131.\end{aligned}$$

□

Example 12.6 Use the three $U(0,1)$ random variates below to simulate values for T_{50} and hence values for the present value of future loss, L_0, for the deferred annuity contract. Assume that the force of interest from age 65 is 6% per year:

$$u_1 = 0.16025, \qquad u_2 = 0.51720, \qquad u_3 = 0.99855.$$

Solution 12.6 Let F_T be the distribution function of T_{50}. Each simulated u_j generates a simulated future lifetime t_j through the inverse transform method, where

$$u_j = F_T(t_j).$$

See Appendix C. Hence

$$\begin{aligned} u &= F_T(t) \\ &= 1 - e^{-(B/\log c)c^{50}(c^t - 1)} \\ \Rightarrow t &= F_T^{-1}(u) \\ &= \frac{1}{\log c}\left(\log\left(1 - \frac{(\log c)(\log(1-u))}{B\,c^{50}}\right)\right). \end{aligned} \tag{12.10}$$

So

$$\begin{aligned} t_1 &= F_T^{-1}(0.16025) = 10.266, \\ t_2 &= F_T^{-1}(0.5172) = 24.314, \\ t_3 &= F_T^{-1}(0.9985) = 53.969. \end{aligned}$$

These simulated lifetimes can be checked by noting in each case that ${}_{t_j}q_{50} = u_j$.

We can convert the sample lifetimes to the corresponding sample of the present value of future loss random variable, L_0, as follows. If (50) dies after exactly 10.266 years, then death occurs during the deferred period, the death benefit is $10.266P$, the present value of the premiums paid is $P\bar{a}_{\overline{10.266}|}$, and so the present value of the future loss is

$$L_0 = 10.266\,P\,e^{-10.266\,\delta} - P\bar{a}_{\overline{10.266}|_\delta} = -\$8383.80.$$

Similarly, the other two simulated future lifetimes give the following losses

$$\begin{aligned} L_0 &= 10\,000e^{-15\delta}\bar{a}_{\overline{9.314}|_{r=6\%}} - P\bar{a}_{\overline{15}|_\delta} = -\$13\,223.09, \\ L_0 &= 10\,000e^{-15\delta}\bar{a}_{\overline{38.969}|_{r=6\%}} - P\bar{a}_{\overline{15}|_\delta} = \$24\,202.36. \end{aligned}$$

The first two simulations generate a profit, and the third generates a loss. □

Example 12.7 Repeat Example 12.6, generating 5 000 values of the present value of future loss random variable. Use the simulation output to:

(a) Estimate the expected value and the standard deviation of the present value of the future loss from a single policy.
(b) Calculate a 95% confidence interval for the expected value of the present value of the loss.

(c) Estimate the probability that the contract generates a loss.
(d) Calculate a 95% confidence interval for the probability that the contract generates a loss.

Solution 12.7 Use an appropriate random number generator to produce a sequence of 5 000 $U(0,1)$ random numbers, $\{u_j\}$. Use equation (12.10) to generate corresponding values of the future lifetime, $\{t_j\}$, and the present value of the future loss for a life with future lifetime t_j, say $\{L_{0,j}\}$, as in Example 12.6.

The result is a sample of 5 000 independent values of the future loss random variable. Let $\bar{l}$ and s_l represent the mean and standard deviation of the sample.

(a) The precise answers will depend on the random number generator (and seed value) used. Our calculations gave

$$\bar{l} = -\$6\,592.74; \qquad s_l = \$15\,733.98.$$

(b) Let μ and σ denote the (true) mean and standard deviation of the present value of the future loss on a single policy. Using the central limit theorem, we can write

$$\frac{1}{5\,000}\sum_{j=1}^{5\,000} L_{0,j} \sim N(\mu, \sigma^2/5\,000).$$

Hence

$$\Pr\left[\mu - 1.96\,\frac{\sigma}{\sqrt{5\,000}} \le \frac{1}{5\,000}\sum_{j=1}^{5\,000} L_{0,j} \le \mu + 1.96\,\frac{\sigma}{\sqrt{5\,000}}\right] = 0.95.$$

Since $\bar{l}$ and s_l are estimates of μ and σ, respectively, a 95% confidence interval for the mean loss is

$$\left(\bar{l} - 1.96\,\frac{s_l}{\sqrt{5\,000}},\ \ \bar{l} + 1.96\,\frac{s_l}{\sqrt{5\,000}}\right).$$

Using the values of $\bar{l}$ and s_l from part (a) gives $(-7028.86,\ -6156.61)$ as a 95% confidence interval for μ.

(c) Let L^- denote the number of simulations which produce a loss, that is, the number for which $L_{0,i}$ is positive. Let p denote the (true) probability that the present value of the loss on a single policy is positive. Then

$$L^- \sim B(5\,000,\ p)$$

and our estimate of p, denoted $\hat{p}$, is given by

$$\hat{p} = \frac{l^-}{5\,000},$$

where l^- is the simulated realization of L^-, that is, the number of losses which are positive out of the full set of 5 000 simulated losses. Using a normal approximation, we have

$$\frac{L^-}{5\,000} \sim N\left(p,\ \frac{p(1-p)}{5\,000}\right)$$

and so an approximate 95% confidence interval for p is

$$\left(\hat{p} - 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{5\,000}},\ \ \hat{p} + 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{5\,000}}\right)$$

where we have replaced p by its estimate $\hat{p}$. Our calculations gave a total of 1563 simulations with a positive value for the present value of the future loss. Hence

$$\hat{p} = 0.3126$$

and an approximate 95% confidence interval for this probability is

$$(0.2998,\ 0.3254).$$

Different sets of random numbers would result in different values for each of these quantities. □

In fact it was not necessary to use simulation to calculate μ or p in this example. As we have seen in Example 12.5, the values of μ and p can be calculated as $-\$6735.38$ and 0.3131, respectively. The 95% confidence intervals calculated in Example 12.7 parts (b) and (d) comfortably span these true values. We used simulation in this example to illustrate the method and to show how accurate we can be with 5 000 simulations.

An advantage of Monte Carlo simulation is that we can easily adapt the simulation to model the effect of a random force of interest from age 65, which would be less tractable analytically. The next example demonstrates this in the case where the force of interest from age 65 is fixed but unknown.

Example 12.8 Repeat Example 12.7, but now assuming that $\mathbf{r}$ is a random variable with a $N(0.06, 0.015^2)$ distribution. Assume the random variables T_{50} and $\mathbf{r}$ are independent.

Solution 12.8 For each of the 5 000 simulations, generate both a value for T_{50}, as in the previous example, and also, independently, a value of $\mathbf{r}$ from the $N(0.06, 0.015^2)$ distribution. Let t_j and r_j denoted the simulated values of T_{50} and $\mathbf{r}$, respectively, for the jth simulation. The simulated value of the present value of the loss for this simulation, $L_{0,j}$, is

$$L_{0,j} = \begin{cases} P\, t_j\, v^{t_j} - P\, \bar{a}_{\overline{t_j}|} & \text{if } t_j \leq 15, \\ 10\,000\, \bar{a}^{*}_{\overline{t_j-15}|}\, v^{15} - P\, \bar{a}_{\overline{15}|} & \text{if } t_j > 15, \end{cases}$$

where * now denotes calculation at the simulated force of interest r_j. The remaining steps in the solution are as in Example 12.7.

Our simulation gave the following results:

$$\bar{l} = -\$6220.5; \quad s_l = \$16\,903.1; \quad L^{-} = 1\,502.$$

Hence, an approximate 95% confidence interval for the mean loss is

$$(-6\,689, \ -5\,752).$$

An estimated probability that a policy generates a loss is

$$\hat{p} = 0.3004,$$

with an approximate 95% confidence interval for this probability of

$$(0.2877, \ 0.3131).$$

Note that allowing for the future interest variability has reduced the expected profit and increased the standard deviation. The probability of loss is not significantly different from the fixed interest case. □

12.6 Notes and further reading

The simple interest rate models we have used in this chapter are useful for illustrating the possible impact of interest rate uncertainty, but developing more realistic interest rate models is a major topic in its own right, beyond the scope of this text. Some models are presented in McDonald (2009) and a comprehensive presentation of the topic is available in Cairns (2004).

We have shown in this chapter that uncertainty in the mortality experience is a source of non-diversifiable risk. This is important because improving mortality has been a feature in many countries and the rate of improvement has been difficult to predict. See, for example, Willets *et al.* (2004). In these circumstances, the assumptions about the survival model in Section 12.4.1 may not be reasonable and so a significant aspect of mortality risk is non-diversifiable. In Chapter 19 we look at some models of longevity that have been used by actuaries to quantify non-diversifiable mortality risk.

Note that in Examples 12.6–12.8 we simulated the future lifetime random variable T_{50} assuming the survival model and its parameters were known. Monte Carlo methods could be used to model uncertainty about the survival model; for example, by assuming that the two parameters in the Gompertz formula were unknown but could be modelled as random variables with specified distributions.

Monte Carlo simulation is a key tool in modern risk management. A general introduction is presented in e.g. Ross (2013), and Glasserman (2013) offers a text more focused on financial modelling. Algorithms for writing your own generators are given in the *Numerical Recipes* reference texts, such as Press *et al.* (2007).

12.7 Exercises

Shorter exercises

Exercise 12.1 You are given the following zero-coupon bond prices:

Term, t (years)	$P(t)$ as % of face value
1	94.35
2	89.20
3	84.45
4	79.95
5	75.79

(a) Calculate the annual effective spot rates for $t = 1, 2, 3, 4, 5$.
(b) Calculate the one-year forward rates, at $t = 0, 1, 2, 3, 4$.
(c) Calculate the EPV of a five-year term life annuity-due of \$1000 per year, assuming that the probability of survival each year is 0.99.

Exercise 12.2 An insurer issues a whole life annuity-due of \$10 000 per year to (60). Mortality is assumed to follow the Standard Ultimate Life Table. The k-year deferred one-year forward rates are denoted $f(k, k+1)$, and are given in the following table.

$f(0,1)$	$f(1,2)$	$f(2,3)$	$f(3,4)$	$f(k, k+1), k \geq 4$
0.03	0.035	0.04	0.045	0.05

Calculate the EPV of the annuity.

Exercise 12.3 An insurer issues a portfolio of identical five-year term insurance policies to independent lives aged 75. One half of all the policies have a sum insured of \$10 000, and the other half have a sum insured of \$100 000. The sum insured is payable immediately on death.

The insurer wishes to measure the uncertainty in the total present value of claims in the portfolio. The insurer uses the Standard Ultimate Survival Model, and assumes an interest rate of 6% per year effective.

(a) Calculate the standard deviation of the present value of the benefit for an individual policy, chosen at random.
(b) Calculate the standard deviation of the total present value of claims for the portfolio assuming that 100 contracts are issued.
(c) By comparing the portfolio of 100 policies with a portfolio of 100 000 policies, demonstrate that the mortality risk is diversifiable.

Longer exercises

Exercise 12.4 (a) The coefficient of variation for a random variable X is defined as the ratio of the standard deviation of X to the mean of X. Let X denote the aggregate loss on a portfolio, so that $X = \sum_{j=1}^{N} X_j$. Assume that, for each j, $X_j > 0$ and X_j has finite mean and variance.

Show that, if the portfolio risk is diversifiable, then the limiting value of the coefficient of variation of aggregate loss X, as $N \to \infty$, is zero.

(b) An insurer issues a portfolio of identical 15-year term insurance policies to independent lives aged 65. The sum insured for each policy is \$100 000, payable at the end of the year of death.

The mortality for the portfolio is assumed to follow Makeham's law with $A = 0.00022$ and $B = 2.7 \times 10^{-6}$. The insurer is uncertain whether the parameter c for Makeham's mortality law is 1.124, as in the Standard Ultimate Survival Model, or 1.114. The insurer models this uncertainty assuming that there is a 75% probability that $c = 1.124$ and a 25% probability that $c = 1.114$. Assume the same mortality applies to each life in the portfolio. The effective rate of interest is assumed to be 6% per year.

(i) Calculate the coefficient of variation of the present value of the benefit for an individual policy.
(ii) Calculate the coefficient of variation of the total present value of benefits for the portfolio assuming that 10 000 policies are issued.
(iii) Demonstrate that the mortality risk is not fully diversifiable, and find the limiting value of the coefficient of variation.

Exercise 12.5 An insurer issues a 25-year endowment insurance policy to (40), with level premiums payable continuously throughout the term of the policy, and with sum insured \$100 000 payable immediately on death or at the end of the term. The insurer calculates the premium assuming an interest rate of 7% per year effective, and using the Standard Ultimate Survival Model.

(a) Calculate the annual net premium payable.
(b) Suppose that the effective annual interest rate is a random variable, **i**, with the following distribution:

$$\mathbf{i} = \begin{cases} 5\% \text{ with probability } 0.5, \\ 7\% \text{ with probability } 0.25, \\ 11\% \text{ with probability } 0.25. \end{cases}$$

Write down the EPV of the net future loss on the policy using the mean interest rate, and the premium calculated in part (a).
(c) Calculate the EPV of the net future loss on the policy using the modal interest rate, and the premium calculated in part (a).
(d) Calculate the EPV and the standard deviation of the present value of the net future loss on the policy. Use the premium from part (a) and assume that the future lifetime is independent of the interest rate.
(e) Comment on the results.

Exercise 12.6 An actuary is concerned about the possible effect of pandemic risk on the term insurance portfolio of her insurer. She assesses that in any year there is a 1% probability that mortality rates at all ages will increase by 25%, for that year only.

(a) State, with explanation, whether pandemic risk is diversifiable or non-diversifiable.
(b) Describe how the actuary might quantify the possible impact of pandemic risk on her portfolio.

Excel-based exercises

Exercise 12.7 Consider an endowment insurance with sum insured \$100 000 issued to a life aged 45 with term 15 years under which the death benefit is payable at the end of the year of death. Premiums, which are payable annually in advance, are calculated using the Standard Ultimate Survival Model, assuming a yield curve of effective annual spot rates given by

$$y_t = 0.035 + \frac{\sqrt{t}}{200}.$$

(a) Show that the net premium for the contract is \$4207.77.
(b) Calculate the net premium determined using a flat yield curve with effective rate of interest $i = y_{15}$ and comment on the result.
(c) Calculate the net policy value for a policy still in force three years after issue, using the rates implied by the original yield curve, using the premium basis.

Exercise 12.8 An insurer issues 15-year term insurance policies to lives aged 50. The sum insured of \$200 000 is payable immediately on death. Level premiums of \$550 per year are payable continuously throughout the term of the policy. The insurer assumes the lives are subject to Gompertz' law of mortality with $B = 3 \times 10^{-6}$ and $c = 1.125$, and that interest rates are constant at 5% per year.

(a) Generate 1000 simulations of the future loss.
(b) Using your simulations from part (a), estimate the mean and variance of the future loss random variable.
(c) Calculate a 90% confidence interval for the mean future loss.
(d) Calculate the true value of the mean future loss. Does it lie in your confidence interval in part (c)?
(e) Repeat the 1000 simulations 20 times. How often does the confidence interval calculated from your simulations not contain the true mean future loss?
(f) If you calculated a 90% confidence interval for the mean future loss a large number of times from 1000 simulations, how often (as a percentage) would you expect the confidence interval not to contain the true mean?
(g) Now assume interest rates are unknown. The insurer models the interest rate on all policies, $\boldsymbol{I}$, as a lognormal random variable, such that

$$1 + \boldsymbol{I} \sim LN(0.0485,\ 0.0241).$$

Re-estimate the 90% confidence interval for the mean of the future loss random variable, using Monte Carlo simulation. Comment on the effect of interest rate uncertainty.

Answers to selected exercises

12.1 (a) (0.05988, 0.05881, 0.05795, 0.05754, 0.05701)
(b) (0.05988, 0.05774, 0.05625, 0.05629, 0.05489)
(c) \$4395.73
12.2 155 413
12.3 (a) \$19 784 (b) \$193 054
12.4 (b)(i) 2.2337 (ii) 0.2204 (iii) 0.2192
12.5 (a) \$1608.13 (b) \$0 (c) \$7325.40 (d) \$2129.80, \$8489.16
12.7 (b) \$4319.50 (c) \$13 548
12.8 (d) −\$184.07 (f) 10%

13

Emerging costs for traditional life insurance

13.1 Summary

In this chapter we introduce emerging costs, or cash flow analysis, for traditional insurance contracts. This is often called **profit testing** when applied to life insurance.

We introduce profit testing in two stages. First we consider only those cash flows generated by the policy, then we introduce reserves to complete the cash flow analysis.

We define several measures of the profitability of a contract: internal rate of return, expected present value of future profit (net present value), profit margin and discounted payback period. We show how cash flow analysis can be used to set premiums to meet a given measure of profit.

We restrict our attention in this chapter to deterministic profit tests, ignoring uncertainty. We introduce stochastic profit tests in Chapter 15.

13.2 Introduction

Traditionally, actuarial analysis has focused on determining the EPV of a cash flow series, usually under a constant interest rate assumption. This emphasis on the EPV was important in an era of manual computation, but with powerful computers available we can do better. In this chapter, we look at techniques for projecting the cash flows emerging from an individual contract in each time period, using some specified assumptions about the interest and demographic experience. The use of cash flow projections offers much more flexibility in the input assumptions than the EPV approach – for example, it is easy to incorporate yield curves, or more sophisticated models of policyholder behaviour – and provides actuaries with a better understanding of the liabilities under their management and the relationship between the liabilities and the corresponding assets. For modern contracts, with variable premiums and complex financial guarantees, traditional valuation techniques are not very useful. Profit testing techniques offer the flexibility to explore risk and return for a wide range of modern and traditional contracts.

The purpose of a profit test is to identify the profit which the insurer can expect from a contract at the end of each time period. There are many reasons why this might be valuable. Some of the ways in which profit tests are applied in practice are described here.

1. **To set premiums**
   The traditional approach to premium calculation in Chapter 6 does not explicitly allow for profit, nor for yield curves. Even when a profit loading is explicitly introduced, the methods of Chapter 6 do not give a picture of how the profit might emerge over time, and do not allow the insurer to determine return on capital from the contract. Profit testing allows premiums to be set to meet specified profit measures, and allows the insurer to stress test the assumptions to consider how sensitive the emerging profit would be to different assumptions.
2. **To set reserves**
   We use the term 'reserve' to indicate the assets that the insurer holds (or is projected to hold in the future), to meet the future net liabilities of a contract. As we discussed in Chapter 7, the term 'reserve' has often been used interchangeably with 'policy value', which was defined, for a policy in force at time t, say, as the expected value at t of the present value of the future loss random variable, L_t.

   The traditional approach to the reserve calculation has been to use a policy value. However, it is not necessary to do so. Reserves can be determined arbitrarily, or by evaluating the capital required to support the liabilities under specified assumptions. Profit test analysis allows reserves to be determined and tested under a range of more complex assumptions for interest rates and policyholder behaviour than is feasible using the traditional calculations described in earlier chapters.
3. **To measure profitability**
   The insurer will be interested in projecting emerging cash flows to assess liquidity needs. For example, new business strain for new contracts creates a need for capital, which may be available from surplus emerging from more mature business. Developing cash flow models allows the insurer to manage portfolios taking different maturities and cash flow patterns into account. The insurer may use cash flow emergence and profitability measures to determine strategies for marketing and product development.
4. **To stress test profitability**
   The assumptions used to project future cash flows can be adjusted to explore the impact of adverse experience. Usually, the insurer would profit test

contracts using a range of assumptions to get a feel for the sensitivity of the cash flows to different adverse scenarios.

5. **To determine distributable surplus**
   Policyholders of participating or with-profit contracts will be entitled to a share of the profits generated within a specified fund. Universal life policyholders will share the investment profits generated by the funds supporting their contracts. It is apparent that in order to plan strategies for profit distribution, it would be helpful to have an idea of how the surplus will emerge. We may also use profit testing to explore risks associated with different methods for distributing surplus.

In this chapter, we look at how profit tests can be used for premium setting, reserve calculations and for measuring profitability, all in the context of traditional insurance. In Chapters 14 and 15 we consider applications to non-traditional insurance.

13.3 Profit testing a term insurance policy

We introduce profit testing by studying in some detail a 10-year term insurance issued to a life aged 60. The details of the policy are as follows. The sum insured, denoted S, is \$100 000, payable at the end of the year of death. Level annual premiums, denoted P, of amount \$1500 are payable throughout the term.

13.3.1 Time step

We will project the cash flows from this policy at discrete intervals throughout its term. It would be very common to choose one month as the interval since premiums are often paid monthly, and the profit test would be regularly updated through the term of the contract. However, to illustrate more clearly the mechanics of profit testing, we use a time interval of one year for this example, taking time 0 to be the moment when the policy is issued.

13.3.2 Profit test basis

To estimate the future cash flows, the insurer needs to make assumptions about the expenses which will be incurred, the survival model for the policyholder, the rate of interest to be earned on cash flows within each time period before the profit is released and possibly other items such as an assessment of the probability that the policyholder surrenders the policy. For ease of presentation, we ignore the possibility of surrender in this example.

The set of assumptions used in the profit test is called the **profit test basis**.

Survival probabilities

We project cash flows using expected values for mortality costs. For example, the expected cost of a death benefit of S paid at the end of the first year, for a life aged 60 at the start of the year is $q_{60}\, S$.

In this example, we assume a survival model for the profit test following

$$q_{60+t} = 0.01 + 0.001\, t \quad \text{for } t = 0, 1, \ldots, 9.$$

The survival model used in a profit test may be different to the premium basis. For example, the insurer may incorporate margins in the premium basis – meaning, adopt more conservative assumptions – to allow for adverse experience. In the profit test, the insurer may be interested in a 'best estimate' picture of the emerging profits, in which case the survival model should not incorporate any margins.

Expenses

In Chapter 6 we discussed how expenses are incorporated into the calculation of the premium for a policy. Typically, the acquisition expenses, incurred at the start of the contract, are high, and the later expenses, associated with record maintenance and premium collection, tend to be smaller.

In profit testing, it is necessary to be more specific about the acquisition expenses. As we project cash flows, we assume that some expenses arise even before the first premium is collected. These expenses are treated as being incurred at the start of the contract, at time $t = 0$. This differs from the treatment of expenses allocated to subsequent time periods, where expenses are combined with all the other sources of income and outgo for the period, and values accumulated to the year end.

The reason for treating the acquisition expenses differently is that prudent capital management requires us to recognize losses as early as possible; surplus may be carried forward, but losses should be accounted for as soon as they are incurred. In this example we are projecting cash flows to the year end before analysing the surplus emerging. It would not generally be prudent to combine the high acquisition costs with the other first year income and outgo, as that would delay recognition of those expenses, and lessen their impact.

In our examples we will identify, specifically, the initial expenses which should be allocated to time 0, as distinct from the expenses which arise at inception, but may be accounted for with the other first year cash flows. The time 0 expenses will be identified as **pre-contract expenses**; other expenses that arise during the first policy year are intended to be included in the first year cash flows. If no distinction is made, it should be assumed that all initial expenses should be allocated to the time 0 cash flows. This is a common approach, because it gives the most conservative result.

For this example, we use the following expense assumptions basis.

Pre-contract expenses:	\$400 plus 20% of the first premium.
Renewal expenses:	3.5% of premiums, including the first.

Interest on insurer assets

In each year that the policy is still in force, the cash flows contributing to the surplus emerging at the end of that year include the premiums, less any premium-related expenses, interest earned on the invested assets, less the expected cost of claims at the end of the year. We therefore require an assumption about the interest rate earned on insurer assets during the projection year. Often, this will be a best estimate, which will differ from the interest assumptions for premiums and reserves which typically incorporate margins for adverse experience, or for implicit profit loading.

The step-by-step process for profit testing makes it very simple to allow for different interest rates in different projection periods, so that a yield curve could easily be accommodated. In this example though, we will assume a constant interest rate of 5.5% per year.

Emerging surplus for the term insurance example, without reserves

The calculations of the emerging surplus, called the **net cash flows** for the policy, are summarized in Table 13.1.

For time $t = 0$ the only entry is the acquisition expense for the policy, which is $400 + 0.2\,P = 700$. This expense is assumed to occur and to be paid at time 0, so no interest accrues. In all our profit test tables throughout this and subsequent chapters, the first row will account for costs (not income) at $t = 0$, and will not be accumulated. In the table, and in the examples following in this and subsequent chapters, we let E_0 denote the pre-contract, acquisition expenses, assumed incurred at time 0, and we let E_t denote the tth year expenses, incurred at the start of the year from $t-1$ to t, for $t = 1, 2, \ldots, 10$.

After the time 0 row for pre-contract costs, each subsequent row shows the income and outgo cash flows for the specified policy year.

The second row refers to cash flows in the first policy year, which we label as $t = 1$, and which runs from time 0 (after the acquisition expenses already accounted for in the time 0 row) to time 1.

There is a premium of 1500 payable at time 0; there are premium expenses of $0.035P = 52.5$ that are not included in the time 0 acquisition expenses. Interest is earned at 5.5% through the year, so the interest income over the year is $0.055(1500 - 52.5) = 79.61$. At the year end, the expected cost of death benefits is

$$EDB_1 = q_{60}\,S = 0.01 \times 100\,000 = 1000.$$

Hence the emerging surplus, or net cash flow, at time 1 is

$$1500 - 52.5 + 79.61 - 1000 = 527.11.$$

For subsequent policy years, we will adopt the convention that the net cash flows are calculated *assuming the policy is still in force at the start of the year*. This means that we are starting each time step with a new assumption. For example, considering the second year of the policy, we project all cash flows assuming the policy is in force at the start of the year, at $t = 1$, but by the end of the year, the policy may be in force (with probability p_{x+1}), or the policyholder may have died (with probability q_{x+1}). When we move to the third policy year, we assume the policy is in force at the start of the year. We discuss this convention in more detail after we work through the examples.

Using this convention, consider, for example, the seventh year of the projection. We assume the policy is in force at the start of the year, and the insurer receives the premium then due, of $P = 1500$; at the same time, the insurer incurs expenses of 3.5% of the premium, $E_7 = 52.5$. The balance is invested for the year at the assumed interest rate of 5.5%, generating investment income of $I_7 = 0.055(P - E_7) = 79.61$. At the year end, the expected cost of death benefits is $EDB_7 = q_{66}\,S = 1600$. Hence, the expected value at time 7 of the net cash flows received during the 7th year, for a policy in force at the start of the year, is

$$P - E_7 + I_7 - EDB_7 = -72.89.$$

13.3.3 Incorporating reserves

Table 13.1 reveals a typical feature of net cash flows, in that several of the net cash flows in later years are negative. This occurs because the level premium

Table 13.1 *Net cash flows for the 10-year term insurance in Section 13.3.*

Time t	Premium at $t-1$	Expenses E_t	Interest I_t	Claims EDB_t	Surplus emerging at t
0		700.00			−700.00
1	1500	52.50	79.61	1000	527.11
2	1500	52.50	79.61	1100	427.11
3	1500	52.50	79.61	1200	327.11
4	1500	52.50	79.61	1300	227.11
5	1500	52.50	79.61	1400	127.11
6	1500	52.50	79.61	1500	27.11
7	1500	52.50	79.61	1600	−72.89
8	1500	52.50	79.61	1700	−172.89
9	1500	52.50	79.61	1800	−272.89
10	1500	52.50	79.61	1900	−372.89

is more than sufficient to pay the expected death claims and expenses in the early years, but, with an increasing probability of death, the premium is not sufficient in the later years. The expected cash flow values in the final column of Table 13.1 show the same general features as the values illustrated in Figures 6.1 and 6.2.

In Chapter 7 we explained why the insurer needs to set aside assets to cover negative expected future cash flows. The policy values that we calculated in that chapter represented the amount that would, in expectation, be sufficient with the future premiums to meet future benefits. In modelling cash flows, we use reserves rather than policy values. The reserve is the actual amount of money held by the insurer to meet future liabilities. The reserve may be equal to a policy value, but does not need to be. For traditional insurance such as the term policy in this example, it is common to use a policy value calculation to set reserves, perhaps using conservative assumptions, or using a net premium approach with a different (hypothetical) premium to the actual gross premium for the contract.

Note that the negative cash flow at time 0 in Table 13.1 does not require a reserve since it will have been paid as soon as the policy was issued.

Suppose that the insurer sets reserves for this policy equal to the net premium policy values on the following (reserve) basis.

Interest:	4% per year effective on all cash flows.
Survival model:	$q_{60+t} = 0.011 + 0.001\,t$ for $t = 0, 1, \dots, 9$.

Then the reserve required at the start of the $(t+1)$th year, i.e. at time t, is

$$100\,000\,A^{\;\;1}_{60+t:\overline{10-t}|} - P^n\,\ddot{a}_{60+t:\overline{10-t}|},$$

where the net premium, P^n, is calculated as

$$P^n = 100\,000\,\frac{A^{\;\;1}_{60:\overline{10}|}}{\ddot{a}_{60:\overline{10}|}} = \$1447.63,$$

and all functions are calculated using the reserve basis. The values for the reserves are shown in Table 13.2. We now include in our profit test the cost of capital arising from the need to allocate the reserves to the policies in force. We do this by following (loosely) the accounting approach, where reserves brought forward are treated as income at the start of each year, and reserves carried forward are treated as a cost at the end of each year.

To illustrate this, consider, for example, the reserve required at time 1, ${}_1V = 410.05$. This amount is required for every policy *still in force at time 1*. The cost to the insurer of setting up this reserve is assigned to the previous

Table 13.2 *Reserves for the 10-year term insurance in Section 13.3.*

t	${}_tV$	t	${}_tV$
0	0.00	5	1219.94
1	410.05	6	1193.37
2	740.88	7	1064.74
3	988.90	8	827.76
4	1150.10	9	475.45

Table 13.3 *Emerging surplus, per policy in force at the start of each year, for the 10-year term insurance in Section 13.3.*

t (1)	${}_{t-1}V$ (2)	P (3)	E_t (4)	I_t (5)	EDB_t (6)	E_tV (7)	Pr_t (8)
0			700.00			0.00	−700.00
1	0.00	1500	52.50	79.61	1000	405.95	121.16
2	410.05	1500	52.50	102.17	1100	732.73	126.99
3	740.88	1500	52.50	120.36	1200	977.04	131.70
4	988.90	1500	52.50	134.00	1300	1135.15	135.26
5	1150.10	1500	52.50	142.87	1400	1202.86	137.61
6	1219.94	1500	52.50	146.71	1500	1175.47	138.68
7	1193.37	1500	52.50	145.25	1600	1047.70	138.41
8	1064.74	1500	52.50	138.17	1700	813.69	136.72
9	827.76	1500	52.50	125.14	1800	466.89	133.52
10	475.45	1500	52.50	105.76	1900	0.00	128.71

time period and the expected cost is

$$ {}_1V\, p_{60} = 410.05 \times (1 - 0.01) = 405.95. $$

The cost includes the factor p_{60} since all costs relating to the previous time period are per policy in force *at the start of that time period*, that is, at time 0. The expected proportion of policyholders surviving to the start of the following time period, i.e. to age 61, is p_{60}. Note that p_{60} is calculated on the profit test basis, not the reserve basis which is used solely for determining the ${}_tV$ values. In general, the cost at the end of the year from $t-1$ to t of setting up a reserve of amount ${}_tV$ at time t for each policy still in force at time t is ${}_tV\, p_{60+t-1}$.

The profit test calculations, including reserves, are set out in Table 13.3, and the individual calculations are described in more detail below.

Column (1) labels the rows. The first row, labelled $t = 0$, is for the cash flows required immediately before the inception of the contract. For subsequent rows, the cash flows in the tth year are those from $t-1$ to t. In each row, we assume the contract is in force at the start of the year.

Column (2) shows the reserve brought forward at the start of each year, at time $t-1$, assuming the contract is in force at that time. The reserve brought forward is treated as an item of income in the cash flows for the tth year.

Column (3) shows the premium paid at the start of the year, assuming the policy is in force at that time, $P = 1500$.

Column (4) shows the expenses. E_0 denotes the initial expenses incurred at time 0 and for $t = 1, 2, \ldots, 10$, E_t denotes the renewal expenses incurred at the start of the year from $t-1$ to t.

Column (5) shows I_t, which denotes the interest earned in the year from $t-1$ to t on the assets invested at the start of the year. In this case, for $t = 1, 2, \ldots, 10$, with an assumed interest rate of 5.5%, we have

$$I_t = 0.055\ \left({}_{t-1}V + P - E_t\right).$$

Column (6) shows the expected cost of death benefits in the tth year, EDB_t, assuming the policy is in force at time $t-1$. So, with a sum insured of $S = 100\,000$, the expected death benefit at time t for a contract in force at time $t-1$ is

$$EDB_t = S\, q_{60+t-1}.$$

Column (7) shows the expected cost of the reserve carried forward at time t, denoted E_tV, for a policy in force at time $t-1$, that is,

$$E_tV = p_{60+t-1}\ {}_tV.$$

Note that, if ${}_0V > 0$ then there is an **initial reserve** requirement. In this case, the initial reserve, ${}_0V$, would be included in the first row, $t = 0$, in column (7), and would be treated as an acquisition cost. Then, in the second row, ${}_0V$ would be in column (2) as an item of income for the year from 0 to 1. Initial reserves can arise if the reserve basis is different from the premium basis, or if premiums are not calculated using the equivalence principle.

Column (8) shows the expected profit emerging at time t given that the policy is in force at time $t-1$, except for the first row, where Pr_0 represents the acquisition costs. That is,

$$\text{Pr}_0 = -E_0 - {}_0V = -700$$

and for $t = 1, 2, \ldots, 10$,

$$\text{Pr}_t = {}_{t-1}V + P - E_t + I_t - EDB_t - E_tV. \tag{13.1}$$

Equation (13.1) is sometimes expressed slightly differently, as

$$\text{Pr}_t = (P - E_t)(1 + i) + \Delta\, {}_tV - EDB_t,$$

where $\Delta\,_tV$ is called the **change in reserve** in year t and is defined as

$$\Delta\,_tV = (1+i)\,_{t-1}V - \,_tV\,p_{60+t-1}.$$

This alternative version reflects the difference between the reserves and the other cash flows. The incoming and outgoing reserves each year are not real income and outgo in the same way as premiums, claims and expenses; they are accounting transfers. It also allows for the use of a different return on assets underlying reserves than on other cash flows.

13.3.4 Profit signature

We have used an important convention in the construction of the Tables 13.1 and 13.3, that is worth emphasizing. The entries in each row are calculated *assuming the policy is in force at the start of the year*. We use this convention because it is very convenient; it makes the profit test more flexible, and (once you are used to it) easier to construct. However, it also means that we cannot simply gather together all the Pr_t entries from the table to analyse the future profits on a contract, since each entry is based on a different assumption about the probability that the policy is still in force.

The vector $\mathrm{Pr} = (\mathrm{Pr}_0, \ldots, \mathrm{Pr}_{10})'$ is called the **profit vector** for the contract. So, the elements of Pr denote the expected profit emerging at the end of each year, given that the policy is in force at the start of the year. For an overall, unconditional projection of the emerging surpluses from a newly issued contract, we need to adjust the Pr_t values to remove the conditioning.

Let Π_t, represent the expected profit emerging at time t from the cash flows in the year $t-1$ to t, given that the contract is in force at time $t = 0$ (i.e. unconditionally). The relationship between Π_t and Pr_t for $t = 1, 2, \ldots, 10$, is

$$\Pi_t = \mathrm{Pr}_t \times \mathrm{Pr}\left[\text{in force at time } t-1 \mid \text{in force at time } 0\right].$$

The vector Π is called the **profit signature** for the contract. We have, for the current example,

$$\Pi_0 = \mathrm{Pr}_0 \quad \text{and} \quad \Pi_t = \,_{t-1}p_{60}\,\mathrm{Pr}_t \quad \text{for } t = 1, 2, \ldots, 10.$$

The profit signature is the key to assessing the profitability of a new contract. The profit signature for the 10-year term example is given alongside the profit vector Pr in Table 13.4. We show them together to emphasize the difference between the two vectors, which is important in applying and interpreting the profit test. In the profit vector, Pr_t represents the profit emerging at time t from the cash flows in the year $t-1$ to t, given that the contract is in force at time $t-1$, for $t = 1, 2, \ldots, 10$. In the profit signature, Π_t represents the unconditional profit emerging at time t for a newly issued contract.

Table 13.4 *Profit vector and profit signature for the 10-year term insurance.*

t	Pr_t	Π_t	t	Pr_t	Π_t
0	−700.00	−700.00	6	138.68	130.56
1	121.16	121.16	7	138.41	128.35
2	126.99	125.72	8	136.72	124.75
3	131.70	128.95	9	133.52	119.76
4	135.26	130.84	10	128.71	113.37
5	137.61	131.39			

13.4 Profit testing principles

13.4.1 Assumptions

In this section we generalize the process described in the example of the previous section. In this description we calculate the emerging profit assuming annual time steps, but the method can be very easily adapted to other frequencies.

We assume a contract with a term of n years, issued to (x), with cash flows dependent on whether the policyholder dies, surrenders or continues in force through to the end of the policy year.

We assume that a policyholder whose policy is in force at time $t-1$, dies in the year $t-1$ to t with probability p^{0d}_{x+t-1}, withdraws or surrenders the contract with probability p^{0w}_{x+t-1}, and remains in force at time t with probability $p^{00}_{x+t-1} = 1 - p^{0d}_{x+t-1} - p^{0w}_{x+t-1}$.

13.4.2 The profit vector

The profit vector is

$$\mathrm{Pr} = (\mathrm{Pr}_0\,, \mathrm{Pr}_1\,, \ldots\,, \mathrm{Pr}_n)'\,.$$

The profit vector elements Pr_t, for $t \geq 1$, represent the expected surplus emerging at each year end for a contract in force at time $t-1$, i.e. at the start of the year. The first element of the vector, Pr_0, has a slightly different interpretation. It represents the value at time $t = 0$ of the pre-contract cash flows, including the acquisition expenses, E_0, and the cost of setting up initial reserves, ${}_0V$, where required. So

$$\mathrm{Pr}_0 = -E_0 - {}_0V$$

and for $t = 1, 2, \ldots, n$,

$$\mathrm{Pr}_t = {}_{t-1}V + P_t - E_t + I_t - EDB_t - ESB_t - EEB_t - E_tV,$$

where

${}_tV$ is the reserve required at time t for a policy in force at that time.
P_t is the tth premium, paid at time $t-1$, for a policy in force at time $t-1$.

E_t is the premium expense incurred at time $t-1$ for a policy in force at time $t-1$.

I_t is the investment income earned on the insurer's funds over $t-1$ to t for a policy in force at time $t-1$. That is, $I_t = i_t\,({}_{t-1}V + P_t - E_t)$ where i_t is the yield on investments from time $t-1$ to t.

EDB_t is the expected cost of death benefits at time t for a policy in force at time $t-1$. That is, $EDB_t = p^{0d}_{x+t-1}\,(S_t + e_t)$, where S_t is the sum insured and e_t is the claim expense.

ESB_t is the expected cost of surrender benefits at time t for a policy in force at time $t-1$. That is, $ESB_t = p^{0w}_{x+t-1}\,CV_t$ where CV_t is the cash or surrender value payable for surrenders at time t.

EEB_t is the expected cost of endowment or survivor benefits at time t for a policy in force at time $t-1$. This would be applicable for a maturity benefit under an endowment policy, or for end-year annuity benefits.

E_tV is the expected cost of setting the reserve at time t for a policy in force at time $t-1$. That is, $E_tV = p^{00}_{x+t-1}\,{}_tV$.

13.4.3 The profit signature

The profit signature is

$$\Pi = (\Pi_0, \Pi_1, \ldots, \Pi_n)'\,.$$

The profit signature elements Π_t, for $t \geq 1$, represent the expected surplus emerging at the tth year-end **for a contract in force at the issue date, i.e at time** $t = 0$. The first term of the vector, Π_0, represents the value at time $t = 0$ of the pre-contract cash flows. So

$$\Pi_0 = \mathrm{Pr}_0$$

and for $t = 1, 2, \ldots, n$, we multiply Pr_t, which is the expected surplus conditional on the contract being in force at time $t-1$, by the probability of being in force at time $t-1$, to get the unconditional expected surplus at time t for a new contract, so

$$\Pi_t = {}_{t-1}p^{00}_x\,\mathrm{Pr}_t.$$

13.4.4 The net present value

Having developed the projected expected year-end emerging surplus for a new contract, it is often convenient to express the values in a single metric. The **net present value** (**NPV**) of the contract is the present value of the projected emerging surplus values. To determine the present values, we discount at an appropriate rate of interest, which is normally higher than the assumed yield on assets which is specified in the profit testing basis. The interest rate for discounting surplus represents the return on capital required by the shareholders, since the emerging surplus can be considered as the return to shareholders on capital supplied to support the contract liability. The rate is sometimes called the **risk discount rate** or **hurdle rate**.

Assuming a risk discount rate of r per year effective, the net present value of a policy is

$$\text{NPV} = \sum_{t=0}^{n} \Pi_t \, v_r^t .$$

13.4.5 Notes on the profit testing method

(1) For a large portfolio of similar policies, the profit signature describes the expected surplus emerging at each year end for each contract issued. This is clearly useful information, and begs the question: why take the intermediate step of calculating the profit vector, for which each term is conditional on the contract being in force at successive policy anniversaries? The answer is that the profit vector is also useful, particularly for a portfolio of in-force contracts at different durations.

Suppose an insurer has a portfolio of 10-year term insurance policies, all issued to lives aged 60 at different times in the previous ten years, and all represented by the policy terms and assumptions used for the example in Section 13.3. The profit vector can be used to analyse expected emerging surplus from each cohort. For example, for each contract still in force after $k \geq 1$ complete years (at age $60 + k$), the profit signature from future surplus can be calculated as

$$\left(\mathrm{Pr}_{k+1}, \quad {}_1p_{60+k}^{00}\,\mathrm{Pr}_{k+2}, \quad {}_2p_{60+k}^{00}\,\mathrm{Pr}_{k+3}, \ldots, {}_{9-k}p_{60+k}^{00}\,\mathrm{Pr}_{10}\right)'$$

and the NPV of the future surplus is

$$\sum_{u=1}^{10-k} {}_{u-1}p_{60+k}^{00}\,\mathrm{Pr}_{k+u}\; v_r^u .$$

(2) We have been rather loose about random variables and expectation in this chapter. In practice, profit testing would be carried out on large portfolios, rather than on individual contracts. Aggregating makes the individual cash flows relatively more predictable, and it may be reasonable to assume that death benefits and reserve costs will be quite close to the expected values. The approach used here is described as deterministic, which is loosely used to mean that we project cash flows without allowing for random variation.

(3) An aspect of profit testing that can be confusing is that we are often working with several different sets of assumptions. For example, we may have a Premium Basis, a Reserve Basis and a Profit Testing Basis that all

use different mortality, interest rate and expense assumptions for the same policy. The reason is that each basis plays a different role; the reserve basis may be constrained by regulation; the premium basis may include implicit or explicit profit loadings; the profit test basis may be a best estimate basis, implying no margins (typically using a median value for variables such as interest or expense inflation), or it may use a series of different assumptions (or scenarios) to assess the impact on profit emergence of more adverse experience. The key is to be clear of the role of each set of assumptions, and to ensure that the correct basis is used for each calculation.

(4) The process described in this section can be used with traditional style contracts, allowing for multiple decrements or multiple states, but assumes that the contract is in force in only one state. When there are two or more states representing the in force contracts, the process requires some extra steps, which we describe in Section 13.9 below.

13.5 Profit measures

Once we have projected the cash flows, we need to assess whether the emerging profit is adequate. There are a number of ways to measure profit, all based on the profit signature.

The net present value is a commonly used measure of profit for project appraisals in all fields. For the example in Section 13.3, if the insurer uses a risk discount rate of 10% per year, then the NPV of the contract is \$74.13.

We also define the **partial net present value**, $\mathrm{NPV}(t)$, for $t \leq n$, as the net present value of all cash flows up to and including time t, so that

$$\mathrm{NPV}(t) = \sum_{k=0}^{t} \Pi_k \, v_r^k.$$

Often the partial NPV is negative in the early years of a contract, reflecting the acquisition costs, and has a single sign change at some point of the contract, assuming the NPV of the contact is positive. The partial NPV values for $t = 0, 1, \ldots, 10$ for the 10-year term insurance example are given in Table 13.5, showing this typical pattern for emerging profit. The NPV for the contract is the final value in the partial NPV vector.

The NPV is closely related to the **internal rate of return (IRR)**, which is the interest rate j such that the net present value is zero. That is, given a profit signature $(\Pi_0, \Pi_1, \ldots, \Pi_n)'$ for an n-year contract, the internal rate of return is j where

$$\sum_{t=0}^{n} \Pi_t \, v_j^t = 0. \tag{13.2}$$

Table 13.5 *Partial NPVs for Section 13.3 example, 10% risk discount rate.*

t	NPV(t)	t	NPV(t)
0	−700.00	6	−144.43
1	−589.85	7	−78.56
2	−485.95	8	−20.37
3	−389.07	9	30.42
4	−299.70	10	74.13
5	−218.12		

The IRR is commonly used as a metric for assessing profitability, with insurers setting a minimum value (the hurdle rate) for the IRR. However, there are several well-documented problems with the IRR as a measure of profit. One is that there may be no real solution to equation (13.2), or there may be many. However, a quick check on the IRR can be determined by using the hurdle rate to calculate the NPV. If the NPV is greater than zero, and if the partial NPV has a single sign change, then there is a unique real solution to the IRR equation, and the IRR is greater than the hurdle rate. If the NPV is negative, then the IRR, if it exists, is less than the hurdle rate.

For the policy in Section 13.3, we know that the IRR is greater than 10%, as, at 10% risk discount rate, the NPV is greater than zero, and the partial NPV has a single sign change. In fact, the internal rate of return in this case is $j = 12.4\%$.

The partial NPV is useful for another profit measure, the **discounted payback period (DPP)**, also known as the break-even period. This is defined as the first time at which the partial NPV is greater than zero, using the risk discount rate. In other words, the DPP is m where

$$m = \min\{t : \text{NPV}(t) \geq 0\}.$$

The DPP represents the time until the insurer starts to make a profit on the contract, based on the hurdle rate of return. For the example in Section 13.3, the DPP is nine years.

The **profit margin** is the NPV expressed as a proportion of the EPV of the premiums, evaluated at the risk discount rate. For a contract with level premiums of P per year payable annually throughout an n year contract issued to a life aged x, the profit margin is

$$\text{Profit Margin} = \frac{\sum_{t=0}^{n} \Pi_t \, v_r^t}{P \sum_{t=0}^{n-1} {}_tp_x^{00} \, v_r^t} \tag{13.3}$$

using the risk discount rate for all calculations.

For the example in Section 13.3, the profit margin using a risk discount rate of 10% is

$$\frac{\text{NPV}}{P\ddot{a}_{60:\overline{10}|}} = \frac{74.13}{9\,684.5} = 0.77\%.$$

Another profit measure is the NPV as a proportion of the acquisition costs. For the example in Section 13.3, the acquisition costs are \$700, so the NPV is 10.6% of the total acquisition costs.

None of these measures of profit explicitly takes into consideration the risk associated with the contract. Most of the inputs we have used in the emerging surplus calculation are, in practice, uncertain. If the experience is adverse, the profit will be smaller, or there could be significant losses.

13.6 Using the profit test to calculate the premium

Setting a premium using the profit test can be achieved by finding the minimum premium that satisfies the insurer's required profit measure.

For example, suppose the insurer requires a profit margin of 5% for the 10-year term insurance from Section 13.3, using the same 10% per year risk discount rate, and the same basis for the profit test and reserves as before. With the premium tested, $P = \$1500$, the profit margin is only 0.77%. Increasing the premium to \$1575.21 gives a NPV of \$508.50 at the 10% per year risk discount rate, and an EPV of premiums of \$10 170.03, which gives the profit margin required.

The revised profit signature is

$$(-715.0, 197.7, 201.5, 203.9, 204.9, 204.5, 202.6, 199.4, 194.6, 188.4, 180.8),$$

which gives a DPP of five years, at 10% risk discount rate, and an IRR of 25.0% per year.

In this example, we can solve the equation for the unknown premium, given a profit margin, because the equation is a linear function of the premium. This will not always be the case, but numerical methods, or appropriate software such as Solver in Excel, usually work well.

13.7 Using the profit test to calculate reserves

In the first part of the example in Section 13.3, we saw that if the insurer holds no reserves, negative surplus emerges in later years of the contract, which is unacceptable under risk management or accounting principles. In the second part of the example, it was assumed that the insurer held net premium reserves, which resulted in positive emerging surplus in each year after the initial outgo from acquisition expenses. This result is acceptable, but using capital to support the liabilities is expensive. The NPV of the emerging profit for the 10-year term insurance example is \$270.39 without reserves, compared with \$74.13 with the net premium reserves from Table 13.2. This means that, even though term insurance is not very demanding on capital (as reserves are relatively small), the NPV without reserves is more than three times the NPV when reserves are taken into account.

Because holding capital reduces profitability, the insurer will not want to hold more than necessary. The objective of the capital is to avoid negative surpluses in later contract years. We can use the profit test to determine the minimum reserve that would be required at each year end to eliminate negative surpluses emerging in any year, after the initial outgo.

We demonstrate this process using the same 10-year term insurance example. We work backwards from the final contract year; for each year, we calculate the reserve required at the start of the year to match exactly the expected outgo in that year with no excess surplus emerging.

Consider the final year of the term insurance contract, $t = 10$. At the start of the year the insurer receives \$1500 premium, of which 3.5% is immediately spent on renewal expenses, leaving \$1447.50. This sum, together with the reserve, and with interest earned on the premium plus reserve less expenses (at 5.5% according to the profit test basis), must be exactly enough to meet the expected year end outgo of \$1900.

Suppose the reserve at the start of the year (i.e. at time $t = 9$) which exactly eliminates a negative emerging cash flow in the final year is denoted ${}_9V^Z$. Then we have

$$1.055\ \left({}_9V^Z + 1447.5\right) = q_{69}\, S = 1900 \quad \Rightarrow \quad {}_9V^Z = 353.45.$$

In other words, putting ${}_9V = 353.45$ into the profit test, in place of the net premium policy value of \$475.45, generates a value of $\mathrm{Pr}_{10} = 0$. Now, this is smaller than the value of Pr_{10} in Table 13.3, so it might look as if this is not going to help, but what is actually happening is that surplus will emerge sooner, which should increase the profitability.

We then move back to the ninth policy year. Now the reserve at the start of the year, together with the premium, net of expenses, and with the interest income, must be sufficient to meet the expected cost of death benefit for a contract in force at the start of the year ($EDB_9 = 1800$) and also to support the cost of carrying forward the final year reserve (${}_9V^Z = \$353.45$). Hence, the reserve equation for the minimum reserve at time $t = 8$ is

$$1.055\ \left({}_8V^Z + 0.965P\right) = q_{68}\, S + p_{68}\, {}_9V^Z$$

$$\Rightarrow {}_8V^Z = (1800 + 0.982 \times 353.45)/1.055 - 0.965P = 587.65.$$

Table 13.6 *Zeroized Reserves.*

t	${}_tV^Z$	t	${}_tV^Z$
9	353.45	5	658.32
8	587.65	4	494.78
7	711.42	3	247.62
6	732.63	2	−78.17

Table 13.7 *Emerging profit after zeroization.*

t	Pr_t	Π_t	NPV(t)
0	−700.00	−700.00	−700.00
1	527.11	527.11	−220.81
2	427.11	422.84	128.65
3	82.47	80.74	189.31
4	0.00	0.00	189.31
5	0.00	0.00	189.31
6	0.00	0.00	189.31
7	0.00	0.00	189.31
8	0.00	0.00	189.31
9	0.00	0.00	189.31
10	0.00	0.00	189.31

Continuing back, at time $t = 7$, we need a reserve of ${}_7V^Z$ where

$$1.055\ \left({}_7V^Z + 0.965P\right) = q_{67}\, S + p_{67}\, {}_8V^Z$$
$$\Rightarrow {}_7V^Z = (1700 + 0.983 \times 587.65)\,/1.055 - 0.965P = 711.42.$$

Continuing in this way we obtain the values in Table 13.6.

We see that ${}_2V^Z$ is negative, but reserves cannot be negative. Policy values can be negative, as expected values, but the capital held for future liabilities cannot be negative (see Section 7.7 for more discussion of this). Rather than allow a negative reserve, we set ${}_2V^Z = 0$. Repeating the process for $t = 1$ and $t = 0$ generates negative values in both cases, so we set both ${}_0V^Z$ and ${}_1V^Z$ equal to 0.

Now we re-do the profit test to see the impact of using these minimum reserves on the profit signature and the NPV. The results are shown in Table 13.7, using a 10% per year risk discount rate for the partial NPVs.

We have set the reserve to be exactly sufficient, together with the premium and interest income, to meet the projected outgo, leaving emerging surplus of zero, for the years from $t = 4$ onwards. This process for determining reserves is called **zeroization**, and the resulting reserves are called **zeroized** reserves. By comparing the partial NPVs using the higher reserves (from Table 13.5) with the zeroized reserves, we have a higher ultimate NPV using the zeroized reserves – increased from \$74.13 per policy to \$189.31. We also see a faster emergence of surplus, with a DPP of two years, down from nine years with the higher reserves.

Holding less capital increases the NPV here, because, as is typical, the interest assumed to be earned on the capital, at 5.5% per year, is less than the risk discount rate, at 10% per year. The risk discount rate indicates the return required on the equity invested. Within the profit test, assets are earning only 5.5% per year, but the high risk discount rate means that (loosely) the capital required for the contract needs to earn 10% per year. If less capital is required, the cost of that capital is lower, allowing more profit in the form of NPV.

We may generalize the algorithm in the example, to develop an expression for the zeroized reserves in principle. Using the assumptions and notation of Section 13.4 above, and given the zeroized reserve at t, ${}_tV^Z$, then the zeroized reserve at time $t-1 \geq 0$ is

$${}_{t-1}V^Z = \max\left(\left(EDB_t + ESB_t + EEB_t + E_tV^Z\right)/(1+i_t) - (P_t - E_t),\, 0\right).$$

So, given that at the maturity of the contract, we can assume ${}_nV^Z = 0$, it is possible to work backwards through the cash flows to determine the schedule of zeroized reserves for any policy.

13.8 Profit testing for participating insurance

Participating, or with-profit insurance was introduced in Section 1.3.3. In this section we show how a profit test can be used to determine distributable surplus for participating whole life policies. There are no new principles involved.

A traditional participating policy would have level premiums and benefits, similar to a non-participating (non-par or without profit) contract. The premium would be set on a fairly conservative basis, meaning that it is designed to be more than adequate to pay the fixed benefits. As surpluses emerge, a portion is returned to the policyholders, and the rest is retained by the company. The policyholders' share of profits can be distributed in different ways, including cash payments, or through increasing the sum insured.

Example 13.1 A life aged 60 purchases a participating whole life contract. The sum insured is \$100 000, payable at the end of the year of death. Premiums of \$2 300 are payable annually in advance.

Reserves are calculated using net premium policy values, modified using the full preliminary term approach, assuming an interest rate of 5% per year.

Cash values are \$0 for the first four years, 10% of the year end reserve for surrenders in the fifth year, 20% in the sixth year, 30% in the seventh year, and

Table 13.8 *Profit vector calculation for the participating whole life policy, with cash dividends; Example 13.1*

t (1)	${}_{t-1}V$ (2)	$P - E_t$ (3)	I_t (4)	$EDB_{t\text{-}}$ (5)	$ECV_{t\text{-}}$ (6)	E_tV (7)	$\Pr_{t\text{-}}$ (8)	Div_t (9)	$\Pr_t$ (10)
0	–	−1 800	–	–	–	–	−1 800	0	−1 800
1	0	2 200	132	340	0	0	1 992	0	1 992
2	0	2 200	132	379	0	1 699	254	228	25
3	1 795	2 200	240	423	0	3 448	364	327	36
4	3 645	2 200	351	473	0	5 246	477	429	48
5	5 548	2 200	465	529	37	7 091	556	500	56
6	7 504	2 200	582	591	95	8 983	618	556	62
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
28	58 149	2 200	3 621	7 224	3 780	51 800	1 166	1 049	120
29	60 361	2 200	3 754	8 080	3 879	53 157	1 199	1 079	120
30	62 518	2 200	3 883	9 033	52 901	0	6 667	6 001	667

continue increasing at the same rate to 90% of the year end reserve in the 13th year. All policies surrendering after the 13th year receive a surrender benefit of 90% of the year end reserve. In addition, all surrendering policyholders receive the cash dividend due for their final contract year.

Construct the profit test and determine the NPV and the profit margin of this contract to the insurer, using the following assumptions.

- The survival model is the Standard Ultimate Survival Model.
- For the first nine years, 5% of the surviving in force policyholders are assumed to surrender their contracts at each year end; from the 10th to the 29th years, 7.5% of survivors surrender at each year end. All surviving policyholders are assumed to surrender at the end of the 30th year.
- Premiums and reserves earn an investment return of 6% per year.
- 90% of surplus emerging each year from the second policy anniversary onwards is distributed to the policyholders as a cash dividend. No dividend is payable in the first policy year. No dividend is declared if the surplus emerging is negative.
- Initial, pre-contract expenses are \$1 800, assumed incurred at time $t = 0$.
- Premium expenses of \$100 are incurred with each premium payment, including the first.
- The risk discount rate for determining the NPV is 10% per year.

Solution 13.1 We show the first and last few rows of the profit test in Table 13.8. Details of the column calculations are given below. Some numbers in the table are rounded for presentation.

Column (1) t denotes the tth policy year, from time $t-1$ to time t, except that $t=0$ denotes time 0 when pre-contract expenses are incurred.

Column (2) gives the reserve brought forward each year. The full preliminary term policy value has ${}_0V = {}_1V = 0$, and for $t = 2, 3, \ldots$ is calculated as the net premium policy value for a contract issued one year later (in this case to a life aged 61). So, letting $S = 100\,000$:

$$ {}_tV = SA_{60+t} - P^*\ddot{a}_{60+t} \quad \text{where } P^* = \frac{SA_{61}}{\ddot{a}_{61}} = 2\,064.47. $$

All annuity and insurance functions are calculated at 5% per year, using the Standard Ultimate Survival Model. Hence, for example

$$ {}_4V = SA_{64} - P^*\ddot{a}_{64} = 0.34113\,S - 13.8363\,P^* = 5\,548. $$

Column (3) shows $P - E_t$, the premium received less expenses incurred at the start of the year.

Column (4) shows I_t, the interest on funds invested in the tth year. The assumed return is 6% per year, so, for example,

$$ I_4 = 0.06\ ({}_3V + P - E_4) = 0.06\,(3\,644.88 + 2\,300 - 100) = 350.69. $$

Column (5) shows the expected death benefit at time t, denoted $EDB_{t\text{-}}$, for a policy in force at the start of the year, i.e. at time $t-1$. We use the subscript t- because the expected death benefit does not include the dividend payable in the tth year, which (according to the terms of the example) would be added to the sum insured at the year end payment. So the actual death benefit paid would be $S + Div_t$, but the cost of the additional payment of dividend is included in Column (9). We have

$$ EDB_{t\text{-}} = S \times p^{0d}_{60+t-1}, $$

where p^{0d}_{x+t-1} is the probability of a death benefit claim in the tth year, given that the policy is in force at the start of the year. In this case, this is equal to the mortality rate q_{x+t-1} under the Standard Ultimate Survival Model, as all withdrawals are assumed to take place at the year end.

Column (6) shows the expected cash value payment at time t, denoted $ECV_{t\text{-}}$, for a policy in force at the start of the year. Again, this excludes the dividend paid to surrendering policyholders, the cost of which is included in Column (9). The probability of surrender is

$$ p^{0w}_{x+t-1} = (1 - p^{0d}_{x+t-1})\, q^{*w}_t, $$

where p^{0d}_{x+t-1} is as above, and q^{*w}_t is the probability of surrender for a policy that is in force at the end of the tth year – that is, $q^{*w}_t = 0.05$ for $t = 1, 2, \ldots, 9$, $q^{*w}_t = 0.075$ for $t = 10, 11, \ldots, 29$ and $q^{*w}_{30} = 1$.

The cash value is, say, $h_t\,{}_tV$ for surrenders at time t, where

$$h_t = 0 \text{ for } 1, 2, 3, 4;$$
$$h_5 = 0.1, h_6 = 0.2, \ldots, h_{12} = 0.8;$$
$$h_t = 0.9 \text{ for } t \geq 13.$$

The expected cash value is the product of the probability of surrender and the cash value paid on surrender under the profit test assumptions, so, for example,

$$ECV_{5\text{-}} = p_{64}^{0w} \times 0.1 \times {}_5V = (0.99471 \times 0.05) \times 0.1 \times 7\,504.02 = 37.32$$

and

$$ECV_{30\text{-}} = p_{89}^{0w} \times 0.9 \times {}_{30}V = 0.90967 \times 0.9 \times 64\,615 = 52\,901.$$

Note that policyholders who surrender at the year end will be eligible for a share of the profit emerging during the year.

Column (7) shows the expected cost of the reserve carried forward, denoted E_tV, for policies remaining in force at the year end, given the policy was in force at the start of the year. For example, the probability that a policy is in force at time 5 given that it is in force at time 4 is

$$p_{64}^{00} = 1 - p_{64}^{0d} - p_{64}^{0w} = 0.94498$$

so that

$$E_5V = p_{64}^{00}\,{}_5V = 0.94498 \times 7\,504.02 = 7\,091.12.$$

Column (8) shows $\Pr_{t\text{-}}$, the profit emerging at time t for a policy in force at time $t-1$, before sharing the profits between the insurer and the policyholder. Hence, for example,

$$\Pr_{5\text{-}} = {}_4V + P - E_5 + I_5 - EDB_5 - ECV_5 - E_5V = 555.68.$$

Column (9) shows Div_t, the share of profits (or dividend) distributed to policyholders, per policy in force at time $t-1$, which is 90% of the pre-dividend profit, or 0 if greater. Hence, for example,

$$Div_5 = 0.90 \times 555.68 = 500.11.$$

Column (10) shows $\Pr_t$, the insurer's surplus emerging at time t per policy in force at time $t-1$, which is the balance of surplus after the policyholder's dividend. Hence, for example,

$$\Pr_5 = \Pr_{5\text{-}} - Div_5 = 55.57.$$

We calculate the NPV and profit margin from the profit vector, Pr, following the usual procedure, giving a NPV of \$303.79, and a profit margin of 1.96%. □

Where the profits of a participating insurance policy are paid as a reversionary bonus, that is, as an addition to the sum insured, we proceed exactly as above, but treat the dividend each year as a single premium for the additional insurance. That is, if S_t is the sum insured in the tth year, and Div_t is the (hypothetical) cash dividend payable, then, for a whole life insurance with sum insured payable at the end of the year of death, the bonus would be $B_t = Div_t/A_{x+t}$, where the insurance function would typically be calculated using the reserve assumption. For continuing policyholders, the sum insured in the $(t+1)$th year would be $S_{t+1} = S_t + B_t$, and the reserve carried forward would include the additional reserve for the bonus, i.e. $B_t\,A_{x+t}$.

13.9 Profit testing for multiple state-dependent insurance

We need to adapt the profit testing methodology for insurance where in-force policies may be in different states of a multiple state model.

In Chapter 8 we saw how policy values for these contracts are state-dependent, where ${}_tV^{(j)}$ denotes the policy value at time t for a policy in State j at that time. Similarly, when we project the cash flows for the contracts, we create different, state-dependent profit vectors for each in-force state, and we can then use transition probabilities to determine the overall profit signature for the contract.

What this means is that for each in-force state j, and for each time period, we generate a profit vector of elements, $\mathrm{Pr}_t^{(j)}$ representing the expected surplus at time t per policy in State j at time $t-1$. Generally, all policies start in State 0, so we calculate $\mathrm{Pr}_t^{(0)}$ for all t, including $t=0$ and $t=1$; for other states $j \neq 0$, we need $\mathrm{Pr}_t^{(j)}$ for $t \geq 2$.

The profit signature entry for the tth year is the unconditional expected surplus, so we combine all the profit vectors using the appropriate probabilities, to get

$$\Pi_t = \sum_j {}_{t-1}p_x^{0j}\,\mathrm{Pr}_t^{(j)}\ .$$

In this section we illustrate the process through an example of a partially accelerated critical illness (CI) and term insurance policy. The policy is in force in the healthy state (State 0) and also after CI diagnosis (State 1), and expires when the policyholder dies.

Table 13.9 *Reserves for Example 13.2.*

t	${}_tV^{(0)}$	${}_tV^{(1)}$	t	${}_tV^{(0)}$	${}_tV^{(1)}$
0	0	0	5	2000	40 000
1	700	43 000	6	1600	38 000
2	1200	43 000	7	1200	34 000
3	1600	42 000	8	1000	27 000
4	2000	42 000	9	500	17 000

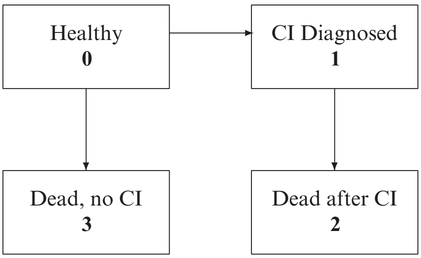


Figure 13.1 Multiple state model for Example 13.2.

Example 13.2 A 10-year partially accelerated CI and term insurance policy is issued to (x). The multiple state model used to analyse the policy is shown in Figure 13.1.

The benefits payable under the contract are

- \$50 000 at the end of the year if (x) is diagnosed with a CI during the year, and survives to the year end.
- \$100 000 at the end of the year if (x) dies during the year, having been healthy at the start of the year.
- \$50 000 at the end of the year if (x) dies during the year, having been diagnosed with a CI before the start of the year.

Premiums of \$2500 are payable annually in advance, whilst in the healthy state.

Reserves at each year end are conditional on whether the policyholder is in State 0 or State 1, and are given in Table 13.9.

One-year transition probabilities for the model are, for $t = 0, 1, 2, \dots, 9$,

$$p^{01}_{x+t} = 0.01, \qquad p^{02}_{x+t} = 0.005 + 0.001t,$$

$$p^{03}_{x+t} = 0.002 + 0.001t, \qquad p^{12}_{x+t} = 0.35.$$

Other profit test assumptions:

Acquisition expenses:	\$250
Premium expenses:	5% of each premium
Renewal expenses in State 1:	\$25

Interest on investments: 6% per year
Risk discount rate: 12% per year

(a) Calculate the profit vector conditional on being in State 0 at the start of the policy year, $\mathrm{Pr}^{(0)}$, and the profit vector conditional on being in State 1 at the start of the policy year, $\mathrm{Pr}^{(1)}$.
(b) Comment on the two profit vectors. Do you see any problem with the conditional emerging cash flows?
(c) Calculate the profit signature and partial net present value vector for a new policy. Does the policy meet a requirement that the IRR exceeds 12%?
(d) Calculate the profit margin for the policy.

Solution 13.2 (a) The profit test table for $\mathrm{Pr}^{(0)}$ is given in Table 13.10, where numbers (other than the elements of $\mathrm{Pr}^{(0)}$) are rounded for presentation. We give a general description of principles here, followed by a more detailed derivation of each column in the description below.

In this table, each row is calculated assuming that the policy is in State 0 at the start of the year. The policy could move to State 1 by the year end, in which case there will be a cash flow corresponding to the CI diagnosis benefit, and it will also be necessary to carry forward a reserve to the following year, which is appropriate for a policy in State 1 at that time. The policy could move to State 2 by the year end, resulting in a benefit payment of \$100 000, and no further cash flows. The policy could move to State 3 by the year end, with the same outcome as for State 2, and the policy could be still in State 0 at the year end, in which case there must be a reserve carried forward for the following year, appropriate for a policy in State 0.

Table 13.10 *Example 13.2 profit vector calculation conditional on the policy being in State 0 at the start of each year.*

t (1)	${}_{t-1}V^{(0)}$ (2)	$P_t - E_t$ (3)	I_t (4)	EB_t^{01} (5)	EB_t^{02} (6)	EB_t^{03} (7)	E_tV^{01} (8)	E_tV^{00} (9)	$\mathrm{Pr}^{(0)}$ (10)
0	0	−250	0					0	−250.00
1	0	2375	143	500	500	200	430	688	199.40
2	700	2375	185	500	600	300	430	1177	252.30
3	1200	2375	215	500	700	400	420	1566	203.10
4	1600	2375	239	500	800	500	420	1954	39.50
5	2000	2375	263	500	900	600	400	1950	287.50
6	2000	2375	263	500	1000	700	380	1557	500.70
7	1600	2375	239	500	1100	800	340	1165	308.30
8	1200	2375	215	500	1200	900	270	969	−49.50
9	1000	2375	203	500	1300	1000	170	483	124.00
10	500	2375	173	500	1400	1100	0	0	47.50

Column (1) shows the time at the year end, except for $t = 0$ which represents the timing of pre-contract cash flows.

Column (2) shows the reserve brought forward at the start of the tth year, for a policy in State 0 at that time. These numbers are taken from Table 13.9.

Column(3) shows the premiums minus expenses at the start of each year, and shows the acquisition expenses at $t = 0$.

Column (4) shows the interest earned on reserves and premiums net of expenses in each year, with interest at 6% per year. That is, for $t = 1, 2, \ldots, 10$,

$$I_t = 0.06({}_{t-1}V^{(0)} + P_t - E_t).$$

Column (5) shows the expected cost of paying the benefit for a life who moves from State 0 to State 1 in the time period from $t-1$ to t, and remains in State 1 at the year end. The benefit is \$50 000 and the transition probability is $p^{01}_{x+t-1} = 0.01$, so

$$EB^{01}_t = 50\,000\, p^{01}_{x+t-1} = 500.$$

Column (6) shows the expected cost of paying the benefit for a life who moves from State 0 to State 2 in the time period from $t-1$ to t. The benefit is \$100 000 and the transition probability is $p^{02}_{x+t-1} = 0.005 + 0.001(t-1)$, so

$$EB^{02}_t = 100\,000\, p^{02}_{x+t-1}.$$

Column (7) shows the expected cost of paying the benefit for a life who moves from State 0 to State 3 in the time period from $t-1$ to t. The benefit is \$100 000 and the transition probability is $p^{03}_{x+t-1} = 0.002 + 0.001(t-1)$, so

$$EB^{03}_t = 100\,000\, p^{03}_{x+t-1}.$$

Column (8) shows the expected cost of the reserve carried forward at the end of the tth year, for a life who is in State 1 at time t, given that the life was in State 0 at time $t-1$, for $t = 1, 2, \ldots, 9$. That is

$$E_tV^{01} = {}_tV^{(1)}\, p^{01}_{x+t-1}.$$

Column (9) shows the expected cost of the reserve carried forward from time t to time $t+1$ for a life who is in State 0 at time t, given that the life was in State 0 at time $t-1$, for $t = 1, 2, \ldots, 9$. That is

$$E_tV^{00} = {}_tV^{(0)}\, p^{00}_{x+t-1}.$$

Table 13.11 *Example 13.2 profit vector calculation conditional on the policy being in State 1 at the start of each year.*

t (1)	${}_{t-1}V^{(1)}$ (2)	E_t (3)	I_t (4)	EB_t^{12} (5)	$E\,{}_tV^{11}$ (6)	$\Pr^{(1)}$ (7)
2	43 000	25	2578.5	17 500	27 950	103.5
3	43 000	25	2578.5	17 500	27 300	753.5
4	42 000	25	2518.5	17 500	27 300	−306.5
5	42 000	25	2518.5	17 500	26 000	993.5
6	40 000	25	2398.5	17 500	24 700	173.5
7	38 000	25	2278.5	17 500	22 100	653.5
8	34 000	25	2038.5	17 500	17 550	963.5
9	27 000	25	1618.5	17 500	11 050	43.5
10	17 000	25	1018.5	17 500	0	493.5

Column (10) shows $\Pr_t^{(0)}$, which is the emerging profit at time t for a policy which is in State 0 at time $t-1$. So, for $t = 1, 2, \ldots, 10$,

$$\Pr_t^{(0)} = {}_{t-1}V^{(0)} + P_t - E_t + I_t - EB_t^{01} - EB_t^{02} - EB_t^{03} - E_t V^{01} - E_t V^{00}.$$

The calculations for $\Pr^{(1)}$, which is the profit vector for policies in State 1 at the start of each year, are given in Table 13.11, and a more detailed explanation is given below.

Column (1) shows the policy year. Note that there are no policies in State 1 at the start of the first year, so the profit vector calculation starts in the second policy year.

Column (2) shows the reserve (from Table 13.9) brought forward for a contract in State 1 at time $t-1$, i.e. ${}_{t-1}V^{(1)}$. This is the same amount whether the policy holder was in State 0 or State 1 in the previous year.

Column (3) shows the renewal expenses for the tth year for a policy in State 1.

Column (4) shows the interest income, for a policy in State 1 at time $t-1$,

$$I_t = 0.06\left({}_{t-1}V^{(1)} - E_t\right).$$

Column (5) shows the expected cost of benefits paid on death during the year $t-1$ to t, for a policy in State 1 at time $t-1$. The benefit payable is \$50 000, and the probability of payment is p_{x+t-1}^{12}, so

$$EB_t^{12} = 50\,000\, p_{x+t-1}^{12} = 17\,500.$$

Column (6) shows the expected cost of the reserve carried forward at the year end, for those policies still in State 1 at time t, so

$$E\,{}_tV^{11} = {}_tV^{(1)}\,p^{11}_{x+t-1}.$$

Column (7) shows the profit emerging at time t for a policy in State 1 at time $t-1$, so

$$\mathrm{Pr}^{(1)}_t = {}_{t-1}V^{(1)} - E_t + I_t - EB^{12}_t - E_tV^{11}.$$

(b) We note that there are some negative expected emerging cash flows, both for the State 1 conditional emerging profit and for the State 0 conditional emerging profit. For each policy in State 1 at time $t = 3$, the expected surplus emerging at the year end is $-\$306.50$. Similarly, for each policy in State 0 at the start of the eighth year, the expected surplus emerging at the year end is $-\$49.50$. A negative emerging profit indicates that inadequate capital is allocated to meet the outgo in those years. Thus, the reserves should be adjusted to avoid the negative values arising.

(c) In Table 13.12 we show the calculation of the profit signature and the partial NPV. More detailed explanations of each column follow.

Column (1) shows the time at the year end, except for $t = 0$ which represents the timing of pre-contract cash flows.

Column (2) shows the State 0 survival probability. Since there are no return transitions to State 0 in this model, we have ${}_0p^{00}_x = 1$, and we can calculate subsequent probabilities recursively as

$${}_tp^{00}_x = {}_{t-1}p^{00}_x\,p^{00}_{x+t-1}.$$

Column (3) shows the probability that a policy which is in State 0 at age x is in State 1 at age $x+t$. We have ${}_0p^{01}_x = 0$, and we can use the

Table 13.12 *Profit signature and NPV function for Example 13.2*

t (1)	${}_tp^{00}_x$ (2)	${}_tp^{01}_x$ (3)	$\mathrm{Pr}^{(0)}$ (4)	$\mathrm{Pr}^{(1)}$ (5)	Π (6)	NPV (7)
0	1.00000	0.00000	−250.00	0.00	−250.00	−250.00
1	0.98300	0.01000	199.40	0.00	199.40	−71.96
2	0.96432	0.01633	252.30	103.50	249.05	126.57
3	0.94407	0.02026	203.10	753.50	208.16	274.74
4	0.92236	0.02261	39.50	−306.50	31.08	294.49
5	0.89930	0.02392	287.50	993.50	287.64	457.70
6	0.87502	0.02454	500.70	173.50	454.43	687.93
7	0.84964	0.02470	308.30	653.50	285.81	817.22
8	0.82330	0.02455	−49.50	963.50	−18.26	809.84
9	0.79614	0.02419	124.00	43.50	103.16	847.04
10	0.76827	0.02369	47.50	493.50	49.76	863.06

Chapman–Kolmogorov equations to calculate ${}_tp_x^{01}$ recursively, as

$$ {}_tp_x^{01} = {}_{t-1}p_x^{00}\; p_{x+t-1}^{01} \;+\; {}_{t-1}p_x^{01}\; p_{x+t-1}^{11}. $$

Columns (4) and (5) show the conditional profit vectors from part (a).

Column (6) is the profit signature vector. Π_t represents the expected profit emerging at time t for a policy issued (and therefore in State 0) at time 0.

Recall that $\mathrm{Pr}_t^{(0)}$ is the emerging profit at time t conditional on being in State 0 at time $t-1$, and $\mathrm{Pr}_t^{(1)}$ is the emerging profit at time t conditional on being in State 1 at time $t-1$, so the profit signature is

$$ \Pi_t = {}_{t-1}p_x^{00}\,\mathrm{Pr}_t^{(0)} \;+\; {}_{t-1}p_x^{01}\,\mathrm{Pr}_t^{(1)}. $$

Column (7) is the partial NPV, at a risk discount rate of $r = 12\%$. So

$$ \mathrm{NPV}(t) = \sum_{k=0}^{t} \Pi_k\, v_r^k. $$

We note that the final NPV, at 12%, is \$863.06, which is greater than 0, and also that there is only one sign change in the partial NPV, so the IRR is uniquely determined, and is greater than 12%.

(d) The profit margin is the NPV divided by the EPV of premiums. As premiums are payable only if the life is in State 0, the EPV of premiums is

$$ 2500 \sum_{t=0}^{9} {}_tp_x^{00}\, v_r^t = 14\,655.31, $$

which gives a profit margin of 5.89%.

□

13.10 Notes

For each of the policies considered in this chapter, benefits are payable at the end of a time period. However, in practice, benefits are usually payable on, or shortly after, the occurrence of a specified event. For example, for the term insurance policy considered in Section 13.3, the death benefit is payable at the end of the year of death. If, instead, the death benefit had been payable immediately on death, then we could allow for this in our profit test by assuming all deaths occurred in the middle of the year. Taking this approach, the expected death claims in Table 13.1 would all be adjusted by multiplying by a factor of $1.055^{1/2}$.

In practice, as we have mentioned, it would be normal to use monthly steps in a profit test, and the assumption that benefits are paid at the end of the month of claim is less artificial than the assumption of payment at the end of the year of death.

Throughout this chapter we have used deterministic assumptions for all the factors. By doing this we gain no insight into the effect of uncertainty on the results. In Chapter 15 we describe how we might use stochastic scenarios for emerging cost analysis for equity-linked contracts. Stochastic scenarios can also be used for traditional insurance.

13.11 Exercises

Shorter exercises

Exercise 13.1 A profit test of a 20-year term insurance issued to (40) is to be carried out on the following basis:

Survival model:	$q_{40+t} = 0.001 + 0.0001t$
Interest:	6% effective per year
Pre-contract expenses:	25% of the first premium
Renewal expenses:	1.5% of each premium after the first
Claim expenses:	\$60

The annual premium is \$270 and the sum insured, payable at the end of the year of death, is \$150 000. Calculate the emerging surplus at the end of the 10th policy year, per policy in force at the start of that year, given that the insurer holds reserves of \$300 per policy in force at the start of each year.

Exercise 13.2 A profit test of a 20-year endowment insurance issued to (45) is to be carried out on the following basis:

Survival model:	$q_{45+t} = 0.0015 + 0.0001t$
Interest:	5% effective per year
Pre-contract expenses:	20% of the first premium
Renewal expenses:	2.5% of each premium after the first
Claim expenses:	\$40

The annual premium is \$8400 and the sum insured, payable at the end of the year of death, or at maturity, is \$250 000. Calculate the emerging surplus for the following two cases:

(a) at the end of the 10th policy year, per policy in force at the start of that year, and
(b) at the end of the 20th policy year, per policy in force at the start of that year,

given that ${}_9V = 88\,129$, ${}_{10}V = 100\,001$ and ${}_{19}V = 232\,012$.

Exercise 13.3 A profit test of a deferred annuity issued to (45) is to be carried out on the following basis:

Survival model:	Standard Ultimate Survival Model
Interest:	5% effective per year
Pre-contract expenses:	20% of the first premium
Renewal expenses:	\$25 on each policy anniversary
Annuity payment expenses:	\$15 each time an annuity payment is made

The annuity is payable annually from age 65; the first annuity payment is \$50 000, and payments increase by 2% each year. The annual premium, payable throughout the deferred period, is \$26 100.

(a) Calculate the emerging surplus at the end of years 1 and 2, per policy in force at the start of each year, and hence calculate NPV(2) using a risk discount rate of 10% per year.

(b) Calculate the emerging surplus in the 30th policy year, per policy in force at the start of that year. Assume the annuity is payable at the start of each policy year.

You are given that ${}_1V = 26\,845$, ${}_2V = 54\,924$, ${}_{29}V = 768\,919$ and ${}_{30}V = 753\,464$.

Exercise 13.4 An insurer is profit testing a fully discrete whole life insurance issued to a select life aged 50, with sum insured \$1 000 000. The annual premium is \$12 000. Death benefits are assumed to be payable in the middle of the year of death.

The reserves at times $t = 9$ and $t = 10$ for policies in force are ${}_9V = 100\,800$ and ${}_{10}V = 115\,500$. Policyholders who surrender during the 10th year receive a cash value of \$85 000. Other profit testing assumptions are:

Renewal expenses:	7% of premiums
Interest:	6% per year on the insurer's funds
Mortality:	Standard Select Life Table
Surrenders:	5% of surviving policyholders at each year end

Calculate Π_{10}.

Exercise 13.5 A five-year term insurance policy with annual cash flows issued to a life (x) produces the profit vector

$$\text{Pr} = (-310,\ 436,\ 229,\ 94,\ -55, -217)',$$

where Pr_0 is the profit at time 0 and Pr_t $(t = 1, 2, \ldots, 5)$ is the profit at time t per policy in force at time $t-1$. This profit vector has been calculated without allowance for reserves.

The survival model used in the profit test is given by $p_{x+t} = 0.987 - 0.001t$, and the interest rate is 5% per year.

The insurer determines reserves by zeroization. Calculate the revised profit vector after allowance for reserves.

Longer exercises

Exercise 13.6 A five-year policy with annual cash flows issued to a life (x) produces the profit vector

$$\text{Pr} = (-360.98, 149.66, 14.75, 273.19, 388.04, 403.00)',$$

where Pr_0 is the profit at time 0 and Pr_t $(t = 1, 2, \ldots, 5)$ is the profit at time t per policy in force at time $t-1$.

The survival model used in the profit test is given by $q_{x+t} = 0.0085 + 0.0005t$.

(a) Calculate the profit signature for this policy.
(b) Calculate the NPV for this policy using a risk discount rate of 10% per year.
(c) Calculate the NPV for this policy using a risk discount rate of 15% per year.
(d) Comment briefly on the difference between your answers to parts (b) and (c).
(e) Calculate the IRR for this policy.

Exercise 13.7 An insurer issues a four-year term insurance contract to a life aged 60. The sum insured, \$100 000, is payable at the end of the year of death. The gross premium for the contract is \$1100 per year. The reserve at each year end is 30% of the gross premium.

The company uses the following assumptions to assess the profitability of the contract:

Survival model:	$q_{60} = 0.008$, $q_{61} = 0.009$, $q_{62} = 0.010$, $q_{63} = 0.012$
Interest:	8% effective per year
Pre-contract expense:	30% of the first gross premium
Renewal expenses:	2% of each gross premium after the first
Claim expenses:	\$60
Lapses:	None

(a) Calculate the profit vector for the contract.
(b) Calculate the profit signature for the contract.
(c) Calculate the net present value of the contract using a risk discount rate of 12% per year.
(d) Calculate the profit margin for the contract using a risk discount rate of 12% per year.

(e) Calculate the discounted payback period using a risk discount rate of 12% per year.
(f) Determine whether the internal rate of return for the contract exceeds 50% per year.
(g) If the insurer has a hurdle rate of 15% per year, is this contract satisfactory?

Exercise 13.8 An insurer issues a special three-year insurance contract to a life aged 60. The death benefit is \$10 000, and is payable at the end of the year of death. The gross premium for the contract is \$75 per year, payable annually in advance.

A benefit of \$120 is payable on survival to the maturity date.

Policyholders who surrender at the end of the first or second year receive a 50% refund of their premiums paid, without interest.

The company uses the following assumptions to analyse the emerging surplus of the contract.

Interest:	7% per year
Pre-contract expenses:	\$5 immediately before first premium
Renewal expenses:	\$2 incurred on each premium date, including the first
Mortality:	Standard Ultimate Life Table
Surrenders:	5% of policyholders in force at the end of the first and second years are assumed to surrender

Assume first that the insurer sets reserves of ${}_0V = 5$, ${}_1V = 40$, ${}_2V = 80$ for each policy in force.

(a) Calculate the profit vector for the contract.
(b) Calculate the profit signature for the contract.
(c) Calculate the NPV assuming a risk discount rate of 10% per year.
(d) The insurer is considering a different reserving method. The reserves would be set by zeroizing the emerging profits under the profit test assumptions.
Calculate ${}_tV^Z$ for $t = 0, 1, 2$.
(e) Calculate the revised NPV using the zeroized reserves.
(f) Explain why the NPV has increased.

Exercise 13.9 A life aged 40 holds a participating, paid-up whole life insurance contract. The sum insured is \$100 000, payable at the end of the year of death. The insurer distributes a cash dividend to holders of policies which are in force at the year end (after surrender and death exits). The cash dividend is determined using 80% of the emerging surplus at each year end, if the surplus is positive.

(a) Calculate the cash dividend projected for this policy at the end of the current policy year, assuming the policy is in force at the year end, and using the following additional assumptions and information.
   (i) The mortality probability for the year is 0.0004, and 8% of surviving policyholders surrender their policies at the year end.
   (ii) Reserves are held equal to the policy value calculated assuming mortality rates from the Standard Ultimate Survival Model, and an interest rate of 4%, ignoring expenses.
   (iii) Surrender values are held equal to the policy value calculated assuming mortality rates from the Standard Ultimate Survival Model, and an interest rate of 5%, ignoring expenses.
   (iv) Interest at 6% per year is earned on the insurer's assets.
   (v) Expenses of \$20 are incurred at the start of each year.

(b) Your colleague suggests that policyholders should not participate in profits arising from surrenders. Calculate the revised dividend ignoring surrender profits, and critique this approach.

Excel-based exercises

Exercise 13.10 A 10-year term insurance issued to a life aged 55 has sum insured \$200 000 payable immediately on death, and monthly premiums of \$100 payable throughout the term.

Initial, pre-contract expenses are \$500 plus 50% of the first monthly premium; renewal expenses are 5% of each monthly premium after the first. The insurer earns interest at 6% per year on all cash flows and assumes the policyholder is subject to the Standard Ultimate Survival Model.

Calculate the profit vector at monthly intervals for this policy, assuming deaths occur at the mid-point of each month.

Exercise 13.11 A life insurer issues a 20-year endowment insurance policy to a life aged 55. The sum insured is \$100 000, payable at the end of the year of death or on survival to age 75. Premiums are payable annually in advance for at most 10 years. The insurer assumes that initial expenses will be \$300, and renewal expenses, which are incurred at the beginning of the second and subsequent years in which a premium is payable, will be 2.5% of the gross premium. The funds invested for the policy are expected to earn interest at 7.5% per year. The insurer holds net premium reserves, using an interest rate of 6% per year. The Standard Ultimate Survival Model is used for the premium and the net premium reserve calculations.

The insurer sets premiums so that the profit margin on the contract is 15%, using a risk discount rate of 12% per year.

Calculate the gross annual premium.

Exercise 13.12 Repeat Exercise 13.11 assuming that the sum insured is payable immediately on death, premiums are payable monthly for at most 10 years and expenses are \$300 initially and then 2.5% of each monthly premium after the first.

Exercise 13.13 A life insurance company issues a special 10-year term insurance policy to two lives aged 50 at the issue date, in return for the payment of a single premium. The following benefits are payable under the contract.

- In the event of either of the lives dying within 10 years, a sum insured of \$100 000 is payable at the year end.
- In the event of the second death within 10 years, a further sum insured of \$200 000 is payable at the year end. (If both lives die within 10 years and in the same year, a total of \$300 000 is payable at the end of the year of death.)

The basis for the calculation of the premium and the reserves is as follows.

Survival model:	Assume the two lives are independent with respect to survival and the model for each follows the Standard Ultimate Survival Model
Interest:	4% per year
Expenses:	3% of the single premium at the start of each year that the contract is in force

(a) Calculate the single premium using the equivalence principle.
(b) Calculate the reserves on the premium basis assuming that
   (i) only one life is alive, and
   (ii) both lives are still alive.
(c) Using the premium and reserves calculated, determine the profit signature for the contract assuming:

Survival model:	As for the premium basis
Interest:	8% per year
Expenses:	1.5% of the premium at issue, increasing at 4% per year

Exercise 13.14 A life insurance company issues a reversionary annuity policy to a husband and wife, both of whom are aged exactly 60. The annuity commences at the end of the year of death of the wife and is payable subsequently while the husband is alive, for a maximum period of 20 years after the commencement date of the policy. The annuity is payable annually at \$10 000 per year. The premium for the policy is payable annually while the wife and husband are both alive and for a maximum of five years.

The basis for calculating the premium and reserves is as follows.

Survival model: Assume the two lives are independent with respect to survival

	and the model for each follows the Standard Ultimate Survival Model
Interest:	4% per year
Expenses:	Initial expense of \$300 and an expense of 2% of each annuity payment whenever an annuity payment is made

(a) Calculate the annual premium.
(b) Calculate the NPV for the policy assuming:

- a risk discount rate of 15% per year,
- expenses and the survival model are as in the premium basis, and
- interest is earned at 6% per year on cash flows.

Exercise 13.15 A life aged 60 purchases a deferred life annuity, with a five-year deferred period. At age 65 the annuity vests, with payments of \$20 000 per year at each year end, so that the first payment is on the 66th birthday. All payments are contingent on survival. The policy is purchased with a single premium.

If the policyholder dies before the first annuity payment, the insurer returns her gross premium, with interest of 5% per year, at the end of the year of her death.

(a) Calculate the single premium using the following premium basis:

Survival model:	Standard Ultimate Survival Model
Interest:	6% per year before vesting; 5% per year thereafter
Expenses:	\$275 at issue plus \$20 with each annuity payment

(b) Gross premium reserves are calculated using the premium basis. Calculate the year end reserves (**after** the annuity payment) for each year of the contract.
(c) The insurer conducts a profit test of the contract assuming the following basis:

Survival model:	Standard Ultimate Survival Model
Interest:	8% per year before vesting; 6% per year thereafter
Expenses:	\$275 at issue plus \$20 with each annuity payment

(i) Calculate the profit signature for the contract.
(ii) Calculate the profit margin for the contract using a risk discount rate of 10% per year.

Exercise 13.16 A special 10-year endowment insurance is issued to a healthy life aged 55. The benefits under the policy are

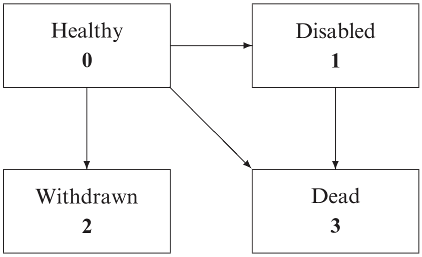


Figure 13.2 Multiple state model for Exercise 13.16.

- \$50 000 if at the end of a month the life is disabled, having been healthy at the start of the month,
- \$100 000 if at the end of a month the life is dead, having been healthy at the start of the month,
- \$50 000 if at the end of a month the life is dead, having been disabled at the start of the month,
- \$50 000 if the life survives as healthy to the end of the term.

On withdrawal at any time, a surrender value equal to 80% of the net premium policy value is payable, and level monthly premiums are payable throughout the term while the life is healthy.

The survival model used for profit testing is shown in Figure 13.2. The transition intensities μ_x^{01}, μ_x^{02}, μ_x^{03} and μ_x^{12} are constant for all ages x with values per year as follows:

$$\mu_x^{01} = 0.01, \quad \mu_x^{02} = 0.015, \quad \mu_x^{03} = 0.01, \quad \mu_x^{12} = 0.03.$$

Other elements of the profit testing basis are as follows.

- Interest: 7% per year.
- Expenses: 5% of each gross premium, including the first, together with an additional initial expense of \$1 000.
- The benefit on withdrawal is payable at the end of the month of withdrawal and is equal to 80% of the sum of the reserve held at the start of the month and the premium paid at the start of the month.
- Reserves are set equal to the net premium policy values.
- The gross premium and net premium policy values are calculated using the same survival model as for profit testing except that withdrawals are ignored, so that $\mu_x^{03} = 0$ for all x.
- The net premium policy values are calculated using an interest rate of 5% per year.

The monthly gross premium is calculated using the equivalence principle on the following basis:

- Interest: 5.25% per year.
- Expenses: 5% of each premium, including the first, together with an additional initial expense of \$1000.

(a) Calculate the monthly premium on the net premium policy value basis.
(b) Calculate the reserves at the start of each month for both healthy lives and for disabled lives.
(c) Calculate the monthly gross premium.
(d) Project the emerging surplus using the profit testing basis.
(e) Calculate the internal rate of return.
(f) Using a risk discount rate of 15% per year, calculate
   (i) the NPV,
   (ii) the profit margin (using the EPV of gross premiums),
   (iii) the NPV as a percentage of the acquisition costs, and
   (iv) the discounted payback period for the contract.

Answers to selected exercises

13.1 \$15.36
13.2 (a) \$773.86 (b) \$2172.10
13.3 (a) \$580.70, \$688.11, −\$4067.02 (b) \$3503.74
13.4 \$1185.08
13.5 $(-310, 436, 89.68, 0, 0, 0)'$
13.6 (a) $(-360.98,\ 149.66,\ 14.62,\ 268.43,\ 377.66,\ 388.29)'$
(b) \$487.88 (c) \$365.69 (e) 42.7%
13.7 (a) $(-330.00,\ 60.16,\ 293.07, 193.34, 319.92)'$
(b) $(-330.00,\ 60.16,\ 290.73, 190.07, 311.36)'$
(c) \$288.64 (d) 7.8% (e) 3 years (f) No (The IRR is 42%.)
(g) Yes
13.8 (a) $(-10,\ 9.74,\ 3.55,\ 1.88)'$
(b) $(-10,\ 9.74,\ 3.36,\ 1.69)'$
(c) 2.89 (d) 0, 35.13, 78.24 (e) 3.18
13.9 (a) \$700.32 (b) \$299.56
13.10 Selected values are $\mathrm{Pr}_{30} = 54.53$ and $\mathrm{Pr}_{84} = 28.75$, measuring time in months
13.11 \$4553.75
13.12 \$394.27 (per month)
13.13 (a) \$4 180.35
(b) Selected values are (i) ${}_4V = \$3126.04$, and (ii) ${}_4V = \$3146.06$

(c) Selected values are $\Pi_0 = -\$62.71$, $\Pi_5 = \$177.03$ and $\Pi_{10} = \$62.52$

13.14 (a) \$1832.79 (b) \$779.26

13.15 (a) \$192 805.84

(b) Selected values are ${}_4V = \$243\,148.51$ and ${}_{10}V = \$226\,245.94$

(c) (i) Selected values are $\Pi_4 = \$4\,538.90$ and $\Pi_{10} = \$2\,429.55$ (ii) 14.8%

13.16 (a) \$452.00

(b) Selected values are ${}_4V^{(0)}$=15 613.44 and ${}_4V^{(1)}$=7157.17, and ${}_8V^{(0)}$=36 761.39 and ${}_8V^{(1)}$=2769.93 (time in years)

(c) \$484.27

(d) Selected values are \$35.48 and \$11.43 at time 4 years, and \$72.27 and \$4.54 at time 8 years, for States 0 and 1 respectively

(e) 32.7% (f) (i) \$992.29 (ii) 3.84% (iii) 97% (iv) 5 years and 5 months

14
Universal life insurance

14.1 Summary

Universal life insurance is a form of whole life (or endowment) insurance, with some profit sharing incorporated in the design, and which also has more flexible payment schedules than traditional insurance.

We demonstrate how to use the profit testing techniques from Chapter 13 to analyse a universal life insurance contract, and we consider the impact of different types of death and surrender benefits.

14.2 Introduction

To create insurance policies that can compete with the flexibility and upside potential of mutual fund type investments, insurers have devised a range of new style, variable contracts, with greater flexibility, greater transparency, and with profit sharing integrated in the policy design. These modern contracts can be placed in two broad categories. The first, which we might call flexible insurance, is developed from the traditional insurance model, with added flexibility in premiums and benefits, and increased emphasis on the investment returns, compared with traditional insurance. The second category, which we call equity-linked insurance, or separate account insurance, uses the mutual fund investment as a starting point, and adds in elements of insurance, such as additional life insurance benefits, and guaranteed minimum payments. We discuss equity-linked insurance in subsequent chapters. In this chapter we describe the form of flexible insurance known as **Universal Life** insurance, which is a very popular product in North America, and we show how profit testing can be used to analyse the policy design.

14.3 Universal life insurance

14.3.1 Introduction

Universal life (UL) insurance is generally issued as a whole life contract, but with transparent account balances allowing policyholders to view the policy, to some extent, as a savings account with built-in life insurance. The policyholder may vary the amount and timing of premiums, within some constraints. The premium is deposited into a notional account, which is used to determine the death and survival benefits. The account is notional because assets are not actually segregated from the insurer's general funds (unlike, equity-linked insurance, which we discuss in Chapter 15).

The insurer shares the profits through the **credited interest rate** which is declared and applied by the insurer at regular intervals (typically monthly). The policy contract specifies a minimum value for the credited interest rate, regardless of the investment performance of the insurer's assets. The notional account, made up of the premiums and credited interest, is subject to monthly deductions (also notional); there is a charge for the cost of life insurance cover, and a separate charge to cover expenses. The **account value** or **account balance** is the balance of funds in the notional account. Note that the cost of insurance and expense charge deductions are set by the insurer, and need not be the best estimate of the anticipated expenses or insurance costs. In the profit test examples that follow in this section, the best estimate assumptions for incurred expenses and for the cost of death benefits are quite different to the charges set by the insurer for expenses and the cost of insurance. The account value represents the insurer's liability, analogous to the reserve under a traditional contract. The account value also represents the cash value for a surrendering policyholder, after an initial period (typically 7–10 years) when surrender charges are applied to ensure recovery of the acquisition costs.

In this section we consider the basic UL policy, which may be viewed as a variation of the traditional participating contract. We have simplified the terms of a standard UL policy to demonstrate the key principles. The most obvious simplification is that we have assumed annual cash flows where monthly would be more common. We have also assumed a fixed term for the UL contracts in the examples, even though UL contracts are generally whole life policies. However, it would be common for policyholders to use the contracts for fixed horizon planning, and the policy design assumes that most policies will be surrendered as the policyholder moves into retirement.

14.3.2 Key design features

Death Benefit

On the policyholder's death the total death benefit payable is the account value of the policy, plus an **additional death benefit** (ADB).

The ADB is required to be a significant proportion of the total death benefit, except at very advanced ages, to justify the policy being considered an insurance contract. The **corridor factor requirement** sets the minimum value for the ratio of the total death benefit (i.e. account value plus ADB) to the account value at death. In the USA the corridor factor is around 2.5 up to age 40, decreasing to 1.05 at age 90, and to 1.0 at age 95 and above.

There are two types of death benefit, Type A and Type B.

Type A offers a level total death benefit, which comprises the account value plus the additional death benefit. As the account value increases, the ADB decreases. However the ADB cannot decline to zero, except at very old ages, because of the corridor factor requirement. For a Type A UL policy, the level death benefit is the **face amount** of the policy.

Type B offers a level ADB. The amount paid on death would be the account value plus the level ADB selected by the policyholder, provided this satisfies the corridor factor requirement.

Premiums

Premiums may be subject to some minimum level and payment term, but otherwise are highly flexible.

Expense Charges

Expense charges are expressed as a percent of account value, or of premiums, and may also include a flat fee. It may be referred to as the **MER**, for Management Expense Rate, a term used more widely for mutual fund investment. The expense charge is deducted from the account value, at rates which are variable at the insurer's discretion, subject to a maximum which is specified in the original contract.

Credited Interest Rate

The interest rate applied to the policyholder's account balance is usually determined at the insurer's discretion, but may be based on published rates, such as yields on government bonds. A minimum guaranteed annual credited interest rate is specified in the policy document.

Cost of Insurance

This is the charge deducted from the policyholder's account balance to cover the cost of the additional death benefit cover. Usually, the CoI is calculated using an estimate (perhaps conservative) of the mortality rate for that period, which is known as the **CoI rate**. As the policyholder ages, the mortality charge (per \$1 of ADB) increases, so the CoI can be interpreted as the single premium for a one-year term insurance with sum insured equal to the ADB, assuming mortality equal to the CoI rate discounted to the start of the period of insurance. The rate of interest used to determine the CoI may differ from the credited rate.

Policyholders may be offered the option of a 'Level Cost of Insurance' charge. In this case, the death benefit cover is treated as a traditional term or whole life insurance for the purpose of determining the CoI, and the CoI deduction is constant through the term of the policy.

Surrender Charge

If the policyholder chooses to surrender the policy, the surrender value paid will be the policyholder's account balance reduced by a surrender charge. The main purpose of the surrender charge is to ensure that the insurer receives enough to pay its acquisition expenses. The total sum available to the policyholder on surrender is the account value minus the surrender charge (or zero if greater), and is referred to as the **cash value** of the contract at each duration.

No Lapse Guarantee

An additional feature of some policies is the no lapse guarantee, under which the death benefit cover continues even if the account value declines to zero, provided that the policyholder pays a pre-specified minimum premium at each premium date. This guarantee could apply if expense and mortality charges increase sufficiently to exceed the minimum premium. The policyholder's account value would support the balance until it is exhausted, at which time the no lapse guarantee would come into effect.

14.3.3 Projecting account values

The insurer must determine appropriate schedules for expense charges and cost of insurance charges to create a contract that is marketable and profitable. An important objective of the UL policy is transparency; the policyholder can see their account value growing, and can identify the expense and cost of insurance deductions. They will see the credited interest rate and will therefore have some measure of the success of the contract as an investment.

We will illustrate how account values accumulate with the simplifying assumption of annual cash flows. First, we introduce some notation.

AV_t denotes the policyholder's account value at time t.

EC_t denotes the expense charge deducted from the account value at the beginning of the tth year.

CoI_t denotes the Cost of Insurance deducted from the account value at the beginning of the tth year.

i_t^c denotes the credited interest rate applied to investments during the tth year.

P_t denotes the premium paid at the start of the tth year.

DB_t denotes the death benefit cover in the tth year.

CV_t denotes the cash value paid on surrender at the end of the tth year.

Then the fundamental equation of a UL policy is the following recursion:

$$(AV_{t-1} + P_t - EC_t - CoI_t)\,(1 + i_t^c) = AV_t. \tag{14.1}$$

It is interesting to consider the insurer's perspective here. The account value represents a reserve for the policy – it is a measure of the capital the insurer needs to hold in respect of the policy liabilities. The expense charge, CoI and credited interest rate are factors used in the development of the account value, but otherwise do not represent real cash flows. That is, it would make no difference to any of the contract cash flows if the insurer charged \$50 less in expense charges and \$50 more in CoI. In fact, EC_t, CoI_t and i_t^c can be changed, jointly, in innumerable ways, but if they generate the same account values it would make no difference to any of the cash flows of the policy. The only important numbers in the contract cash flows are the premiums, the account values, the death benefits and the cash surrender values. The only purpose of specifying EC_t, CoI_t and i_t^c is to derive the account values.

An analogy with traditional insurance is that we may assume a premium basis to determine premiums and a policy value basis to determine the reserves. Once the premiums and reserves are calculated, the assumptions in these two bases do not impact the policy cash flows. In fact, when we profit test a traditional product, we generally use different assumptions for mortality, expenses, surrender rates and interest than we use in either the premium basis or the reserve basis. Similarly, here, the expense charges, CoI and credited interest impact the cash flows only through the account values. They do not represent actual cash flows into or out of the insurer's funds.

14.3.4 Profit testing Universal life policies

Universal life policies are best analysed using profit testing. The process is similar to profit testing traditional insurance, even though the contracts appear different.

For the first step, before the profit test, we project the annual account values of the policy **assuming the policy remains in force to the final projection date**, just as, for traditional insurance, we have to calculate reserves at each time point, assuming the policy is still in force at that time, before we can profit test the policy.

To project the account values, we need an assumed schedule of premiums, P_t, a specification of the expense charges, EC_t, and we need to calculate the cost of insurance charges, CoI_t.

The additional death benefit in the tth year is the total death benefit minus the year end account value (similar to the sum at risk for a traditional policy), that is

$$ADB_t = DB_t - AV_t.$$

The CoI charge is a single premium for a one-year term insurance for a death benefit of ADB_t. The CoI basis will be specified. Let q^*_{x+t} denote the CoI mortality rate and i_q denote the CoI interest rate. Expenses are ignored. Then

$$CoI_t = q^*_{x+t-1}\, v_q\, ADB_t. \tag{14.2}$$

The CoI pays for the Additional Death Benefit; it is not based on the full Death Benefit. This is because the account value is available to fund the balance, as $DB_t - ADB_t = AV_t$.

For Type B policies the ADB is fixed. For Type A policies, the total death benefit is fixed (except for corridor factor adjustments), which means the ADB is a function of the account value, which makes the CoI calculation a little more complicated.

Once the AV_t values are determined through the account value projection, the profit test proceeds, very similarly to a traditional policy, except that AV_t takes the role of the reserve, and the sum insured (DB_t) is variable, depending on the account values.

In the following sections we demonstrate profit testing for some UL policies. The first is a Type B policy, which is the simpler case. The second is a Type A policy, which is the more common contract design in practice.

14.3.5 Universal life Type B profit test

Example 14.1 (Step 1: account value projection) A UL policy is sold to a 45-year-old man. The initial premium is \$2250 and the ADB is \$100 000. The policy charges are:

Cost of Insurance:	120% of the Standard Select Survival Model,
	$i_q = 5\%$ per year interest.
Expense Charges:	\$48 + 1% of premium at the start of each year.

Surrender penalties at each year end are the lesser of the full account value and the following surrender penalty schedule:

Year of surrender	1	2	3–4	5–7	8–10	> 10
Penalty	\$4500	\$4100	\$3500	\$2500	\$1200	\$0

Assume

(i) the policy remains in force for 20 years,
(ii) interest is credited to the account at 5% per year,
(iii) all cash flows occur at policy anniversaries, and
(iv) there is no corridor factor requirement for the policy.

Project the account value and the cash value at each year end for the 20-year projected term, given that the policyholder pays the full premium of \$2250 for six years, and then pays no further premiums.

Solution 14.1 The key formulae for UL account values are (14.1) and (14.2):

$$(AV_{t-1} + P_t - EC_t - CoI_t)\,(1 + i_t^c) = AV_t$$

and

$$CoI_t = q^*_{x+t-1}\, v_q\, ADB_t.$$

Applying these formulae give the projected account values in Table 14.1. Specifically, the columns in Table 14.1 are calculated as follows:

Column (1) denotes the term at the end of the policy year.
Column (2) is the tth premium, P_t, assumed paid at time $t-1$.

Table 14.1 *Projected account values for the Type B UL policy in Example 14.1, assuming level premiums for six years.*

Year t (1)	P_t (2)	EC_t (3)	CoI_t (4)	Credited interest (5)	AV_t (6)	CV_t (7)
1	2250	70.50	75.34	105.21	2 209.37	0.00
2	2250	70.50	91.13	214.89	4 512.63	412.63
3	2250	70.50	104.71	329.37	6 916.79	3416.79
4	2250	70.50	114.57	449.09	9 430.80	5 930.80
5	2250	70.50	125.66	574.23	12 058.87	9 558.87
6	2250	70.50	138.12	705.01	14 805.27	12 305.27
7	0	48.00	152.12	730.26	15 335.41	12 835.41
⋮	⋮	⋮	⋮	⋮	⋮	⋮
19	0	48.00	540.59	1 034.47	21 723.82	21 723.82
20	0	48.00	604.34	1 053.57	22 125.05	22 125.05

Column (3) is the tth year expense charge deduction, $EC_t = 48 + 0.01\,P_t$.

Column (4) is the CoI for the year from time $t-1$ to time t, assumed to be deducted at the start of the year. The mortality rate (or CoI rate) assumed is

$$q^*_{[45]+t-1} = 1.2\,q^d_{[45]+t-1}$$

where $q^d_{[x]+t}$ is taken from the Standard Select Survival Model. Multiply by the ADB, and discount from the year end payment date, to get the CoI as

$$CoI_t = 100\,000\,q^*_{[45]+t-1}\,v_{5\%}.$$

Note that the interest rate is specified in the CoI pricing assumptions – in this case it is the same as the crediting rate, but it may be different.

Column (5) is the credited interest at time t, assuming a 5% level crediting rate applied to the account value from the previous year, plus the premium, minus the expense loading and CoI.

Column (6) is the year end account value, from recursion formula (14.1) for AV_t.

Column (7) is the year end cash value, which is the account value minus the applicable surrender penalty, with a minimum value of \$0.

In more detail, the first two rows are calculated as follows:

First Year

AV_0:	0
P_1:	2250
EC_1:	$48 + 0.01 \times 2250 = 70.50$
CoI rate:	$q^*_{[45]} = 1.2 \times 0.000659$
CoI_1:	$100\,000 \times q^*_{[45]} \times v_{5\%} = 75.34$
Credited interest:	$0.05\,(2250 - 70.50 - 75.34) = 105.21$
AV_1:	$2250 - 70.50 - 75.34 + 105.21 = 2209.37$
CV_1:	$\max(2209.37 - 4500, 0) = 0$

Second Year

AV_1:	2209.37
P_2:	2250
EC_2:	$48 + 0.01 \times 2250 = 70.50$
CoI rate:	$q^*_{[45]+1} = 1.2 \times 0.000797$
CoI_1:	$100\,000 \times q^*_{[45]+1} \times v_{5\%} = 91.13$
Credited interest:	$0.05\,(2209.37 + 2250 - 70.50 - 91.13) = 214.89$
AV_2:	$2209.37 + 2250 - 70.50 - 91.13 + 214.89 = 4512.63$
CV_2:	$4512.63 - 4100 = 412.63$

□

Example 14.2 (Step 2: the profit test) For the scenario described below, calculate the profit signature, the discounted payback period and the net present value, using a hurdle interest rate of 10% per year effective, for the UL policy described in Example 14.1.

Assume

- Policies remain in force for a maximum of 20 years.
- Premiums of \$2 250 are payable for six years, and no premiums are payable thereafter.
- The insurer does not change the CoI rates or expense charges from the values given in Example 14.1.
- Interest is credited to the policyholder's account value in the tth year using a 2% interest spread, with a minimum credited interest rate of 2%. In other words, if the insurer earns more than 4%, the credited interest will be the earned interest rate less 2%. If the insurer earns less than 4%, the credited interest rate will be 2%.
- The ADB remains at \$100 000 throughout.
- Interest earned on all insurer's funds at 7% per year.
- Mortality experience is 100% of the Standard Select Survival Model.
- Incurred expenses are \$2000 at inception, \$45 plus 1% of premium at renewal, \$50 on surrender (even if no cash value is paid), \$100 on death.
- Surrenders occur at year ends. The surrender rate given in the following table is the proportion of in force policyholders surrendering at each year end.

Duration at year end	Surrender rate q^w_{45+t-1}
1	5%
2–5	2%
6–10	3%
11	10%
12–19	15%
20	100%.

- The insurer holds the full account value as reserve for this contract.

Solution 14.2 We use the account values from Example 14.1, as the credited interest rate of 5% used there corresponds to the profit test assumption for the credited rate (7% earned rate, minus the 2% spread, with a 2% minimum).

Note that the expense charge, CoI and credited interest rate used in the AV calculation are not needed in the profit test. The expenses, mortality and earned interest rate assumptions for the profit test are different to the expense charge, CoI mortality, and credited interest rate used in the AV projection, much as the profit test assumptions for a traditional contract usually differ from the premium basis.

The cash flows for the profit test in the tth year, $t \geq 1$, assuming the policy is in force at the start of the year, are as follows.

AV_{t-1} is the account value (reserve) brought forward.

P_t is the premium payable at the start of the tth year.

E_t is the incurred expenses at the start of the tth year.

I_t is the interest earned through the year on the invested funds.

EDB_t is the expected cost of death benefits paid at the end of the tth year. If $p^{0d}_{[45]+t-1}$ is the profit test mortality rate assumed, and DB_t is the total death benefit payable, then

$$DB_t = AV_t + ADB \quad \text{and} \quad EDB_t = p^{0d}_{[45]+t-1}\,(DB_t + 100),$$

where $ADB = \$100\,000$ is the fixed additional death benefit, and the \$100 allows for claims expenses.

ESB_t is the expected cost of surrender benefits at time t:

$$ESB_t = p^{0w}_{45+t-1}\,(CV_t + 50),$$

where $p^{0w}_{[45]+t-1} = (1 - p^{0d}_{[45]+t-1})\, q^{w}_{45+t-1}$ is the probability that the life survives the year, and then withdraws at the year end; CV_t is the cash value in the tth year, which is AV_t minus the surrender penalty at time t, and \$50 is the associated expense.

EAV_t is the expected cost of the account value carried forward at the year end for policies that continue in force. The probability that a policy which is in force at time $t-1$ remains in force at time t is

$$p^{00}_{[45]+t-1} = 1 - p^{0d}_{[45]+t-1} - p^{0w}_{[45]+t-1}$$

and

$$EAV_t = p^{00}_{[45]+t-1}\,AV_t.$$

The net surplus at the end of each year, assuming the policy is in force at the start of the year, is the profit vector entry

$$\mathrm{Pr}_t = AV_{t-1} + P_t - E_t + I_t - EDB_t - ESB_t - EAV_t.$$

The profit test table is presented (partially) in Table 14.2. As usual, the first row represents the pre-contract outgo, given in the example as \$2 000. The subsequent rows are determined using the formulae described above.

In Table 14.3 we show the profit signature and partial NPV calculations, exactly following the methodology of Chapter 12. To help understand the derivation of these tables, we show here the detailed calculations for the first two years' cash flows.

Table 14.2 *UL Type B policy from Example 14.2: calculating the profit vector.*

Year t	AV_{t-1}	P_t	E_t	I_t	EDB_t	ESB_t	EAV_t	Pr_t
0	0	0	2 000					−2 000
1	0	2 250	0	158	67	2	2 098	240
2	2 209	2 250	68	307	83	9	4 419	188
3	4 513	2 250	68	469	98	69	6 772	224
4	6 917	2 250	68	637	110	119	9 233	274
5	9 431	2 250	68	813	123	192	11 805	306
6	12 059	2 250	68	997	139	370	14 344	385
7	14 805	0	45	1 033	154	386	14 856	398
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
18	20 797	0	45	1 453	514	3 186	18 010	495
19	21 278	0	45	1 486	576	3 251	18 378	514
20	21 724	0	45	1 518	646	22 058	–	492

Table 14.3 *UL Type B policy from Example 14.2: profit signature, NPV and DPP at 10% per year risk discount rate.*

t	${}_{t-1}p^{00}_{[45]}$	Pr_t	Π_t	$NPV(t)$
0	1.00000	−2000.00	−2000.00	−2000.00
1	1.00000	240.04	240.04	−1781.78
2	0.94937	187.79	178.28	−1634.44
⋮	⋮	⋮	⋮	⋮
16	0.34500	457.56	157.86	−6.29
17	0.29225	475.98	139.11	21.23
18	0.24747	494.96	122.49	43.26
19	0.20946	514.47	107.76	60.88
20	0.17720	492.22	87.22	73.84

At time $t = 0$

Initial Expenses, E_0:	2000
Profit Vector, Pr_0:	−2000
Profit Signature, Π_0:	−2000
Partial NPV, NPV(0):	−2000

First Year

AV_0:	0
P_1:	2250
E_1:	0 (all accounted for in Pr_0)
I_1:	$0.07 \times 2250 = 157.50$

EDB_1	$0.000659 \times (100\,000 + 2209.37 + 100) = 67.44$
ESB_1:	$0.999341 \times 0.05 \times (0 + 50) = 2.50$
EAV_1:	$0.999341 \times 0.95 \times 2209.37 = 2097.52$
Pr_1:	$2250 + 157.50 - 67.44 - 2.50 - 2097.52 = 240.04$
Π_1:	240.04
NPV(1):	$-2000 + 240.04\, v_{10\%} = -1781.78$

Second Year

AV_1:	2209.37
P_2:	2250
E_2:	$45 + 0.01 \times 2250 = 67.50$
I_2:	$0.07 \times (2209.37 + 2250 - 67.50) = 307.43$
EDB_2:	$0.000797 \times (100\,000 + 4512.63 + 100) = 83.41$
ESB_2:	$0.999203 \times 0.02 \times (412.63 + 50) = 9.25$
EAV_2:	$0.999203 \times 0.98 \times 4512.63 = 4418.85$
Pr_2:	$2209.37 + 2250 - 67.50 + 307.43 - 83.41 - 9.25 - 4418.85$ $= 187.79$
Π_2:	$0.999341 \times 0.95 \times 187.79 = 178.28$
NPV(2):	$-1781.78 + 178.28\, v_{10\%}^2 = -1634.44$

From the final column of Table 14.3, we see that the NPV of the emerging profit, using the 10% per year risk discount rate, is \$73.84. The table also shows that the discounted payback period is 17 years. □

14.3.6 Universal life Type A profit test

The Type A contract is a little more complicated than Type B. The total death benefit is set at the face amount (*FA*), so the ADB is the excess of the face amount over the account value (*AV*); however, there is also, generally, a corridor factor requirement. The corridor factor is a guaranteed minimum ratio of the total death benefit to the account value. Given a corridor factor of γ, say, the total death benefit is the greater of *FA* and $\gamma\, AV$. The ADB is the difference between the total death benefit and the account value, that is,

$$ADB = \max\left(FA - AV_t,\ (\gamma_t - 1)AV_t\right).$$

So the ADB depends on the account value at the end of the year, and the account value at the end of the year depends on the ADB. How we solve this is to calculate the account value, and therefore the ADB, assuming first that $ADB = FA - AV_t$, and second that $ADB = (\gamma_t - 1)AV_t$, and choose the one that generates a higher ADB, or, equivalently, the lower account value.

Let ADB^f and AV_t^f denote the additional death benefit and end-year account value assuming that the ADB is the excess of the face amount over the account value, that is

$$ADB_t^f = FA - AV_t^f$$

and let $ADB_t{}^c$ and AV_t^c denote the additional death benefit and end-year account value based on the corridor factor, so that

$$ADB_t^c = (\gamma_t - 1)\, AV_t^c.$$

Then $ADB_t = \max(ADB_t^f, ADB_t^c)$ and because AV_t is a decreasing function of the ADB, we also have $AV_t = \min(AV_t^f,\, AV_t^c)$.

Using equations (14.1) and (14.2),

$$AV_t^f = \left(AV_{t-1} + P_t - EC_t - q^*_{x+t-1}\, v_q \left(FA - AV_t^f\right)\right)(1 + i_t^c)$$

$$\Rightarrow AV_t^f = \frac{\left(AV_{t-1} + P_t - EC_t - q^*_{x+t-1}\, v_q\, FA\right)(1 + i_t^c)}{1 - q^*_{x+t-1}\, v_q\, (1 + i_t^c)} \tag{14.3}$$

and similarly,

$$AV_t^c = \left(AV_{t-1} + P_t - EC_t - q^*_{x+t-1}\, v_q (\gamma_t - 1) AV_t^c\right)(1 + i_t^c)$$

$$\Rightarrow AV_t^c = \frac{\left(AV_{t-1} + P_t - EC_t\right)(1 + i_t^c)}{1 + q^*_{x+t-1}\, v_q (\gamma_t - 1)(1 + i_t^c)}. \tag{14.4}$$

We illustrate with a simple example. Suppose a Type A UL contract, issued some time ago to a life now aged 50, has face amount $FA = \$100\,000$. The assumed AV credited rate is $i_t^c = 5\%$, the CoI (mortality) rate for the year is $q^*_{50} = 0.004$, and the CoI interest rate is $i_q = 0\%$. The account value at the start of the year is $AV_{t-1} = \$50\,000$. The corridor factor for the year is $\gamma_t = 2.2$. There is no premium paid and no expense deduction from the account value in the year. From formulae (14.3) and (14.4) we obtain AV_t^f and AV_t^c respectively as

$$AV_t^f = \frac{(50\,000 - 0.004 \times 100\,000)\, 1.05}{1 - 0.004 \times 1.05} = 52\,300,$$

$$AV_t^c = \frac{50\,000 \times 1.05}{1 + 0.004 \times 1.2 \times 1.05} = 52\,237.$$

So the two possible values for the ADB are $ADB^f = \$100\,000 - AV_t^f = \$47\,700$ and $ADB^c = 1.2 \times \$52\,237 = \$62\,684$. Since the ADB is the larger of these two values, we have $ADB = \$62\,684$ and $AV_t = \$52\,237$, which means that the corridor factor has come into play.

Although we have derived formulae for AV_t^c and AV_t^f, the resulting equations are not the main point here. Minor changes to the standard UL premium and benefit conditions will result in different formulae. The important message here is how the formulae are derived – assuming either the face amount total death benefit, or the corridor factor total death benefit, solving the AV_t equation in both cases, and selecting whichever is smaller.

Example 14.3 (Type A UL profit test) Consider the following UL policy issued to a life aged 45:

- Face amount \$100 000.
- Type A death benefit with corridor factors (γ_t) applying to benefits payable in respect of deaths in the tth year, as follows:

t	1	2	3	4	5	6	7	8	9	10
γ_t	2.15	2.09	2.03	1.97	1.91	1.85	1.78	1.71	1.64	1.57

t	11	12	13	14	15	16	17	18	19	20
γ_t	1.50	1.46	1.42	1.38	1.34	1.30	1.28	1.26	1.24	1.24

- CoI based on: 120% of mortality rates from the Standard Select Survival Model, and 4% interest; the CoI is calculated assuming the fund earns 4% interest during the year.
- Expense charges: 20% of the first premium plus \$200, 3% of subsequent premiums.
- Initial premium: \$3500.
- Surrender penalties:

Year of surrender	1	2	3–4	5–7	≥ 8
Penalty	\$2 500	\$2 100	\$1 200	\$600	\$0

(a) Project the account and cash values for this policy assuming level premiums of \$3 500 are paid annually in advance, that the policyholder surrenders the contract after 20 years, and that the credited interest rate is 4% per year.

(b) Profit test the contract using the basis below. Use annual steps, and determine the NPV and DPP using a risk discount rate of 10% per year. Assume

- Level premiums of \$3 500 paid annually in advance.
- Insurer's funds earn 6% per year.
- Policyholders' accounts are credited at 4% per year.
- Surrender rates are as in Example 14.2 above. All surviving policyholders surrender after 20 years.

- Mortality follows the Standard Select Survival Model.
- Incurred expenses are:
  - pre-contract expenses of 60% of the premium due immediately before the issue date,
  - maintenance expenses of 2% of premium at each premium date including the first,
  - \$50 on surrender,
  - \$100 on death.
- The insurer holds reserves equal to the policyholder's account value.

Solution 14.3 (a) Following the same methodology as for the Type B policy, we first project the AV_t and CV_t values, assuming that the policy stays in force throughout the term. The results are shown in Table 14.4.

We describe the calculations for the first two years and the last year in more detail to clarify the table.

First Year

P_1: 3 500

EC_1: $200 + 20\% \times 3\,500 = 900$

$q^*_{[45]}$: 0.000791

From formulae (14.3) and (14.4), we obtain

$$AV_1^f = \frac{(3500 - 900 - 0.000791v(100\,000))\ 1.04}{1 - 0.000791} = 2626.97,$$

$$AV_1^c = \frac{(3500 - 900)\ 1.04}{1 + 0.000791(1.15)} = 2701.54.$$

So, taking the smaller value as AV_1,

$$AV_1 = 2626.97,\ DB_1 = FA = 100\,000,\ CV_1 = 2626.97 - 2500 = 126.97.$$

Table 14.4 *Type A UL account value and cash value projection for Example 14.3.*

t	AV_{t-1}	P_t	EC_t	AV_t	DB_t	CV_t
1	0	3 500	900	2 626.97	100 000.00	126.97
2	2 626.97	3 500	105	6 173.08	100 000.00	4 073.08
3	6 173.08	3 500	105	9 851.68	100 000.00	8 651.68
4	9 851.68	3 500	105	13 672.70	100 000.00	12 472.70
⋮	⋮	⋮	⋮	⋮	⋮	⋮
19	86 055.68	3 500	105	92 902.15	115 198.67	92 902.15
20	92 902.15	3 500	105	99 996.75	123 995.97	99 996.75

Second Year

P_2: 3 500

EC_2: $3\% \times 3\,500 = 105$

$q^*_{[45]+1}$: 0.000957

Again using formulae (14.3) and (14.4), we obtain

$$AV_2^f = \frac{(2626.97 + 3500 - 105 - 0.000957\, v\,(100\,000))\; 1.04}{1 - 0.000957} = 6173.08,$$

$$AV_2^c = \frac{(2626.97 + 3500 - 105)\; 1.04}{1 + 0.000957(1.09)} = 6256.33.$$

Taking the smaller value as AV_2,

$$AV_2 = 6173.08,\;\; DB_2 = FA = 100\,000,\;\; CV_2 = 6173.08 - 2100 = 4073.08.$$

Twentieth Year

AV_{19}: 92 902.15

P_{20}: 3 500

EC_{20}: 105

q^*_{64}: 0.006346

Again using formulae (14.3) and (14.4),

$$AV_{20}^f = \frac{(92\,902.15 + 3500 - 105 - (0.006346)\, v\,(100\,000))\; 1.04}{1 - 0.006346}$$
$$= 100\,149.99,$$

$$AV_{20}^c = \frac{(92\,902.15 + 3500 - 105)\; 1.04}{1 + 0.006346(0.24)} = 99\,996.75.$$

Proceeding as for the first and second years,

$$AV_{20} = 99\,996.75, \quad DB_{20} = \gamma_{20}\, AV_{20} = 123\,995.97,$$
$$CV_{20} = AV_{20} = 99\,996.75.$$

(b) The profit test results are presented in Tables 14.5 and 14.6. In Table 14.5, we derive the profit vector, and in Table 14.6 we show the profit signature and the emerging NPV using the 10% per year risk discount rate.

Given the Standard Select Survival Model mortality rate $q_{[45]+t-1}$, and q_t^w, the probability that a life aged $45 + t$ surrenders at the end of the tth year, given that the policy is still in force at that time, the probabilities for mortality, surrender and surviving in force in the tth year are

$$\text{Mortality: } p^{0d}_{[45]+t-1} = q_{[45]+t-1},$$

$$\text{Surrender: } p^{0w}_{[45]+t-1} = (1 - q_{[45]+t-1})\, q_t^w,$$

$$\text{Surviving: } p^{00}_{[45]+t-1} = 1 - p^{0d}_{[45]+t-1} - p^{0w}_{[45]+t-1}.$$

The profit vector is, as in the previous example,

$$\mathrm{Pr}_t = AV_{t-1} + P_t - E_t + I_t - EDB_t - ESB_t - EAV_t$$

where

$$EDB_t = p^{0d}_{[45]+t-1}(DB_t + 100),$$
$$ESB_t = p^{0w}_{[45]+t-1}(CV_t + 50),$$
$$EAV_t = p^{00}_{[45]+t-1}\,AV_t.$$

For a more detailed explanation, we show here the calculations for the cash flows in the final two years, represented by the final two rows in Table 14.5. Values for AV_t and CV_t are taken from Table 14.4. Values in Table 14.5 have been rounded to the nearest dollar for presentation only.

Table 14.5 *Emerging profit for Type A UL policy in Example 14.3.*

t	AV_{t-1}	P_t	E_t	I_t	EDB_t	ESB_t	EAV_t	Pr_t
0			2100					−2100
1	0	3500	70	206	66	9	2 494	1067
2	2 627	3500	70	363	80	82	6 045	213
3	6 173	3500	70	576	92	174	9 646	268
4	9 852	3500	70	797	100	250	13 386	342
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
19	86 056	3500	70	5369	545	13 877	78 593	1839
20	92 902	3500	70	5780	656	99 518	–	1938

Table 14.6 *Profit signature and emerging NPV at 10% per year risk discount rate for Type A UL policy in Example 14.3.*

t	Π_t	NPV_t
0	−2100.00	−2100.00
1	1066.99	−1130.01
2	202.59	−962.58
3	249.02	−775.49
4	311.49	−562.73
⋮	⋮	⋮
19	385.26	2159.49
20	343.44	2210.54

19th Year

AV_{18}: 86 055.68 (from Table 14.4)

P_{19}: 3500

E_{19}: $0.02 \times 3500 = 70$

I_{19}: $0.06 \times (86\,056 + 3500 - 70) = 5\,369.14$

EDB_{19}: $p_{63}^{0d} \times (DB_{19} + 100)$ (DB from Table 14.4)

$= 0.004730 \times (115\,199 + 100) = 545.38$

ESB_{19}: $p_{63}^{0w} \times (CV_{19} + 50)$ (CV from Table 14.4)

$= 0.995270 \times 0.15 \times 92\,952.15 = 13\,876.87$

EAV_{19}: $0.995270 \times 0.85 \times 92\,902.15 = 78\,593.30$

Pr_{19}: $86\,055.68 + 3\,500 - 70 + 5\,369.14$

$-545.38 - 13\,876.87 - 78\,593.30 = 1\,839.27$

20th Year

AV_{19}: 92 902.15 (from Table 14.4)

P_{20}: 3500

E_{20}: $0.02 \times 3500 = 70$

I_{20}: $0.06 \times (92\,902 + 3500 - 70) = 5\,779.93$

EDB_{20}: $p_{64}^{0d} \times (DB_{20} + 100)$ (DB from Table 14.4)

$= 0.005288 \times (123\,996 + 100) = 656.22$

ESB_{20}: $p_{64}^{0w} \times (CV_{20} + 50)$ (CV from Table 14.4)

$= 0.994712 \times 1 \times 100\,046.75 = 99\,517.70$

EAV_{20}: 0

Pr_{20}: $92\,902.15 + 3\,500 - 70 + 5\,779.93$

$-656.22 - 99\,517.70 = 1\,938.16$

From Table 14.6, we see that the NPV of the contract is \$2 210. The DPP is seven years.

□

14.3.7 No lapse guarantees

UL policies are often offered with ‘secondary guarantees’, a term that refers to a range of benefits additional to the basic contract terms, sometimes offered as optional riders. The most common secondary guarantee for a UL policy is the no lapse guarantee. With this benefit, once a specified number of premiums have been paid in full, the death benefit cover remains in place even if no further premiums are paid, and even if the account value is insufficient to support the CoI in any future year.

For a Type A policy, ignoring corridor factors, the value of the no lapse guarantee can be considered by analogy with a traditional paid-up whole life insurance, as discussed in Chapter 7.

Suppose the UL policy has been issued to (x), and has been in force for t years. The policyholder has the right to cease premiums and maintain their death benefit insurance. Suppose that the face value is S and the account value is AV_t. Assuming annual cash flows, the EPV of the death benefit is SA_{x+t}. If this is less than the account value, then the assets of the policy are sufficient to support the no lapse guarantee. If SA_{x+t} is greater than the account value, then the insurer must set aside additional reserves to cover the additional costs.

It follows that the reserve for the no lapse guarantee at time t can be set as

$${}_tV^{nlg} = \max(SA_{x+t} - AV_t,\ 0).$$

Note that the expense charges and CoI are not needed for this calculation.

The no lapse guarantee may have an expiry date – that is, the death benefit continues up to a specified age, say $x+n$, without further premiums. In this case the reserve would be calculated using a term insurance factor

$${}_tV^{nlg} = \max(S\,A^{\;1}_{x+t:\overline{n-t|}} - AV_t,\ 0).$$

In the UL profit tests above we have assumed that the insurer holds the full account value of the policy, but does not hold any additional reserve. The account value takes the role of the reserve in the profit test, with the account value brought forward entering the profit test as an item of income at the start of each time step, and the expected cost of the account value carried forward as an item of outgo at the end of each time step. When there are potential costs in excess of the account value, then there will be additional reserves brought forward and carried forward.

14.3.8 Comments on UL profit testing

As discussed in the previous section, we assume that UL reserves are the Account Values, together, if necessary, with additional reserves for no lapse guarantees or other ancillary benefits. Additional reserves will be required if the cost of insurance is set at a level amount.

If there are no secondary guarantees, it might be possible to hold a reserve less than the full account value, to allow for the reduced payouts on surrender, and perhaps to take advantage of future profits from the interest spread.

From a risk management perspective, allowing for the surrender penalty in advance by holding less than the account value is not ideal; surrenders are notoriously difficult to predict. History does not always provide a good model, as economic circumstances and variations in policy conditions have a significant impact on policyholder behaviour. In addition, surrenders are not as diversifiable as deaths; that is, the impact of the general economy on surrenders is a systematic risk, impacting the whole portfolio at the same time.

The worked examples in this chapter are simplified to provide a better illustration of the key features of a UL contract. In particular, we have not addressed the fact that the expense charge, CoI and credited interest are changeable at the discretion of the insurer. However, there will be maximum, guaranteed rates set out at issue for expense and CoI charges, and a minimum guaranteed credited interest rate. The profit test would be conducted using several assumptions for these charges, including the guaranteed rates. However, it may be unwise to set the reserves assuming the future charges and credited interest are at the guaranteed level. Although the insurer has the right to move charges up and interest down, it may be difficult, commercially, to do so unless other insurers are moving in the same direction. When there is so much discretion, both for the policyholder and the insurer, it would be usual to conduct a large number of profit tests with different scenarios to assess the full range of potential profits and losses.

14.4 Notes and further reading

We have assumed annual time steps for the examples in this chapter. As in Chapter 13, this is a simplification, applied to make the examples easier to follow. Typically, UL expense and CoI charges would be deducted monthly.

For UL insurance, the insurer has significant discretion about the crediting rate (for UL). It is common for the insurer to apply some smoothing, so that changes in crediting rates are not sudden or dramatic. The methods and impact of smoothing are beyond the scope of this text, but some information is available, for example, in Atkinson and Dallas (2000).

14.5 Exercises

Note: Several of the exercises are adapted from exam questions used by the Society of Actuaries (SOA) for their MLC examination. These questions are copyrighted to the Society of Actuaries, and are reproduced with permission.

The convention for the SOA questions is that the ‘CoI rate’ refers to the mortality rate used in the CoI calculation, and that the rate of interest for the CoI calculation is the same as the credited rate unless otherwise indicated.

Shorter exercises

Exercise 14.1 You are calculating asset shares for a portfolio of UL insurance policies with a death benefit of \$1000 on (x), payable at the end of the year of death.

You are given, for an individual policy in force throughout the fifth year:

- The account value at the end of year 4 is \$30.
- The asset share at the end of year 4 is \$20.

- During the fifth year:
  - A premium of \$20 is paid at the start of the year.
  - Annual cost of insurance charges of \$2 and annual expense charges of \$7 are deducted from the account value at the start of the year.
  - The insurer incurs expenses of \$2 at the start of the year.
  - The mortality rate for the universal life portfolio was 0.1%.
  - The withdrawal rate for the portfolio was 5%.
  - The credited interest rate was 6%.
  - The investment return experienced by the insurer was 8%.
- All withdrawals occur at the end of the policy year; the withdrawal benefit is the account value less a surrender charge of \$20.

Calculate the asset share at the end of year 5.

(*Copyright 2012. The Society of Actuaries, Schaumburg, Illinois. Reproduced with permission.*)

Exercise 14.2 For a UL insurance policy with a death benefit of \$50 000 plus the account value, you are given the following information.

(i)

Policy Year	Monthly Premium	% Premium Charge	CoI Rate per month	Monthly Expense Charge	Surrender Charge
1	300	W%	0.2%	10	500
2	300	15%	0.3%	10	125

(ii) The credited interest rate is $i^{(12)} = 0.054$.
(iii) The cash surrender value at the end of month 11 is \$1200.00.
(iv) The cash surrender value at the end of month 13 is \$1802.94.

Calculate W%, the percent of premium charge in policy year 1.

(*Copyright 2012. The Society of Actuaries, Schaumburg, Illinois. Reproduced with permission.*)

Exercise 14.3 You are given the following about a UL insurance policy on (60).

(i) The death benefit equals the account value plus \$200 000.
(ii)

Age x	Annual Premium	Annual CoI Rate per 1000	Annual Expense Charges
60	5000	5.40	100
61	5000	6.00	100

(iii) Interest is credited at 6% per year.
(iv) Surrender value equals 93% of account value during the first two years. Surrenders occur at the end of the policy year.
(v) Surrenders are 6% per year of those who survive.
(vi) Mortality rates are $q_{60} = 0.00340$ and $q_{61} = 0.00380$.
(vii) $i = 7\%$.

Calculate the present value at issue of the insurer's expected surrender benefits paid in the second year.

(*Copyright 2012. The Society of Actuaries, Schaumburg, Illinois. Reproduced with permission.*)

Exercise 14.4 A Type A UL policy was issued $t-1$ years ago to (x). The face amount of the policy is \$100 000. In the tth year, the corridor factor is 1.95; the CoI mortality rate is 0.005, the interest rate for the CoI is 4%, and the crediting rate is 5%. Expense charges of \$25 + 0.4% of the account value are deducted from the account value at the start of each year. The account value at the start of the year is \$49 500. No premium is paid.

Calculate the cost of insurance charge, the additional death benefit and the year end account value in the tth year.

Exercise 14.5 For a UL insurance policy with death benefit of \$100 000 issued to (40), you are given:

- The account value at the end of year 5 is \$2 029.
- A premium of \$200 is paid at the start of year 6.
- Expense charges in renewal years are \$40 per year plus 10% of premium.
- The cost of insurance charge for year 6 is \$400.
- Expense and cost of insurance charges are payable at the start of the year.
- Under a no lapse guarantee, after the premium at the start of year 6 is paid, the insurance is guaranteed to continue until the insured reaches age 65.
- If the expected present value of the guaranteed insurance coverage is greater than the account value, the company holds a reserve for the no lapse guarantee equal to the difference. The expected present value is based on the Standard Ultimate Survival Model at 5% interest and no expenses.

Calculate the reserve for the no lapse guarantee, immediately after the premium and charges have been accounted for at the start of year 6.

(*Copyright 2012. The Society of Actuaries, Schaumburg, Illinois. Reproduced with permission.*)

Exercise 14.6 For a Type A UL policy with a death benefit of \$500 000, you are given:

(i) The account value at time 5 is \$200 000.
(ii) Premiums of \$25 000 are payable annually at the beginning of each year.
(iii) Expense charges are 2% of premium.
(iv) The CoI rate per \$1000 in year 6 is \$30.
(v) $i_c = i_q = 0.05$.
(vi) The policy is subject to a corridor factor of 2.5.

Calculate the additional death benefit for this policy at time 6.

(*Copyright 2017. The Society of Actuaries, Schaumburg, Illinois. Reproduced with permission.*)

Exercise 14.7 For a Type B UL policy you are given:

- $AV_4 = 5\,000$.
- The additional death benefit is \$1 000 000.
- The CoI rate in year 5 is \$10 per \$1000 of insurance.
- Expense charges are \$50 per year plus 5% of premiums.
- $i_q = 0.045$.
- The policy does not have a no lapse guarantee.

Calculate the minimum premium to be paid at the beginning of year 5 so that the policy does not lapse before the next premium is paid at the beginning of year 6.

Longer exercises

Exercise 14.8 A life insurance company is profit testing a Type B UL policy with an additional death benefit of \$150 000, issued to lives aged 35. Some policy information is given in the following table, based on a credited rate of 4% per year.

Policy year	Premium	Expense charge	CoI Mortality	CoI Interest	Surrender charge
1	4800	960	0.00231	4%	2500
2	4800	240	0.00246	4%	1500
3	4800	240	0.00262	4%	0

The profit testing basis is

Mortality:	$q_{35+t} = 0.0015$ for $t = 0,\ 1,\ 2$
Surrender:	10% of in-force policies at the end of the first year
	20% of in-force policies at the end of the second year
	100% of in-force policies at the end of the third year
Interest:	8% on the insurer's funds
Pre-contract expenses:	\$800
Other expenses:	25% of premium in year 1, 7% in years 2 and 3;
	\$500 on death
Hurdle rate:	10% per year

(a) Calculate the projected account value at the end of each of the first three years, for a policy that remains in force.
(b) Calculate the profit vector.
(c) Calculate the NPV.
(d) Calculate the profit margin.

Excel-based exercises

Exercise 14.9 An insurer issues a Type B UL policy with a death benefit of \$50 000 plus the account value, to (50). Premiums of \$300 per month are payable at the start of each month.

Expense charges in the first year are 20% of premium plus \$20 each month. In the second year, the charges are reduced to 15% of premium plus \$10 each month.

The CoI is calculated assuming a mortality rate of ${}_{\frac{1}{12}}q_x = 0.002$ for $50 \leq x \leq 50\frac{11}{12}$, and ${}_{\frac{1}{12}}q_x = 0.003$ for $51 \leq x \leq 51\frac{11}{12}$, and interest of 4% per year effective.

The cash value is determined by deducting a surrender charge from the account value, with a minimum cash value of \$0. The surrender charge is \$500 for surrenders during the first year, and \$125 for surrenders in the second year.

You are given that the credited interest rate is 0.45% per month throughout the first two years, and that the cash value is \$860 at the end of the 11th month.

Calculate the cash value for the policy at the end of the 13th month.

Exercise 14.10 A life insurance company issues a four-year UL policy to (65). The main features of the contract are as follows.

Premiums: \$3 000 per year, payable yearly in advance.
Expense charges: 4% of premium is deducted at the start of the first year; \$100 plus 0.4% of the account value (before premium) is deducted at the start of each subsequent policy year.
CoI: \$25 is deducted from the account value at the start of each year.

Death benefit: greater of \$12 000 or 1.5 times the account value at the year end.

Maturity benefit: 100% of the account value.

The company uses the following assumptions in carrying out a profit test of this contract.

Interest rate: 4.5% per year in year 1, 5.5% per year in year 2, and 6.5% per year in years 3 and 4.

Credited interest: Earned rate minus 1%, with a 4% minimum.

Survival model: Standard Ultimate Survival Model.

Withdrawals: None.

Initial expenses: \$200 pre-contract expenses.

Renewal expenses: payable annually at each premium date, initial cost (with first premium) \$50, increasing with inflation of 2% per year.

Risk discount rate: 8% per year.

There are no reserves held other than the account value.

(a) Calculate the profit signature and NPV of a newly issued contract.
(b) Calculate the profit signature and NPV for the policy given that the policyholder dies in the first year of the contract.
(c) Calculate the profit signature and NPV for the policy given that the policyholder survives to the contract end.
(d) Calculate the profit signature and NPV for the policy given that the policyholder surrenders at the end of the second year, assuming (i) that the cash value is 100% of the year end account value, and then (ii) that the cash value is 90% of the year end account value.
(e) Calculate the surrender penalty at time 2, as a proportion of AV_2, which gives the same profit margin for surrendering policyholders as for policyholders who remain in force throughout.
(f) Comment on your results.

Answers to selected exercises

14.1 \$40.96
14.2 25%
14.3 \$380.01
14.4 CoI: \$235.19, ADB: \$48 919.20, AV: \$51 493.90
14.5 \$622.29
14.6 \$338 361
14.7 \$4 862.50

14.8 (a) $AV_1 = 3\,647.10, \quad AV_2 = 8\,166.38, \quad AV_3 = 12\,842.44$
(b) $(-800,\ 264.78,\ 667.40,\ 572.63)'$
(c) \$245.21 (d) 2.12%

14.9 \$1 464.60

14.10 (a) $(-200.00,\ 60.14,\ 109.19,\ 156.20,\ 182.16)'$, \$207.19
(b) $(-200,\ -8\,917.25,\ 0,\ 0,\ 0)'$, $-\$8\,456.71$
(c) $(-200,\ 113.55,\ 148.92,\ 193.14,\ 239.57)'$, \$362.23
(d) (i) $(-200,\ 113.55,\ 148.92,\ 0,\ 0)'$, \$32.82
(ii) $(-200,\ 113.55,\ 758.40,\ 0,\ 0)'$, \$555.35
(e) 3.1%

15
Emerging costs for equity-linked insurance

15.1 Summary

In this chapter we introduce equity-linked insurance contracts. We explore deterministic emerging costs techniques with examples, and demonstrate that deterministic profit testing cannot adequately model these contracts.

We introduce stochastic cash flow analysis, which gives a fuller picture of the characteristics of the equity-linked cash flows, particularly when guarantees are present, and we demonstrate how stochastic cash flow analysis can be used to determine better contract design.

Finally we discuss the use of quantile and conditional tail expectation reserves for equity-linked insurance.

15.2 Equity-linked insurance

In Chapter 1 we described some modern insurance contracts where the main purpose of the contract is investment. These contracts generally include some death benefit, predominantly as a way of distinguishing them from pure investment products, but they are designed to emphasize the investment opportunity, with a view to competing with pure investment products sold by banks and other financial institutions. Equity-linked insurance can be viewed as a natural development from the traditional participating insurance and Universal Life products, which offer both insurance and investment benefits.

The equity-linked insurance which we explore in this chapter differs from Universal Life and participating insurance, in that the assets of the policyholders are kept separate from the insurer's main funds. In contrast, Universal Life premiums are combined with the general assets of the insurer, and there is no identifiable 'policyholder account'. Essentially, for equity-linked contracts, policyholders may select the funds in which they invest, making the policies look very similar to collective investment products such as mutual funds in North America, and unit trusts in the UK, Australia and elsewhere.

The contracts we consider are called **unit-linked policies** in the UK and parts of Europe, **variable annuities** in the USA (though there is not necessarily any annuity component) and **segregated fund policies** in Canada. All fall under the generic title of equity-linked insurance. The basic premise of these contracts is that a policyholder pays a single or regular premium which, after deducting expenses, is invested on the policyholder's behalf in specified collective investment funds. These form the **policyholder's fund**. The value of the policyholder's fund moves up or down each month, just like a mutual fund investment. Regular **management charges** are deducted from the fund by the insurer and paid into the **insurer's fund** to cover expenses and insurance charges.

On survival to the end of the contract term the benefit may be just the policyholder's fund and no more, or there may be a **guaranteed minimum maturity benefit** (GMMB).

On death during the term of the policy, the policyholder's estate would receive the policyholder's fund, possibly with an extra amount – for example, a death benefit of 110% of the policyholder's fund means an additional death benefit of 10% of the policyholder's fund at the time of death. There may also be a **guaranteed minimum death benefit** (GMDB).

Some conventions and jargon have developed around these contracts, particularly in the UK where the policyholder is deemed to buy units in an underlying asset fund (hence 'unit-linked'). One example is the **bid-offer spread**. If a contract is sold with a bid-offer spread of, say, 5%, only 95% of the premium paid is actually invested in the policyholder's fund; the remainder goes to the insurer's fund. There may also be an **allocation percentage**; if 101% of the premium is allocated to units at the offer price, and there is a 5% bid-offer spread, then 101% of 95% of the premium (that is 95.95%) goes to the policyholder's fund and the rest goes to the insurer's fund. The bid-offer spread mirrors the practice in unitized investment funds that are major competitors for policyholders' investments.

Because the policyholder's funds are not mixed with the general assets of the company, but are held separately, another general term for this type of contract is **Separate Account Insurance**. The fact that the policyholder's assets are held separately from the insurer's assets makes a difference to our analysis of these policies; the policyholder's funds do not directly contribute to the insurer's profit or loss; investment gains and losses are all passed straight to the policyholder. However, the policyholder's fund contributes indirectly. The insurer receives income from the regular management charges that depend on the policyholder's fund value; the additional death benefit cover will be a function of the fund value, and the cost of any guarantees offered will also depend on the fund value.

15.3 Deterministic profit testing for equity-linked insurance

Equity-linked insurance policies are usually analysed using emerging surplus techniques. The process is similar to the UL profit test, except that we separate the cash flows into those that are in the policyholder's fund and those that are income or outgo for the insurer. It is the insurer's cash flows that are important in pricing, reserving, and profit projections, but since the insurer's income and outgo depend on how much is in the policyholder's fund, we must first project the cash flows for the policyholder's fund and use these to project the cash flows for the insurer's fund. The projected cash flows for the insurer's fund can then be used to calculate the profitability of the contract using the profit vector, profit signature, and perhaps the NPV, IRR, profit margin and discounted payback period, in the same way as in Chapters 13 and 14. We show two examples in this section. The first assumes annual cash flows, to make the calculations easier to follow. The second uses monthly time steps, which is more realistic.

Before we present the examples, we introduce some notation and key relationships. For annual time steps, the t subscript refers to the cash flows in the tth policy year. In order to project the emerging profit from the insurer's cash flows, we first need to project the policyholder's fund through the term of the contract, as the insurer's cash flows depend on the fund values. The relevant cash flows, with notation, are described here.

Policyholder's fund: F_t is the amount in the policyholder's separate account at time t.

Premium: P_t is the total premium paid by the policyholder at time $t-1$; the insurer will make some deductions for expenses and contingencies. The remainder is invested in the policyholder's fund.

Allocated premium: AP_t is the part of the tth premium that is invested in the policyholder's fund.

Interest on policyholder's assets: i_t^f is the assumed rate of interest earned on the policyholder's fund in the tth year. It will depend on the type of assets available, and in practice is highly variable. In this section we adopt simple deterministic assumptions for i_t^f, but in Section 15.4 we explore a stochastic approach.

Management charge: MC_t is the management charge deducted from the policyholder's fund during the tth year. The management charge may be deducted at the start or the end of the year; we generally assume the year end in our examples. This passes to the insurer's assets.

So, assuming that the management charge is deducted from the policyholder's fund at the year end, we have

$$F_t = (F_{t-1} + AP_t)\,(1 + i_t^f) - MC_t. \tag{15.1}$$

The first step in a profit test of an equity-linked policy is the projection of the policyholder's fund, *assuming that the policyholder stays in force throughout the contract*. This is exactly the process we used for Universal Life profit testing in Chapter 14. We are not directly interested in the policyholder's fund. We project the fund values because the insurer cash flows in, say, the tth policy year depend on the fund values in respect of policies in force during the tth year.

Following the conventions of profit testing, we first calculate the profit vector by projecting the insurer's cash flows each year, assuming that the policy is in force at the start of the year. We use the following notation for the insurer's cash flows and profit. Some of these terms have been introduced in previous chapters, and are repeated here for convenience.

Reserve, ${}_{t-1}V$**:** Often the policyholder's funds are sufficient for the policyholder's benefits, as maturity or surrender values. The insurer's reserve, which is additional to the separate account holding the policyholder's funds is only required if there are potential additional future liabilities that need advance reserves. If required, ${}_{t-1}V$ is the reserve brought forward to the tth year in respect of a policy in force at the start of the tth year.

Unallocated premium, UAP_t**:** This is $P_t - AP_t$, which the difference between the full premium paid and the allocated premium paid into the policyholder's fund. The unallocated premium is paid into the insurer's funds.

Expenses, E_t**:** this refers to the projected incurred expenses. Pre-contract expenses will be allocated to time 0, as usual. Other expenses are assumed to be incurred at the start of each policy year (i.e. at time $t-1$).

Interest, I_t**:** this is the interest income on the insurer's assets invested through the tth year.

Expected cost of death benefit, EDB_t**:** this covers any additional death benefit not covered by the policyholder's fund. That is, if the benefit paid at the end of the year of death is DB_t, the policyholder's fund will cover F_t, the additional death benefit is $DB_t - F_t$. If the mortality probability for the tth year is p^{0d}_{x+t-1}, then

$$EDB_t = p^{0d}_{x+t-1}\left(DB_t - F_t\right).$$

Expected cost of cash values, ECV_t**:** this covers any additional cash value paid on surrender or at maturity, not covered by the policyholder's fund. If there is a surrender penalty, so that the surrendering policyholder receives less than their fund value, then ECV_t will be negative, i.e. an item of income not outgo. In the final year of the contract, there may be a sum payable at maturity, additional to the policyholder's fund – for example, if a guaranteed minimum maturity benefit applies. If the probability of surrender in the tth year is p^{0w}_{x+t-1}, then

$$ECV_t = p^{0w}_{x+t-1}\,(CV_t - F_t)\,,$$

where, in the final year, CV_t refers to the payment at maturity.

Expected cost of year end reserve, E_tV: as used throughout Chapters 13 and 14, if p^{00}_{x+t-1} is the probability that a policy in force at the start of the tth year is still in force at the start of the $(t+1)$th year, then

$$E_tV = p^{00}_{x+t-1}\ {}_tV.$$

Assuming that the management charge is paid from the policyholder's fund to the insurer at each year end, for a policy in force at the start of the year, we have the profit vector calculation, that is, profit emerging at time t for a policy in force at time $t-1$,

$$\mathrm{Pr}_t = {}_{t-1}V + UAP_t - E_t + I_t + MC_t - EDB_t - ECV_t - E_tV. \qquad (15.2)$$

In practice, not all of these terms may be needed. Some policies will not carry reserves, and it would be common for the cash value to be equal to the policyholder's fund, which would mean that the cost to the insurer (ECV_t) would be zero. Also, formula (15.2) may need some adjustment, for example, if management charges are deducted at the start of the year rather than the end.

Once the profit vector has been calculated, the profit signature, NPV and profit margin can all be determined using the techniques from Chapter 13.

The following two examples illustrate the calculations.

Example 15.1 A 10-year equity-linked contract is issued to a life aged 55 with the following terms.

The policyholder pays an annual premium of \$5000. The insurer deducts a 5% expense allowance from the first premium and a 1% allowance from subsequent premiums. The remainder is invested into the policyholder's fund.

At the end of each year a management charge of 0.75% of the policyholder's fund is transferred from the policyholder's fund to the insurer's fund.

If the policyholder dies during the contract term, a benefit of 110% of the value of the policyholder's year end fund (after management charge deductions) is paid at the end of the year of death.

If the policyholder surrenders the contract, he receives the value of the policyholder's fund at the year end, after management charge deductions.

If the policyholder holds the contract to the maturity date, he receives the greater of the value of the policyholder's fund and the total of the premiums paid.

(a) Assume the policyholder's fund earns interest at 9% per year. Project the year end fund values for a contract that remains in force for 10 years.

(b) Calculate the profit vector for the contract using the following basis.

Survival model: The probability of dying in any year is 0.005.

Lapses: 10% of lives in force at the year end surrender in the first year of the contract, 5% in the second year and none in subsequent years. All surrenders occur at the end of a year immediately after the management charge deduction.

Initial expenses: 10% of the first premium plus \$150, incurred before the first premium payment.

Renewal expenses: 0.5% of the second and subsequent premiums.

Interest: The insurer's funds earn interest at 6% per year.

Reserves: The insurer holds no reserves for the contract.

(c) Calculate the profit signature for the contract.

(d) Calculate the NPV using a risk discount rate of 15% per year effective.

Solution 15.1 (a) The first step in a profit test is the projection of F_t, assuming the policy is in force for the full 10-year contract. We extract from all the information above the parts that relate to the policyholder's fund.

We are given that the annual premium is \$5000; 5% is deducted from the first premium, giving an allocated premium of $AP_1 = 4\,750$. In subsequent years, the allocated premium is 99% of the premium, so that for $t = 2, 3, \ldots, 10$, $AP_t = 4\,950$.

We are also given the assumption that $i_t^f = 0.09$.

The management charge in the tth year is

$$MC_t = 0.0075 \times ((F_{t-1} + AP_t) \times 1.09),$$

and, following equation (15.1), for $t = 1, 2, 3, \ldots, 10$, we have

$$F_t = (F_{t-1} + AP_t) \times 1.09 - MC_t = (F_{t-1} + AP_t) \times 1.09 \times 0.9925.$$

The projection of the policyholder's fund is shown in Table 15.1. The key to the columns of Table 15.1 is as follows.

(1) The entries for t are the years of the contract, from time $t-1$ to time t.

(2) This shows the allocated premium, AP_t, invested in the policyholder's fund at time $t-1$.

(3) This shows the fund brought forward from the previous year end.

(4) This shows the amount in the policyholder's fund at the year end, just before the annual management charge is deducted.

(5) This shows the management charge, at 0.75% of the previous column.

(6) This shows the remaining fund, which is carried forward to the next year.

(b) The sources of income and outgo for the insurer's funds, for a contract in force at the start of the year, are:

Unallocated premium: $UAP_t = 5\,000 - AP_t$. This is the amount the insurer takes when the premiums are paid. The rest goes into the policyholder's fund.

Expenses, E_t:

Initial expenses (pre-contract) of $E_0 = 0.1 \times 5000 + 150 = 650$.
First year expenses of $E_1 = 0$ (all included in E_0).
For $t = 2, 3, \ldots, 10$, $E_t = 0.005 \times 5000 = 25$.

Interest, I_t: earned at 6% per year, so for $t = 1, 2, \ldots, 10$,

$$I_t = 0.06\,(UAP_t - E_t)\,.$$

Note that there are no reserves required for this policy.

Management Charge: MC_t is assumed to be received at the year end. The values are taken from Table 15.1.

Expected cost of deaths: the death benefit is greater than the policyholder's fund value, which means there is a cost to the insurer if the policyholder dies. The death benefit is 110% of F_t, so the insurer's liability if the policyholder dies in the tth year is 10% of F_t (the rest is paid from the policyholder's fund). The mortality probability is given as 0.005, so the expected cost of the additional death benefit is

$$EDB_t = 0.005 \times (1.10F_t - F_t) = 0.005 \times 0.10F_t.$$

Expected cost of cash values: There is no cost to the insurer if the policyholder surrenders the contract early, but there is a potential cost

Table 15.1 *Projection of policyholder's fund for Example 15.1.*

t (1)	AP_t (2)	F_{t-1} (3)	$F_{t\text{-}}$ (4)	MC_t (5)	F_t (6)
1	4750	0.00	5 177.50	38.83	5 138.67
2	4950	5 138.67	10 996.65	82.47	10 914.17
3	4950	10 914.17	17 291.95	129.69	17 162.26
4	4950	17 162.26	24 102.36	180.77	23 921.60
5	4950	23 921.60	31 470.04	236.03	31 234.01
6	4750	31 234.01	39 440.58	295.80	39 144.77
7	4950	39 144.77	48 063.30	360.47	47 702.83
8	4950	47 702.83	57 391.58	430.44	56 961.14
9	4950	56 961.14	67 483.15	506.12	66 977.02
10	4950	66 977.02	78 400.45	588.00	77 812.45

from the GMMB at maturity. The fund value at maturity is F_{10} which is projected to be \$77 812.45. The GMMB requires a final payment of at least 10×5000. Since this is smaller than F_{10}, there is no projected cost from the GMMB.

Following equation (15.2), we have

$$\mathrm{Pr}_t = UAP_t - E_t + I_t + MC_t - EDB_t.$$

The emerging surplus is shown in Table 15.2.

(c) For the profit signature, we multiply the tth element of the profit vector, Pr_t, by the probability that the contract is still in force at the start of the year for $t = 1, 2, \ldots, 10$. (For $t = 0$, the required probability is 1.) The values are shown in Table 15.3.

Table 15.2 *Emerging surplus for Example 15.1.*

t	Unallocated premium UAP_t	Expenses E_t	Interest I_t	Management charge MC_t	Expected death benefit EDB_t	Pr_t
0	0.00	650.00	0.00	0.00	0.00	−650.00
1	250.00	0.00	15.00	38.83	2.57	301.26
2	50.00	25.00	1.50	82.47	5.46	103.52
3	50.00	25.00	1.50	129.69	8.58	147.61
4	50.00	25.00	1.50	180.77	11.96	195.31
5	50.00	25.00	1.50	236.03	15.62	246.91
6	50.00	25.00	1.50	295.80	19.57	302.73
7	50.00	25.00	1.50	360.47	23.85	363.12
8	50.00	25.00	1.50	430.44	28.48	428.46
9	50.00	25.00	1.50	506.12	33.49	499.14
10	50.00	25.00	1.50	588.00	38.91	575.60

Table 15.3 *Calculation of the profit signature for Example 15.1.*

t	Probability in force	Π_t	t	Probability in force	Π_t
0	1.00000	−650.00	6	0.83384	252.43
1	1.00000	301.26	7	0.82967	301.27
2	0.89550	92.70	8	0.82552	353.70
3	0.84647	124.95	9	0.82139	409.99
4	0.84224	164.50	10	0.81729	470.43
5	0.83803	206.92			

(d) The NPV is calculated by discounting the profit signature at the risk discount rate of interest, $r = 15\%$, so that

$$\text{NPV} = \sum_{t=0}^{10} \Pi_t \, (1+r)^{-t} = \$531.98.$$

□

Example 15.2 The terms of a five-year equity-linked insurance policy issued to a life aged 60 are as follows.

The policyholder pays a single premium of \$10 000. The insurer deducts 3% of the premium for expenses. The remainder is invested in the policyholder's fund.

At the start of the second and subsequent months, a management charge of 0.06% of the policyholder's fund is transferred to the insurer's fund.

If the policyholder dies during the term, the policy pays out 101% of all the money in her fund. In addition, the insurer guarantees a minimum benefit. The guaranteed minimum death benefit in the tth year is $10\,000\,(1.05^{t-1})$, where $t = 1, 2, \ldots, 5$.

If the policyholder surrenders the contract during the first year, she receives 90% of the money in the policyholder's fund. In the second year a surrendered contract pays 95% of the policyholder's fund. If the policyholder surrenders the contract after the second policy anniversary, she receives 100% of the policyholder's fund.

If the policyholder holds the contract to the maturity date, she receives the money in the policyholder's fund with a guarantee that the payout will not be less than \$10 000.

The insurer assesses the profitability of the contract by projecting cash flows on a monthly basis using the following assumptions.

Survival model: The force of mortality is constant for all ages and equal to 0.006 per year.

Death benefit: This is paid at the end of the month in which death occurs.

Lapses: Policies are surrendered only at the end of a month. The probability of surrendering at the end of any particular month is 0.004 in the first year, 0.002 in the second year and 0.001 in each subsequent year.

Interest: The policyholder's fund earns interest at 8% per year effective. The insurer's fund earns interest at 5% per year effective.

Initial expenses: 1% of the single premium plus \$150.

Renewal expenses: 0.008% of the single premium plus 0.01% of the policyholder's funds at the end of the previous month. Renewal expenses are payable at the start of each month after the first.

(a) Calculate the probabilities that a policy in force at the start of a month is still in force at the start of the next month.
(b) Construct a table showing the projected policyholder's fund assuming the policy remains in force throughout the term.
(c) Construct a table showing the projected insurer's fund.
(d) Calculate the NPV for the contract using a risk discount rate of 12% per year.

Solution 15.2 (a) The probability of not dying in any month is

$$\exp\{-0.006/12\} = 0.9995.$$

Hence, allowing for lapses, the probability that a policy in force at the start of a month, at time t, say, is still in force at the start of the following month is as follows, where $h = \frac{1}{12}$ is the timestep for this example:

$$
\begin{aligned}
{}_hp^{00}_{60+t-h} &= (1-0.004)\ \exp\{-0.006/12\} = 0.9955 \quad \text{in the first year,}\\
{}_hp^{00}_{60+t-h} &= (1-0.002)\ \exp\{-0.006/12\} = 0.9975 \quad \text{in the second year,}\\
{}_hp^{00}_{60+t-h} &= (1-0.001)\ \exp\{-0.006/12\} = 0.9985 \quad \text{in subsequent years.}
\end{aligned}
$$

(b) Table 15.4 shows the projected policyholder's fund at selected durations assuming the policy remains in force throughout the five years. Note that in this example the management charge is deducted at the start of the month rather than the end. The guaranteed minimum death benefit is also given in this table – in the first year this is the full premium and it increases by 5% at the start of each year.
(c) The projected cash flows for the insurer's fund are shown in Table 15.5.
UAP_t: the unallocated premium is \$300 in the first month, and \$0 thereafter, as this is a single premium policy.
MC_t: the management charge is taken from Table 15.4. In this example, it is assumed to be paid at the start of each month.
E_t: the expenses are described in the example; the pre-contract expenses are allocated to time 0.
I_t is calculated as $(1.05^{1/12} - 1)(UAP_t + MC_t - E_t)$.
EDB_t is the expected cost of additional death benefits. The death benefit at time t is $DB_t = \max(1.01 \times F_t, GMDB_t)$, where $GMDB_t$ is the guaranteed minimum death benefit shown in Table 15.4. Hence, the additional death benefit is

$$ADB_t = DB_t - F_t = \max\left(0.01 \times F_t, GMDB_t - F_t\right),$$

Table 15.4 *Deterministic projection of the policyholder's fund for Example 15.2.*

t	AP_t	F_{t-1}	MC_t	F_t	GMDB
$\frac{1}{12}$	9 700	0.00	0.00	9 762.41	10 000.00
$\frac{2}{12}$	0	9 762.41	5.86	9 819.33	10 000.00
$\frac{3}{12}$	0	9 819.33	5.89	9 876.57	10 000.00
$\frac{4}{12}$	0	9 876.57	5.93	9 934.16	10 000.00
$\frac{5}{12}$	0	9 934.16	5.96	9 992.07	10 000.00
$\frac{6}{12}$	0	9 992.07	6.00	10 050.33	10 000.00
⋮	⋮	⋮	⋮	⋮	⋮
1	0	10 346.74	6.21	10 407.07	10 000.00
$1\frac{1}{12}$	0	10 407.07	6.24	10 467.74	10 500.00
⋮	⋮	⋮	⋮	⋮	⋮
2	0	11 094.29	6.66	11 158.97	10 500.00
$2\frac{1}{12}$	0	11 158.97	6.70	11 224.03	11 025.00
⋮	⋮	⋮	⋮	⋮	⋮
3	0	11 895.85	7.14	11 965.20	11 025.00
$3\frac{1}{12}$	0	11 965.20	7.18	12 034.96	11 576.25
⋮	⋮	⋮	⋮	⋮	⋮
4	0	12 755.32	7.65	12 829.68	11 576.25
$4\frac{1}{12}$	0	12 829.68	7.70	12 904.48	12 155.06
⋮	⋮	⋮	⋮	⋮	⋮
5	0	13 676.89	8.21	13 756.62	12 155.06

and the expected cost of the additional death benefit in the month from time $t-h$ to time t is

$$EDB_t = {}_hp^{0d}_{60+t-h} \max\left(0.01 \times F_t, GMDB_t - F_t\right),$$

where ${}_hp^{0d}_{60+t-h} = 1 - e^{-0.006/12}$ is the probability of death in the month. ECV_t is the expected cost of surrender and maturity payments. In this case, expected profits on surrenders in the first two years are a source of income for the insurer's fund since, on surrendering her policy, the policyholder receives less than the full amount of the policyholder's fund. Let ${}_hp^{0w}_{60+t-h}$ denote the lapse probability for the month, and let CV_t denote the total cash value paid on surrender at time t. Then

$$ECV_t = {}_hp^{0w}_{60+t-h}\left(CV_t - F_t\right).$$

Table 15.5 *Deterministic projection of the insurer's fund for Example 15.2.*

t	UAP_t	MC_t	E_t	I_t	EDB_t	ECV_t	$\Pr_t$
0	0	0.00	250.00	0.00	0.00	0.00	−250.00
$\frac{1}{12}$	300	0.00	0.80	1.22	0.12	−3.90	304.20
$\frac{2}{12}$	0	5.86	1.78	0.02	0.09	−3.93	7.93
$\frac{3}{12}$	0	5.89	1.78	0.02	0.06	−3.95	8.01
$\frac{4}{12}$	0	5.93	1.79	0.02	0.05	−3.97	8.08
$\frac{5}{12}$	0	5.96	1.79	0.02	0.05	−3.99	8.13
$\frac{6}{12}$	0	6.00	1.80	0.02	0.05	−4.02	8.18
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	0	6.21	1.83	0.02	0.05	−4.16	8.50
$1\frac{1}{12}$	0	6.24	1.84	0.02	0.05	−1.05	5.42
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2	0	6.66	1.91	0.02	0.06	−1.12	5.83
$2\frac{1}{12}$	0	6.70	1.92	0.02	0.06	0.00	4.74
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
3	0	7.14	1.99	0.02	0.06	0.00	5.11
$3\frac{1}{12}$	0	7.18	2.00	0.02	0.06	0.00	5.14
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
4	0	7.65	2.08	0.02	0.06	0.00	5.54
$4\frac{1}{12}$	0	7.70	2.08	0.02	0.06	0.00	5.57
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
5	0	8.21	2.17	0.02	0.00	0.00	5.99

In the first year (i.e. for $t = \frac{1}{12}, \frac{2}{12}, \ldots, 1$), $CV_t = 0.9\,F_t$, and in the second year $CV_t = 0.95\,F_t$. For $t = 2\frac{1}{12}, 2\frac{2}{12}, \ldots$, $CV_t = F_t$. For example, for $t = \frac{3}{12}$, ${}_hp^{0w}_{60+t-h} = 0.004 \times e^{-0.006/12}$, so

$$ECV_t = 0.004 \times e^{-0.006/12} \times (-0.10\,F_t)\,.$$

There is no projected maturity benefit other than the policyholder's fund. $\Pr_t$: the expected profit at the end of the month per policy in force at the start of the month is calculated, following equation (15.2), as

$$\Pr_t = UAP_t - E_t + MC_t + I_t - EDB_t - ECV_t.$$

(d) Table 15.6 shows, for selected durations, the expected profit at the end of the month per policy in force at the start of the tth month, $\Pr_t$, the probability that the policy is in force at the start of the month (given only

Table 15.6 *Calculation of the profit signature for Example 15.2.*

t	$\Pr_t$	Probability in force	Π_t
0	−250.00	1.0000	−250.00
$\frac{1}{12}$	304.20	1.0000	304.20
$\frac{2}{12}$	7.93	0.9955	7.90
$\frac{3}{12}$	8.01	0.9910	7.94
$\frac{4}{12}$	8.08	0.9866	7.97
$\frac{5}{12}$	8.13	0.9821	7.98
$\frac{6}{12}$	8.18	0.9777	8.00
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	8.50	0.9516	8.09
$1\frac{1}{12}$	5.42	0.9492	5.14
$\vdots$	$\vdots$	$\vdots$	$\vdots$
2	5.83	0.9235	5.38
$2\frac{1}{12}$	4.74	0.9221	4.37
$\vdots$	$\vdots$	$\vdots$	$\vdots$
3	5.11	0.9070	4.63
$3\frac{1}{12}$	5.14	0.9056	4.66
$\vdots$	$\vdots$	$\vdots$	$\vdots$
4	5.54	0.8908	4.93
$4\frac{1}{12}$	5.57	0.8895	4.96
$\vdots$	$\vdots$	$\vdots$	$\vdots$
5	5.99	0.8749	5.24

that it was in force at time 0) and the profit signature, Π_t, which is the product of these two elements.

The net present value for this policy is calculated by summing the elements of the profit signature discounted to time 0 at the risk discount rate, r. Hence

$$\text{NPV} = \sum_{k=0}^{60} \Pi_{\frac{k}{12}} (1+r)^{-\frac{k}{12}} = \$302.42.$$

□

In both the examples in this section, the benefit involved a guarantee. In the first example the guarantee had no effect at all on the calculations, and in the second the effect was negligible. This does not mean that the guarantees are cost-free. In practice, even though the policyholder's fund may earn on average a return of 9% or more, the return could be very volatile. A few years of poor returns could generate a significant cost for the guarantee. We can explore the sensitivity of the emerging profit to adverse scenarios by using **stress testing**.

In Example 15.1 there is a GMMB – the final payout is guaranteed to be at least the total amount invested, \$50 000. Assume, as an adverse scenario, that the return on the policyholder's fund is only 5% rather than 9%. The result is that the GMMB still has no effect, and the NPV changes from \$531.98 to \$417.45. We must reduce the return assumption to 1% or lower for the guarantee to have any cost. However, under the deterministic model there is no way to turn this analysis into a price for the guarantee.

Furthermore, the deterministic approach does not reflect the potentially huge uncertainty involved in the income and outgo for equity-linked insurance. The insurer's cash flows depend on the policyholder's fund, and the policyholder's fund depends on market conditions.

The deterministic profit tests described in this section can be quite misleading. The investment risks in equity-linked insurance cannot be treated deterministically. It is crucial that the uncertainty is properly taken into consideration for adequate pricing, reserving and risk management. In the next section we develop the methodology introduced in this section to allow appropriately for uncertainty.

15.4 Stochastic profit testing

For traditional insurance policies we often assume that the demographic uncertainty dominates the investment uncertainty – which may be a reasonable assumption if the underlying assets are invested in low-risk fixed interest securities of appropriate duration. The uncertainty involved in equity-linked insurance is very different. The mortality element is assumed diversifiable and is not usually the major factor. The uncertainty in the investment performance is a far more important element, and it is not diversifiable. Selling 1000 equity-linked contracts with GMMBs to identical lives is almost the same as issuing one big contract; when one policyholder's fund dips in value, they all dip, increasing the chance that the GMMB will cost the insurer money for every contract.

Using a deterministic profit test does not reflect the reality of the situation adequately in most cases. The EPV of future profit – expected in terms of demographic uncertainty only – does not contain any information about the uncertainty from investment returns. The profit measure for an equity-linked contract is modelled more appropriately as a random variable rather than a single number. This is achieved by stochastic profit testing.

The good news is that we have done much of the work for stochastic profit testing in the deterministic profit testing of the previous section. The difference is that in the earlier section we assumed deterministic interest and demographic scenarios. In this section we replace the deterministic investment scenarios with stochastic scenarios. The most common practical way to do this is with Monte Carlo simulation, which we introduced in Section 12.5, and used already for this purpose with interest rates in Chapter 12.

Using Monte Carlo simulation, we generate a large number of outcomes for the investment return on the policyholder's fund. The simulated returns are used in place of the constant investment return assumption in the deterministic case. The profit test proceeds exactly as described in the deterministic approach, except that we repeat the test for each simulated investment return outcome, so we generate a random sample of outcomes for the contract, which we can use to determine the probability distribution for each profit measure for a contract.

Typically, the policyholder's fund may be invested in a mixed fund of equities or equities and bonds. The policyholder may have a choice of funds available, involving greater or lesser amounts of uncertainty.

A very common assumption for returns on equity portfolios is the independent lognormal assumption. This assumption, which is very important in financial modelling, can be expressed as follows. Let $R_1, R_2, \ldots$ be a sequence of random variables, where R_t represents the accumulation at time t of a unit amount invested in an equity fund at time $t-1$, so that $R_t - 1$ is the rate of interest earned in the year. These random variables are assumed to be mutually independent, and each R_t is assumed to have a lognormal distribution (see Appendix A). Note that if R_t has a lognormal distribution with parameters μ_t and σ_t, then

$$\log R_t \sim N(\mu_t, \sigma_t^2).$$

Hence, values for R_t can be simulated by simulating values for $\log R_t$ and exponentiating.

We demonstrate stochastic profit testing for equity-linked insurance by considering further the 10-year policy discussed in Example 15.1. In the discussion of Example 15.1 in Section 15.3 we assumed a rate of return of 9% per year on the policyholder's fund. This resulted in a zero cost for the GMMB. We now assume that the accumulation factor for the policyholder's fund over the tth policy year is R_t, where the sequence $\{R_t\}_{t=1}^{10}$ satisfies the independent lognormal assumption. To simplify our presentation we further assume that these random variables are identically distributed, with $R_t \sim LN(\mu, \sigma)$, where $\mu = 0.074928$ and $\sigma = 0.15$. Note that the expected accumulation factor each year is

Table 15.7 *A single simulation of the profit test.*

t (1)	Simulated z_t (2)	Simulated r_t (3)	Management charge (4)	Fund c/f (5)	$\Pr_t$ (6)	Π_t (7)
0					−650.00	−650.00
1	0.95518	1.24384	44.31	5 863.94	306.38	306.38
2	−2.45007	0.74633	60.53	8 010.27	83.03	74.35
3	−1.23376	0.89571	87.07	11 521.61	107.80	90.80
4	0.55824	1.17194	144.78	19 159.03	161.70	135.51
5	−0.62022	0.98206	177.57	23 498.89	192.32	160.37
6	0.01353	1.08000	230.44	30 494.26	241.69	200.52
7	−1.22754	0.89655	238.33	31 539.16	249.06	205.61
8	0.07758	1.09042	298.41	39 490.18	305.17	250.66
9	−0.61893	0.98225	327.38	43 323.89	332.22	271.52
10	−0.25283	1.03770	375.70	49 717.95	96.71	78.64

$$E[R_t] = e^{\mu+\sigma^2/2} = 1.09,$$

which is the same as under the deterministic assumption in Section 15.3.

Table 15.7 shows the results of a single simulation of the investment returns on the policyholder's fund for the policy in Example 15.1.

The values in column (2), labelled $z_1, \ldots, z_{10}$, are simulated values from a $N(0, 1)$ distribution. These values are converted to simulated values from the specified lognormal distribution using $r_t = \exp\{0.074928 + 0.15z_t\}$, giving the annual accumulation factors shown in column (3). The values $\{r_t\}_{t=1}^{10}$ are a single simulation of the random variables $\{R_t\}_{t=1}^{10}$. These simulated annual accumulation factors should be compared with the value 1.09 used in the calculation of Table 15.1. The values in columns (4) and (5) are calculated in the same way as those in columns (5) and (6) in Table 15.1, using the annual interest rate $r_t - 1$ in place of 0.09. Note that in some years, for example the second policy year, the accumulation factor for the policyholder's fund is less than one. The values in column (6) are calculated in the same way as the final column in Table 15.2 except that there is an extra deduction in the calculation of $\Pr_{10}$ of amount

$$p_{54}^{00} \max(50\,000 - F_{10}, 0)$$

where F_{10} denotes the final fund value. This deduction was not needed in our calculations in Section 15.3 since, with the deterministic interest assumption, the final fund value, \$77 812.45, was greater than the GMMB. For this simulation, F_{10} is less than the GMMB so there is a deduction of amount

$$0.995 \times (50\,000 - 49\,717.95) = \$280.64.$$

The values for Π_t are calculated by multiplying the corresponding value of Pr_t by the probability of the policy being in force, as shown in Table 15.3. The values for Pr_t and Π_t shown in Table 15.7 should be compared with the corresponding values in Tables 15.2 and 15.3, respectively. Using a risk discount rate of 15% per year, the NPV using this single simulation of the investment returns on the policyholder's fund is \$232.09.

To measure the effect of the uncertainty in rates of return, we generate a large number, N, of sets of rates of return and for each set carry out a profit test as above. Let NPV_i denote the net present value calculated from the ith simulation, for $i = 1, 2, \ldots, N$. Then the net present value for the policy, NPV, is being modelled as a random variable and $\{\mathrm{NPV}_i\}_{i=1}^N$ is a set of N independent values sampled from the distribution of NPV. From this sample we can estimate the mean, standard deviation and percentiles of this distribution. We can also count the number of simulations for which NPV_i is negative, denoted N^-, and the number of simulations, denoted N^*, for which the final fund value is greater than \$50 000, so that there is no liability for the GMMB.

Let m and s be the estimates of the mean and standard deviation of NPV. Since N is large, we can appeal to the central limit theorem to say that a 95% confidence interval (CI) for E[NPV] is given by

$$\left(m - 1.96 \, \frac{s}{\sqrt{N}}, \; m + 1.96 \, \frac{s}{\sqrt{N}} \right).$$

It is important whenever reporting summary results from a stochastic simulation to give some measure of the variability of the results, such as a standard deviation or a confidence interval.

Calculations by the authors using $N = 1000$ gave the results shown in Table 15.8. To calculate the median and the percentiles we arrange the simulated values of NPV in ascending or descending order. Let $\{\mathrm{NPV}_{(i)}\}_{i=1}^{1\,000}$ denote the simulated values for NPV arranged in ascending order. Then the median is estimated as $(\mathrm{NPV}_{(500)} + \mathrm{NPV}_{(501)})/2$, so that 50% of the observations lie above the estimated median, and 50% lie above. This would be true for any value lying between $\mathrm{NPV}_{(500)}$ and $\mathrm{NPV}_{(501)}$, and taking the mid-point is a conventional approach. Similarly the fifth percentile value is estimated as $(\mathrm{NPV}_{(50)} + \mathrm{NPV}_{(51)})/2$ and the 95th percentile is estimated as $(\mathrm{NPV}_{(950)} + \mathrm{NPV}_{(951)})/2$.

The results in Table 15.8 put a very different light on the profitability of the contract. Under the deterministic analysis, the profit test showed no liability for the guaranteed minimum maturity benefit, and the contract appeared to be profitable overall – the net present value was \$531.98. Under the stochastic analysis, the GMMB plays a very important role. The value of N^* shows that in most cases the GMMB liability is zero and so it does not affect the median.

Table 15.8 *Results from 1 000 simulations of the net present value.*

E[NPV]	380.91
SD[NPV]	600.61
95% CI for E[NPV]	(343.28, 417.74)
5th percentile	–859.82
Median of NPV	498.07
95th percentile	831.51
N^-	87
N^*	897

However, it does have a significant effect on the mean, which is considerably lower than the median. From the fifth percentile figure, we see that very large losses are possible; from the 95th percentile we see that there is somewhat less upside potential with this policy. Note also that an estimate of the probability that the net present value is negative, calculated using a risk discount rate of 15% per year, is

$$N^-/N = 0.087,$$

indicating a probability of around 9% that this apparently profitable contract actually makes a loss.

This profit test reveals what we are really doing with the deterministic test, which is, approximately at least, projecting the median result. Notice how close the median value of NPV is to the deterministic value.

15.5 Stochastic pricing

Recall from Chapter 6 that the equivalence principle premium is defined such that the expected value of the present value of the future loss at the issue of the policy is zero. In fact, the expectation is usually taken over the future lifetime uncertainty (given fixed values for the mortality rates), not the uncertainty in investment returns or non-diversifiable mortality risk. This is an example of an expected value premium principle, where premiums are set considering only the expected value of future loss, not any other characteristics of the loss distribution.

The example studied in Section 15.4 above demonstrates that incorporating a guarantee may add significant risk to a contract and that this only becomes clear when modelled stochastically. The risk cannot be quantified deterministically. Using the mean of the stochastic output is generally not adequate as it fails to protect the insurer against significant non-diversifiable risk of loss.

For this reason it is not advisable to use the equivalence premium principle when there is significant non-diversifiable risk. Instead we can use stochastic simulation with different premium principles.

The quantile premium principle is similar to the portfolio percentile premium principle in Section 6.6. This principle is based on the requirement that the policy should generate a profit with a given probability. We can extend this principle to the pricing of equity-linked policies. For example, we might be willing to write a contract if, using a given risk discount rate, the lower fifth percentile point of the net present value is positive and the expected net present value is at least 65% of the acquisition costs.

The example studied throughout Section 15.4 meets neither of these requirements; the lower fifth percentile point is −\$859.82 and the expected net present value, \$380.91, is 58.6% of the acquisition costs, \$650.

We cannot determine a premium analytically for this contract which would meet these requirements. However, we can investigate the effects of changing the structure of the policy. For the example studied in Section 15.4, Table 15.9 shows results in the same format as in Table 15.8 for four changes to the policy structure. These changes are as follows.

(1) Increasing the premium from \$5 000 to \$5 500, and hence increasing the GMMB to \$55 000 and the acquisition costs to \$700.
(2) Increasing the annual management charge from 0.75% to 1.25%.
(3) Increasing the expense deductions from the premiums from 5% to 6% in the first year and from 1% to 2% in subsequent years.
(4) Decreasing the GMMB from 100% to 90% of premiums paid.

In each of the four cases, the remaining features of the policy are as described in Example 15.1.

Table 15.9 *Results from changing the structure of the policy in Example 15.1.*

	Change			
	Increase *P* (1)	Increase *MC* (2)	Increase *UAP* (3)	Decrease *GMMB* (4)
E[*NPV*]	433.56	939.60	594.68	460.33
SD[*NPV*]	660.67	725.97	619.75	384.96
95% CI	(392.61, 474.51)	(894.60, 984.60)	(556.27, 633.09)	(436.47, 484.19)
5%-ile	−930.81	−617.22	−724.40	145.29
Median of *NPV*	562.87	1 065.66	721.74	500.00
95%-ile	929.66	1 625.44	1 051.78	831.51
N^-	86	78	80	46
N^*	897	882	894	939

Increasing the premium, change (1), makes little difference in terms of our chosen profit criterion. The lower fifth percentile point is still negative – the increase in the GMMB means that even larger losses can occur – and the expected net profit is still less than 65% of the increased acquisition costs. The premium for an equity-linked contract is not like a premium for a traditional contract, since most of it is unavailable to the insurer. The role of the premium in a traditional policy – to compensate the insurer for the risk coverage offered – is taken in equity-linked insurance by the management charge on the policyholder's funds and any loading taken from the premium before it is invested.

Increasing the management charge, change (2), or the expense loadings, change (3), does increase the expected net present value to the required level but the probability of a loss is still greater than 5%.

The one change that meets both parts of our profit criterion is change (4), reducing the level of the maturity guarantee. This is a demonstration of the important principle that risk management begins with the design of the benefits.

An alternative, and in many ways more attractive, method of setting a premium for such a contract is to use modern financial mathematics to both price the contract and reduce the risk of making a loss. We return to this topic in Chapter 17.

15.6 Stochastic reserving

15.6.1 Reserving for policies with non-diversifiable risk

In Chapter 7 we defined a policy value as the EPV of the future loss from the policy (using a deterministic interest rate assumption). This, like the use of the equivalence principle to calculate a premium, is an example of the application of the expected value principle. When the risk is almost entirely diversifiable, the expected value principle works adequately. When the risk is non-diversifiable, which is usually the case for equity-linked insurance, the expected value principle is inadequate both for pricing, as discussed in Section 15.5, and for calculating appropriate reserves.

Consider the further discussion of Example 15.1 in Section 15.4. On the basis of the assumptions in that section, there is a 5% chance that the insurer will make a loss in excess of \$859.82, in present value terms calculated using the risk discount rate of 15% per year, on each policy issued. If the insurer has issued a large number of these policies, such losses could have a disastrous effect on its solvency, unless the insurer has anticipated the risk by reserving for it, by hedging it in the financial markets (which we explain in Chapter 17) or by reinsuring it (which means passing the risk on by taking out insurance with another insurer).

Calculating reserves for policies with significant non-diversifiable risk requires a methodology that takes account of more than just the expected value of the loss distribution. Such methodologies are called **risk measures**. A risk measure is a functional that is applied to a random loss to give a reserve value that reflects the riskiness of the loss.

There are two common risk measures used to calculate reserves for non-diversifiable risks: the quantile reserve and the conditional tail expectation reserve.

15.6.2 Quantile reserving

A quantile reserve (also known as **Value-at-Risk**, or VaR) is defined in terms of a parameter α, where $0 \le \alpha \le 1$. Suppose we have a random loss L. The quantile reserve with parameter α represents the amount which, with probability α, will not be exceeded by the loss.

If L has a continuous distribution function, F_L, the α-quantile reserve is Q_α, where

$$\Pr[L \le Q_\alpha] = \alpha, \tag{15.3}$$

so that

$$Q_\alpha = F_L^{-1}(\alpha).$$

If F_L is not continuous, so that L has a discrete or a mixed distribution, Q_α needs to be defined more carefully. In the example below (which continues in the next section) we assume that F_L is continuous.

To see how to apply this in practice, consider again Example 15.1 as discussed in Section 15.4. Suppose that immediately after issuing the policy, and paying the acquisition costs of \$650, the insurer wishes to set up a 95% quantile reserve, denoted ${}_0V$. In other words, after paying the acquisition costs the insurer wishes to set aside an amount of money, ${}_0V$, so that, with probability 0.95, it will be able to pay its liabilities.

We need some notation. Let j denote the rate of interest per year assumed to be earned on reserves. In practice, j will be a conservative rate of interest, probably much lower than the risk discount rate. Let ${}_tp_{55}^{00}$ denote the probability that a policy is still in force at duration t. This is consistent with our notation from Chapter 8 since our underlying model for the policy contains three states: in force (which we denote by 0), lapsed and dead.

The reserve, ${}_0V$, is calculated by simulating N sets of future accumulation factors for the policyholder's fund, exactly as in Section 15.4, and for each of these we calculate $\Pr_{t,i}$, the profit emerging at time t, $t = 1, 2, \ldots, 10$ for simulation i, per policy in force at duration $t-1$. For simulation i, we calculate the EPV of the future loss, say L_i, as

$$L_i = -\sum_{t=1}^{10} \frac{{}_{t-1}p_{55}^{00}\,\mathrm{Pr}_{t,i}}{(1+j)^t}. \tag{15.4}$$

Note that in the definition of L_i we are considering *future* profits at times $t = 1, 2, \ldots, 10$, and we have not included $\mathrm{Pr}_{0,i}$ in the definition.

Then ${}_0V$ is set equal to the upper 95th percentile point of the empirical distribution of L obtained from our simulations, provided that the upper 95th percentile is positive, so that the reserve is positive. If the upper 95th percentile point is negative, ${}_0V$ is set equal to zero.

Calculations by the authors, with $N = 1000$ and $j = 0.06$, gave a value for ${}_0V$ of \$1259.56. Hence, if, after paying the acquisition costs, the insurer sets aside a reserve of \$1259.56 for each policy issued, it will be able to meet its future liabilities with probability 0.95 **provided** all the assumptions underlying this calculation are realized. These assumptions relate to

- expenses,
- lapse rates,
- the survival model, and, in particular, the diversification of the mortality risk,
- the interest rate earned on the insurer's fund,
- the interest rate earned on the reserve,
- the interest rate model for the policyholder's fund,
- the accuracy of our estimate of the upper 95th percentile point of the loss distribution.

The reasoning underlying this calculation assumes that no adjustment to this reserve will be made during the course of the policy. In practice, the insurer will review its reserves at regular intervals, possibly annually, during the term of the policy and adjust the reserve if necessary. For example, if after one year the rate of return on the policyholder's fund has been low and future expenses are now expected to be higher than originally estimated, the insurer may need to increase the reserve. On the other hand, if the experience in the first year has been favourable, the insurer may be able to reduce the reserve. The new reserve would be calculated by simulating the present value of the future loss from time $t = 1$, using the information available at that time, and setting the reserve equal to the greater of zero and the upper 95th percentile of the simulated loss distribution.

In our example, the initial reserve, ${}_0V = \$1259.56$, is around 25% of the annual premium, \$5000. This amount is expected to earn interest at a rate, 6%, considerably less than the insurer's risk discount rate, 15%. Setting aside substantial reserves, which may not be needed when the policy matures, will have a serious effect on the profitability of the policy.

15.6.3 CTE reserving

The quantile reserve assesses the 'worst-case' loss, where worst case is defined as the event with a $1-\alpha$ probability. One problem with the quantile approach is that it does not take into consideration what the loss will be if that $1-\alpha$ worst-case event actually occurs. In other words, the loss distribution above the quantile does not affect the reserve calculation. The Conditional Tail Expectation (or CTE) was developed to address some of the problems associated with the quantile risk measure. It was proposed more or less simultaneously by several research groups, so it has a number of names, including Tail Value at Risk (or Tail-VaR), Tail Conditional Expectation (or TCE) and Expected Shortfall.

As for the quantile reserve, the CTE is defined using some confidence level α, where $0 \leq \alpha \leq 1$, and is typically 90%, 95% or 99% for reserving.

In words, CTE_α is the expected loss given that the loss falls in the worst $1-\alpha$ part of the distribution of L. The worst $1-\alpha$ part of the loss distribution is the part above the α-quantile, Q_α. If Q_α falls in a continuous part of the loss distribution, that is, not in a probability mass, then we can define the CTE at confidence level α as

$$\mathrm{CTE}_\alpha = \mathrm{E}\left[L|L > Q_\alpha\right]. \tag{15.5}$$

If L has a discrete or a mixed distribution, then more care needs to be taken with the definition. If Q_α falls in a probability mass, that is, if there is some $\epsilon > 0$ such that $Q_{\alpha+\epsilon} = Q_\alpha$, then, if we consider only losses strictly greater than Q_α, we are using less than the worst $1-\alpha$ of the distribution; if we consider losses greater than or equal to Q_α, we may be using more than the worst $1-\alpha$ of the distribution. We therefore adapt the formula of equation (15.5) as follows. Define $\beta' = \max\{\beta : \ Q_\alpha = Q_\beta\}$. Then

$$\mathrm{CTE}_\alpha = \frac{(\beta'-\alpha)Q_\alpha + (1-\beta')\,E[L|L > Q_\alpha]}{1-\alpha}. \tag{15.6}$$

It is worth noting that, given that the CTE_α is the mean loss given that the loss lies above the VaR at level α (at least when the VaR does not lie in a probability mass) then CTE_α is always greater than or equal to Q_α, and usually strictly greater. Hence, for a given value of α, the CTE_α reserve is generally considerably more conservative than the Q_α quantile reserve.

Suppose the insurer wishes to set a $\mathrm{CTE}_{0.95}$ reserve, just after paying the acquisition costs, for the policy studied in Example 15.1 and throughout Sections 15.4, 15.5 and 15.6.2. We proceed by simulating a large number of times the present value of the future loss using formula (15.4), with the rate of interest j per year we expect to earn on reserves, exactly as in Section 15.6.2. From our calculations in Section 15.6.2 with $N = 1000$ and

$j = 0.06$, the 50 worst losses, that is, the 50 highest values of L_i, ranged in value from \$1260.76 to \$7512.41, and the average of these 50 values is \$3603.11. Hence we set the $\mathrm{CTE}_{0.95}$ reserve at the start of the first year equal to \$3603.11.

The same remarks that were made about quantile reserves apply equally to CTE reserves.

(1) The CTE reserve in our example has been estimated using simulations based only on information available at the start of the policy.
(2) In practice, the CTE reserve would be updated regularly, perhaps yearly, as more information becomes available, particularly about the rate of return earned on the policyholder's fund. If the returns are good in the early years of the contract, then it is possible that the probability that the guarantee will cost anything reduces, and part of the reserve can be released back to the insurer before the end of the term.
(3) Holding a large CTE reserve, which earns interest at a rate lower than the insurer's risk discount rate, and which may not be needed when the policy matures, will have an adverse effect on the profitability of the policy.

15.6.4 Comments on reserving

The examples in this chapter illustrate an important general point. Financial guarantees are risky and can be expensive. Several major life insurance companies have found their solvency at risk through issuing guarantees that were not adequately understood at the policy design stage, and were not adequately reserved for thereafter. The method of covering that risk by holding a large quantile or CTE reserve reduces the risk, but at great cost in terms of tying up amounts of capital that are huge in terms of the contract overall. This is a passive approach to managing the risk and is usually not the best way to manage solvency or profitability.

Using modern financial theory we can take an active approach to financial guarantees that for most equity-linked insurance policies offers less risk, and, since the active approach requires less capital, it generally improves profitability when the required risk discount rate is large enough to make carrying capital very expensive.

The active approach to risk mitigation and management comes from option pricing theory. We utilize the fact that the guarantees in equity-linked insurance are financial options embedded in insurance contracts. There is an extensive literature available on the active risk management of financial options. In Chapter 16 we review the science of option risk management, at an introductory level, and in Chapter 17 we apply the science to equity-linked insurance.

15.7 Notes and further reading

A practical feature of equity-linked contracts in the UK which complicates the analysis a little is **capital** and **accumulation units**. The premiums paid at the start of the contract, which are notionally invested in capital units, are subject to a significantly higher annual management charge than later premiums, which are invested in accumulation units. This contract design has been developed to defray the insurer's acquisition costs at an early stage.

Stochastic profit testing can also be used for traditional insurance. We would generally simulate values for the interest earned on assets, and we might also simulate expenses and withdrawal rates. Exercise 15.3 demonstrates this.

For shorter term insurance, the sensitivity of the profit to the investment assumptions may not be very great. The major risk for such insurance is mis-estimation of the underlying mortality rates. This is also non-diversifiable risk, as underestimating the mortality rates affects the whole portfolio. It is therefore useful with term insurance to treat the force of mortality as a stochastic input.

The CTE has become a very important risk measure in actuarial practice. It is intuitive, easy to understand and to apply with simulation output. As a mean, it is more robust with respect to sampling error than a quantile. The CTE is used for stochastic reserving and solvency testing for Canadian and US equity-linked life insurance.

Hardy (2003) discusses risk measures, quantile reserves and CTE reserves in the context of equity-linked life insurance. In particular, she gives full definitions of quantile and CTE reserves, and shows how to simulate the emerging costs and calculate profit measures when stochastic reserving is used.

15.8 Exercises

Shorter exercises

Exercise 15.1 An insurer used 1 000 simulations to estimate the present value of future loss distribution for a segregated fund contract. Table 15.10 shows the largest 100 simulated values of L_0.

(a) Estimate $\Pr[L_0 > 10]$.
(b) Calculate an approximate 99% confidence interval for $\Pr[L_0 > 10]$.
(c) Estimate $Q_{0.99}(L_0)$.
(d) Estimate $\mathrm{CTE}_{0.99}(L_0)$.

Excel-based exercises

Exercise 15.2 An insurer sells a one-year variable annuity contract. The policyholder deposits \$100, and the insurer deducts 3% for expenses and profit. The expenses incurred at the start of the year are 2.5% of the premium.

Table 15.10 *Largest 100 values from 1000 simulations.*

6.255	6.321	6.399	6.460	6.473	6.556	6.578	6.597	6.761	6.840
6.865	6.918	6.949	7.042	7.106	7.152	7.337	7.379	7.413	7.430
7.585	7.614	7.717	7.723	7.847	7.983	8.051	8.279	8.370	8.382
8.416	8.508	8.583	8.739	8.895	8.920	8.981	9.183	9.335	9.455
9.477	9.555	9.651	9.675	9.872	9.972	10.010	10.199	10.216	10.268
10.284	10.814	10.998	11.170	11.287	11.314	11.392	11.546	11.558	11.647
11.840	11.867	11.966	12.586	12.662	12.792	13.397	13.822	13.844	14.303
14.322	14.327	14.404	14.415	14.625	14.733	14.925	15.076	15.091	15.343
15.490	15.544	15.617	15.856	16.369	16.458	17.125	17.164	17.222	17.248
17.357	17.774	18.998	19.200	21.944	21.957	22.309	24.226	24.709	26.140

The remainder of the premium is invested in an investment fund. At the end of one year the policyholder receives the fund proceeds; if the proceeds are less than the initial \$100 investment the insurer pays the difference.

Assume that a unit investment in the fund accumulates to R after 1 year, where $R \sim LN(0.09, 0.18)$.

Let F_1 denote the fund value at the year end. Let L_0 denote the present value of future outgo minus the margin offset income random variable, assuming a force of interest of 5% per year, i.e.

$$L_0 = \max(100 - 97\,R, 0)\,e^{-0.05} - (3 - 2.5).$$

(a) Calculate $\Pr[F_1 < 100]$.
(b) Calculate $\mathrm{E}[F_1]$.
(c) Show that the fifth percentile of the distribution of R is 0.81377.
(d) Hence, or otherwise, calculate $Q_{0.95}(L_0)$.
(e) Let f be the probability density function of a lognormal random variable with parameters μ and σ. Use the result (which is derived in Appendix A)

$$\int_0^A x f(x)\,dx = e^{\mu+\sigma^2/2}\,\Phi\left(\frac{\log A - \mu - \sigma^2}{\sigma}\right),$$

where Φ is the standard normal distribution function, to calculate
(i) $\mathrm{E}[L_0]$, and
(ii) $\mathrm{CTE}_{0.95}(L_0)$.
(f) Now simulate the year end fund, using 100 projections. Compare the results of your simulations with the accurate values calculated in parts (a)–(e).

Exercise 15.3 A life insurer issues a special five-year endowment insurance policy to a life aged 50. The death benefit is \$10 000 and is payable at the end of the year of death, if death occurs during the five-year term. The maturity benefit on survival to age 55 is \$20 000. Level annual premiums are payable in advance.

Reserves are required at integer durations for each policy in force, are independent of the premium, and are as follows:

$$ {}_0V = 0,\ {}_1V = 3\,000,\ {}_2V = 6\,500,\ {}_3V = 10\,500,\ {}_4V = 15\,000,\ {}_5V = 0. $$

The company determines the premium by projecting the emerging cash flows according to the projection basis given below. The profit objective is that the EPV of future profit must be 1/3 of the gross annual premium, using a risk discount rate of 10% per year.

Projection basis

Initial expenses:	10% of the gross premium plus \$100
Renewal expenses:	6% of the second and subsequent gross premiums
Survival model:	Standard Ultimate Survival Model
Interest on all funds:	8% per year effective

(a) Calculate the annual premium.

(b) Generate 500 different scenarios for the cash flow projection, assuming a premium of \$3740, and assuming interest earned follows a lognormal distribution, such that if I_t denotes the return in the tth year,

$$ (1 + I_t) \sim LN(0.07, 0.13). $$

(i) Estimate the probability that the policy will make a loss in the final year, and calculate a 95% confidence interval for this probability.

(ii) Calculate the exact probability that the policy will make a loss in the final year, assuming mortality exactly follows the projection basis, so that the interest rate uncertainty is the only source of uncertainty. Compare this with the 95% confidence interval for the probability determined from your simulations.

(iii) Estimate the probability that the policy will achieve the profit objective, and calculate a 95% confidence interval for this probability.

Exercise 15.4 An insurer issues an annual premium unit-linked contract with a five-year term. The policyholder is aged 60 and pays an annual premium of \$100. A management charge of 3% per year of the policyholder's fund is deducted annually in advance.

The death benefit is the greater of \$500 and the amount of the fund, payable at the end of the year of death. The maturity benefit is the greater of \$500 and the amount of the fund, paid on survival to the end of the five-year term.

Mortality rates assumed are: $q_{60} = 0.0020$, $q_{61} = 0.0028$, $q_{62} = 0.0032$, $q_{63} = 0.0037$ and $q_{64} = 0.0044$. There are no lapses.

(a) Assuming that interest of 8% per year is earned on the policyholder's fund, project the policyholder's fund values for the term of the contract and hence calculate the insurer's management charge income.

(b) Assume that the insurer's fund earns interest of 6% per year. Expenses of 2% of the policyholder's funds are incurred by the insurer at the start of each year. Calculate the profit signature for the contract assuming that no reserves are held.

(c) Explain why reserves may be established for the contract even though no negative cash flows appear after the first year in the profit test.

(d) Explain how you would estimate the 99% quantile reserve and the 99% CTE reserve for this contract.

(e) The contract is entering the final year. Immediately before the final premium payment the policyholder's fund is \$485.

Assume that the accumulation factor for the policyholder's fund each year is lognormally distributed with parameters $\mu = 0.09$ and $\sigma = 0.18$. Let L_4 represent the present value of future loss random variable at time 4, using an effective rate of interest of 6% per year.

(i) Calculate the probability of a payment under either of the guarantees.

(ii) Calculate $Q_{99\%}(L_4)$ assuming that insurer's funds earn 6% per year as before.

Exercise 15.5 A life insurance company issues a five-year unit-linked endowment policy to a life aged 50 under which level premiums of \$750 are payable yearly in advance throughout the term of the policy or until earlier death.

In the first policy year, 25% of the premium is allocated to the policyholder's fund, followed by 102.5% in the second and subsequent years. The units are subject to a bid-offer spread of 5% and an annual management charge of 1% of the bid value of units is deducted at the end of each policy year. Management charges are deducted from the unit fund before death, surrender and maturity benefits are paid.

If the policyholder dies during the term of the policy, the death benefit of \$3000 or the bid value of the units, whichever is higher, is payable at the end of the policy year of death. The policyholder may surrender the policy only at the end of each policy year. On surrender, the bid value of the units is payable at the end of the policy year of exit. On maturity, 110% of the bid value of the units is payable. The company uses the following assumptions in carrying out profit tests of this contract:

Rate of growth on assets in the policyholder's fund:	6.5% per year
Rate of interest on insurer's fund cash flows:	5.5% per year

Survival model:	Standard Ultimate Survival Model
Initial expenses:	\$150
Renewal expenses:	\$65 per year on the second and subsequent premium dates
Initial commission:	10% of first premium
Renewal commission:	2.5% of the second and subsequent years' premiums
Risk discount rate:	8.5% per year
Surrenders:	10% of policies in force at the end of each of the first three years.

(a) Calculate the profit margin for the policy on the assumption that the company does not hold reserves.

(b) (i) Explain briefly why it would be appropriate to establish reserves for this policy.

(ii) Calculate the effect on the profit margin of a reserve requirement of \$400 at the start of the second, third and fourth years, and \$375 at the start of the fifth year. There is no initial reserve requirement.

(c) An actuary has suggested the profit test should be stochastic, and has generated a set of random accumulation factors for the policyholder's funds. The first stochastic scenario of annual accumulation factors for each of the five years is generated under the assumption that the accumulation factors are lognormally distributed with parameters $\mu = 0.07$ and $\sigma = 0.2$. Using the random standard normal deviates given below, conduct the profit test using your simulated accumulation factors, and hence calculate the profit margin, allowing for the reserves as in part (b):

$$-0.71873, \quad -1.09365, \quad 0.08851, \quad 0.67706, \quad 1.10300.$$

Answers to selected exercises

15.1 (a) 0.054 (b) (0.036, 0.072) (c) \$17.30 (d) \$21.46

15.2 (a) 0.37040 (b) \$107.87 (d) \$19.54
(e) (i)\$3.46 (ii) \$24.83

15.3 (a) \$3739.59
(b) Based on one set of 500 projections:
(i) 0.526, (0.482, 0.570) (ii) 0.519 (iii) 0.508, (0.464, 0.552)

15.4 (a) $(3.00,\ 6.14,\ 9.44,\ 12.88,\ 16.50)'$
(b) $(0.27,\ 1.37,\ 2.77,\ 4.33,\ 5.76)'$
(e)(i) 0.114 (ii) \$80.50

15.5 (a) 1.56% (b)(ii) Reduces to 0.51% (c) −1.43%

16
Option pricing

16.1 Summary

In this chapter we review the basic financial mathematics behind option pricing. First, we discuss the no arbitrage assumption, which is the foundation for all modern financial mathematics. We present the binomial model of option pricing, and illustrate the principles of the risk neutral and real world measures, and of pricing by replication.

We discuss the Black–Scholes–Merton option pricing formula, and, in particular, demonstrate how it may be used both for pricing and risk management.

16.2 Introduction

In Section 15.4 we discussed the problem of non-diversifiable risk in connection with equity-linked insurance policies. A methodology for managing this risk, stochastic pricing and reserving, was set out in Sections 15.5 and 15.6. However, as we explained there, this methodology is not entirely satisfactory since it often requires the insurer to set aside large amounts of capital as reserves to provide some protection against adverse experience. At the end of the contract, the capital may not be needed, but having to maintain large reserves is expensive for the insurer. If experience *is* adverse, there is no assurance that reserves will be sufficient.

Since the non-diversifiable risks in equity-linked contracts and some pension plans typically arise from financial guarantees on maturity or death, and since these guarantees are very similar to the guarantees in exchange traded financial options, we can use the Black–Scholes–Merton theory of option pricing to price and actively manage these risks. When a financial guarantee is a part of the benefits under an insurance policy, we call it an **embedded option**.

There are several reasons why it is very helpful for an insurance company to understand option pricing and financial engineering techniques. The insurer may buy options from a third party such as a bank or a reinsurer to offset the embedded options in their liabilities; a good knowledge of derivative pricing will be useful in the negotiations. Also, by understanding financial engineering methods an insurer can make better risk management decisions. In particular, when an option is embedded in an insurance policy, the insurer must make an informed decision whether to hedge the products in-house or subcontract the task to a third party.

There are many different types of financial guarantees in insurance contracts. This chapter contains sufficient introductory material on financial engineering to enable us to study in Chapter 17 the valuation and hedging of options embedded within insurance policies that can be viewed as relatively straightforward European put or call options.

16.3 The 'no arbitrage' assumption

The 'no arbitrage' assumption is the foundation of modern valuation methods in financial mathematics. The assumption is more colloquially known as the 'no free lunch' assumption, and states quite simply that you cannot get something for nothing.

An **arbitrage** opportunity exists if an investor can construct a portfolio that costs zero at inception and generates positive profits with a non-zero probability in the future, with no possibility of incurring a loss at any future time.

If we assume that there are no arbitrage opportunities in a market, then it follows that **any two securities or combinations of securities that give exactly the same payments must have the same price**. For example, consider two assets priced at $\$A$ and $\$B$ which produce the same future cash flows. If $A \neq B$, then an investor could buy the asset with the lower price and sell the more expensive one. The cash flows purchased at the lower price would exactly match the cash flows sold, so the investor would make a risk-free profit of the difference between A and B.

The no arbitrage assumption is very simple and very powerful. It enables us to find the price of complex financial instruments by 'replicating' the payoffs. Replication is a crucial part of the framework. This means that if we can construct a portfolio of assets with exactly the same payments as the investment in which we are interested, then the price of the investment must be the same as the price of the '**replicating portfolio**'.

For example, suppose an insurer incurs a liability, under which it must deliver the price of one share in Superior Life Insurance Company in one year's time, and the insurer wishes to value this liability. The traditional way to value this might be by constructing a probability distribution for the future value – suppose the current value is \$400 and the insurer assumes the share price in one year's time will follow a lognormal distribution, with parameters

$\mu = 6.07$ and $\sigma = 0.16$. Then the mean value of the share price in one year's time is $e^{\mu+\sigma^2/2} = \$438.25$.

The next step is to discount to current values, at, say 6% per year (perhaps using the long-term bond yield), to give a present value of \$413.45.

So we have a value for the liability, with an implicit risk management plan of putting the \$413.45 in a bond, which in one year will pay \$438.25, which may or may not be sufficient to buy the share to deliver to the creditor. It will almost surely be either too much or not enough.

A better approach is to replicate the payoff, and value the cost of replication. In this simple case, that means holding a replicating portfolio of one share in Superior Life Insurance Company. The cost of this now is \$400. In one year, the portfolio is exactly sufficient to pay the creditor, whatever the outcome. So, since it costs \$400 to replicate the payoff, that is how much the liability is worth. It cannot be worth \$413.45 – that would allow the company to sell the liability for \$413.45, and replicate it for \$400, giving a risk-free profit (or arbitrage) of \$13.45.

Replication does not require a model; we have eliminated the uncertainty in the payoff, and we implicitly have a risk management strategy – buy the share and hold it until the liability falls due.

Although this is an extreme example, the same argument will be applied in this chapter and the next, even when finding the replicating portfolio is a more complicated process.

In practice, in most securities markets, arbitrage opportunities arise from time to time and are very quickly eliminated as investors spot them and trade on them. Since they exist only for very short periods, assuming that they do not exist at all is sufficiently close to reality for most purposes.

16.4 Options

Options are very important financial contracts, with billions of dollars of trades in options daily around the world. In this section we introduce the language of options and explain how some option contracts operate. European options are perhaps the most straightforward type of options, and the most basic forms of these are a **European call option** and a **European put option**.

The holder of a European call option on a stock has the right (but not the obligation) to buy an agreed quantity of that stock at a fixed price, known as the **strike price**, at a fixed date, known as the **expiry** or **maturity** date of the contract.

Let S_t denote the price of the stock at time t. The holder of a European call option on this stock with strike price K and maturity date T would exercise the option only if $S_T > K$, in which case the option is worth $S_T - K$ to the option holder at the maturity date. The option would not be exercised at the maturity date in the case when $S_T < K$, since the stock could then be bought for a lower price in the market at that time. Thus, the payoff at time T under the option is

$$(S_T - K)^+ = \max(S_T - K, 0).$$

The holder of a European put option on a stock has the right (but not the obligation) to sell an agreed quantity of that stock at a fixed strike price, at the maturity date of the contract. The holder of a European put option would exercise the option only if $S_T < K$, since the holder of the option could sell the stock at time T for K then buy the stock at the lower price of S_T in the market and hence make a profit of $K - S_T$. In this case the option is worth $K - S_T$ to the option holder at the maturity date. The option would not be exercised at the maturity date in the case when $S_T > K$, since the option holder would then be selling stock at a lower price than could be obtained by selling it in the market. Thus, the payoff at time T under a European put option is

$$(K - S_T)^+ = \max(K - S_T, 0).$$

In making all of the above statements, we are assuming that people act rationally when they exercise options. We can think of options as providing guarantees on prices. For example, a call option guarantees that the holder of the option pays no more than the strike price to buy the underlying stock at the maturity date.

American options are defined similarly, except that the option holder has the right to exercise the option at any time before the maturity date. The names 'European' and 'American' are historical conventions, and do not signify where these options are sold – both European and American options are sold worldwide. In this book we are concerned only with European options which are significantly more straightforward to price than American options. Many of the options embedded in life insurance contracts are European-style.

If, at any time t prior to the maturity date, the stock price S_t is such that the option would mature with a non-zero value if the stock price did not change, we say that the option is 'in-the-money'; so, a call option is in-the-money when $S_t > K$, and a put option is in-the-money when $K > S_t$. When $K = S_t$, or even when K is close to S_t, we say the option is 'at-the-money'. Otherwise it is 'out-of-the-money'.

16.5 The binomial option pricing model

16.5.1 Assumptions

Throughout Section 16.5 we use the no arbitrage principle together with a simple discrete time model of a stock price process called the binomial model to price options.

Although the binomial model is simple, and not very realistic, it is useful because the techniques we describe below carry through to more complicated models for a stock price process.

We make the following assumptions.

- There is a financial market in which there exists a risk-free asset (such as a zero-coupon bond) and a risky asset, which we assume here to be a stock. The market is free of arbitrage.
- The financial market is modelled in discrete time. Trades occur only at specified time points. Changes in asset prices and the exercise date for an option can occur only at these same dates.
- In each unit of time the stock price either moves up by a predetermined amount or moves down by a predetermined amount. This means there are just two possible states one period later if we start at a given time and price.
- Investors can buy and sell assets without cost. These trades do not impact the prices.
- Investors can *short sell* assets, so that they can hold a negative amount of an asset. This is achieved by selling an asset they do not own, so the investor 'owes' the asset to the lender. We say that an investor is *long* in an asset if the investor has a positive holding of the asset, and is *short* in the asset if the investor has a negative holding.

We start by considering the pricing of an option over a single time period. We then extend this to two time periods.

16.5.2 Pricing over a single time period

To illustrate ideas numerically, consider a stock whose current price is \$100 and whose price at time $t = 1$ will be either \$105 or \$90. We assume that the continuously compounded risk-free rate of interest is $r = 0.03$ per unit of time. Note that we must have

$$90 < 100e^r < 105$$

since otherwise arbitrage is possible. To see this, suppose $100e^r > 105$. In this case an investor could receive \$100 by short selling one unit of stock at time $t = 0$ and invest this for one unit of time at the risk-free rate of interest. At time $t = 1$ the investor would then have $\$100e^r$, part of which would be used to buy one unit of stock in the market to wipe out the negative holding, leaving a profit of either $\$(100e^r - 105)$ or $\$(100e^r - 90)$, both of which are positive. Similarly, if $100e^r < 90$ (which means a negative risk-free rate) selling the risk-free asset short and buying the stock will generate an arbitrage.

Now, consider a put option on this stock which matures at time $t = 1$ with a strike price of $K = \$100$. The holder of this option will exercise the option at time $t = 1$ only if the stock price goes down, since by exercising the option the option holder will get \$100 for a stock worth \$90. As we are assuming that there are no trading costs in buying and selling stocks, the option holder could use the sale price of \$100 to buy stock at \$90 at time $t = 1$ and make a profit of \$10.

The seller of the put option will have no liability at time $t = 1$ if the stock price rises, since the option holder will not sell a stock for \$100 when it is worth \$105 in the market. However, if the stock price falls, the seller of the put option has a liability of \$10.

We use the concept of replication to price this put option. This means that we look for a portfolio of assets at time $t = 0$ that will exactly match the payoff under the put option at time $t = 1$. Since our market comprises only the risk-free asset and the stock, any portfolio at time $t = 0$ must consist of some amount, say \$$a$, in the risk-free asset and some amount, \$100b, in the stock (so that b units of stock are purchased). Then at time $t = 1$, the portfolio is worth

$$ae^r + 105b$$

if the stock price goes up, and is worth

$$ae^r + 90b$$

if the stock price goes down. If this portfolio replicates the payoff under the put option, then the portfolio must be worth 0 at time $t = 1$ if the stock price goes up, and \$10 at time $t = 1$ if the stock price goes down. To achieve this we require that

$$\begin{aligned} ae^r + 105b &= 0, \\ ae^r + 90b &= 10. \end{aligned}$$

Solving these equations we obtain $b = -2/3$ and $a = 67.9312$. We have shown that a portfolio consisting of \$67.9312 of the risk-free asset and a short holding of $-2/3$ units of stock exactly matches the payoff under the put option at time $t = 1$, regardless of the stock price at time $t = 1$. This portfolio is called the **replicating**, or **hedge**, portfolio.

The no arbitrage principle tells us that if the put option and the replicating portfolio have the same value at time $t = 1$, they must have the same value at time $t = 0$, and this then must be the price of the option, which is

$$a + 100b = \$1.26.$$

We can generalize the above arguments to the case when the stock price at time $t = 0$ is S_0, the stock price at time $t = 1$ is uS_0 if the stock price goes up and dS_0 if the stock price goes down, and the strike price for the put option is K. We note here that under the no arbitrage assumption, we must have $dS_0 < S_0e^r < uS_0$. Similarly, we must also have $dS_0 < K < uS_0$ for a contract to be feasible.

The hedge portfolio consists of \$$a$ in the risk-free asset and \$$bS_0$ in stock. Since the payoff at time $t = 1$ from this portfolio replicates the option payoff, we must have

$$ae^r + buS_0 = 0,$$
$$ae^r + bdS_0 = K - dS_0,$$

giving

$$a = \frac{ue^{-r}(K - dS_0)}{u - d} \qquad \text{and} \qquad b = \frac{dS_0 - K}{S_0(u - d)}.$$

The option price at time 0 is $a + bS_0$, the value of the hedge portfolio, which we can write as

$$e^{-r}q\,(K - dS_0) \tag{16.1}$$

where

$$q = \frac{u - e^r}{u - d}. \tag{16.2}$$

Note that, from our earlier assumptions,

$$0 < q < 1.$$

An interesting feature of expression (16.1) for the price of the put option is that, if we were to treat q as the probability of a downward movement in the stock price and $1 - q$ as the probability of an upward movement, then formula (16.1) could be thought of as the discounted value of the expected payoff under the option. If the stock price moves down, the payoff is $K - dS_0$, with discounted value $e^{-r}(K - dS_0)$. If q were the probability of a downward movement in the stock price, then $qe^{-r}(K - dS_0)$ would be the EPV of the option payoff. Recall that these parameters, q and $1 - q$ are not the true 'up' and 'down' probabilities. In fact, nowhere in our determination of the price of the put option have we needed to know the probabilities of the stock price moving up or down. The parameter q comes from the binomial framework, but it is not the 'real' probability of a downward movement; it is just convenient to treat it as such, as it allows us to use the conventions and notation of probability. It is important to remember though that we have not used a probabilistic argument here, we have used instead a replication argument.

It turns out that the price of an option in the binomial framework can *always* be expressed as the discounted value of the option's 'expected' payoff using the artificial probabilities of upward and downward price movements, $1-q$ and q, respectively. The following example demonstrates this for a general payoff.

Example 16.1 Consider an option over one time period which has a payoff C_u if the stock price at the end of the period is uS_0, and has a payoff C_d if the stock price at the end of the period is dS_0. Show that the option price is

$$e^{-r}\left(C_u(1-q)+C_d\,q\right)$$

where q is given by formula (16.2).

Solution 16.1 We construct the replicating portfolio which consists of \$$a$ in the risk-free asset and \$$bS_0$ in stock so that

$$\begin{aligned} ae^r + buS_0 &= C_u, \\ ae^r + bdS_0 &= C_d, \end{aligned}$$

giving

$$b = \frac{C_u - C_d}{(u-d)\,S_0}$$

and

$$\begin{aligned} a &= e^{-r}\left(C_u - u\,\frac{C_u - C_d}{u-d}\right) \\ &= e^{-r}\left(\frac{u}{u-d}\,C_d - \frac{d}{u-d}\,C_u\right). \end{aligned}$$

Hence the option price is

$$\begin{aligned} a + bS_0 &= e^{-r}\left(\frac{u}{u-d}\,C_d - \frac{d}{u-d}\,C_u\right) + \frac{C_u - C_d}{u-d} \\ &= C_u\left(\frac{1-de^{-r}}{u-d}\right) + C_d\left(\frac{ue^{-r}-1}{u-d}\right) \\ &= e^{-r}\left(C_u\left(\frac{e^r-d}{u-d}\right) + C_d\left(\frac{u-e^r}{u-d}\right)\right) \\ &= e^{-r}\left(C_u(1-q)+C_d\,q\right). \end{aligned}$$

□

In the above example, if we treat q as the probability that the stock price at time $t=1$ is dS_0, then the expected payoff under the option at time $t=1$ is

$$C_u(1-q)+C_d\,q,$$

and so the option price is the discounted expected payoff. Note that q has not been defined as the probability that the stock price is equal to dS_0 at time

$t = 1$, and, in general, will not be equal to this probability. We emphasize that the probability q is an artificial construct, but a very useful one.

Under the binomial framework that we use here, there is some real probability that the stock price moves down or up. We have not needed to identify it here. The true distribution is referred to by different names, the **physical measure**, the **real world measure**, the **subjective measure** or **nature's measure**. In the language of probability theory, it is called the P-measure. The artificial distribution that arises in our pricing of options is called the **risk neutral measure**, and in the language of probability theory is called the Q-measure. The term 'measure' can be thought of as interchangeable with 'probability distribution'. In what follows, we use E^Q to denote expectation with respect to the Q-measure. The Q-measure is called the risk neutral measure since, under the Q-measure, the expected return on every asset in the market (risky or not) is equal to the risk-free rate of interest, as if investors in this hypothetical world were neutral as to the risk in the assets. We know that in the real world investors require extra expected return for extra risk. We demonstrate risk neutrality in the following example.

Example 16.2 Show that if S_1 denotes the stock price at time $t = 1$, then under our model $\mathrm{E}^Q[e^{-r}S_1] = S_0$.

Solution 16.2 Under the Q-measure,

$$S_1 = \begin{cases} uS_0 & \text{with probability } 1-q, \\ dS_0 & \text{with probability } q. \end{cases}$$

Then

$$\begin{aligned} \mathrm{E}^Q[e^{-r}S_1] &= e^{-r}\left((1-q)uS_0 + qdS_0\right) \\ &= e^{-r}\left(\left(\frac{e^r - d}{u-d}\right)uS_0 + \left(\frac{u-e^r}{u-d}\right)dS_0\right) \\ &= S_0. \end{aligned}$$

□

The result in Example 16.2 shows that under the risk neutral measure, the stock price at time $t = 0$ is the EPV under the Q-measure of the stock price at time $t = 1$. We also see that the expected accumulation factor of the stock price over a unit time interval is e^r, the same as the risk-free accumulation factor. Under the P-measure we expect the accumulation factor to exceed e^r on average, as a reward for the extra risk.

16.5.3 Pricing over two time periods

In the previous section we considered a single period of time and priced the option by finding the replicating portfolio at time $t = 0$. We now extend this idea to pricing an option over two time periods. This involves the idea of **dynamic hedging**, which we introduce by extending the numerical example of the previous section.

Let us now assume that in each of our two time periods, the stock price can either increase by 5% of its value at the start of the time period, or decrease by 10% of its value. We assume that the stock price movement in the second time period is independent of the movement in the first time period.

As before, we consider a put option with strike price \$100, but this time the exercise date is at the end of the second time period. As illustrated in Figure 16.1, the stock price at time $t = 2$ is \$110.25 if the stock price moves up in each time period, \$94.50 if the stock price moves up once and down once, and \$81.00 if the stock price moves down in each time period. This means that the put option will be exercised if at time $t = 2$ the stock price is \$94.50 or \$81.00.

In order to price the option, we use the same replication argument as in the previous section, but now we must work backwards from time $t = 2$. Suppose first that at time $t = 1$ the stock price is \$105. We can establish a portfolio at time $t = 1$ that replicates the payoff under the option at time $t = 2$. Suppose this portfolio contains $\$a_u$ of the risk-free asset and b_u units of stock, so that the replicating portfolio is worth $\$(a_u + 105b_u)$. Then at time $t = 2$, the value of the portfolio should be 0 if the stock price moves up in the second time period since the option will not be exercised, and the value should be \$5.50 if the stock price moves down in the second time period since the option will be exercised in this case. The equations that determine a_u and b_u are

$$a_u e^r + 110.25 b_u = 0,$$
$$a_u e^r + 94.5 b_u = 5.50,$$

giving $b_u = -0.3492$ and $a_u = 37.3622$. This shows that the replicating portfolio at time $t = 1$, if the stock price at that time is 105, has value $P_u = a_u + 105b_u = \$0.70$.

Similarly, if at time $t = 1$ the stock price is \$90, we can find the replicating portfolio whose value at time $t = 1$ is $\$(a_d + 90b_d)$, where the equations that determine a_d and b_d are

$$a_d e^r + 94.5 b_d = 5.50,$$
$$a_d e^r + 81 b_d = 19,$$

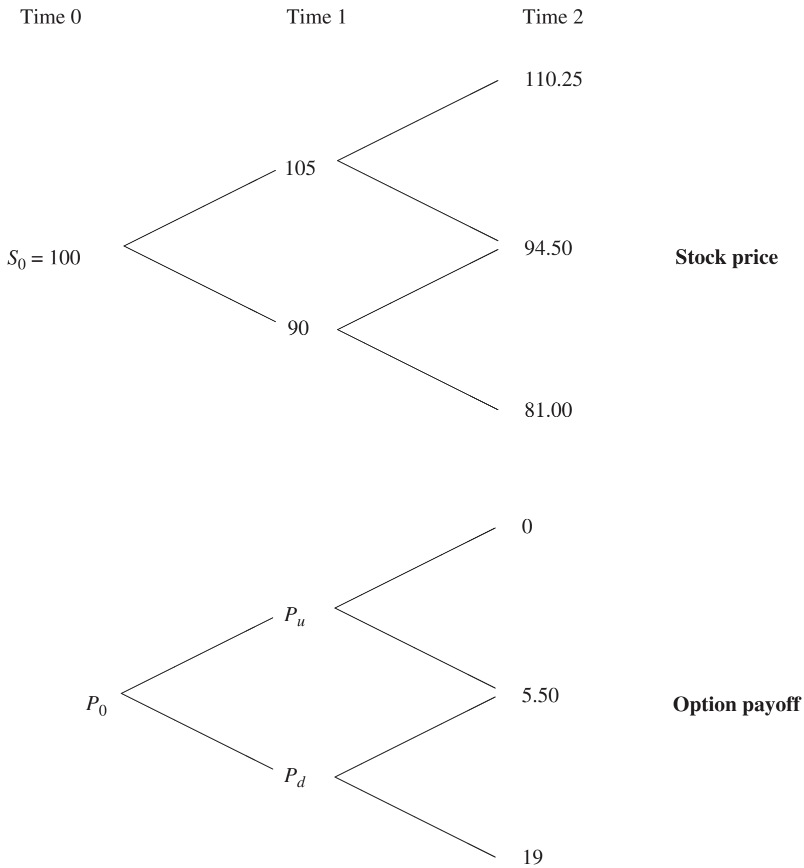


Figure 16.1 Two-period binomial model.

since if the stock price rises to \$94.50, the payoff under the put option is \$5.50, and if the stock price falls to \$81, the payoff under the option is \$19. Solving these two equations we find that $b_d = -1$ and $a_d = 97.0446$. Thus, the replicating portfolio at time $t = 1$, if the stock price at that time is \$90, has value $P_d = a_d + 90b_d = \$7.04$.

We now move back to time $t = 0$. At this time point we want to find a portfolio that replicates the possible amounts required at time $t = 1$, namely \$0.70 if the stock price goes up to \$105 in the first time period, and \$7.04 if it goes down to \$90. This portfolio consists of \$$a$ in the risk-free asset and b units of stock, so that the equations that determine a and b are

$$ae^r + 105b = 0.70,$$

$$ae^r + 90b = 7.04,$$

giving $b = -0.4233$ and $a = 43.8049$. The replicating portfolio has value P_0 at time $t = 0$, where

$$P_0 = a + 100b = \$1.48$$

and, by the no arbitrage principle, this is the price of the option.

There are two important points to note about the above analysis. The first is a point we noted about option pricing over a single period – we do not need to know the true probabilities of the stock price moving up or down in any time period in order to find the option price. The second point is that the replicating portfolio is **self-financing**. The initial portfolio of \$43.80 in the risk-free asset and a short holding of -0.4233 units of stock is exactly sufficient to provide the replicating portfolio at time $t = 1$ regardless of the stock price movement in the first time period. The replicating portfolio at time $t = 1$ then matches exactly the option payoff at time $t = 2$. Thus, once the initial portfolio has been established, no further injection of funds is required to match the option payoff at time $t = 2$.

What we have done in this process is an example of dynamic hedging. At time $t = 1$ we established what portfolios were required to replicate the possible payoffs at time $t = 2$, then at time $t = 0$ we worked out what portfolio was required to provide the portfolio values required at time $t = 1$. This process works for any number of steps, but if there is a large number of time periods it is a time-consuming process to work backwards through time to construct all the hedging strategies. However, if all we want to work out is the option price, the result we saw for a single time period, that the option price is the discounted value of the expected payoff at the expiry date under the Q-measure, also holds when we are dealing with multiple time periods.

In our analysis we have $u = 1.05$, $d = 0.9$ and $r = 0.03$. From formula (16.2), the probability of a downward movement in the stock price under the Q-measure is

$$q = \frac{1.05 - e^{0.03}}{1.05 - 0.9} = 0.1303,$$

and so the expected payoff at the expiry date is

$$19q^2 + 5.5 \times 2(1 - q)q = \$1.5962.$$

This gives the option price as

$$1.5962e^{-0.06} = \$1.48.$$

16.5.4 Summary of the binomial model option pricing technique

- We use the principle of replication; we construct a portfolio that replicates the option's payoff at maturity. The value of the option is the cost of purchasing the replicating (or hedge) portfolio.
- We use dynamic hedging – replication requires us to rebalance the portfolio at each time step according to the movement in the stock price in the previous time step.
- We do not use any argument involving the true probabilities of upward or downward movements in the stock price. However, there are important links between the real world (P-measure) model and the risk neutral (Q-measure) model. We started by assuming a two-point distribution for the stock price after a single time period in the real world. From this we showed that in the risk neutral world the stock price after a single time period also has a two-point distribution with the same possible values, uS_0 and dS_0, but the probabilities of moving up or down are not linked to those of the real world model.
- Our valuation can be written in the form of an EPV, using artificial probabilities that are determined by the possible changes in the stock price. This artificial distribution is called the risk neutral measure because the mean accumulation of a unit of stock under this distribution is exactly the accumulated value of a unit investment in the risk-free asset. Thus, an investor would be indifferent between investment in the risk-free asset and investment in the stock, under the risk neutral measure.

The binomial model option pricing framework is clearly not very realistic, but we can make it more flexible by increasing the number of steps in a unit of time, as discussed below. If we do this, the binomial model converges to the Black–Scholes–Merton model, which is described in the following section.

16.6 The Black–Scholes–Merton model

16.6.1 The model

Under the Black–Scholes–Merton model, we make the following assumptions.

- The market consists of zero-coupon bonds (the risk-free asset) and stocks (the risky asset).
- The stock does not pay any dividends, or, equivalently, any dividends are immediately reinvested in the stock. This assumption simplifies the presentation but can easily be relaxed if necessary.
- Portfolios can be rebalanced (that is, stocks and bonds can be bought and sold) in continuous time. In the two-period binomial example we showed

how the replicating portfolio was rebalanced (costlessly) after the first time unit. In the continuous time model the stock price moves are continuous, so the rebalancing is (at least in principle) continuous.

- There are no transactions costs associated with trading the stocks and bonds.
- The continuously compounded risk-free rate of interest, r per unit time, is constant and the yield curve is flat.
- Stocks and bonds can be bought or sold in any quantities, positive or negative; we are not restricted to integer units of stock, for example. Selling or buying can be transacted at any time without restrictions on the amounts available, and the amount bought or sold does not affect the price.
- In the real world, the stock price, denoted S_t at time t, follows a continuous time lognormal process with some parameters μ and σ. This process, also called geometric Brownian motion, is the continuous time version of the lognormal model for one year accumulation factors introduced in Chapter 15.

Clearly these are not realistic assumptions. Continuous rebalancing is not feasible, and although major financial institutions like insurance companies can buy and sell assets cheaply, transactions costs will arise. We also know that yield curves are rarely flat. Despite all this, the Black–Scholes–Merton model works remarkably well, both for determining the price of options and for determining risk management strategies. The Black–Scholes–Merton theory is extremely powerful and has revolutionized risk management for non-diversifiable financial risks.

A lognormal stochastic process with parameters μ and σ has the following characteristics.

- Over any fixed time interval, say $(t, t+\tau)$ where $\tau > 0$, the stock price accumulation factor, $S_{t+\tau}/S_t$, has a lognormal distribution with parameters $\mu\tau$ and $\sigma\sqrt{\tau}$, so that

$$\frac{S_{t+\tau}}{S_t} \sim_P LN(\mu\tau, \sigma\sqrt{\tau}), \tag{16.3}$$

which implies that

$$\log \frac{S_{t+\tau}}{S_t} \sim_P N(\mu\tau, \sigma^2\tau).$$

We have added the subscript P as a reminder that these statements refer to the real world, or P-measure, model. Our choice of parameters μ and σ here uses the standard parameterizations. Some authors, particularly in financial mathematics, use the same σ, but use a different location parameter μ', say, such that $\mu' = \mu + \sigma^2/2$. It is important to check what μ represents when it is used as a parameter of a lognormal distribution.

We call $\log(S_{t+\tau}/S_t)$ the log-return on the stock over the time period $(t, t+\tau)$. The parameter μ is the mean log-return over a unit of time, and σ is the standard deviation of the log-return over a unit of time. We call σ the **volatility**, and it is common for the unit of time to be one year so that these parameters are expressed as annual rates. Some information on the lognormal distribution is given in Appendix A.

- Stock price accumulation factors over non-overlapping time intervals are independent of each other. (This is the same as in the binomial model, where the stock price movement in any time interval is independent of the movement in any other time interval.) Thus, if S_u/S_t and S_w/S_v represent the accumulation factors over the time intervals (t, u) and (v, w) where $t < u \leq v < w$, then these accumulation factors are independent of each other.

The lognormal process assumed in the Black–Scholes–Merton model can be derived as the continuous time limit, as the number of steps increases, of the binomial model of the previous sections. The proof requires mathematics beyond the scope of this book, but we give some references in Section 16.7 for interested readers.

16.6.2 The Black–Scholes–Merton option pricing formula

Under the Black–Scholes–Merton model assumptions we have the following important results.

- There is a unique risk neutral distribution, or Q-measure, for the stock price process, under which the stock price process, $\{S_t\}_{t\geq 0}$, is a lognormal process with parameters $r - \sigma^2/2$ and σ.
- For any European option on the stock, with payoff function $h(S_T)$ at maturity date T, the value of the option at time $t \leq T$ denoted $v(t)$, can be found as the expected present value of the payoff under the risk neutral distribution (Q-measure)

$$\boxed{v(t) = \mathrm{E}_t^Q\left[e^{-r(T-t)}\, h(S_T)\right],} \tag{16.4}$$

where E_t^Q denotes expectation using the risk neutral (or Q) measure, using all the information available up to time t. This means, in particular, that valuation at time t assumes knowledge of the stock price at time t.

Important points to note about this result are:

- Over any fixed time interval, say $(t, t+\tau)$ where $\tau > 0$, the stock price accumulation factor, $S_{t+\tau}/S_t$, has a lognormal distribution in the risk neutral world with parameters $(r - \sigma^2/2)\tau$ and $\sigma\sqrt{\tau}$, so that

$$\frac{S_{t+\tau}}{S_t} \sim_Q LN\big((r-\sigma^2/2)\tau, \sigma\sqrt{\tau}\big), \tag{16.5}$$

which implies that

$$\log\frac{S_{t+\tau}}{S_t} \sim_Q N\big((r-\sigma^2/2)\tau, \sigma^2\tau\big).$$

We have added the subscript Q as a reminder that these statements refer to the risk neutral, or Q-measure model.

- The expected Q-measure present value (at rate r per year) of the future stock price, $S_{t+\tau}$, is the stock price now, S_t. This follows from the previous point since

$$\mathrm{E}_t^Q[S_{t+\tau}/S_t] = \exp((r-\sigma^2/2)\tau + \tau\,\sigma^2/2) = e^{r\,\tau}.$$

  This is the result within the Black–Scholes–Merton framework which corresponds to the result in Example 16.2 for the binomial model.
- The Q-measure is related to the corresponding P-measure in two ways:
  - Under the Q-measure, the stock price follows a lognormal process, as it does in the real world.
  - The volatility parameter, σ, is the same for both measures.
- The first of these connections should not surprise us since the real world model, the lognormal process, can be regarded as the limit of a binomial process, for which, as we have seen in Section 16.5, the corresponding risk neutral model is also binomial; the limit as the number of steps increases in the (risk neutral) binomial model is then also a lognormal process. The second connection does not have any simple explanation. Note that the parameter μ, the mean log-return per unit time for the P-measure, does not appear in the specification of the Q-measure. This should not surprise us: the real world probabilities of upward and downward movements in the binomial model did not appear in the corresponding Q-measure probability, q.
- Formula (16.4) is the continuous-time extension of the same result for the single period binomial model (Example 16.1) and the two-period binomial model (Section 16.5.3). In both the binomial and Black–Scholes–Merton models, we take the expectation under the Q-measure of the payoff discounted at the risk-free force of interest.
- A mathematical derivation of the Q-measure and of formula (16.4) is beyond the scope of this book. Interested readers should consult the references in Section 16.7.

Now consider the particular case of a European call option with strike price K. The option price at time t is $c(t)$, where

$$c(t) = \mathrm{E}_t^Q\left[e^{-r(T-t)}(S_T-K)^+\right]. \tag{16.6}$$

To evaluate this price, first we write it as

$$c(t) = e^{-r(T-t)} S_t \, \mathrm{E}_t^Q \left[(S_T/S_t - K/S_t)^+ \right].$$

Now note that, under the Q-measure,

$$S_T/S_t \sim LN((r - \sigma^2/2)(T-t),\ \sigma\sqrt{(T-t)}).$$

So, letting f and F denote the lognormal probability density function and distribution function, respectively, of S_T/S_t, under the Q-measure, we have

$$\begin{aligned} c(t) &= e^{-r(T-t)} S_t \int_{K/S_t}^{\infty} (x - K/S_t) f(x)\, dx \\ &= e^{-r(T-t)} S_t \left(\int_{K/S_t}^{\infty} x f(x)\, dx - \frac{K}{S_t}(1 - F(K/S_t)) \right). \end{aligned} \tag{16.7}$$

In Appendix A we derive the formula

$$\int_0^a x f_{LN}(x)\, dx = \exp\{\mu + \sigma^2/2\}\, \Phi\left(\frac{\log a - \mu - \sigma^2}{\sigma} \right),$$

where f_{LN} is the density function of a lognormal random variable with parameters μ and σ, and Φ denotes the standard normal distribution function. Since the mean of this random variable is

$$\int_0^\infty x f_{LN}(x)\, dx = \exp\{\mu + \sigma^2/2\},$$

we have

$$\begin{aligned} \int_a^\infty x f_{LN}(x)\, dx &= \exp\{\mu + \sigma^2/2\} \left(1 - \Phi\left(\frac{\log a - \mu - \sigma^2}{\sigma} \right) \right) \\ &= \exp\{\mu + \sigma^2/2\}\, \Phi\left(\frac{-\log a + \mu + \sigma^2}{\sigma} \right). \end{aligned}$$

Applying this to formula (16.7) for $c(t)$ gives

$$\begin{aligned} c(t) = e^{-r(T-t)} S_t\, e^{r(T-t)}\, \Phi\left(\frac{-\log(K/S_t) + (r - \sigma^2/2)(T-t) + \sigma^2(T-t)}{\sigma\sqrt{T-t}} \right) \\ - e^{-r(T-t)} K \left(1 - \Phi\left(\frac{\log(K/S_t) - (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \right) \right) \end{aligned}$$

$$= S_t\,\Phi\left(\frac{\log(S_t/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right)$$
$$- e^{-r(T-t)} K\,\Phi\left(\frac{\log(S_t/K) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right),$$

which we usually write as

$$\boxed{c(t) = S_t \Phi\left(d_1(t)\right) - K e^{-r(T-t)} \Phi\left(d_2(t)\right),} \tag{16.8}$$

where

$$\boxed{d_1(t) = \frac{\log(S_t/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2(t) = d_1(t) - \sigma\sqrt{T-t}.} \tag{16.9}$$

Since the stock price S_t appears (explicitly) only in the first term of formula (16.8) and r appears only in the second term, this formula *suggests* that the replicating portfolio at time t for the call option comprises

- $\Phi\left(d_1(t)\right)$ units of the stock, with total value at time t

$$S_t\,\Phi\left(d_1(t)\right),$$

  plus
- a **short** holding of $\Phi\left(d_2(t)\right)$ units of zero-coupon bonds with face value K, maturing at time T, with a value at time t of

$$-K e^{-r(T-t)} \Phi\left(d_2(t)\right).$$

Indeed, this **is** the self-financing replicating portfolio required at time t. We note though that the derivation is not quite as simple as it looks, as $\Phi\left(d_1(t)\right)$ and $\Phi\left(d_2(t)\right)$ both depend on the current stock price and time.

If the strike price is very small relative to the stock price we see that $\Phi\left(d_1(t)\right)$ tends to one and $\Phi\left(d_2(t)\right)$ tends to zero. The replicating portfolio tends to a long position in the stock and zero in the bond.

For a European put option, with strike price K, the option price at time t is $p(t)$, where

$$p(t) = \mathrm{E}_t^Q\left[e^{-r(T-t)}(K - S_T)^+\right],$$

which, after working through the integration, becomes

$$\boxed{p(t) = K e^{-r(T-t)} \Phi\left(-d_2(t)\right) - S_t \Phi\left(-d_1(t)\right),} \tag{16.10}$$

where $d_1(t)$ and $d_2(t)$ are defined as before. The replicating portfolio for the put option comprises

- $\Phi\left(-d_2(t)\right)$ units of zero-coupon bonds with face value K, maturing at time T, with value at time t

$$Ke^{-r(T-t)}\,\Phi\left(-d_2(t)\right),$$

  plus
- a **short** holding of $\Phi\left(-d_1(t)\right)$ units of the stock, with total value at time t

$$-S_t\,\Phi\left(-d_1(t)\right).$$

For the European call and put options, we can show that

$$S_t\,\frac{d}{dS_t}\,c(t) = S_t\,\Phi(d_1(t)) \quad \text{and} \quad S_t\,\frac{d}{dS_t}\,p(t) = -S_t\,\Phi(-d_1(t)).$$

You are asked to prove the first of these formulae as Exercise 16.2. These two formulae show that, for these options, the replicating portfolio has a portion $S_t\,dv(t)/dS_t$ invested in the stock, and hence a portion $v(t) - S_t\,dv(t)/dS_t$ invested in the bond, where $v(t)$ is the value of the option at time t.

This result holds generally for any option valued under the Black–Scholes–Merton framework. The quantity $dv(t)/dS_t$ is known as the **delta** of the option at time t. The portfolio is the **delta hedge**.

Example 16.3 Let $p(t)$ and $c(t)$ be the prices at time t for a European put and call, respectively, both with strike price K and remaining term to maturity $T-t$.

(a) Use formulae (16.8) and (16.10) to show that, using the Black–Scholes–Merton framework,

$$c(t) + K\,e^{-r(T-t)} = p(t) + S_t. \tag{16.11}$$

(b) Use a no-arbitrage argument to show that formula (16.11) holds *whatever the model for stock price movements between times t and T*.

Solution 16.3 (a) From formulae (16.8) and (16.10), and using the fact that $\Phi(z) = 1 - \Phi(-z)$ for any z, we have

$$\begin{aligned} c(t) &= S_t\,\Phi(d_1(t)) - Ke^{-r(T-t)}\,\Phi(d_2(t)) \\ &= S_t(1 - \Phi(-d_1(t))) - Ke^{-r(T-t)}(1 - \Phi(-d_2(t))) \\ &= S_t - Ke^{-r(T-t)} + p(t), \end{aligned}$$

which proves (16.11).

(b) To prove this result without specifying a model for stock price movements, consider two portfolios held at time t. The first comprises the call option plus a zero-coupon bond with face value K maturing at time T; the second comprises the put option plus one unit of the stock. These two portfolios have current values

$$c(t) + K\,e^{-r(T-t)} \quad \text{and} \quad p(t) + S_t,$$

respectively. At time T the first portfolio will be worth K if $S_T \leq K$, since the call option will then be worthless and the bond will pay K, and it will be worth S_T if $S_T > K$, since then the call option would be exercised and the proceeds from the bond would be used to purchase one unit of stock. Now consider the second portfolio at time T. This will be worth K if $S_T \leq K$, since the put option would be exercised and the stock would be sold at the exercise price, K, and it will be worth S_T if $S_T > K$, since the put option will then be worthless and the stock will be worth S_T. Since the two portfolios have the same payoff at time T under all circumstances, they must have the same value at all other times, in particular at time t. This gives equation (16.11).

This important result is known as **put–call parity**. □

Example 16.4 An insurer offers a two-year contract with a guarantee under which the policyholder invests a premium of \$1000. The insurer keeps 3% of the premium to cover all expenses, then invests the remainder in a mutual fund. (A mutual fund is an investment that comprises a diverse portfolio of stocks and bonds. In the UK similar products are called unit trusts or investment trusts.) The mutual fund investment value is assumed to follow a lognormal process, with parameters $\mu = 0.085$ and $\sigma = 0.2$ per year. The mutual fund does not pay out dividends; any dividends received from the underlying portfolio are reinvested. The risk-free rate of interest is 5% per year compounded continuously. The insurer guarantees that the payout at the maturity date will not be less than the original \$1000 investment.

(a) Show that the 3% expense loading is not sufficient to fund the guarantee.
(b) Calculate the real world probability that the guarantee applies at the maturity date.
(c) Calculate the expense loading that would be exactly sufficient to fund the guarantee.

Solution 16.4 (a) The policyholder has, through the insurer, invested \$970 in the mutual fund. This will accumulate over the two years of the contract to some random amount, S_2, say. If $S_2 <$ \$1000 then the insurer's guarantee bites, and the insurer must make up the difference. In other words, the policyholder has the right at the maturity date to receive a price of \$1000 from the insurer for the mutual fund stocks. This is a two-year put option, with payoff at time $T = 2$ of

$$(1\,000 - S_2)^+.$$

If the mutual fund stocks are worth more than \$1000, then the policyholder just takes the proceeds and the insurer has no further liability.

In terms of option pricing, we have a strike price $K = \$1000$, a mutual fund stock price at time $t = 0$ of $S_0 = \$970$, and a risk-free rate of interest of 5%. So the price of the put option at inception is

$$p(0) = Ke^{-rT}\Phi\left(-d_2(0)\right) - S_0\Phi\left(-d_1(0)\right)$$

where

$$d_1(0) = \frac{\log(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} = 0.3873 \Rightarrow \Phi(-d_1(0)) = 0.3493,$$

$$d_2(0) = d_1(0) - \sigma\sqrt{T} = 0.1044 \Rightarrow \Phi(-d_2(0)) = 0.4584,$$

giving

$$p(0) = 414.786 - 338.794 = \$75.99.$$

So the 3% expense charge, \$30, is insufficient to fund the guarantee cost. The cost of the guarantee is actually 7.599% of the initial investment. However, if we actually set 7.599% as the expense loading, the price of the guarantee would be even greater, as we would invest less money in the mutual fund at inception whilst keeping the same strike price.

(b) The real world distribution of S_2/S_0 is $LN(2\mu, \sqrt{2}\sigma)$. This means that

$$\log(S_2/S_0) \sim N(2\mu, 2\sigma^2),$$

and hence

$$\log S_2 \sim N(\log S_0 + 2\mu, 2\sigma^2).$$

Then $S_2 \sim LN(\log S_0 + 2\mu, \sqrt{2}\sigma)$, which implies that

$$\Pr[S_2 < 1\,000] = \Phi\left(\frac{\log 1\,000 - \log S_0 - 2\mu}{\sigma\sqrt{2}}\right) = 0.311.$$

That is, the probability of a payoff under the guarantee is 0.311.

(c) Increasing the expense loading increases the cost of the guarantee, and there is no analytic method to find the expense loading, \$$E$, which pays for the guarantee with an initial investment of \$$(1\,000-E)$. Figure 16.2 shows a plot of the expense loading against the cost of the guarantee (shown as a solid line). Where this line crosses the line $x = y$ (shown as a dotted line) we have a solution. From this plot we see that the solution is around 10.72% (i.e. the expense loading is around \$107.2). Alternatively, Excel Solver gives the solution that an expense loading of 10.723% exactly funds the resulting guarantee. □

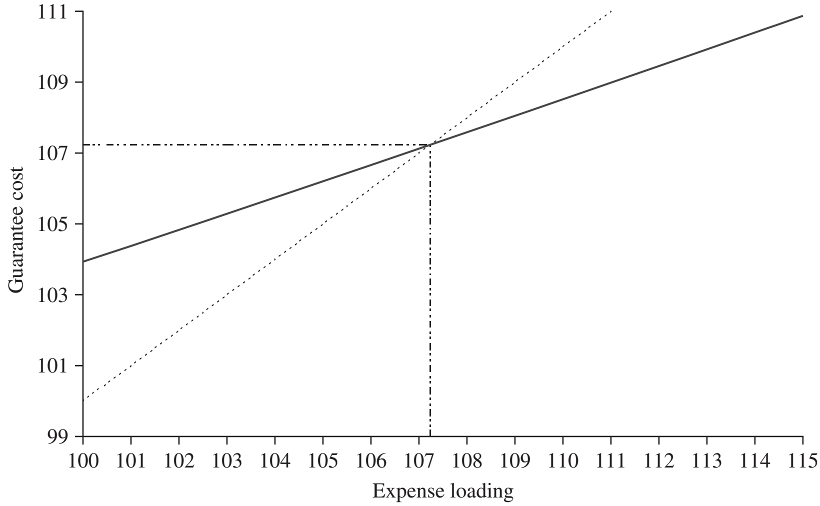


Figure 16.2 Expense loading plotted against guarantee cost for Example 16.4.

Finding the price is only the first step in the process. The beauty of the Black–Scholes–Merton approach is that it gives not only the price but also directs us in what we can do with the price to manage the guarantee risk. In part (a) of Example 16.4, the guarantee payoff can be replicated by investing \$414.79 in two-year zero-coupon bonds and short selling \$338.79 of the mutual fund stock, with a net cost of \$75.99. If we continuously rebalance such that at any time t the bond position has value $1\,000e^{-r(2-t)}\Phi(-d_2(t))$ and the short stock position has value $-\,S_t\Phi(-d_1(t))$, then this will exactly pay off the guarantee liability at the maturity date.

In practice, continuous rebalancing is impossible. Rebalancing at discrete intervals is possible but introduces some additional cash flows, and in the next example we explore this issue.

Example 16.5 Let us continue Example 16.4 above, where an insurer has issued a guarantee which matures in two years. The initial investment (net of expenses) is \$970 and the maturity guarantee is \$1 000.

In Table 16.1 you are given the monthly values for the underlying mutual fund stock price for the two-year period, assuming a starting price of \$970.

Assume, as in Example 16.4, that the continuously compounded risk-free rate, r, is 5% per year. Determine the cash flows arising assuming that the insurer

(a) invests the entire option cost in the risk-free asset,
(b) invests the entire option cost in the mutual fund asset,

Table 16.1 *Table of mutual fund stock prices for Example 16.5.*

Time, t (months)	S_t \$
0	970.00
1	950.07
2	959.99
3	940.93
4	921.06
5	967.25
6	1045.15
7	1007.59
8	945.97
9	913.77
10	932.99
11	951.11
12	906.11
13	824.86
14	831.08
15	797.99
16	785.86
17	724.36
18	707.43
19	713.87
20	715.14
21	690.74
22	675.80
23	699.71
24	766.66

(c) allocates the initial option cost to bonds and the mutual fund at the outset, according to the Black–Scholes–Merton model, that is \$414.79 to zero-coupon bonds and −\$338.79 to the mutual fund shares, and then
  (i) never subsequently rebalances the portfolio,
  (ii) rebalances only once, at the end of the first year, and
  (iii) rebalances at the end of each month.

Solution 16.5 We note that the guarantee ends in-the-money, with a liability under the put option of \$(1 000 − 766.66) = \$233.34 at the maturity date.

(a) If the option cost is invested in the risk-free asset, it accumulates to $75.99e^{2r}$ = \$83.98. This leaves a shortfall at maturity of \$(233.34 − 83.98) = \$149.36.

(b) If the option cost is invested in the mutual fund asset, it will accumulate to $75.99 \times (766.66/970) = \$60.06$ leaving a shortfall at maturity of \$173.28.

(c) (i) If the insurer invests in the initial hedge portfolio, but never rebalances,

- the bond part of the hedge accumulates at the risk-free rate for the whole two-year period to an end value of \$458.41;
- the stock part of the hedge accumulates in proportion to the mutual fund share price, with final value $-338.79 \times (766.66/970) = -\$267.77$; and
- the hedge portfolio value at maturity is then worth $458.41 - 267.77 = \$190.64$, which means that the insurer is liable for an additional cash flow at maturity of \$42.70, as the hedge portfolio value is less than the option guarantee cost. In this case the total cost of the guarantee is the initial hedge cost of \$75.99 plus a final balancing payment of \$42.70.

(ii) If the insurer rebalances only once, at the end of the first year, the value of the initial hedge portfolio at that time is

Bonds: $414.79e^r = \$436.05$.
Mutual fund: $-338.79\,(906.11/970) = -\$316.48$.

So the value of the hedge portfolio immediately before rebalancing is \$119.57.

The rebalanced hedge is found from formula (16.10) with $t = 1$ year as

$$\begin{aligned} p(1) &= Ke^{-r(2-1)}\Phi\left(-d_2(1)\right) - S_1\Phi\left(-d_1(1)\right) \\ &= 603.26 - 504.56 \\ &= 98.70. \end{aligned}$$

This means there is a cash flow of $\$(119.57 - 98.70) = \$20.87$ back to the insurer, as the value of the initial hedge more than pays for the rebalanced hedge.

We now track the new hedge through to the maturity date.
Bonds: $603.26e^r = \$634.19$.
Mutual fund: $-504.56 \times (766.66/906.11) = -\$426.91$.
Total hedge portfolio value: \$207.28.
We need \$233.34 to pay the guarantee liability, so the insurer is liable for an additional cash flow of \$26.06.

So, in tabular form we have the following cash flows, where a positive value is a cash flow out and negative value is a cash flow back to the insurer.

Time (years)	Value of hedge brought forward	Cost of new hedge	Final guarantee cost	Net cash flow $
0	0.00	75.99	–	75.99
1	119.57	98.70	–	–20.87
2	207.28	–	233.34	26.06

(iii) Here, we repeat the exercise in part (b) but we now accumulate and rebalance each month. The results are given in Table 16.2. The second, third and fourth columns show the bond part, the mutual fund part and the total cost of the hedge required at the start of each month. In the final month, the total reflects the cost of the guarantee payoff. The fifth column shows the value of the hedge brought forward, and the difference between the new hedge cost and the hedge brought forward is the cash flow required at that time.

We see how the rebalancing frequency affects the cash flows; with a monthly rebalancing frequency, all the cash flows required are relatively small, after the initial hedge cost. The fact that these cash flows are non-zero indicates that the original hedge is not self-financing with monthly rebalancing. However, the amounts are small, demonstrating that if the insurer follows this rebalancing strategy, there is little additional cost involved after the initial hedge cost, even though the final guarantee payout is substantial. The total of the additional cash flows after the initial hedge cost is −\$12.26 in this case. It can be shown that the expected value of the additional cash flows using the P-measure is zero. □

This example demonstrates that in this case, where the option matures in-the-money, the dynamic hedge is remarkably efficient at converging to the payoff with only small adjustments required each month. If we were to rebalance more frequently still, the rebalancing cash flows would converge to zero. In practice, many hedge portfolios are rebalanced daily or even several times a day.

Of course, this guarantee might well end up out-of-the-money, in which case the hedge portfolio would be worth nothing at the maturity date, and the insurer would lose the cost of establishing the hedge portfolio in the first place. The hedge is a form of insurance, and, as with all insurance, there is a cost even when there is no claim.

16.7 Notes and further reading

This chapter offers a very brief introduction to an important and exciting area. For a much more comprehensive introduction, see for example Hull (2017)

Table 16.2 *Cash flow calculations for Example 16.5.*

Time	New hedge portfolio			Old hedge	Net cash flow
(months)	Bonds	Mutual Fund	Total	brought forward	
0	414.79	−338.79	75.99	0.00	–75.99
1	446.09	−363.17	82.92	84.68	–1.76
2	437.17	−358.37	78.80	80.99	–2.19
3	469.69	−383.83	85.86	87.74	–1.88
4	505.72	−411.67	94.05	95.93	–1.88
5	441.15	−366.59	74.56	75.52	–0.96
6	332.07	−283.53	48.54	46.88	1.66
7	388.22	−329.43	58.79	60.11	–1.33
8	492.86	−411.51	81.35	80.56	0.79
9	557.18	−461.25	95.94	97.41	–1.48
10	531.28	−445.30	85.97	88.56	–2.59
11	505.60	−428.81	76.78	79.54	–2.76
12	603.26	−504.56	98.70	99.18	–0.48
13	769.54	−617.58	151.96	146.46	5.50
14	776.58	−628.41	148.17	150.52	–2.35
15	847.22	−671.33	175.88	176.43	–0.55
16	882.11	−693.88	188.22	189.62	–1.40
17	948.97	−700.74	248.24	246.21	2.03
18	965.94	−697.59	268.35	268.58	–0.23
19	973.54	−707.77	265.76	266.03	–0.27
20	981.09	−712.67	268.42	268.57	–0.15
21	987.44	−690.59	296.84	296.83	0.01
22	991.70	−675.80	315.90	315.90	0.00
23	995.84	−699.71	296.13	296.13	0.00
24			233.34	233.34	0.00

or McDonald (2009). For a description of the history of options and option pricing, see Boyle and Boyle (2001).

The proof that the binomial model converges to the lognormal model as the time unit, h, tends to zero is somewhat technical. The original proof is given in Cox *et al.* (1979).

We assumed from Section 16.6.1 onwards that the stock did not pay any dividends. Adapting the model and results for dividends is explained in Hull (2017) and McDonald (2009).

16.8 Exercises

Shorter exercises

Exercise 16.1 A non-dividend paying stock has a current price of \$8.00. In any unit of time $(t, t+1)$ the price of the stock either increases by 25% or decreases by 20%. \$1 held in cash between times t and $t+1$ receives interest to become \$1.04 at time $t+1$. The stock price after t time units is denoted by S_t.

(a) Calculate the risk-neutral probability measure for the model.
(b) Calculate the price (at time $t=0$) of a derivative contract written on the stock with expiry date $t=2$ which pays \$10.00 if and only if S_2 is not \$8.00 (and otherwise pays 0).

Longer exercises

Exercise 16.2 Let $c(t)$ denote the price of a call option on a non-dividend paying stock, using equation (16.6). Show that

$$\frac{dc(t)}{dS_t} = \Phi(d_1(t)).$$

Hint: Remember that $d_1(t)$ is a function of S_t.

Exercise 16.3 (a) Show that, under the binomial model of Section 16.5,

$$E^Q[S_n] = S_0\, e^{rn}.$$

(b) Show that, under the Black–Scholes–Merton model,

$$E^Q[S_n] = S_0\, e^{rn}.$$

Exercise 16.4 A binomial model for a non-dividend paying security with price S_t at time t is as follows:

$$S_0 = 100,$$

$$S_{t+1} = \begin{cases} 1.1S_t & \text{if the stock price rises,} \\ 0.9S_t & \text{if the stock price falls.} \end{cases}$$

Zero-coupon bonds are available for all integer durations, with a risk-free rate of interest of 6% per time period compounded continuously.

A derivative security pays \$20 at a specified maturity date if the stock price has increased from the start value, and pays \$0 if the stock price is at or below the start value at maturity.

(a) Find the price and the replicating portfolio for the option assuming it is issued at $t=0$ and matures at $t=1$.
(b) Now assume the option is issued at $t=0$ and matures at $t=2$. Find the price and the replicating portfolio at $t=0$ and at $t=1$.

Exercise 16.5 Consider a two-period binomial model for a non-dividend paying security with price S_t at time t, where $S_0 = 1.0$,

$$S_{t+1} = \begin{cases} 1.2S_t & \text{if the stock price rises,} \\ 0.95S_t & \text{if the stock price falls.} \end{cases}$$

At time $t = 2$ option A pays \$3 if the stock price has risen twice, \$2 if it has risen once and fallen once and \$1 if it has fallen twice.

At time $t = 2$ option B pays \$1 if the stock price has risen twice, \$2 if it has risen once and fallen once and \$3 if it has fallen twice.

The risk-free force of interest is 4.879% per period. You are given that the true probability that the price rises each period is 0.5.

(a) Calculate the EPV (under the P-measure) of option A and show that it is the same as the EPV of option B.
(b) Calculate the price of option A and show that it is different to the price of option B.
(c) Comment on why the prices differ even though the expected payout is the same.

Exercise 16.6 A binomial model for a non-dividend paying security with price S_t at time t is as follows: the price at time $t + 1$ is either $1.25\,S_t$ or $0.8\,S_t$. The risk-free rate of interest is 10% per time unit effective.

(a) Calculate the risk neutral probability measure for this model.

The value of S_0 is 100. A derivative security with price D_t at time t pays the following returns at time 2:

$$D_2 = \begin{cases} 1 & \text{if } S_2 = 156.25, \\ 2 & \text{if } S_2 = 100, \\ 0 & \text{if } S_2 = 64. \end{cases}$$

(b) Determine D_1 when $S_1 = 125$ and when $S_1 = 80$ and hence calculate the value of D_0.
(c) Derive the corresponding hedging strategy, i.e. the combination of the underlying security and the risk-free asset required to hedge an investment in the derivative security.
(d) Comment on your answer to (c) in the light of your answer to part (b).

Excel-based exercises

Exercise 16.7 A stock is currently priced at \$400. The price of a six-month European call option with a strike price of \$420 is \$41. The risk-free rate of interest is 7% per year, compounded continuously.

Assume the Black–Scholes pricing formula applies.

(a) Calculate the current price of a six-month European put option with the same exercise price. State the assumptions you make in the calculation.
(b) Estimate the implied volatility of the stock (Hint: use Excel Solver)
(c) Calculate the delta of the option.
(d) Find the hedging portfolio of stock and risk-free zero-coupon bonds that a writer of 10 000 units of the call option should hold.

Answers to selected exercises

16.1 (a) Probability of increase is 0.5333, probability of decrease is 0.4667
(b) \$4.6433

16.4 (a) \$15.24 (b) \$11.61

16.5 (a) \$1.81 (b) Option A: \$1.633, Option B: \$1.995

16.6 (a) Probability of increase is $\frac{2}{3}$, probability of decrease is $\frac{1}{3}$
(b) $D_0 = 1.1019$

16.7 (a) \$46.55 (b) 38.6% (c) 53.42%
(d) Long 5 342.5 shares of stock and short 17 270 bonds, where each bond is worth \$100 at time zero

17
Embedded options

17.1 Summary

In this chapter we describe financial options embedded in insurance contracts, focusing in particular on the most straightforward options which appear as guaranteed minimum death and maturity options in equity-linked life insurance policies effected by a single premium. We investigate pricing, valuation and risk management for these guarantees, performing our analysis under the Black–Scholes–Merton framework described in Chapter 16.

17.2 Introduction

The guaranteed minimum payments under an equity-linked contract usually represent a relatively minor aspect of the total payout under the policy, because the guarantees are designed to apply only in the most extreme situation of very poor returns on the policyholders' funds. Nevertheless, these guarantees are not negligible – failure to manage the risk from apparently innocuous guarantees has led to significant financial problems for some insurers.

In Chapter 15 we described profit testing of equity-linked contracts with guarantees, where the only risk management involved was a passive strategy of holding capital reserves in case the experience is adverse – or, even worse, holding no capital in the expectation that the guarantee will never apply. However, in the case when the equity-linked contract incorporates financial guarantees that are essentially the same as the financial options discussed in Chapter 16, we can use the more sophisticated techniques of Chapter 16 to price and manage the risks associated with the guarantees. These techniques are preferable to those of Chapter 15 because they mitigate the risk that the insurer will have insufficient funds to pay for the guarantees when necessary.

To show how the guarantees can be viewed as options, recall Example 15.2 in Chapter 15, where we described an equity-linked insurance contract, purchased by a single premium P, with a guaranteed minimum maturity benefit

(GMMB) and a guaranteed minimum death benefit (GMDB). Consider, for now, the GMMB only. After some expense deductions a single premium is invested in an equity fund, or perhaps a mixed equity/bond fund. The fund value is variable, moving up and down with the underlying assets. At maturity, the insurer promises to pay the greater of the actual fund value and the original premium amount.

Let F_t denote the value of the policyholder's fund at time t. Suppose that, as in Example 15.2, the benefit for policies still in force at the maturity date, say at time n, (the term is $n = 5$ years in Example 15.2, but more typically it would be 10 years or longer) is $\max(P, F_n)$. As the policyholder's fund contributes the amount F_n, the insurer's additional liability is $h(n)$, where

$$h(n) = \max(P - F_n, 0).$$

The total benefit paid for such a contract in force at the maturity date is

$$F_n + h(n).$$

Recognizing that the fund value process $\{F_t\}_{t\geq 0}$ may be considered analogous to a stock price process, and that P is a fixed, known amount, the guarantee payoff $h(n)$ is identical to the payoff under an n-year European put option with strike price $\$P$, as described in Section 16.4. So, while in Chapter 15 we modelled this contract with cash flow projection, we have a more appropriate technique for pricing and valuation from Chapter 16, using the Black–Scholes–Merton framework.

Similarly, the guaranteed minimum death benefit in an equity-linked insurance contract offers a payoff that can be viewed as an option – often a put option similar to that under a GMMB.

There are a few differences between the options embedded in equity-linked contracts and standard options traded in markets. Two important differences are as follows.

(1) The options embedded in equity-linked contracts have random terms to maturity. If the policyholder surrenders the contract, or dies, before the expiry date, the GMMB will never be paid. The GMDB expires on the death of the policyholder, if that occurs during the term of the contract.
(2) The options embedded in equity-linked contracts depend on the value of the policyholder's fund at death or maturity. The underlying risky asset process represents the value of a traded stock or stock index. The fund value at time t, F_t, is related to the risky asset price, S_t, since we assume the policyholder's fund is invested in a fund with returns following traded stocks, but with the important difference that regular management charges are being deducted from the policyholder's fund.

These differences mean that we must adapt the Black–Scholes–Merton theory of Chapter 16 in order to apply it to equity-linked insurance.

Throughout this chapter we consider equity-linked contracts purchased by a single premium, P, which, after the deduction of any initial charges, is invested in the policyholder's fund. This fund, before allowing for the deduction of any management charges, earns returns following the underlying stock price process, $\{S_t\}_{t\geq 0}$. We make all the assumptions in Section 16.6.1 relating to the Black–Scholes–Merton framework. In particular, we assume the stock price process is a lognormal process with volatility σ per year, and also that there is a risk-free rate of interest, r per year, continuously compounded.

17.3 Guaranteed minimum maturity benefit

17.3.1 Pricing

From Chapter 16 we know that the price of an option is the EPV of the payoff under the risk neutral probability distribution, discounting at the risk-free rate. Suppose a GMMB under a single premium contract guarantees that the payout at maturity, n years after the issue date of the contract, will be at least equal to the single premium, P. Then the option payoff, as mentioned above, is $h(n) = \max(P - F_n, 0)$, because the remainder of the benefit, F_n, will be paid from the policyholder's fund. This payoff is conditional on the policy remaining in force until the maturity date. In order to price the guarantee we assume that the survival of a policyholder for n years, taking account of mortality and lapses, is independent of the fund value process and is a diversifiable risk. For simplicity here we ignore surrenders and assume all policyholders are aged x at the commencement of their policies, and are all subject to the same survival model. Under these assumptions, the probability that a policy will still be in force at the end of the term is ${}_np_x$.

Consider the situation at the issue of the contract. If the policyholder does not survive n years, the GMMB does not apply at time n, and so the insurer does not need to fund the guarantee in this case. If the policyholder does survive n years, the GMMB does apply at time n, and we know that the amount required at the issue of the contract to fund this guarantee is

$$\mathrm{E}_0^Q\left[e^{-rn}\left(P - F_n\right)^+\right].$$

Thus, the expected amount (with respect to mortality and lapses) required by the insurer at the time of issue *per contract issued* is $\pi(0)$, where

$$\pi(0) = {}_np_x\,\mathrm{E}_0^Q\left[e^{-rn}\left(P - F_n\right)^+\right].$$

Note that we are adopting a mixture of two different methodologies here. The non-diversifiable risk from the stock price process, which channels through to

F_n, is priced using the methodology of Chapter 16, whereas the mortality risk, which we have assumed to be diversifiable, is priced using the expected value principle.

Suppose that the total initial expenses are a proportion e of the single premium, and the management charge is a proportion m of the policyholder's fund, deducted at the start of each year after the first. Then

$$F_n = P(1-e)(1-m)^{n-1}\,\frac{S_n}{S_0}.$$

Since we are interested in the relative increase in S_t, we can assume $S_0 = 1$ without any loss of generality. (We interpret the stock price process $\{S_t\}_{t\geq 0}$ as an index for the fund assets; as an index, we can arbitrarily set S_0 to any convenient value.) Then

$$F_n = P(1-e)(1-m)^{n-1}S_n.$$

The value of the guarantee can be written

$$\begin{aligned}\pi(0) &= {}_np_x\,\mathrm{E}_0^Q\left[e^{-rn}(P - P(1-e)(1-m)^{n-1}S_n)^+\right]\\ &= P\,{}_np_x\,\mathrm{E}_0^Q\left[e^{-rn}\left(1-(1-e)(1-m)^{n-1}S_n\right)^+\right]\\ &= P\,{}_np_x\,\xi\,\mathrm{E}_0^Q\left[e^{-rn}\left(\xi^{-1}-S_n\right)^+\right]\end{aligned}$$

where the expense factor $\xi = (1-e)(1-m)^{n-1}$ is a constant. We can now apply formula (16.10) for the price of a put option, setting the strike price for the option per unit of guarantee, $K = \xi^{-1}$. Then the price at the issue date of a GMMB, guaranteeing a return of at least the premium P, is

$$\begin{aligned}\pi(0) &= P\,{}_np_x\,\xi\,\left(\xi^{-1}\,e^{-rn}\Phi(-d_2(0)) - \Phi(-d_1(0))\right) \qquad (17.1)\\ &= P\,{}_np_x\,\left(e^{-rn}\Phi(-d_2(0)) - \xi\,\Phi(-d_1(0))\right)\end{aligned}$$

where

$$d_1(0) = \frac{\log\xi + (r+\sigma^2/2)n}{\sigma\sqrt{n}} \quad \text{and} \quad d_2(0) = d_1(0) - \sigma\sqrt{n}.$$

The return of premium guarantee is a common design for a GMMB, but many other designs are sold. Any guarantee can be viewed as a financial option. Suppose $h(n)$ denotes a general payoff function for a GMMB when it matures at time n years. In equation (17.1) the payoff function is $h(n) = (P - F_n)^+$. In other cases when the only random quantity in the payoff function is the fund value at maturity, we can use exactly the same approach as in equation (17.1), so that the value of the GMMB is always

$$\pi(0) = {}_np_x \, \mathrm{E}_0^Q \left[e^{-rn} h(n) \right].$$

Example 17.1 Consider a 10-year equity-linked contract issued to a life aged 60, with a single premium of $P = \$10\,000$. After a deduction of 3% for initial expenses, the premium is invested in an equity fund. An annual management charge of 0.5% is deducted from the fund at the start of every year except the first.

The contract carries a guarantee that the maturity benefit will not be less than the single premium, P.

The risk-free rate of interest is 5% per year, continuously compounded, and stock price volatility is 25% per year.

(a) Calculate the cost at issue of the GMMB as a percentage of the single premium, assuming there are no lapses and that the survival model is Makeham's law with $A = 0.0001$, $B = 0.00035$ and $c = 1.075$.
(b) Now suppose that, allowing for mortality and lapses, the insurer expects only 55% of policyholders to maintain their policies to maturity. Calculate the revised cost at issue of the GMMB as a percentage of the single premium, commenting on any additional assumptions required.

Solution 17.1 (a) With $n = 10$, we have

$$\xi = (1 - 0.03)(1 - 0.005)^9 = 0.927213,$$

$$d_1(0) = \frac{\log \xi + (r + \sigma^2/2)n}{\sigma\sqrt{n}} = 0.932148,$$

$$d_2(0) = d_1(0) - \sigma\sqrt{n} = 0.141579,$$

$$\mathrm{E}_0^Q \left[e^{-10r} \, h(10) \right] = 0.106275 \, P$$

and ${}_{10}p_{60} = 0.673958$ so that

$$\pi(0) = 0.0716P.$$

That is, the option cost, assuming no lapses, is 7.16% of the single premium.

(b) If we assume that precisely 55% of policies issued reach maturity, the option value per policy issued is reduced to $0.55 \, \mathrm{E}_0^Q \left[e^{-10r} \, h(10) \right] = 0.55 \times 0.106275P = 5.85\%$ of the single premium.

The assumption that 55% of policies reach maturity is reasonable if we assume that survival, allowing for mortality and lapses, is a diversifiable risk which is independent of the stock price process. In practice, lapse rates may depend on the fund's performance so that this assumption may not be reasonable. □

17.3.2 Reserving

We have already defined the reserve for an insurance contract as the capital set aside during the term of a policy to meet future obligations under the policy. In Chapter 15 we demonstrated a method of reserving for financial guarantees using a stochastic projection of the net present value of future outgo minus income, where we set the reserve to provide adequate resources in the event that investment experience for the portfolio was adverse.

Using the Black–Scholes–Merton approach, the value of the guarantee is interpreted as the value of the portfolio of assets that hedges, or replicates, the payoff under the guarantee. The insurer may use the cost of the guarantee to purchase appropriate options from another financial institution. If the mortality and lapse experiences follow the basis assumptions, the payoffs from the options will be precisely the amounts required for the guarantee payments. There is usually no need to hold further reserves since any reserve would cover only the future net expenses of maintaining the contract, which are, usually, fully defrayed by the future management charge income.

Increasingly, insurers are hedging their own guarantees. This should be less expensive than buying options from a third party, but requires the insurer to have the necessary expertise in financial risk management. When the insurer retains the risk, the contribution to the policy reserve for the guarantee will be the cost of maintaining the replicating portfolio. We saw in Chapter 16 that the cost of the replicating portfolio at time t, before an option matures, is the price of the option at time t.

Suppose we consider the GMMB from Section 17.3.1, where the guarantee liability for the insurer at maturity, time n, is $(P - F_n)^+$, and where the issue price was $\pi(0)$ from equation (17.1). The contribution to the reserve at time t, where $0 \le t \le n$, for the GMMB, assuming the contract is still in force at time t, is the value at t of the option, which is

$$\pi(t) = P \; {}_{n-t}p_{x+t} \left(e^{-r(n-t)} \Phi(-d_2(t)) - \xi S_t \Phi(-d_1(t)) \right),$$

where

$$d_1(t) = \frac{\log(\xi\, S_t) + (r + \sigma^2/2)(n-t)}{\sigma\sqrt{n-t}} \quad \text{and} \quad d_2(t) = d_1(t) - \sigma\sqrt{n-t}.$$

Note here that the expense factor $\xi = (1-e)(1-m)^{n-1}$ does not depend on t, but the reserve at time t does depend on the stock price at time t, S_t.

For a more general GMMB, with payoff $h(n)$ on survival to time n, the contribution to the reserve is

$$\pi(t) = {}_{n-t}p_{x+t}\, \mathrm{E}_t^Q \left[e^{-r(n-t)} h(n) \right],$$

where E_t^Q denotes the expectation at time t with respect to the Q-measure. In particular, E_t^Q assumes knowledge of the stock price process at t, S_t.

In principle, the hedge for the maturity guarantee will (under the basis assumptions) exactly pay off the guarantee liability, so there should be no need to apply stochastic reserving methods. In practice though, it is not possible to hedge the guarantee perfectly, as the assumptions of the Black–Scholes–Merton formula do not apply exactly. The insurer may hold an additional reserve over and above the hedge cost to allow for unhedgeable risk and for the risk that lapses, mortality and volatility do not exactly follow the basis assumptions. Determining an appropriate reserve for the unhedgeable risk is beyond the scope of this book, but could be based on the stochastic methodology described in Chapter 15.

Example 17.2 Assume that the policy in Example 17.1 is still in force six years after it was issued to a life aged 60. Assuming there are no lapses, calculate the contribution to the reserve from the GMMB at this time given that, since the policy was purchased, the value of the stock has

(a) increased by 45%, and
(b) increased by 5%.

Solution 17.2 (a) Recall that in the option valuation we have assumed that the return on the fund, before management charge deductions, is modelled by the index $\{S_t\}_{t\geq 0}$, where $S_0 = 1$. We are given that $S_6 = 1.45$. Then

$$\pi(6) = P\ {}_4p_{66}\left(e^{-4r}\Phi(-d_2(6)) - \xi S_6\Phi(-d_1(6))\right)$$

where

$$d_1(t) = \frac{\log(\xi S_t) + (r + \sigma^2/2)(10-t)}{\sigma\sqrt{10-t}} \quad \text{and} \quad d_2(t) = d_1(t) - \sigma\sqrt{10-t},$$

and $\xi = 0.927213$ as in Example 17.1. So

$$d_1(6) = 1.241983 \quad \text{and} \quad d_2(6) = 0.741983,$$

and as ${}_4p_{66} = 0.824935$,

$$\pi(6) = 0.035892P = \$358.92.$$

(b) For $S_6 = 1.05$, we have $\pi(6) = \$905.39$.

A lower current fund value means that the guarantee is more likely to mature in-the-money and so a larger reserve is required. □

17.4 Guaranteed minimum death benefit

17.4.1 Pricing

Not all equity-linked insurance policies carry GMMBs, but most carry GMDBs of some kind to distinguish them from regular investment products. The most common guarantees on death are a fixed or an increasing minimum death benefit. In Canada, for example, contracts typically offer a minimum death benefit of the total amount of premiums paid. In the USA, the guaranteed minimum payout on death might be the accumulation at some fixed rate of interest of all premiums paid. In the UK, the benefit might be the greater of the total amount of premiums paid and, say, 101% of the policyholder's fund.

We approach GMDBs in the same way as we approached GMMBs. Consider an n-year policy issued to a life aged x under which the payoff under the GMDB is $h(t)$ if the life dies at age $x+t$, where $t < n$. If the insurer knew at the issue of the policy that the life would die at age $x+t$, the insurer could cover the guarantee by setting aside

$$v(0,t) = \mathrm{E}_0^Q\left[e^{-rt}\, h(t)\right]$$

at the issue date, where Q is again the risk neutral measure for the stock price process that underlies the policyholder's fund.

We know from Chapter 2 that the probability density associated with death at age $x+t$ for a life now aged x is ${}_tp_x\mu_{x+t}$, and so the amount that should be set aside to cover the GMDB, denoted $\pi(0)$, is found by averaging over the possible ages at death, $x+t$, so that

$$\pi(0) = \int_0^n v(0,t)\, {}_tp_x\, \mu_{x+t}\, dt. \tag{17.2}$$

If the death benefit is payable at the end of the month of death rather than immediately, the value of the guarantee becomes

$$\pi(0) = \sum_{j=1}^{12n} v\,(0, j/12)\;\, {}_{\frac{j-1}{12}|\frac{1}{12}}q_x. \tag{17.3}$$

Notice that (17.2) and (17.3) are similar to formulae we have met in earlier chapters. For example, the EPV of a term insurance benefit of \$$S$ payable immediately on the death within n years of a life currently aged x is

$$\int_0^n S\, v^t\, {}_tp_x\, \mu_{x+t}\, dt. \tag{17.4}$$

There are similarities and differences between (17.2) and (17.4). In each expression we are finding the expected amount required at time 0 to provide a death benefit (and in each case we require 0 at time n with probability ${}_np_x$). In expression (17.4) the amount required if death occurs at time t is the present value of the payment at time t, namely Sv^t, whereas in expression (17.2) $v(0,t)$ is the amount required at time 0 in order to replicate the (possible) payment at time t.

Example 17.3 An insurer issues a five-year equity-linked insurance policy to a life aged 60. A single premium of $P = \$10\,000$ is invested in an equity fund. Management charges of 0.25% are deducted at the start of each month. At the end of the month of death before age 65, the death benefit is the accumulated amount of the investment with a GMDB equal to the accumulated amount of the single premium, with interest at 5% per year compounded continuously.

Calculate the value of the guarantee on the following basis.

- Survival model: Makeham's law with $A = 0.0001$, $B = 0.00035$ and $c = 1.075$
- Risk-free rate of interest: 5% per year, continuously compounded
- Volatility: 25% per year

Solution 17.3 As in previous examples, let $\{S_t\}_{t\geq 0}$ be an index of prices for the equity fund, with $S_0 = 1$, and let $m = 0.0025$ denote the monthly management charge. Then the payoff if death occurs in the month from time $k - \frac{1}{12}$ to k, for $k = \frac{1}{12}, \frac{2}{12}, \ldots, \frac{60}{12}$, is

$$h(k) = \max(Pe^{0.05k} - F_k, 0)$$

where

$$F_k = P\,S_k\,(1-m)^{12k},$$

so that

$$h(k) = P(1-m)^{12k} \max\left(\frac{e^{0.05k}}{(1-m)^{12k}} - S_k, 0\right).$$

For any value of k ($k = \frac{1}{12}, \frac{2}{12}, \ldots, \frac{60}{12}$), the payoff is a multiple of the payoff under a put option with strike price $e^{0.05k}/(1-m)^{12k}$. Before applying formula (16.10) to value this option, it is convenient to extend the notation for $d_1(t)$ and $d_2(t)$ in formula (16.9) to include the maturity date, so we now write these as $d_1(t,T)$ and $d_2(t,T)$ where T is the maturity date.

We can now apply formula (16.10) with strike price $e^{0.05k}/(1-m)^{12k}$, which we discount at the risk-free rate of $r = 0.05$, to obtain the first term in formula (16.10) as $\Phi(-d_2(0,k))/(1-m)^{12k}$. Thus, if $v(0,k)$ denotes the value at time 0 of the guarantee at time k, then

$$\begin{aligned} v(0,k) &= P(1-m)^{12k}\left(\frac{\Phi(-d_2(0,k))}{(1-m)^{12k}} - S_0\Phi(-d_1(0,k))\right) \\ &= P\left(\Phi(-d_2(0,k)) - (1-m)^{12k}\Phi(-d_1(0,k))\right) \end{aligned}$$

where, from (16.9),

$$d_1(0,k) = \frac{\log((1-m)^k/e^{0.05k}) + (r+\sigma^2/2)\,k}{\sigma\sqrt{k}}$$

and

$$d_2(0,k) = d_1(0,k) - \sigma\sqrt{k},$$

with $\sigma = 0.25$.

Table 17.1 shows selected values from a spreadsheet containing deferred mortality probabilities and option prices for each possible month of death. Using these values in formula (17.3), the value of this GMDB is 2.7838% of the single premium, or \$278.38. □

17.4.2 Reserving

We now apply the approach of the previous section to reserving for a GMDB on the assumption that the insurer is internally hedging. Consider a policy issued to a life aged x with a term of n years and with a GMDB which is payable immediately on death if death occurs at time s where $0 < s < n$. Suppose that the payoff function under the guarantee at time s is $h(s)$. Let $v(t,s)$ denote

Table 17.1 *Spreadsheet excerpt for the GMDB in Example 17.3.*

k (years)	$d_1(0,k)$	$d_2(0,k)$	$v(0,k)$	${}_{\frac{k-1}{12}\mid\frac{1}{12}}q_x$
1/12	0.001400	−0.070769	300.16	0.002248
2/12	0.001980	−0.100082	431.43	0.002257
3/12	0.002425	−0.122575	534.79	0.002265
4/12	0.002800	−0.141538	623.65	0.002273
5/12	0.003130	−0.158244	703.20	0.002282
6/12	0.003429	−0.173347	776.12	0.002290
7/12	0.003704	−0.187237	843.99	0.002299
⋮	⋮	⋮	⋮	⋮
56/12	0.010477	−0.529585	2708.30	0.002702
57/12	0.010570	−0.534293	2735.70	0.002709
58/12	0.010662	−0.538959	2762.88	0.002717
59/12	0.010754	−0.543585	2789.86	0.002725
60/12	0.010844	−0.548173	2816.63	0.002732

the price at time t for an option with payoff $h(s)$ at time s, where $0 \leq t \leq s$, assuming the policyholder dies at age $x+s$. Then

$$v(t,s) = \mathrm{E}_t^Q \left[e^{-r(s-t)}\, h(s) \right].$$

Hence, the value of the GMDB for a policy in force at time t $(< n)$ is $\pi(t)$, where

$$\begin{aligned}\pi(t) &= \int_t^n v(t,s)\; {}_{s-t}p_{x+t}\, \mu_{x+s}\, ds \\ &= \int_0^{n-t} v(t, w+t)\; {}_{w}p_{x+t}\, \mu_{x+t+w}\, dw,\end{aligned}$$

when the benefit is payable immediately on death, and

$$\pi(t) = \sum_{j=1}^{12(n-t)} v(t, t+j/12)\; {}_{\frac{j-1}{12}|\frac{1}{12}}q_{x+t},$$

when the benefit is payable at the end of the month of death.

Example 17.4 Assume that the policy in Example 17.3 is still in force three years and six months after the issue date. Calculate the contribution of the GMDB to the reserve if the stock price index of the underlying fund assets

(a) has grown by 50% since inception, so that $S_{3.5} = 1.5$, and
(b) is the same as the initial value, so that $S_{3.5} = 1$.

Solution 17.4 Following the solution to Example 17.3, the strike price for an option expiring at time s is $e^{0.05s}/(1-m)^{12s}$. Since we are valuing the option at time $t < s$, the time to expiry is now $s-t$. Thus, applying formula (16.10) we have

$$\begin{aligned}v(t,s) &= P(1-m)^{12s} \left(\frac{e^{0.05s}\, e^{-0.05(s-t)}}{(1-m)^{12s}}\, \Phi(-d_2(t,s)) - S_t \Phi(-d_1(t,s)) \right) \\ &= P \left(e^{0.05t}\, \Phi(-d_2(t,s)) - S_t (1-m)^{12s} \Phi(-d_1(t,s)) \right)\end{aligned}$$

where

$$d_1(t,s) = \frac{\log(S_t (1-m)^{12s}/e^{0.05s}) + (r+\sigma^2/2)(s-t)}{\sigma\sqrt{s-t}}$$

and

$$d_2(t,s) = d_1(t,s) - \sigma\sqrt{s-t}.$$

For the valuation at time $t = 3.5$, we calculate $v(3.5, s)$ for $s = 3\frac{7}{12}, 3\frac{8}{12}, 3\frac{9}{12}, \ldots, 5$ and multiply each value by the mortality probability, ${}_{s-t-\frac{1}{12}|\frac{1}{12}}q_{63.5}$. The resulting valuation is

(a) \$30.55 when $S_{3.5} = 1.5$, and
(b) \$172.05 when $S_{3.5} = 1$.

□

Example 17.5 An insurer offers a 10-year equity-linked policy with a single premium. An initial expense deduction of 4% of the premium is made, and the remainder of the premium is invested in an equity fund. Management charges are deducted daily from the policyholder's account at a rate of 0.6% per year. On death before the policy matures a death benefit of 110% of the fund value is payable. There is no guaranteed minimum maturity benefit.

(a) Calculate the price at issue of the excess amount of the death benefit over the fund value at the date of death for a life aged 55 at the purchase date, as a percentage of the single premium.
(b) Calculate the value of the excess amount of the death benefit over the fund value at the date of death six years after the issue date, as a percentage of the policyholder's fund at that date. You are given that the policy is still in force at the valuation date.

Basis:

- Survival model: Makeham's law, with $A = 0.0001$, $B = 0.00035$ and $c = 1.075$
- Risk-free rate of interest: 5% per year, continuously compounded
- Volatility: 25% per year

Solution 17.5 (a) First, we note that the daily management charge can be treated as a continuous deduction from the fund, so that, for a unit premium,

$$F_t = 0.96e^{-0.006t}S_t.$$

Second, we note that the excess amount of the death benefit over the fund value at the date of death can be viewed as a GMDB equal to 10% of the fund value at the date of death. For a unit premium, the payoff function $h(s)$ if death occurs at time s, is

$$h(s) = 0.1\,F_s = 0.096\,e^{-0.006s}S_s.$$

The value at issue of the death benefit payable if the policyholder dies at time s is

$$v(0, s) = \mathrm{E}_0^Q\left[e^{-rs}h(s)\right] = \mathrm{E}_0^Q\left[e^{-rs}\,0.096\,e^{-0.006s}S_s\right].$$

In the previous chapter we saw that under the risk neutral measure the EPV of a stock price at a future point in time is the stock price now. Thus

$$E_0^Q\left[e^{-rs}\,S_s\right] = S_0.$$

Since $S_0 = 1$, we have

$$v(0,s) = S_0 \times 0.096\,e^{-0.006s} = 0.096\,e^{-0.006s}.$$

The GMDB value at issue is then

$$\begin{aligned}\pi(0) &= \int_0^{10} v(0,s)\,{}_sp_{55}\,\mu_{55+s}\,ds = 0.096 \int_0^{10} e^{-0.006s}\,{}_sp_{55}\,\mu_{55+s}\,ds \\ &= 0.096\,\bar{A}^{\,1}_{55:\overline{10|}\,\delta=0.6\%} \qquad (17.5)\\ &= 0.02236.\end{aligned}$$

So the value of the GMDB at the inception of the policy is 2.24% of the single premium.

(b) The value at time $t < s$ of the option that would be needed to fund the GMDB if the policyholder were to die at time s, given that the policy is in force at t, is, for a unit premium,

$$v(t,s) = \mathrm{E}_t^Q\left[e^{-r(s-t)}\,h(s)\right] = 0.1 \times 0.96 S_t\,e^{-0.006s}.$$

The total contribution to the reserve for the GMDB for a policy still in force at time t, with original premium P, is then

$$\begin{aligned}\pi(t) &= P \int_0^{10-t} v(t, w+t)\,{}_wp_{55+t}\,\mu_{55+t+w}\,dw \\ &= 0.096\,P\,S_t \int_0^{10-t} e^{-0.006(w+t)}\,{}_wp_{55+t}\,\mu_{55+t+w}\,dw \\ &= 0.096\,P\,S_t\,e^{-0.006t} \int_0^{10-t} e^{-0.006\,w}\,{}_wp_{55+t}\,\mu_{55+t+w}\,dw \\ &= 0.096\,P\,S_t\,e^{-0.006t}\,\bar{A}^{\,1}_{55+t:\overline{10-t|}\,\delta=0.6\%}.\end{aligned}$$

So, at time $t = 6$, given that the policy is still in force, the contribution to the reserve from the GMDB, per unit premium, is

$$\begin{aligned}\pi(6) &= 0.096\,P\,S_6\,e^{-0.006\times 6}\,\bar{A}^{\,1}_{61:\overline{4|}\,\delta=0.6\%} \\ &= 0.096\,P\,S_6\,e^{-0.036} \times 0.12403.\end{aligned}$$

The fund value at time $t = 6$ is

$$F_6 = 0.96\,P\,S_6 \times e^{-0.036},$$

and so the reserve, as a proportion of the fund value, is

$$\frac{0.096\,P\,S_6\,e^{-0.036}\,\bar{A}^{\;1}_{61:\overline{4}|\,\delta=0.6\%}}{0.96\,P\,S_6\,e^{-0.036}} = 0.1\bar{A}^{\;1}_{61:\overline{4}|\,\delta=0.6\%} = 0.0124.$$

That is, the GMDB reserve would be 1.24% of the policyholder's fund value. □

17.5 Funding methods for embedded options

In discussing pricing above, we have expressed the price of a GMMB and a GMDB as a percentage of the initial premium. This is appropriate if the option is funded by a deduction from the premium at the inception of the policy. That is, the price of the option would come from the initial deduction of eP in the notation of Section 17.3.1 above. This sum could then be invested in the hedge portfolio for the option.

A relatively large expense deduction at inception, called a **front-end-load**, is common for UK policies, but less common in North America. A more common expense loading in North America is a management charge, applied as a regular percentage deduction from the policyholder's fund.

If the guarantee is to be funded through a regular management charge, rather than a deduction from the single premium as in Sections 17.3.1 and 17.4.1, we need a way to express the cost in terms of this charge.

Consider a single premium equity-linked policy with a term of n years issued to a life aged x. We assume, for simplicity, that there are no lapses and no initial expenses, so that $e = 0$ in the notation of Section 17.3.1. Also, we assume that mortality is a diversifiable risk which is independent of the stock price process.

Let $\pi(0)$ denote the cost at inception of the guarantees embedded in the policy, as derived in Sections 17.3.1 and 17.4.1. Suppose these guarantees consist of a payment of amount $h(t)$ if the life dies at time t ($< n$) and a payment $h(n)$ if the life survives to the end of the term. The value of each of these guarantees is

$$\mathrm{E}_0^Q\left[h(t)\,e^{-rt}\right]$$

given that the life does die at time t, and

$$\mathrm{E}_0^Q\left[h(n)\,e^{-rn}\right]$$

given that the life does survive to time n. Allowing for the probabilities of death and survivorship, we have

$$\pi(0) = \int_0^n \mathrm{E}_0^Q \left[h(t)\, e^{-rt} \right] {}_t p_x\, \mu_{x+t}\, dt + {}_n p_x\, \mathrm{E}_0^Q \left[h(n)\, e^{-rn} \right].$$

We interpret $\pi(0)$ as the cost at time 0 of setting up the replicating portfolios to pay the guarantees.

Let c denote the component of the management charge that is required to fund the guarantees from a total (fixed) management charge of m ($> c$) per year. We call c the **risk premium** for the guarantees.

Assume that the management charge is deducted daily, which we treat as a continuous deduction. With these assumptions, the fund value at time t for a policy still in force at that time, F_t, can be written

$$F_t = P\, S_t\, e^{-mt}.$$

Hence, the risk premium received in the time interval t to $t + dt$ for a policy still in force is (loosely) $c\, P\, S_t\, e^{-mt}\, dt$. Ignoring survivorship for the moment, the value at time 0 of this payment can be calculated as the cost of setting up a replicating portfolio which will pay this amount at time t. This cost is $c\, P\, e^{-mt}\, dt$ since an investment of this amount at time 0 in the stock will accumulate to $c\, P\, S_t\, e^{-mt}\, dt$ at time t (recall that $S_0 = 1$). Allowing for survivorship, the value at time 0 of the risk premium received in the time interval t to $t + dt$ is $c\, P\, e^{-mt}\, dt\, {}_t p_x$. So the value at time 0 of the total risk premiums to be received is

$$\int_0^n c\, P\, e^{-mt}\, {}_t p_x\, dt = c\, P\, \bar{a}_{x:\overline{n}|\,\delta=m}.$$

The risk premium c is chosen so that the value to the insurer of the risk premiums to be received is equal to the cost at time 0 of setting up the replicating portfolios to pay the guarantees, so that

$$c = \frac{\pi(0)}{P\, \bar{a}_{x:\overline{n}|\,\delta=m}}.$$

Calculating c from this formula is a slightly circular process. The risk premium c is a component of the total management charge m, but we need to know m to calculate the right-hand side of this equation for c. In practice, we may need to iterate through the calculations a few times to determine the value of c. In some cases there may be no solution. For example, increasing the total management charge m may increase the cost of the guarantees, therefore requiring a higher value for the risk premium c, which may in turn require a higher value for m.

If the management charge is deducted less frequently, say annually in advance, we can use the same principles as above to derive the value of the risk premiums. The cost at time 0 of setting up the replicating portfolios which will provide exactly for the guarantees is still $\pi(0)$. Ignoring survivorship, the amount of the risk premium to be received at time t ($t = 0, 1, \ldots, n-1$) is $c\,F_t = c\,P\,(1-m)^t\,S_t$ and the value of this at time 0 is $c\,P\,(1-m)^t$. Allowing for survivorship, this value is $c\,P\,(1-m)^t\,{}_tp_x$ and so the value at time 0 of all the risk premiums to be received is

$$\sum_{t=0}^{n-1} {}_tp_x\,c\,P\,(1-m)^t = c\,P\,\ddot{a}_{x:\overline{n}|\,i^*},$$

where $i^* = m/(1-m)$ so that $1/(1+i^*) = 1-m$.

Example 17.6 In Example 17.3 the monthly management charge, m, was 0.25% of the fund value and the GMDB option price was determined to be 2.7838% of the single premium.

You are given that 0.20% per month is allocated to commission and administrative expenses. Determine whether the remaining 0.05% per month is sufficient to cover the risk premium for the option.

Use the same basis as in Example 17.3.

Solution 17.6 The risk neutral value of the risk premium of c per month is

$$\mathrm{E}_0^Q\left[cF_0 + cF_{1/12}\,e^{-r/12}\,{}_{1/12}p_{60} + \cdots + c\,F_{59/12}\,e^{-59r/12}\,{}_{59/12}p_{60}\right]$$

$$= c\,P\,S_0\left(1 + (1-m)\,{}_{1/12}p_{60} + (1-m)^2\,{}_{2/12}p_{60} + \cdots + (1-m)^{59}\,{}_{59/12}p_{60}\right)$$

$$= 12\,c\,P\,S_0\,\ddot{a}^{(12)}_{60:\overline{5}|}$$

where the annuity interest rate is i such that

$$v_i^{1/12} = (1-m) \Rightarrow i = (1-m)^{-12} - 1 = 3.0493\% \text{ per year.}$$

The annuity value is 4.32662, so the value of the risk premium of 0.05% per month is \$259.60.

The value of the guarantee at the inception date, from Example 17.3, is \$278.38 so the risk premium of 0.05% per month is not sufficient to pay for the guarantee. The insurer needs to revise the pricing structure for this product. □

17.6 Risk management

The option prices derived in this chapter are the cost of either buying the appropriate options in the market, or internally hedging the options. If the insurer does not plan to purchase or hedge the options, then the price or reserve amount calculated may be inadequate. It would be inappropriate to charge an option premium using the Black–Scholes–Merton framework, and then invest the premium in bonds or stocks with no consideration of the dynamic hedging implicit in the calculation of the cost. Thus, the decision to use Black–Scholes–Merton pricing carries with it the consequential decision either to buy the options or to hedge using the Black–Scholes–Merton framework.

Under the assumptions of the Black–Scholes–Merton model, and provided the mortality and lapse experience is as assumed, the hedge portfolio will mature to the precise cost of the guarantee. In reality the match will not be exact but will usually be very close. So hedging is a form of risk mitigation. Choosing not to hedge may be a very risky strategy – with associated probabilities of severe losses. Generally, if the risk is not hedged, the reserves required using the stochastic techniques of Chapter 15 will be considerably greater than the hedge costs.

One of the reasons why the hedge portfolio will not exactly meet the cost of the guarantee is that under the Black–Scholes–Merton assumptions, the hedge portfolio should be continuously rebalanced. In reality, the rebalancing will be less frequent. A large portfolio might be rebalanced daily, a smaller one at weekly or even monthly intervals.

If the hedge portfolio is rebalanced at discrete points in time (e.g. monthly), there will be small costs (positive or negative) incurred as the previous hedge portfolio is adjusted to create the new hedge portfolio. See Example 16.5.

The hedge portfolio value required at time t for an n-year GMMB is, from Section 17.3.2,

$$\pi(t) = {}_{n-t}p_{x+t}\, E_t^Q[e^{-r(n-t)}\, h(n)] = {}_{n-t}p_{x+t}\, v(t,n)$$

where, as above, $v(t,n)$ is the value at time t of the option maturing at time n, unconditional on the policyholder's survival.

The hedge portfolio is invested partly in zero-coupon bonds, maturing at time n, and partly (in fact, a negative amount, i.e. a short sale) in stocks. The value of the stock part of the hedge portfolio is

$${}_{n-t}p_{x+t} \left(\frac{d}{dS_t}\, v(t,n) \right) S_t$$

and the value of the zero-coupon bond part of the hedge portfolio is

$$\pi(t) - {}_{n-t}p_{x+t} \left(\frac{d}{dS_t}\, v(t,n) \right) S_t.$$

For a GMDB, the approach is identical, but the option value is a weighted average of options of all possible maturity dates, so the hedge portfolio is a mixture of zero coupon bonds of all possible maturity dates, and (short positions in) stocks. For example, when the benefit is payable immediately on death, the value at time t of the option is $\pi(t)$, where

$$\pi(t) = \int_0^{n-t} v(t, w+t)\, {}_wp_{x+t}\, \mu_{x+t+w}\, dw.$$

The stock part of the hedge portfolio has value

$$\int_0^{n-t} S_t \left(\frac{d}{dS_t} v(t, w+t) \right) {}_wp_{x+t}\, \mu_{x+t+w}\, dw.$$

The value of the bond part of the hedge portfolio is the difference between $\pi(t)$ and the value of the stock part, so that the amount invested in a w-year zero-coupon bond at time t is (loosely)

$$\left(v(t, t+w) - S_t \frac{d}{dS_t} v(t, t+w) \right) {}_wp_{x+t}\, \mu_{x+t+w}\, dw.$$

The hedge strategy described in this section, which is called a delta-hedge, uses only zero-coupon bonds and stocks to replicate the guarantee payoff. More complex strategies are also possible, bringing options and futures into the hedge, but these are beyond the scope of this book.

The Black–Scholes–Merton valuation can be interpreted as a **market-consistent valuation**, by which we mean that the option sold in the financial markets as a stand-alone product (rather than embedded in life insurance) would have the same value. Many jurisdictions are moving towards market consistent valuation for accounting purposes, even where the insurers do not use hedging.

17.7 Profit testing

Whether the insurer is hedging internally or buying the options to hedge, the profit testing of an equity-linked policy proceeds as described in Chapter 15. The insurer might profit test deterministically, using best estimate scenarios, and then stress test using different scenarios, or might test stochastically, using Monte Carlo simulation to generate the scenarios for the increase in the stock prices in the policyholder's fund. In this section, we first explore deterministic profit testing, and then discuss how to make the profit test stochastic.

The cash flow projection depends on the projected fund values. Suppose we are projecting the emerging cash flows for a single premium equity-linked policy with a term of n years and with a GMDB and/or a GMMB, for a given stock price scenario. We assume all cash flows occur at intervals of $1/m$ years.

Assuming the insurer hedges the options internally, the income to and outgo from the insurer's fund for this contract arise as follows:

Income: + Initial front-end-load expense deduction.
+ Regular management charge income.
+ Investment return on income over the $1/m$ year period.

Outgo: – Expenses.
– Initial hedge cost, at time $t = 0$.
– After the first month, the hedge portfolio needs to be rebalanced; the cost is the difference between the hedge value brought forward and the hedge required to be carried forward.
– If the policyholder dies, there may be a GMDB liability.
– If the policyholder survives to maturity, there may be a GMMB liability.

The part of this that differs from Chapter 15 is the cost of rebalancing the hedge portfolio. In Example 16.5, for a standard put option, we looked at calculating rebalancing errors for a hedge portfolio adjusted monthly. The hedge portfolio adjustment in this chapter follows the same principles, but with the complication that the option is contingent on survival. As in Example 16.5, we assume that the hedge portfolio value is invested in a delta hedge. If rebalancing is continuous (in practice, one or more times daily), then the hedge adjustment will be (in practice, close to) zero, and the emerging guarantee cost will be zero given that the experience in terms of stock price movements and survival is in accordance with the models used. Under the model assumptions, the hedge is self-financing and exactly meets the guarantee costs. Also, if the hedge cost is used to buy options in the market, there will be no hedge adjustment cost and no guarantee cost once the options are purchased.

If the rebalancing takes place every $1/m$ years, then we need to model the rebalancing costs. We break the hedge portfolio down into the stock part, assumed to be invested in the underlying index $\{S_t\}_{t\geq 0}$, and the bond part, invested in a portfolio of zero-coupon bonds. Suppose the values of these two parts are $\Psi_t\, S_t$ and Υ_t, respectively, so that

$$\pi(t) = \Upsilon_t + \Psi_t\, S_t\,.$$

Then $1/m$ years later, the bond part of the hedge portfolio has appreciated by a factor $e^{r/m}$ and the stock part by a factor $S_{t+1/m}/S_t$. This means that, before rebalancing, the value of the hedge portfolio is, say, $\pi^{bf}(t+\frac{1}{m})$, where

$$\pi^{bf}(t+\tfrac{1}{m}) = \Upsilon_t\, e^{r/m} + \Psi_t\, S_{t+1/m}\,.$$

The rebalanced hedge portfolio required at time $t + 1/m$ has value $\pi(t+\frac{1}{m})$, but is required only if the policyholder survives. If the policyholder dies, the guarantee payoff is $h(t+\frac{1}{m})$. So the total cost at time $t+1/m$ of rebalancing the hedge, given that the policy was in force at time t, is

$$\pi\left(t+\tfrac{1}{m}\right)\ {}_{\frac{1}{m}}p_{x+t}-\pi^{bf}\left(t+\tfrac{1}{m}\right)$$

and the cost of the GMDB is

$$h(t+\tfrac{1}{m})\ {}_{\frac{1}{m}}q_{x+t}.$$

Note that these formulae need to be adjusted for the costs at the final maturity date, n: $\pi(n)$ is zero since there is no longer any need to set up a hedge portfolio, and the cost of the GMMB is $h(n)\ {}_{\frac{1}{m}}p_{x+n-\frac{1}{m}}$.

If lapses are explicitly allowed for, then the mortality probability would be replaced by an in-force survival probability.

In the following example, all of the concepts introduced in this chapter are illustrated as we work through the process of pricing and profit-testing an equity-linked contract with both a GMDB and a GMMB.

Example 17.7 An insurer issues a five-year equity-linked policy to a life aged 60. The single premium is $P=\$1000\,000$. The benefit on maturity or death is a return of the policyholder's fund, subject to a minimum of the initial premium. The death benefit is payable at the end of the month of death and is based on the fund value at that time.

Management charges of 0.3% per month are deducted from the fund at the start of each month.

(a) Calculate the monthly risk premium (as part of the overall management charge) required to fund the guarantees, assuming
  (i) volatility is 25% per year, and
  (ii) volatility is 20% per year.

  Basis:

  Survival model: Makeham's law with $A=0.0001$, $B=0.00035$ and $c=1.075$
  Lapses: None
  Risk-free rate of interest: 5% per year, continuously compounded

(b) The insurer is considering purchasing the options for the guarantees in the market; in this case the price for the options would be based on the 25% volatility assumption. Assuming that the monthly risk premium based on the 25% volatility assumption is used to purchase the options for the GMDB and GMMB liabilities, profit test the contract for the two stock price scenarios below, using a risk discount rate of 10% per year effective, and using monthly time intervals. Use the basis from part (a), assuming, additionally, that expenses incurred at the start of each month are 0.01% of the fund, after deducting the management charge, plus \$20. The two stock price scenarios are

(i) stock prices in the policyholder's fund increase each month by 0.65%, and
(ii) stock prices in the policyholder's fund decrease each month by 0.05%.

(c) The alternative strategy for the insurer is to hedge internally. Calculate all the cash flows to and from the insurer's fund at times 0, $\frac{1}{12}$ and $\frac{2}{12}$ per policy issued for the following stock price scenarios:
(i) stock prices in the policyholder's fund increase each month by 0.65%,
(ii) stock prices in the policyholder's fund decrease each month by 0.05%, and
(iii) $S_{\frac{1}{12}} = 1.0065$, $S_{\frac{2}{12}} = 0.9995$.
Assume that:

- the hedge cost is based on the 20% volatility assumption,
- the hedge portfolio is rebalanced monthly,
- expenses incurred at the start of each month are 0.025% of the fund, after deducting the management charge, and
- the insurer holds no additional reserves apart from the hedge portfolio for the options.

Solution 17.7 (a) The payoff function, $h(t)$, for $t = \frac{1}{12}, \frac{2}{12}, \dots, \frac{59}{12}, \frac{60}{12}$, is

$$h(t) = (P - F_t)^+$$

where $F_t = P\,S_t\,(1-m)^{12t}$ and $m = 0.003$.
Let $v(t,s)$ denote the value at t of the option given that it matures at s $(> t)$. Then

$$\begin{aligned} v(t,s) &= \mathrm{E}_t^Q\left[e^{-r(s-t)}h(s)\right] \\ &= \mathrm{E}_t^Q\left[e^{-r(s-t)}\left(P - P\,S_s(1-m)^{12s}\right)^+\right] \\ &= P\left(e^{-r(s-t)}\Phi(-d_2(t,s)) - S_t(1-m)^{12s}\Phi(-d_1(t,s))\right) \end{aligned}$$

where

$$d_1(t,s) = \frac{\log(S_t(1-m)^{12s}) + (r+\sigma^2/2)(s-t)}{\sigma\sqrt{s-t}}$$

and

$$d_2(t,s) = d_1(t,s) - \sigma\sqrt{s-t}.$$

The option price at issue per unit of premium is

$$\pi(0) = v\left(0, \tfrac{1}{12}\right)\ {}_{\frac{1}{12}}q_x + v\left(0, \tfrac{2}{12}\right)\ {}_{\frac{1}{12}|\frac{1}{12}}q_x + v\left(0, \tfrac{3}{12}\right)\ {}_{\frac{2}{12}|\frac{1}{12}}q_x + \cdots$$
$$+ v\left(0, \tfrac{60}{12}\right)\ {}_{\frac{59}{12}|\frac{1}{12}}q_x + v\left(0, \tfrac{60}{12}\right)\ {}_{\frac{60}{12}}p_x.$$

This gives the option price as
(i) $0.145977\,P$ for $\sigma = 0.25$ per year, and
(ii) $0.112710\,P$ for $\sigma = 0.20$ per year.
Next, we convert the premium to a regular charge on the fund, c, using

$$\pi(0) = 12\,c\,P\,\ddot{a}^{(12)}_{60:\overline{5}|},$$

where the interest rate for the annuity is $i = (1-m)^{-12} - 1 = 3.6712\%$, which gives $\ddot{a}^{(12)}_{60:\overline{5}|} = 4.26658$. The charge on the fund is then
(i) $c = 0.00285$ for $\sigma = 0.25$, and
(ii) $c = 0.00220$ for $\sigma = 0.20$.

(b) Following the convention of Chapter 15, we use the stock price scenarios to project the policyholder's fund value assuming that the policy stays in force throughout the five-year term of the contract. From this projection we can project the management charge income to the insurer's fund at the start of each month. Outgo at the start of the month comprises the risk premium for the option (which is paid to the option provider), and the expenses.
The steps in this calculation are as follows. At time $t = k/12$, where $k = 0, 1, \ldots, 59$, assuming the policy is still in force:

- The policyholder's fund, just before the deduction of the management charge, is F_t, where

$$F_t = P\,(1+g)^k\,(1-0.003)^k$$

and g is the rate of growth of the stock price.
- The amount transferred to the insurer's fund in respect of the management charge is

$$0.003\,F_t.$$

- The insurer's expenses, excluding the risk premium, are

$$0.0001\,(1-0.003)\,F_t + 20.$$

- The risk premium is

$$0.00285\,(1-0.003)\,F_t.$$

- The profit to the insurer is

$$\Pr_t = (0.003 - (1-0.003)(0.0001 + 0.00285))\,F_t - 20.$$

- The profit to the insurer, allowing for survivorship to time t, is

$$\Pi_t = {}_tp_{60}\,((0.003 - (1 - 0.003)(0.0001 + 0.00285))\,F_t - 20).$$

- The net present value of the profit using a risk discount rate of 10% per year is

$$\mathrm{NPV} = \sum_{k=0}^{59} \Pi_{\frac{k}{12}}\, 1.1^{-\frac{k}{12}}.$$

Because the insurer is buying the options, there is no outgo for the insurer in respect of the guarantees on death or maturity – the purchased options are assumed to cover any liability. As there is no residual liability for the insurer for the contract, there is no need to hold reserves. There are no end-of-month cash flows, so we calculate the profit vector using cash flows at the start of the month. Hence, Pr_t is the profit to the insurer at time t, assuming the policy is in force at that time, and Π_t is the profit at time t assuming only that the policy was in force at time 0.

Some of the calculations for the scenario where the stock price grows at 0.65% per month are presented in Table 17.2.

The NPV for this contract, using the 10% risk discount rate and the first stock price scenario, is \$1940.11.

The second stock price scenario, with stock prices falling by 0.05% each month, gives a NPV of \$1463.93.

(c) The items of cash flow for the insurer's fund at times 0, $\frac{1}{12}$ and $\frac{2}{12}$, per policy in force at the start of each month, are shown in Table 17.3. The individual items are as follows:

Income: the management charge (1).
Outgo:
the insurer's expenses (2),

Table 17.2 *Profit test for Example 17.7 part (b), Scenario (i).*

Time, t (months)	Management charge	Expenses	Risk premium	Pr_t	${}_{t/12}p_{60}$	Π_t
0	3000.00	119.70	2842.63	37.67	1	37.67
1	3010.44	120.05	2852.52	37.87	0.99775	37.79
2	3020.92	120.40	2862.45	38.08	0.99550	37.90
⋮	⋮	⋮	⋮	⋮	⋮	
58	3669.78	141.96	3477.27	50.55	0.85582	43.26
59	3682.55	142.38	3489.37	50.79	0.85309	43.33

Table 17.3 *Profit vector calculations for Example 17.7 part (c).*

Scenario	Time, t	Management charge (1)	Expenses (2)	GMDB and GMMB (3)	Cost of hedge (4)	Pr_t (5)	Π_t (6)
	0	3 000	249	0.0	112 710	−109 959	−109 959
(i)	$\frac{1}{12}$	3 010	250	0.0	−1 381	4 141	4 141
	$\frac{2}{12}$	3 021	251	0.0	−1 392	4 162	4 152
	0	3 000	249	0.0	112 710	−109 959	−109 959
(ii)	$\frac{1}{12}$	2 989	248	7.9	−1 380	4 114	4 114
	$\frac{2}{12}$	2 979	248	15.8	−1 397	4 113	4 104
	0	3 000	249	0.0	112 710	−109 959	−109 959
(iii)	$\frac{1}{12}$	3 010	250	0.0	−1 381	4 141	4 141
	$\frac{2}{12}$	2 981	248	14.7	−1 355	4 074	4 064

the amount, if any, needed to increase death or maturity benefits to the guaranteed amount (3),
the amount needed to set up, or rebalance, the hedge portfolio (4).

In column (5) we have the profit vector, that is, the expected profit per policy in force at the start of the month

$$(5) = (1) - (2) - (3) - (4).$$

In column (6) we have the profit signature

$$(6)_t = (5)_t \times {}_{t-1/12}p_{60}.$$

The individual cash flows at time t, per policy in force at the start of the month, are calculated as follows.

(1) Management charge: $P\,S_t \times 0.997^{12t} \times 0.003$.
(2) Expenses: $P\,S_t \times 0.997^{12t+1} \times 0.00025$.
(3) The expected cost of the GMDB, for $t = \frac{1}{12}, \frac{2}{12} \ldots, 5$, per policy in force at the start of the month, is

$$_{\frac{1}{12}}q_{60+t-\frac{1}{12}} \max(0,\, P - F_t).$$

The expected cost of the GMMB at $t = 5$, per policy in force at $t = 4\frac{11}{12}$ is

$$_{\frac{1}{12}}p_{64\frac{11}{12}} \max(0,\, P - F_5).$$

(4) The cost of setting up the hedge portfolio at time 0 is the same for each stock price scenario and is equal to $P\,\pi(0)$.

Table 17.4 *Hedge portfolios for Example 17.7 part (c).*

Time t		Investment scenario		
		(i)	(ii)	(iii)
0	$\pi(t)$	112 710	112 710	112 710
	Υ_t	417 174	417 174	417 174
	$\Psi_t\, S_t$	−304 465	−304 465	−304 465
$\frac{1}{12}$	$\pi(t)$	111 342	113 478	111 342
	Υ_t	415 700	421 369	415 700
	$\Psi_t\, S_t$	−304 358	−307 891	−304 358
$\frac{2}{12}$	$\pi(t)$	109 956	114 253	114 097
	Υ_t	414 172	425 626	425 216
	$\Psi_t\, S_t$	−304 216	−311 373	−311 119

At time $t = \frac{1}{12}$ the value of the hedge portfolio brought forward is

$$P(\Upsilon_0\, e^{0.05/12} + \Psi_0\, S_{\frac{1}{12}}).$$

The cost of setting up the new hedge portfolio for each policy still in force is $P\pi(\frac{1}{12})$. Hence, the expected cost per policy in force at the start of the month is $P\ {}_{\frac{1}{12}}p_{60}\ \pi(1/12)$. Hence, the net cost of rebalancing the hedge portfolio at this time, allowing for the hedge brought forward, is

$$P\left({}_{\frac{1}{12}}p_{60}\ \pi(1/12) - \left(\Upsilon_0\, e^{0.05/12} + \Psi_0\, S_{\frac{1}{12}}\right)\right).$$

Similarly, the net cost of rebalancing the hedge portfolio at time $t = \frac{2}{12}$, per policy in force at time $\frac{1}{12}$, is

$$P\left({}_{\frac{1}{12}}p_{60+\frac{1}{12}}\ \pi(2/12) - \left(\Upsilon_{\frac{1}{12}}\ e^{0.05/12} + \Psi_{\frac{1}{12}}\ S_{\frac{2}{12}}\right)\right).$$

The values of $\pi(t)$, Υ_t and Ψ_t are shown in Table 17.4. □

We note several important points about this example.

(1) Stock price scenarios (i) and (ii) used in parts (b) and (c) are not realistic, and lead to unrealistic figures for the NPV. This is particularly true for the internal hedging case, part (c). The NPV values for scenarios (i) and (ii), assuming internal hedging and a risk discount rate of 10% per year, can be shown to be \$98 500 and \$72 700, respectively. If the lognormal model for stock prices is appropriate, then the expected present value (under the P-measure) of the hedge rebalancing costs will be close to zero. Under both scenarios (i) and (ii) in Example 17.7 the present value is significant and negative, meaning that the hedge portfolio value brought forward each

month is more than sufficient to pay for the guarantee and new hedge portfolio at the month end. This is because more realistic scenarios involve far more substantial swings in stock price values, and it is these that generate positive hedge portfolio rebalancing costs.

(2) The comment above is more clearly illustrated when the profit test is used with stochastic stock price scenarios. In the table below we show some summary statistics for 500 simulations of the NPV for part (c), again calculated using a risk discount rate of 10% per year. The stock price scenarios were generated using a lognormal model, with parameters $\mu = 0.08$ per year, and volatility $\sigma = 0.2$ per year.

Mean NPV	Standard deviation	5% quantile	50% quantile	95% quantile
\$31 684	\$37 332	−\$23 447	\$28 205	\$99 861

We note that the NPV value for scenario (i) falls outside the 90% confidence interval for the net present value generated by stochastic simulation. This is because this scenario is highly unrepresentative of the true stock price process. Over-reliance on deterministic scenarios can lead to poor risk management.

(3) If we run a stochastic profit test under part (b), where the option is purchased in the market, the variability of simulated NPVs is very small. The net management charge income is small, and the variability arising from the guarantee cost has been passed on to the option provider. The mean NPV over 500 simulations is approximately \$2137, and the standard deviation of the NPV is approximately \$766, assuming the same parameters for the stock price process as for (c) above.

(4) If we neither hedge nor reserve for this option, and instead use the methods from Chapter 15, the two deterministic scenarios give little indication of the variability of the net present value. Using the first scenario (increasing prices) generates a NPV of \$137 053 and using the second gives \$2381. Using stochastic simulation generates a mean NPV of around \$100 000 with a 5% quantile of approximately −\$123 000.

17.8 Notes and further reading

There is a wealth of literature on pricing and hedging embedded options. Hardy (2003) gives some examples and information on practical ways to manage the risks. The options illustrated here are relatively straightforward. Much more convoluted options are sold, particularly in association with variable annuity policies. For example, a guaranteed minimum lifetime withdrawal benefit allows the policyholder the right to withdraw a specified proportion of the initial premium annually for their remaining lifetime, after an introductory period, even if the fund is exhausted. Other complicating features include resets where the policyholder has the right to set the guarantee at the current fund value at certain times during the contract. New variants are being created regularly, reflecting the strong interest in these products in the market.

In Section 17.2 we noted three differences between options embedded in insurance policies and standard options commonly traded in financial markets. The first was the life contingent nature of the benefit and the second was the fact that the option is based on the fund value rather than the underlying stocks. Both of these issues have been addressed in this chapter. The third issue is the fact that embedded options are generally much longer term than traded options. One of the implications is that the standard models for short-term options may not be appropriate over longer terms. The most important area of concern here is the lognormal model for stock prices. There is considerable empirical evidence that the lognormal model is not a good fit for stock prices in the long run. This issue is not discussed further here, but is important for a more advanced treatment of equity-linked insurance risk management. Sources for further information include Hardy (2003) and Møller (1998).

Early applications of modern financial mathematics to equity-linked insurance can be found in Brennan and Schwartz (1976) and Boyle and Schwartz (1977), and a comparison of the methods of Chapters 15 and 17 was first presented in Boyle and Hardy (1997); Hardy (2000) specifically considered modern variable annuities.

In some countries annual premium equity-linked contracts are common. We have not discussed these in this chapter, as the valuation and risk management is more complicated and requires more advanced financial mathematics. Bacinello (2003) discusses an Italian style annual premium policy.

17.9 Exercises

Longer exercises

Exercise 17.1 An insurer is designing a 10-year single premium variable annuity policy with a guaranteed maturity benefit of 85% of the single premium.

(a) Calculate the value of the GMMB at the issue date for a single premium of \$100.
(b) Calculate the value of the GMMB as a regular annual deduction from the fund.

(c) Calculate the value of the GMMB two years after issue, assuming that the policy is still in force, and that the underlying stock prices have decreased by 5% since inception.

Basis and policy information:

Age at issue:	60
Front end expense loading:	2%
Annual management charge:	2% at each year end (including the first)
Survival model:	Standard Ultimate Survival Model
Lapses:	5% at each year end except the final year
Risk-free rate:	4% per year, continuously compounded
Volatility:	20% per year

Exercise 17.2 An insurer issues a 10-year equity-linked insurance policy to a life aged 60. A single premium of \$10 000 is invested in an equity fund. Management charges at a rate of 3% per year are deducted daily. At the end of the month of death before age 70, the death benefit is 105% of the policyholder's fund subject to a minimum of the initial premium.

(a) Calculate the price of the death benefit at issue.
(b) Express the cost of the death benefit as a continuous charge on the fund.

Basis:

Survival model:	Standard Ultimate Survival Model
Risk-free rate:	4% per year, continuously compounded
Volatility:	25% per year
Lapses:	None

Excel-based exercises

Exercise 17.3 An insurer issues a range of 10-year variable annuity guarantees. Assume an investor deposits a single premium of \$100 000. The policy carries a guaranteed minimum maturity benefit of 100% of the premium.

(a) Calculate the probability that the guaranteed minimum maturity benefit will mature in-the-money (i.e. the probability that the fund at the maturity date is worth less than 100% of the single premium) under the P-measure.
(b) Calculate the probability that the guaranteed minimum maturity benefit will mature in-the-money under the Q-measure.
(c) Calculate the EPV of the option payoff under the P-measure, discounting at the risk-free rate.
(d) Calculate the price of the option.
(e) A colleague has suggested the value of the option should be the EPV of the guarantee under the P-measure, analogous to the value of term insurance liabilities. Explain why this value would not be suitable.

(f) For options that are complicated to value analytically we can use Monte Carlo simulation to find the value. We simulate the payoff under the risk neutral measure, discount at the risk-free rate and take the mean value to estimate the Q-measure expectation. Use Monte Carlo simulation to estimate the value of this option with 1000 scenarios, and comment on the accuracy of your estimate.

Basis:

Survival model:	No mortality
Stock price appreciation:	Lognormally distributed, with $\mu = 0.08$ per year, $\sigma = 0.25$ per year
Risk-free rate of interest:	4% per year, continuously compounded
Management charges:	3% of the fund per year, in advance

Exercise 17.4 An insurer issues a single premium variable annuity contract with a 10-year term. There is a guaranteed minimum maturity benefit equal to the initial premium of \$100.

After five years the policyholder's fund value has increased to 110% of the initial premium. The insurer offers the policyholder a reset option, under which the policyholder may reset the guarantee to the current fund level, in which case the remaining term of the policy will be increased to 10 years.

(a) Determine which of the original guarantee and the reset guarantee has greater value at the reset date.
(b) Determine the threshold value for F_5 (i.e. the fund at time 5) at which the option to reset becomes more valuable than the original option.

Basis:

Survival model:	No mortality
Volatility:	$\sigma = 0.18$ per year
Risk-free rate of interest:	5% per year, continuously compounded
Management charges:	1% of the fund per year, in advance
Front-end-load charge:	3%

Exercise 17.5 An insurer issues a five-year single premium equity-linked insurance policy to (60) with guaranteed minimum maturity benefit of 100% of the initial premium. The premium is \$100 000. Management fees of 0.25% of the fund are deducted at the start of each month.

(a) Verify that the guarantee cost expressed as a monthly deduction is 0.19% of the fund.
(b) The actuary is profit testing this contract using a stochastic profit test. The actuary first works out the hedge rebalancing cost each month then inserts that into the profit test.

Table 17.5 *Single scenario of stock prices for stochastic profit test for Exercise 17.5.*

t	S_t	t	S_t	t	S_t	t	S_t	t	S_t
0	1.00000								
1	0.95449	13	0.92420	25	1.09292	37	1.09203	49	1.34578
2	0.96745	14	0.95545	26	1.17395	38	1.10988	50	1.42368
3	0.97371	15	1.02563	27	1.27355	39	1.05115	51	1.50309
4	1.01158	16	1.13167	28	1.32486	40	1.05659	52	1.63410
5	1.01181	17	1.25234	29	1.31999	41	1.18018	53	1.45134
6	0.93137	18	1.10877	30	1.24565	42	1.20185	54	1.46399
7	0.98733	19	1.10038	31	1.20481	43	1.34264	55	1.40476
8	0.89062	20	0.99481	32	1.18405	44	1.37309	56	1.44512
9	0.91293	21	1.04213	33	1.23876	45	1.39327	57	1.39672
10	0.90374	22	1.07980	34	1.15140	46	1.40633	58	1.30130
11	0.88248	23	1.14174	35	1.09478	47	1.41652	59	1.25762
12	0.92712	24	1.12324	36	1.03564	48	1.43076	60	1.19427

Table 17.6 *Hedge rebalance table for Exercise 17.5, in \$100 000s.*

Time (months)	S_t	Option cost at t	Stock part of hedge at t	Bond part of hedge at t	Hedge b/f	Hedging rebalance cost
0	1.00000	10.540	−27.585	38.125	–	–
1	0.95449	11.931	−29.737	41.668	11.955	−0.024
2	0.96745	11.592	−29.528	41.120	11.701	−0.109
⋮		⋮	⋮	⋮	⋮	⋮
59	1.25762	0.200	−7.658	7.858	0.526	−0.326
60	1.19427	0.000	–	–	0.619	−0.619

The stock price figures in Table 17.5 represent one randomly generated scenario. The table shows the stock price index values for each month in the 60-month scenario.

(i) Table 17.6 shows the first two rows of the hedge rebalancing cost table. Use the stock price scenario in Table 17.5 to complete this table. Calculate the present value of the hedge rebalance costs at an effective rate of interest of 5% per year.

(ii) Table 17.7 shows the first two rows of the profit test for this scenario. The insurer uses the full cost of the option at the start of the contract to pay for the hedge portfolio.
Complete the profit test and determine the profit margin (NPV as a percentage of the single premium) for this scenario.

Table 17.7 *Profit test table for Exercise 17.5, in $s.*

Time, t (months)	F_t	Management costs	Expenses	Hedge costs	Pr_t
0	100 000.00	250.00	1000.00	10 540.21	−11 290.21
1	95 210.38	238.03	61.89	−23.99	200.13
2	96 261.88	240.65	62.57	−109.16	287.24
⋮					

(iii) State with reasons whether you would expect this contract to be profitable, on average, over a large number of simulations.

Basis for hedging and profit test calculations:

Survival:	Standard Ultimate Survival Model
Lapses:	None
Risk-free rate:	5% per year, continuously compounded
Volatility:	20% per year
Incurred expenses – initial:	1% of the premium
Incurred expenses – renewal:	0.065% of the fund before management charge deduction, monthly in advance from the second month
Risk discount rate:	10% per year

Answers to selected exercises

17.1 (a) \$4.61 (b) 0.68% (c) \$6.08

17.2 (a) \$107.75 (b) 0.13%

17.3 (a) 0.26545 (b) 0.60819 (c) \$6033 (d) \$18 429

17.4 (a) The original option value is \$4.85 and the reset option value is \$6.46.
(b) At $F_5 = 103.4$ both options have value \$6.07.

17.5 (b) (i) The PV of rebalancing costs is −\$1092.35 (ii) −1.23%
(iii) We note that the initial hedge cost converts to a monthly outgo of 0.19% of the fund; adding the monthly incurred expenses, this comes to 0.255%, compared with income of 0.25% of the fund. Overall we would not expect this contract to be profitable on these terms.

18
Estimating survival models

18.1 Summary

In this chapter we introduce some actuarial approaches to estimation and inference used to construct the life tables and survival models that we have been using in previous chapters. We start with a discussion of typical characteristics of lifetime data for actuarial applications. We then show how to use lifetime data to fit survival models, including parametric and non-parametric approaches.

We next move to the Markov models from Chapter 8. Starting with the alive–dead model, and assuming a piecewise constant force of mortality, we derive the maximum likelihood estimator for the force of mortality for each age-year. We then extend the methodology to multiple state models with piecewise constant transition intensities.

18.2 Introduction

In previous chapters we have made extensive use of life tables and other parametric models of survival, as well as multiple state models for more complex state-contingent benefits. In this chapter we consider some of the ways in which we may use data to calibrate these models. The statistical basis for this chapter largely comes from **survival analysis**, a branch of statistics concerned with modelling times to failure (or other event) in different contexts. For example, engineers may be interested in failure times of machine components; medical statisticians study disease-specific event times. Problems we have considered, such as modelling the time to death of life insurance policyholders or transition times for state-dependent insurance benefits, clearly fall within the realm of survival analysis.

This chapter represents a shift in focus. In previous chapters, we have used models to determine premiums and policy values, to assess the mean and variance of present value random variables, using tools from probability theory.

In this chapter, we use results from statistics, where the main focus is the collection and interpretation of data. In particular, we make extensive reference to maximum likelihood estimation, one of the most important tools of statistics. A brief review of the key facts about maximum likelihood estimation is given in Section A.5 of Appendix A.

It is fairly common in statistical analysis to be somewhat loose with language around data. Suppose we gather a random sample of 1 000 policyholders aged 50, whom we intend to observe until they die. At the start, the time to death of each life in the sample is a random variable. After the data have been collected, the time to death for each life is a fixed, known data point, each one representing a single observation of an underlying random variable. Sometimes, we use the same notation for the underlying random variable and the data point, and assume that the context adequately signals which interpretation we are using. If the context is not clear, we may emphasize the difference by using different notation (for example, upper case for random variables, lower case for observed values), or by using the phrase *ex ante*, which means 'before the event', to signal that we are considering the random variable, while the phrase *ex post*, which means 'after the event', is used to signal that we are referring to the observed data.

For actuarial applications, the lifetime data being analysed come from information about policyholders or pension plan members. Individual life insurance companies use their records to assess their mortality experience and to compare it with benchmark models of mortality. Historically, there would have been substantial grouping of data, as thousands of lives would be involved in insurers' portfolios. However, as computational limitations become much less of an issue, it is now more common to collect and analyse data policy by policy. This is called the **seriatim** approach. Seriatim data collection offers the richest and most accurate information. For each data point, the record would include all important dates, including the date of birth and the date of purchase of insurance. If the policy is no longer active, the record would include the date that the policy expired, and whether the policyholder died, surrendered the policy, or survived to the end of the policy term. In almost all the development in this chapter (Section 18.4.2 is the exception), we assume that seriatim data are available.

18.3 Actuarial lifetime data

Suppose we observe a sample of independent lives over some time period. We are interested in the random variable T_0, which is the age at death random variable for each life in the underlying population. As in Chapter 2, let f_0 denote the density function of T_0, and let F_0 and S_0 denote the distribution function and survival function, respectively. In order to fit and test models for the age at death, we are interested in the likelihood function for the distribution of T_0, which is the joint probability function of all the observations in the sample. Because we assume independence of the lives observed, the likelihood function is the product of the individual *ex ante* probability, or probability density function, for each observed value in the sample.

Let t_j denote the age at death of the jth life, for $j = 1, 2, \ldots, n$. If all the lives in the sample are observed from birth to death, then the likelihood function is

$$L = \prod_{j=1}^{n} f_0(t_j) = \prod_{j=1}^{n} {}_{t_j}p_0\ \mu_{t_j},$$

which is the product of the *ex ante* density function of each observation. We say that the jth life contributes $f_0(t_j)$ to the likelihood. However, it would be very unusual for data to be used in exactly this form, as it requires decades of observation to track lives from birth to death. More commonly the data are collected over a relatively short time interval, so not all lives are observed from birth, and not all lives are observed until death. In this case the data are **left truncated** and **right censored**.

18.3.1 Left truncation

Truncation of data means that information is conditional, because there are some data points that are unseen. Actuarial lifetime data are usually left truncated because we observe lives only from some age $x > 0$; if a life had died before that age, then that life would have been unseen, and hence would have been truncated from the data. When we start observing a life from age $x > 0$, then any inference is conditional on survival to age x.

This means that if a life enters observation at age x, and dies at age $x + t$, then the contribution to the likelihood is conditional on survival to age x; that is, the density function at t, which depends on x, is

$$f_x(t) = \frac{f_0(x+t)}{S_0(x)} = \frac{{}_{x+t}p_0\ \mu_{x+t}}{{}_{x}p_0} = {}_tp_x\mu_{x+t}.$$

This makes sense, as t is an observation of the future lifetime of (x), and ${}_tp_x\ \mu_{x+t}$ is the density function of the future lifetime of (x) at time t.

Less commonly, we may deal with data which are **right truncated**, when older lives are excluded from observation. This can arise when we use death records from a relatively recent period to record age at death, which means that lives who have not yet died are not observed.

18.3.2 Right censoring

Censored data occur when we know a range for a variable, but not an exact figure. Typically, mortality data are right censored. Right censoring occurs when an individual is still alive when the period of observation ends, so all we know for that individual is that the time to death is greater than the time under observation.

There are different forms of censoring. The most common in lifetime data is **Type 1 censoring**. In this case, lives are observed for a fixed period (or until earlier death). The period may differ for different lives, but it is known when the life enters the study. **Random censoring** differs from Type 1, as the censoring times are random rather than fixed. For example, suppose an insurer is collecting lifetime data from a portfolio of 10-year term insurance policyholders. The data would be Type 1 censored, but would also be randomly censored in the cases of lapsing policyholders who leave observation before the Type 1 end date.

If the random time to censoring is independent of the time to death, then there is no effective difference in how we handle and interpret the data under random and Type 1 censoring. However, note that this condition is unlikely to be true in the case of the term insurance policyholders, as the lives who are very unwell, and therefore most likely to die, are also least likely to lapse. Nevertheless, in the rest of this chapter, we assume independence of the time to death and the time to the censored exit from observation.

The pertinent information from the jth life in a study of mortality consists of (i) age at entry to the study, x_j (which is not a random variable), (ii) the time until exit from the study, t_j, and (iii) the mode of exit, death or censorship, which we indicate with the **death indicator variable** δ_j, which is defined as

$$\delta_j = \begin{cases} 1 & \text{if } (x_j) \text{ died at time } t_j, \\ 0 & \text{if } (x_j) \text{ was right censored at time } t_j. \end{cases}$$

We are recording observed values of two *ex ante* random variables; the time until exit, and an indicator variable for death. Note that the time until exit has a mixed distribution if the study is over a fixed time period; the distribution is mixed because there is a mass of probability representing the probability that the life survives from entry to the study until the end date of the study. The indicator variable δ_j is discrete. For convenience, when we refer to their joint distribution below, we refer to the *joint probability function*.

So, if the jth life enters observation at age x (at time 0) and dies at time t, the data record will be $(x, t, \delta = 1)$, and the contribution to the likelihood function is the joint probability function of the death time and $\delta = 1$, which is just the density function of T_x at t, i.e. $f_x(t, \delta = 1) = {}_tp_x\ \mu_{x+t}$.

If instead this life is censored at age $x + t$, then the data record is $(x,\ t,\ \delta = 0)$ and the contribution to the likelihood function is the joint probability function of the exit time and $\delta = 0$, which is just the probability of survival to age $x + t$, so that $f_x(t, \delta = 0) = {}_tp_x$.

We can summarize the contribution to the likelihood function of any data point (x, t, δ) as the joint probability function of the variables t and δ, which can be written as

$$f_x(t, \delta) = {}_tp_x \, \mu_{x+t}^{\delta}. \tag{18.1}$$

It is also common for lifetime data to be **interval censored**. Interval censoring occurs when we observe lower and upper limits for lifetime, whereas right censoring implies a lower limit only. Interval censoring arises when we observe data at discrete intervals, say annually. So, for example, if we observe the jth life annually from age x, and discover that they died between ages $x + k$ and $x + k + 1$, then the contribution to the likelihood function is

$${}_kp_x - {}_{k+1}p_x.$$

Type 2 censoring is less common for actuarial lifetime data. Under Type 2 censoring, a population is observed until a pre-specified number of failures (deaths in our case) has been recorded. In this case the censoring time is random and is not independent of the times to failure. This would be commonly seen in quality control, where engineering components are tested to failure.

Example 18.1 You are given the following data from a study of mortality at old ages.

Six lives were recruited to the study at the start of the observation period of six years. The individual records are as follows.

Life, (j)	x_j	t_j	δ_j
1	94.0	6.0	0
2	92.0	6.0	0
3	96.5	0.5	1
4	97.0	between 1 and 2	1
5	93.5	4.5	1
6	99.0	4.0	0

For each life, identify the form of censoring in the data, and write down the contribution of the data point to the likelihood function in terms of actuarial functions.

Solution 18.1 Lives (1) and (2) are Type 1 right censored, as they survived to the end of the observation period. Life (4) is interval censored, as the time of death is within an interval, but not known precisely. Life (6) is censored, but not Type 1, so we assume that it is a random censorship.

The contributions to the likelihood function are

Life (1): $f_{94.0}(6,0) = {}_6p_{94.0}$

Life (2): $f_{92.0}(6,0) = {}_6p_{92.0}$

Life (3): $f_{96.5}(0.5,1) = {}_{0.5}p_{96.5}\ \mu_{97.0}$

Life (4): $f_{97.0}(1 < t < 2,\ 1) = {}_1p_{97.0} - {}_2p_{97.0}$

Life (5): $f_{93.5}(4.5,1) = {}_{4.5}p_{93.5}\ \mu_{98.0}$

Life (6): $f_{99.0}(4.0,0) = {}_4p_{99.0}$

□

There are two different ways to use the likelihood information to estimate the underlying mortality distribution.

(1) The parametric approach uses data to fit a parametric lifetime distribution, such as the Makeham model. In this case, we would express the logarithm of the likelihood (log-likelihood hereafter) as a function of the model parameters, and then find the values of the parameters that maximize the log-likelihood function. Generally, we use numerical methods, as the distributions suitable for lifetime modelling are not tractable for analytic likelihood maximization. We can also use numerical methods to estimate the second derivatives of the likelihood function, with respect to the parameters, which gives an estimate of the asymptotic covariance matrix for the parameter estimates.
(2) The non-parametric approach uses the empirical distribution generated by the data, which implicitly means that the parameters of the distribution are the ${}_tp_x$ (or μ_{x+t}) values themselves.

We give an example of a parametric approach here. In the following sections we outline some non-parametric approaches.

Example 18.2 An insurance company is analysing the mortality of its whole life policyholders. There are n lives in the study, and for each life you are given the data (x_j, t_j, δ_j), for $j = 1, 2, \ldots, n$, where x_j is the age at purchase of the policy of the jth life, t_j is the time from entry to exit, and $\delta_j = 0$ if the policy terminated through surrender, and $\delta_j = 1$ if the policy terminated through death.

The company decides to fit a Gompertz model, $\mu_x = Bc^x$, to the data.

(a) Write down the log-likelihood function in terms of the Gompertz parameters.
(b) By differentiating the log-likelihood function, write down the two equations that give the maximum likelihood estimates for the parameters.

Solution 18.2 The likelihood function is

$$L(B,\ c) = \prod_{j=1}^{n} {}_{t_j}p_{x_j}\, \mu_{x_j+t_j}^{\delta_j}$$

and so the log-likelihood function, denoted $l(B, c)$, is

$$l(B, c) = \log L(B, c) = \sum_{j=1}^{n} \left(\log {}_{t_j}p_{x_j} + \delta_j \log \mu_{x_j+t_j}\right).$$

For the Gompertz model,

$${}_tp_x = \exp\left\{\frac{-B}{\log c}\, c^x \left(c^t - 1\right)\right\},$$

so the log-likelihood is

$$l(B, c) = -\sum_{j=1}^{n} \frac{B}{\log c}\, c^{x_j}(c^{t_j}-1) + \sum_{j=1}^{n} \delta_j \left(\log B + (x_j + t_j)\log c\right).$$

The equations for the maximum likelihood estimates are found by differentiating the log-likelihood with respect to the parameters, and then setting the derivatives to zero. We have

$$\frac{\partial l}{\partial B} = \frac{-1}{\log c} \sum_{j=1}^{n} c^{x_j}(c^{t_j}-1) + \frac{1}{B}\sum_{j=1}^{n} \delta_j,$$

$$\frac{\partial l}{\partial c} = -B \sum_{j=1}^{n} \left(\frac{(x_j + t_j)\, c^{x_j+t_j} - x_j\, c^{x_j}}{c\ \log c} - \frac{c^{x_j}(c^{t_j} - 1)}{c\,(\log c)^2}\right) + \frac{1}{c}\sum_{j=1}^{n} \delta_j(x_j + t_j).$$

Hence, the equations satisfied by the maximum likelihood estimates $\hat{B}$ and $\hat{c}$ are

$$0 = \frac{-1}{\log \hat{c}} \sum_{j=1}^{n} \hat{c}^{x_j}(\hat{c}^{t_j}-1) + \frac{1}{\hat{B}}\sum_{j=1}^{n} \delta_j,$$

$$0 = -\hat{B} \sum_{j=1}^{n} \left(\frac{(x_j + t_j)\, \hat{c}^{x_j+t_j} - x_j\, \hat{c}^{x_j}}{\hat{c}\ \log \hat{c}} - \frac{\hat{c}^{x_j}(\hat{c}^{t_j} - 1)}{\hat{c}\,(\log \hat{c})^2}\right) + \frac{1}{\hat{c}}\sum_{j=1}^{n} \delta_j(x_j + t_j),$$

and these equations would have to be solved numerically to obtain $\hat{B}$ and $\hat{c}$. □

18.4 Non-parametric survival function estimation

18.4.1 The empirical distribution for seriatim data

Suppose we have a sample of n distinct failure times, $t_1, t_2, \ldots, t_n$, with no censoring or truncation. We are interested in estimating a survival function, S, from these data, and we denote our estimate by $\hat{S}$. As the empirical distribution is discrete, we denote by $\hat{f}$ the estimated probability function.

The empirical distribution of failure times for this sample is the discrete distribution with sample space $\{t_1, t_2, \ldots, t_n\}$, and with probability function $\hat{f}(t_j) = 1/n$ for $j = 1, 2, \ldots, n$. That means that the empirical survival function is a step function with

$$\hat{S}(t) = \begin{cases} 1 & \text{for } 0 \leq t < t_1, \\ (n-1)/n & \text{for } t_1 \leq t < t_2, \\ (n-2)/n & \text{for } t_2 \leq t < t_3, \\ \vdots & \end{cases}$$

In general, suppose we have complete data of the times to death for all n lives, with no censoring or truncation. Let n_t denote the number of survivors at time t from the n lives at time 0. Then the empirical survival function is

$$\hat{S}(t) = \frac{\text{Number of survivors at time } t}{n} = \frac{n_t}{n},$$

and $\hat{S}(t)$ is an estimate of the underlying population survival probability to time t. We can estimate the uncertainty involved by considering the *ex ante* random variable N_t, which is the random number of survivors at time t from the n lives at time 0. N_t has a binomial distribution, $N_t \sim B(n, S(t))$, and so $\hat{S}(t) = N_t/n$ has variance

$$\mathrm{V}[\hat{S}(t)] = \frac{\mathrm{V}[N_t]}{n^2} = \frac{nS(t)(1-S(t))}{n^2} = \frac{S(t)(1-S(t))}{n}.$$

Although $S(t)$ is the unknown, underlying survival probability, we can estimate it with the observed value of $\hat{S}(t) = n_t/n$, giving an approximate variance of

$$\mathrm{V}[\hat{S}(t)] \approx \frac{n_t(n-n_t)}{n^3} = \hat{S}(t)^2\left(\frac{1}{n_t} - \frac{1}{n}\right).$$

18.4.2 The empirical distribution for grouped data

When data are grouped it means that we have interval censoring, which must be allowed for in the empirical distribution. For example, suppose we observe 100 000 lives from exact age 50 for 30 years, and collect the grouped information on deaths given in the table below.

These data are interval censored, as we do not have exact information on times of death, but we do have bounded information. We can construct the empirical survival function at the end of each age interval, as follows:

Age last birthday	Deaths
50–59	1 700
60–69	4 650
70–74	5 520
75–79	9 680

$$\hat{S}_{50}(10) = \frac{100\,000 - 1\,700}{100\,000} = 0.9830,$$

$$\hat{S}_{50}(20) = \frac{100\,000 - (1\,700 + 4\,650)}{100\,000} = 0.9365,$$

$$\hat{S}_{50}(25) = \frac{100\,000 - (1\,700 + 4\,650 + 5\,520)}{100\,000} = 0.8813,$$

$$\hat{S}_{50}(30) = \frac{100\,000 - (1\,700 + 4\,650 + 5\,520 + 9\,680)}{100\,000} = 0.7845.$$

For values of t between the end points of the intervals, we need to make some assumption about the distribution of deaths within the interval. The simplest assumption is that deaths are uniformly distributed within each interval, which means that $\hat{S}_{50}(t)$ can be estimated using linear interpolation between the survival function values at the end points. This is called the **ogive empirical survival function**, and is the result of assuming that $\hat{S}_{50}(t)$ follows a straight line between the interval end points.

That is, assume that, in estimating S from grouped data, we have lower and upper end points for one interval of t_L and t_U. We can calculate $\hat{S}(t_L)$ and $\hat{S}(t_U)$. Now suppose that we want to estimate $\hat{S}(t)$ for some t where $t_L \le t < t_U$. Then, using linear interpolation, the ogive empirical survival function at t is

$$\hat{S}(t) = \frac{(t_U - t)\hat{S}(t_L) + (t - t_L)\hat{S}(t_U)}{t_U - t_L}.$$

Example 18.3 For the grouped data listed above, calculate $\hat{S}_{50}(t)$ using the ogive empirical survival function at (a) $t = 2$, (b) $t = 17$ and (c) $t = 28$.

Solution 18.3 (a) $\hat{S}_{50}(0) = 1$ and $\hat{S}_{50}(10) = 0.9830$, so

$$\hat{S}_{50}(2) = \frac{8 \times 1 + 2 \times 0.9830}{10} = 0.9966.$$

(b) We now interpolate between $\hat{S}(10)$ and $\hat{S}(20)$ to get

$$\hat{S}_{50}(17) = \frac{3 \times 0.9830 + 7 \times 0.9365}{10} = 0.9505.$$

(c) Similarly, interpolating between $\hat{S}(25)$ and $\hat{S}(30)$ gives

$$\hat{S}_{50}(28) = \frac{2 \times 0.8813 + 3 \times 0.7845}{5} = 0.8232.$$

□

18.4.3 The Kaplan–Meier estimate

In practice, we need to adapt the empirical probability function to allow for right censoring and left truncation. A popular approach using seriatim data is to use the **Kaplan–Meier** estimate, also known as the **product limit** estimate.

Suppose we have a seriatim data set that includes information on censored values. For example, using $^+$ to indicate a censored observation, consider the following data set of exit times $\{t_k\}$, which includes five death times and five censored observations:

$$4^+,\ 5,\ 10,\ 10^+,\ 14,\ 15^+,\ 16^+,\ 17,\ 19^+,\ 20.$$

In this case, we can use the Kaplan–Meier method to determine a non-parametric survival function. The Kaplan–Meier estimate is a maximum likelihood estimate of a survival function, assuming a discrete underlying probability model, with deaths occurring only at the death times in the data (the same underlying model as used for the empirical survival function). So, if we let $t_{(j)}$ denote the jth time of death, then the parameters of this model are the survival probabilities for each death time,

$$\Pr[\text{alive at } t_{(j)}^{+} | \text{alive at } t_{(j)}^{-}],$$

which we denote $p_{(j)}$. Given this notation, the Kaplan–Meier assumptions and approach can be summarized as follows.

1. Deaths can occur only at specific times, $t_{(j)}$, which are exactly the times at which deaths occurred in the data.
2. The probability that a life alive at time $t_{(j)}^{-}$ (i.e. just before the death(s) at that time) survives to time $t_{(j)}^{+}$ (i.e. just after the death(s) at that time), is $p_{(j)}$. Hence, the probability of dying at time $t_{(j)}$ is $1 - p_{(j)}$.
3. The probability of surviving from time $t_{(j-1)}^{+}$ to time $t_{(j)}^{-}$, that is, from just after the death(s) at time $t_{(j-1)}$ to just before the death(s) at time $t_{(j)}$, is 1.
4. The survival function at time t is
$$S(t) = \prod_{j:t_{(j)} \leq t} p_{(j)}.$$
Note that the survival function is constant between death times.
5. The parameters $p_{(j)}$, $j = 1, 2, \ldots, m$, are unknown. We use maximum likelihood to estimate the values from the data, and the estimated values are denoted $\hat{p}_{(j)}$.

6. The maximum likelihood estimate of the survival function is the product of the maximum likelihood estimates of the parameters $\{p_{(j)}\}$, that is

$$\hat{S}(t) = \prod_{j:t_{(j)} \leq t} \hat{p}_{(j)}.$$

To illustrate, consider the 10 lives whose t_k values are listed above.

There are five death times, so the parameters of the Kaplan–Meier survival model are $p_{(j)}$, for $j = 1, 2, \ldots, 5$.

For the first life, $t_1^+ = 4$ gives us no information about any of the parameters.

For the second life, $t_2 = t_{(1)} = 5$ (second exit time, first death time), so the contribution to the likelihood is $1 - p_{(1)}$.

For the third life, $t_3 = t_{(2)} = 10$, so the life survived at time 5, and died at time 10. The contribution to the likelihood is $p_{(1)}(1-p_{(2)})$. An important point here is that if a death time matches a censoring time, we always assume that the death(s) occur first.

For the fourth life $t_4 = 10^+$, so the life survived to time 5, and then to time 10, but this life provides no information on the subsequent $\{p_{(j)}\}$ $(j = 3, 4, 5)$. The contribution to the likelihood is $p_{(1)}\, p_{(2)}$.

Proceeding similarly for the other lives, we end up with the contributions to the likelihood for each life summarized in Table 18.1. In each case, the life contributes $p_{(j)}$ if they survive through time $t_{(j)}$, and $1 - p_{(j)}$ if they die at time $t_{(j)}$, but makes no contribution if they are not observed at time $t_{(j)}$.

The product of each row gives the contribution to the likelihood for that life, and the product of each life's contribution gives the overall likelihood, shown in the final row of Table 18.1.

Table 18.1 *Table of likelihood for the illustrative Kaplan–Meier data in Section 18.4.3.*

		Contribution to likelihood				
t_k	δ_k	at $t_{(1)} = 5$	at $t_{(2)} = 10$	at $t_{(3)} = 14$	at $t_{(4)} = 17$	at $t_{(5)} = 20$
4	0	–	–	–	–	–
5	1	$1 - p_{(1)}$	–	–	–	–
10	1	$p_{(1)}$	$1 - p_{(2)}$	–	–	–
10	0	$p_{(1)}$	$p_{(2)}$	–	–	–
14	1	$p_{(1)}$	$p_{(2)}$	$1 - p_{(3)}$	–	–
15	0	$p_{(1)}$	$p_{(2)}$	$p_{(3)}$	–	–
16	0	$p_{(1)}$	$p_{(2)}$	$p_{(3)}$	–	–
17	1	$p_{(1)}$	$p_{(2)}$	$p_{(3)}$	$1 - p_{(4)}$	–
19	0	$p_{(1)}$	$p_{(2)}$	$p_{(3)}$	$p_{(4)}$	–
20	1	$p_{(1)}$	$p_{(2)}$	$p_{(3)}$	$p_{(4)}$	$1 - p_{(5)}$
Likelihood		$(1 - p_{(1)})p_{(1)}^8$	$(1 - p_{(2)})p_{(2)}^7$	$(1 - p_{(3)})p_{(3)}^5$	$(1 - p_{(4)})p_{(4)}^2$	$1 - p_{(5)}$

As usual when deriving maximum likelihood estimates, we maximize the log-likelihood, which in this case is

$$l = \log(1-p_{(1)}) + 8\log p_{(1)} + \log(1-p_{(2)}) + 7\log p_{(2)} + \log(1-p_{(3)}) \\ + 5\log p_{(3)} + \log(1-p_{(4)}) + 2\log p_{(4)} + \log(1-p_{(5)}).$$

The maximum likelihood estimates are found by setting the partial derivatives equal to zero, giving the following results for $p_{(1)}$ to $p_{(4)}$:

$$\frac{8}{\hat{p}_{(1)}} - \frac{1}{1-\hat{p}_{(1)}} = 0 \Rightarrow \hat{p}_{(1)} = \frac{8}{9},$$
$$\frac{7}{\hat{p}_{(2)}} - \frac{1}{1-\hat{p}_{(2)}} = 0 \Rightarrow \hat{p}_{(2)} = \frac{7}{8},$$
$$\frac{5}{\hat{p}_{(3)}} - \frac{1}{1-\hat{p}_{(3)}} = 0 \Rightarrow \hat{p}_{(3)} = \frac{5}{6},$$
$$\frac{2}{\hat{p}_{(4)}} - \frac{1}{1-\hat{p}_{(4)}} = 0 \Rightarrow \hat{p}_{(4)} = \frac{2}{3}.$$

For $p_{(5)}$, the derivative of l with respect to $p_{(5)}$ is $1/(1-p_{(5)})$, which does not equal zero for any feasible value of $\hat{p}_{(5)}$ (that is $0 \le \hat{p}_{(5)} \le 1$), because there is no turning point for the likelihood as a function of $p_{(5)}$. However, the objective is to maximize the log-likelihood, and, as the log-likelihood is a decreasing function of $p_{(5)}$, it is maximized when $\hat{p}_{(5)} = 0$.

The maximum likelihood estimate of the survival function is

$$\hat{S}(t) = \prod_{j:t_{(j)} \le t} \hat{p}_{(j)}.$$

So, for example, $\hat{S}(12) = (8 \times 7 \times 5)/(9 \times 8 \times 6) = 0.6481$. Note that $\hat{S}(t)$ is constant between values of $t_{(j)}$.

You may notice a pattern to the maximum likelihood estimates of the survival probabilities. In each case,

$$\hat{p}_{(j)} = \frac{\text{number of lives surviving to time } t_{(j)}^{+}}{\text{number of lives surviving to time } t_{(j)}^{-}}.$$

This is a natural estimate for $p_{(j)}$, as it is the empirical survival probability at time $t_{(j)}$. We use this observation to determine a general form for the maximum likelihood estimates of the survival probabilities for any group of lives, without having to work through the individual likelihood contributions for each life.

We define the **risk set** at time $t_{(j)}$, denoted r_j, to be the number of lives observed to have survived to time $t_{(j)}^{-}$. If any lives left observation through death or right censorship before time $t_{(j)}^{-}$, they are not included in r_j. If any lives entered observation (left-truncated) in the same period, they *are* included in r_j. Also, let d_j denote the number of deaths at time $t_{(j)}$. In principle, as deaths are happening in continuous time and the lives are independent, we would expect $d_j = 1$ for each j, but granularity in recording (e.g., we may record to the nearest week, or day) can lead to simultaneous deaths. Let m denote the number of death times in the data.

Then the likelihood and log-likelihood can be written as

$$L = \prod_{j=1}^{m} p_{(j)}^{r_j - d_j} \, (1 - p_{(j)})^{d_j}$$

and

$$l = \sum_{j=1}^{m} \left((r_j - d_j) \log p_{(j)} + d_j \log(1 - p_{(j)}) \right).$$

Taking the partial derivatives with respect to $p_{(j)}$ for $j = 1, 2, \ldots, m$ gives

$$\frac{\partial\, l}{\partial\, p_{(j)}} = \frac{r_j - d_j}{p_{(j)}} - \frac{d_j}{1 - p_{(j)}}, \tag{18.2}$$

and setting these partial derivatives equal to zero gives the maximum likelihood estimates as

$$\hat{p}_{(j)} = \frac{r_j - d_j}{r_j}.$$

Returning to our illustrative example, we have the data and calculations summarised in Table 18.2. (In this, and other, tables, it is convenient to define $t_{(0)} = 0$, with r_0 denoting the number of lives initially under observation.) Note how much more concise this format is, compared with the seriatim approach in Table 18.1. Also interesting to note is that, if there are no censored or truncated data, the Kaplan–Meier method will give the empirical survival function from Section 18.4.1.

Because the exact times of censoring do not affect the results, and because we may have a large amount of data, it is common for the data for the Kaplan–

Table 18.2 *Data summary based on death times.*

j	$t_{(j)}$	r_j	d_j	$\hat{p}_{(j)}$	$\hat{S}(t)$, $t_{(j)} \le t < t_{(j+1)}$
0	0	10			1
1	5	9	1	8/9	8/9
2	10	8	1	7/8	7/9
3	14	6	1	5/6	35/54
4	17	3	1	2/3	70/162
5	20	1	1	0/1	0

Table 18.3 *Excerpt from mortality study data for Example 18.4.*

j	$t_{(j)}$	Deaths at $t_{(j)}$	Exits in $(t^+_{(j)}, t^-_{(j+1)})$ (censored)	Entrants in $(t^+_{(j)}, t^-_{(j+1)})$ (truncated)
0	0		10	2
1	23.5	1	5	8
2	44.5	1	5	3
3	57.0	2	1	6
4	59.0	1	6	2

Table 18.4 *Solution with risk set calculations for Example 18.4.*

j	$t_{(j)}$	r_j	d_j	$\hat{p}_{(j)}$	$\hat{S}(t)$, $t_{(j)} \leq t < t_{(j+1)}$	c_j
0	0	100			1	8
1	23.5	92	1	91/92	0.9891	−3
2	44.2	94	1	93/94	0.9786	2
3	57.0	91	2	89/91	0.9571	−5
4	59.0	94	1	93/94	0.9469	4

Meier estimator to be presented in grouped form, similar to the first four columns of Table 18.2. Alternatively, the data may give the number of exiting (right-censored) and incoming (left-truncated) observations instead of directly giving the risk set. Let c_j denote the number of exits (right-censored) minus the number of new entrants (left-truncated) to the observation set, between times $t^+_{(j)}$ and $t^-_{(j+1)}$. Then, assuming that we know the initial risk set, r_0, the risk set at time $t_{(1)}$ is $r_1 = r_0 - c_0$, and for $j = 1, 2, \ldots$, we have

$$r_{j+1} = r_j - d_j - c_j.$$

Example 18.4 Initially, 100 lives are included in an observation of ages at death. You are given excerpted information from the study data in Table 18.3. Calculate the size of the risk set at each of the first four times of death, and estimate the conditional and unconditional survival probability up to age 59, using the Kaplan–Meier method.

Solution 18.4 We repeat the table with the risk set calculations included. We are given that $r_0 = 100$, and we have c_j = exits − entrants, so $r_1 = r_0 - c_0 = 92$, $r_2 = r_1 - d_1 - c_1 = 94$, and so on. The c_j values are in the final column, since they are not required for $\hat{S}(t_{(j)})$, but are required for calculating the risk set at the following death date, $t_{(j+1)}$. The results are shown in Table 18.4.

□

Greenwood's formula

Because $\hat{S}(t)$ is a function of the random observations, we can derive a variance, which can be approximated using information from the data. The result is Greenwood's formula:

$$\mathrm{V}[\hat{S}(t)] \approx \hat{S}(t)^2 \left(\sum_{j:t_{(j)} \leq t} \frac{d_j}{r_j(r_j - d_j)} \right). \tag{18.3}$$

The derivation of this starts with the estimates $\hat{p}_{(j)} = 1 - d_j/r_j$. Assume at each death date $t_{(j)}$ that r_j is known, and that d_j is the random number of deaths from the risk set r_j. Then d_j is a binomial random variable, $d_j \sim B(r_j, 1 - p_{(j)})$, with

$$\mathrm{E}[d_j] = r_j\,(1 - p_{(j)}) \quad \text{and} \quad \mathrm{V}[d_j] = r_j\,(1 - p_{(j)})\,p_{(j)}.$$

Then, as $\hat{p}_{(j)} = 1 - d_j/r_j$,

$$\mathrm{E}[\hat{p}_{(j)}] = 1 - \frac{\mathrm{E}[d_j]}{r_j} = p_{(j)}$$

and

$$V[\hat{p}_{(j)}] = \frac{\mathrm{V}[d_j]}{r_j^2} = \frac{p_{(j)}\,(1 - p_{(j)})}{r_j}.$$

Now, under the Kaplan–Meier approach, the $\{\hat{p}_{(j)}\}$ are independent of each other. We know this because the derivatives of the partial likelihood with respect to each parameter in equation (18.2) did not involve any of the other parameters (a sufficient, but not necessary condition for independence). The expected value of $\hat{S}(t)$ is

$$\mathrm{E}[\hat{S}(t)] = \mathrm{E}\left[\prod_{j:t_{(j)} \leq t} \hat{p}_{(j)} \right] = \prod_{j:t_{(j)} \leq t} \mathrm{E}[\hat{p}_{(j)}] = \prod_{j:t_{(j)} \leq t} p_{(j)} = S(t),$$

where we can move the expectation inside the product because of independence. The variance is

$$\begin{aligned} \mathrm{V}[\hat{S}(t)] &= \mathrm{E}\left[\left(\prod_{j:t_{(j)} \leq t} \hat{p}_{(j)} \right)^2 \right] - \mathrm{E}\left[\prod_{j:t_{(j)} \leq t} \hat{p}_{(j)} \right]^2 \\ &= \mathrm{E}\left[\prod_{j:t_{(j)} \leq t} \hat{p}_{(j)}^2 \right] - S(t)^2. \end{aligned}$$

Again using independence of the $\{\hat{p}_{(j)}\}$,

$$\begin{aligned}
\mathrm{E}\left[\prod_{j:t_{(j)}\le t} \hat{p}_{(j)}^2\right] &= \prod_{j:t_{(j)}\le t} \mathrm{E}\left[\hat{p}_{(j)}^2\right] = \prod_{j:t_{(j)}\le t} \left(\mathrm{V}\left[\hat{p}_{(j)}\right] + \mathrm{E}\left[\hat{p}_{(j)}\right]^2\right) \\
&= \prod_{j:t_{(j)}\le t} \left(\frac{p_{(j)}\,(1-p_{(j)})}{r_j} + p_{(j)}^2\right) \\
&= \prod_{j:t_{(j)}\le t} p_{(j)}^2 \left(\frac{1-p_{(j)}}{p_{(j)}\,r_j} + 1\right) \\
&= S(t)^2 \prod_{j:t_{(j)}\le t} \left(\frac{1-p_{(j)}}{p_{(j)}\,r_j} + 1\right).
\end{aligned}$$

Now we note that $(1-p_{(j)})/(r_j\,p_{(j)})$ is expected to be fairly small, so we approximate the product of the $1 + (1-p_{(j)})/(r_j\,p_{(j)})$ terms by ignoring all cross products involving $(1-p_{(j)})(1-p_{(k)})$, giving the approximation

$$\prod_{j:t_{(j)}\le t} \left(\frac{1-p_{(j)}}{p_{(j)}\,r_j} + 1\right) \approx 1 + \sum_{j:t_{(j)}\le t} \frac{1-p_{(j)}}{p_{(j)}\,r_j}.$$

So our approximation to the variance is

$$\begin{aligned}
\mathrm{V}[\hat{S}(t)] &\approx S(t)^2 \left(1 + \sum_{j:t_{(j)}\le t} \frac{1-p_{(j)}}{p_{(j)}\,r_j}\right) - S(t)^2 \\
&\approx S(t)^2 \sum_{j:t_{(j)}\le t} \frac{1-p_{(j)}}{p_{(j)}\,r_j}.
\end{aligned}$$

Now we approximate $S(t)$ with $\hat{S}(t)$, and $p_{(j)}$ with $\hat{p}_{(j)} = 1 - d_j/r_j$, to give

$$\mathrm{V}[\hat{S}(t)] \approx \hat{S}(t)^2 \sum_{j:t_{(j)}\le t} \frac{d_j}{r_j(r_j - d_j)}.$$

Example 18.5 Calculate the standard deviation of the $\hat{S}(t)$ values in Example 18.4 using Greenwood's formula.

Solution 18.5 For $j = 1$, we have

$$\mathrm{SD}[\hat{S}(23.5)] \approx 0.9891\sqrt{\frac{1}{92 \times 91}} = 0.01081.$$

Similarly, for $j = 2$, we have

$$\mathrm{SD}[\hat{S}(44.2)] \approx 0.9786\sqrt{\frac{1}{92 \times 91} + \frac{1}{94 \times 93}} = 0.01496.$$

Proceeding similarly for the subsequent values of $\hat{S}(t)$ gives the values in the following table.

j	$t_{(j)}$	r_j	d_j	$\hat{p}_{(j)}$	$\hat{S}(t)$, $t_{(j)} \leq t < t_{(j+1)}$	$\mathrm{SD}[\hat{S}(t)]$, $t_{(j)} \leq t < t_{(j+1)}$
1	23.5	92	1	91/92	0.9891	0.01081
2	44.2	94	1	93/94	0.9786	0.01496
3	57.0	91	2	89/91	0.9571	0.02099
4	59.0	94	1	93/94	0.9469	0.02310

□

Confidence intervals for $\hat{S}(t)$

We saw in the derivation of Greenwood's formula that $\mathrm{E}[\hat{S}(t)] = S(t)$. We also know from the properties of maximum likelihood estimators that, subject to certain conditions, the maximum likelihood estimator is asymptotically normally distributed, so an approximate 95% confidence interval for $S(t)$ is

$$\hat{S}(t) \pm 1.96\sqrt{\mathrm{V}[\hat{S}(t)]}. \tag{18.4}$$

So, in Example 18.4, an approximate 95% confidence interval for $S(23.5)$ would be

$$0.9891 \pm 1.96 \times 0.01081 = (0.9679,\ 1.0103).$$

Now, the upper bound of the confidence interval in this example is not very useful, as we know that $S(t) \leq 1$. We also find that, when $\hat{S}(t)$ is small, formula (18.4) can result in negative lower bounds.

One way of ensuring that the bounds are meaningful is to use a log-transformation of the estimator. In our case, we have $S(t) \in [0, 1]$. It then follows that $-\log S(t) \in [0, \infty]$, and hence $\log(-\log S(t)) \in [-\infty, \infty]$. What this means is that, if we use $g(\hat{S}(t)) = \log(-\log \hat{S}(t))$ to transform the estimator, then we can construct a confidence interval for an unbounded statistic. When we obtain this confidence interval for $g(S)$, we can transform it back to a confidence interval for S with the inverse of the function $\log(-\log S)$, i.e. $g^{-1}(S) = \exp\{-\exp\{g(S)\}\}$. We use two characteristics of functions of maximum likelihood estimators for this. Suppose $\hat{\theta}$ is the maximum likelihood estimator of θ, and that the variance of $\hat{\theta}$ can be estimated from $\hat{\theta}$ (and does not require any other estimates of parameters). Then the following statements are true for any monotonic and differentiable function g.

(i) The maximum likelihood estimate of $g(\theta)$ is $g(\hat{\theta})$.
(ii) The variance of $g(\hat{\theta})$ can be approximated as

$$\mathrm{V}[g(\hat{\theta})] \approx \left(\frac{dg(\theta)}{d\theta}\right)^2 \mathrm{V}[\hat{\theta}].$$

This is an application of the **delta method**. In applications of this result, the derivative $g'(\theta)$ is estimated by $g'(\hat{\theta})$.

Applying the delta method to the function $g(s) = \log(-\log s)$, and using $\hat{S}(t)$ in place of $S(t)$, we have

$$g'(s) = -\frac{1}{s \log s} \Rightarrow \mathrm{V}[g(\hat{S}(t))] \approx \left(\frac{1}{\hat{S}(t)\log\hat{S}(t)}\right)^2 \mathrm{V}[\hat{S}(t)].$$

This allows us to construct, say, an approximate 95% confidence interval for $g(S(t)) = \log(-\log S(t))$, which can be transformed back to give bounds for $S(t)$, as follows.

Let g_{L} and g_{U} represent the lower and upper bounds of the confidence interval of $\hat{g}(S(t))$, and let s_{L} and s_{U} represent the lower and upper bounds of the confidence interval for $\hat{S}(t)$ after transforming the data back using g^{-1}. Then

$$\begin{aligned} g(\hat{S}(t)) \pm 1.96\sqrt{\mathrm{V}[\hat{g}(S(t))]} &\approx g(\hat{S}(t)) \pm 1.96\sqrt{\left(\frac{dg(S(t))}{dS(t)}\right)^2 \mathrm{V}[S(\hat{t})]} \\ &= \log(-\log\hat{S}(t)) \pm 1.96\sqrt{\left(\frac{1}{\hat{S}(t)\log\hat{S}(t)}\right)^2 \mathrm{V}[\hat{S}(t)]} \\ &= (g_{\mathrm{L}},\, g_{\mathrm{U}}). \end{aligned}$$

Thus, our confidence interval for $S(t)$ is $(s_{\mathrm{L}}, s_{\mathrm{U}})$ where

$$s_{\mathrm{U}} = \exp\{-\exp\{g_{\mathrm{L}}\}\} \quad \text{and} \quad s_{\mathrm{L}} = \exp\{-\exp\{g_{\mathrm{U}}\}\}.$$

Note that $g(s)$ is a decreasing function of s here, so the bounds switch when the data are transformed in either direction.

To see this in practice, consider $S(23.5)$ from Example 18.5. We have $\hat{S}(23.5) = 0.9891$ and $\mathrm{V}[\hat{S}(23.5)] \approx 0.01081^2$. So the estimated 95% confidence interval for the transformed estimator is $(g_{\mathrm{L}},\, g_{\mathrm{U}})$, where

$$g_{\mathrm{L}} = \log(-\log 0.9891) - 1.96\sqrt{\left(\frac{1}{0.9891\ \log 0.9891}\right)^2 (0.01081)^2} = -6.4763,$$

$$g_{\mathrm{U}} = \log(-\log 0.9891) + 1.96\sqrt{\left(\frac{1}{0.9891\ \log 0.9891}\right)^2 (0.01081)^2} = -2.5564.$$

We now use $g^{-1}(g_{\mathrm{L}})$ and $g^{-1}(g_{\mathrm{U}})$ to find the bounds of the log-confidence interval:

$$s_{\mathrm{L}} = \exp\{-\exp\{-2.5564\}\} = 0.9253$$

and

$$s_{\mathrm{U}} = \exp\{-\exp\{-6.4763\}\} = 0.9985.$$

If it is not clear from the context, we refer to the confidence interval without transformation (e.g. $\hat{S}(t) \pm 1.96\sqrt{\mathrm{V}[\hat{S}(t)]}$) as a **linear confidence interval**, and the confidence interval using the $\log(-\log)$ transformation, $(s_{\mathrm{L}}, s_{\mathrm{U}})$, as the **log-confidence interval**.

Notes on Kaplan–Meier calculations

(1) We have described the Kaplan–Meier estimate of S in terms of ages at death, which is the most useful interpretation in the context of developing practical survival models. In other survival analysis contexts, such as survival after diagnosis of a disease, the age variable would be a time variable, and that is the usual presentation.

(2) If the final value in the data is censored, then $\hat{S}(t)$ is undefined beyond that point. If we need values for $\hat{S}(t)$ beyond the final death time, then we must extrapolate from the data. Popular methods include linear extrapolation to some selected maximum age ω, or **exponential extrapolation**, such that if the last death occurs at age $t_{(\max)}$, we could set

$$\hat{S}(t) = (\hat{S}(t_{(\max)}))^{t/t_{\max}} \quad \text{for } t > t_{(\max)}. \tag{18.5}$$

(3) If there is a very large data set, with deaths at very short intervals of, say h, the $1 - \hat{p}_{(j)}$ are estimates of ${}_hq_{t_{(j)}}$, which will be very close to $h\,\mu_{t_{(j)}}$.

(4) In principle, the Kaplan–Meier survival function could be used as the basis of a life table. It would probably be smoothed (or graduated) first.

(5) Another application of the Kaplan–Meier survival function is to assess whether the mortality data used are consistent with an underlying model. For example, an insurer would check how closely their mortality experience matches the mortality tables provided by their professional bodies, or required by regulation.

18.4.4 The Nelson–Aalen estimator

We define the cumulative hazard function, which we denote by H, as

$$H(x) = \int_0^x \mu_y\, dy,$$

and so $S_0(x) = e^{-H(x)}$.

The Nelson–Aalen estimator is a non-parametric estimator of the cumulative hazard function, which can be used to construct a non-parametric estimator for the survival function.

The maximum likelihood estimator of the cumulative hazard function, using the same notation as we used for the Kaplan–Meier estimator, is

$$\hat{H}(x) = \sum_{j:t_{(j)} \leq x} \frac{d_j}{r_j},$$

and, using similar arguments to the Kaplan–Meier estimator, we can derive

$$\mathrm{V}[\hat{H}(x)] \approx \sum_{j:t_{(j)} \leq x} \frac{d_j(r_j - d_j)}{r_j^3}.$$

The Nelson–Aalen estimate of the survival function is $\hat{S}(x) = e^{-\hat{H}(x)}$. Using the delta method, with $g(H) = e^{-H}$, we have

$$\begin{aligned}\mathrm{V}[\hat{S}(x)] &\approx \left(e^{-H(x)}\right)^2 \mathrm{V}[\hat{H}(x)] = (S(x))^2\, \mathrm{V}[\hat{H}(x)] \\ &\approx (\hat{S}(x))^2 \sum_{j:t_{(j)} \leq x} \frac{d_j(r_j - d_j)}{r_j^3}.\end{aligned}$$

We can construct an approximate 95% confidence interval for $H(x)$ as

$$\hat{H}(x) \pm 1.96\sqrt{\mathrm{V}[\hat{H}(x)]}$$

and use this to construct a confidence interval for $\hat{S}(x)$. However, it may be preferable to transform the estimator to avoid problems with potentially negative bounds for $H(x)$. Because there is no upper bound for $H(x)$, a single log-transformation of $g(H) = \log H$ transforms the bounds from $[0, \infty]$ to $[-\infty, \infty]$.

If the estimate of $H(x)$ is based on a large sample, such that the intervals between deaths are small, then a survival distribution might be derived by smoothing the cumulative hazard function, giving, say, $\tilde{H}(x)$, and then (numerically) differentiating it to get $\mu_x = \frac{d\tilde{H}(x)}{dx}$.

18.5 The alive–dead model

The non-parametric approaches described in the previous section focus on the survival function for future lifetimes. However, the more basic building block of survival models is the force of mortality, and it is useful to consider methods that use the mortality data to estimate μ_x directly, rather than through the survival function or the cumulative hazard function. In addition, the methodology extends to multiple state models, where we are interested in estimating forces of transition between states.

We assume that the mortality data are partitioned into age-years, and that the force of mortality is constant within each age-year; that is, let $\mu_{x+t} = \mu_x$ (say), for integer x and $0 \le t < 1$. That means that we fit a piecewise linear model to the force-of-mortality function. To do this, we essentially treat the lifetime data for each year of age as separate data sets.

For example, suppose we have a life who entered observation at age 40.2 and died at age 41.4. For the parametric approach, we described this observation as $x = 40.2$, $t = 1.2$, and $\delta = 1$, but for the piecewise constant model for the force of mortality we split the observation into individual age-years. For the year of age 40 to 41, this life enters observation at age $x = 40.2$ and is censored at age 41, so the data point for estimating μ_{40} is $x = 40.2$, $t = 0.8$, and $\delta = 0$. To estimate μ_{41}, we consider only the information between ages 41 and 42: the life entered observation at age $x = 41$, and was observed for $t = 0.4$ years, at which point they died, so $\delta = 1$.

More generally, for the year of age x to $x + 1$, let n denote the number of lives who contribute some information; that is, who are under observation for at least some part of the year. For each of the n observations, we know the age at entry, x_j, where $x \le x_j < x + 1$, and we observe values for the following two random variables.

1. The period of observation, $t_j > 0$, is called the **waiting time**. As we are working within the age-year x to $x + 1$, we have $x_j + t_j \le x + 1$.
2. The death indicator variable, $\delta_j = 1$ if the life dies before age $x_j + t_j$, and $\delta_j = 0$ if the life is censored at age $x_j + t_j$.

We know that the waiting time and indicator random variables (t_j and δ_j) are *not* independent, because, if $x_j + t_j = x + 1$, then we know that the life survived to the end of the observation period, so $\delta_j = 0$.

We also know, from equation (18.1), that the joint probability function of (t_j, δ_j) is

$$f_{x_j}(t_j, \delta_j) = {}_{t_j}p_{x_j} \left(\mu_{x_j+t_j}\right)^{\delta_j}.$$

Using the fact that $t_j \le 1$, we have $\mu_{x_j+t_j} = \mu_x$ under the piecewise constant assumption, and thus ${}_{t_j}p_{x_j} = e^{-t_j\,\mu_x}$. So the joint probability function can be written as

$$f_{x_j}(t_j, \delta_j) = e^{-t_j\,\mu_x} \left(\mu_x\right)^{\delta_j}.$$

Assuming that the n observed lives are independent, we have a likelihood function for μ_x of

$$L(\mu_x) = \prod_{j=1}^{n} f_{x_j}(t_j, \delta_j) = \prod_{j=1}^{n} e^{-t_j \mu_x} \; (\mu_x)^{\delta_j}$$
$$= \exp\left\{-\mu_x \sum_{j=1}^{n} t_j\right\} \left(\mu_x\right)^{\sum_{j=1}^{n} \delta_j}.$$

Let $w_x = \sum_{j=1}^{n} t_j$ denote the total waiting time, which is the aggregate number of years that the n lives spent alive and not right censored, between ages x and $x+1$, and let $d_x = \sum_{j=1}^{n} \delta_j$ denote the total number of deaths observed from the n lives, between ages x and $x+1$. Then we can estimate μ_x as follows:

$$\begin{aligned} L(\mu_x) &= e^{-w_x \mu_x} \, \mu_x^{d_x}, \\ l(\mu_x) &= \log L(\mu_x) = -w_x \, \mu_x + d_x \log \mu_x \,, \qquad (18.6) \\ \frac{\partial \, l}{\partial \, \mu_x} &= -w_x + \frac{d_x}{\mu_x}. \end{aligned}$$

Setting this derivative equal to zero gives the maximum likelihood estimate of μ_x,

$$\hat{\mu}_x = \frac{d_x}{w_x} = \frac{\text{Total number of deaths}}{\text{Total waiting time}}.$$

As $\hat{\mu}_x$ is the maximum likelihood estimate of μ_x, we can derive its asymptotic variance. In this context, we now treat $\hat{\mu}_x$ as a random variable, equal to D_x/W_x, where W_x is the *ex ante* random total waiting time between ages x and $x+1$ for the n lives, and D_x is the *ex ante* random number of deaths between ages x and $x+1$ for the n lives. So, from equation (18.6), we have

$$l(\hat{\mu}_x) = -W_x \, \mu_x + D_x \log \mu_x,$$

where μ_x is the unknown parameter (and is not a random variable). The asymptotic variance of $\hat{\mu}_x$, which we use to approximate $\mathrm{V}[\hat{\mu}_x]$, is

$$-\left(\mathrm{E}\left[\frac{\partial^2 \, l}{\partial \, \mu_x^2}\right]\right)^{-1}.$$

Now

$$\frac{\partial^2 \, l}{\partial \, \mu_x^2} = -\frac{D_x}{\mu_x^2} \;\Rightarrow \left(\mathrm{E}\left[-\frac{\partial^2 \, l}{\partial \, \mu_x^2}\right]\right)^{-1} = \frac{\mu_x^2}{\mathrm{E}[D_x]}.$$

We use our observed value of d_x to approximate $\mathrm{E}[D_x]$ and our maximum likelihood estimate, $\hat{\mu}_x = d_x/w_x$, to approximate μ_x, giving

$$V[\hat{\mu}_x] \approx \left(\frac{d_x}{w_x}\right)^2 \frac{1}{d_x} = \frac{d_x}{w_x^2}\,.$$

18.5.1 Notes on the alive–dead model

(1) The aggregate waiting time between ages x and $x+1$ is traditionally called the **central exposed to risk** and is denoted E_x^{c}.

(2) Conducting a completely separate estimation for each year of age loses some information, in the sense that we expect μ_x to be a relatively smooth function of x, so, for example, the values of μ_{x-1} and μ_{x-2} might be assumed to give some information about μ_x that is not captured here. However, we can use graduation techniques (e.g. kernel density smoothing) to transform the raw $\hat{\mu}_x$ estimates into a more realistic smooth function of x.

(3) If we are using the piecewise constant estimates of μ_x as the basis for a smooth, continuous function, it would be more appropriate to set $\mu_{x+\frac{1}{2}} = \hat{\mu}_x$, i.e. to set the estimator to be the mid-year force of mortality, rather than the value at the start of the year of age. This gives an estimator that is well known to actuaries, namely

$$\hat{\mu}_{x+\frac{1}{2}} = \frac{d_x}{E_x^{\mathrm{c}}} \,.$$

(4) To construct life tables at integer ages, we use mortality rates. Given the estimated values for μ_x, and the assumption that μ_x is constant between ages x and $x+1$, we have

$$\hat{q}_x = 1 - e^{-\hat{\mu}_x} \tag{18.7}$$

as the maximum likelihood estimate of q_x under this model.

(5) Using the delta method again, we can derive the variance of the estimate for q_x in equation (18.7). Recall that for a monotonic, differentiable function g, and a maximum likelihood estimate $\hat{\mu}$ of a parameter μ,

$$\mathrm{V}[g(\hat{\mu})] \approx \left(\frac{dg(\mu)}{d\mu}\right)^2 V[\hat{\mu}].$$

Let $g(\mu) = 1 - e^{-\mu}$ so that $g'(\mu) = e^{-\mu}$, then

$$\begin{aligned} \mathrm{V}[\hat{q}_x] = \mathrm{V}[g(\hat{\mu}_x)] &\approx \exp(-2\hat{\mu}_x)\mathrm{V}[\hat{\mu}_x] \\ &\approx \exp\left(-2\,\frac{d_x}{E_x^{\mathrm{c}}}\right) \frac{d_x}{(E_x^{\mathrm{c}})^2}. \end{aligned}$$

(6) A different estimator for q_x that is commonly used is $\tilde{q}_x$, where

$$\tilde{q}_x = \frac{d_x}{E_x},$$

in which E_x is the **initial exposed to risk**, which, unfortunately, has several different definitions. Roughly, the initial exposed to risk represents the risk set for the year of age x to $x+1$, such that we may regard the number of deaths as a binomial random variable, with $D_x \sim B(E_x, q_x)$. It is simple to show that, under this model, the maximum likelihood estimate of q_x is $\hat{q}_x = d_x/E_x$. If we observe n lives from age x to the earlier of age $x+1$ or death, then this estimator is valid, and $E_x = n$. However, there are potentially significant technical problems when we have truncation and censoring between the integer ages, and the use of the initial exposed to risk is generally deprecated now.

(7) The **actuarial estimate** of q_x is defined as

$$\tilde{q}_x = \frac{d_x}{E_x^c + \frac{1}{2}d_x} ,$$

and the denominator here can be viewed as an estimate of the initial exposed to risk. However, the actuarial estimate can be justified without using the binomial distribution, using $\mu_{x+\frac{1}{2}} = d_x/E_x^c$, and assuming that deaths are uniformly distributed between integer ages. Under UDD, it is straightforward to show that

$$q_x = \frac{\mu_{x+\frac{1}{2}}}{1+\frac{1}{2}\mu_{x+\frac{1}{2}}}.$$

Substituting $\mu_{x+\frac{1}{2}} = d_x/E_x^c$ gives

$$\tilde{q}_x = \frac{d_x}{E_x^c + \frac{1}{2}d_x}. \tag{18.8}$$

For small values of μ_x, the difference between $\hat{q}_x$ from equation (18.7) and $\tilde{q}_x$ from equation (18.8) will be very small, but the maximum likelihood estimate, $\hat{q}_x$, is more consistent, and is therefore generally preferred. The problem with $\tilde{q}_x$ is that we have assumed constant force of mortality between integer ages to derive the maximum likelihood estimate for μ, and then used UDD, which is a different, inconsistent assumption, to calculate $\tilde{q}_x$.

18.6 Estimation of transition intensities in multiple state models

To estimate transition intensities for models with more than two states, we use a very similar approach to that of the previous section for the alive–dead model. As before, we consider each age-year separately, and we assume that all transition intensities are constant within each age-year. We then construct the likelihood function, using the observed data for each life consisting of (i) the time spent in each state, and (ii) the number of transitions between

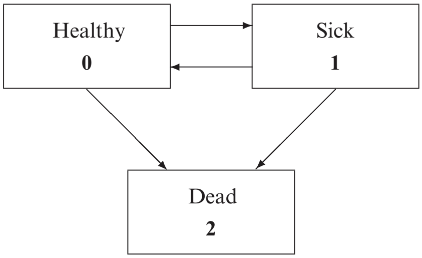


Figure 18.1 The sickness–death model for disability income insurance.

states, during the age-year. Note that the likelihood is now a function of all the possible transition intensities.

If the life under consideration stayed in the same state throughout the observation period, or until a fixed, Type 1 censorship point, then the contribution to the likelihood is the *ex ante* probability of staying in that state. If the life moves between states, then the contribution to the likelihood is the joint *ex ante* probability function of the observed movements. We again use the term probability function to cover both cases.

We develop our results in the context of the sickness–death model introduced in Chapter 8, represented by Figure 18.1. The piecewise constant transition intensities are represented by $\mu_x^{ij} = \mu_{x+t}^{ij}$ for integer x and for $0 \leq t < 1$.

Suppose we observe the history listed in the table below for a life under observation between integer ages x and $x + 1$. Time, t, is measured in years from age x.

Time	Event
$t = 0$	In State 0
$t = 0.1$	Moves to State 1
$t = 0.5$	Moves to State 0
$t = 0.8$	Moves to State 1
$t = 1$	Still in State 1

To construct the contribution to the likelihood function for this life, we partition this history into individual intervals, where each interval ends with a transition or with censorship. Note that the probabilities and transition intensities within the age-year do not depend on the age at transition, because we assume constant transition intensities between integer ages.

The first interval is (0, 0.1]. The probability function is based on the joint distribution of the time of exit from State 0 and the mode of exit. The density of the time of exit from State 0, given constant transition intensities, is

$$ {}_{t}p_x^{\overline{00}} \left(\mu_x^{01} + \mu_x^{02} \right) $$

and the probability that the transition at time t is to State 1, rather than State 2, (given that there is a transition) is, by proportion,

$$ \frac{\mu_x^{01}}{\mu_x^{01} + \mu_x^{02}}. $$

So the contribution to the likelihood of this part of the history, given that the transition is at time $t = 0.1$, is

$$ {}_{t}p_x^{\overline{00}} \left(\mu_x^{01} + \mu_x^{02} \right) \frac{\mu_x^{01}}{\mu_x^{01} + \mu_x^{02}} = {}_{0.1}p_x^{\overline{00}} \mu_x^{01}. $$

The second interval, (0.1, 0.5], is similar, except that now the probability function combines the density associated with leaving State 1 after 0.4 years, and the probability that the exit is to State 0, rather than State 2, giving a contribution of

$$ {}_{0.4}p_x^{\overline{11}} \mu_x^{10}. $$

The contribution for the third interval is similar to that for the first interval, namely

$$ {}_{0.3}p_x^{\overline{00}} \mu_x^{01}. $$

As observation is censored at the end of the fourth interval, the contribution to the likelihood is

$$ {}_{0.2}p_x^{\overline{11}}. $$

We can use the time-line diagram in Figure 18.2 to demonstrate the likelihood for this history. This is exactly the same idea as the time-line diagrams in previous chapters – compare it with Figure 8.5, for example. The difference here is that we omit the dt terms which are used to derive instantaneous probabilities for setting up integrals, but are not included in the probability functions.

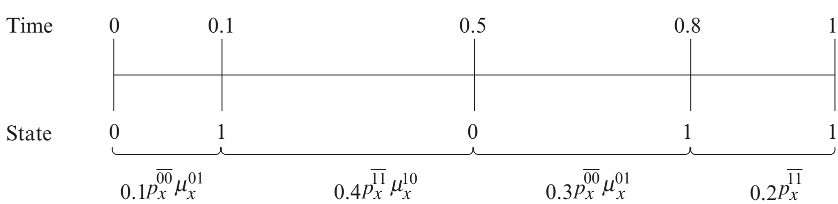


Figure 18.2 Time-line diagram for sickness–death example transition history.

So the observation of this life contributes the following to the likelihood:

$$ {}_{0.1}p_x^{\overline{00}}\ \mu_x^{01}\ {}_{0.4}p_x^{\overline{11}}\ \mu_x^{10}\ {}_{0.3}p_x^{\overline{00}}\ \mu_x^{01}\ {}_{0.2}p_x^{\overline{11}}. \tag{18.9} $$

Constant transition intensities within each year of age give us

$$ {}_tp_x^{\overline{00}} = e^{-t\left(\mu_x^{01}+\mu_x^{02}\right)} \quad \text{and} \quad {}_tp_x^{\overline{11}} = e^{-t\left(\mu_x^{10}+\mu_x^{12}\right)}. $$

So the contribution to the likelihood in (18.9) can be expressed entirely in terms of the transition intensities as

$$ \begin{aligned} & e^{-0.1\left(\mu_x^{01}+\mu_x^{02}\right)}\ \mu_x^{01}\ e^{-0.4\left(\mu_x^{10}+\mu_x^{12}\right)}\ \mu_x^{10}\ e^{-0.3\left(\mu_x^{01}+\mu_x^{02}\right)}\ \mu_x^{01}\ e^{-0.2\left(\mu_x^{10}+\mu_x^{12}\right)} \\ & \quad = e^{-0.4\left(\mu_x^{01}+\mu_x^{02}\right)}\ e^{-0.6\left(\mu_x^{10}+\mu_x^{12}\right)}\ \left(\mu_x^{01}\right)^2\ \left(\mu_x^{10}\right). \end{aligned} $$

Note that the precise timing of the transitions does not matter here. The only information required is

(1) the life spent 0.4 years in State 0,
(2) the life spent 0.6 years in State 1,
(3) the number of transitions from State 0 to State 1 was 2,
(4) the number of transitions from State 1 to State 0 was 1, and
(5) there were no other transitions during the year.

This is sufficient information because of the assumption of constant transition intensities within each age-year.

For a more general formula for the contribution to the likelihood of a single policy history, assuming the sickness–death model with piecewise constant transition intensities, we adopt the following notation, where we consider all events occurring between ages x and $x+1$.

Let $t_j^{(k)}$ denote the total time spent by the jth life in State k between ages x and $x+1$, for $k=0,1$. This is the **waiting time in State *k*** for the life.

Let d_j^{ik} denote the total number of direct transitions from State i to State k between ages x and $x+1$ for the jth life, for $i=0$, 1 and for $k=0$, 1, 2, where $i \neq k$.

Then the contribution to the likelihood for the jth life is

$$ e^{-t_j^{(0)}\left(\mu_x^{01}+\mu_x^{02}\right)}\ e^{-t_j^{(1)}\left(\mu_x^{10}+\mu_x^{12}\right)}\ \left(\mu_x^{01}\right)^{d_j^{01}}\ \left(\mu_x^{02}\right)^{d_j^{02}}\ \left(\mu_x^{10}\right)^{d_j^{10}}\ \left(\mu_x^{12}\right)^{d_j^{12}}. $$

Example 18.6 An insurer is analysing its experience using the sickness–death model. You are given the following records of all transitions between ages x and $x+1$ for three lives. For each life, identify the values of $t^{(0)}$, $t^{(1)}$, d^{01}, d^{02}, d^{10}, and d^{12}, and write down the contribution to the likelihood function, in terms of the transition intensities.

Life 1		Life 2		Life 3	
$t = 0$	In State 0	$t = 0$	In State 1	$t = 0$	In State 0
$t = 1$	In State 0	$t = 0.3$	Moves to State 0	$t = 0.4$	Moves to State 1
		$t = 0.5$	Moves to State 1	$t = 0.6$	Moves to State 0
		$t = 0.8$	Moves to State 2	$t = 0.9$	Moves to State 1
				$t = 1$	In State 1

Solution 18.6 We can tabulate the information as follows:

Life	$t^{(0)}$	$t^{(1)}$	d^{01}	d^{02}	d^{10}	d^{12}
1	1	0	0	0	0	0
2	0.2	0.6	1	0	1	1
3	0.7	0.3	2	0	1	0

Then the contributions to the likelihood function are:

$$\text{Life 1:} \quad e^{-(\mu_x^{01}+\mu_x^{02})}$$

$$\text{Life 2:} \quad e^{-0.2(\mu_x^{01}+\mu_x^{02})} e^{-0.6(\mu_x^{10}+\mu_x^{12})} \left(\mu_x^{01}\right)\left(\mu_x^{10}\right)$$

$$\text{Life 3:} \quad e^{-0.7(\mu_x^{01}+\mu_x^{02})} e^{-0.3(\mu_x^{10}+\mu_x^{12})} \left(\mu_x^{01}\right)^2\left(\mu_x^{10}\right)$$

□

We can collect together the contributions to the likelihood from, say, n lives under observation between ages x and $x+1$, to give the full likelihood function as

$$L = \prod_{j=1}^{n} e^{-t_j^{(0)}(\mu_x^{01}+\mu_x^{02})}\, e^{-t_j^{(1)}(\mu_x^{10}+\mu_x^{12})} \left(\mu_x^{01}\right)^{d_j^{01}} \left(\mu_x^{02}\right)^{d_j^{02}} \left(\mu_x^{10}\right)^{d_j^{10}} \left(\mu_x^{12}\right)^{d_j^{12}}$$

and hence the log-likelihood is

$$\begin{aligned} l = \sum_{j=1}^{n} \Big(-t_j^{(0)} \left(\mu_x^{01}+\mu_x^{02}\right) - t_j^{(1)} \left(\mu_x^{10}+\mu_x^{12}\right) \\ + d_j^{01}\log\mu_x^{01} + d_j^{02}\log\mu_x^{02} + d_j^{10}\log\mu_x^{10} + d_j^{12}\log\mu_x^{12} \Big) \\ = -\left(\mu_x^{01}+\mu_x^{02}\right)\sum_{j=1}^{n} t_j^{(0)} - \left(\mu_x^{10}+\mu_x^{12}\right)\sum_{j=1}^{n} t_j^{(1)} \\ + \log\mu_x^{01}\sum_{j=1}^{n} d_j^{01} + \log\mu_x^{02}\sum_{j=1}^{n} d_j^{02} + \log\mu_x^{10}\sum_{j=1}^{n} d_j^{10} + \log\mu_x^{12}\sum_{j=1}^{n} d_j^{12}. \end{aligned}$$

For $i = 0, 1$ and $k = 0, 1, 2$ with $i \neq k$, let

$$T^{(i)} = \sum_{j=1}^{n} t_j^{(i)} \quad \text{and} \quad D^{ik} = \sum_{j=1}^{n} d_j^{ik}.$$

Then $T^{(0)}$ is the total waiting time in State 0 for the n lives, $T^{(1)}$ is the total waiting time in State 1 for the n lives, D^{01} is the total number of transitions from State 0 to State 1 by the n lives, and so on.

Using these definitions, the log-likelihood is

$$\begin{aligned} l = &-T^{(0)}\left(\mu_x^{01} + \mu_x^{02}\right) - T^{(1)}\left(\mu_x^{10} + \mu_x^{12}\right) \\ &+ D^{01}\log\mu_x^{01} + D^{02}\log\mu_x^{02} + D^{10}\log\mu_x^{10} + D^{12}\log\mu_x^{12}, \end{aligned}$$

and taking the partial derivative with respect to each μ_x^{ij} we obtain

$$\begin{aligned} \frac{\partial\, l}{\partial\, \mu_x^{01}} &= -T^{(0)} + \frac{D^{01}}{\mu_x^{01}}, & \frac{\partial\, l}{\partial\, \mu_x^{02}} &= -T^{(0)} + \frac{D^{02}}{\mu_x^{02}}, \\ \frac{\partial\, l}{\partial\, \mu_x^{10}} &= -T^{(1)} + \frac{D^{10}}{\mu_x^{10}}, & \frac{\partial\, l}{\partial\, \mu_x^{12}} &= -T^{(1)} + \frac{D^{12}}{\mu_x^{12}}. \end{aligned}$$

Setting these partial derivatives equal to zero yields the maximum likelihood estimates

$$\hat{\mu}_x^{01} = \frac{D^{01}}{T^{(0)}}, \qquad \hat{\mu}_x^{02} = \frac{D^{02}}{T^{(0)}}, \qquad \hat{\mu}_x^{10} = \frac{D^{10}}{T^{(1)}}, \qquad \hat{\mu}_x^{12} = \frac{D^{12}}{T^{(1)}}.$$

In each case, the maximum likelihood estimate of the transition intensity from State i to State k is the number of transitions from State i to State k, divided by the total waiting time in State i.

To estimate the variance of the maximum likelihood estimates, we first consider the second derivatives, which are

$$\begin{aligned} \frac{\partial^2\, l}{\partial\, (\mu_x^{01})^2} &= \frac{-D^{01}}{(\mu_x^{01})^2}, & \frac{\partial^2\, l}{\partial\, (\mu_x^{02})^2} &= \frac{-D^{02}}{(\mu_x^{02})^2}, \\ \frac{\partial^2\, l}{\partial\, (\mu_x^{10})^2} &= \frac{-D^{10}}{(\mu_x^{10})^2}, & \frac{\partial^2\, l}{\partial\, (\mu_x^{12})^2} &= \frac{-D^{12}}{(\mu_x^{12})^2}. \end{aligned}$$

We notice that all cross derivatives are zero, which means that the maximum likelihood estimates are independent, and that, for example,

$$\mathrm{V}\left[\hat{\mu}_x^{01}\right] \approx \left(-\mathrm{E}\left[\frac{\partial^2\, l}{\partial\, (\mu_x^{01})^2}\right]\right)^{-1} = \frac{\left(\mu_x^{01}\right)^2}{\mathrm{E}\left[D^{01}\right]}. \tag{18.10}$$

Substituting $\hat{\mu}_x^{01} = D^{01}/T^{(0)}$ for μ_x^{01}, and D^{01} for $\mathrm{E}[D^{01}]$, we have

$$\mathrm{V}\left[\hat{\mu}_x^{01}\right] \approx \frac{D^{01}}{\left(T^{(0)}\right)^2}.$$

The results for the sickness–death model are easily generalized to any Markov multiple state model. Given aggregated data such that D^{ik} is the total number of observed transitions from State i directly to State k, and $T^{(i)}$ is the total waiting time in State i, then the maximum likelihood estimate for μ_x^{ik} and the approximate variance of the maximum likelihood estimate for μ_x^{ik} are

$$\hat{\mu}_x^{ik} = \frac{D^{ik}}{T^{(i)}} \quad \text{and} \quad \mathrm{V}\left[\hat{\mu}_x^{ik}\right] \approx \frac{D^{ik}}{\left(T^{(i)}\right)^2}.$$

18.7 Comments

1. Throughout this chapter, we have assumed that the individual data points are independent. However, a common problem with life insurance data is that a single individual may hold more than one policy. To ensure that the data are as independent as possible, multiple policies insuring the same individual should be collected together as a single data point. We call this **deduplication**.
2. Generally, before analysing a mortality experience we separate the data into reasonably homogeneous groups. In life insurance, data would typically be classified by the sex and smoking status of the insured life. In addition, we would generally analyse individuals who buy annuities separately from individuals who buy life insurance, and we may separate shorter term insurance policyholders from whole life, as self-selection can create different underlying lifetime distributions for these groups. This separation recognizes that premiums for these groups should be based on the most representative data possible.

   In analysing the mortality experience of pension plan members, we generally do not have information on smoking status, and in any case, we do not separate smokers and non-smokers in the valuation process. We would analyse the data by sex, and if the plan offers spousal benefits we might analyse data from spouses separately from data from members.
3. Another source of heterogeneity in mortality data is the socio-economic status of the individual life. It is well documented that wealthier lives experience lighter mortality than less wealthy lives. In mortality data of insured lives, experience can differ quite significantly between individuals with small death benefits and individuals with large death benefits, even after other factors have been allowed for. Similarly with pension plan

data, there may be a significant difference in experience if the data are stratified by benefit amount. However, stratifying the data means that the separated data sets are smaller, which leads to potential for very high uncertainty, particularly at older ages when the exposure is low, or at very young ages where few deaths are observed. An alternative is to use a more complex regression approach, where the sum insured would be included as a regression coefficient.

4. The piecewise constant assumptions used in Sections 18.5 and 18.6 are unlikely to generate smooth functions, due to sampling variability. Typically, transition intensities would be graduated by a suitable smoothing process before being used.
5. In this chapter, we have assumed that the underlying survival function is not changing over time. However, in Chapter 3 we noted that mortality rates do change over time, and in Chapter 19 we introduce more sophisticated models of time-varying mortality. If the data used for calibrating a mortality model are collected over a reasonably short period, the impact of mortality changes will be small. If data are collected over decades, then consideration of trends in mortality should be incorporated in the analysis.

18.8 Notes and further reading

Model selection, goodness-of-fit tests and graduation are important steps in survival analysis, but are beyond the scope of this chapter. For a general actuarial coverage, see Klugman *et al.* (2012), and for a more specific coverage relating to mortality modelling, see Macdonald *et al.* (2018).

Brown *et al.* (1974) first suggested exponential extrapolation for the non-parametric survival function given in equation (18.5).

In Section 18.5.1 we noted that the initial exposed to risk measure is obsolescent. The Continuous Mortality Investigation Bureau, in CMI (2008), offers a detailed explanation of the problems with the initial exposed to risk, and all subsequent CMI reports use only the central exposed to risk.

18.9 Exercises

Shorter exercises

Exercise 18.1 You are given the following survival data; assume all nine lives are observed from birth:

$$27, \quad 30^+, \quad 34, \quad 58^+, \quad 68, \quad 68^+, \quad 70, \quad 77, \quad 78^+.$$

(a) Calculate $\hat{S}(69)$ using the Kaplan–Meier approach, together with an estimate of the standard deviation of $\hat{S}(69)$.

(b) Calculate $\hat{S}(69)$ using the Nelson–Aalen approach, together with an estimate of the standard deviation of $\hat{S}(69)$.

Exercise 18.2 You are given the following information with respect to a Kaplan–Meier estimate of a survival function S.

- $\hat{S}(t_{(5)}) = 0.87500$, $\hat{S}(t_{(6)}) = 0.8500$ and $\hat{S}(t_{(7)}) = 0.82875$.
- At each time of death $t_{(j)}$, there is one death.

(a) Calculate the risk set at time $t_{(6)}$.
(b) There were two exits from the risk set between times $t_{(6)}$ and $t_{(7)}$. How many new entrants were there in this period?

Exercise 18.3 Initially, 500 lives, each aged 20, were included in study of times to death. You are given the following information about the study participants at the first five death times.

j	$t_{(j)}$	Number of deaths at $t_{(j)}$	Exits in $\left(t^+_{(j)}, t^-_{(j+1)}\right)$
0	0	0	4
1	22.0	1	10
2	31.5	1	35
3	50.7	1	0
4	51.3	1	0
5	53.0	1	0

(a) Calculate the risk sets at each of the death times, and estimate the conditional and unconditional survival probability estimates up to age 53, using the Kaplan–Meier method. Put your answer in the format of a table, similar to the solution to Example 18.4, i.e. with headers

j	$t_{(j)}$	r_j	d_j	$\hat{p}_{(j)}$	$\hat{S}(t)$, $t_{(j)} \leq t < t_{(j+1)}$	c_j

(b) The data were later combined with another study. In the new study, 100 lives were observed from age 50. No lives were censored before age 53, and there were individual deaths observed at ages 51.5 and 51.8. Calculate the revised table in part (a) given this additional information.

Exercise 18.4 You are given that the estimate of a survival function at time 80 is $\hat{S}(80) = 0.3$, and that $V[\hat{S}(80)] \approx 0.03$.

(a) Calculate the 99% linear confidence interval for $S(80)$.
(b) Calculate the 99% log-confidence interval for $S(80)$.

Exercise 18.5 You are given that the estimate of the cumulative hazard function at time 30 is $\hat{H}(30) = 0.004$, and that $\mathrm{SD}[\hat{H}(30)] \approx 0.0025$.

(a) Calculate the 95% linear confidence interval for $H(30)$.
(b) Calculate the 95% log-confidence interval for $H(30)$.

Longer exercises

Exercise 18.6 The following data are taken from the employment records between ages 58 and 59 of 1 000 pension plan members. The possible decrements from active employment (State 0) are by early retirement (State 1), transfer to another position (State 2) or death in Service (State 3).

Total waiting time in active employment	785 years
Total number of early retirements	150
Total number of withdrawals	200
Total number of deaths in service	20

(a) Estimate ${}_1p_{58}^{00}$.
(b) Estimate ${}_{0.5}p_{58}^{02}$.
(c) Calculate a 95% log-confidence interval for μ_{58}^{01}.

Exercise 18.7 A large number of individuals were followed for 30 years to assess the age at which a disease symptom first appeared.

(a) For the individuals described below, describe the censoring and/or truncation involved in the observation:
  (i) The first life enrolled in the study with the disease already present at age 45.
  (ii) Two individuals enrolled at ages 30 and 42 and never showed the symptoms.
  (iii) The next two healthy individuals both enrolled in the study at age 35. The first developed the disease sometime between ages 40 and 41, and the second sometime between ages 55 and 56.
  (iv) One life enrolled in the study at age 41 and died of unrelated causes, with no symptoms of the disease, at age 65.2.
  (v) Three individuals, who joined the study at ages 36, 42 and 44, moved away from the community, and therefore left the study, at ages 55.1, 47.4 and 62.9, respectively.
(b) In terms of the probability function of the random variable which represents the age at onset of the disease, write down the contribution to the likelihood function for these nine individuals.

Exercise 18.8 You are given the following five observations of age at death:

$$20.2, \quad 42.7, \quad 46.8, \quad 59.7, \quad 60.5.$$

(a) Write down the log-likelihood assuming a Gompertz model with unknown parameters B and c.
(b) Verify that the maximum likelihood estimates are $\hat{B} = 0.00082$ and $\hat{c} = 1.0936$.
(c) You are given the further information that five more lives were observed up to age 65, at which point all were alive, when the study was terminated. Write down the revised log-likelihood function, and show that the revised maximum likelihood estimates are $\hat{B} = 0.00109$ and $\hat{c} = 1.0568$.
(d) Explain briefly which parameter estimates (those from part (b) or those from part (c)) are to be preferred.

Exercise 18.9 (a) Prove that the Nelson–Aalen estimate of the survival function is always greater than or equal to the Kaplan–Meier estimate.
(b) Stating clearly your assumptions, show that

$$\mathrm{V}[\hat{H}(x)] \approx \sum_{j:t_{(j)} \leq x} \frac{d_j(r_j - d_j)}{r_j^3}.$$

Exercise 18.10 You are given the following 18 observations of survival time data, (t_j, δ_j); the data are left truncated at $t = 10$. Times listed are from age 0.

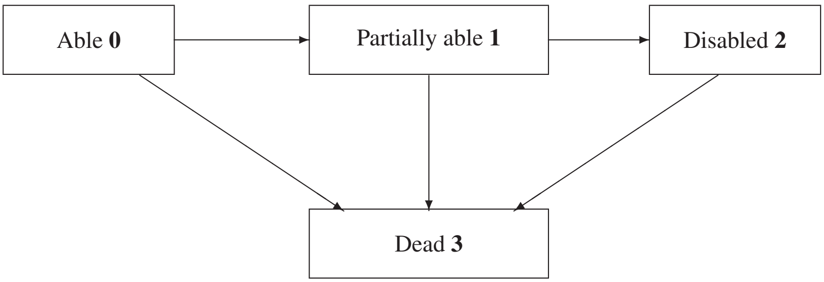
(12.8, 1)	(15.0, 0)	(30.0, 0)	(35.3, 1)	(35.6, 1)	(40.6, 1)
(43.3, 1)	(45.0, 0)	(48.0, 1)	(48.8, 1)	(53.8, 1)	(54.4,1)
(58.9,1)	(59.5, 1)	(64.5, 1)	(67.2, 1)	(69.5, 1)	(88.8, 1)

(a) (i) Calculate the Kaplan–Meier estimate of the probability that a life aged 10 survives another 40 years, i.e. $S_{10}(40)$.
(ii) Calculate an approximate standard deviation for $\hat{S}_{10}(40)$.
(b) Assume an exponential model for the time to failure, that is,

$$f_0(x) = \frac{1}{\theta}\, e^{-x/\theta} \quad \text{and} \quad F_0(x) = 1 - e^{-x/\theta}.$$

(i) Calculate the maximum likelihood estimate of θ. You are given that $\sum_{j=1}^{18} t_j = 871$.
(ii) Use the asymptotic variance of the maximum likelihood estimate to estimate the standard deviation of $\hat{\theta}$.
(iii) Calculate the maximum likelihood estimate of $S_{10}(40)$.
(iv) Using the delta method, estimate the standard deviation of $\hat{S}_{10}(40)$.
(c) Plot the Kaplan–Meier and fitted exponential survival functions, and comment on your results.

Exercise 18.11 A long-term care insurance policy is modelled by the following four-state model in which the transition intensities are all assumed to be constant between integer ages.



(a) Write down the contribution to the likelihood for μ_x^{ij} for each of the life histories described in the following table, each of whom enters observation at age x. Time t is counted from age x.

Life 1	At $t = 0$	In State 1
	At $t = 1$	In State 1
Life 2	At $t = 0$	In State 0
	At $t = 0.25$	Moves to State 1
	At $t = 0.75$	Moves to State 2
Life 3	At $t = 0$	In State 1
	At $t = 0.5$	Moves to State 2
	At $t = 0.8$	Dies

(b) The actuary has the following summary of all observations:

Total waiting time in state **0**	4 521.2 years
Total waiting time in state **1**	357.9 years
Total waiting time in state **2**	69.6 years
Total number of transitions **0** $\rightarrow$ **1**	163
Total number of transitions **0** $\rightarrow$ **3**	125
Total number of transitions **1** $\rightarrow$ **2**	154
Total number of transitions **1** $\rightarrow$ **3**	42
Total number of transitions **2** $\rightarrow$ **3**	25

(i) Derive maximum likelihood estimates of all the transition intensities along with estimates of the associated standard errors.

(ii) Give a general reasoning explanation for why the estimates of the transition intensities out of State 1 have higher standard errors than those out of State 0.

Excel-based exercises

Exercise 18.12 You are given the following 18 observations of survival time data, (t_j, δ_j); the data are left truncated at $t = 10$. (These are the same data as used in Exercise 18.10.)

(12.8, 1)	(15.0, 0)	(30.0, 0)	(35.3, 1)	(35.6, 1)	(40.6, 1)
(43.3, 1)	(45.0, 0)	(48.0, 1)	(48.8, 1)	(53.8, 1)	(54.4,1)
(58.9,1)	(59.5, 1)	(64.5, 1)	(67.2, 1)	(69.5, 1)	(88.8, 1)

Assume a Weibull model for the time to failure, that is,

$$f_0(x) = \frac{\gamma\, x^{\gamma-1}}{\theta^\gamma} \exp\left\{-\left(\frac{x}{\theta}\right)^\gamma\right\} \quad \text{and} \quad S_0(x) = \exp\left\{-\left(\frac{x}{\theta}\right)^\gamma\right\}.$$

(a) Use Excel Solver to find the maximum likelihood estimates of γ and θ.

(b) Using Excel, create a graph of the Nelson–Aalen cumulative hazard function against the Weibull cumulative hazard function (on the same axes). Comment on the fit.

Answers to selected exercises

18.1 (a) 0.6095, 0.1805 (b) 0.6351, 0.1562

18.2 (a) 35 (b) 8

18.3 (a) $\hat{S}(53.0) = 0.98927$ (b) $\hat{S}(53.0) = 0.98686$

18.4 (a) (0, 0.7461) (b) (0.0159, 0.7046)

18.5 (a) (0, 0.0089) (b) (0.0012, 0.0136)

18.6 (a) 0.6242 (b) 0.1135 (c) (0.1628, 0.2242)

18.10 (a) (i) 0.55407 (ii) 0.12697
(b) (i) 46.0667 (ii) 11.8944 (iii) 0.4197 (iv) 0.09409

18.11 (b) (i) $\hat{\mu}_x^{01} = 0.0361$, SD ≈ 0.0028; $\hat{\mu}_x^{03} = 0.0276$, SD ≈ 0.0025
$\hat{\mu}_x^{12} = 0.4303$, SD ≈ 0.0347; $\hat{\mu}_x^{13} = 0.1174$, SD ≈ 0.0181
$\hat{\mu}_x^{23} = 0.3592$, SD ≈ 0.0718

18.12 (a) $\hat{\gamma} = 3.383$, $\hat{\theta} = 58.57$

19

Stochastic longevity models

19.1 Summary

Longevity models allow for stochastic variation in the underlying force of mortality, so that instead of assuming, for example, that μ_x follows a Gompertz model, we now assume that μ_x changes with time, and can be modelled as a stochastic process. In this chapter we introduce the Lee–Carter and Cairns–Blake–Dowd models for longevity. We illustrate some of the structural assumptions of the models, and demonstrate key features. We also discuss briefly how the models are applied in actuarial risk management.

19.2 Introduction

In this chapter we introduce a more advanced topic of considerable interest to actuaries involved in long-term life insurance and annuities. Population mortality changes over time, with a history of generally decreasing mortality rates of younger and middle-aged lives, leading to longer life expectancy. For some time, the changes were quite smooth, and deterministic models, such as the mortality reduction factors in Chapter 3, adequately captured the trends. More recently, changes have been more dramatic and less predictable. The curves in Figure 3.3 in Chapter 3 illustrate this fact. In this chapter we introduce some of the stochastic models developed to help actuaries and demographers understand and prepare for increasing uncertainty in the survival models that they rely on.

In Chapter 8 we defined a stochastic process as a collection of random variables indexed by a time variable. A stochastic longevity model is a stochastic process for the rate (or force) of mortality experienced by lives at different ages for different future dates. We generally use discrete time stochastic processes, so that we generate stochastic mortality rates suitable for each age in each future calendar year.

19.3 The Lee–Carter model

The Lee–Carter model is probably the most famous stochastic longevity model of the past 30 years. It models central death rates, so we first define these.

Given a mortality model, defined by the force of mortality μ_x, say, the central death rate is m_x, where

$$m_x = \frac{q_x}{\int_0^1 {}_r p_x \, dr} = \frac{\int_0^1 {}_r p_x \, \mu_{x+r} \, dr}{\int_0^1 {}_r p_x \, dr}$$

so that m_x is a weighted average of the force of mortality between ages x and $x+1$. If the force of mortality is reasonably linear over each year of age, then $m_x \approx \mu_{x+\frac{1}{2}}$, and if the force of mortality is assumed to be constant from age x to age $x+1$, equal to μ_x^*, say, then $m_x = \mu_x^*$, and $q_x = 1 - e^{-m_x}$.

Taking longevity changes into consideration, we assume that the central death rate varies by age x and by calendar year t, so we write it as $m(x,t)$. In the Lee–Carter model, a stochastic process is defined for $\log m(x,t)$, which we denote as $lm(x,t)$. For each integer age x, $lm(x,t)$ is assumed to follow a discrete-time stochastic process as follows:

$$\log m(x,t) = lm(x,t) = \alpha_x + \beta_x \, K_t + \epsilon_{x,t}, \tag{19.1}$$

where

- α_x and β_x are parameters depending only on the attained age x.
- Past values of K_t are parameters found by fitting data to the model. Future values of K_t are modelled as a time series that is fitted to the estimated historical values. The series $\{K_t\}$ is not age-dependent.

  The fitted values for K_t often appear to follow a linear trend, and the usual forecasting model is a random walk with drift, which we also assume in this chapter. This means that we assume

  $$K_t = K_{t-1} + c + \sigma_k \, Z_t, \tag{19.2}$$

  where c is a constant drift term, σ_k is the standard deviation of the annual change in K_t, and the $\{Z_t\}$ are assumed to be independent and identically distributed random variables, with a standard normal distribution.
- $\epsilon_{x,t}$ is a random error term which is assumed to be sufficiently small to be negligible, and is often ignored in the definition and analysis of the Lee–Carter model. In this chapter we follow this custom, which means we assume that all of the uncertainty in the model is generated by the stochastic process $\{K_t\}$.

So, ignoring the $\epsilon_{x,t}$ term, and assuming that $\{K_t\}$ follows a random walk with a given starting value of K_0, we can write the Lee–Carter model as

$$lm(x,t) = \alpha_x + \beta_x K_t \tag{19.3}$$
$$\Rightarrow m(x,t) = \exp\{\alpha_x + \beta_x K_t\}$$

where

$$K_t = K_{t-1} + c + \sigma_k Z_t \quad \text{and} \quad Z_t \sim N(0,1).$$

The key to the model is the separation of age effects and year effects. The time series $\{K_t\}$ introduces random year effects, and the factor β_x allows for the year effects having a different impact on different ages.

The model has an identifiability problem in the form presented above. We could, for example, multiply each β_x by 2 and divide each K_t by 2 and end up with the same values for $m(x,t)$. To deal with this issue, we add two constraints which depend on the data used to fit the model parameters.

Suppose the data cover ages x_o to x_w, and calendar years t_0 to t_n. Then the constraints are

$$\sum_{x=x_o}^{x_w} \beta_x = 1 \quad \text{and} \quad \sum_{t=t_0}^{t_n} K_t = 0.$$

By applying these constraints we can see how the α_x parameters can be interpreted, as, for a given age x, if we sum equation (19.3) over t,

$$\sum_{t=t_0}^{t_n} lm(x,t) = (t_n - t_0)\alpha_x + \beta_x \sum_{t=t_0}^{t_n} K_t \Rightarrow \alpha_x = \frac{\sum_{t=t_0}^{t_n} lm(x,t)}{t_n - t_0}.$$

Hence α_x represents the average of the log-central death rates at age x over the period of the data.

The estimation process for the model is quite complex, and is beyond the scope of this chapter. Figure 19.1 shows typical plots for the fitted parameters for the Lee–Carter model for male lives aged 40–89, based on England and Wales population mortality data from 1961 to 2011. For this data set we see a decreasing mortality trend from the negative slope of the $\{K_t\}$, and also that the 55–70 age group experienced more benefit from the decreasing mortality trend than did older or younger ages, which is evident from the higher $\{\beta_x\}$ values for this age group.

The following examples show how the Lee–Carter model can be used to determine estimates and ranges for future rates of mortality, and also give some insight into the model characteristics.

Example 19.1 You are given the following parameters for the Lee–Carter model:

$$\alpha_{70} = -2.684, \quad \beta_{70} = 0.04, \quad K_{t-1} = -10, \quad c = -0.4, \quad \sigma_k = 0.7.$$

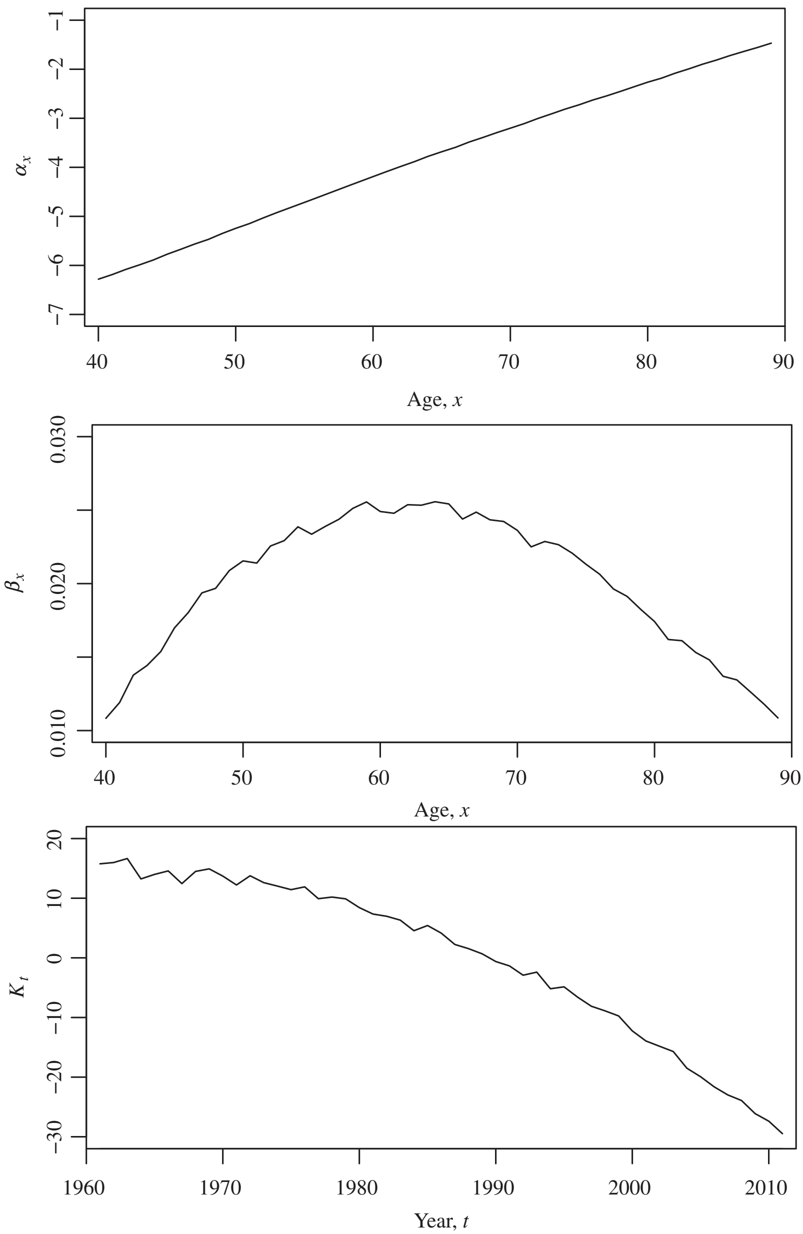


Figure 19.1 Example of fitted parameters for the Lee–Carter model (England and Wales, male mortality).

(a) (i) Calculate the mean and standard deviation of $m(70,t)$.
(ii) Calculate the median and the 5% quantile of $m(70,t)$.

(b) Let $p(x,t)$ denote the one-year survival probability at age x in year t.
(i) Calculate $p(70,t)$ assuming that the central death rate takes the mean value from part (a), and assuming a constant force of mortality between integer ages. Explain why this is not the mean value of $p(70,t)$.
(ii) Calculate the median and the 95% quantile of $p(70,t)$ assuming a constant force of mortality between integer ages.

(c) (i) Calculate $p(70,t)$ assuming that the central death rate takes the mean value from part (a), and assuming UDD between integer ages.
(ii) Calculate the median and the 95% quantile of $p(70,t)$ assuming UDD.

Solution 19.1 (a) (i) Starting from equation (19.3), then applying (19.2), we get

$$\begin{aligned} lm(70,t) &= \alpha_{70} + \beta_{70}\, K_t \\ &= \alpha_{70} + \beta_{70}\,(K_{t-1} + c + \sigma_k\, Z_t) \\ &= -2.684 + 0.04(-10 - 0.4 + 0.7Z_t) \\ &= -3.1 + 0.028Z_t \\ \Rightarrow m(70,t) &= e^{-3.1+0.028Z_t}. \end{aligned} \tag{19.4}$$

As $Z_t \sim N(0,1)$, it follows that $lm(70,t) \sim N(-3.1,\, 0.028^2)$. Furthermore, we know that for any normal random variable, say, $X \sim N(\mu,\sigma^2)$, the random variable $Y = e^X$ has a lognormal distribution, with parameters μ and σ. We have (from Appendix A)

$$\mathrm{E}[Y] = e^{\mu+\sigma^2/2} \quad \text{and} \quad \mathrm{V}[Y] = \mathrm{E}[Y]^2\left(e^{\sigma^2} - 1\right).$$

So $m(70,t) = e^{lm(70,t)} \sim LN(-3.1, 0.028)$ and

$$\mathrm{E}[m(70,t)] = e^{-3.1+0.028^2/2} = 0.04507,$$
$$\mathrm{V}[m(70,t)] = (0.04507)^2(e^{0.028^2} - 1) = 0.00126^2.$$

(ii) As m is a strictly increasing function of Z_t, we can find the q-quantile of m by replacing Z_t in equation (19.4) with its q-quantile.

We can show this symbolically. Let $Q_q(m)$ denote the desired q-quantile of $m(70,t)$, and let z_q denote the q-quantile of the standard normal distribution (so that $\Phi(z_q) = q$). Then we have

$$
\begin{aligned}
& \Pr\left[m(70,t) \le Q_q(m)\right] = q \\
\Rightarrow\ & \Pr\left[lm(70,t) \le \log Q_q(m)\right] = q \\
\Rightarrow\ & \Pr\left[-3.1 + 0.028Z_t \le \log Q_q(m)\right] = q \\
\Rightarrow\ & \Pr\left[Z_t \le \frac{\log Q_q(m) + 3.1}{0.028}\right] = q \\
\Rightarrow\ & \frac{\log Q_q(m) + 3.1}{0.028} = z_q \\
\Rightarrow\ & Q_q(m) = e^{-3.1+0.028z_q}
\end{aligned}
$$

which, as required, shows that the q-quantile of $m(70,t)$ can be found by substituting z_q for Z_t in equation (19.4).

The median corresponds to $q = 0.5$; we have $z_{0.5} = 0$ (the median of the standard normal distribution), which means that the median of $m(70,t)$ is

$$Q_{0.5}(m(70,t)) = e^{-3.1} = 0.04505.$$

Similarly, as $z_{0.05} = -1.645$, we have

$$Q_{0.05}(m(70,t)) = e^{-3.1+0.028(-1.645)} = 0.04302.$$

(b) (i) With the constant force assumption, the force of mortality between ages x and $x+1$ in year t is $m(x,t)$, so

$$p(x,t) = e^{-m(x,t)}. \tag{19.5}$$

Hence, a central death rate of $\mathrm{E}[m(70,t)] = 0.04507$ (from part (a)) corresponds to a survival probability of $p(70,t) = e^{-0.04507} = 0.95493$.

This is not the expected value of $p(70,t)$; for this we need

$$\mathrm{E}[p(70,t)] = \mathrm{E}\left[e^{-m(70,t)}\right] = \mathrm{E}\left[e^{-\left(e^{-3.1+0.028Z_t}\right)}\right],$$

which cannot be evaluated analytically, but can be found using numerical integration to be 0.95593.

(ii) We see from formula (19.5) that the survival probability is a decreasing function of the central death rate, so that the q-quantile for $p(70,t)$ corresponds to the $(1-q)$-quantile for $m(70,t)$. Again, we can demonstrate this mathematically. Let $Q_q(p)$ denote the q-quantile of $p(70,t)$, and let $Q_q(m)$ again denote the q-quantile of $m(70,t)$. Then

$$
\begin{aligned}
&\Pr[p(70,t) \le Q_q(p)] = q \\
&\Rightarrow \Pr[-m(70,t) \le \log Q_q(p)] = q \\
&\Rightarrow \Pr[m(70,t) > -\log Q_q(p)] = q \\
&\Rightarrow \Pr[m(70,t) \le -\log Q_q(p)] = 1-q \\
&\Rightarrow -\log Q_q(p) = Q_{1-q}(m) \\
&\Rightarrow Q_q(p) = e^{-Q_{1-q}(m)}.
\end{aligned}
$$

So the q-quantile of $p(70,t)$ can be found by substituting the (1 - q)-quantile of $m(70,t)$ for $m(70,t)$ in equation (19.5), that is

$$
\begin{aligned}
Q_{0.5}(p(70,t)) &= e^{-Q_{0.5}(m(70,t))} = 0.95595, \\
Q_{0.95}(p(70,t)) &= e^{-Q_{0.05}(m(70,t))} = 0.95789.
\end{aligned}
$$

(c) (i) Under UDD we have ${}_r q_x = r\,q_x$ for $0 \le r \le 1$. Then

$$
m_x = \frac{q_x}{\int_0^1 {}_r p_x \, dr} = \frac{q_x}{\int_0^1 (1-r\,q_x) dr} = \frac{q_x}{1-q_x/2}
$$

$$
\Rightarrow q_x = \frac{m_x}{1+m_x/2} \Rightarrow p_x = \frac{1-m_x/2}{1+m_x/2}.
$$

So assuming $m(70,t) = 0.04507$, we have $p(70,t) = 0.95593$.

(ii) Following the same process as in part (b), we have

$$
\begin{aligned}
Q_{0.5}(p(70,t)) &= \frac{1-Q_{0.5}(m(70,t))/2}{1+Q_{0.5}(m(70,t))/2} = 0.95594, \\
Q_{0.95}(p(70,t)) &= \frac{1-Q_{0.05}(m(70,t))/2}{1+Q_{0.05}(m(70,t))/2} = 0.95788.
\end{aligned}
$$

□

The quantiles calculated in the example above can be used to give some measure of longevity risk, but it should be noted that there is significant uncertainty in the parameters that is not captured in these calculations. We have also ignored the error term $\epsilon_{x,t}$ from equation (19.1).

Example 19.2 Define the central death rate improvement factor as the random variable

$$
\varphi^m(x,t) = 1 - \frac{m(x,t)}{m(x,t-1)}.
$$

(a) Show that the distribution of $\varphi^m(x,t)$ does not depend on t.

(b) Calculate the mean, standard deviation, median and the 95% quantile of $\varphi^m(70,t)$, using the parameters given in Example 19.1.

Solution 19.2 (a) Consider first

$$\log\left(\frac{m(x,t)}{m(x,t-1)}\right) = lm(x,t) - lm(x,t-1) = \beta_x(K_t - K_{t-1})$$
$$= \beta_x(c + \sigma_k Z_t)$$

using equation (19.2). As $Z_t \sim N(0,1)$, it follows that

$$\beta_x(c + \sigma_k Z_t) \sim N(\beta_x c, \beta_x^2 \sigma_k^2).$$

As $\exp\{\beta_x(c + \sigma_k Z_t)\} = 1 - \varphi^m(x,t)$, it then follows that

$$1 - \varphi^m(x,t) \sim LN\,(\beta_x c,\ \beta_x \sigma_k)\,,$$

which demonstrates that the distribution of $\varphi^m(x,t)$ does not depend on t.

(b) We have $1 - \varphi^m(70,t) \sim LN\,(-0.016, 0.028)$, so

$$\mathrm{E}\left[\varphi^m(70,t)\right] = 1 - e^{-0.016+0.028^2/2} = 0.01549,$$
$$\mathrm{V}\left[\varphi^m(70,t)\right] = \mathrm{V}\left[1 - \varphi^m(70,t)\right] = \left(e^{-0.016+0.028^2/2}\right)^2\left(e^{0.028^2} - 1\right)$$
$$= 0.02757^2,$$
$$Q_{0.5}(\varphi^m(70,t)) = 1 - e^{-0.016} = 0.01587,$$
$$Q_{0.95}(\varphi^m(70,t)) = 1 - Q_{0.05}(\varphi^m(70,t)) = 1 - e^{-0.016-1.645(0.028)}$$
$$= 0.06017.$$

□

Example 19.3 The log-improvement factor is defined as

$$R(x,t) = \log\left(\frac{m(x,t)}{m(x,t-1)}\right).$$

Show that $R(x,t)$ and $R(y,t)$ are perfectly correlated for $y \neq x$.

Solution 19.3 The correlation coefficient is

$$\rho = \frac{\mathrm{E}[R(x,t)R(y,t)] - \mathrm{E}[R(x,t)]\mathrm{E}[R(y,t)]}{\mathrm{SD}[R(x,t)]\,\mathrm{SD}[R(y,t)]},$$

where

$$R(x,t) = (\alpha_x + \beta_x K_t) - (\alpha_x + \beta_x K_{t-1}) = \beta_x\,(K_t - K_{t-1})$$

and $K_t - K_{t-1} = c + \sigma_k Z_t$ so that

$$\mathrm{E}[K_t - K_{t-1}] = c \quad \text{and} \quad \mathrm{SD}[K_t - K_{t-1}] = \sigma_k.$$

It then follows that

$$\mathrm{E}[R(x,t)] = \beta_x c \quad \text{and} \quad \mathrm{SD}[R(x,t)] = \beta_x \sigma_k,$$

and, as $R(x,t)\,R(y,t) = \beta_x\,\beta_y\,(K_t - K_{t-1})^2$,

$$\mathrm{E}[R(x,t)R(y,t)] = \mathrm{E}[\beta_x\beta_y\,(K_t - K_{t-1})^2] = \beta_x\beta_y\left(c^2 + \sigma_k^2\right).$$

Substituting into our expression for ρ gives

$$\rho = \frac{\beta_x\beta_y(c^2+\sigma_k^2) - \beta_x\beta_y c^2}{(\beta_x\sigma_k)\left(\beta_y\sigma_k\right)} = 1,$$

and so we have perfect correlation. □

Although the Lee–Carter model has been widely applied, there are some problems with it, particularly for actuarial applications. The most important is that the fit to data tends not to be very good. This is partly because the model assumes (implicitly) perfect correlation between log-improvement factors at different ages, as we showed in Example 19.3, but mortality data show that log-improvements may be far from perfectly correlated. Also, the model does not allow for any cohort effect, yet we often see cohort effects in both actuarial and population data. In Figure 19.2 we show a plot of some of the residuals generated from fitting the Lee–Carter model to the England and Wales male mortality data used in Figure 19.1. If the fit is good, we expect the residual plot to look like white noise, with no systematic patterns. Instead we see clear diagonal trends, indicating a cohort effect in these data that is not captured by the Lee–Carter model.

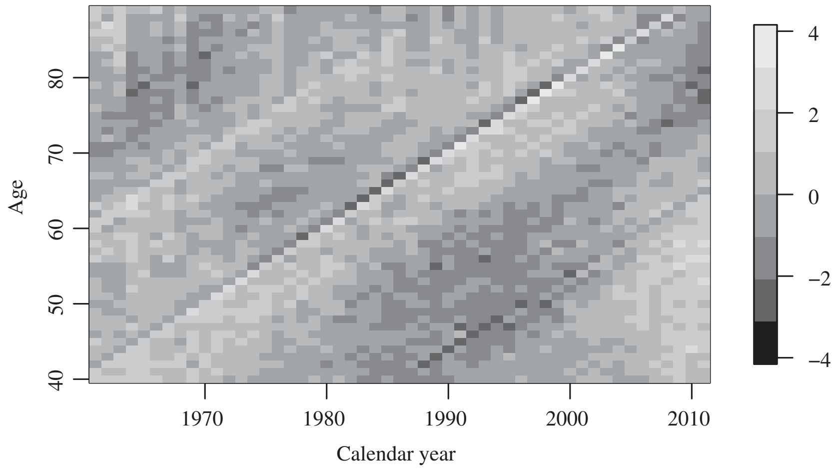


Figure 19.2 Heatmap of residuals, Lee–Carter model fitted to England and Wales, males data.

19.4 The Cairns–Blake–Dowd models

The Cairns–Blake–Dowd (CBD) family of models has become very popular for actuarial applications. The original model fits a two-factor time series to the logit of the mortality rate; the model has subsequently been extended in a number of ways. In this section we describe the original model, and briefly discuss a popular extension.

The model works with the logit function of mortality rates. The logit function is defined as $\mathrm{logit}(x) = \log(x/(1-x))$. In this section we let

$$lq(x,t) = \log\left(\frac{q(x,t)}{1-q(x,t)}\right),$$

where $q(x,t) = 1 - p(x,t)$ is the mortality rate at age x in year t.

19.4.1 The original CBD model

The original CBD model is defined as

$$lq(x,t) = K_t^{(1)} + K_t^{(2)}\,(x-\bar{x})\,, \tag{19.6}$$

where

- $\bar{x}$ is the average age in the data set.
- $\{K_t^{(1)}\}$ and $\{K_t^{(2)}\}$ are correlated time series. Usually, each is assumed to follow a random walk with drift, so that

$$\begin{aligned} K_t^{(1)} &= K_{t-1}^{(1)} + c^{(1)} + \sigma_{k_1} Z_t^{(1)}, \\ K_t^{(2)} &= K_{t-1}^{(2)} + c^{(2)} + \sigma_{k_2} Z_t^{(2)}. \end{aligned}$$

- $Z_t^{(1)}$ and $Z_t^{(2)}$ are standard normal random variables, which are correlated with each other in each year, but are independent from year to year. This means that

$$\begin{aligned} \mathrm{E}[Z_t^{(1)}] &= \mathrm{E}[Z_t^{(2)}] = 0, \\ \mathrm{V}[Z_t^{(1)}] &= \mathrm{V}[Z_t^{(2)}] = 1, \\ \mathrm{E}[Z_t^{(1)} Z_t^{(2)}] &= \rho, \ \text{ where } -1 \le \rho \le 1, \\ \mathrm{E}[Z_t^{(i)} Z_u^{(j)}] &= 0 \text{ for } t \ne u \ \text{ and } i,j = 1,2. \end{aligned}$$

Example 19.4 Suppose that

$$K_{t-1}^{(1)} = -3.2, \quad K_{t-1}^{(2)} = 0.01, \quad c^{(1)} = -0.02, \quad c^{(2)} = 0.0006,$$
$$\sigma_{k_1} = 0.03, \quad \sigma_{k_2} = 0.005, \quad \rho = 0.2, \quad \bar{x} = 70.$$

(a) Calculate the mean and variance of $lq(65,t)$.
(b) Calculate $\Pr[q(65,t) > 0.036]$.

(c) Using numerical integration, calculate $\mathrm{E}[q(65,t)]$.
(d) Calculate the median and the 95% quantile of $p(65,t)$.

Solution 19.4 (a) From (19.6) we have

$$lq(65,t) = K_t^{(1)} + K_t^{(2)}(65-70)$$

where

$$\begin{aligned} K_t^{(1)} &= -3.2 - 0.02 + 0.03Z_t^{(1)}, \\ K_t^{(2)} &= 0.01 + 0.0006 + 0.005Z_t^{(2)}. \end{aligned}$$

Hence

$$lq(65,t) = -3.273 + 0.03\,Z_t^{(1)} - 0.025\,Z_t^{(2)}.$$

On the right-hand side we have a linear function of correlated normal random variables, which means that $lq(65,t)$ is also normally distributed, with mean $\mathrm{E}[lq(65,t)] = -3.273$ and

$$\begin{aligned} \mathrm{V}[lq(65,t)] &= 0.03^2 + 0.025^2 - 2(0.03)(0.025)(0.2) \\ &= 0.001225 = 0.035^2. \end{aligned}$$

(b) We turn the probability statement about $q(65,t)$ into a probability statement about $lq(65,t)$, since we know from part (a) that $lq(65,t) \sim N(-3.273, 0.035^2)$. We have

$$\begin{aligned} \Pr[q(65,t) > 0.036] &= \Pr\left[\frac{q(65,t)}{1-q(65,t)} > \frac{0.036}{1-0.036}\right] \\ &= \Pr\left[lq(65,t) > \log\left(\frac{0.036}{1-0.036}\right)\right] \\ &= \Pr\left[Z > \frac{1}{0.035}\left(\log\left(\frac{0.036}{1-0.036}\right) + 3.273\right)\right] \\ &= \Pr\left[Z > -0.4164\right] = 0.6614, \end{aligned}$$

where $Z \sim N(0,1)$.

(c) We make use of the fact that if X is a random variable distributed on (0,1), then $\mathrm{E}[X] = \int_0^1 \Pr[X > x]\,dx$. So

$$\mathrm{E}[q(65,t)] = \int_0^1 \Pr[q(65,t) > x]\,dx,$$

and, generalizing the argument from part (b), we calculate this as

$$\mathrm{E}[q(65,t)] = \int_0^1 \Pr\left[Z > \frac{1}{0.035}\left(\log\left(\frac{x}{1-x}\right) + 3.273\right)\right] dx.$$

Numerical integration gives $\mathrm{E}[q(65,t)] = 0.03653$.

(d) As $lq(x,t)$ is an increasing function of $q(x,t)$, it is therefore a decreasing function of $p(x,t)$. So the α-quantile of $p(x,t)$ corresponds to the $(1-\alpha)$-quantile of $q(x,t)$. To find the median, we have

$$\begin{aligned} & Q_{0.5}(lq(65,t)) = -3.273 \\ \Rightarrow\ & Q_{0.5}\left(\frac{q(65,t)}{1-q(65,t)}\right) = e^{-3.273} = 0.03789 \\ \Rightarrow\ & Q_{0.5}(q(65,t)) = \frac{0.03789}{1+0.03789} = 0.03651 \\ \Rightarrow\ & Q_{0.5}(p(65,t)) = 0.96349. \end{aligned}$$

To find the 95% quantile, we have

$$\begin{aligned} & Q_{0.05}(lq(65,t)) = -3.273 - 1.645(0.035) = -3.33057 \\ \Rightarrow\ & Q_{0.05}\left(\frac{q(65,t)}{1-q(65,t)}\right) = e^{-3.33057} = 0.03577 \\ \Rightarrow\ & Q_{0.05}(q(65,t)) = \frac{0.03577}{1+0.03577} = 0.03454 \\ \Rightarrow\ & Q_{0.95}(p(65,t)) = 0.96546. \end{aligned}$$

□

The CBD M7 model

The CBD model has some advantages over the Lee–Carter model, with fewer parameters, and less parameter uncertainty in practice, but the fit to population mortality data of the original model is not significantly better than under Lee–Carter, and in some cases is worse.

By adding one or two terms to the CBD model the fit can be significantly improved, while the advantages of the model are mostly retained. A popular extension is the CBD M7 model, which is defined as

$$lq(x,t) = K_t^{(1)} + K_t^{(2)}(x-\bar{x}) + K_t^{(3)}\left((x-\bar{x})^2 - s_x^2\right) + G_{t-x}\,.$$

- There are two additional terms that are not in the original model. The first is an extra year-effect time series, $\{K_t^{(3)}\}$, which has a quadratic impact across the age groups. The s_x term is the standard deviation of the age range used. So, if the age range used to fit the model runs from age 50 to age 90, then

$$\bar{x} = \frac{1}{41}\sum_{x=50}^{90} x = 70 \quad \text{and} \quad s_x^2 = \frac{1}{41}\sum_{x=50}^{90}(x-70)^2 = 140.$$

$\{K_t^{(3)}\}$ models a year effect that has most impact on the youngest and oldest ages. Typically this is an increasing function.

- The second is G_{t-x} which introduces a cohort effect time series. For a life aged, say, 65 in 2018, the lq function would use G_{1953}, where 1953 is the birth year of the cohort, and that same G_{1953} term would appear in the subsequent lq functions for the cohort (that is $lq(66, 2019)$, $lq(67, 2020)$, ...). For G_y where y lies beyond the range of the data, we fit a time series, but it is typically more cyclical than $\{K_t^{(j)}\}$ ($j = 1, 2, 3$), and would usually be fitted to an ARIMA-type time series model, scaled such that the average value over all relevant birth years is zero. Lives born in birth years with $G_y < 0$ are expected to have lighter mortality, and cohorts with $G_y > 0$ are expected to experience heavier mortality, compared with the average.
- The impact of adding the $K^{(3)}$ and G terms is to change the typical path for $K^{(2)}$, which for the England and Wales data is relatively flat.

In Figures 19.3 to 19.5 we show some results from fitting the M7 model to the same England and Wales mortality data previously fitted to the Lee–Carter model. In Figure 19.3 we show the fitted paths for the $\{K_t^{(j)}\}$ time series; in Figure 19.4 we show the fitted path for the cohort time series, $\{G_y\}$, by birth year of the cohort, and in Figure 19.5 we show a heatmap of the residuals of the model fit.

We note that $\{K_t^{(1)}\}$ appears consistent with a random walk with decreasing trend, $\{K_t^{(2)}\}$ is quite flat, but with a slight parabolic shape, and $\{K_t^{(3)}\}$ is consistent with a random walk with an increasing trend. The fitted values of the cohort series $\{G_y\}$ appear cyclical.

Comparing the residuals for the M7 model in Figure 19.5 with the residuals from the Lee–Carter model in Figure 19.2 indicates a clear improvement in the overall fit.

19.4.2 Actuarial applications of stochastic longevity models

In the examples in Section 19.2 we calculated some measures of risk or variability relating to one-period-ahead mortality. The impact of stochastic longevity on the cash flows of a portfolio of annuities projected much farther into the future is not so analytically tractable, especially for more complex models such as CBD M7.

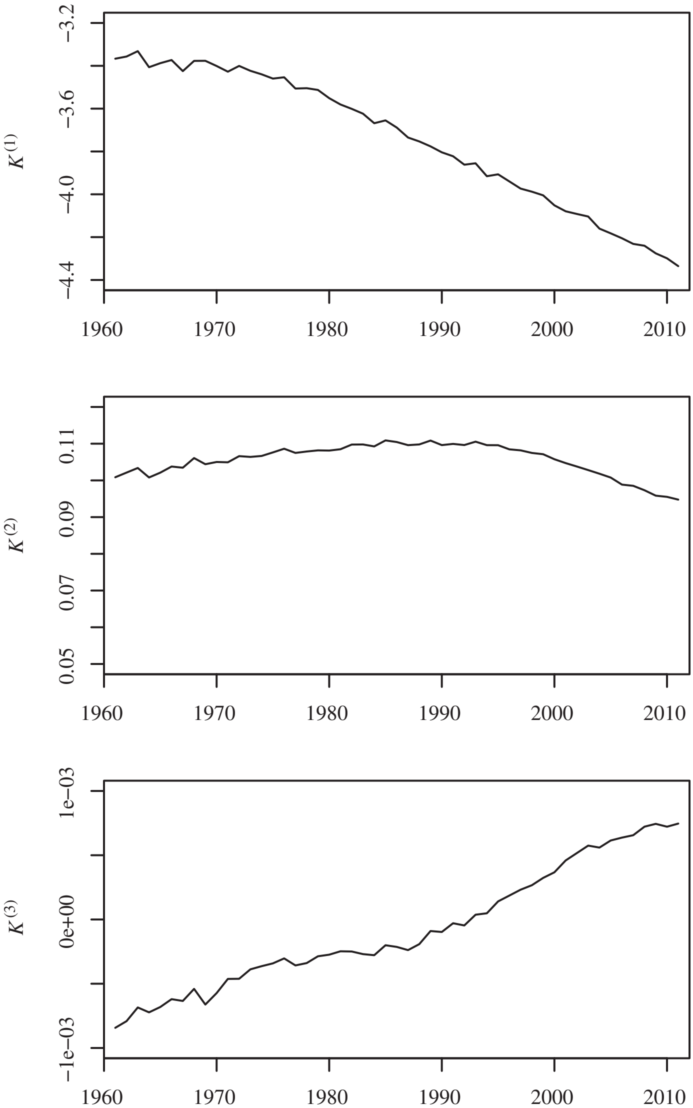


Figure 19.3 $\{K^{(1)}\}$, $\{K^{(2)}\}$, and $\{K^{(3)}\}$ fitted values for the CBD M7 model, fitted to England and Wales male mortality, ages 40–89, years 1961–2011.

Commonly, actuaries use Monte Carlo simulation to assess the potential impact of longevity risk. The method can be used to generate a large number of (pseudo-) random paths for $p(x, t)$ into the future. We can use these paths to estimate distributions of cash flows and present values.

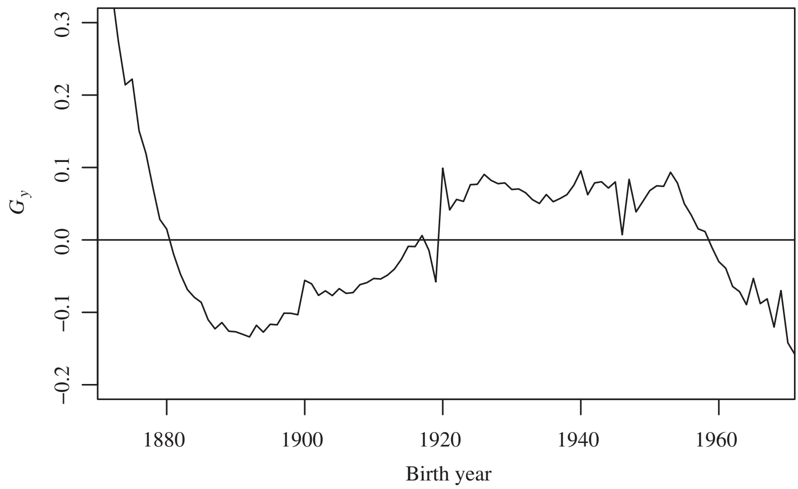


Figure 19.4 Cohort parameter G_y by birth year for the CBD M7 model fitted to England and Wales, males, 1961–2011.

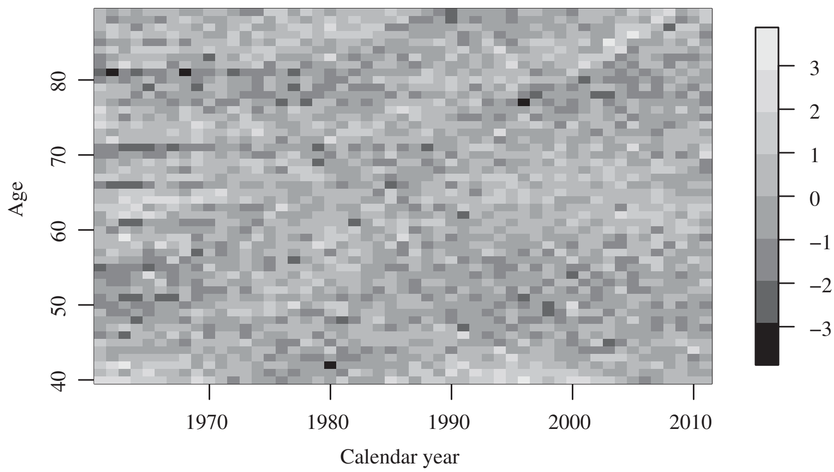


Figure 19.5 Heatmap of residuals for the CBD M7 model fitted to England and Wales, males, 1961–2011.

For example, suppose an insurer uses Monte Carlo simulation to generate 10 000 different 50-year paths for $p(x,k)$, for $x = 60, 61, \ldots, 109$, and for k in years measured from the most recent information, when $k = 0$. We assume 110 is the ultimate age attainable. This means that, for each path we are simulating survival probabilities for all ages, for each of the 50 years. We repeat this 10 000 times, using random number generators that create independent paths from the underlying process that are equally likely, and that can be treated as a random sample from the distribution. Let $p_j(x,k)$ denote the simulated value for $p(x,k)$ from the jth simulated path. Given a single path of survival probabilities starting at age 60, for example, that is, given $p_j(60,0)$, $p_j(61,1)$, ..., we can calculate the EPV of an immediate annuity issued to (60) at time $k=0$, conditional on that path. Let $a(60,0)_j$ denote the EPV given the jth path. Then

$$a_j(60,0) = \sum_{k=1}^{50} v^k \prod_{i=0}^{k-1} p_j(60+i,i).$$

Repeating this for all of the 10 000 paths for the survival probabilities gives us a sample of 10 000 values for the present value of the immediate annuity issued to a life aged 60, taking longevity risk into consideration. From this sample, we can calculate moments such as the mean and variance, and we can assess the exposure to longevity risk by considering the impact if the annuity value takes an adverse value, such as the 95% quantile of the distribution.

Since longevity changes tend to impact an entire portfolio, not just a single age range, an insurer would not look separately at values for each age group, but would generate valuations for the whole portfolio of annuities at different ages, valued along each separate path for $p(x,k)$ for all ages x. The results could be used to assess the adequacy of reserves and of pricing, taking longevity risk into consideration. The ability of the insurer to survive extreme scenarios could be investigated by considering, for example, the worst-case quantiles of the simulated distribution of the portfolio's present value.

19.4.3 Notes on stochastic longevity models

1. Different populations can display very different combinations of age, year and cohort effects. In some populations cohort effects are not at all strong, while in others they are crucial to model fit. We also need to consider sub-populations. We often use data from a national census or social security records to fit stochastic longevity models, as a large amount of data is needed, due to the large number of parameters to be fitted, and because at much older ages we have less reliable parameter estimates as we may have very few lives in the database.

   There is a problem, though, if the population statistics are then used to model improvement factors for insured lives and annuitants. Typically, people who buy annuities and insurance are healthier and wealthier than the population as a whole, and they tend to be the first to benefit from the medical and social advances that improve longevity. The whole-population models may underestimate the longevity risk from current annuitants, if population and annuitant longevity continue to improve. On the other hand, we may overestimate the risk if the annuitant population mortality trends slow down before the population as a whole. For example, suppose we use annuitant mortality from 2015 as our base mortality table, and then use estimated population mortality improvement factors to project the annuitant mortality rates into the future. Suppose further that population longevity improvement is generated by improved population access to new drugs such as statins. If the annuitant population already benefited from early access to the drugs, based on their relatively privileged social position, then applying the population improvement factors to the base annuitant mortality will double count the impact of statins, and will overestimate the longevity of the group.

2. Stochastic longevity models can be used to produce deterministic improvement scales, as described in Chapter 3. Typically, we would use the median or mean improvement factors from the stochastic model to generate scales to apply to current tables, although we may also adjust the factors to create a more conservative basis.
3. The modelling process for the series $\{K_t\}$ starts with estimating the values for all the years in the data, since we do not directly observe these values. These are the values plotted in Figures 19.2 and 19.5. Then the estimated values are analysed using standard time series methods. For the Lee–Carter and original CBD models, applied to population data from the USA or the UK, the estimated values appear to follow a reasonably straight line, but for more complex models such as the CBD M7, and for some other populations, a better fit might be obtained using an ARIMA-type time series model.
4. We have not given much indication of whether or why one stochastic longevity model is better than another. We can perform statistical analysis of goodness of fit, but the model uncertainty and parameter uncertainty are typically very large, and it is not unusual for all the models to fail standard tests of fit. We need to find ways of choosing an acceptable model, but this is never an automatic process. Some models do well on relative fit, but demonstrate extreme parameter uncertainty – for example, generating very different values when we use slightly different historic periods to estimate the parameters.

Ultimately, there is still a very large amount of parameter and model uncertainty in each of the popular stochastic longevity models. Nevertheless, they are becoming essential tools for actuarial risk management of annuity portfolios and pension plans, because they are so much better than no model at all. At least we can generate some indication of the range of possible outcomes for an annuity portfolio's cash flows. Still, it is essential for actuaries using these models to understand the significant limitations of the models.

5. Given the sometimes conflicting information from the statistical metrics for model selection, we might want to assess the reasonableness of the models intuitively. The CBD models generate smooth survival rates across ages within each year, because of the $(x - \bar{x})$ and $((x - \bar{x})^2 - s_x^2)$ terms. The Lee–Carter model may generate rather less smoothness across the ages depending on the parameters.

19.5 Notes and further reading

The Lee–Carter model is introduced in Lee and Carter (1992); a summary of the model and its features is presented in Lee (2000). A catalogue of several stochastic longevity models, including the Lee–Carter and CBD models, with discussion of their characteristics and their fit to data is given in Cairns *et al.* (2009). This is where the M7 terminology comes from for the CBD M7 model.

Li *et al.* (2009) give more information on the fitting process for the Lee–Carter model, and Li *et al.* (2010) present a case study, applying the longevity model to develop deterministic mortality improvement scales, with associated measures of uncertainty.

The figures in this chapter were generated using the R package STMoMo. For details, see Villegas *et al.* (2015).

19.6 Exercises

Shorter exercises

Exercise 19.1 In year Y you are given the following information for a Lee–Carter longevity model:

$$K_Y = -3.5, \quad c = -0.5, \quad lm(40, Y) = -6.830, \quad \mathrm{E}[lm(40, Y+1)] = -6.845.$$

(a) Identify α_{40} and β_{40}.
(b) You are also given that $\sigma_k = 2$. Calculate the expected value and the median of the improvement factor $\varphi^{(m)}(40, Y+1)$.

Exercise 19.2 You are given that $m(x,t) = 0.025$. Calculate $\operatorname{logit} q(x,t)$ assuming

(a) a constant force of mortality between integer ages, and
(b) UDD.

Exercise 19.3 Assume that mortality follows the Lee–Carter model such that, for integer age x with $50 \le x \le 110$,

$$\alpha_x = -11 + 0.1x,$$

$$\beta_x = \begin{cases} 0.03 & \text{for } 50 \le x \le 79, \\ 0.01 & \text{for } 80 \le x \le 89, \\ 0 & \text{for } x \ge 90. \end{cases}$$

You are also given that $K_0 = -2$, $c = -0.4$ and $\sigma_k = 1.5$.

(a) Calculate $m(70, 0)$.
(b) Calculate $\Pr[m(70, 1) > m(70, 0)]$, ignoring the $\epsilon_{x,t}$ term in equation (19.1).
(c) Calculate $\Pr[m(71, 1) < m(70, 0)]$, ignoring the $\epsilon_{x,t}$ term in equation (19.1).
(d) Repeat part (b), assuming that $\epsilon_{x,t} \sim N(0,\, 0.03^2)$ for all x and for $t = 1, 2, \ldots$, and that $\epsilon_{x,t}$ is independent of K_t.

Longer exercises

Exercise 19.4 Consider the CBD model. For a dataset covering ages 40 to 80, you are given that, at time Y,

$$K_Y^{(1)} = -3, \quad K_Y^{(2)} = 0.02, \quad c^{(1)} = -0.04, \quad c^{(2)} = 0.001,$$

$$\sigma_{k_1} = 0.05, \quad \sigma_{k_2} = 0.005, \quad \rho(Z_t^{(1)}, Z_t^{(2)}) = 0.3.$$

(a) Calculate (i) the mean and (ii) the standard deviation of $lq(65, Y{+}1)$.
(b) Calculate (i) the median and (ii) the 95% quantile of $p(65, Y{+}1)$.

Exercise 19.5 Consider the CBD M7 model. For a dataset covering ages 40 to 80, you are given that

$$K_t^{(3)} = K_{t-1}^{(3)} + c^{(3)} + \sigma_{k_3} Z_t^{(3)},$$

$$K_Y^{(1)} = -5, \quad K_Y^{(2)} = 0.1, \quad K_Y^{(3)} = 0.0001, \quad c^{(1)} = -0.04, \quad c^{(2)} = c^{(3)} = 0,$$

$$\sigma_{k_1} = 0.03, \quad \sigma_{k_2} = 0.001, \quad \sigma_{k_3} = 0.0006,$$

$$\rho(Z_t^{(1)}, Z_t^{(2)}) = 0.5, \quad \rho(Z_t^{(1)}, Z_t^{(3)}) = \rho(Z_t^{(2)}, Z_t^{(3)}) = 0, \quad G_{Y-64} = -0.05.$$

(a) Calculate (i) the mean and (ii) the standard deviation of $lq(65, Y{+}1)$.
(b) Calculate (i) the median and (ii) the 95% quantile of $p(65, Y{+}1)$.

Exercise 19.6 Show that, under the Lee–Carter model (with random walk K), the median one-year survival probability for a life aged x at time $T{+}s$, denoted $p_m(x, T + s)$, can be written in terms of the survival probability at time T as

$$p_m(x, T + s) = (p(x, T))^{\exp\{\beta_x c s\}}.$$

Exercise 19.7 You are given the following parameters for the Lee–Carter model:

$$\alpha_{50} = -6, \quad \beta_{50} = 0.08, \quad K_Y = -10, \quad c = -0.4, \quad \sigma_k = 3.$$

(a) (i) Calculate the mean and standard deviation of $m(50, Y+2)$.
(ii) Calculate the median and the 75% quantile of $m(50, Y+2)$.
(b) Calculate the median and the 75% quantile of $q(50, Y+2)$ assuming a constant force of mortality between integer ages.
(c) Calculate the median and 75% quantile of $q(50, Y+2)$ assuming UDD between integer ages.

Excel-based exercises

Exercise 19.8 Using the parameters in Exercise 19.3, ignoring $\epsilon_{x,t}$, simulate 1000 paths for $m(70+t, t)$ for and $t = 1, 2, \ldots, 40$.

(a) For each path, calculate the path of values for the time t survival probabilities, $p(70+t, t)$, assuming that the force of mortality is constant between integer ages.
(b) For each path, calculate the EPV of an annuity-due of one per year payable to a life aged 70 in year 0. Assume an interest rate of 5% per year.
(c) Plot the histogram of the annuity EPVs, and report the mean and variance.

Answers to selected exercises

19.1 (a) $\alpha_{40} = -6.725$, $\beta_{40} = 0.03$ (b) Mean: 1.311%, median: 1.489%
19.2 (a) -3.676353 (b) -3.676301
19.3 (a) 0.017249 (b) 0.3949 (c) 0.0253 (d) 0.4122
19.4 (a)(i) -2.395 (ii) 0.06225
(b)(i) 0.94955 (ii) 0.95424
19.5 (a)(i) -4.6015 (ii) 0.07639
(b)(i) 0.99006 (ii) 0.99123
19.7 (a)(i) 0.0011067, 0.0003867 (ii) 0.001045, 0.001313
(b) 0.001044, 0.001313 (c) 0.001044, 0.001313

Appendix A
Probability and statistics

A.1 Probability distributions

In this appendix we give a very brief description of the more important probability distributions used in this book. Derivations of the results quoted in this appendix can be found in standard introductory textbooks on probability theory. Information about other distributions mentioned in this book can be found in Klugman *et al.* (2012).

A.1.1 Binomial distribution

If a random variable X has a binomial distribution with parameters n and p, where n is a positive integer and $0 < p < 1$, then its probability function is

$$\Pr[X = x] = \binom{n}{x} p^x (1 - p)^{n-x}$$

for $x = 0, 1, 2, \ldots, n$. This distribution has mean np and variance $np(1 - p)$, and we write $X \sim B(n, p)$.

The special case when $n = 1$ is called the Bernoulli distribution.

A.1.2 Uniform distribution

If a random variable X has a uniform distribution on the interval (a, b), then the distribution function is

$$\Pr[X \leq x] = F(x) = \frac{x - a}{b - a}$$

for $a \leq x \leq b$, and the probability density function is

$$f(x) = F'(x) = \frac{1}{b - a}$$

for $a < x < b$. This distribution has mean $\frac{a+b}{2}$ and variance $\frac{(b-a)^2}{12}$, and we write $X \sim U(a, b)$.

A common version of the uniform distribution is $U(0, 1)$, in which case $F(x) = x$ and $f(x) = 1$.

A.1.3 Normal distribution

If a random variable X has a normal distribution with parameters μ and σ^2, where $-\infty < \mu < \infty$ and $\sigma > 0$, then its probability density function is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{\frac{-(x-\mu)^2}{2\sigma^2}\right\}$$

for $-\infty < x < \infty$. This distribution has mean μ and variance σ^2, and we write $X \sim N(\mu, \sigma^2)$.

The random variable Z defined by the transformation $Z = (X - \mu)/\sigma$ is said to have a **standard normal distribution**, and $Z \sim N(0, 1)$.

The distribution function of the standard normal distribution is denoted Φ. It does not have a closed analytic form, but is available from statistical tables or quantitative software packages, including Excel, where it is referenced using the NORM.S.DIST function.

It is sometimes useful to note that, as the standard normal distribution is symmetric about 0, $\Phi(-z) = 1 - \Phi(z)$.

We use the standard normal distribution function, Φ, to calculate probabilities when $X \sim N(\mu, \sigma^2)$ as

$$\Pr[X \le x] = \Pr\left[Z \le \frac{x-\mu}{\sigma}\right] = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

The q-quantile of the standard normal distribution, for any $0 < q < 1$, is z_q such that

$$\Phi(z_q) = q \Rightarrow z_q = \Phi^{-1}(q).$$

Commonly used quantiles of the standard normal distribution are often tabulated in statistical tables, and can be found from Excel using the NORM.S.INV function.

The q-quantile of the $N(\mu, \sigma^2)$ distribution is $\mu + \sigma z_q$, where z_q is the q-quantile of the standard normal distribution.

A.1.4 Lognormal distribution

If $Y \sim N(\mu, \sigma^2)$, then $X = e^Y$ is said to have a lognormal distribution, with parameters μ and σ. We write $X \sim LN(\mu, \sigma)$

The probability density function of the $LN(\mu, \sigma)$ distribution is

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{\frac{-(\log x-\mu)^2}{2\sigma^2}\right\}$$

for $x > 0$, where, as for the normal distribution, $-\infty < \mu < \infty$ and $\sigma > 0$.

The mean and variance of the $LN(\mu, \sigma)$ distribution are

$$\mathrm{E}[X] = e^{\mu+\sigma^2/2} \quad \text{and} \quad \mathrm{V}[X] = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right).$$

We can calculate probabilities for the lognormal distribution using the standard normal distribution function, as follows. Note that if $X \sim LN(\mu,\sigma)$ then $Y = \log X \sim N(\mu,\sigma^2)$, so

$$\Pr[X \le x] = \Pr[\log X \le \log x] = \Pr[Y \le \log x] = \Phi\left(\frac{\log x - \mu}{\sigma}\right).$$

In Chapters 13 and 16 we used the result that if $X \sim LN(\mu,\sigma)$ then

$$\int_0^a x f(x)\, dx = \exp\{\mu + \sigma^2/2\}\Phi\left(\frac{\log a - \mu - \sigma^2}{\sigma}\right). \tag{A.1}$$

To show this, we first note that

$$\int_0^a x f(x)\, dx = \int_0^a \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{\frac{-(\log x - \mu)^2}{2\sigma^2}\right\} dx,$$

and the substitution $z = \log x$ gives

$$\int_0^a x f(x)\, dx = \int_{-\infty}^{\log a} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{\frac{-(z-\mu)^2}{2\sigma^2}\right\} \exp\{z\} dz.$$

Combining the exponential terms, the exponent becomes

$$\begin{aligned}
z - \frac{(z-\mu)^2}{2\sigma^2} &= \frac{-1}{2\sigma^2}\left(z^2 - 2\mu z + \mu^2 - 2\sigma^2 z\right) \\
&= \frac{-1}{2\sigma^2}\left(z^2 - 2(\mu+\sigma^2)z + \mu^2\right) \\
&= \frac{-1}{2\sigma^2}\left(z^2 - 2(\mu+\sigma^2)z + (\mu+\sigma^2)^2 + \mu^2 - (\mu+\sigma^2)^2\right) \\
&= \frac{-1}{2\sigma^2}\left((z-(\mu+\sigma^2))^2 - 2\mu\sigma^2 - \sigma^4\right) \\
&= \frac{-\left(z-(\mu+\sigma^2)\right)^2}{2\sigma^2} + \mu + \frac{\sigma^2}{2}.
\end{aligned}$$

This technique is known as 'completing the square' and is very useful in problems involving normal or lognormal random variables. We can now write

$$\int_0^a x f(x)\,dx = \int_{-\infty}^{\log a} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{\frac{-\left(z-(\mu+\sigma^2)\right)^2}{2\sigma^2}\right\} \exp\left\{\mu+\frac{\sigma^2}{2}\right\} dz$$

$$= \exp\left\{\mu+\frac{\sigma^2}{2}\right\} \int_{-\infty}^{\log a} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{\frac{-\left(z-(\mu+\sigma^2)\right)^2}{2\sigma^2}\right\} dz.$$

Now the integrand is the probability density function of normal random variable with mean $\mu+\sigma^2$ and variance σ^2, and so

$$\int_{-\infty}^{\log a} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{\frac{-\left(z-(\mu+\sigma^2)\right)^2}{2\sigma^2}\right\} dz = \Phi\left(\frac{\log a-\mu-\sigma^2}{\sigma}\right),$$

giving formula (A.1). We note that

$$\lim_{a\to\infty} \Phi\left(\frac{\log a-\mu-\sigma^2}{\sigma}\right) = 1,$$

and from this result and formula (A.1) we see that the mean of the lognormal distribution with parameters μ and σ^2 is

$$\exp\left\{\mu+\frac{\sigma^2}{2}\right\}.$$

A.2 The central limit theorem

The central limit theorem is a very important result in probability theory. Suppose that $X_1, X_2, X_3, \ldots$ is a sequence of independent and identically distributed random variables, each having mean μ and variance σ^2. Now define the sum $S_n = \sum_{i=1}^n X_i$ so that $\mathrm{E}[S_n] = n\mu$ and $\mathrm{V}[S_n] = n\sigma^2$. The mean of the $\{X_i\}$ is $\bar{X}_n = S_n/n$.

The central limit theorem states that

$$\lim_{n\to\infty} \Pr\left[\frac{S_n-n\mu}{\sigma\sqrt{n}} \le x\right] = \Phi(x).$$

We can also write

$$\lim_{n\to\infty} \Pr\left[\frac{\bar{X}_n-\mu}{\sigma/\sqrt{n}} \le x\right] = \Phi(x).$$

This means that whenever we sum n independent, identically distributed random variables, provided that n is a reasonably large number, their mean

$\bar{X}$ is approximately normally distributed, with mean μ, and variance σ^2/n. This result holds whether the individual X_i have a discrete distribution (for example, the binomial distribution) or a continuous distribution (for example, the lognormal distribution).

A.3 Functions of a random variable

In many places in this book we have considered functions of a random variable. For example, in Chapter 4 we considered v^{T_x} where T_x is a random variable representing future lifetime. We have also evaluated the expected value and higher moments of functions of a random variable. Here, we briefly review the theory that we have applied, considering separately random variables that follow discrete, continuous and mixed distributions. We quote results only, giving references for these results in Section A.6.

A.3.1 Discrete random variables

We first consider a discrete random variable, X, with probability function $p_X(x) = \Pr[X = x]$ for $x = 0, 1, 2, \ldots$. Let g be a function and let $Y = g(X)$, so that the possible values for Y are $g(0), g(1), g(2), \ldots$. Then Y takes the value $g(x)$ if X takes the value x. Thus,

$$\Pr[Y = g(x)] = \Pr[X = x] = p_X(x),$$

and so

$$\mathrm{E}[Y] = \sum_{x=0}^{\infty} g(x) \Pr[Y = g(x)] = \sum_{x=0}^{\infty} g(x)\, p_X(x). \tag{A.2}$$

Thus, we can compute $\mathrm{E}[Y]$ in terms of the probability function of X. Higher moments are similarly computed. For $r = 1, 2, 3, \ldots$, we have

$$\mathrm{E}[Y^r] = \sum_{x=0}^{\infty} g(x)^r\, p_X(x).$$

A.3.2 Continuous random variables

We next consider the situation of a continuous random variable, X, distributed on $(0, \infty)$ with probability density function $f_X(x)$ for $x > 0$. Consider a function g, let g^{-1} denote the inverse of this function, and define $Y = g(X)$. Then we can compute the expected value of Y as

$$\mathrm{E}[Y] = \mathrm{E}[g(X)] = \int_0^\infty g(x) f_X(x)\, dx. \tag{A.3}$$

As in the case of discrete random variables, the expected value of Y can be found without explicitly stating the distribution of Y, and higher moments can be found similarly. Note the analogy with equation (A.2); the probability function has been replaced by probability density function, and summation by integration.

It can be shown that Y has a probability density function, which we denote f_Y, given by

$$f_Y(y) = f_X\left(g^{-1}(y)\right) \left| \frac{d}{dy} g^{-1}(y) \right| \tag{A.4}$$

provided that g is a monotone function. However, formula (A.3) allows us to compute the expected value of Y without finding its probability density function.

For example, suppose that X has an exponential distribution with parameter λ. Now define $Y = e^{-\delta X}$, where $\delta > 0$. Then by formula (A.3) with $g(y) = e^{-\delta y}$,

$$\mathrm{E}[Y] = \int_0^\infty e^{-\delta y}\, \lambda\, e^{-\lambda y}\, dy = \frac{\lambda}{\lambda + \delta}.$$

The alternative (and more complicated) approach to finding $\mathrm{E}[Y]$ is to first identify the distribution of Y, then find its mean. To follow this approach, we first note that if $g(y) = e^{-\delta y}$, then $g^{-1}(y) = (-1/\delta) \log y$ and so

$$\frac{d}{dy}\, g^{-1}(y) = \frac{-1}{\delta y}.$$

By formula (A.4), Y has probability density function $f_Y(y)$, which is defined for $0 < y < 1$ (since $X > 0$ implies that $0 < e^{-\delta X} < 1$ as $\delta > 0$), with

$$\begin{aligned} f_Y(y) &= \lambda \exp\{(\lambda/\delta) \log y\} \frac{1}{\delta y} \\ &= \frac{\lambda}{\delta}\, y^{(\lambda/\delta)-1}. \end{aligned}$$

Thus

$$\mathrm{E}[Y] = \int_0^1 y f_Y(y)\, dy = \frac{\lambda}{\delta} \int_0^1 y^{\lambda/\delta} dy = \frac{\lambda}{\delta}\, \frac{y^{(\lambda/\delta)+1}}{(\lambda/\delta)+1}\Bigg|_0^1 = \frac{\lambda}{\lambda+\delta}.$$

We could also have evaluated this integral by noting that Y has a beta distribution on (0,1) with parameters λ/δ and 1. The key point is that a function of a random variable is itself a random variable with its own distribution, but because of formula (A.3) it is not necessary to find this distribution to evaluate its moments.

A.3.3 Mixed random variables

Most of the mixed random variables we have encountered in this book have a probability density function over an interval and a mass of probability at one point only. For example, under an endowment insurance with term n years, there is probability density associated with payment of the sum insured at time t for $0 < t < n$, and a mass of probability associated with payment at time n. In that situation we defined the random variable (see Section 4.4.7)

$$Z = \begin{cases} v^{T_x} & \text{if } T_x < n, \\ v^n & \text{if } T_x \geq n. \end{cases}$$

More generally, suppose that X is a random variable with probability density function f over some interval (or possibly intervals) which we denote by $\mathcal{I}$, and has probability masses, $\Pr[X = x_i]$, at points $x_1, x_2, x_3, \ldots$. Then if we define $Y = g(X)$, we have

$$\mathrm{E}[Y] = \int_{\mathcal{I}} g(x) f(x)\, dx + \sum_i g(x_i) \Pr[X = x_i].$$

For example, suppose that $\Pr[X \leq x] = 1 - e^{-\lambda x}$ for $0 < x < n$, and that $\Pr[X{=}n] = e^{-\lambda n}$. Then X has probability density function $f(x) = \lambda e^{-\lambda x}$ for $0 < x < n$, and has a mass of probability of amount $e^{-\lambda n}$ at n. If we define $Y = e^{-\delta X}$, then

$$\begin{aligned} \mathrm{E}[Y] &= \int_0^n e^{-\delta x} \lambda e^{-\lambda x}\, dx + e^{-\delta n} e^{-\lambda n} \\ &= \frac{\lambda}{\lambda + \delta} \left(1 - e^{-(\lambda+\delta)n}\right) + e^{-(\lambda+\delta)n} \\ &= \frac{1}{\lambda + \delta} \left(\lambda + \delta e^{-(\lambda+\delta)n}\right). \end{aligned}$$

A.4 Conditional expectation and conditional variance

Consider two random variables X and Y whose first two moments exist. We can find the mean and variance of Y in terms of the conditional mean and variance of Y given X. In particular,

$$\mathrm{E}[Y] = \mathrm{E}\big[\mathrm{E}[Y|X]\big] \tag{A.5}$$

and

$$\mathrm{V}[Y] = \mathrm{E}\big[\mathrm{V}[Y|X]\big] + \mathrm{V}\big[\mathrm{E}[Y|X]\big]. \tag{A.6}$$

These formulae hold generally, but to prove them we restrict ourselves here to the situation when both X and Y are discrete random variables. Consider first expression (A.5). We note that for a function g of X and Y, we have

$$\mathrm{E}[g(X, Y)] = \sum_x \sum_y g(x, y) \Pr[X = x, Y = y] \tag{A.7}$$

(this is just the bivariate version of formula (A.2)). By the rules of conditional probability,

$$\Pr[X = x, Y = y] = \Pr[Y = y|X = x] \Pr[X = x]. \tag{A.8}$$

Then setting $g(X, Y) = Y$ and using (A.7) and (A.8) we obtain

$$\begin{aligned}
\mathrm{E}[Y] &= \sum_x \sum_y y \Pr[Y = y|X = x] \Pr[X = x] \\
&= \sum_x \Pr[X = x] \sum_y y \Pr[Y = y|X = x] \\
&= \sum_x \Pr[X = x]\mathrm{E}[Y|X = x] \\
&= \mathrm{E}\big[\mathrm{E}[Y|X]\big].
\end{aligned}$$

To obtain formula (A.6) we have

$$\begin{aligned}
\mathrm{V}[Y] &= \mathrm{E}[Y^2] - \mathrm{E}[Y]^2 \\
&= \mathrm{E}\big[\mathrm{E}[Y^2|X]\big] - \mathrm{E}[Y]^2 \\
&= \mathrm{E}\big[\mathrm{V}[Y|X] + \mathrm{E}[Y|X]^2\big] - \mathrm{E}[Y]^2 \\
&= \mathrm{E}\big[\mathrm{V}[Y|X]\big] + \mathrm{E}\big[\mathrm{E}[Y|X]^2\big] - \mathrm{E}\big[\mathrm{E}[Y|X]\big]^2 \\
&= \mathrm{E}\big[\mathrm{V}[Y|X]\big] + \mathrm{V}\big[\mathrm{E}[Y|X]\big].
\end{aligned}$$

A.5 Maximum likelihood estimation

A.5.1 The likelihood function

Maximum likelihood estimation (MLE) is a method for estimating parameters of a distribution, or a function of the parameters, from a set of observations drawn from a random sample. The intuition behind maximum likelihood estimation is that we find the parameters that maximize the joint probability distribution of the observed data, which we call the likelihood function. We denote the likelihood function by $L(\boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is a vector of unknown parameters. Generally, we work with the log-likelihood, $l(\boldsymbol{\theta}) = \log L(\boldsymbol{\theta})$, rather than the likelihood. Since log is an increasing function, maximizing the log-likelihood gives parameters that also maximize the likelihood.

Suppose we have a random sample, $X_1, X_2, \ldots, X_n$, with observed values $x_1, x_2, \ldots, x_n$.

- If the $\{X_j\}$ are independent and identically distributed, with probability function f (which is a function of $\boldsymbol{\theta}$), then the likelihood and log-likelihood functions are

$$L(\boldsymbol{\theta}) = \prod_{j=1}^{n} f(x_j) \quad \text{and} \quad l(\boldsymbol{\theta}) = \sum_{j=1}^{n} \log f(x_j).$$

- The $\{X_j\}$ may be independent but not identically distributed. This arises in survival analysis when some observations are censored or truncated, but all the $\{X_j\}$ have a probability function which depends on the same underlying $\boldsymbol{\theta}$. In this case, letting f_{X_j} denote the probability function of X_j, the likelihood function is

$$L(\boldsymbol{\theta}) = \prod_{j=1}^{n} f_{X_j}(x_j) \quad \text{and} \quad l(\boldsymbol{\theta}) = \sum_{j=1}^{n} \log f_{X_j}(x_j). \tag{A.9}$$

  This is the form assumed for the rest of this section.
- The $\{X_j\}$ may not be independent. In this case, the likelihood must take the dependence into consideration. For example, suppose we have joint lifetime data $x_1, x_2, \ldots, x_{2n-1}, x_{2n}$, where x_1 and x_2 are data for a couple, x_3 and x_4 are data for a different couple, and so on. It would not be appropriate to multiply together the individual, marginal probability functions for each of the x_j. Instead, assuming independence between couples, the likelihood function would be the product of the joint probability functions $f(x_1, x_2)$, $f(x_3, x_4)$ and so on. In general, the joint probability function for a dependent random sample is

$$L(\boldsymbol{\theta}) = f(x_1) f(x_2|x_1) f(x_3|x_1, x_2) \ldots f(x_n|x_1, \ldots, x_{n-1}).$$

  Although we do not deal with dependent data in this book, it is important to be aware of this general form.

A.5.2 Finding the maximum likelihood estimates

We may be able to maximize $l(\boldsymbol{\theta})$ by taking partial derivatives with respect to each of the parameters, and then setting the partial derivatives equal to zero. If there are k parameters, this gives us a system of k equations.

In practice, software is used to solve for the parameters when we cannot solve the k equations analytically. For example, we can maximize the log-likelihood in Excel with the Solver Add-In. This requires an initial estimate of the parameter values, say $\tilde{\boldsymbol{\theta}}$, entered as variables in Excel. We then calculate the contribution to the log-likelihood of each data point (i.e. $\log f_{X_j}(x_j; \tilde{\boldsymbol{\theta}})$) using the estimates $\tilde{\boldsymbol{\theta}}$ of the parameters. Sum the contributions for the total log-likelihood evaluated at $\tilde{\boldsymbol{\theta}}$. Then set Solver to maximize the total log-likelihood by changing the values in the cells containing $\tilde{\boldsymbol{\theta}}$. It may be necessary to add constraints to the parameters. In most cases (smooth likelihood, not too many parameters) Solver finds the maximum likelihood estimates even if the first guess values for $\boldsymbol{\theta}$ are not particularly good.

A.5.3 Properties of maximum likelihood estimates

The MLE of parameter θ_i is denoted $\hat{\theta}_i$, which depends on the random sample. As a function of the random variables $X_1, X_2, \ldots, X_n$, $\hat{\theta}_i$ is a random variable. When we substitute the observed x_j for the random variable X_j, $\hat{\theta}_i$ becomes a point estimate of the unknown parameter θ_i. When we talk about the expected value or variance of $\hat{\boldsymbol{\theta}}$, or an individual parameter $\hat{\theta}_i$, we are referencing the random variable.

One of the reasons why MLE is a popular tool in statistics is that it has many appealing properties; some key ones are listed below.

1. The MLE is asymptotically unbiased. This means that as the sample size increases, the expected value of $\hat{\theta}_i$ converges to the true, unknown value θ_i. In many cases, the MLE is unbiased for all sample sizes.
2. The MLE is asymptotically minimum variance. This means that for large sample sizes, no other estimator has a smaller variance than the maximum likelihood estimator.

   For a distribution dependent on a single parameter θ, or when $\hat{\theta}$ is independent of all other parameter estimates, the asymptotic variance of $\hat{\theta}$ is

   $$\sigma^2_{\theta^a} = \left(-\mathrm{E}\left[\frac{\partial^2 l}{\partial \theta^2} \right] \right)^{-1} .$$

   This is a function of the unknown θ, but we can estimate this variance by substituting the point estimate $\hat{\theta}$.

   For dependent parameter estimators, the asymptotic covariance matrix of the parameter vector $\boldsymbol{\theta}$ is

   $$\Sigma_{\theta^a} = \left(-\mathrm{E}\left[\frac{\partial^2 l}{\partial \boldsymbol{\theta}^2} \right] \right)^{-1} .$$

Note that the estimators $\hat{\theta}_i$ and $\hat{\theta}_j$ are independent if $\mathrm{E}\left[\frac{\partial^2 l}{\partial\theta_i\,\partial\theta_j}\right] = 0$, so a sufficient, but not necessary condition for independence is that

$$\frac{\partial^2 l}{\partial\theta_i\,\partial\theta_j} = 0.$$

3. For a monotonic, differentiable function g, and for independent $\hat{\theta}$, the MLE of $g(\theta)$ is $g(\hat{\theta})$ (this is the **invariance property** of MLEs), and the asymptotic variance of $g(\hat{\theta})$ is

$$\sigma^2_{g(\theta)^a} = \left(\frac{\partial g}{\partial\theta}\right)^2 \sigma^2_{\theta^a}.$$

We estimate this variance by substituting $\hat{\theta}$ for θ. This is the one parameter version of the **delta method**. It can be generalised to multiple, dependent parameters; see Klugman *et al.* (2012) for details and examples.
4. For independent $\hat{\theta}$, the asymptotic distribution of $\hat{\theta}$ is

$$\hat{\theta} \to N(\theta, \sigma^2_{\theta^a}).$$

This allows us to use the normal distribution to construct approximate confidence intervals for θ, provided the sample size is large enough. However, when the estimator is near a bound for θ, we need a very large sample size for the normal assumption to become valid. For example, if θ is a variance parameter, and its point estimate is close to zero, then the estimator (as a random variable) $\hat{\theta}$ will be positively skewed, as it cannot be less than zero.
5. We can use the invariance property of the MLE to avoid the problem of bounded parameters. If $0 < \theta < \infty$, then $g(\theta) = \log\theta$ is unbounded. The delta method can be used to find the approximate variance of $g(\hat{\theta})$, and this can be used to construct a confidence interval for $g(\theta)$. We then apply the inverse transformation, g^{-1}, to the confidence interval bounds for $g(\theta)$ to give a confidence interval for θ that will be within the feasible range for θ.

   If θ is a probability, then $0 < \theta < 1$, in which case $g(\theta) = \log(-\log\theta)$ is unbounded, and we can proceed as above.

   The resulting interval (in both cases) is called the **log-confidence interval**. See Section 18.7 for an example.

A.6 Notes and further reading

Further details on the probability theory contained in this appendix can be found in texts such as Grimmett and Welsh (2014) and Hogg *et al.* (2014). Maximum likelihood estimation is covered in Klugman *et al.* (2012).

Appendix B
Numerical techniques

B.1 Numerical integration

In this section we illustrate two methods of numerical integration. The first, the trapezium rule has the advantage of simplicity, but its main disadvantage is the amount of computation involved for the method to be very accurate. The second, repeated Simpson's rule, is not quite as straightforward, but is usually more accurate. We now outline each method, and give numerical illustrations of both. Further details can be found in the references in Section B.3.

Our aim in the next two sections is to evaluate numerically

$$I = \int_a^b f(x)\, dx$$

for some function f.

B.1.1 The trapezium rule

Under the trapezium rule, the interval (a, b) is split into n intervals, each of length $h = (b - a)/n$. Thus, we can write I as

$$\begin{aligned} I &= \int_a^{a+h} f(x)\, dx + \int_{a+h}^{a+2h} f(x)\, dx + \cdots + \int_{a+(n-1)h}^{a+nh} f(x)\, dx \\ &= \sum_{j=0}^{n-1} \int_{a+jh}^{a+(j+1)h} f(x)\, dx. \end{aligned}$$

We obtain the value of I under the trapezium rule by assuming that f is a linear function in each interval so that under this assumption

$$\int_{a+jh}^{a+(j+1)h} f(x)\, dx = \frac{h}{2}\left(f(a+jh) + f(a+(j+1)h)\right),$$

Table B.1 *Values of I^* under the trapezium rule and repeated Simpson's rule.*

n	Trapezium I^*	Repeated Simpson I^*
10	12.6529448	12.6424116
20	12.6450449	12.6424112
40	12.6430696	12.6424112
80	12.6425758	12.6424112
160	12.6424523	12.6424112
320	12.6424215	12.6424112

and hence

$$I = h\left(\tfrac{1}{2}f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h) + \tfrac{1}{2}f(b)\right)$$
$$= h\left(\tfrac{1}{2}f(a) + \sum_{j=1}^{n-1} f(a+jh) + \tfrac{1}{2}f(b)\right).$$

To illustrate the application of the trapezium rule, consider

$$I^* = \int_0^{20} e^{-0.05x}\, dx.$$

We have chosen this integral as we can evaluate it exactly as

$$I^* = \frac{1}{0.05}\left(1 - e^{-0.05\times 20}\right) = 12.6424112,$$

and hence we can compare evaluation by numerical integration with the true value. We have $a = 0$ and $b = 20$, and for our numerical illustration we have set $n = 10, 20, 40, 80, 160$ and 320, so that the values of h are 2, 1, 0.5, 0.25, 0.125 and 0.0625. Table B.1 shows the results. We see that in this example we need a small value of h to obtain an answer that is correct to four decimal places, but we note that the percentage error is fairly small in all cases.

B.1.2 Repeated Simpson's rule

This rule is based on Simpson's rule which gives the following approximation:

$$\int_a^{a+2h} f(x)\, dx \approx \frac{h}{3}\left(f(a) + 4f(a+h) + f(a+2h)\right).$$

This approximation arises by approximating the function f by a quadratic function that goes through the three points $(a, f(a))$, $(a+h, f(a+h))$ and

$(a+2h, f(a+2h))$. Repeated application of this result leads to the repeated Simpson's rule, namely

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left(f(a) + 4\sum_{j=1}^{n} f(a+(2j-1)h) + 2\sum_{j=1}^{n-1} f(a+2jh) + f(b)\right)$$

where $h = (b-a)/2n$.

Let us again consider

$$I^* = \int_0^{20} e^{-0.05x}\,dx.$$

Table B.1 shows numerical values for I^*.

We see from Table B.1 that the calculations are considerably more accurate than under the trapezium rule. The reason for this is that the error in applying the trapezium rule is

$$\frac{(b-a)^3}{12n^2} f''(c)$$

for some c, where $a < c < b$, whilst under repeated Simpson's rule the error is

$$\frac{(b-a)^5}{2880n^4} f^{(4)}(c)$$

for some c, where $a < c < b$.

B.1.3 Integrals over an infinite interval

Many situations arise under which we have to find the numerical value of an integral over the interval $(0, \infty)$. For example, we saw in Chapter 2 that the complete expectation of life is given by

$$\mathring{e}_x = \int_0^{\infty} {}_tp_x\,dt.$$

To evaluate such integrals numerically, it usually suffices to take a pragmatic approach. For example, looking at the integrand in the above expression, we might say that the probability of a life aged x surviving to age 120 is very small, and so we might replace the upper limit of integration by $120-x$, and perform numerical integration over the finite interval $(0, 120-x)$. We could then assess our answer by considering a wider interval, say $(0, 130-x)$.

To illustrate this idea, consider the following integral from Section 2.5 where we computed $\mathring{e}_x$ for a range of values for x. Table B.2 shows values of

Table B.2 *Values of I_m.*

m	I_m
60	34.67970
70	34.75059
80	34.75155
90	34.75155
100	34.75155

$$I_m = \int_0^m {}_tp_{40}\, dt$$

for a range of values for m. These values have been calculated using repeated Simpson's rule. We set $n = 120$ for $m = 60$, then changed the value of n for each subsequent value of m in such a way that the value of h was unchanged. For example, with $m = 70$, setting $n = 140$ results in $h = 0.25$, which is the same value of h obtained when $m = 60$ and $n = 120$. This maintains consistency between successive calculations of I_m values. For example,

$$I_{70} = I_{60} + \int_{60}^{70} {}_tp_{40}\, dt,$$

and setting $n = 140$ to compute I_{70} then gives the value we computed for I_{60} with $n = 120$. From this table our conclusion is that, to five decimal places, $\mathring{e}_{40} = 34.75155$.

B.2 Woolhouse's formula

Woolhouse's formula was introduced in Chapter 5. Here we give an indication of how this formula arises. We use the Euler–Maclaurin formula which is concerned with numerical integration. This formula gives a series expansion for the integral of a function, assuming that the function is differentiable a certain number of times. For a function f, the Euler–Maclaurin formula can be written as

$$\int_a^b f(x)\, dx = h\left(\sum_{i=0}^{N} f(a+ih) - \tfrac{1}{2}\left(f(a) + f(b)\right)\right) + \tfrac{h^2}{12}\left(f'(a) - f'(b)\right) - \tfrac{h^4}{720}\left(f'''(a) - f'''(b)\right) + \cdots, \tag{B.1}$$

where $h = (b-a)/N$, N is an integer, and the terms we have omitted involve higher derivatives of f. We shall apply this formula twice, in each case ignoring third and higher-order derivatives of f.

First, setting $a = 0$ and $b = N = n$ (so that $h = 1$), the right-hand side of (B.1) is

$$\sum_{i=0}^{n} f(i) - \tfrac{1}{2}\left(f(0) + f(n)\right) + \tfrac{1}{12}\left(f'(0) - f'(n)\right). \tag{B.2}$$

Second, setting $a = 0$, $b = n$ and $N = mn$ for some integer $m > 1$ (so that $h = 1/m$), the right-hand side of (B.1) is

$$\frac{1}{m}\left(\sum_{i=0}^{mn} f(i/m) - \tfrac{1}{2}\left(f(0) + f(n)\right)\right) + \frac{1}{12m^2}\left(f'(0) - f'(n)\right). \tag{B.3}$$

As each of (B.2) and (B.3) approximates the same quantity, we can obtain an approximation to $\frac{1}{m}\sum_{i=0}^{mn} f(i/m)$ by equating them, so that

$$\frac{1}{m}\sum_{i=0}^{mn} f(i/m) \approx \sum_{i=0}^{n} f(i) - \frac{m-1}{2m}\left(f(0) + f(n)\right) + \frac{m^2-1}{12m^2}\left(f'(0) - f'(n)\right). \tag{B.4}$$

The right-hand side of formula (B.4) gives the first three terms of Woolhouse's formula, and in actuarial applications it usually suffices to apply only these terms.

B.3 Notes and further reading

A list of numerical integration methods is given in Abramowitz and Stegun (1965). Details of the derivation of the trapezium rule and repeated Simpson's rule can be found in standard texts on numerical methods such as Burden *et al.* (2015) and Ralston and Rabinowitz (2001).

Appendix C

Monte Carlo simulation

C.1 The inverse transform method

The inverse transform method allows us to simulate observations of a random variable, X, when we have a uniform $U(0, 1)$ random number generator available.

The method states that if $F(x) = \Pr[X \leq x]$ and u is a random drawing from the $U(0, 1)$ distribution, then

$$x = F^{-1}(u)$$

is our simulated value of X.

The result follows for the following reason: if $U \sim U(0, 1)$, then $F^{-1}(U)$ has the same distribution as X. To show this, we assume for simplicity that the distribution function F is continuous – this is not essential for the method, it just gives a simpler proof. First, we note that as the distribution function F is continuous, it is a monotonic increasing function. Next, we know from the properties of the uniform distribution on $(0, 1)$ that for $0 \leq y \leq 1$,

$$\Pr[U \leq y] = y.$$

Now let $\tilde{X} = F^{-1}(U)$. Then

$$\begin{aligned}\Pr[\tilde{X} \leq x] &= \Pr[F^{-1}(U) \leq x] \\ &= \Pr[U \leq F(x)]\end{aligned}$$

since F is a monotonic increasing function. So

$$\Pr[\tilde{X} \leq x] = \Pr[U \leq F(x)] = F(x) = \Pr[X \leq x]$$

which shows that $\tilde{X}$ and X have the same distribution function.

Example C.1 Simulate three values from an exponential distribution with mean 100 using the three random drawings

$$u_1 = 0.1254, \quad u_2 = 0.4529, \quad u_3 = 0.7548,$$

from the $U(0, 1)$ distribution.

Solution C.1 Let F denote the distribution function of an exponentially distributed random variable with mean 100, so that

$$F(x) = 1 - \exp\{-x/100\}.$$

Then setting $u = F^{-1}(x)$ gives

$$x = -100 \log(1 - u),$$

and hence our three simulated values from this exponential distribution are

$$-100 \log 0.8746 = 13.399,$$

$$-100 \log 0.5471 = 60.312,$$

$$-100 \log 0.2452 = 140.57. \qquad \square$$

C.2 Simulation from a normal distribution

In Chapter 12 we used Excel to generate random numbers from a normal distribution. In many situations, for example if we wish to create a large number of simulations of an insurance portfolio over a long time period, it is much more effective in terms of computing time to use a programming language rather than a spreadsheet. Most programming languages do not have an in-built function to generate random numbers from a normal distribution, but do have a random number generator, that is they have an in-built function to generate (pseudo-)random numbers from the $U(0, 1)$ distribution.

Without going into details, we now state the two most common approaches to simulating values from a standard normal distribution. The detail behind these 'recipes' can be found in the references in Section C.3.

C.2.1 The Box–Muller method

The Box–Muller method is to first simulate two values, u_1 and u_2, from a $U(0, 1)$ distribution, then to compute the pair

$$x = \sqrt{-2 \log u_1}\ \cos(2\pi u_2),$$

$$y = \sqrt{-2 \log u_1}\ \sin(2\pi u_2),$$

which are random drawings from the standard normal distribution.

For example, if $u_1 = 0.643$ and $u_2 = 0.279$, we find that $x = -0.1703$ and $y = 0.9242$.

C.2.2 The polar method

From a computational point of view, the weakness of the Box–Muller method is that we have to compute trigonometric functions to apply it. This issue can be avoided by using the polar method which says that if u_1 and u_2 are as above, then set

$$v_1 = 2u_1 - 1,$$
$$v_2 = 2u_2 - 1,$$
$$s = v_1^2 + v_2^2.$$

If $s < 1$, we compute

$$x = v_1\sqrt{\frac{-2\log s}{s}},$$
$$y = v_2\sqrt{\frac{-2\log s}{s}},$$

which are random drawings from the standard normal distribution. However, should the computed value of s exceed 1, we discard the random drawings from the $U(0, 1)$ distribution and repeat the procedure until the computed value of s is less than 1.

For example, if $u_1 = 0.643$ and $u_2 = 0.279$, we find that $v_1 = 0.2860$, $v_2 = -0.4420$ and hence $s = 0.2772$. As the value of s is less than 1, we proceed to compute $x = 0.8703$ and $y = -1.3450$.

C.3 Notes and further reading

Details of all the above methods can be found in standard texts on simulation, e.g. Ross (2013).

Appendix D
Tables

D.1 The Standard Select and Ultimate Life Tables

In this appendix we show tables of selected functions for the Standard Ultimate and Select Survival Models. These are used extensively throughout the book for examples and exercises.

The **Standard Ultimate Survival Model** follows Makeham's Law, parameterized as follows:

$$\mu_x = A + Bc^x \quad \text{where } A = 0.00022, B = 2.7 \times 10^{-6},\ c = 1.124. \tag{D.1}$$

The **Standard Select Survival Model** is defined as follows:

◇ The select period is two years.
◇ The ultimate part of the model is the Standard Ultimate Survival Model.
◇ For the select part of the model, for $0 \leq s \leq 2$,

$$\mu_{[x]+s} = 0.9^{2-s}\mu_{x+s}. \tag{D.2}$$

This model was introduced in Example 3.13.

We also present, for convenience of access, the pension service table derived and used for examples and exercises in Chapter 11, and the Standard Sickness–Death tables used in Chapter 8.

In general, when an exercise or example references the Standard Ultimate Life Table, or Standard Select Life Table, the solutions have been prepared using tabular values, rather than using the full underlying model. Often, the basis will reference, for example, the Standard Ultimate Life Table with uniform distribution of deaths, and the result will be different from the answer obtained using the full Makeham's distribution which defines the Standard Ultimate Survival Model. The objective of providing tables, and of asking questions that require fractional age assumptions, is to prepare readers for the use of fractional age assumptions in other contexts where the full underlying model is unavailable.

There are also many examples and exercises in the text based on the full select and ultimate models that cannot be answered using these tables. It is recommended that readers create their own spreadsheets, which can be used when the parameters, ages or interest rates required are different to those used here.

Table D.1 *Standard Select and Ultimate Survival Model.*

x	$l_{[x]}$	$l_{[x]+1}$	l_{x+2}	$x+2$	x	$l_{[x]}$	$l_{[x]+1}$	l_{x+2}	$x+2$
			100000.00	20	60	96568.13	96287.48	95940.60	62
			99975.04	21	61	96232.34	95920.27	95534.43	63
20	99995.08	99973.75	99949.71	22	62	95858.91	95511.80	95082.53	64
21	99970.04	99948.40	99923.98	23	63	95443.51	95057.36	94579.73	65
22	99944.63	99922.65	99897.79	24	64	94981.34	94551.72	94020.33	66
23	99918.81	99896.43	99871.08	25	65	94467.11	93989.16	93398.05	67
24	99892.52	99869.70	99843.80	26	66	93895.00	93363.38	92706.06	68
25	99865.69	99842.38	99815.86	27	67	93258.63	92667.50	91936.88	69
26	99838.28	99814.41	99787.20	28	68	92551.02	91894.03	91082.43	70
27	99810.20	99785.70	99757.71	29	69	91764.58	91034.84	90133.96	71
28	99781.36	99756.17	99727.29	30	70	90891.07	90081.15	89082.09	72
29	99751.69	99725.70	99695.83	31	71	89921.62	89023.56	87916.84	73
30	99721.06	99694.18	99663.20	32	72	88846.72	87852.03	86627.64	74
31	99689.36	99661.48	99629.26	33	73	87656.25	86555.99	85203.46	75
32	99656.47	99627.47	99593.83	34	74	86339.55	85124.37	83632.89	76
33	99622.23	99591.96	99556.75	35	75	84885.49	83545.75	81904.34	77
34	99586.47	99554.78	99517.80	36	76	83282.61	81808.54	80006.23	78
35	99549.01	99515.73	99476.75	37	77	81519.30	79901.17	77927.35	79
36	99509.64	99474.56	99433.34	38	78	79584.04	77812.44	75657.16	80
37	99468.12	99431.02	99387.29	39	79	77465.70	75531.88	73186.31	81
38	99424.18	99384.82	99338.26	40	80	75153.97	73050.22	70507.19	82
39	99377.52	99335.62	99285.88	41				67614.60	83
40	99327.82	99283.06	99229.76	42				64506.50	84
41	99274.69	99226.72	99169.41	43				61184.88	85
42	99217.72	99166.14	99104.33	44				57656.68	86
43	99156.42	99100.80	99033.94	45				53934.73	87
44	99090.27	99030.10	98957.57	46				50038.65	88
45	99018.67	98953.40	98874.50	47				45995.64	89
46	98940.96	98869.96	98783.91	48				41841.05	90
47	98856.38	98778.94	98684.88	49				37618.56	91
48	98764.09	98679.44	98576.37	50				33379.88	92
49	98663.15	98570.40	98457.24	51				29183.78	93
50	98552.51	98450.67	98326.19	52				25094.33	94
51	98430.98	98318.95	98181.77	53				21178.30	95
52	98297.24	98173.79	98022.38	54				17501.76	96
53	98149.81	98013.56	97846.20	55				14125.89	97
54	97987.03	97836.44	97651.21	56				11102.53	98
55	97807.07	97640.40	97435.17	57				8469.73	99
56	97607.84	97423.18	97195.56	58				6248.17	100
57	97387.05	97182.25	96929.59	59					
58	97142.13	96914.80	96634.14	60					
59	96870.22	96617.70	96305.75	61					

Table D.2 *Standard Select Life Table, $i = 5\%$ per year.*

x	$\ddot{a}_{[x]}$	$A_{[x]}$	${}^2A_{[x]}$	${}_5E_{[x]}$	${}_{10}E_{[x]}$	${}_{20}E_{[x]}$	x
20	19.9673	0.04918	0.00576	0.78255	0.61227	0.37441	20
21	19.9206	0.05140	0.00610	0.78254	0.61223	0.37431	21
22	19.8717	0.05373	0.00648	0.78252	0.61218	0.37419	22
23	19.8203	0.05618	0.00689	0.78249	0.61213	0.37406	23
24	19.7665	0.05874	0.00734	0.78247	0.61208	0.37392	24
25	19.7100	0.06143	0.00783	0.78244	0.61201	0.37375	25
26	19.6509	0.06424	0.00837	0.78241	0.61194	0.37356	26
27	19.5889	0.06720	0.00895	0.78237	0.61186	0.37336	27
28	19.5239	0.07029	0.00959	0.78233	0.61177	0.37312	28
29	19.4558	0.07353	0.01028	0.78229	0.61167	0.37286	29
30	19.3845	0.07693	0.01104	0.78224	0.61156	0.37256	30
31	19.3098	0.08049	0.01186	0.78218	0.61143	0.37223	31
32	19.2316	0.08421	0.01276	0.78211	0.61128	0.37186	32
33	19.1496	0.08811	0.01373	0.78204	0.61112	0.37144	33
34	19.0639	0.09220	0.01479	0.78196	0.61094	0.37097	34
35	18.9742	0.09647	0.01594	0.78187	0.61074	0.37044	35
36	18.8802	0.10094	0.01720	0.78176	0.61051	0.36985	36
37	18.7820	0.10562	0.01856	0.78165	0.61025	0.36919	37
38	18.6793	0.11051	0.02004	0.78152	0.60996	0.36844	38
39	18.5718	0.11563	0.02164	0.78137	0.60963	0.36761	39
40	18.4596	0.12097	0.02338	0.78121	0.60927	0.36667	40
41	18.3422	0.12656	0.02527	0.78102	0.60886	0.36562	41
42	18.2197	0.13240	0.02731	0.78082	0.60840	0.36444	42
43	18.0917	0.13849	0.02952	0.78058	0.60788	0.36312	43
44	17.9581	0.14485	0.03191	0.78032	0.60730	0.36165	44
45	17.8188	0.15149	0.03450	0.78003	0.60664	0.35999	45
46	17.6734	0.15841	0.03730	0.77970	0.60591	0.35815	46
47	17.5219	0.16563	0.04032	0.77932	0.60509	0.35608	47
48	17.3640	0.17314	0.04358	0.77891	0.60416	0.35377	48
49	17.1995	0.18098	0.04709	0.77844	0.60313	0.35120	49

Table D.2 *(Cont.)*

x	$\ddot{a}_{[x]}$	$A_{[x]}$	${}^2A_{[x]}$	${}_5E_{[x]}$	${}_{10}E_{[x]}$	${}_{20}E_{[x]}$	x
50	17.0284	0.18913	0.05087	0.77791	0.60196	0.34832	50
51	16.8503	0.19761	0.05495	0.77732	0.60066	0.34512	51
52	16.6651	0.20642	0.05933	0.77665	0.59919	0.34156	52
53	16.4728	0.21558	0.06404	0.77591	0.59755	0.33760	53
54	16.2730	0.22509	0.06909	0.77507	0.59572	0.33320	54
55	16.0658	0.23496	0.07451	0.77413	0.59366	0.32832	55
56	15.8509	0.24519	0.08031	0.77307	0.59135	0.32293	56
57	15.6283	0.25579	0.08653	0.77189	0.58877	0.31697	57
58	15.3979	0.26677	0.09317	0.77056	0.58588	0.31041	58
59	15.1596	0.27811	0.10025	0.76907	0.58265	0.30319	59
60	14.9134	0.28984	0.10781	0.76739	0.57904	0.29528	60
61	14.6593	0.30194	0.11586	0.76552	0.57501	0.28663	61
62	14.3972	0.31442	0.12441	0.76341	0.57051	0.27721	62
63	14.1274	0.32727	0.13350	0.76105	0.56550	0.26700	63
64	13.8497	0.34049	0.14313	0.75841	0.55992	0.25596	64
65	13.5644	0.35407	0.15333	0.75545	0.55371	0.24411	65
66	13.2717	0.36801	0.16411	0.75214	0.54682	0.23143	66
67	12.9717	0.38230	0.17548	0.74844	0.53917	0.21797	67
68	12.6647	0.39692	0.18746	0.74429	0.53070	0.20377	68
69	12.3510	0.41186	0.20005	0.73966	0.52134	0.18891	69
70	12.0309	0.42710	0.21326	0.73450	0.51102	0.17350	70
71	11.7050	0.44262	0.22709	0.72873	0.49966	0.15767	71
72	11.3735	0.45840	0.24154	0.72230	0.48719	0.14160	72
73	11.0371	0.47442	0.25662	0.71515	0.47355	0.12548	73
74	10.6963	0.49065	0.27229	0.70719	0.45867	0.10954	74
75	10.3517	0.50706	0.28856	0.69835	0.44250	0.09403	75
76	10.0040	0.52362	0.30541	0.68854	0.42501	0.07920	76
77	9.6540	0.54029	0.32280	0.67768	0.40618	0.06531	77
78	9.3023	0.55704	0.34072	0.66568	0.38600	0.05258	78
79	8.9497	0.57382	0.35912	0.65245	0.36451	0.04121	79
80	8.5972	0.59061	0.37797	0.63789	0.34179	0.03133	80

Table D.3 *Standard Ultimate Life Table, $i = 5\%$ per year.*

x	$\ddot{a}_x$	A_x	${}^2A_x$	${}_5E_x$	${}_{10}E_x$	${}_{20}E_x$	x
20	19.9664	0.04922	0.00580	0.78252	0.61224	0.37440	20
21	19.9197	0.05144	0.00614	0.78250	0.61220	0.37429	21
22	19.8707	0.05378	0.00652	0.78248	0.61215	0.37417	22
23	19.8193	0.05622	0.00694	0.78245	0.61210	0.37404	23
24	19.7655	0.05879	0.00739	0.78243	0.61205	0.37390	24
25	19.7090	0.06147	0.00788	0.78240	0.61198	0.37373	25
26	19.6499	0.06429	0.00841	0.78236	0.61191	0.37354	26
27	19.5878	0.06725	0.00900	0.78233	0.61183	0.37334	27
28	19.5228	0.07034	0.00964	0.78229	0.61174	0.37310	28
29	19.4547	0.07359	0.01033	0.78224	0.61163	0.37284	29
30	19.3834	0.07698	0.01109	0.78219	0.61152	0.37254	30
31	19.3086	0.08054	0.01192	0.78213	0.61139	0.37221	31
32	19.2303	0.08427	0.01281	0.78206	0.61124	0.37183	32
33	19.1484	0.08817	0.01379	0.78199	0.61108	0.37141	33
34	19.0626	0.09226	0.01486	0.78190	0.61090	0.37094	34
35	18.9728	0.09653	0.01601	0.78181	0.61069	0.37041	35
36	18.8788	0.10101	0.01727	0.78170	0.61046	0.36982	36
37	18.7805	0.10569	0.01863	0.78158	0.61020	0.36915	37
38	18.6777	0.11059	0.02012	0.78145	0.60990	0.36841	38
39	18.5701	0.11571	0.02173	0.78130	0.60957	0.36757	39
40	18.4578	0.12106	0.02347	0.78113	0.60920	0.36663	40
41	18.3403	0.12665	0.02536	0.78094	0.60879	0.36558	41
42	18.2176	0.13249	0.02741	0.78072	0.60832	0.36440	42
43	18.0895	0.13859	0.02963	0.78048	0.60780	0.36307	43
44	17.9558	0.14496	0.03203	0.78021	0.60721	0.36159	44
45	17.8162	0.15161	0.03463	0.77991	0.60655	0.35994	45
46	17.6706	0.15854	0.03744	0.77956	0.60581	0.35809	46
47	17.5189	0.16577	0.04047	0.77918	0.60498	0.35601	47
48	17.3607	0.17330	0.04374	0.77875	0.60404	0.35370	48
49	17.1960	0.18114	0.04727	0.77827	0.60299	0.35112	49

Table D.4 *Standard Ultimate Life Table, $i = 5\%$ per year.*

x	$\ddot{a}_x$	A_x	${}^2A_x$	${}_5E_x$	${}_{10}E_x$	${}_{20}E_x$	x
50	17.0245	0.18931	0.05108	0.77772	0.60182	0.34824	50
51	16.8461	0.19780	0.05517	0.77711	0.60050	0.34503	51
52	16.6606	0.20664	0.05957	0.77643	0.59902	0.34146	52
53	16.4678	0.21582	0.06430	0.77566	0.59736	0.33749	53
54	16.2676	0.22535	0.06938	0.77479	0.59550	0.33308	54
55	16.0599	0.23524	0.07483	0.77382	0.59342	0.32819	55
56	15.8444	0.24550	0.08067	0.77273	0.59109	0.32279	56
57	15.6212	0.25613	0.08692	0.77151	0.58848	0.31681	57
58	15.3901	0.26714	0.09360	0.77014	0.58556	0.31024	58
59	15.1511	0.27852	0.10073	0.76860	0.58229	0.30300	59
60	14.9041	0.29028	0.10834	0.76687	0.57864	0.29508	60
61	14.6491	0.30243	0.11644	0.76493	0.57457	0.28641	61
62	14.3861	0.31495	0.12506	0.76276	0.57003	0.27698	62
63	14.1151	0.32785	0.13421	0.76033	0.56496	0.26674	63
64	13.8363	0.34113	0.14392	0.75760	0.55932	0.25569	64
65	13.5498	0.35477	0.15420	0.75455	0.55305	0.24381	65
66	13.2557	0.36878	0.16507	0.75114	0.54609	0.23112	66
67	12.9542	0.38313	0.17654	0.74732	0.53836	0.21764	67
68	12.6456	0.39783	0.18862	0.74305	0.52981	0.20343	68
69	12.3302	0.41285	0.20133	0.73828	0.52036	0.18856	69
70	12.0083	0.42818	0.21467	0.73295	0.50994	0.17313	70
71	11.6803	0.44379	0.22864	0.72701	0.49848	0.15730	71
72	11.3468	0.45968	0.24324	0.72039	0.48590	0.14122	72
73	11.0081	0.47580	0.25847	0.71303	0.47215	0.12511	73
74	10.6649	0.49215	0.27433	0.70483	0.45715	0.10918	74
75	10.3178	0.50868	0.29079	0.69574	0.44085	0.09368	75
76	9.9674	0.52536	0.30783	0.68566	0.42323	0.07887	76
77	9.6145	0.54217	0.32544	0.67450	0.40427	0.06500	77
78	9.2598	0.55906	0.34359	0.66217	0.38396	0.05230	78
79	8.9042	0.57599	0.36224	0.64859	0.36235	0.04096	79
80	8.5484	0.59293	0.38134	0.63365	0.33952	0.03113	80

D.2 Joint life functions

Table D.5 *Standard Ultimate Life Table, independent lives, $i = 5\%$ per year.*

x	$\ddot{a}_{x:x}$	$\ddot{a}_{x:x:\overline{10\rvert}}$	$A_{x:x}$	${}^2A_{x:x}$	${}_{10}E_{x:x}$
50	15.8195	8.0027	0.24669	0.08187	0.58996
51	15.5982	7.9916	0.25723	0.08806	0.58738
52	15.3690	7.9792	0.26814	0.09468	0.58449
53	15.1318	7.9653	0.27944	0.10175	0.58125
54	14.8867	7.9496	0.29111	0.10929	0.57764
55	14.6336	7.9321	0.30316	0.11732	0.57361
56	14.3725	7.9125	0.31559	0.12586	0.56911
57	14.1035	7.8906	0.32840	0.13494	0.56409
58	13.8266	7.8660	0.34159	0.14457	0.55851
59	13.5419	7.8386	0.35515	0.15477	0.55230
60	13.2497	7.8080	0.36906	0.16555	0.54540
61	12.9500	7.7738	0.38333	0.17694	0.53775
62	12.6432	7.7357	0.39794	0.18893	0.52928
63	12.3296	7.6932	0.41288	0.20155	0.51991
64	12.0094	7.6459	0.42812	0.21480	0.50959
65	11.6831	7.5934	0.44366	0.22868	0.49822
66	11.3511	7.5351	0.45947	0.24320	0.48576
67	11.0140	7.4704	0.47552	0.25834	0.47211
68	10.6722	7.3989	0.49180	0.27410	0.45723
69	10.3265	7.3199	0.50826	0.29047	0.44107
70	9.9774	7.2329	0.52488	0.30743	0.42358
71	9.6257	7.1371	0.54163	0.32496	0.40475
72	9.2722	7.0321	0.55847	0.34302	0.38459
73	8.9175	6.9173	0.57536	0.36159	0.36311
74	8.5627	6.7922	0.59225	0.38062	0.34041
75	8.2085	6.6563	0.60912	0.40007	0.31658
76	7.8559	6.5093	0.62591	0.41989	0.29178
77	7.5057	6.3510	0.64258	0.44002	0.26621
78	7.1590	6.1812	0.65910	0.46040	0.24014
79	6.8166	6.0002	0.67540	0.48097	0.21388
80	6.4794	5.8083	0.69146	0.50165	0.18776

Table D.6 *Standard Ultimate Life Table, independent lives, $i = 5\%$ per year.*

x	$\ddot{a}_{x:x+10}$	$\ddot{a}_{x:x+10:\overline{10}\rvert}$	$A_{x:x+10}$	${}^2A_{x:x+10}$	${}_{10}E_{x:x+10}$
50	14.2699	7.9044	0.32048	0.12929	0.56724
51	13.9979	7.8815	0.33344	0.13858	0.56201
52	13.7180	7.8559	0.34676	0.14842	0.55620
53	13.4304	7.8272	0.36046	0.15885	0.54973
54	13.1352	7.7953	0.37451	0.16986	0.54255
55	12.8328	7.7596	0.38891	0.18148	0.53459
56	12.5233	7.7199	0.40365	0.19372	0.52578
57	12.2071	7.6756	0.41871	0.20658	0.51606
58	11.8845	7.6264	0.43407	0.22007	0.50534
59	11.5560	7.5717	0.44972	0.23419	0.49356
60	11.2220	7.5110	0.46562	0.24895	0.48065
61	10.8830	7.4438	0.48176	0.26433	0.46654
62	10.5396	7.3694	0.49811	0.28033	0.45117
63	10.1925	7.2874	0.51464	0.29693	0.43450
64	9.8423	7.1971	0.53132	0.31411	0.41649
65	9.4898	7.0978	0.54810	0.33185	0.39715
66	9.1358	6.9892	0.56496	0.35011	0.37647
67	8.7810	6.8704	0.58186	0.36886	0.35452
68	8.4263	6.7412	0.59875	0.38806	0.33136
69	8.0726	6.6011	0.61559	0.40766	0.30714
70	7.7208	6.4497	0.63234	0.42760	0.28202
71	7.3718	6.2870	0.64896	0.44783	0.25622
72	7.0267	6.1129	0.66540	0.46830	0.23004
73	6.6862	5.9276	0.68161	0.48892	0.20379
74	6.3513	5.7316	0.69756	0.50963	0.17784
75	6.0229	5.5256	0.71320	0.53036	0.15260
76	5.7019	5.3106	0.72848	0.55103	0.12847
77	5.3891	5.0878	0.74338	0.57158	0.10588
78	5.0852	4.8588	0.75785	0.59191	0.08519
79	4.7910	4.6254	0.77186	0.61196	0.06672
80	4.5071	4.3896	0.78538	0.63165	0.05070

D.3 Standard Sickness–Death tables

The model is the sickness–death model illustrated in Figure 8.4, and the transition intensities are as follows:

$$\mu_x^{01} = a_1 + b_1\, e^{c_1 x}, \qquad \mu_x^{02} = a_2 + b_2\, e^{c_2 x},$$
$$\mu_x^{10} = b_1\, e^{c_1(110-x)}, \qquad \mu_x^{12} = 1.4\, \mu_x^{02},$$

where

$$a_1 = 4 \times 10^{-4}, \qquad b_1 = 3.47 \times 10^{-6}, \qquad c_1 = 0.138,$$
$$a_2 = 5 \times 10^{-4}, \qquad b_2 = 7.58 \times 10^{-5}, \qquad c_2 = 0.087.$$

Table D.7 *Probabilities for the Standard Sickness–Death Model.*

x	${}_1p_x^{00}$	${}_1p_x^{01}$	${}_1p_x^{11}$	${}_1p_x^{10}$	${}_{10}p_x^{00}$	${}_{10}p_x^{01}$	${}_{10}p_x^{11}$	${}_{10}p_x^{10}$
50	0.98935	0.00403	0.97819	0.01258	0.83930	0.06557	0.81211	0.06057
51	0.98826	0.00456	0.97904	0.01095	0.82309	0.07382	0.81017	0.05210
52	0.98704	0.00518	0.97961	0.00954	0.80526	0.08301	0.80636	0.04469
53	0.98568	0.00588	0.97993	0.00830	0.78569	0.09321	0.80076	0.03823
54	0.98415	0.00669	0.98000	0.00723	0.76425	0.10447	0.79344	0.03260
55	0.98244	0.00762	0.97985	0.00629	0.74083	0.11685	0.78444	0.02771
56	0.98052	0.00867	0.97948	0.00547	0.71532	0.13039	0.77380	0.02348
57	0.97838	0.00989	0.97889	0.00476	0.68763	0.14509	0.76156	0.01981
58	0.97597	0.01127	0.97810	0.00414	0.65770	0.16096	0.74774	0.01664
59	0.97328	0.01286	0.97710	0.00360	0.62550	0.17793	0.73237	0.01391
60	0.97025	0.01467	0.97590	0.00313	0.59106	0.19590	0.71546	0.01157
61	0.96686	0.01674	0.97449	0.00272	0.55444	0.21473	0.69704	0.00956
62	0.96305	0.01911	0.97286	0.00236	0.51579	0.23418	0.67712	0.00785
63	0.95878	0.02181	0.97101	0.00205	0.47533	0.25396	0.65574	0.00640
64	0.95399	0.02489	0.96893	0.00178	0.43338	0.27366	0.63293	0.00517
65	0.94860	0.02840	0.96660	0.00154	0.39035	0.29283	0.60873	0.00414
66	0.94257	0.03240	0.96402	0.00134	0.34677	0.31089	0.58322	0.00328
67	0.93579	0.03695	0.96118	0.00116	0.30325	0.32720	0.55645	0.00257
68	0.92819	0.04212	0.95804	0.00100	0.26048	0.34107	0.52853	0.00199
69	0.91967	0.04799	0.95460	0.00087	0.21924	0.35178	0.49956	0.00151
70	0.91013	0.05465	0.95082	0.00075	0.18030	0.35865	0.46969	0.00114
71	0.89944	0.06217	0.94670	0.00065	0.14442	0.36105	0.43907	0.00084
72	0.88749	0.07068	0.94220	0.00056	0.11225	0.35853	0.40789	0.00061
73	0.87414	0.08027	0.93730	0.00048	0.08431	0.35084	0.37635	0.00043
74	0.85925	0.09105	0.93197	0.00042	0.06090	0.33798	0.34468	0.00030
75	0.84266	0.10315	0.92617	0.00036	0.04207	0.32028	0.31315	0.00020
76	0.82423	0.11667	0.91988	0.00031	0.02763	0.29836	0.28201	0.00013
77	0.80379	0.13175	0.91305	0.00026	0.01712	0.27309	0.25155	0.00009
78	0.78119	0.14848	0.90565	0.00023	0.00994	0.24555	0.22205	0.00005
79	0.75628	0.16697	0.89763	0.00019	0.00535	0.21686	0.19380	0.00003
80	0.72892	0.18731	0.88896	0.00016	0.00264	0.18815	0.16706	0.00002

Table D.8 *Standard Sickness–Death Model, $i = 5\%$ per year.*

x	$\bar{a}_x^{00}$	$\bar{a}_x^{01}$	$\bar{a}_x^{11}$	$\bar{a}_x^{10}$	$\bar{A}_x^{01}$	$\bar{A}_x^{02}$	$\bar{A}_x^{10}$	$\bar{A}_x^{12}$
50	11.7446	1.9622	12.3918	0.6668	0.24144	0.33124	0.06550	0.36287
51	11.4319	2.0306	12.2392	0.5621	0.25195	0.34316	0.05702	0.37542
52	11.1128	2.0995	12.0670	0.4727	0.26284	0.35537	0.04958	0.38819
53	10.7879	2.1684	11.8774	0.3965	0.27410	0.36786	0.04307	0.40115
54	10.4575	2.2373	11.6725	0.3318	0.28574	0.38062	0.03737	0.41431
55	10.1221	2.3057	11.4539	0.2770	0.29774	0.39364	0.03240	0.42765
56	9.7823	2.3734	11.2233	0.2306	0.31012	0.40692	0.02806	0.44116
57	9.4385	2.4400	10.9821	0.1916	0.32285	0.42045	0.02428	0.45483
58	9.0915	2.5051	10.7317	0.1587	0.33594	0.43420	0.02099	0.46865
59	8.7419	2.5684	10.4733	0.1312	0.34938	0.44817	0.01813	0.48260
60	8.3904	2.6293	10.2081	0.1081	0.36314	0.46235	0.01565	0.49667
61	8.0378	2.6876	9.9369	0.0889	0.37722	0.47671	0.01349	0.51084
62	7.6850	2.7427	9.6609	0.0728	0.39160	0.49123	0.01162	0.52509
63	7.3328	2.7943	9.3809	0.0595	0.40626	0.50590	0.01000	0.53940
64	6.9821	2.8417	9.0977	0.0485	0.42117	0.52070	0.00860	0.55375
65	6.6338	2.8847	8.8122	0.0394	0.43631	0.53559	0.00738	0.56813
66	6.2888	2.9227	8.5250	0.0320	0.45165	0.55057	0.00633	0.58250
67	5.9482	2.9553	8.2370	0.0258	0.46716	0.56560	0.00543	0.59686
68	5.6128	2.9821	7.9488	0.0208	0.48282	0.58065	0.00465	0.61116
69	5.2836	3.0028	7.6611	0.0167	0.49858	0.59571	0.00397	0.62540
70	4.9615	3.0170	7.3745	0.0133	0.51440	0.61073	0.00339	0.63954
71	4.6474	3.0244	7.0897	0.0106	0.53026	0.62569	0.00290	0.65357
72	4.3420	3.0249	6.8072	0.0085	0.54611	0.64057	0.00247	0.66746
73	4.0463	3.0181	6.5276	0.0067	0.56192	0.65533	0.00210	0.68119
74	3.7608	3.0041	6.2514	0.0053	0.57764	0.66994	0.00179	0.69474
75	3.4863	2.9827	5.9792	0.0042	0.59323	0.68438	0.00152	0.70807
76	3.2232	2.9541	5.7114	0.0033	0.60867	0.69861	0.00128	0.72118
77	2.9721	2.9184	5.4486	0.0025	0.62390	0.71260	0.00109	0.73404
78	2.7333	2.8756	5.1911	0.0020	0.63889	0.72634	0.00092	0.74663
79	2.5071	2.8262	4.9393	0.0015	0.65362	0.73979	0.00078	0.75893
80	2.2935	2.7703	4.6937	0.0012	0.66805	0.75293	0.00065	0.77094

D.4 Pension plan service table

Table D.9 *Pension plan service table, Chapter 11.*

x	l_x	w_x	i_x	r_x	d_x	x	l_x	w_x	i_x	r_x	d_x
20	1 000 000	95 104	951	0	237	44	137 656	6 708	134	0	95
21	903 707	85 946	859	0	218	45	130 719	2 586	129	0	100
22	816 684	77 670	777	0	200	46	127 904	2 530	127	0	106
23	738 038	70 190	702	0	184	47	125 140	2 476	124	0	113
24	666 962	63 430	634	0	170	48	122 428	2 422	121	0	121
25	602 728	57 321	573	0	157	49	119 763	2 369	118	0	130
26	544 677	51 800	518	0	145	50	117 145	2 317	116	0	140
27	492 213	46 811	468	0	134	51	114 572	2 266	113	0	151
28	444 800	42 301	423	0	125	52	112 042	2 216	111	0	163
29	401 951	38 226	382	0	117	53	109 553	2 166	108	0	176
30	363 226	34 543	345	0	109	54	107 102	2 118	106	0	190
31	328 228	31 215	312	0	102	55	104 688	2 070	103	0	206
32	296 599	28 207	282	0	96	56	102 308	2 023	101	0	224
33	268 014	25 488	255	0	91	57	99 960	1 976	99	0	243
34	242 181	23 031	230	0	86	58	97 642	1 930	96	0	264
35	218 834	10 665	213	0	83	59	95 351	1 884	94	0	288
36	207 872	10 131	203	0	84	60^-	93 085	0	0	27 926	0
37	197 455	9 623	192	0	84	60^+	65 160	0	62	6 188	210
38	187 555	9 141	183	0	85	61	58 700	0	56	5 573	212
39	178 147	8 682	174	0	86	62	52 860	0	50	5 018	213
40	169 206	8 246	165	0	87	63	47 579	0	45	4 515	214
41	160 708	7 832	157	0	89	64	42 805	0	41	4 061	215
42	152 631	7 438	149	0	90	65^-	38 488	0	0	38 488	0
43	144 954	7 064	141	0	93						

References

Abramowitz, M. and Stegun, I. A. (1965). *Handbook of Mathematical Functions*. New York: Dover.

Atkinson, D. B. and Dallas, J. W. (2000). *Life Insurance Products and Finance: Charting a Clear Course*. Itasca: Society of Actuaries.

Bacinello, A. R. (2003). Fair valuation of a guaranteed life insurance participating contract embedding a surrender option. *Journal of Risk and Insurance* **70**, 3, 461–487.

Blake, D. (2006). *Pension Finance*. Chichester: John Wiley & Sons.

Bowers, N. L., Gerber, H. U., Hickman, J. C., Jones, D. A. and Nesbitt, C. J. (1997). *Actuarial Mathematics*, 2nd edition. Itasca: Society of Actuaries.

Boyle, P. P. and Boyle, F. (2001). *Derivatives: The Tools that Changed Finance*. London: Risk Books.

Boyle, P. P. and Hardy, M. R. (1997). Reserving for maturity guarantees: Two approaches. *Insurance: Mathematics and Economics* **21**, 2, 113–27.

Boyle, P. P. and Schwartz, E. S. (1977). Equilibrium prices of guarantees under equity-linked contracts. *Journal of Risk and Insurance* **44**, 639–60.

Brennan, M. J. and Schwartz, E. S. (1976). The pricing of equity-linked life insurance policies with an asset value guarantee. *Journal of Financial Economics* **3**, 3, 195–213.

Brown, B. W. J., Hollander, M. and Korwar, R. M. (1974). Nonparametric tests of independence for censored data, with applications to heart transplant studies, in *Reliability and Biometry: Statistical Analysis of Life Length*, ed. F. Proschan and R. J. Serfling. Philadelphia: Society for Industrial and Applied Mathematics, pp. 291–302.

Burden, R. L., Faires, J. D. and Burden, A. M. (2015). *Numerical Analysis*, 10th edition. Boston: Cengage Learning.

Cairns, A. J. G. (2004). *Interest Rate Models: An Introduction*. Princeton: Princeton University Press.

Cairns A. J. G., Blake, D., Dowd, K., Coughlan, G. D., Epstein, D., Ong, A. and Balevich, I. (2009). A quantitative comparison of stochastic mortality models using data from England and Wales and the United States. *North American Actuarial Journal* **13**, 1, 1–35.

Charles, J., Ellis, P., Boit, F., Maher, J., Cresswell, C. and Richard, P. (2000). *Report of the Structured Settlements Working Party*. The Institute of Actuaries and Faculty of Actuaries.

CMI (1991). *Continuous Mortality Investigation Report*, Number 12. London and Edinburgh: The Institute of Actuaries and the Faculty of Actuaries.

CMI (2006). *The Graduation of the CMI 1999–2002 Mortality Experience: Final 00 Series Mortality Tables Assured Lives*. London and Edinburgh: The Institute of Actuaries and Faculty of Actuaries.

CMI (2008). *Comparison of Approaches for Calculating Initial Exposure*. London and Edinburgh: The Institute and Faculty of Actuaries.

Coleman, D. and Salt, J. (1992). *The British Population: Patterns, Trends, and Processes*. Oxford: Oxford University Press.

Cox, D. and Miller, H. (1977). *The Theory of Stochastic Processes*. Boca Raton: CRC Press.

Cox, J. C., Ross, S. A. and Rubinstein, M. (1979). Options pricing: A simplified approach. *Journal of Financial Economics* **7**, 3, 229–63.

Dickson, D. C. M. (2006). Premiums and reserves for life insurance products. *Australian Actuarial Journal* **12**, 2, 259.

Forfar, D. O., McCutcheon, J. J. and Wilkie, A. D. (1988). On graduation by mathematical formula. *Journal of the Institute of Actuaries* **115**, 1, 1–149.

Glasserman, P. (2013). *Monte Carlo Methods in Financial Engineering*. New York: Springer Science & Business Media.

Gompertz, B. (1825). XXIV. On the nature of the function expressive of the law of human mortality, and on a new mode of determining the value of life contingencies. In a letter to Francis Baily, Esq. FRS &c. *Philosophical Transactions of the Royal Society of London* **115**, 513–83.

Graham, R. L., Knuth, D. E. and Patashnik, O. (1994). *Concrete Mathematics*, 2nd edition. Upper Saddle River, NJ: Addison-Wesley.

Grimmett, G. and Welsh, D. (2014) *Probability: An Introduction*. Oxford: Oxford University Press.

Hardy, M. R. (2000). Hedging and reserving for single-premium segregated fund contracts. *North American Actuarial Journal* **4**, 2, 63–74.

Hardy, M. R. (2003). *Investment Guarantees: Modeling and Risk Management for Equity-Linked Life Insurance*. New York: John Wiley & Sons.

Hoem, J. M. (1983). The reticent trio: Some little–known early discoveries in insurance mathematics by L. H. F. Opperman, T. N. Thiele and J. P. Gram. *International Statistical Review / Revue Internationale de Statistique* **51**, 213–21.

Hoem, J. M. (1988). New avenues in modelling life insurance and other insurance of persons. *Transactions of the XXIII International Congress of Actuaries*, Volume R, pp. 171–202.

Hogg, R. V., Tanis, E. A. and Zimmerman, D. (2014). *Probability and Statistical Inference*. New York: Pearson.

Hull, J. (2017). *Options, Futures, and Other Derivatives*, 10th edition. New York: Pearson.

Ji, M., Hardy, M. R. and Li, J. S.-H. (2012) A semi-Markov multiple state model for reverse mortgage terminations. *Annals of Actuarial Science* **6**, 2, 235–57.

Klugman, S. A., Panjer, H. H. and Willmot, G. E. (2012). *Loss Models: From Data to Decisions*. New York: John Wiley & Sons.

Lauritzen, S. L., Thiele, T. N., Thiele, J. and Hald, A. (2002). *Thiele: Pioneer in Statistics*. Oxford: Clarendon Press.

Lee, R. (2000). The Lee–Carter method for forecasting mortality, with various extensions and applications. *North American Actuarial Journal* **4**, 1, 80–91.

Lee, R. D. and Carter, L. R. (1992). Modeling and forecasting U.S. mortality. *Journal of the American Statistical Association* **87**, 419, 659–71.

Li, J. S.-H., Hardy, M. R. and Tan, K. S. (2009). Uncertainty in mortality forecasting: An extension to the classical Lee–Carter approach. *ASTIN Bulletin: The Journal of the IAA* **39**, 1, 137–64.

Li, J. S.-H., Hardy, M. R. and Tan, K. S. (2010). Developing mortality improvement formulas: The Canadian insured lives case study. *North American Actuarial Journal* **14**, 4, 381–99.

Macdonald, A. S., Richards, S. J. and Currie, I. D. (2018). *Modelling Mortality with Actuarial Applications*. Cambridge: Cambridge University Press.

Macdonald, A. S., Waters, H. R. and Wekwete, C. T. (2003). The genetics of breast and ovarian cancer I: A model of family history. *Scandinavian Actuarial Journal*, 1, 1–27.

McDonald, R. L. (2009). *Fundamentals of Derivatives Markets*. New York: Pearson.

McGill, D. M., Brown, K. N., Schieber, S. J. and Haley, J. J. (2005). *Fundamentals of Private Pensions*. New York: Oxford University Press.

Møller, T. (1998). Risk-minimizing hedging strategies for unit-linked life insurance contracts. *ASTIN Bulletin: The Journal of the IAA* **28**, 1, 17–47.

Neill, A. (1977). *Life Contingencies*. London: Heinemann.

Norberg, R. (1995). Differential equations for moments of present values in life insurance. *Insurance: Mathematics and Economics* **17**, 2, 171–80.

Press, W. H., Teukolsy, S. A., Vetterling, W. T. and Flannery, B. P. (2007). *Numerical Recipes: The Art of Scientific Computing*, 3rd edition. New York: Cambridge University Press.

Ralston, A. and Rabinowitz, P. (2001). *A First Course in Numerical Analysis*, 2nd edition. New York: Dover.

Renn, D. F. (ed.) (1998). *Life, Death and Money: Actuaries and the Creation of Financial Security*. Oxford: Blackwell.

Rolski, T., Schmidli, H., Schmidt, V. and Teugels, J. L. (2009). *Stochastic Processes for Insurance and Finance*. Chichester: John Wiley & Sons.

Ross, S. M. (2013). *Simulation*, 5th edition. Cambridge: Academic Press.

Ross, S. M., Kelly, J. J., Sullivan, R. J., Perry, W. J., Mercer, D., Davis, R. M., Washburn, T. D., Sager, E. V., Boyce, J. B. and Bristow, V. L. (1996). *Stochastic Processes*, Volume 2. New York: Wiley.

Shampine, L. F. (2018). *Numerical Solution of Ordinary Differential Equations*. Abingdon: Routledge.

Sverdrup, E. (1965). Estimates and test procedures in connection with stochastic models for deaths, recoveries and transfers between different states of health. *Scandinavian Actuarial Journal*, 3–4, 184–211.

Villegas, A. M., Kaishev, V. and Millossovich, P. (2015). StMoMo: An R package for stochastic mortality modelling, in *7th Australasian Actuarial Education and Research Symposium*, 38pp.

Waters, H. R. (1984). An approach to the study of multiple state models. *Journal of the Institute of Actuaries* **111**, 2, 363–74.

Waters, H. R. and Wilkie, A. D. (1987). A short note on the construction of life tables and multiple decrement models. *Journal of the Institute of Actuaries* **114**, 3, 569–80.

Willets, R., Gallop, A., Leandro, P., Lu, J., Macdonald, A., Miller, K., Richards, S., Robjohns, N., Ryan, J. and Waters, H. (2004). Longevity in the 21st century. *British Actuarial Journal* **10**, 4, 685–832.

Woolhouse, W. (1869). On an improved theory of annuities and assurances. *Journal of the Institute of Actuaries and Assurance Magazine* **15**, 2, 95–125.

Yamamoto, D. (2015). *Fundamentals of Retiree Group Benefits*, 2nd edition. Winsted: Actex Publications.

Index