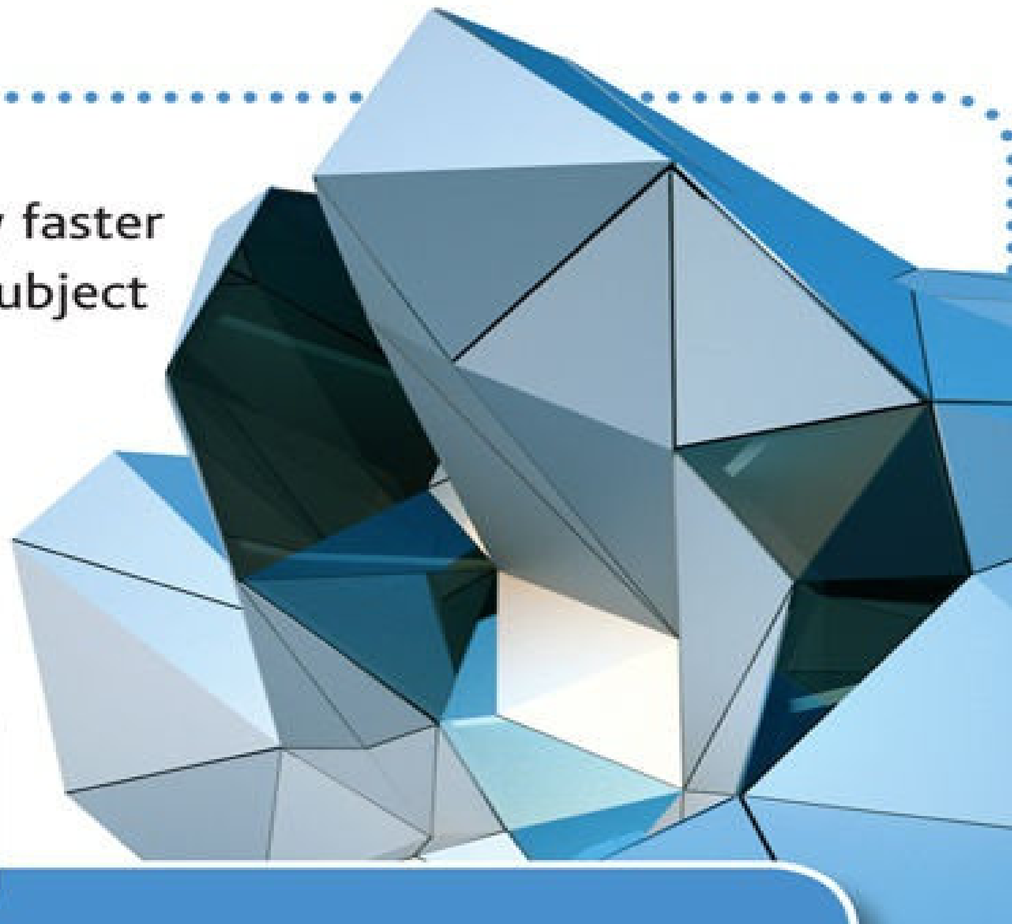
Teach Yourself®

TRIGONOMETRY

A complete introduction



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Hugh Neill Chief Examiner

TRIGONOMETRY

A complete introduction

Paul Abbott

Revised by Hugh Neill

Contents

Welcome to *Trigonometry: A complete introduction*

Teach Yourself Trigonometry has been substantially revised and rewritten to take account of modern needs and recent developments in the subject.

It is anticipated that every reader will have access to a scientific calculator which has sines, cosines and tangents, and their inverses. It is also important that the calculator has a memory, so that intermediate results can be stored accurately. No support has been given about how to use the calculator, except in the most general terms. Calculators vary considerably in the keystrokes which they use, and what is appropriate for one calculator may be inappropriate for another.

There are many worked examples in the book, with complete, detailed answers to all the questions. At the end of each worked example, you will find the symbol ∎ to indicate that the example has been completed, and what follows is text.

Some of the exercises from the original *Teach Yourself Trigonometry* have been used in this revised text, but all the answers have been reworked to take account of the greater accuracy available with calculators.

I would like to thank Linda Moore for her help in reading and correcting the text. But the responsibility for errors is mine.

Hugh Neill

Introduction

Trigonometry is the study of the relationships between the sides and angles in a triangle. It is one of the most practical branches of pure mathematics and it has many applications in the real world. Trigonometry is based on the principle that the ratio between two sides of any similar right-angled triangle is a constant: this enables you to calculate the size of any missing sides or angles in a right-angled triangle. The trigonometric functions (sine, cosine and tangent) are defined as the ratio between two sides of a right-angled triangle. Using these functions, you can calculate the area of any triangle or find missing angles or sides – these skills have obvious applications in surveying and civil engineering. Trigonometry enables surveyors to work out the height of buildings or use triangulation to work out the exact location of a fixed point, which is vital for map making. The triangle is the strongest shape to use in a structure and so trigonometry underpins many of the calculations needed in civil engineering. Many bridges, roofs and other structures are held up by a system of triangular supports.

Trigonometry has many other real-life applications, for example it is used extensively in mechanics to describe the motion of objects. To work out the trajectory of a bullet fired from a gun or the motion of a simple pendulum requires trigonometry – even the way that light or sound waves move can be described using trigonometry.

Trigonometry was first used in astronomy and still has many applications in this field today. For instance, it is used to accurately work out the distance to nearer stars using the phenomenon of parallax, which is the apparent motion of nearby stars relative to more distant stars. You can see parallax for yourself by holding out a finger and then looking at it with first just your right eye and then just your left. The position of your finger will appear to shift, relative to the more distant background. A nearby star appears to move slightly against the background of more distant stars when viewed twice, using observations made six months apart – once the earth has made half of its orbit around the sun. Using the distance that the earth has moved in this time as a base line, it is possible to construct a right-angled triangle using the sun, the earth and the star in question as the three vertices. The distance to the star can then be worked out using the angle of parallax and the distance between the sun and the earth. In fact, the earliest practical uses of trigonometry were in the fields of astronomy and hence navigation.

However, it is when you extend the definitions of the trigonometric functions so that they apply to angles of any size that even more applications emerge. Trigonometry is the mathematics of oscillations and waves – the graphs of sine and cosine are periodic (repeating) waves and so these functions can be used to model waves that occur in real-life situations. Modern technology is hugely reliant on waves; electromagnetic waves such as radio waves, x-rays and microwaves can all be modelled using sine and cosine functions. In optics, the sine function is used in Snell’s law to work out the angle of refraction of light entering a different medium. Modern power lines use alternating current (where the flow of electric charge periodically reverses direction) to deliver power over long distances, the voltage is described mathematically using the sine function.

The periodic, wave-like nature of the trigonometric functions means that they are incredibly useful in mathematical modelling: almost any oscillating system can be described using a combination of sine or cosine functions. For example, a musical note can be modelled by a sine wave, and a chord (several notes played together) can be modelled by several sine functions added together. Combining sine and cosine functions allows us to produce louder or quieter tones, and functions can even be added to together in order to cancel out unwanted sound completely. Trigonometry is fundamental to the principle of sound compression used in MP3 players. The sine function also has applications in climatology as it can be used to model the seasonal fluctuations of carbon dioxide in the atmosphere. In fact, many real-life situations that display seasonal fluctuations, including temperature, can be approximated by trigonometric functions.

There is a seemingly endless list of uses for trigonometry. It is used in medicine (it forms the basis of the mathematics behind CT (computed tomography) scanning), cartography, astronomy, engineering, surveying – even psychology and probability theory make use of trigonometry. Trigonometry is not just to do with triangles, it is the mathematics of waves and oscillations as well. Any problem to do with angles, oscillations or waves can be modelled using trigonometry, which must make trigonometry the most relevant branch of mathematics there is.

1
The tangent

In this chapter you will learn:

- *what a tangent is*
- *the meanings of 'opposite', 'adjacent' and 'hypotenuse' in right-angled triangles*
- *how to solve problems using tangents.*

1.1 Introduction

The method used by Thales to find the height of the pyramid in ancient times is essentially the same as the method used today. It is therefore worth examining more closely.

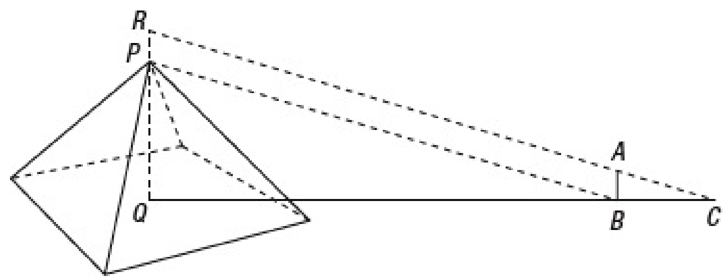


Figure 1.1

You can assume that the sun's rays are parallel because the sun is a long way from the earth. In Figure 1.1, it follows that the lines *RC* and *PB*, which represent the rays falling on the tops of the objects, are parallel.

Therefore, angle *PBQ* = angle *ACB* (they are corresponding angles). These angles each represent the altitude of the sun.

As angles *PQB* and *ABC* are right angles, triangles *PQB* and *ABC* are similar, so $\frac{PQ}{QB} = \frac{AB}{BC}$ or $\frac{PQ}{AB} = \frac{QB}{BC}$.

The height *PQ* of the pyramid is independent of the length of the stick *AB*. If you change the length *AB* of the stick, the length of its shadow will be changed in proportion. You can therefore make the following important general deduction.

For the given angle *ACB*, the ratio $\frac{AB}{BC}$ stays constant whatever the length of *AB*. You can calculate this ratio beforehand for any angle *ACB*. If you do this, you do not need to use the stick, because if you know the angle and the value of the ratio, and you have measured the length *QB*, you can calculate *PQ*.

Thus if the angle of elevation is 64° and the value of the ratio for this angle had been previously found to be 2.05, then you have

$$\frac{PQ}{QB} = 2.05$$

Therefore $$PQ = 2.05 \times QB.$$

1.2 The idea of the tangent ratio The idea of a

constant ratio for every angle is the key to the development of trigonometry.

Let *POQ* (Figure 1.2) be any acute angle $\theta°$. From points *A*, *B*, *C* on one arm, say *OQ*, draw perpendiculars *AD*, *BE*, *CF* to the other arm, *OP*. As these perpendiculars are parallel, the triangles *AOD*, *BOE* and *COF* are similar.

Therefore $$\frac{AD}{OD} = \frac{BE}{OE} = \frac{CF}{OF}.$$

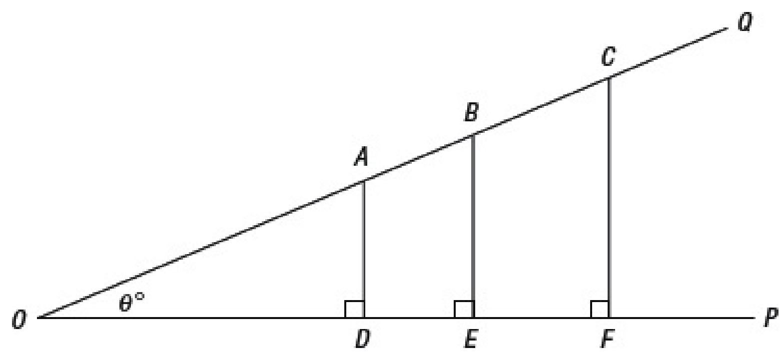


Figure 1.2

Nugget

So if *OE* is double the length of *OD* then BE will be double the length of *AD*.

Now take any point *Y*, it does not matter which, on the arm *OQ*. For that angle $\theta°$ the ratio of the perpendicular *XY* drawn from *Y* on the arm *OQ* to the distance *OX* intercepted on the other arm *OP* is constant (see Figure 1.3).

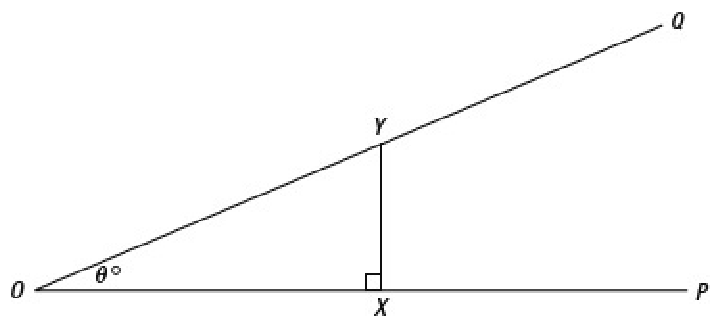


Figure 1.3

This is true for any angle; each angle $\theta°$ has its own particular ratio corresponding to it. This ratio is called the **tangent** of the angle $\theta°$. In practice, the name tangent is abbreviated to **tan.**

Thus for $\theta°$ in Figures 1.2 and 1.3 you can write $\tan\theta° = \frac{XY}{OX}$.

1.3 A definition of tangent

There was a general discussion of the idea of the tangent ratio in Section 1.2, but it is important to refine that discussion into a formal definition of the tangent of an angle.

In Figure 1.4, the origin O is the centre of a circle of radius 1 unit. Draw a radius OP at an angle $\theta°$ to the x-axis, where $0 \leq \theta° < 90$. Let the coordinates of P be (x, y).

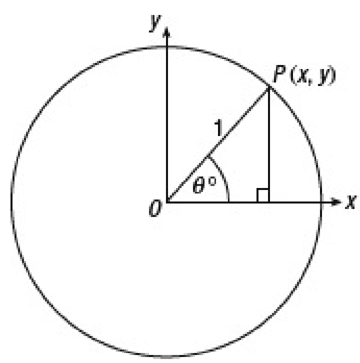


Figure 1.4

Then the tangent of the angle $\theta°$, written tan $\theta°$, is defined by $\tan\theta° = \frac{y}{x}$.

You can see from the definition that if $\theta = 0$. the y-coordinate of P is 0, so tan $\theta = 0$. If $\theta° = 45$, then $x = y$, so $\tan 45° = 1$.

As θ increases, y increases and x decreases, so the tangents of angles close to 90° are very large. You will see that when $\theta° = 90$, the value of x is 0, so $\frac{y}{x}$ is not defined; it follows that tan 90° does not exist, and is undefined.

1.4 Values of the tangent

You can find the value of the tangent of an angle by using your calculator. Try using it. You should find that the tangent of 45°, written tan 45°, is 1, and tan 60° = 1.732…. If you have difficulty with this, you should consult your calculator handbook, and make sure that you can find the tangent of any angle quickly and easily.

Your calculator must be in the correct mode. There are other units, notably radians or rads, for measuring angle, and you must ensure that your calculator is in degree mode, rather than radian or rad mode. Radians are widely used in calculus, and are the subject of Chapter 4.

Some calculators also give tangents for grades, another unit for angle. There are 100 grades in a right angle; this book will not use grades.

Your calculator will also reverse this process of finding the tangent of an angle. If you need to know which angle has a tangent of 0.9, you look up the **inverse tangent**. This is often written as $\tan^{-1} 0.9$, or sometimes as arctan 0.9. Check that $\tan^{-1} 0.9 = 41.987…°$. If it does not, consult your calculator handbook.

In the work that follows, the degree sign will always be included, but you might wish to leave it out in your work, provided there is no ambiguity. Thus you would write tan 45° = 1 and tan 60° = 1.732….

Exercise 1.1

In questions 1 to 6, use your calculator to find the values of the tangents of the angles. Give your answers correct to three decimal places.

1 tan 20°
2 tan 30°
3 tan 89.99°
4 tan 40.43°
5 tan 62°
6 tan 0.5°

In questions 7 to 12, use your calculator to find the angles with the following tangents. Give your answer correct to the nearest one hundredth of a degree.

7 0.342
8 2
9 6.123
10 0.0001
11 1
12 $\sqrt{3}$

1.5 Notation for angles and sides Using notation such as *ABC* for an angle is cumbersome. It is often more convenient to refer to an angle by using only the middle letter of the three that define it. Thus, if there is no ambiguity, tan *B* will be used in preference for tan *ABC*.

Single Greek letters such as α (alpha), β (beta), θ (theta) and Φ (phi) are often used for angles.

Similarly, it is usually easier to use a single letter such as h to represent a distance along a line, rather than to give the beginning and end of the line as in the form AB.

1.6 Using tangents

Here are some examples which illustrate the use of tangents and the technique of solving problems with them.

Example 1.1

A surveyor who is standing at a point 168 m horizontally distant from the foot of a tall tower measures the angle of elevation of the top of the tower as 38.25°. Find the height above the ground of the top of the tower.

You should always draw a figure. In Figure 1.5, *P* is the top of the tower and *Q* is the bottom. The surveyor is standing at *O* which is at the same level as *Q*. Let the height of the tower be *h* metres.

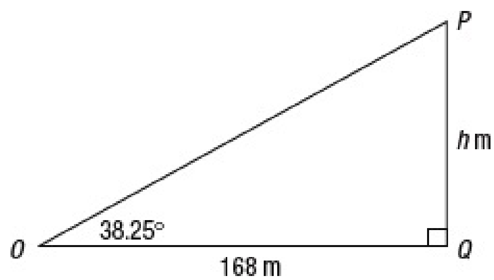


Figure 1.5

Then angle *POQ* is the angle of elevation and equals 38.25°.

Then
$$\begin{aligned}\frac{h}{168} &= \tan 38.25^\circ\\ h &= 168\times\tan 38.25^\circ\\ &= 168\times 0.7883364\ldots\\ &= 132.44052\ldots\,.\end{aligned}$$

The height of the tower is 132 m, correct to three significant figures. ∎

In practice, if you are using a calculator, there is no need to write down all the steps given above. You should write down enough so that you can follow your own working, but you do not need to write down the value of the tangent as an intermediate step. It is entirely enough, and actually better practice, to write the calculation above as

$$\begin{aligned}\frac{h}{168} &= \tan 38.25^\circ\\ h &= 168\times\tan 38.25^\circ\\ &= 132.44052\ldots\,.\end{aligned}$$

However, in this chapter and the next, the extra line will be inserted as a help to the reader.

Example 1.2

A person who is 168 cm tall had a shadow that was 154 cm long. Find the angle of elevation of the sun.

In Figure 1.6 let *PQ* be the person and *OQ* be the shadow. Then *PO* is the sun's ray and θ is the angle of elevation of the sun.

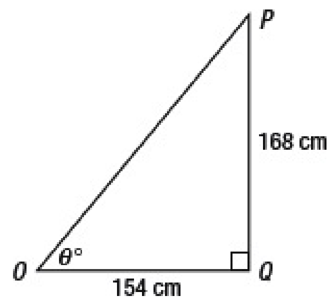


Figure 1.6

Then

$$\begin{aligned}\tan\theta^\circ &= \frac{168}{154}\\ &= 1.09090\ldots\\ \theta^\circ &= \tan^{-1}1.09090\ldots\\ &= 47.489\ldots.\end{aligned}$$

Therefore the angle of elevation of the sun is approximately 47.49°. ∎

Note once again that you can use the calculator and leave out a number of steps, provided that you give enough explanation to show how you obtain your result. Thus you could write

$$\begin{aligned}\tan\theta^\circ &= \frac{168}{154}\\ \theta^\circ &= \tan^{-1}\left(\frac{168}{154}\right)\\ &= 47.489\ldots.\end{aligned}$$

Example 1.3

Figure 1.7 represents a cross-section of a symmetrical roof in which *AB* is the span, and *OP* the rise. *P* is the mid-point of *AB*.

The rise of the roof is 7 m and its angle of slope is 32°. Find the roof span.

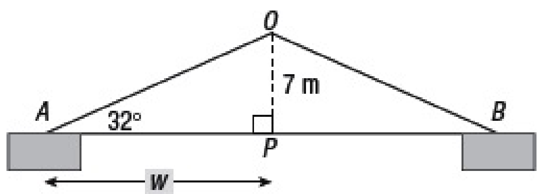


Figure 1.7

As the roof is symmetrical, *OAB* is an isosceles triangle, so *OP* is perpendicular to *AB*. Call the length *AP* *w* metres.

Therefore $\tan 32^\circ = \dfrac{7}{w}$,

so
$$
\begin{aligned}
w &= \frac{7}{\tan 32^\circ}\\
&= \frac{7}{0.624869\ldots}\\
&= 11.2023\ldots .
\end{aligned}
$$

The roof span is $2w$ metres, that is approximately 22.4 m. $\blacksquare$

Exercise 1.2

1 The angle of elevation of the sun is 48.4°. Find the height of a flag staff whose shadow is 7.42 m long.

2 A boat leaving a harbour travels 4 miles east and 5 miles north. Find the bearing of the boat from the harbour.

3 A boat that is on a bearing of 038° from a harbour is 6 miles north of the harbour. How far east is the boat from the harbour?

4 A ladder resting against a wall makes an angle of 69° with the ground. The foot of the ladder is 7.5 m from the wall. Find the height of the top of the ladder.

5 From the top window of a house that is 1.5 km away from a tower, it is observed that the angle of elevation of the top of the tower is 3.6° and the angle of depression of the bottom is 1.2°. Find the height of the tower in metres.

6 From the top of a cliff 32 m high, it is noted that the angles of depression of two boats lying in the line due east of the cliff are 21° and 17°. How far are the boats apart?

7 Two adjacent sides of a rectangle are 15.8 cm and 11.9 cm. Find the angles that a diagonal of the rectangle makes with the sides.

8 P and Q are two points directly opposite to one another on the banks of a river. A distance of 80 m is measured along one bank at right angles to PQ. From the end of this line, the angle subtended by PQ is 61°. Find the width of the river.

9 A ladder that is leaning against a wall makes an angle of 70° with the ground and reaches 5 m up the wall. The foot of the ladder is then moved 50 cm closer to the wall. Find the new angle that the ladder makes with the ground.

1.7 Opposite and adjacent sides Sometimes the triangle with which you have to work is not conveniently situated on the page. Figure 1.8a shows an example of this.

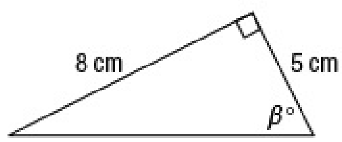


Figure 1.8a

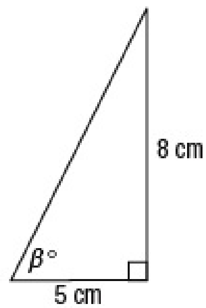


Figure 1.8b

In this case, there is no convenient pair of axes involved. However, you could rotate the figure, either actually or in your imagination, to obtain Figure 1.8b.

You can now see that $\tan \beta° = \frac{AB}{BC} = 1.6$, and you can calculate β, but how could you see that $\tan \beta° = \frac{AB}{BC} = 1.6$ easily from the diagram in Figure 1.8a, without going through the process of getting to Figure 1.8b?

When you are using a right-angled triangle you will always be interested in one of the angles other than the right angle. For the moment, call this angle the ‘angle of focus’. One of the sides will be opposite this angle; call this side the **opposite**. One of the other sides will join the angle in which you are interested; call this side the **adjacent**.

Then $$\text{tangent} = \frac{\text{opposite}}{\text{adjacent}}.$$

This works for all right-angled triangles. In the two cases in Figures 1.8a and 1.8b, the opposite and adjacent sides are labelled in Figures

1.9a and 1.9b.

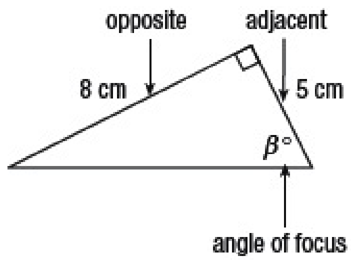


Figure 1.9a

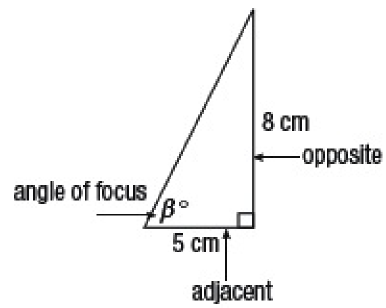


Figure 1.9b

As you can see, in both cases

$$\tan \beta° = \frac{\text{opposite}}{\text{adjacent}} = \frac{8}{5}.$$

Many people find this method the most convenient when using the tangent.

Nugget

The other side of the right-angled triangle, the longest side, is called the **hypotenuse.** The hypotenuse will feature in Chapter 2. It can be confusing to determine which side is the opposite and which the adjacent. Remember adjacent means ‘next to’ – so the adjacent is the side which is **next to** the angle of focus. The angle of focus is between two sides: the adjacent and the longest side (hypotenuse).

Example 1.4

In a triangle *ABC*, angle *B* = 90°, *AB* = 5 cm and *BC* = 7 cm. Find the size of angle *A*.

Draw a diagram (Figure 1.10).

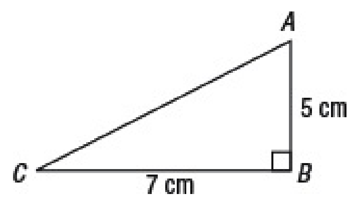


Figure 1.10

In triangle *ABC*, focus on angle *A*. The opposite is 7 cm and the adjacent is 5 cm.

Therefore $\tan A° = \frac{7}{5}$

and angle A = 54.46°. ■

Note that in this case you could find angle *C* first using $\tan C° = \frac{5}{7}$, and then use the fact that the sum of the angles of triangle *ABC* is 180° to find angle *A*.

Example 1.5

Find the length a in Figure 1.11.

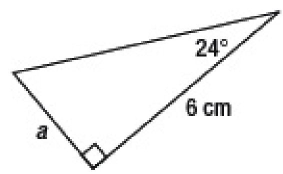


Figure 1.11

Focus on the angle 24°. The opposite side is *a* and the adjacent side is 6 cm.

Then

$$\begin{aligned}\tan 24° &= \frac{a}{6}\\ a &= 6\ \tan\ 24°\\ &= 6\times 0.4452\ldots\\ &= 2.671\ldots.\end{aligned}$$

Example 1.6

Find the length x cm in Figure 1.12.

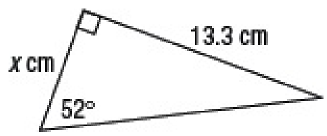


Figure 1.12

Focus on the angle 52°. The opposite side is 13.3 cm and the adjacent side is *x* cm.

Then

$$\begin{aligned}\tan 52^\circ &= \frac{13.3}{x}\\ x &= \frac{13.3}{\tan 52^\circ}\\ &= \frac{13.3}{1.2799\ldots}\\ &= 10.391\ldots \quad \blacksquare\end{aligned}$$

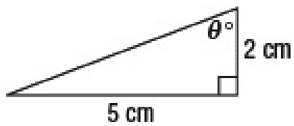
Nugget

You may find it helps to rotate the page so that the triangle looks like ◣ or ◢.

Exercise 1.3

In qustions 1 to 10, find the side or angle indicated by the letter.

1



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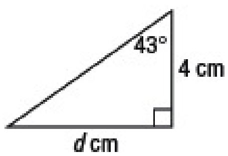

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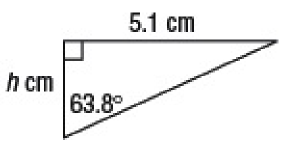

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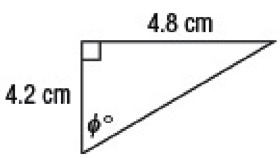

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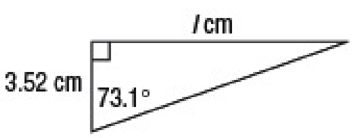

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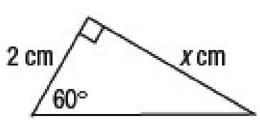

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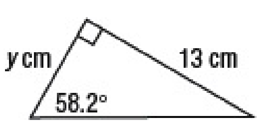

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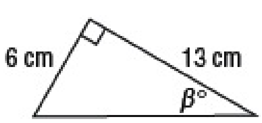

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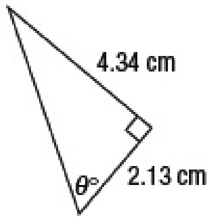

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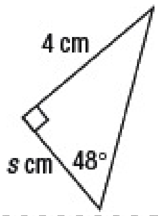
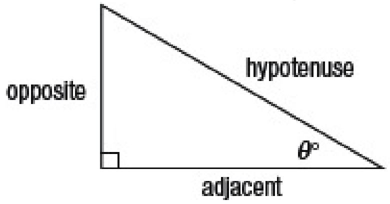
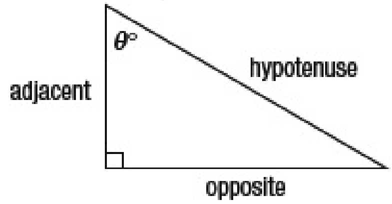


Key ideas

- Angles are normally denoted by the **Greek letters:** α (alpha), β (beta), θ (theta) and Φ (phi).
- The sides of a right-angled triangle can be labelled like this:





  The hypotenuse is the longest side.
  The angle of focus, θ, is the angle between the adjacent and the hypotenuse.

- The ratio $\frac{opposite}{adjacent}$ is called the tangent of the angle θ.
  Tangent is abbreviated to tan.

$$\text{So } \tan\theta° = \frac{opposite}{adjacent}.$$

- To find the angle which has a tangent of, say, 0.5, you need to use the **inverse tangent.** Find $\tan^{-1} 0.5$ or arctan 0.5 using your calculator and check that you get an answer of 26.565…° so tan 26.565…° = 0.5.
- You can rearrange the formula for the tangent ratio in the following ways: $\text{opposite} = \text{adjacent} \times \tan\theta°$ and $\text{adjacent} = \frac{\text{opposite}}{\tan\theta°}$.
- tan 45° = 1 and tan 90° is undefined.

2
Sine and cosine

In this chapter you will learn:

- *what the sine and cosine are*
- *how to use the sine and cosine to find lengths and angles in right-angled triangles*
- *how to solve multistage problems using sines, cosines and tangents.*

2.1 Introduction

In Figure 2.1 a perpendicular is drawn from *A* to *OB*.

You saw on page 4 that the ratio $\frac{AB}{OB} = \tan\theta^\circ$.

Now consider the ratios of each of the lines *AB* and *OB* to the hypotenuse *OA* of triangle *OAB*.

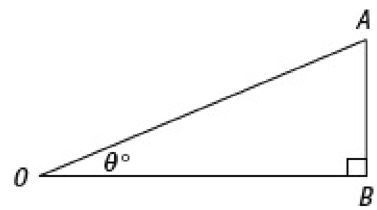
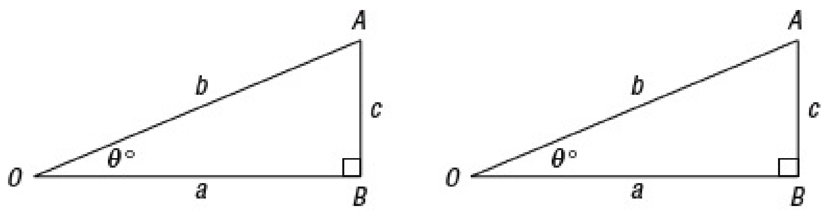


Figure 2.1

Just as for a fixed angle $\theta°$ the ratio $\frac{AB}{OB}$ is constant (and equal to tan $\theta°$), wherever A is, the ratio $\frac{AB}{OA}$, that is $\frac{\text{opposite}}{\text{hypotenuse}}$ is constant.

This ratio is called the **sine of the angle $\boldsymbol{\theta}$°** and is written sin $\theta°$.



$$\sin\theta° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{c}{b} \qquad \cos\theta° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{b}$$

Figure 2.2

Similarly, the ratio $\frac{OB}{OA}$, that is $\frac{\text{adjacent}}{\text{hypotenuse}}$, is also constant for the angle $\theta°$.

This ratio is called the **cosine of the angle $\theta°$** and is written cos $\theta°$.

Thus $$\sin\theta° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{c}{b},\ \cos\theta° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{b}.$$

2.2 Definition of sine and cosine

In Section 2.1 there is a short discussion introducing the sine and cosine ratios. In this section there is a more formal definition.

Draw a circle with radius 1 unit, and centre at the origin O. Draw the radius OP at an angle $\theta°$ to the x-axis in an anticlockwise direction (see Figure 2.3).

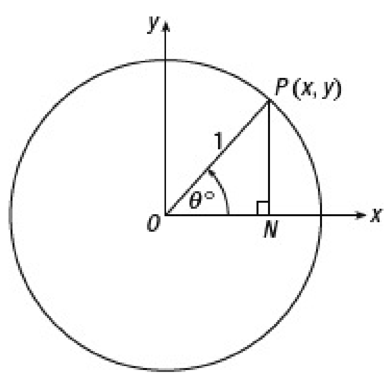


Figure 2.3

Then let P have coordinates (x, y).

Then $\sin \theta° = y$ and $\cos \theta° = x$ are the definitions of sine and cosine which will be used in the remainder of the book.

Note the arrow labelling the angle $\theta°$ in Figure 2.3; this is to emphasize that angles are measured positively in the anticlockwise direction.

Note also two other properties of $\sin \theta°$ and $\cos \theta°$.

- In the triangle OPN, angle $OPN = (90 - \theta)°$, and

$$\sin(90-\theta)° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{x}{1} = \cos\theta°$$

that is $$\sin(90-\theta)° = \cos\theta°.$$

Similarly $$\cos(90-\theta)° = \sin\theta°.$$

- Using Pythagoras's theorem on triangle OPN gives $x^2 + y^2 = 1$.

  Therefore $\sin^2 \theta° + \cos^2 \theta° = 1$, where $\sin^2 \theta°$ means $(\sin \theta°)^2$ and means $\cos^2 \theta°$ means $(\cos \theta°)^2$.

The equation $\sin^2 \theta° + \cos^2 \theta° = 1$ is often called the Pythagorean identity.

Finding the values of the sine and cosine of angles is similar to finding the tangent of an angle. Use your calculator in the way that you would expect. You can use the functions $\sin^{-1}$ and $\cos^{-1}$ to find the inverse sine and cosine in the same way that you used $\tan^{-1}$ to find the inverse tangent.

Nugget

$\sin(90 - \theta)° = \cos\theta°$, $\cos(90 - \theta°) = \sin\theta°$ and $\cos^2\theta° + \sin^2\theta° = 1$ are identities which means they are true for any value of θ. For example, $\sin(90 - 30)° = \cos 30°$ and $\cos(90 - 40)° = \sin 40°$. Also $\cos^2 60° + \sin^2 60° = 1$. Use your calculator to check these and try some other values of θ as well.

2.3 Using the sine and cosine

In the examples which follow there is a consistent strategy for starting the problem.

- Look at the angle (other than the right angle) involved in the problem.
- Identify the sides, adjacent, opposite and hypotenuse, involved in the problem.
- Decide which trigonometric ratio is determined by the two sides.
- Make an equation which starts with the trigonometric ratio for the angle concerned, and finishes with the division of two lengths.
- Solve the equation to find what you need.

Here are some examples which use this strategy.

Nugget

You can use the word '*sohcahtoa*' to help you remember which ratio you need. The middle letter of *soh, cah* and *toa* is on the numerator (top) of each ratio. So *soh* gives $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$, *cah* gives $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$, and *toa* gives $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$

For example, if you know the adjacent and the hypotenuse then you have ***cah*** so use $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$

Example 2.1

Find the length marked x cm in the right-angled triangle in Figure 2.4.

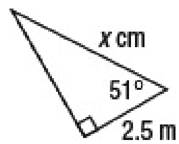


Figure 2.4

The angle concerned is 51°; relative to the angle of 51°, the side 2.5 cm is the adjacent, and the side marked *x* cm is the hypotenuse. The ratio concerned is the cosine.

Start by writing $\cos 51° = \frac{2.5}{x}$.

Then solve this equation for *x*.

$$x = \frac{2.5}{\cos 51°} = 3.972\ldots$$

The length of the side is 3.97 cm approximately. ∎

Example 2.2

The length of each leg of a step ladder is 2.5 m. When the legs are opened out, the distance between their feet is 2 m. Find the angle between the legs.

In Figure 2.5, let *AB* and *AC* be the legs of the ladder. As there is no right angle, you have to make one by dropping the perpendicular *AO* from *A* to the base *BC*. The triangle *ABC* is isosceles, so *AO* bisects the angle *BAC* and the base *BC*.

Therefore $\quad BO = OC = 1$ m.

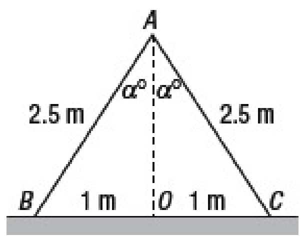


Figure 2.5

You need to find angle *BAC*. Call it $2\alpha°$, so angle $BAO = \alpha°$.

The sides of length 1 m and 2.5 m are the opposite and the hypotenuse for the angle $\boldsymbol{\alpha}°$, so you need the sine ratio.

Then $$\sin\alpha° = \frac{1}{2.5} = 0.4$$

so $$\alpha° = 23.578\ldots,$$

and $$2\alpha° = 47.156\ldots.$$

Therefore the angle between the legs is 47.16° approximately. ∎

Example 2.3

A 30 m ladder on a fire engine has to reach a window 26 m from the ground that is horizontal and level. What angle, to the nearest degree, must it make with the ground and how far from the building must it be placed?

Let the ladder be *AP* (Figure 2.6), let $\theta°$ be the angle that the ladder makes with the ground and let *d* metres be the distance of the foot of the ladder from the window.

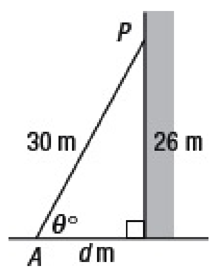


Figure 2.6

As the sides 26 m and 30 m are the opposite and the hypotenuse for the angle θ°, you need the sine ratio.

$$\text{Then} \qquad \sin\theta° = \frac{26}{30} = 0.8666\ldots$$

$$\text{so} \qquad \theta° = 60.073\ldots\,.$$

The ladder is placed at an angle of 60° to the ground.

To find the distance *d*, it is best to use Pythagoras's theorem.

$$d^2 = 30^2 - 26^2 = 900 - 676 = 224,$$

$$\text{so} \qquad d = 14.966\ldots\,.$$

The foot of the ladder is 14.97 m from the wall. ▮

Example 2.4

The height of a cone is 18 cm, and the angle at the vertex is 88°.

Find the slant height.

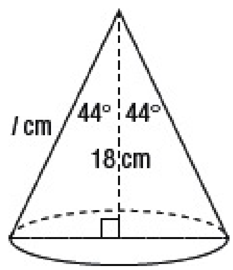


Figure 2.7

In Figure 2.7, let *l* cm be the slant height of the cone. Since the perpendicular to the base bisects the vertical angle of the cone, each part is 44°.

The sides 18 cm and *l* cm are the adjacent and hypotenuse for the angle 44°, so the ratio concerned is the cosine.

Then $$\cos 44° = \frac{18}{l},$$

so $$l = \frac{18}{\cos 44°} = \frac{18}{0.71933\ldots}$$

$$l = 25.022\ldots.$$

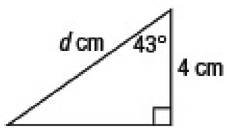
The slant height is approximately 25.0 cm.

Exercise 2.1

In questions 1 to 10, find the side or angle indicated by the letter.

1

2



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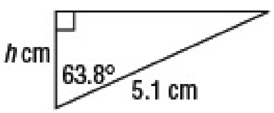

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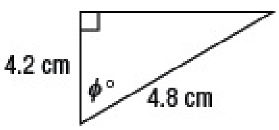

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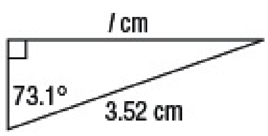

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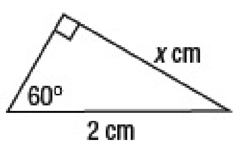

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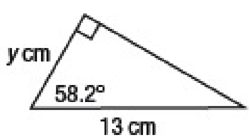

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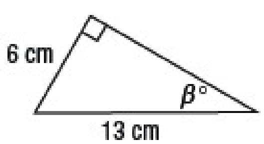

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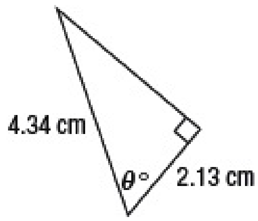

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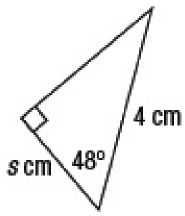


11 A circle of radius 45 mm has a chord of length 60 mm. Find the sine and the cosine of the angle at the centre of the circle subtended by this chord.

12 In a circle with radius 4 cm, a chord is drawn subtending an angle of 80° at the centre. Find the length of this chord and its distance from the centre.

13 The sides of a triangle are 135 mm, 180 mm and 225 mm. Prove that the triangle is right-angled, and find its angles.

14 In a right-angled triangle, the hypotenuse has length 7.4 cm, and one of the other sides has length 4.6 cm. Find the smallest angle of the triangle.

15 A boat travels a distance of 14.2 km on a bearing of 041°. How far east has it travelled?

16 The height of an isosceles triangle is 3.8 cm, and the equal angles are 52°. Find the length of the equal sides.

17 A chord of a circle is 3 m long, and it subtends an angle of 63° at the centre of the circle. Find the radius of the circle.

18 A person is walking up a road angled at 8° to the horizontal. How far must the person walk along the road to rise a height of 1 km? ; **19** In a right-angled triangle, the sides creating the right angle are 4.6 m and 5.8 m. Find the angles and the length of the hypotenuse.

2.4 Trigonometric ratios of 45°, 30° and 60°

You can calculate the trigonometric ratios exactly for some simple angles.

SINE, COSINE AND TANGENT OF 45°

Figure 2.8 shows an isosceles right-angled triangle whose equal sides are each 1 unit.

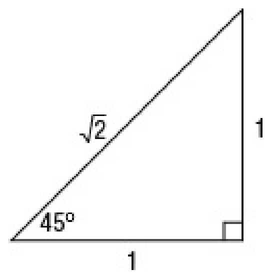


Figure 2.8

Using Pythagoras's theorem, the hypotenuse has length $\sqrt{2}$ units.

Therefore the trigonometric ratios for 45° are given by

$$\sin 45° = \frac{1}{\sqrt{2}}, \cos 45° = \frac{1}{\sqrt{2}}, \tan 45° = 1.$$

You can, if you wish, use the equivalent values

$$\sin 45° = \frac{\sqrt{2}}{2}, \cos 45° = \frac{\sqrt{2}}{2}, \tan 45° = 1.$$

These values are obtained from their previous values by noting that

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

SINE, COSINE AND TANGENT OF 30° AND 60°

Figure 2.9 shows an equilateral triangle of side 2 units.

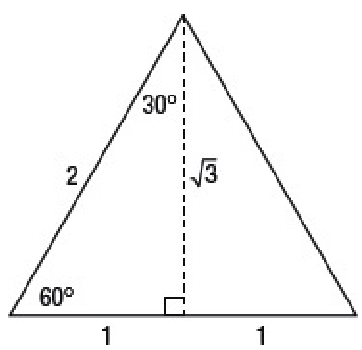


Figure 2.9

The perpendicular from the vertex bisects the base, dividing the original triangle into two triangles with angles of 30°, 60° and 90° and sides of length 1 unit, 2 units and, using Pythagoras's theorem, $\sqrt{3}$ units.

Therefore the trigonometric ratios for 30° are given by

The same ratios for 60° are given by

$$\sin 30^\circ = \frac{1}{2}, \; \cos 30^\circ = \frac{\sqrt{3}}{2}, \; \tan 30^\circ = \frac{1}{\sqrt{3}} \text{ or } \frac{\sqrt{3}}{3}.$$

The same ratios for 60° are given by

$$\sin 60^\circ = \frac{\sqrt{3}}{2}, \; \cos 60^\circ = \frac{1}{2}, \; \tan 60^\circ = \sqrt{3}.$$

It is useful either to remember these results, or to be able to get them quickly.

Nugget

Using these values enables you to work out exact values when solving problems.

2.5 Using the calculator accurately

When you use a calculator to solve an equation such as $\sin\theta° = \frac{23}{37}$, it is important to be able to get as accurate an answer as you can.

WRONG METHOD

$$\sin\theta° = \frac{23}{37} = 0.622$$
$$\theta° = 38.46.$$

CORRECT METHOD

$$\sin\theta° = \frac{23}{37} = 0.6216216216$$
$$\theta° = 38.43.$$

What has happened? The problem is that in the wrong method the corrected answer, 0.622, a three-significant-figure approximation, has been used in the second part of the calculation to find the angle, and has introduced an error.

You can avoid the error by not writing down the three-significant-figure approximation and by using the calculator in the following way.

$$\sin\theta° = \frac{23}{37}$$
$$\theta° = 38.43.$$

In this version, the answer to $\frac{23}{37}$ was used directly to calculate the angle, and therefore all the figures were preserved in the process.

Sometimes it may be necessary to use a calculator memory to store an intermediate answer to as many figures as you need.

It is not necessary in this case, but you could calculate $\frac{23}{37}$ and put the result into memory A. Then you can calculate $\sin^{-1} A$ to get an accurate answer.

You need to be aware of this point for the multistage problems in Section 2.8, and especially so in Chapter 6.

2.6 Slope and gradient

Figure 2.10 represents a side view of the section of a rising path AC. AB is horizontal and BC is the vertical rise.

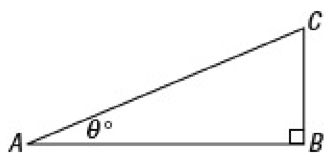


Figure 2.10

Let the angle between the path and the horizontal be $\theta°$.

Then $\theta°$ is called the **angle of slope** or simply **the slope** of the path. The ratio $\tan \theta°$ is called the **gradient** of the path.

Sometimes, especially by the side of railways, the gradient is given in the form 1 in 55. This means $\text{gradient} = \frac{1}{55}$.

When the angle of slope $\theta°$ is very small, as in the case of a railway and most roads, it makes little practical difference if you take $\sin \theta°$ instead of $\tan \theta°$ as the gradient.

Also, in practice it is easier to measure $\sin \theta°$ (by measuring BC and AC), and the difference between AC and AB is relatively small provided that $\theta°$ is small.

You can use your calculator to see just how small the difference is between sines and tangents for small angles.

2.7 Projections

In Figure 2.11, let l be a straight line, and let AB be a straight line segment which makes an angle $\theta°$ with l.

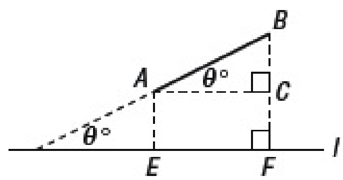


Figure 2.11

Perpendiculars are drawn from A and B to l, meeting l at E and F. Then EF is called the **projection** of AB on l.

You can see from Figure 2.11 that the lengths AC and EF are equal.

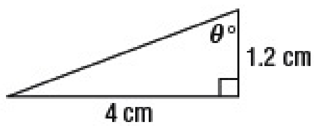
$$\begin{aligned}
\text{As} \qquad \cos\theta^\circ &= \frac{AC}{AB}, \\
AC &= AB\cos\theta^\circ \\
\text{so} \qquad EF &= AB\cos\theta^\circ.
\end{aligned}$$

Exercise 2.2

This is a miscellaneous exercise involving sines, cosines and tangents.

In questions 1 to 10, find the marked angle or side.

1



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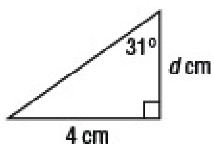

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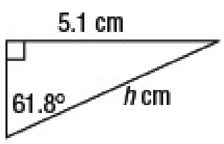

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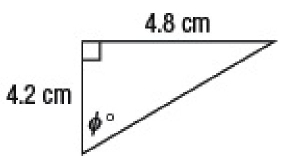

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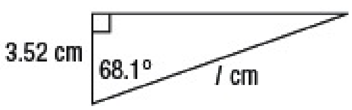

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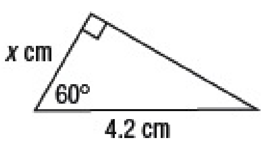

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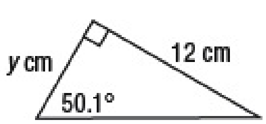

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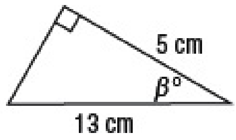

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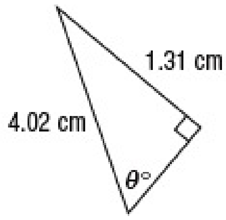

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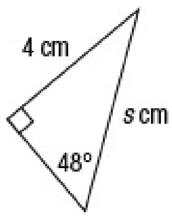


11 In a right-angled triangle the two sides creating the right angle have lengths of 2.34 m and 1.64 m. Find the smallest angle and the length of the hypotenuse.

12 In the triangle *ABC*, *C* is a right angle, *AC* is 122 cm and *AB* is 175 cm. Calculate angle *B*.

13 In triangle *ABC*, angle $C = 90°$, and $A = 37.35°$ and $AB = 91.4$ mm. Find the lengths of *BC* and *CA*.

14 *ABC* is a triangle. Angle *C* is a right angle, *AC* is 21.32 m and *BC* is 12.56 m. Find the angles *A* and *B*.

15 In a triangle *ABC*, *AD* is the perpendicular from *A* to *BC*. The lengths of *AB* and *BC* are 3.25 cm and 4.68 cm and angle *B* is 55°. Find the lengths of *AD*, *BD* and *AC*.

16 *ABC* is a triangle, right-angled at *C*. The lengths of *BC* and *AB* are 378 mm and 543 mm. Find angle *A* and the length of *CA*.

17 A ladder 20 m long rests against a vertical wall. Find the inclination of the ladder to the horizontal when the foot of the ladder is 7 m from the wall.

18 A ship starting from *O* travels 18 kmh^{-1} in the direction 35° north of east. How far will it be north and east of *O* after an hour?

19 A pendulum of length 20 cm swings on either side of the vertical through an angle of 15° on each side. Through what height does the bob rise?

20 The side of an equilateral triangle is *x* metres. Find in terms of *x* the altitude of the triangle. Hence find sin 60° and sin 30°.

21 A straight line 3.5 cm long makes an angle of 42° with the *x*-axis. Find the lengths of its projections on the *x*- and *y*-axes.

22 When you walk 1.5 km up the line of greatest slope of a hill you rise 94 m. Find the gradient of the hill.

23 A ship starts from a given point and sails 15.5 km on a bearing of 319°. How far has it gone west and north respectively?

24 A point *P* is 14.5 km north of *Q* and *Q* is 9 km west of *R*. Find the bearing of *P* from *R* and its distance from *R*.

2.8 Multistage problems

This section gives examples of multistage problems where you need to think out a strategy before you start. See also the advice given in Section 2.5 about the accurate use of a calculator.

Nugget

Always draw a diagram when solving problems. Look for any symmetry or ways to form right-angled triangles.

Example 2.5

ABCD is a kite in which $AB = AD = 5$ cm and $BC = CD = 7$ cm. Angle $DAB = 80°$. Calculate angle *BCD*.

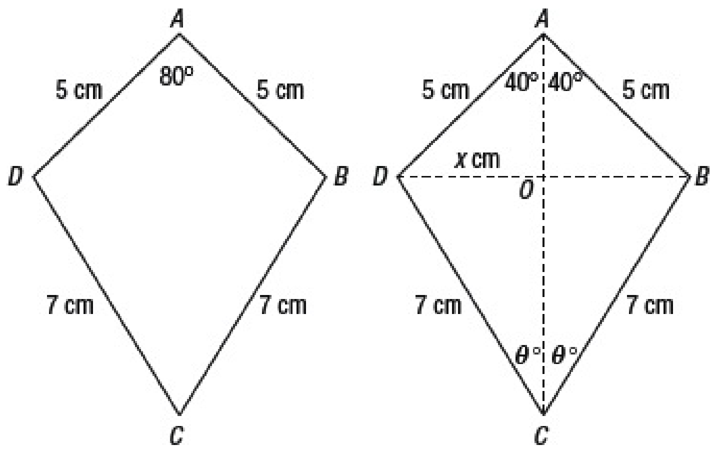


Figure 2.12

The left-hand diagram in Figure 2.12 shows the information. In the right-hand diagram, the diagonals, which cut at right angles at *O*, have been drawn, and the line *AC*, which is an axis of symmetry, bisects the kite.

Let *DO* be *x* cm, and let angle *DCA* = θ°.

$$\begin{array}{ll} \text{Then} & \sin 40° = \dfrac{x}{5} \\ \text{so} & x = 5 \sin 40°. \\ \text{Also} & \sin \theta° = \dfrac{x}{7}. \\ \text{substituting for } x & \sin \theta° = \dfrac{5 \sin 40°}{7} \\ \text{so} & \theta° = 27.33\ldots \end{array}$$

Angle *BCD* = 2θ° = 54.66°, correct to two decimal places. ▮

Example 2.6

Figure 2.13 represents part of a symmetrical roof frame. *PA* = 28 m, *AB* = 6 m and angle *OPA* = 21°. Find the lengths of *OP* and *OA*.

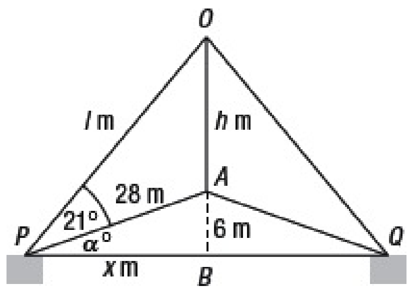


Figure 2.13

Let $OP = l$ m, $PB = x$ m and $OA = h$ m.

To find l you need to find angle OPB; to do this you need first to find angle APB. Let angle $APB = \alpha°$.

$$\begin{aligned}
&\text{Then} && \sin\alpha° = \frac{6}{28} = 0.21428\ldots \\
&\text{so} && \alpha° = 12.373\ldots. \\
&\text{Therefore} && \text{angle } OPB = \alpha° + 21° \\
& && = 33.373\ldots°.
\end{aligned}$$

Next you must find the length x.

To find x, use Pythagoras's theorem in triangle APB.

$$\begin{aligned}
& && x^2 = 28^2 - 6^2 = 784 - 36 = 748 \\
&\text{so} && x = 27.349\ldots. \\
&\text{Then} && \cos 33.373\ldots° = \frac{27.349\ldots}{l} \\
&\text{so} && l = \frac{27.349\ldots}{\cos 33.373\ldots°} \\
& && = 32.7500\ldots.
\end{aligned}$$

To find h, you need to use $h = OB - AB = OB - 6$.

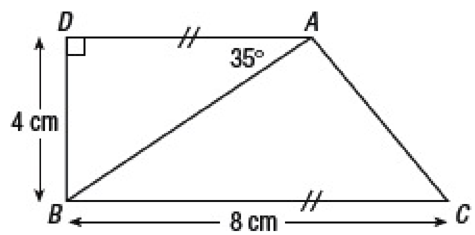
$$\begin{aligned}
&\text{To find } OB, && \sin 33.373\ldots° = \frac{OB}{32.7500\ldots} \\
&\text{so} && OB = 32.7500\ldots \times \sin 33.373\ldots° \\
& && = 18.0156\ldots, \\
&\text{and} && h = 18.0156\ldots - 6 = 12.0156\ldots
\end{aligned}$$

Therefore $OP = 32.75$ m and $OA = 12.02$ m approximately. ∎

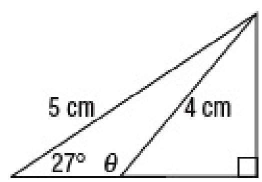
Exercise 2.3

1 *ABCD* is a kite in which $AB = AD = 5$ cm, $BC = CD = 7$ cm and angle $DAB = 80°$. Calculate the length of the diagonal *AC*.

2 In the diagram, find the length of *AC*.



3 In the diagram, find the angle θ.



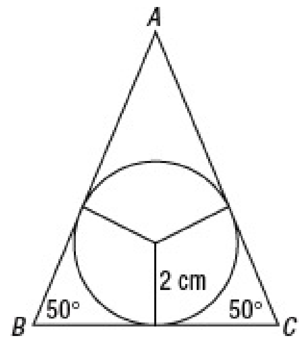
4 *PQRS* is a rectangle. A semicircle drawn with *PQ* as diameter cuts *RS* at *A* and *B*. The length *PQ* is 10 cm, and angle *BQP* is 30°. Calculate the length *PS*.

5 A ship sails 5 km on a bearing of 45° and then 6 km on a bearing of 60°. Find its distance and bearing from its starting point.

6 The lengths *AB* and *AC* of a triangle *ABC* are 5 cm and 6 cm respectively. The length of the perpendicular from *A* to *BC* is 4 cm. Calculate the angle *BAC*.

7 In a triangle *ABC*, the angles at *A* and *C* are 20° and 30° respectively. The length of the perpendicular from *B* to *AC* is 10 cm. Calculate the length of *AC*.

8 In the isosceles triangle *ABC*, the equal angles at *B* and *C* are each 50°. The sides of the triangle each touch a circle of radius 2 cm.



Calculate the length *BC*.

9 A ladder of length 5 m is leaning against a vertical wall at an angle of 60° to the horizontal. The foot of the ladder moves in by 50 cm. By how much does the top of the ladder move up the wall?

10 *PQRS* is a trapezium, with *PQ* parallel to *RS*. The angles at *P* and *Q* are 120° and 130° respectively. The length *PQ* is 6.23 cm and the distance between the parallel sides is 4.92 cm. Calculate the length of *RS*.

Key ideas

- cos $\boldsymbol{\theta}$ is short for the cosine of the angle $\boldsymbol{\theta}$.
- sin $\boldsymbol{\theta}$ is short for the sine of the angle $\boldsymbol{\theta}$.
- $\sin\theta° = \frac{opposite}{hypotenuse}$

  $\cos\theta° = \frac{adjacent}{hypotenuse}$

  $\tan\theta° = \frac{opposite}{adjacent}$
- You can use the word '*sohcahtoa*' to help you remember the trigonometric ratios.
- $\sin(90-\theta)° = \cos\theta°$
  $\cos(90-\theta)° = \sin\theta°$
- The Pythagorean identity is $\cos^2\theta + \sin^2\theta = 1$.

This identity is true for all angles measured in either degrees or radians (see Chapter 7).

-

Angle, θ	0°	30°	45°	60°	90°
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan θ	0	$\frac{1}{\sqrt{3}}$ or $\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

- When a path is inclined at angle of $\boldsymbol{\theta}$ to the horizontal, the gradient is given by tan $\boldsymbol{\theta}$. When $\boldsymbol{\theta}$ is small then the gradient is approximately equal to sin $\boldsymbol{\theta}$.

3
In three dimensions

In this chapter you will learn:

- *the importance of good diagrams in solving three-dimensional problems*
- *how to break down three-dimensional problems into two-dimensional problems*
- *how to solve problems by using pyramids, boxes and wedges.*

3.1 Introduction

Working in three dimensions introduces no new trigonometric ideas, but you do need to be able to think in three dimensions and to be able to visualize the problem clearly. To do this, it is a great help to be able to draw a good figure.

You can solve all the problems in this chapter by picking out right-angled triangles from a three-dimensional figure, drawn, of course, in two dimensions.

This chapter will consist mainly of examples, which will include certain types of diagram which you should be able to draw quickly and easily.

3.2 Pyramid problems

The first type of diagram is the pyramid diagram. This diagram will work for all problems that involve pyramids with a square base.

Example 3.1

ABCD is the square base of side 5 cm of a pyramid whose vertex *V* is 6 cm directly above the centre *O* of the square. Calculate the angle *AVC*.

The diagram is drawn in Figure 3.1.

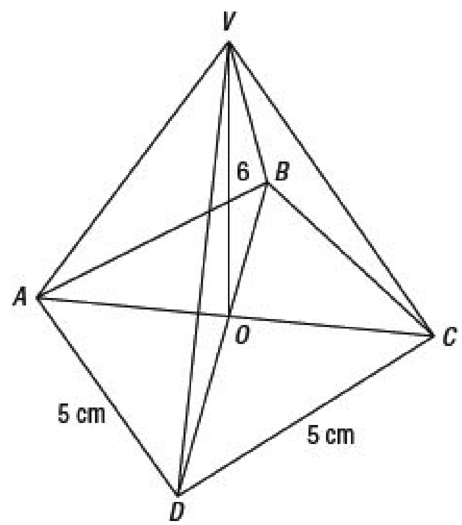


Figure 3.1

There are a number of features you should notice about this diagram: • The diagram is large enough to avoid points being 'on top of one another'.

- The vertex *V* should be above the centre of the base. It is best to draw the base first, draw the diagonals intersecting at *O*, and then put *V* vertically above *O*, siting *V* so that the edge *VC* of the pyramid does not appear to pass through *B*.
- Note that the dimensions have not been put on the measurements where, to do so, would add clutter.
- To solve the problem, you must develop a strategy that involves creating or recognizing right-angled triangles.
- The angle *AVC* is the vertical angle of the isosceles triangle *AVC* that is bisected by *VO*. If you can find the length *AO*, you can find angle *AVC*. You can find *AO* by using the fact that *ABCD* is a square (see Figure 3.2).

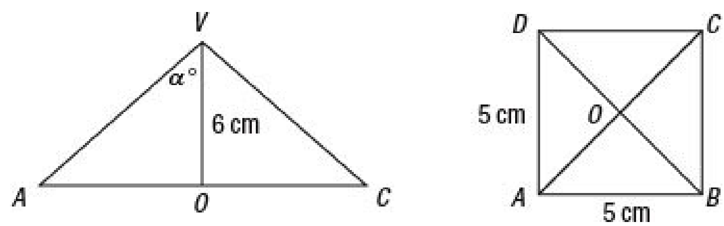


Figure 3.2

Find *AC* by using Pythagoras's theorem.

$$AC^2 = 5^2 + 5^2 = 50,$$

so $$AC = \sqrt{50}\ \text{cm and } AO = \tfrac{1}{2}\sqrt{50}\ \text{cm.}$$

Let angle *AVO* = $\alpha°$.

then $$\tan\alpha° = \frac{\frac{1}{2}\sqrt{50}}{6}$$

and $$\alpha° = 30.5089\ldots$$

Therefore $$\text{angle } AVC = 2\alpha° = 61.02°. \blacksquare$$

If you find that you can solve the problem without drawing the subsidiary diagrams in Figure 3.2, then do so. But many people find that it helps to see what is happening.

All square-based pyramid problems can be solved using this diagram.

Example 3.2

PQRSV is a pyramid with vertex *V*, which is situated symmetrically above the mid-point *O* of the rectangular base *PQRS*. The lengths of *PS* and *RS* are 6 cm and 5 cm, and the slant height *VP* is 8 cm. Find the angle that the edge *VP* makes with the ground.

Figure 3.3 shows the situation.

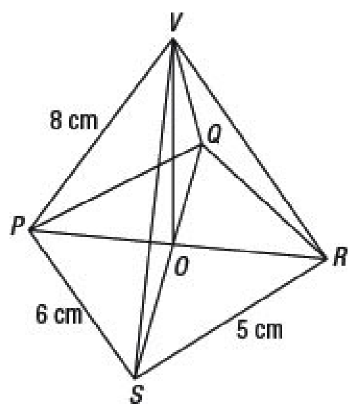


Figure 3.3

Let the required angle *VPO* be ***α***°. Then you can find ***α***° from triangle *VPO* if you can find either *PO* or *VO*. You can find *PO* from the rectangle *PQRS*.

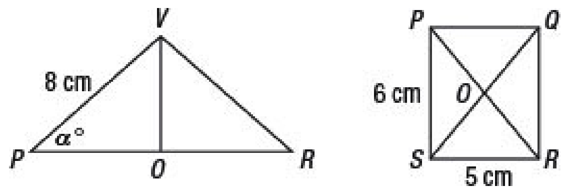


Figure 3.4

As $PO = \frac{1}{2}PR$, and using Pythagoras's theorem gives $PR = \sqrt{PS^2 + RS^2} = \sqrt{6^2 + 5^2} = \sqrt{61}$,

you find $PO = \frac{1}{2}\sqrt{61}\,\text{cm}$

Then $\cos\alpha° = \frac{PO}{VP} = \frac{\frac{1}{2}\sqrt{61}}{8}$,

so $\alpha° = 60.781\ldots$

Therefore, the edge *VP* makes an angle of approximately 60.78° with the horizontal. ∎

3.3 Box problems

The second type of problem involves drawing a box.

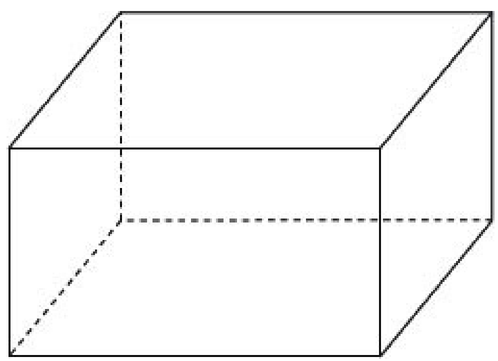


Figure 3.5

Nugget

The usual way to draw the box is to draw two identical rectangles, one ‘behind’ the other, and then to join up the corners appropriately. It can be useful to make some of the lines dotted to make it clear which face is in front.

Example 3.3

A room has length 4 m, width 3 m and height 2 m. Find the angle that a diagonal of the room makes with the floor.

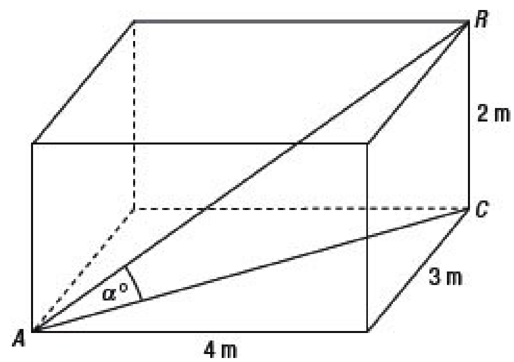


Figure 3.6

Let the diagonal of the room be *AR*, and let *AC* be the diagonal of the floor. Let the required angle be $\alpha°$.

You can use Pythagoras's theorem to find the diagonal *AC* of the floor and so find the angle $\alpha°$.

$$AC = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

so $$\tan \alpha° = \frac{2}{5}$$

and $$\alpha = 21.8014....$$

Therefore, the diagonal makes an angle of approximately 21.80° with the floor.

Sometimes the problem is a box problem, but may not sound like one.

Example 3.4

The top, *P*, of a pylon standing on level ground subtends an angle of 10° at a point *S*, which is 50 m due south, and 5° at a point *W*, lying west of the pylon. Calculate the distance *SW* correct to the nearest metre.

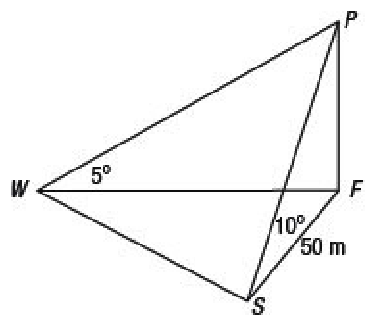


Figure 3.7

This figure is part of the box diagram in Figure 3.5, with some lines removed. *F* is the foot of the pylon.

To calculate *SW*, start by finding the height of the pylon, then the length *FW*, and then use Pythagoras's theorem to find *SW*.

In triangle *PFS*, $\tan 10° = \frac{PF}{50}$, so $PF = 50\tan 10°$

In triangle *PFW*, $\tan 5° = \frac{PF}{FW} = \frac{50\tan 10°}{FW}$

so $FW = \frac{50\tan 10°}{\tan 5°}$

Finally $SW^2 = FW^2 + SF^2 = \left(\frac{50\tan 10°}{\tan 5°}\right)^2 + 50^2$

$= 12654.86\ldots$

so $SW = 112$ m, correct to the nearest metre. ∎

3.4 Wedge problems

The third type of problem involves drawing a wedge. This wedge is really only part of a box, so you could think of a wedge problem as a special case of a box problem, but it is easier to think of it in a separate category.

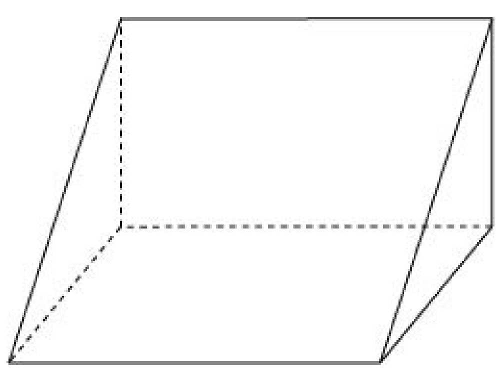


Figure 3.8

Example 3.5

The line of greatest slope of a flat hillside slopes at an angle of 20° to the horizontal. To reduce the angle of climb, a walker walks on a path on the hillside, which makes an angle of 50° with the line of greatest slope. At what angle to the horizontal does the walker climb on this path?

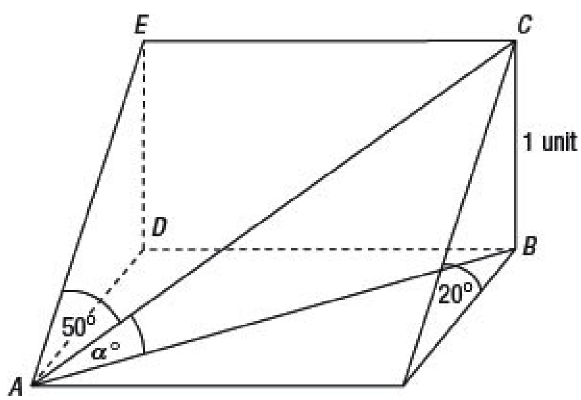


Figure 3.9

Let *AE* be the line of greatest slope, and *AC* be the path of the walker. The angle that you need to find is angle *BAC*. Call this angle $\alpha°$. As there are no units to the problem, let the height *BC* be 1 unit.

To find α you need to find another length (apart from *BC*) in triangle *BAC*. This will come first from the right-angled triangle *DAE*, and then from triangle *AEC*, shown in Figure 3.10. Note that angle *DAE* = 20° as *AE* is a line of greatest slope.

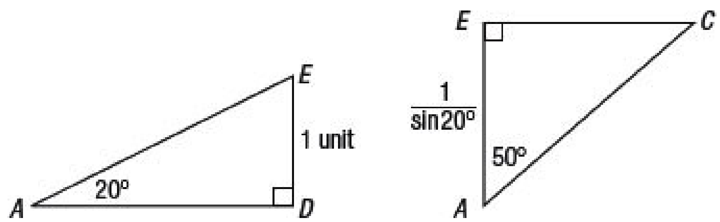


Figure 3.10

In triangle *AED* $\quad \sin 20° = \dfrac{1}{AE}$, so $AE = \dfrac{1}{\sin 20°}$.

In triangle *EAC* $\quad \cos 50° = \dfrac{EA}{AC} = \dfrac{\frac{1}{\sin 20°}}{AC}$,

so $\quad AC = \dfrac{1}{\sin 20° \cos 50°}$.

In triangle ABC $\quad \sin\alpha = \dfrac{1}{AC} = \dfrac{1}{\dfrac{1}{\sin 20° \cos 50°}} = \sin 20° \cos 50°$

so $\quad \alpha° = 12.700....$ ∎

The walker walks at 12.7° approximately to the line of greatest slope.

You may find it easier to follow if you evaluate *AE* and *AC* as you go along, but it is better practice to avoid it if you can.

Exercise 3.1

1 A pyramid has its vertex directly above the centre of its square base. The edges of the base are each 8 cm, and the vertical height is 10 cm. Find the angle between the slant face and the base, and the angle between a slant edge and the base.

2 A symmetrical pyramid stands on a square base of side 8 cm. The slant height of the pyramid is 20 cm. Find the angle between the slant edge and the base, and the angle between a slant face and the base.

3 A square board is suspended horizontally by four equal ropes attached to a point *P* directly above the centre of the board. Each rope has length 15 m and is inclined at an angle of 10° to the vertical. Calculate the length of the side of the square board.

4 A pyramid has its vertex directly above the centre of its square base. The edges of the base are each 6 cm, and the vertical height is 8 cm. Find the angle between two adjacent slant faces.

5 Find the angle that a main diagonal of a cube makes with the base. (Assume that the cube has sides of length 1 unit.) 6 A pylon is situated at a corner of a rectangular field with dimensions 100 m by 80 m. The angle subtended by the pylon at the furthest corner of the field is 10°. Find the angles subtended by the pylon at the other two corners of the field.

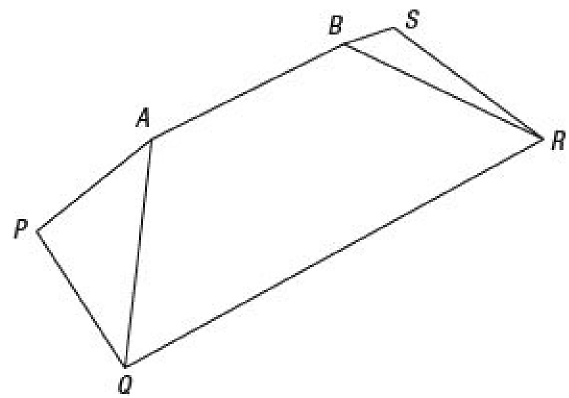
7 A regular tetrahedron has all its edges 8 cm in length. Find the angles that an edge makes with the base.

8 All the faces of a square-based pyramid of side 6 cm slope at an angle of 60° to the horizontal. Find the height of the pyramid, and the angle between a sloping edge and the base.

9 A vertical flag pole standing on horizontal ground has six ropes attached to it at a point 6 m from the ground. The other ends of the ropes are attached

to points on the ground that lie in a regular hexagon with sides 4 m. Find the angle that a rope makes with the ground.

10 The diagram shows a roof structure. *PQRS* is a horizontal rectangle. The faces *ABRQ*, *ABSP*, *APQ* and *BRS* all make an angle of 45° with the horizontal.



Find the angle made by the sloping edges with the horizontal.

Key ideas

- To solve a three-dimensional problem, you need to pick out two-dimensional right-angled triangles. Always draw a diagram.
- There are three main types of three-dimensional problem: pyramids, boxes and wedges.
- If the resulting triangles are not right-angled then look out for any symmetry or for the possibility of dividing a triangle into two right-angled triangles.
- In order to avoid errors never round until you reach the final answer – always store any intermediate values in the memory of your calculator.

4
Angles of any magnitude

In this chapter you will learn:

- *how to extend the definitions of sine, cosine and tangent to angles greater than 90° and less than 0°*
- *the shape of the graphs of sine, cosine and tangent*
- *the meaning of the terms 'period' and 'periodic'.*

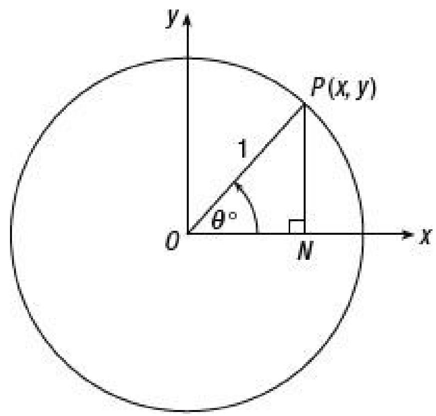
4.1 Introduction If, either by accident or by experimenting with your calculator, you have tried to find sines, cosines and tangents of angles outside the range from 0 to 90°, you will have found that your calculator gives you a value. But what does this value mean, and how is it used? This chapter gives some answers to those questions.

If you think of sine, cosine and tangent only in terms of ratios of ‘opposite’, ‘adjacent’ and ‘hypotenuse’, then it is difficult to give meanings to trigonometric ratios of angles outside the interval 0 to 180° – after all, angles of triangles have to lie within this interval. However, the definition of tangent given in Section 1.3 and the definition of sine and cosine in Section 2.2 both extend naturally to angles of any magnitude.

It is convenient to start with the sine and cosine.

4.2 Sine and cosine for any angle In Section 2.2, the following construction was given as the basis of the definition of sine and cosine.

Draw a circle with radius 1 unit, and centre at the origin *O*. Draw the radius *OP* at an angle $\theta°$ to the *x*-axis in an anticlockwise direction (see Figure 4.1). Let *P* have coordinates (x, y).



y
P (x, y)
1
θ°
O
N
x

Figure 4.1

Then sin $\theta°$ = y and cos $\theta°$ = x are the definitions of sine and cosine for any size of the angle $\theta°$.

The arrow labelling the angle $\theta°$ in Figure 4.1 emphasizes that angles are measured positively in the anticlockwise direction, and negatively in the clockwise direction.

It is useful to divide the plane into four quadrants, called 1, 2, 3 and 4, as shown in Figure 4.2.

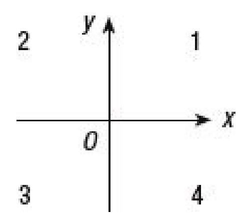


Figure 4.2

Then for any given angles such as 60°, 210° and −40°, you can associate a quadrant, namely, the quadrant in which the radius corresponding to the angle lies.

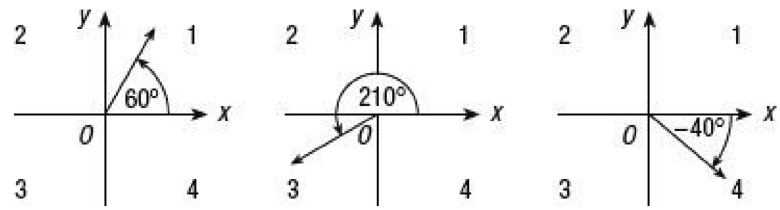


Figure 4.3

In Figure 4.3, you can see that 60° is in quadrant 1, and is called a first quadrant angle; 210° is a third quadrant angle; −40° is a fourth quadrant angle.

You can have angles greater than 360°. For example, you can check that 460° is a second quadrant angle, and −460° is a third quadrant angle.

The definition of sin $\theta°$, namely sin $\theta° = y$, shows that if $\theta°$ is a first or second quadrant angle, then the y-coordinate of P is positive so sin $\theta° > 0$; if $\theta°$ is a third or fourth quadrant angle, sin $\theta° < 0$.

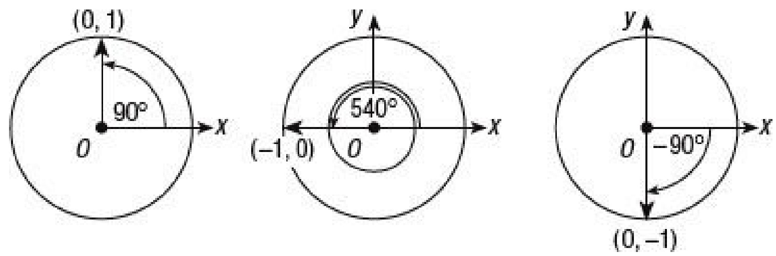
You can see from this definition, and from Figure 4.3, that sin $60° > 0$, that sin $210° < 0$ and that $\sin(-40°) < 0$; you can easily check these with your calculator.

Similarly, the definition of cos $\theta°$, namely cos $\theta° = x$, shows that if 0° is a first or fourth quadrant angle then the x-coordinate of P is positive so cos $\theta° > 0$; if $\theta°$ is a second or third quadrant angle, cos $\theta° < 0$.

You can also see that cos $60° > 0$, that cos $210° < 0$ and that $\cos(-40°) > 0$. Again, you can easily check these with your calculator.

SINE AND COSINE FOR MULTIPLES OF 90°

The easiest way to find the sine and cosine of angles such as 90°, 540° and −90° is to return to the definitions, that is sin $\theta° = y$ and cos $\theta° = x$ (see Figure 4.4).



(0, 1)
90°
x
O
y
540°
x
(−1, 0)
O
y
x
O
−90°
(0, −1)

Figure 4.4

Then you see from the left-hand diagram that the radius for 90° ends at (0, 1), so sin 90° = 1 and cos 90° = 0.

Similarly, the radius for 540° ends up at (−1, 0), so sin 540° = 0 and cos 540° = −1.

Finally, the radius for −90° ends up at (0, −1), so sin(−90°) = −1 and cos(−90°) = 0.

Once again, you can check these results with your calculator.

4.3 Graphs of sine and cosine functions As the sine and cosine functions are defined for all angles, you can draw their graphs.

Figure 4.5 shows the graph of $y = \sin \theta°$ drawn for values of $\theta°$ from −90 to 360.

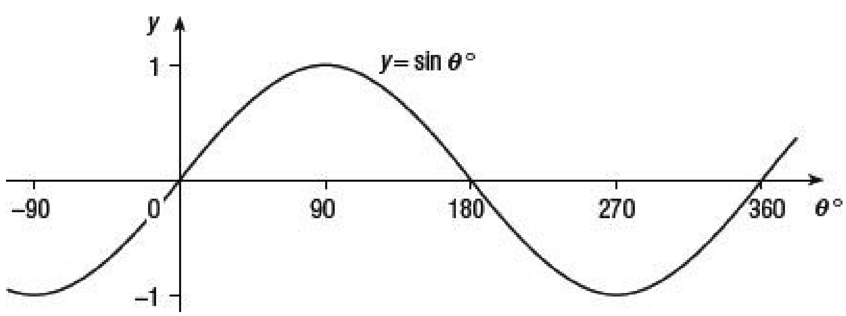


Figure 4.5

You can see that the graph of $y = \sin \theta°$ has the form of a wave. As it repeats itself every 360°, it is said to be **periodic,** with **period** 360°. As you would expect from Section 4.2, the value of $\sin \theta°$ is positive for first and second quadrant angles and negative for third and fourth quadrant angles.

Nugget

So $\sin 30° = \sin(30° + 360°) = \sin(30° + 2 \times 360°)$ and so on.

Also $\sin 30° = \sin(30° - 360°) = \sin(30° - 2 \times 360°)$ and so on.

Figure 4.6 shows the graph of $y = \cos \theta°$ drawn for values of $\theta°$ from -90 to 360.

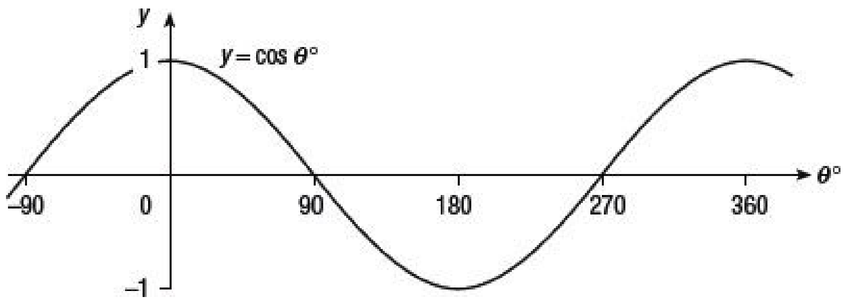


Figure 4.6

As you can see, the graph of $y = \cos\theta°$ also has the form of a wave. It is also **periodic,** with **period** 360°. The value of $\cos\theta°$ is positive for first and fourth quadrant angles and negative for second and third quadrant angles.

It is the wave form of these graphs and their periodic properties that make the sine and cosine so useful in applications. This point is taken further in physics and engineering.

Nugget

Notice that the graph of $y = \sin\theta°$ has rotational symmetry about the origin. So $\sin\theta° = -\sin(-\theta°)$ and, for example, $\sin 45° = -\sin(-45°)$.

The graph of $y = \cos\theta°$ is symmetrical about the y-axis. So $\cos\theta° = \cos(-\theta°)$, for example, $\cos 45° = \cos(-45°)$.

As the two graphs are just translations of each other it can be hard to remember which is which - here is a memory aid to help you: on the graph of $\sin\theta°$ there is an elongated S shape at the origin.

Exercise 4.1

In questions 1 to 8, use your calculator to find the following sines and cosines.

1 sin 130°
2 cos 140°
3 sin 250°
4 cos 370°
5 sin(−20)°
6 cos 1000°
7 sin 36000°
8 cos(−90)°

In questions 9 to 14, say in which quadrant the given angle lies.

9 200°
10 370°
11 (−300)°
12 730°
13 −600°
14 1000°

In questions 15 to 20, find the following sines and cosines without using your calculator **15** cos 0°
16 sin 180°
17 cos 270°
18 sin(−90)°
19 cos(−180)°
20 sin 450°

4.4 The tangent of any angle In Section 1.3 you saw that the definition of the tangent for

an acute angle was given by $\tan\theta° = \frac{y}{x}$. This definition is extended to all angles, positive and negative (see **Figure 4.7**).

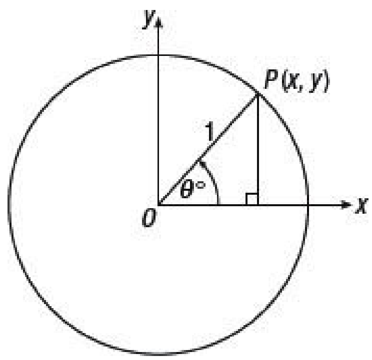


Figure 4.7

If the angle $\theta°$ is a first quadrant angle, tan $\theta°$ is positive. For a second quadrant angle, y is positive and x is negative, so tan $\theta°$ is negative. For a third quadrant angle, y and x are both negative, so tan $\theta°$ is positive. And for a fourth quadrant angle, y is negative and x is positive, so tan $\theta°$ is negative.

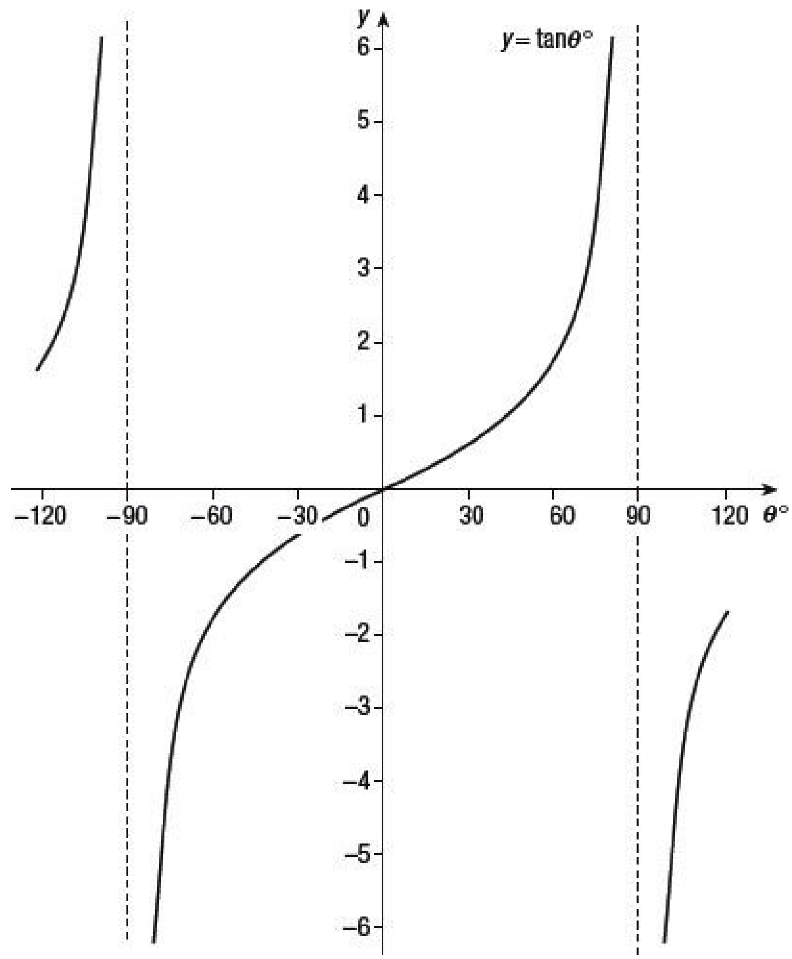
Nugget

To remember which function is positive in which quadrant use the memory aid '**A**ll **S**tudents **T**ake **C**offee'

2 Students	1 All		2 Sin	1 All
3 Take	4 Coffee		3 Tan	4 Cos

4.5 Graph of the tangent function Just as you can draw graphs of the sine and cosine functions, you can draw a graph of the tangent function. Its graph is shown in Figure 4.8.

You can see from Figure 4.8 that, like the sine and cosine functions, the tangent function is periodic, but with period 180°, rather than 360°.



y
y = tanθ°
6
5
4
3
2
1
0
−1
−2
−3
−4
−5
−6
−120
−90
−60
−30
30
60
90
120
θ°

Figure 4.8

You can also see that, for odd multiples of 90°, the tangent function is not defined. You cannot talk about tan 90°. It does not exist.

Nugget

Notice that the graph of $y = \tan \theta°$ has rotational symmetry about the origin. So $\tan \theta° = -\tan(-\theta°)$ and, for example, $\tan 60° = -\tan(-60°)$.

4.6 Sine, cosine and tangent There is an important relation between the sine, cosine and tangent, which you can deduce immediately from their definitions.

From the definitions

$$\sin\theta° = y,$$
$$\cos\theta° = x,$$
$$\tan\theta° = \frac{y}{x},$$

you can see that

$$\tan\theta° = \frac{\sin\theta°}{\cos\theta°}. \qquad \text{Equation 1}$$

Equation 1 will be used repeatedly throughout the remainder of the book.

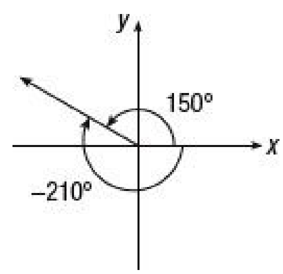
Exercise 4.2

In questions 1 to 4, use your calculator to find the following tangents.

1 tan 120°
2 tan(−30)°
3 tan 200°
4 tan 1000°
5 Attempt to find tan 90° on your calculator. You should find that it gives some kind of error message.
6 Calculate the value of $\frac{\sin 12°}{\cos 12°}$.
7 Calculate the value of $\frac{\cos 1000°}{\sin 1000°}$.

Key ideas

- You can find the values of sin θ°, cosθ°, and tan θ° for any value of θ° – this includes negative values and angles greater than 90°.
- The angle $\boldsymbol{\theta}$° is measured from the *x*-axis. Positive values of $\boldsymbol{\theta}$° are measured in an anticlockwise direction and negative in a clockwise direction from the *x*-axis.



- The graph of y = sin $\boldsymbol{\theta}$° is a wave with period 360°. The graph of y = sin $\boldsymbol{\theta}$° has a maximum value of 1 and a minimum value of −1, so −1 ≤ sin $\boldsymbol{\theta}$° ≥ 1 (see Figure 4.5).
- The graph of y = cos $\boldsymbol{\theta}$° is a wave with period 360°. The graph of y = cos $\boldsymbol{\theta}$° has a maximum value of 1 and a minimum value of −1, so −1 ≤ cos $\boldsymbol{\theta}$° ≤ 1 (see Figure 4.6).
- The graph of y = tan $\boldsymbol{\theta}$° has period 180°. tan90° is undefined and, likewise, the tangent of odd multiples of 90 is undefined (see Figure 4.8).
- $\tan\theta° = \dfrac{\sin\theta°}{\cos\theta°}$

  Note this equation is true for angles measured in radians – see Chapter 7.

5
Solving simple equations

In this chapter you will learn:

- *how to solve simple equations involving sine, cosine and tangent*
- *the meaning of ‘principal angle’*
- *how to use the principal angle to find all solutions of the equation.*

5.1 Introduction This chapter is about solving equations of the type sin $\theta°$ = 0.4, cos $\theta°$ = 0.2 and tan $\theta°$ = 0.3.

It is easy, using a calculator, to find the sine of a given angle. It is also easy, with a calculator, to find one solution of an equation such as sin $\theta°$ = 0.4. You use the $\sin^{-1}$ key and find $\theta°$ = 23.57…. So far so good.

The problem is that Figure 5.1 shows there are many angles, infinitely many in fact, for which sin $\theta°$ = 0.4. You have found one of them – how do you find the others from the angle that you have found?

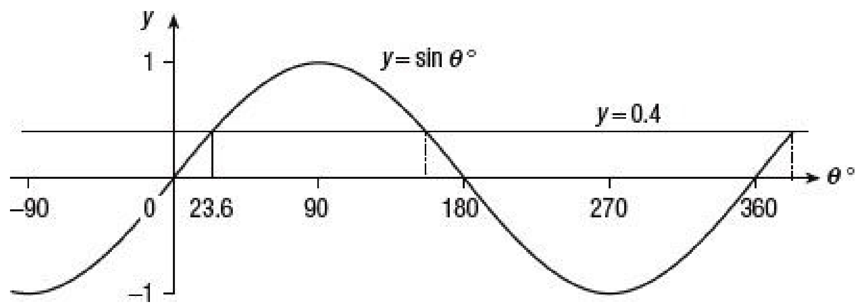


Figure 5.1

Figure 5.1 shows that there is another angle lying between 90° and 180° satisfying the equation sin θ° = 0.4, and then (infinitely) many others, repeating every 360°.

5.2 Solving equations involving sines

PRINCIPAL ANGLES

The angle given by your calculator when you press the $\sin^{-1}$ key is called the **principal angle.**

For the sine function the principal angle lies in the interval $-90 \le \theta° \le 90$.

If you draw the graph of $y = \sin^{-1} x$ using your calculator values you get the graph shown in Figure 5.2.

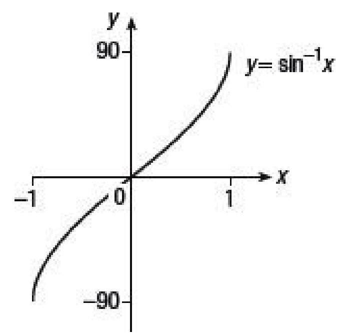


Figure 5.2

So the question posed in Section 5.1 is: ‘Given the principal angle for which $\sin \theta° = 0.4$, how do you find all the other angles?’

Look at the sine graph in Figure 5.1. Notice that it is symmetrical about the 90° point on the θ-axis. This shows that for any angle $\alpha°$

$$\sin(90 - \alpha)° = \sin(90 + \alpha)°. \qquad \text{Equation 1}$$

If you write $x = 90 - \alpha$, then $\alpha = 90 - x$, so $90 + \alpha = 180 - x$. Equation 1 then becomes, for any angle $\alpha°$, $\sin \theta° = \sin(180 - \theta)°.$ Equation 2

Nugget

For example, $\sin 30° = \sin 150°$ since $180° - 30° = 150°$. This equation is true when $\alpha°$ is negative. So $\sin(-20°) = \sin 200°$ since $180° - (-20)° = 200°$.

Equation 2 is the key to solving equations which involve sines.

Returning to the graph of $y = \sin\theta°$ in Figure 5.1, and using Equation 2, you can see that the other angle between 0 and 180 with $\sin\theta° = 0.4$ is $180 - \sin^{-1}0.4 = 180 - 23.57\ldots = 156.42\ldots°$.

Now you can add (or subtract) multiples of 360θ° to find all the other angles solving $\sin\theta° = 0.4$, and obtain $\theta = 23.57, 156.42, 383.57, 516.42, \ldots°$

correct to two decimal places.

SUMMARY

To solve an equation of the form $\sin x° = c$ where c is given:

- find the principal angle
- use Equation 2 to find another angle for which $\sin x° = c$
- add or subtract any multiple of 360.

Example 5.1

Solve the equation sin $x°$ = −0.2, giving all solutions in the interval from −180 to 180.

The principal angle is −11.54°.

From Equation 2, 180 −(−11.54) = 191.54° is also a solution, but this is outside the required range. However, as you can add and subtract any multiple of 360, you can find the solution between −180 and 180 by subtracting 360.

Therefore the required solution is

$$191.54 - 360 = -168.46°.$$

Therefore the solutions are −168.46° and −11.54°. ∎

Example 5.2

Solve the equation $\sin 2x° = 0.5$, giving all solutions from 0 to 360°.

Start by letting $y = 2x$. Then you have first to solve for y the equation $\sin y° = 0.5$. Note also that if x lies between 0 and 360, then y, which is $2x$, lies between 0 and 720.

The principal angle for the solution of $\sin y° = 0.5$ is $\sin^{-1} 0.5 = 30°$.

From Equation 2, $180 - 30 = 150°$ is also a solution.

Adding multiples of 360 shows that 390° and 510° are also solutions for y.

Thus $y = 2x = 30°, 150°, 390°, 510°$

so $\quad x = 15°, 75°, 195°, 255°$. ∎

Example 5.3

Find the smallest positive root of the equation $\sin(2x + 50)° = 0.1$.

Substitute $y = 2x + 50$, so you solve $\sin y° = 0.1$.

From $y = 2x + 50$ you find that $x = \frac{1}{2}(y - 50)$

As $x > 0$ for a positive solution $\frac{1}{2}(y - 50) > 0$, so $y > 50$.

The principal angle solving $\sin y° = 0.1$ is 5.74°.

From Equation 2, $180 - 5.74 = 174.26°$ is the first solution greater than 50.

As $$x = \frac{1}{2}(y - 50),$$

$$x = \frac{1}{2}(174.26 - 50) = 62.13°. \blacksquare$$

Exercise 5.1

In questions 1 to 8, find the solutions of the given equation in the interval from 0 to 360.

1 $\sin \theta° = 0.3$
2 $\sin \theta° = 0.45$
3 $\sin \theta° = 0$
4 $\sin \theta° = 1$
5 $\sin \theta° = -1$
6 $\sin \theta° = -0.1$
7 $\sin \theta° = -0.45$
8 $\sin \theta° = -0.5$

In questions 9 to 16, find the solutions of the given equation in the interval from −180 to 180.

9 $\sin \theta° = -0.15$
10 $\sin \theta° = -0.5$
11 $\sin \theta° = 0$
12 $\sin \theta° = 1$
13 $\sin \theta° = -1$
14 $\sin \theta° = 0.9$
15 $\sin \theta° = -0.9$
16 $\sin \theta° = -0.766$

In questions 17 to 24, find the solutions of the given equation in the interval from 0 to 360.

17 $\sin 2x° = 0.5$
18 $\sin 2\theta° = 0.45$
19 $\sin 3\theta° = 0$
20 $\sin 2\theta° = -1$

21 $\sin \frac{1}{2}\theta° = 0.5$

22 $\sin\frac{1}{2}\theta° = 1$

23 $\sin 3\theta° = -0.5$

24 $3\sin 2x° = 2$

25 The height h in metres of the water in a harbour t hours after the water is at its mean level is given by $h = 6 + 4\sin(30t)°$. Find the first positive value of t for which the height of the water first reaches 9m.

26 The length l hours of a day t days after the beginning of the year is given approximately by $l = 12 - 6\cos\left(\frac{360}{365}t\right)°$

Find the approximate number of days per year that the length of day is longer than 15 hours.

5.3 Solving equations involving cosines To solve an equation of the form cos $\theta°$ = 0.2 it is helpful to look at the graph of the cosine function in Figure 5.3.

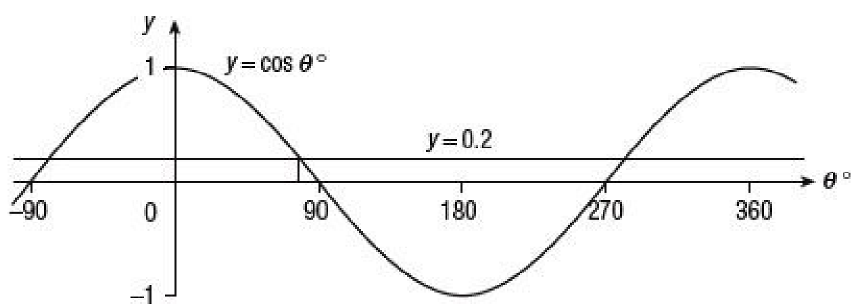


Figure 5.3

For the cosine the principal angle is in the interval $0 \le \theta° \le 180$.

For the equation $\cos \theta° = 0.2$, the principal angle is 78.46…°.

If you draw the graph of $y = \cos^{-1} x$ using your calculator values, you get the graph shown in Figure 5.4.

The symmetry of the cosine graph in Figure 5.3 shows that

$\cos(-\theta°) = \cos \theta°$. Equation 3

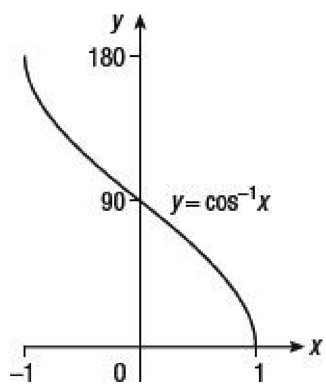


Figure 5.4

If you use Equation 3, you find that cos(−78.46…)° = 0.2.

If the interval from which you need the solution is from −180 to 180, you have two solutions, −78.46…° and 78.46…°.

If you need solutions between 0 and 360, you can subtract the first solution from 360 and obtain 78.46° and 281.54°, correct to two decimal places.

SUMMARY

To solve an equation of the form cos θ° = c where c is given: • find the principal angle

- use Equation 3 to find another angle for which cos θ° = c
- add or subtract any multiple of 360.

Example 5.4

Solve the equation $\cos \boldsymbol{\theta}° = -0.1$ giving all solutions in the interval -180 to 180, correct to two decimal places.

For $\cos \theta° = -0.1$, the principal angle is $95.74°$.

Using Equation 3, the other angle in the required interval is $-95.74°$.

The solutions, correct to two decimal places, are $-95.74°$ and $95.74°$. ∎

Example 5.5

Find all the solutions of the equation 2 cos $3x°$ = 1 in the interval 0 to 360.

The equation 2 cos $3x°$ = 1 can be written in the form $\cos y° = \frac{1}{2}$, where $y = 3x$. If x lies in the interval 0 to 360, then y, which is $3x$, lies in the interval 0 to 1080.

The principal angle for the solution of $\cos y° = \frac{1}{2}$ is $\cos^{-1} \frac{1}{2} = 60$.

From Equation 3, −60 is also a solution of $\cos y° = \frac{1}{2}$.

Adding multiples of 360 shows that 300, 420, 660, 780 and 1020 are also solutions for y.

Thus $y = 3x$ = 60, 300, 420, 660, 780, 1020

so x = 20, 100, 140, 220, 260, 340. ▮

5.4 Solving equations involving tangents To solve an equation of the form tan $\theta°$ = 0.3 it is helpful to look at the graph of the tangent function, Figure 4.8.

The symmetry of the tangent graph shows that

$$\tan \theta° = \tan(180 + \theta)°. \qquad \text{Equation 4}$$

For the tangent the principal angle is in the interval $-90 \le \theta \le 90$. For the equation tan $\theta°$ = 0.3, the principal angle is 16.69….

Using Equation 4, you find that the other angle between 0 and 360 satisfying tan $\theta°$ = 0.3 is 180 + 16.69… = 196.69….

SUMMARY

To solve an equation of the form tan $\theta^\circ = c$ where c is given: • find the principal angle

- use Equation 4 to find another angle for which tan $\theta^\circ = c$
- add or subtract any multiple of 180.

Nugget

Remember that the graph of $y = \tan \theta°$ has a period of 18θ°. In other words it repeats every 180°, so once you have the principal angle you can find all the other solutions by adding or subtracting multiples of 180°.

Example 5.6

Solve the equation tan $\theta°$ = −0.6 giving all solutions in the interval −180 to 180 correct to two decimal places.

For tan $\theta°$ = −0.6, the principal angle is −30.96.

Using Equation 4, the other angle in the required interval, −180 to 180 is 180 + (−30.96) = 149.04.

Therefore the solutions, correct to two decimal places, are −30.96 and 149.04.

Example 5.7

Find all the solutions of the equation $\tan \frac{3}{2}\theta° = -1$ in the interval 0 to 360.

The equation $\tan \frac{3}{2}\theta° = -1$ can be written in the form $\tan y° = -1$, where $y = \frac{3}{2}\theta$. If θ lies in the interval 0 to 360, then y, which is $\frac{3}{2}\theta$, lies in the interval 0 to 540.

The principal angle for the solution of $\tan y° = -1$ is $\tan^{-1}(-1) = -45$.

From Equation 4, 135 is also a solution of $\tan y° = -1$.

Adding multiples of 180 shows that 315 and 495 are also solutions for y in the interval from 0 to 540.

Thus $\quad y° = \frac{3}{2}\theta° = 135, 315, 495$

so $\quad \theta° = 90, 210, 330.$ ▮

Exercise 5.2

In questions 1 to 10, find all the solutions to the given equation in the interval 0 to 360 inclusive.

1 $\cos\theta° = -\frac{1}{3}$

2 $\tan x° = 2$

3 $\cos\alpha° = \frac{3}{4}$

4 $\tan\beta° = -0.5$

5 $\cos 2\theta° = \frac{1}{2}$

6 $\tan 2\theta° = 1$

7 $\cos \frac{1}{2}\theta° = -0.2$

8 $\tan \frac{1}{3}x° = 1.1$

9 $\cos 2\theta° = -0.766$

10 $\tan 2x° = -0.1$

In questions 11 to 16, find all the solutions to the given equation in the interval −180 to 180 inclusive.

11 $\cos 2x° = -0.3$

12 $\tan 2x° = -0.5$

13 $\sin 2\theta° = 0.4$

14 $\cos \frac{2}{3}x° = 0.5$

15 $\tan \frac{3}{2}x° = 1$

16 $\sin \frac{2}{3}x° = -0.5$

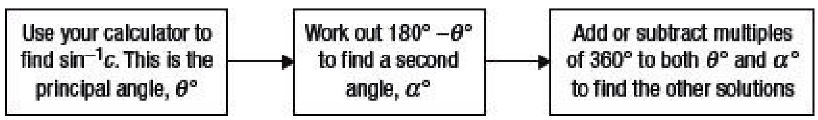
17 The height h in metres of water in a harbour above low tide is given by the equation $h = 14 - 10\cos(30t)°$ where t is measured in hours from midday. A ship can enter the harbour when the water is greater than 20m. Between what times can the boat first enter the harbour?

Key ideas

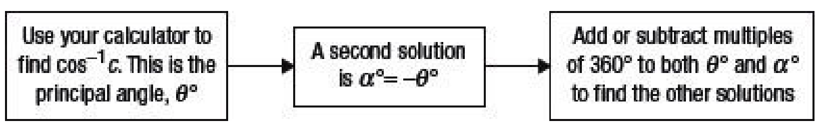
- The **principal angle** is the angle that your calculator gives you.

Function	Interval for principal angle
$\sin\theta°$	$-90° \le \theta \le 90°$
$\cos\theta°$	$0° \le \theta \le 180°$
$\tan\theta°$	$-90° < \theta < 90°$

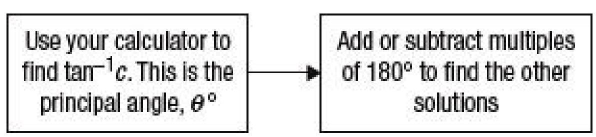
- To solve $\sin\theta° = c$



- To solve $\cos\theta° = c$



- To solve $\tan\theta° = c$



- $\sin(90 - \theta)° = \sin(90 + \theta)°$
- $\sin\theta° = \sin(180 - \theta)°$
- $\cos(-\theta) = \cos\theta$

  Note that the above equation is also true for angles measured in radians – see Chapter 7.

- $\tan \theta° = \tan(180 + \theta)°$

7
Radians

In this chapter you will learn:

- *that you can measure an angle in radians as an alternative to degrees*
- *the formulae for length of a circular arc and the area of a circular sector*
- *how to convert from radians to degrees and vice versa.*

7.1 Introduction

Who decided that there should be 360 degrees in a full circle, and therefore 90 degrees in a right angle? It is not the answer to this question which is important − it was actually the Babylonians − it is the fact that the question exists at all. Someone, somewhere did make the decision that the unit for angle should be the degree as we now know it. However, it could just have equally been 80 divisions which make a right angle or 100 divisions. So it seems worth asking, is there a best unit for measuring angle? Or is there a better choice for this unit than the degree? It turns out that the answer is yes. A better unit is the radian.

7.2 Radians

A **radian** is the angle subtended at the centre of a circle by a circular arc equal in length to the radius (see Figure 7.1).

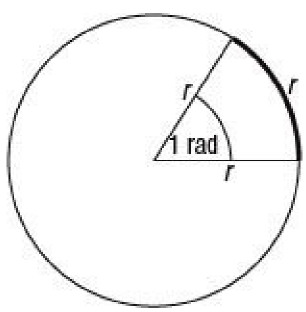


Figure 7.1

The angle of 1 radian is written as 1 rad, but if no units are given for angles you should assume that the unit is radians.

If you are using radians with a calculator, you will need to make sure that the calculator is in radian mode. If necessary, look up how to use radian mode in the manual.

7.3 Length of a circular arc The right-hand diagram in Figure 7.2 shows a circle of radius r cm with an angle of *θ* rad at the centre. The left-hand diagram shows a circle with the same radius but with an angle of 1 rad at the centre.

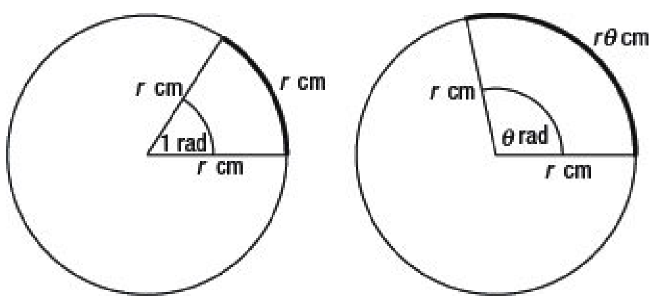


Figure 7.2

Look at the relationship between the left-and right-hand diagrams. As the angle at the centre of the circle in the left-hand diagram has been multiplied by a factor $\boldsymbol{\theta}$ to get the right-hand diagram, so has the arc length. The new arc length is therefore $\boldsymbol{\theta}$ × the original arc length r cm and therefore $r\theta$ cm.

If you call the arc length s cm, then

$$s = r\theta. \qquad \text{Equation 1}$$

Nugget

The equation $s = r\boldsymbol{\theta}$ is only true when $\boldsymbol{\theta}$ is in radians. Rearranging this equation gives $\boldsymbol{\theta} = \frac{s}{r}$. As the units of length for *s* and *r* will cancel out in this equation when an angle is measured in radians it is a pure number and doesn't have any units. However, sometimes 'rads' or a superscript c, ^{c}, is used to show that an angle is in radians.

Example 7.1

O is the centre of a circle of radius 3 cm. The points *A* and *B* lie on its circumference and angle *AOB* = 2 rad. Find the length of the perimeter of the segment bounded by the arc *AB* and the chord *AB*.

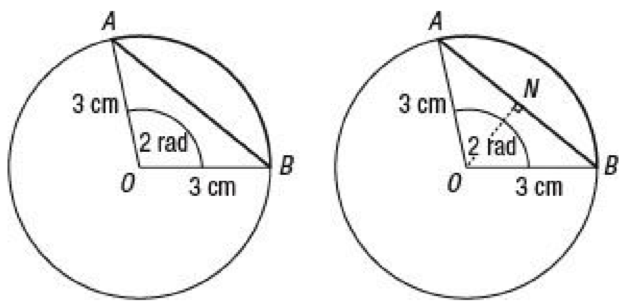


Figure 7.3

The left-hand diagram in Figure 7.3 shows this situation. The perimeter must be found in two sections, the arc *AB* and the chord *AB*.

The length of the arc *AB* is given by using Equation 1,

$$\text{length of arc } AB = 3 \times 2 = 6 \text{ cm.}$$

To find the length of the chord *AB*, drop the perpendicular from *O* to *AB* meeting *AB* at *N*, shown in the right-hand part of Figure 7.3.

Then $$AB = 2AN = 2 \times (3 \sin 1) = 5.05.$$

The perimeter is then given by

$$\text{perimeter} = \text{arc } AB + AB = (6 + 5.05) \text{ cm} = 11.05 \text{ cm.} \blacksquare$$

7.4 Converting from radians to degrees Consider the case when the arc of a circle of radius *r* cm is actually the complete circumference of the circle. In this case, the arc length is $2\pi r$ cm.

Suppose that the angle at the centre of this arc is θ rad. Then, using Equation 1, the length of the arc is $r\theta$ cm.

Then it follows that $2\pi r = r\theta$, so that the angle at the centre is 2π rad.

But, as the angle at the centre of the circle is 360°,

$$2\pi \text{ rad} = 360°.$$

Therefore $$\pi \text{ rad} = 180°.$$

This equation, π rad = 180°, is the one you should remember when you need to change from degrees to radians, and vice versa.

In many cases, when angles such as 45° and 60° are given in radians they are given as multiples of π. That is $\frac{1}{4}\pi \text{ rad} = 45°$ and $\frac{1}{3}\pi \text{ rad} = 60°$.

You can also work out 1 radian in degrees from the equation π rad = 180°. You find that $1 \text{ rad} = 57.296°$.

This equation is very rarely used in practice. When you need to convert radians to degrees or vice versa, use the fact that π rad = 180° and use either $\frac{180}{\theta}$ or $\frac{\pi}{180}$ as a conversion factor.

Thus, for example,

$$10° = 10 \times \frac{\pi}{180} \text{ rad} = \frac{1}{18}\pi \text{ rad}.$$

7.5 Area of a circular sector Figure 7.4 shows a shaded sector of a circle with radius *r* units and an angle at the centre of *θ* rad. Let the area of the shaded region be *A* units^2.

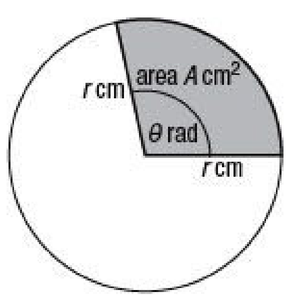


Figure 7.4

The area A units^2 is a fraction of the area of the whole circle. As the area of the whole circle is πr^2 units^2, and the angle at the centre is 2π rad, when the angle at the centre is $\boldsymbol{\theta}$ rad, the shaded area is a fraction $\frac{\theta}{2\pi}$ of the total area of the circle.

Therefore the area, in units^2, of the shaded sector is

$$\frac{\theta}{2\pi} \times \pi r^2 = \frac{1}{2} r^2 \theta. \qquad \text{Equation 2}$$

Nugget

If you wanted to work out the area of a quarter circle you would find $\frac{1}{4} \times \pi r^2$ or $\frac{\pi/2}{2\pi} \times \pi r^2$.

Example 7.2

O is the centre of a circle of radius 3 cm. The points *A* and *B* lie on its circumference and angle *AOB* = 2 rad. Find the area of the segment bounded by the arc *AB* and the chord *AB*.

Figure 7.5 shows this situation. The area of the segment must be found by finding the area of the whole sector *OAB*, and then subtracting the area of the triangle *OAB*.

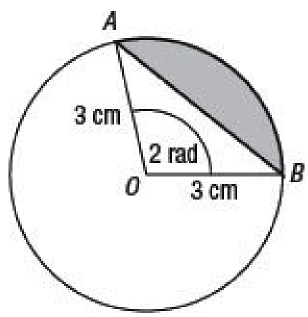


Figure 7.5

The area A cm^2 of the sector OAB is given by using Equation 2, $A = \frac{1}{2}r^2\theta = \frac{1}{2} \times 3^2 \times 2 = 9$.

To find the area of triangle OAB, use the formula $\frac{1}{2}ab \sin C$, given on page 71.

Then

$$\text{area of triangle } OAB = \tfrac{1}{2}ab \sin C = \tfrac{1}{2} \times 3^2 \times \sin 2 = 4.092.$$

The area of the shaded segment is then given by

$$\begin{aligned} \text{area of segment} &= \text{area of sector } OAB - \text{area of triangle } OAB \\ &= (9 - 4.092)\ \text{cm}^2 = 4.908\ \text{cm}^2. \end{aligned}$$ ∎

Exercise 7.1

In questions 1 to 6, write down the number of degrees in each of the angles that are given in radians.

1 $\frac{1}{3}\pi$

2 $\frac{1}{12}\pi$

3 $\frac{3}{2}\pi$

4 $\frac{2}{3}\pi$

5 $\frac{3}{4}\pi$

6 4π

In questions 7 to 1 2, find the values of the given ratios.

7 $\sin \frac{1}{5}\pi$

8 $\cos \frac{1}{8}\pi$

9 $\sin \frac{1}{10}\pi$

10 $\cos \frac{3}{8}\pi$

11 $\sin(\frac{1}{3}\pi + \frac{1}{4}\pi)$

12 $\sin \frac{1}{6}\pi$

13 Give the angle 0.234 rad in degrees correct to two decimal places.

In questions 14 to 17, express the following angles in radians, using fractions of π.

14 15°

15 72°

16 66°

17 105°

18 Find in radians the angle subtended at the centre of a circle of radius 2.4 cm by a circular arc of length 11.4 cm.

19 Find the length of the circular arc that subtends an angle of 0.31 rad at the centre of a circle of radius 3.6 cm.

20 Find the area of the circular sector that subtends an arc of 2.54 rad at the centre of a circle of radius 2.3 cm.

21 Find in radians the angle that a circular sector of area 20 cm^2 subtends at the centre of a circle of radius 5 cm.

22 A circular arc is 154 cm long and the radius of the arc is 252 cm. Find the angle subtended at the centre of the circle, in radians and degrees.

23 The angles of a triangle are in the ratio of 3 : 4 : 5. Express them in radians.

24 A chord of length 8 cm divides a circle of radius 5 cm into two parts. Find the area of each part.

25 Two circles each of radius 4 cm overlap, and the length of their common chord is also 4 cm. Find the area of the overlapping region.

26 A new five-sided coin is to be made in the shape of Figure 7.6.

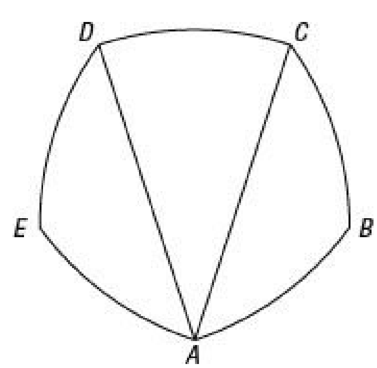
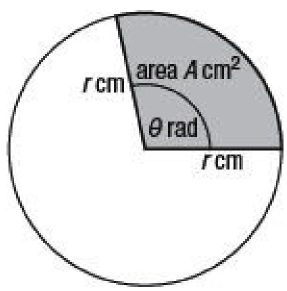


Figure 7.6

The point *A* on the circumference of the coin is the centre of the arc *CD*, which has a radius of 2 cm. Similarly *B* is the centre of the arc *DE*, and so on. Find the area of one face of the coin.

Key ideas

- A **radian** is the angle subtended at the centre of a circle by a circular arc equal in length to the radius.
- $s = r\theta$ where s = arc length, r = radius and $\boldsymbol{\theta}$ is the angle in radians.
- π radians = 180°.
- To convert from degrees to radians, multiply by $\frac{\pi}{180°}$.
- To convert from radians to degrees, multiply by $\frac{180°}{\pi}$.
- The area of a sector is $A = \frac{1}{2}r^2\boldsymbol{\theta}$.



8
Relations between the ratios

In this chapter you will learn:

- *some relations between the sine and cosine of an angle*
- *the trigonometric form of Pythagoras's theorem*
- *the meaning of secant, cosecant and cotangent.*

8.1 Introduction

$$\sin(90-\theta)^\circ = \cos\theta^\circ,$$

$$\cos(90-\theta)^\circ = \sin\theta^\circ,$$

In Section 2.2 you saw that for any angle θ°: $\sin^2\theta + \cos^2\theta = 1.$

The third of these relations is a form of Pythagoras's theorem, and it sometimes goes by that name.

In Section 4.6 you saw that

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

$$\sin\theta^\circ = \sin(180-\theta)^\circ,$$

$$\cos\theta^\circ = \cos(-\theta)^\circ,$$

In Sections 5.2 to 5.4 you saw that $\tan\theta^\circ = \tan(180+\theta)^\circ.$

In this chapter, you will explore these and other relations, as well as meeting the new ratios secant, cosecant and cotangent.

You will also learn to solve a wider variety of trigonometric equations, using these rules to help.

Some of the relations given above hold whatever units are used for measuring angles. Examples are $\sin^2\theta^\circ + \cos^2\theta^\circ = 1$ and $\frac{\sin\theta}{\cos\theta} = \tan\theta$. When this is the case no units for angle are given.

However, for some of the relations the angles must be measured in degrees for the relation to be true. This is the case for $\sin(90-\theta)^\circ = \cos\theta^\circ$ and $\sin(180-\theta)^\circ = \sin\theta^\circ$. In these cases, degree signs will be used.

8.2 Secant, cosecant and cotangent The three relations secant, cosecant and cotangent, usually abbreviated to sec, cosec and cot, are defined by the rules

$$\sec\theta = \frac{1}{\cos\theta},$$
$$\operatorname{cosec}\theta = \frac{1}{\sin\theta},$$
$$\cot\theta = \frac{1}{\tan\theta},$$

provided cos θ, sin θ and tan θ are not zero.

Nugget

These are sometimes called reciprocal trigonometric functions. You can use the third letter of each function to help you remember them:
$\mathrm{se\underline{c}}\,\theta = \frac{1}{\mathrm{\underline{c}os}\,\theta}$, $\mathrm{co\underline{s}ec}\,\theta = \frac{1}{\mathrm{\underline{s}in}\,\theta}$, and $\mathrm{co\underline{t}}\,\theta = \frac{1}{\mathrm{\underline{t}an}\,\theta}$.

In the early part of the twentieth century, tables were used to find values of the trigonometric ratios. There used to be tables for sec, cosec and cot, but these have now all but disappeared, and if you want their values from a calculator, you need to use the definitions above.

You can write Pythagoras's theorem, $\sin^2 \theta + \cos^2 \theta = 1$, in terms of these new ratios.

Divide every term of $\sin^2 \theta + \cos^2 \theta = 1$ by $\cos^2 \theta$ to obtain
$\frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$,

which simplifies to $\tan^2 \theta + 1 = \sec^2 \theta$.

Similarly, by dividing $\sin^2 \theta + \cos^2 \theta = 1$ by $\sin^2 \theta$ you can show that
$1 + \cot^2 \theta = \mathrm{cosec}^2\, \theta$.

Example 8.1

Let cos $x = \frac{3}{5}$. Find the possible values of tan *x*, sec *x* and cosec *x*. Using Pythagoras's theorem, $\sin^2 x + \cos^2 x = 1$, the value of sin *x* is $\sin x = \pm\sqrt{1-\left(\frac{3}{5}\right)^2} = \pm\frac{4}{5}$.

Then, using $\tan x = \frac{\sin x}{\cos x}$, $\tan x = \frac{\sin x}{\cos x} = \frac{\pm\frac{4}{5}}{\frac{3}{5}} = \pm\frac{4}{3}$.

As $\sec x = \frac{1}{\cos x}$, $\sec x = \frac{1}{\frac{3}{5}} = \frac{5}{3}$.

As $\operatorname{cosec} x = \frac{1}{\sin x}$, $\operatorname{cosec} x = \frac{1}{\pm\frac{4}{5}} = \pm\frac{5}{4}$.

Example 8.2

Solve the equation $3\cos^2\theta° = 1 - 2\sin\theta°$ giving solutions in the interval −180 to 180.

If you substitute $\cos^2\theta° = 1 - \sin^2\theta°$ you obtain an equation in which every term, except the constant, is a multiple of a power of $\sin\theta°$, that is, a polynomial equation in $\sin\theta°$. You can solve this by the usual methods.

Thus
$$\begin{aligned} 3\cos^2\theta° &= 1 - 2\sin\theta° \\ 3(1 - \sin^2\theta°) &= 1 - 2\sin\theta° \\ 3\sin^2\theta° - 2\sin\theta° - 2 &= 0. \end{aligned}$$

This is a quadratic equation in $\sin\theta$. Using the quadratic equation formula,
$$\sin\theta° = \frac{2 \pm \sqrt{28}}{6},$$
so
$$\sin\theta° = 1.215\ldots \text{ or } \sin\theta° = -0.5485\ldots.$$

The first of these is impossible. The principal angle corresponding to the second is −33.27°.

Then 180° − (−33.27°) = 213.27° is also a solution (see Section 5.2). But this is outside the required range, so subtract 360 to get $213.27 - 360 = -146.73$.

Thus the solutions are −33.27 and −146.73. ∎

Nugget

You may find solving the equation $3 \sin^2 \theta° - 2 \sin \theta° - 2 = 0$ easier if you let $y = \sin \theta$ and then solve $3y^2 - 2y - 2 = 0$. If you use this method don't forget to then solve $\sin \theta° = 1.215\ldots$ and $\sin \theta° = -0.5485\ldots$.

Example 8.3

Solve the equation cos $\theta°$ = 1 + sec $\theta°$ giving all the solutions in the interval from 0 to 360.

Notice that if you write sec $\theta° = \frac{1}{\cos\theta°}$ all the terms in the equation will involve cos $\theta°$. Therefore

$$
\begin{aligned}
\cos\theta° &= 1 + \frac{1}{\cos\theta°} \\
\cos^2\theta° &= \cos\theta° + 1 \\
\cos^2\theta° - \cos\theta° - 1 &= 0 \\
\cos\theta° &= \frac{1 \pm \sqrt{5}}{2} = 1.618\ldots \text{ or } -0.618\ldots .
\end{aligned}
$$

The first of these solutions is impossible. The principal angle corresponding to the second is 128.17.

Then 360 − 128.17 = 231.83 is also a solution (see Section 5.4). Thus the solutions are 128.17 and 231.83. ▮

Example 8.4

Solve the equation 2 sec $\theta°$ = 2 + tan^2 $\theta°$ giving all the solutions in the interval from −180 to 180.

If you use Pythagoras's theorem in the form tan^2 $\theta°$ = sec^2 $\theta°$ − 1 to substitute for tan^2 $\theta°$ all the terms in the equation will involve sec $\theta°$.

$$\text{Therefore} \qquad \begin{aligned} 2\sec\theta° &= 2+\tan^2\theta° \\ &= 2+(\sec^2\theta°-1) \end{aligned}$$

$$\begin{aligned} \sec^2\theta° - 2\sec\theta° + 1 &= 0 \\ (\sec\theta° - 1)^2 &= 0 \\ \sec\theta° &= 1. \end{aligned}$$

$$\text{Therefore} \qquad \cos\theta° = \frac{1}{\sec\theta°} = 1$$

$$\text{so} \qquad \theta = 0.$$

Thus the solution is 0. ∎

Example 8.5

Solve the equation tan θ = 2 sin θ giving all solutions between $-\pi$ and π inclusive.

It is often useful to write equations in terms of the sine and cosine functions, because there are so many more simplifying equations which you can use.

So
$$\tan\theta = 2\sin\theta$$
$$\frac{\sin\theta}{\cos\theta} = 2\sin\theta.$$

Multiplying both sides of this equation by cos θ gives

$$\sin\theta - 2\sin\theta\cos\theta = 0.$$

Factorizing,
$$\sin\theta(1-2\cos\theta) = 0$$

So
$$\sin\theta = 0 \text{ or } \cos\theta = \tfrac{1}{2}.$$

Using the methods of Sections 5.2 and 5.3, solve these two equations for θ.

When sin θ = 0, $\theta = -\pi$ or 0 or π; when $\cos\theta = \frac{1}{2}, \theta = -\frac{1}{3}\pi$ or $\frac{1}{3}\pi$.

Therefore $\theta = -\pi, -\frac{1}{3}\pi, 0, \frac{1}{3}\pi$ or π. ∎

Nugget

The aim when solving a trigonometric equation is to rewrite it – using relationships such as $\sin^2\theta + \cos^2\theta = 1$ or $\tan\theta = \frac{\sin\theta}{\cos\theta}$ – into a simpler equation (often one that is just in terms of sin, cos or tan) which you can solve directly.

Exercise 8.1

1 Find the value of $\cos\theta$ given that $\sin\theta = 0.8192$, and that θ is obtuse.

2 Find the possible values of $\tan\theta$ given that $\cos\theta = 0.3$.

3 Find the possible values of $\sec\theta$ when $\tan\theta = 0.4$.

4 The angle α is acute, and $\sec\alpha = k$. Find in terms of k the value of $\operatorname{cosec}\alpha$.

5 Let $\tan\theta° = t$, where θ lies between 90 and 180. Calculate, in terms of t, the values of $\sec\theta°$, $\cos\theta°$ and $\sin\theta°$.

6 Let $\sec\theta = s$, where θ is acute. Find the values of $\cot\theta$ and $\sin\theta$ in terms of s.

In questions 7 to 15, solve the given equation for $\theta°$, giving your answers in the interval from −180 to 180.

7 $\cos^2\theta° = \frac{1}{4}$

8 $2\sin\theta° = \operatorname{cosec}\theta°$

9 $2\sin^2\theta° - \sin\theta° = 0$

10 $2\cos^2\theta° = 3\sin\theta° + 2$

11 $\tan\theta° = \cos\theta°$

12 $\sin\theta° = 2\cos\theta°$

13 $2\sec\theta° = \operatorname{cosec}\theta°$

14 $5(1 - \cos\theta°) = 4\sin^2\theta°$

15 $4\sin\theta°\cos\theta° + 1 = 2(\sin\theta° + \cos\theta°)$

Key ideas

- The relationships $\cos\theta = \cos(-\theta)$ and $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ are true for all angles in both radians and degrees.
- You can write Pythagoras's theorem as:

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

  These relationships are true for all angles in both radians and degrees.

- $\sec\theta = \dfrac{1}{\cos\theta}$ provided $\cos\theta \neq 0$

  $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$ provided $\sin\theta \neq 0$

  $\cot\theta = \dfrac{1}{\tan\theta}$ provided $\tan\theta \neq 0$

  These relationships are true for all angles in both radians and degrees.

- To solve $\sin\theta = c$ for θ in radians:

  Use your calculator, in radians mode, to find $\sin^{-1}c$. This is the principal angle, θ → Work out $\pi - \theta$ to find a second angle, α → Add or subtract multiples of 2π to both θ and α to find the other solutions

- To solve $\cos\theta = c$ for θ in radians:

  Use your calculator, in radians mode, to find $\cos^{-1}c$. This is the principal angle, θ → A second solution is $\alpha = -\theta$ → Add or subtract multiples of 2π to both θ and α to find the other solutions

- To solve $\tan\theta = c$:

  Use your calculator to find $\tan^{-1}c$. This is the principal angle, θ → Add or subtract multiples of π to find the other solutions

9
Ratios and compound angles

In this chapter you will learn:

- *how to find the values of sin*(A + B), *cos* (A + B) *and tan*(A + B) *knowing the values of the sine, cosine and tangent of* A *and* B
- *how to modify these formulae for sin* (A − B), *cos* (A − B) *and tan*(A − B)
- *how to find the values of sin*2A, *cos*2A *and tan*2A *knowing the values of the sine, cosine and tangent of* A.

9.1 Compound angles

A compound angle is an angle of the form $A + B$ or $A - B$. This chapter is about finding the sine, cosine and tangent of $A + B$ or $A - B$ in terms of the sine, cosine and tangent, as appropriate, of the individual angles A and B.

Notice immediately that $\sin(A + B)$ is not equal to $\sin A + \sin B$. You can try this for various angles, but if it were true, then

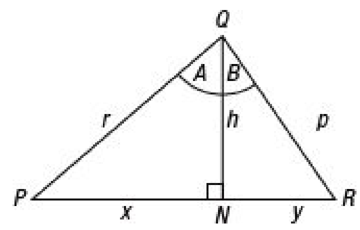
$$\begin{aligned}\sin 180° &= \sin 90° + \sin 90° \\ &= 1 + 1 = 2,\end{aligned}$$

which is clearly false.

It is difficult to give general proofs of formulae for $\sin(A + B)$ and $\cos(A + B)$, and this is not attempted in this book. Proofs that apply only to angles in a restricted range are given. The formulae obtained will then be assumed to be true for all angles.

9.2 Formulae for $\sin(A + B)$ and $\sin(A - B)$ Suppose that angles A and B are both between 0 and 90°. In Figure 9.1, the angles A and B are drawn at the point Q, and the line QN is drawn of length h. PR is perpendicular to QN, and meets QP and QR at P and R respectively.

Let the lengths of PQ, QR, PN and NR be r, p, x and y respectively, as shown in the diagram.



Q
A
B
r
h
p
P
x
N
y
R

Figure 9.1

The strategy for deriving the formula for sin(*A* + *B*) is to say that the area of triangle *PQR* is the sum of the areas of triangles *PQN* and *RQN*.

As the formula for the area of a triangle is $\frac{1}{2}ab\sin C$, the area of triangle *PQR* is $\frac{1}{2}rp\sin(A+B)$ and of triangles *PQN* and *RQN* are $\frac{1}{2}rh\sin A$ and $\frac{1}{2}ph\sin B$ respectively.

Then area of triangle *PQR* = area of triangle *PQN* + area of triangle *RQN* so $\frac{1}{2}rp\sin(A+B)=\frac{1}{2}rh\sin A+\frac{1}{2}ph\sin B.$

Then, multiplying both sides of the equation by 2, and dividing both sides by *rp* gives $\sin(A+B)=\frac{h}{p}\sin A+\frac{h}{r}\sin B.$

Noticing that $\frac{h}{p}=\cos B$ and $\frac{h}{r}=\cos A$ the formula becomes $\sin(A+B)=\cos B\times\sin A+\cos A\times\sin B.$

This equation is usually written as

$$\sin(A+B)=\sin A\cos B+\cos A\sin B. \qquad \text{Equation 1}$$

Although Equation 1 has been proved only for angles *A* and *B*, which are acute, the result is actually true for all angles *A* and *B*, positive and negative. From now on you may assume this result.

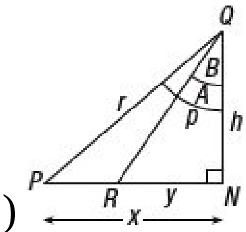
You can use a similar method based on the difference of two areas to derive a formula for sin(*A* − *B*) from Figure 9.2. (You are asked to derive this formula in Exercise 9.1, question 15.) 

Figure 9.2

You would then get the formula

$$\sin(A - B) = \sin A \cos B - \cos A \sin B. \qquad \textbf{Equation 2}$$

Nugget

You can use these formulae to work out the exact value of some other angles. For example, you can find sin 75° exactly by writing it as sin (45° + 30°) = sin 45° cos 30° + cos 45° sin 30°.

9.3 Formulae for cos(A + B) and cos(A − B) Suppose that angles A and B are both between 0 and 90°. In **Figure 9.3**, which is the same as **Figure 9.1**, the angles A and B are drawn at the point Q, and the line QN is drawn of length h. PR is perpendicular to QN, and meets QP and QR at P and R respectively.

Let the lengths of PQ, QR, PN and NR be r, p, x and y respectively, as shown in the diagram.

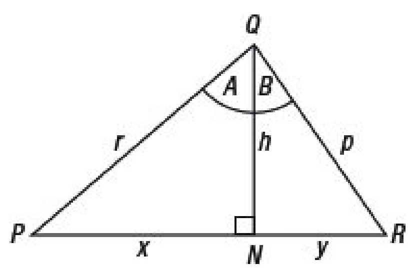


Figure 9.3

The strategy in this case is to use the cosine formula to derive an expression for cos(A + B).

In triangle PQR

$$(x + y)^2 = r^2 + p^2 - 2rp\cos(A + B),$$

so simplifying, and using Pythagoras's theorem,

$$\begin{aligned} 2rp\cos(A + B) &= r^2 + p^2 - (x + y)^2 \\ &= r^2 + p^2 - x^2 - 2xy - y^2 \\ &= (r^2 - x^2) + (p^2 - y^2) - 2xy \\ &= h^2 + h^2 - 2xy = 2h^2 - 2xy. \end{aligned}$$

After dividing both sides by 2rp you obtain

$$\cos(A + B) = \frac{h^2}{rp} - \frac{xy}{rp}.$$

Noticing that $\frac{h}{r} = \cos A$, $\frac{h}{p} = \cos B$, $\frac{x}{r} = \sin A$ and $\frac{y}{p} = \sin B$, the formula becomes
$\cos(A + B) = \cos A \cos B - \sin A \sin B.$ Equation 3

You can use a similar method based on Figure 9.2 to derive a formula for cos(A − B). (You are asked to derive this formula in Exercise 9.1, question 16.) You would then get the formula

$$\cos(A - B) = \cos A \cos B + \sin A \sin B. \qquad \text{Equation 4}$$

9.4 Formulae for tan(A + B) and tan(A − B) You can use Equations 1 and 3 to derive a formula for tan(A + B), starting from the

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

Then
$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

formula

$$= \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}.$$

Now divide the numerator and denominator of this fraction by cos A cos B. Then

$$\begin{aligned}\tan(A+B) &= \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}\\ &= \frac{\dfrac{\sin A\cos B}{\cos A\cos B} + \dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B} - \dfrac{\sin A\sin B}{\cos A\cos B}}\\ &= \frac{\tan A + \tan B}{1 - \tan A\tan B}.\end{aligned}$$

Therefore $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$. Equation 5

As with the formulae for sin(A − B) and cos(A − B), you can use a similar method to derive a formula for tan(A − B). (You are asked to derive this formula in Exercise 9.1, question 17.) You would then get the formula

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}. \qquad \text{Equation 6}$$

9.5 Worked examples

Example 9.1

Using the values of the sines and cosines of 30° and 45° in Section 2.4, find the exact values of sin 75° and cos 15°.

$$
\begin{aligned}
&\text{Using} & \sin(A+B) &= \sin A\cos B + \cos A\sin B \\
&\text{and substituting} & A &= 45^\circ \text{ and } B = 30^\circ, \\
&\text{you find} & \sin 75^\circ &= \sin 45^\circ\cos 30^\circ + \cos 45^\circ\sin 30^\circ \\
& & &= \frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\times\frac{1}{2} \\
& & &= \frac{\sqrt{6}+\sqrt{2}}{4}.
\end{aligned}
$$

To find cos 15°, you could either note that cos θ° = sin(90 − θ)°, and therefore cos 15° = sin 75°, or you could say that

$$
\begin{aligned}
\cos 15^\circ &= \cos(45^\circ - 30^\circ) \\
&= \cos 45^\circ\cos 30^\circ + \sin 45^\circ\sin 30^\circ \\
&= \frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2}\times\frac{1}{2} \\
&= \frac{\sqrt{6}+\sqrt{2}}{4}. \blacksquare
\end{aligned}
$$

Example 9.2

Let the angles α and β be acute, such that $\cos\alpha = 0.6$ and $\cos\beta = 0.8$. Calculate the exact values of $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$.

First you need the values of $\sin\alpha$ and $\sin\beta$. You can do this by using Pythagoras's theorem in the form

$$\sin^2\alpha + \cos^2\alpha = 1.$$

Then

$$\begin{aligned}\sin^2\alpha &= 1 - \cos^2\alpha \\ &= 1 - 0.36 = 0.64.\end{aligned}$$

As α is acute, $\sin\alpha$ is positive, so $\sin\alpha = 0.8$.

Similarly, as β is acute,

$$\begin{aligned}\sin\beta &= \sqrt{1 - \cos^2\beta} \\ &= \sqrt{1 - 0.64} = \sqrt{0.36} \\ &= 0.6.\end{aligned}$$

Then, using the formula $\sin(A + B) = \sin A \cos B + \cos A \sin B$,

$$\begin{aligned}\sin(\alpha + \beta) &= 0.8 \times 0.8 + 0.6 \times 0.6 \\ &= 0.64 + 0.36 \\ &= 1.\end{aligned}$$

Similarly

$$\begin{aligned}\cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= 0.6 \times 0.8 - 0.8 \times 0.6 \\ &= 0.48 - 0.48 \\ &= 0.\end{aligned}$$

So $\sin(\alpha + \beta) = 1$ and $\cos(\alpha + \beta) = 0$. ∎

Example 9.3

Use the formula for $\cos(A - B)$ to show that $\cos(270 - \theta)° = -\sin\theta°$.

Put $A = 270$ and $B = \theta$.

Then

$$\begin{aligned} \cos(270 - \theta)° &= \cos 270° \cos \theta° + \sin 270° \sin \theta° \\ &= 0 \times \cos\theta° + (-1) \times \sin\theta° \\ &= -\sin\theta°. \blacksquare \end{aligned}$$

Exercise 9.1

1 If cos $A = 0.2$ and cos $B = 0.5$, and angles A and B are acute, find the values of $\sin(A + B)$ and $\cos(A + B)$.

2 Use the exact values of sine and cosine of 30° and 45° to find the exact values of sin 15° and cos 75°.

3 Use the formula for $\sin(A - B)$ to show that $\sin(90 - \theta)^\circ = \cos \theta^\circ$.

4 Calculate the value of $\sin(A - B)$ when $\cos A = 0.309$ and $\sin B = 0.23$, given that angle A is acute and angle B is obtuse.

5 Let $\sin A = 0.71$ and $\cos B = 0.32$ where neither A nor B is a first quadrant angle. Find $\sin(A + B)$ and $\tan(A + B)$.

6 Use the formula for $\tan(A + B)$ to find the exact value, in terms of $\sqrt{2}$ and $\sqrt{3}$, of tan 75°.

7 Find $\tan(A + B)$ and $\tan(A - B)$ given that $\tan A = 1.2$ and $\tan B = 0.4$.

8 By using the formula for $\tan(A - B)$, prove that $\tan(180 - \theta)^\circ = -\tan \theta^\circ$.

9 Find the value of sin 52° cos 18° − cos 52° sin 18°.

10 Find the value of cos 73° cos 12° + sin 73° sin 12°.

11 Find the value of $\dfrac{\tan 52^\circ + \tan 16^\circ}{1 - \tan 52^\circ \tan 16^\circ}$.

12 Find the value of $\dfrac{\tan 64^\circ - \tan 25^\circ}{1 + \tan 64^\circ \tan 25^\circ}$.

13 Prove that $\sin(\theta + 45)^\circ = \dfrac{1}{\sqrt{2}}(\sin\theta^\circ + \cos\theta^\circ)$.

14 Prove that $\tan(\theta + 45)^\circ = \dfrac{1 + \tan\theta^\circ}{1 - \tan\theta^\circ}$.

15 Use the method of Section 9.2 to prove that $\sin(A-B) = \sin A \cos B - \cos A \sin B$.

16 Use the method of Section 9.3 to prove that $\cos(A-B) = \cos A \cos B + \sin A \sin B$.

17 Use the method of Section 9.4 to prove that $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$.

9.6 Multiple angle formulae

From Equations 1, 3 and 5 you can deduce other important formulae.

In the formula $\sin(A + B) = \sin A \cos B + \cos A \sin B$ put $B = A$.

Then $\sin(A + A) = \sin A \cos A + \cos A \sin A$

so $\sin 2A = 2 \sin A \cos A.$ Equation 7

You may sometimes need to use this formula with $2A$ replaced by θ.

Then you obtain

$$\sin\theta = 2\sin\tfrac{1}{2}\theta \, \cos\tfrac{1}{2}\theta. \qquad \text{Equation 8}$$

Equations 7 and 8 are really the same formula. Use whichever form is more convenient for the problem in hand.

Nugget

Equation 7 is sometimes called a double-angle formula and Equation 8 is a half-angle formula. Both equations are so easily derived from Equation 1 that you may find it easier to derive them than learn them.

Again, if you put $B = A$ in the formula

$$\cos(A + B) = \cos A \cos B - \sin A \sin B,$$

you get $$\cos(A + A) = \cos A \cos A - \sin A \sin A,$$

which simplifies to

$$\cos 2A = \cos^2 A - \sin^2 A \qquad \text{Equation 9}$$

Note the way of writing $\cos A \times \cos A$ or $(\cos A)^2$ as $\cos^2 A$. This is used for positive powers, but is not usually used for writing powers such as $(\cos A)^{-1}$ because the notation $\cos^{-1} x$ is reserved for the angle whose cosine is x.

You can put Equation 9 into other forms using Pythagoras's theorem, $\sin^2 A + \cos^2 A = 1$. Writing $\sin^2 A = 1 - \cos^2 A$ in Equation 9 you

$$\begin{aligned} \cos 2A &= \cos^2 A - \sin^2 A \\ &= \cos^2 A - (1 - \cos^2 A) \\ &= 2\cos^2 A - 1 \end{aligned}$$

obtain $$\cos 2A = 2\cos^2 A - 1. \qquad \text{Equation 10}$$

On the other hand, if you put $\cos^2 A = 1 - \sin^2 A$ in Equation 9 you get

$$\begin{aligned} \cos 2A &= \cos^2 A - \sin^2 A \\ &= (1 - \sin^2 A) - \sin^2 A \\ &= 1 - 2\sin^2 A \end{aligned}$$

so $$\cos 2A = 1 - 2\sin^2 A \qquad \text{Equation 11}$$

You can also write Equations 10 and 11 in the forms

	$1 + \cos 2A = 2\cos^2 A$	Equation 12
and	$1 - \cos 2A = 2\sin^2 A.$	Equation 13

If you write Equations 9, 10 and 11 in half-angle form, you get

	$\cos\theta = \cos^2 \tfrac{1}{2}\theta - \sin^2 \tfrac{1}{2}\theta,$	Equation 14
	$\cos\theta = 2\cos^2 \tfrac{1}{2}\theta - 1,$	Equation 15
and	$\cos\theta = 1 - 2\sin^2 \tfrac{1}{2}\theta.$	Equation 16

If you put $B = A$ in the formula

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B},$$

you obtain

$$\tan 2A = \frac{\tan A + \tan A}{1 - \tan A \tan A} = \frac{2\tan A}{1 - \tan^2 A}$$

so

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}. \qquad \text{Equation 17}$$

In half-angle form, this is

$$\tan\theta = \frac{2\tan\frac{1}{2}\theta}{1 - \tan^2\frac{1}{2}\theta}. \qquad \text{Equation 18}$$

Exercise 9.2

1 Given that $\sin A = \frac{3}{5}$, and that A is acute, find the values of sin 2A, cos 2A and tan 2A.

2 Given that $\sin A = \frac{3}{5}$, and that A is obtuse, find the values of sin 2A, cos 2A and tan 2A.

3 Find sin2θ, cos2θ and tan2θ when sin θ = 0.25 and θ is acute.

4 Given the values of sin 45° and cos 45°, use the formulae of the previous sections to calculate sin 90° and cos 90°.

5 Given that cos B = 0.66, and that B is acute, find the values of sin 2B and cos 2B.

6 Given that cos B = 0.66, and that B is not acute, find the values of sin 2B and cos 2B.

7 Find the values of 2 sin 36° cos 36° and $2\cos^2 36° - 1$.

8 Given that $\cos 2A = \frac{3}{5}$, find the two possible values of tan A.

9 Prove that $\sin\frac{1}{2}\theta = \pm\sqrt{\frac{1-\cos\theta}{2}}$ and $\cos\frac{1}{2}\theta = \pm\sqrt{\frac{1+\cos\theta}{2}}$.

10 Given that $\cos\theta = \frac{1}{2}$, find $\sin\frac{1}{2}\theta$ and $\cos\frac{1}{2}\theta$.

11 Given that cos 2θ = 0.28, find sin θ.

12 Find the value of $\sqrt{\frac{1-\cos 40°}{1+\cos 40°}}$.

9.7 Identities

It is often extremely useful to be able to simplify a trigonometric expression, or to be able to prove that two expressions are equal for all possible values of the angle or angles involved.

An equation that is true for all possible values of the angle or angles is called an **identity**.

For example, $\sin(A - B) = \sin A \cos B - \cos A \sin B$ is an example of an identity, as are all the formulae given in Equations 1 to 18. So also is $1 + \sin\theta = (\sin\frac{1}{2}\theta + \cos\frac{1}{2}\theta)^2$ but the latter needs to be proved to be an identity.

To prove that a trigonometric equation is an identity, you can choose one of two possible methods.

Method 1 Start with the side of the identity you believe to be the more complicated, and manipulate it, using various formulae including those in Equations 1 to 18, until you arrive at the other side.

Method 2 If you do not see how to proceed with Method 1, then it may help to take the right-hand side from the left-hand side and to prove that the result is zero.

Example 9.4

Prove the identity $1+\sin\theta=(\sin\frac{1}{2}\theta+\cos\frac{1}{2}\theta)^2$.

The more complicated side is the right-hand side, so the strategy will be to use Method 1 and to prove that this is equal to the left-hand side.

In the work which follows, LHS will be used to denote the left-hand side of an equation and RHS the right-hand side.

$$\begin{aligned}\text{RHS} &= \sin^2\tfrac{1}{2}\theta+2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta+\cos^2\tfrac{1}{2}\theta\\ &= (\sin^2\tfrac{1}{2}\theta+\cos^2\tfrac{1}{2}\theta)+2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta\end{aligned}$$

Now use Pythagoras's theorem, and Equation 8. Then

$$\begin{aligned}\text{RHS} &= 1+2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta\\ &= 1+\sin\theta=\text{LHS}.\end{aligned}$$

As RHS = LHS, the identity is true. ▮

Example 9.5

Prove the identity $\frac{\sin A}{1-\cos A}=\frac{1+\cos A}{\sin A}$.

It is not clear which side is the more complicated, so use Method 2. The advantage with Method 2 is that there is then an obvious way to proceed, that is change the resulting expression for LHS − RHS into a single fraction.

$$\begin{aligned}\text{LHS}-\text{RHS}&=\frac{\sin A}{1-\cos A}-\frac{1+\cos A}{\sin A}\\&=\frac{\sin^2 A-(1-\cos A)(1+\cos A)}{\sin A(1-\cos A)}\\&=\frac{\sin^2 A-(1-\cos^2 A)}{\sin A(1-\cos A)}\\&=\frac{\sin^2 A-\sin^2 A}{\sin A(1-\cos A)}\\&=0.\end{aligned}$$

The last step follows from Pythagoras's theorem,

$$\sin^2 A+\cos^2 A=1.$$

Since RHS = LHS, the identity is true. ∎

Example 9.6

Prove the identity $\cos^4 \phi - \sin^4 \phi = \cos 2\phi$.

Starting from the left-hand side, and using Method 1,

$$\begin{aligned}
\text{LHS} &= \cos^4\phi - \sin^4\phi \\
&= (\cos^2\phi - \sin^2\phi)(\cos^2\phi + \sin^2\phi) \\
&= \cos 2\phi \times 1 \\
&= \cos 2\phi = \text{RHS}.
\end{aligned}$$

Equation 9 and Pythagoras's theorem are used in the second step of the argument.

As RHS = LHS, the identity is true. ▮

You must be careful not to use an illogical argument when proving identities. Here is an example of an illogical argument.

Example 9.7

Prove that 2 = 3.

If	$2 = 3$
then	$3 = 2$

Adding the left-hand sides and right-hand sides of these equations gives $5 = 5$.

As this is true, the original statement is true, so 2 = 3. ∎

This is obviously absurd, so the argument itself must be invalid. Nevertheless, many students use just this argument when attempting to prove that something is true. Beware! However, if you stick to the arguments involved with Methods 1 and 2, you will be safe.

Exercise 9.3

In questions 1 to 7, prove the given identities.

1 $\sin(A + B) + \sin(A - B) = 2 \sin A \cos B$

2 $\cos(A + B) + \cos(A - B) = 2 \cos A \cos B$

3 $\dfrac{\cos 2A}{\cos A + \sin A} = \cos A - \sin A$

4 $\sin 3A = 3 \sin A - 4 \sin^3 A$

5 $\dfrac{\sin A}{\cos A} + \dfrac{\cos A}{\sin A} = \dfrac{2}{\sin 2A}$

6 $\dfrac{1}{\sin\theta} + \dfrac{\cos\theta}{\sin\theta} = \dfrac{\cos\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}$

7 $\dfrac{\sin\theta}{\sin\phi} + \dfrac{\cos\theta}{\cos\phi} = \dfrac{2\sin(\theta+\phi)}{\sin 2\phi}$

9.8 More trigonometric equations Sometimes you can use some of the formulae on earlier pages to help you to solve equations. Here are some examples.

Example 9.8

Solve the equation $\cos 2\theta° = \sin \theta°$ giving all solutions between −180 and 180 inclusive.

You can replace the $\cos 2\theta°$ term by $1 - 2\sin^2 \theta°$ (Equation 11) and you will then have an equation in $\sin \theta$.

Then $\cos 2\theta° = \sin\theta°$

so $1 - 2\sin^2\theta° = \sin\theta°$

so $2\sin^2\theta° + \sin\theta° - 1 = 0.$

Factorizing $(2\sin\theta° - 1)(\sin\theta° + 1) = 0$

so $\sin\theta° = 0.5$ or $\sin\theta° = -1.$

Using the methods of Section 6.2, you can solve these equations for θ.

When $\sin \theta° = 0.5$, $\theta = 30$ or 150, and when $\sin \theta° = -1$, $\theta = -90$.

Therefore $\theta = -90$, 30 or 150. ∎

Nugget

Make sure that you find all of the solutions in the given interval – it is a very easy to miss some. You can use a graphics calculator to find how many solutions you need. For example, to find the number of solutions for $\cos 2\theta° = \sin \theta° \cos \theta°$ in the interval $-180°$ to $180°$ you can draw the graphs of $y = \cos 2\theta°$ and $y = \sin \theta° \cos \theta°$ on your calculator and then count the number of points of intersection in the required interval.

Exercise 9.4

In questions 1 to 10, solve the given equation for θ, giving your answers in the interval from −180 to 180.

1 $\sin 2\theta° = \cos \theta°$

2 $2\cos^2 x° - 1 = \frac{1}{2}$

3 $\cos 2\theta° = \sin \theta° \cos \theta°$

4 $4 \sin \theta° \cos \theta° = 1$

5 $1 - 2\sin^2 \theta° = 2 \sin \theta° \cos \theta°$

6 $\frac{1-\tan^2\theta°}{1+\tan^2\theta°} = \frac{1}{2}$

7 $\cos 2\theta° = \cos \theta°$

8 $3 \sin \theta° = 4 \sin^3 \theta°$

9 $4 \cos^3\theta° = 3 \cos \theta°$

10 $2 \tan \theta° = 1 - \tan^2 \theta°$

Key ideas

- $\sin(A+B)=\sin A\cos B+\cos A\sin B$
  $\sin(A-B)=\sin A\cos B-\cos A\sin B$
- $\cos(A+B)=\cos A\cos B-\sin A\sin B$
  $\cos(A-B)=\cos A\cos B+\sin A\sin B$
- $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$
  $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$
- Double angle: $\sin 2A=2\sin A\cos A$

  Half angle: $\sin\theta=2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$
- Double angle: $\cos 2A=\cos^2 A-\sin^2 A$
  $\cos 2A=1-2\sin^2 A$
  $\cos 2A=2\cos^2 A-1$

  Half angle: $\cos\theta=\cos^2\frac{1}{2}\theta-\sin^2\frac{1}{2}\theta$
  $\cos\theta=1-2\sin^2\frac{1}{2}\theta$
  $\cos\theta=2\cos^2\frac{1}{2}\theta-1$
- Double angle: $\tan 2A=\dfrac{2\tan A}{1-\tan^2 A}$

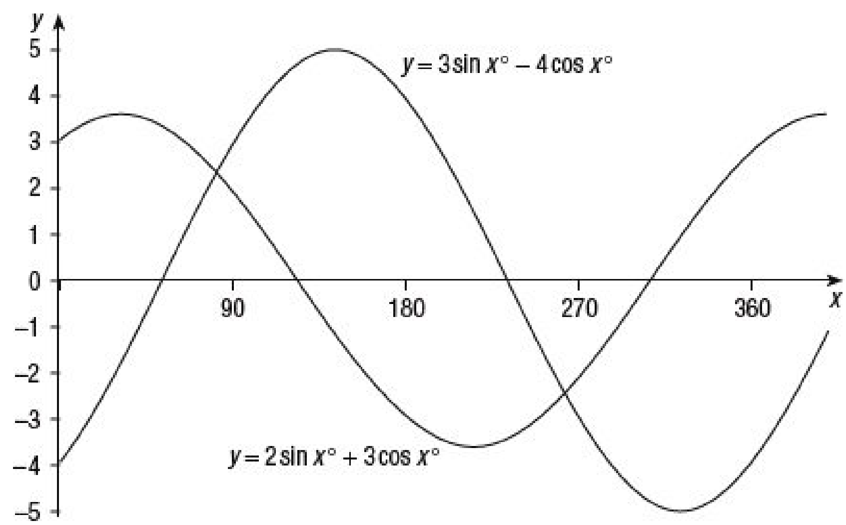
  Half angle: $\tan\theta=\dfrac{2\tan\frac{1}{2}\theta}{1-\tan^2\frac{1}{2}A}$

10

The forms *a* sin *x* and *b* cos *x*

In this chapter you will learn:

- *that the graph of* $y = a$ *sin* $x + b$ *cos* x *is like the graph of sine or cosine*
- *how to express* a *sin* $x + b$ *cos* x *in the form* R *sin(*$x + \alpha$*), and find* R *and* α *in terms of* a *and* b
- *how to use the form* R *sin(*$x + \alpha$*) in applications*.



y
5
4
3
2
1
0
–1
–2
–3
–4
–5
90
180
270
360
x
y = 3 sin x° – 4 cos x°
y = 2 sin x° + 3 cos x°

Figure 10.1

10.1 Introduction

If you have a graphics calculator available, try drawing the graphs of functions of the form $y = 2 \sin x + 3 \cos x$ and $y = 3 \sin x - 4 \cos x$. These two graphs are shown in Figure 10.1.

Both graphs have the characteristic wave properties of the sine and cosine functions. They have been enlarged in the *y*-direction, by different amounts, and translated in the *x*-direction, by different amounts.

This suggests that you may be able to write both of these functions in the form $y = R\sin(x+\alpha)^\circ$

for suitable values of the constants R and α, where the value of R is positive.

This idea is pursued in the next section.

10.2 The form $y = a \sin x + b \cos x$

If you try to choose the values of R and α so that the function $y = R \sin(x + \alpha)$ is identical with $y = a \sin x + b \cos x$, you can start by

$$y = R\sin x\cos\alpha + R\cos x\sin\alpha$$

expanding $\sin(x + \alpha)$ so that or

$$y = (R\cos\alpha)\sin x + (R\sin\alpha)\cos x.$$

If this is the same function as $y = a \sin x + b \cos x$ for all values of x

$$R\cos\alpha = a \text{ and } R\sin\alpha = b,$$

then that is

$$\cos\alpha = \frac{a}{R} \text{ and } \sin\alpha = \frac{b}{R}.$$

You can interpret these two equations by thinking of *a* and *b* as the adjacent and opposite of a triangle which has *R* as its hypotenuse.

Therefore $$R = \sqrt{a^2 + b^2}$$

Alternatively, if you square these equations and add the two equations, you get

$$R^2 \cos^2 \alpha + R^2 \sin^2 \alpha = a^2 + b^2,$$

so $$R^2(\cos^2 \alpha + \sin^2 \alpha) = a^2 + b^2,$$

that is $$R^2 = a^2 + b^2 \text{ and } R = \pm\sqrt{a^2 + b^2}.$$

The value of *R* is always chosen to be positive.

Therefore $$R = \sqrt{a^2 + b^2}$$

Then $$\cos\alpha = \frac{a}{\sqrt{a^2 + b^2}} \text{ and } \sin\alpha = \frac{b}{\sqrt{a^2 + b^2}}.$$

These three equations,

$$R = \sqrt{a^2 + b^2} \qquad \text{Equation 1}$$

and $$\cos\alpha = \frac{a}{\sqrt{a^2 + b^2}} \qquad \text{Equation 2}$$

and $$\sin\alpha = \frac{b}{\sqrt{a^2 + b^2}} \qquad \text{Equation 3}$$

enable you to determine the values of *R* and α.

Example 10.1

Express 2 sin $x°$ + 3 cos $x°$ in the form R sin(x + α)°, where the angles are in degrees.

For the function 2 sin $x°$ + 3 cos $x°$, a = 2 and b = 3. From Equation 1, $R=\sqrt{13}$.

Using this value in Equation 2, $\cos\alpha° = \frac{2}{\sqrt{13}}$ and $\sin\alpha° = \frac{3}{\sqrt{13}}$.

These equations, in which cos $\alpha°$ and sin $\alpha°$ are both positive, show that $\alpha°$ is a first-quadrant angle, and that $\alpha° \approx 56.31$.

Therefore $\quad 2\sin x° + 3\cos x° \equiv \sqrt{13}\sin(x+\alpha)$

where $\quad \alpha° \approx 56.31$. ∎

Note that the symbol '≡' is used to mean 'identically equal to'.

Nugget

You can verify you are right by graphing both forms of the function ($2\sin x° + 3\cos x°$ and $R\sin(x + a)°$) and checking that the two curves coincide.

Example 10.2

Express 3 sin x − 4 cos x in the form R sin(x + α) with α in radians.

For the function 3 sin x − 4 cos x, a = 3 and b = −4. From Equation 1, R = 5. Using this value in Equation 2, $\cos\alpha = \frac{3}{5}$ and $\sin\alpha = \frac{-4}{5}$.

These equations, in which cos α is positive and sin α is negative, show that α is a fourth-quadrant angle, and that:

$$\alpha \approx 5.36.$$

Therefore $$3\sin x - 4\cos x \equiv 5\sin(x+\alpha)$$

where $$\alpha \approx 5.36. \blacksquare$$

Example 10.3

Express sin $x°$ − 2 cos $x°$ in the form R cos($x°$ + $\alpha°$), where $R > 0$.

This is a different form from the original, but it is not difficult to adapt the methods from the beginning of this section.

Comparing the expanded form $R \cos x° \cos \alpha° - R \sin x° \sin \alpha°$ with the form $\sin x° - 2 \cos x°$, gives $R \cos \alpha° = -2$ and $R \sin \alpha° = -1$.

Squaring and adding, as before, gives

$$R = \sqrt{5}$$

and

$$\cos\alpha° = \frac{-2}{\sqrt{5}} \text{ and } \sin\alpha° = \frac{-1}{\sqrt{5}}.$$

Thus, as cos $\alpha°$ and sin $\alpha°$ are both negative, $\alpha°$ is a third-quadrant angle, and

$$\alpha° \approx 153.43$$

Thus

$$\sin x - 2\cos x = \sqrt{5}\cos(x + \alpha)$$

where

$$\alpha° \approx 153.43.$$

Exercise 10.1

In questions 1 to 6, write the given function in the form $R \sin(x + \alpha)°$.

1 $\sin x° + \cos x°$

2 $5 \sin x° + 12 \cos x°$

3 $2 \cos x° + 5 \sin x°$

4 $\cos x° - \sin x°$

5 $\sin x° - 3 \cos x°$

6 $3 \cos x° - \sin x°$

In questions 7 to 9, give α in radians.

7 Write the function $\sin x + \cos x$ in the form $R \cos(x + \alpha)$.

8 Write the function $\sin x + \cos x$ in the form $R \cos(x - \alpha)$.

9 Write the function $\sin x + \cos x$ in the form $R \sin(x - \alpha)$.

10 By writing the functions $7 \cos x + \sin x$ and $5 \cos x - 5 \sin x$ in the form $R \sin(x + \alpha)$, show that they have the same maximum value.

10.3 Using the alternative form There are two main advantages in writing something like $\sin x + \cos x$ in any one of the four forms $R \sin(x + \alpha)$, $R \sin(x - \alpha)$, $R \cos(x + \alpha)$ and $R \cos(x - \alpha)$.

It enables you to solve equations easily, and to find the maximum and minimum values of the function without further work.

Nugget

The sine and cosine functions oscillate between -1 and 1. Therefore the forms $R \sin(x \pm \alpha)$ and $R \cos(x \pm \alpha)$ oscillate between $-R$ and R.

Example 10.4

Solve the equation 5 sin *x*° + 8 cos *x*° = 3 giving all the solutions between 0 and 360.

Using the method of Section 10.2, you can write 5 sin *x*° + 8 cos *x*° as $\sqrt{89}\,\sin(x+57.99)°$.

The equation then becomes

$$\sqrt{89}\,\sin(x+57.99)° = 3$$

or

$$\sin(x+57.99)° = \frac{3}{\sqrt{89}}.$$

Let *z*° = *x*° + 57.99°. Then you require the solutions of $\sin z° = \frac{3}{\sqrt{89}}$ for values of *z*° between 57.99 and 417.99.

The principal angle is 18.54°, and the other angle between 0 and 360 is the second quadrant angle (see Section 5.2), 180° − 18.54° = 161.46°.

You now have to add 360 to the first of these to find the value of *z*° in the required interval. Then the two solutions for *z*° are $z° = 378.54 \text{ and } 161.46.$

You find the solutions for *x* by substituting *z*° = *x*° + 57.99, and you find that

$$x° = 378.54 - 57.99 = 320.55$$

or

$$x° = 161.46 - 57.99 = 103.47.$$

Thus the solutions are 103.47° and 320.55°. ∎

Example 10.5

Find the maximum and minimum values of sin x − 3 cos x and the values of x, in radians, for which they occur.

Writing sin x − 3 cos x in the form R sin(x − α) using the methods in Section 10.2 gives $\sin x - 3\cos x \equiv \sqrt{10}\sin(x - 1.249)$.

The question now becomes: find the maximum and minimum values of $\sqrt{10}\sin(x - 1.249)$ and the values of x for which they occur.

You know that the maximum of a sine function is 1 and that it occurs when the angle is $\frac{1}{2}\pi$.

Thus the maximum value of $\sqrt{10}\sin(x - 1.249)$ is $\sqrt{10}$, and this occurs when $x - 1.249 = \frac{1}{2}\pi$, that is when x = 2.820.

Similarly the minimum value of a sine function is −1, and this occurs when the angle is $\frac{2}{3}\pi$.

Thus the minimum value of $\sqrt{10}\sin(x - 1.259)$ is $-\sqrt{10}$, and this occurs when $x - 1.259 = \frac{3}{2}\pi$, that is, when x = 5.961. ▮

Example 10.6

Solve the equation 3 cos 2x° − 4 sin 2x° = 2 giving all solutions in the interval −180 to 180.

Begin by writing y = 2x: the equation becomes

$$3\cos y^\circ - 4\sin y^\circ = 2$$

with solutions for y needed in the interval −360 to 360.

Writing 3 cos y° − 4 sin y° in the form R cos(y + α)° using the methods in Section 10.2 gives $3\cos y^\circ - 4\sin y^\circ \equiv 5\cos(y + 53.13)^\circ$.

Solving the equation 5 cos(y + 53.13)° = 2 gives

$$y + 53.13^\circ = 66.42\ldots^\circ.$$

The principal angle is 66.42°.

Using the methods of Section 5.3, the angles between −413.13 and 413.13 satisfying this equation are $-293.58^\circ, -66.42^\circ, 66.42^\circ, 293.58^\circ.$

Thus $y^\circ + 53.13^\circ = -293.58^\circ, -66.42^\circ, 66.42^\circ, 293.58^\circ$

so $y^\circ = -346.71^\circ, -119.55^\circ, 13.29^\circ, 240.45^\circ.$

Finally, dividing by 2 as y° = 2x° gives

$$x^\circ = -173.35^\circ, -59.78^\circ, 6.65^\circ, 120.22^\circ.$$

Note that the decision about whether to round up or to round down the final figure on dividing by 2 was made by keeping more significant figures on the calculator. ∎

Example 10.7

Show that the equation 2 sin x + 3 cos x = 4 has no solutions.

You can write this equation in the form

$$\sqrt{13}\sin(x+\alpha)=4$$

for a suitable value of α.

You can then rewrite the equation in the form

$$\sin(x+\alpha)=\frac{4}{\sqrt{13}}.$$

As $\frac{4}{\sqrt{13}}>1$, there is no solution to this equation. ∎

Exercise 10.2

In questions 1 to 6, solve the given equation for $\theta°$, giving the value of $\theta°$ in the interval 0 to 360 inclusive.

1 $\sin\theta° + \cos\theta° = 1$

2 $\sin\theta° + \sqrt{3}\cos\theta° = 1$

3 $3\cos\theta° - 2\sin\theta° = 1$

4 $12\sin\theta° - 5\cos\theta° = 5$

5 $-8\cos\theta° - 7\sin\theta° = 5$

6 $\cos 2\theta° - \sin 2\theta° = -1$

In questions 7 to 12, solve the given equation for $\theta°$, giving the value of $\theta°$ in the interval −180 to 180 inclusive.

7 $\cos\theta° + \sin\theta° = -1$

8 $\sqrt{3}\sin\theta° + \cos\theta° = -1$

9 $3\cos\theta° - \sin\theta° = 2$

10 $-2\cos\theta° - 3\sin\theta° = 3$

11 $6\sin\theta° - 7\cos\theta° = -8$

12 $\sqrt{3}\cos 2\theta° - \sin 2\theta° = -1$

In questions 13 to 18, find the maximum and minimum values of the function and the values of $x°$ in the interval between −180 and 180 inclusive, for which they occur.

13 $y = 2\sin x° - \cos x°$

14 $y = 3\cos x° - 4\sin x°$

15 $y = \sqrt{3}\cos 2x° - \sin 2x°$

16 $y = \cos 2x° - \sin 2x°$

17 $y = 3\sin x° + 4\cos x° + 2$

18 $y = \sqrt{2}\cos 2x° - \sin 2x° + 3$

Key ideas

- $a\sin x + b\cos x = R\sin(x+\alpha)$

  $a\sin x - b\cos x = R\sin(x-\alpha)$

  $a\cos x + b\sin x = R\cos(x-\alpha)$

  $a\cos x - b\sin x = R\cos(x+\alpha)$

  where $R = \sqrt{a^2+b^2}$, $\cos\alpha = \dfrac{a}{\sqrt{a^2+b^2}}$ and $\sin\alpha = \dfrac{b}{\sqrt{a^2+b^2}}$

- Using the forms $R\sin(x \pm \alpha)$ or $R\cos(x \pm \alpha)$ enables you to solve equations in the form $a\sin x + b\cos x = c$.

- To solve an equation in the form $a\sin x + b\cos x = c$, rewrite $a\sin x + b\cos x$ in the form $R\sin(x \pm \alpha)$ or $R\cos(x \pm \alpha)$ then solve the equation $\sin(x \pm \alpha) = \frac{c}{R}$ or $\cos(x \pm \alpha) = \frac{c}{R}$

- The maximum value of the function $R\sin(x \pm \alpha)$ is R

  The minimum value of the function $R\sin(x \pm \alpha)$ is $-R$

11
The factor formulae

In this chapter you will learn:

- *how to express the sum and difference of two sines or cosines in an alternative form as a product*
- *how to do this process in reverse*
- *how to use both processes in solving problems.*

11.1 The first set of factor formulae In Exercise 9.3, question 1, you were asked to prove the identity $\sin(A+B)+\sin(A-B)=2\sin A\cos B$.

The proof of this identity relies on starting with the left-hand side and expanding the terms using Equations 1 and 2 of Chapter 9 to get

$$\sin(A+B)=\sin A\cos B+\cos A\sin B$$

and $$\sin(A-B)=\sin A\cos B-\cos A\sin B.$$

Adding the left-hand sides of these two equations you obtain the required result $\sin(A+B)+\sin(A-B)=2\sin A\cos B$,

which will be used in the rewritten form

$$2\sin A\cos B=\sin(A+B)+\sin(A-B). \qquad \text{Equation 1}$$

This is the first formula of its type. These formulae enable you to move from a product of sines and cosines to a sum or difference of sines and cosines equal to it.

Example 11.1

Use Equation 1 to simplify 2 sin 30° cos 60°.

$$\begin{aligned} 2\sin 30^\circ \cos 60^\circ &= \sin(30+60)^\circ + \sin(30-60)^\circ \\ &= \sin 90^\circ + \sin(-30)^\circ \\ &= 1 - \sin 30^\circ \\ &= 1 - \tfrac{1}{2} = \tfrac{1}{2}. \blacksquare \end{aligned}$$

In the middle of the example, the fact that $\sin(-\theta) = -\sin\theta$ for any angle θ was used to change $\sin(-30)^\circ$ to $-\sin 30^\circ$.

If you subtract the equations

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

and $$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

you obtain $$\sin(A+B) - \sin(A-B) = 2\cos A \sin B,$$

that is $$2\cos A \sin B = \sin(A+B) - \sin(A-B). \qquad \text{Equation 2}$$

If you had used Equation 2 to solve Example 11.1, you would say

$$\begin{aligned} 2\sin 30^\circ \cos 60^\circ &= 2\cos 60^\circ \sin 30^\circ \\ &= \sin(60+30)^\circ - \sin(60-30)^\circ \\ &= \sin 90^\circ - \sin 30^\circ \\ &= 1 - \sin 30^\circ \\ &= 1 - \tfrac{1}{2} = \tfrac{1}{2}. \end{aligned}$$

Two other formulae come from the equivalent formulae for $\cos(A+B)$ and $\cos(A-B)$,

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

and $$\cos(A-B) = \cos A \cos B + \sin A \sin B.$$

First adding, and then subtracting, these equations gives

$$\cos(A+B) + \cos(A-B) = 2\cos A \cos B$$

and $$\cos(A+B) - \cos(A-B) = -2\sin A \sin B.$$

When you rewrite these equations you have

$$2\cos A\cos B = \cos(A+B) + \cos(A-B) \qquad \text{Equation 3}$$

and $$2\sin A\sin B = \cos(A-B) - \cos(A+B). \qquad \text{Equation 4}$$

Note the form of these four equations, which are gathered together for convenience.

$$2\sin A\cos B = \sin(A+B) + \sin(A-B) \qquad \text{Equation 1}$$

$$2\cos A\sin B = \sin(A+B) - \sin(A-B) \qquad \text{Equation 2}$$

$$2\cos A\cos B = \cos(A+B) + \cos(A-B) \qquad \text{Equation 3}$$

$$2\sin A\sin B = \cos(A-B) - \cos(A+B) \qquad \text{Equation 4}$$

Nugget

These formulae are often remembered as:

$$2 \times \sin \times \cos = \sin(\mathit{sum}) + \sin(\mathit{difference})$$
$$2 \times \cos \times \sin = \sin(\mathit{sum}) - \sin(\mathit{difference})$$
$$2 \times \cos \times \cos = \cos(\mathit{sum}) + \cos(\mathit{difference})$$
$$2 \times \sin \times \sin = \cos(\mathit{difference}) - \cos(\mathit{sum}).$$

Note the following points.

- In Equations 1 and 2, it is important that the ‘difference’ is found by subtracting B from A.
- In Equations 3 and 4, it is not important whether you take the difference as $A - B$ or as $B - A$; the equation $\cos(-\theta) = \cos\theta$ for all angles θ ensures that $\cos(B - A) = \cos(A - B)$.
- The order of the right-hand side in Equation 4 is different from the other formulae.

Example 11.2

Express sin 5θ cos 3θ as the sum of two trigonometric ratios.

Using Equation 1, 2 sin A cos B = sin(A + B) + sin(A − B), gives

$$\begin{aligned}\sin 5\theta \cos 3\theta &= \tfrac{1}{2}(2\sin 5\theta \cos 3\theta)\\ &= \tfrac{1}{2}(\sin(5\theta + 3\theta) + \sin(5\theta - 3\theta))\\ &= \tfrac{1}{2}(\sin(8\theta) + \sin(2\theta)). \blacksquare\end{aligned}$$

Example 11.3

Change sin 70° sin 20° into a sum.

Using Equation 4, 2 sin *A* sin *B* = cos(*A* − *B*) − cos(*A* + *B*), gives

$$\begin{aligned}\sin 70^\circ \cos 20^\circ &= \tfrac{1}{2}(2\sin 70^\circ \cos 20^\circ)\\ &= \tfrac{1}{2}(\cos(70^\circ - 20^\circ) - \cos(70^\circ + 20^\circ))\\ &= \tfrac{1}{2}(\cos 50^\circ - \cos 90^\circ)\\ &= \tfrac{1}{2}(\cos 50^\circ - 0)\\ &= \tfrac{1}{2}\cos 50^\circ. \blacksquare\end{aligned}$$

Exercise 11.1

In questions 1 to 8, express the given expression as the sum or difference of two trigonometric ratios.

1 $\sin 3\theta \cos \theta$

2 $\sin 35° \cos 45°$

3 $\cos 50° \cos 30°$

4 $\cos 5\theta \sin 3\theta$

5 $\cos(C + 2D) \cos(2C + D)$ **6** $\cos 60° \sin 30°$

7 $2 \sin 3A \sin A$

8 $\cos(3C + 5D) \sin(3C - 5D)$ **9** In Equation 1, put $A = 90 - C$ and $B = 90 - D$ and simplify both sides of the resulting identity. What equation results?

10 In Equation 1, put $A = 90 - C$ and simplify both sides of the resulting identity. What equation results?

11.2 The second set of factor formulae The second set of factor formulae are really a rehash of the first set that enable you to work the other way round, that is to write the sum or difference of two sines or two cosines as a product of sines and cosines.

Starting from

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B),$$

write it the other way round as

$$\sin(A+B) + \sin(A-B) = 2 \sin A \cos B.$$

Put $A + B = C$ and $A - B = D$.

Then the identity becomes

$$\sin C + \sin D = 2 \sin A \cos B.$$

If you can write A and B in terms of C and D you will obtain a formula for the sum of two sines.

From the equations

$$A + B = C$$

and

$$A - B = D,$$

you can use simultaneous equations to deduce that

$$A = \frac{C+D}{2} \quad \text{and} \quad B = \frac{C-D}{2}.$$

Then $$\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}. \qquad \textbf{Equation 5}$$

You can deduce a second formula in a similar way from Equation 2, by a similar method.

Equation 2 then becomes

$$\sin C - \sin D = 2 \sin \frac{C-D}{2} \cos \frac{C+D}{2}. \qquad \textbf{Equation 6}$$

Similar methods applied to Equations 3 and 4 give

$$\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2} \qquad \textbf{Equation 7}$$

and $$\cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2}. \qquad \textbf{Equation 8}$$

Nugget

Equations 5 to 8 are often remembered as:

$$\sin + \sin = 2 \times \sin(\textit{semisum}) \times \cos(\textit{semidifference})$$

$$\sin - \sin = 2 \times \cos(\textit{semisum}) \times \sin(\textit{semidifference})$$

$$\cos + \cos = 2 \times \cos(\textit{semisum}) \times \cos(\textit{semidifference})$$

$$\cos - \cos = 2 \times \sin(\textit{semisum}) \times \sin(\textit{semidifference reversed}).$$

Example 11.4

Transform sin 25° + sin 18° into a product.

Using Equation 5, $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$, gives

$$\begin{aligned}\sin 25° + \sin 18° &= 2\sin\frac{25°+18°}{2}\cos\frac{25°-18°}{2}\\ &= 2\sin 21.5°\cos 3.5°.\end{aligned}$$

Example 11.5

Change cos 3θ − cos 7θ into a product.

Using Equation 8, $\cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$, gives

$$\begin{aligned}\cos 3\theta - \cos 7\theta &= 2\sin\frac{3\theta + 7\theta}{2}\sin\frac{7\theta - 3\theta}{2}\\ &= 2\sin 5\theta\ \sin 2\theta. \blacksquare\end{aligned}$$

Example 11.6

Solve the equation sin $\theta°$ − sin $3\theta°$ = 0, giving solutions in the interval −180 to 180.

$$\begin{aligned}\sin\theta° - \sin 3\theta° &= 2\cos 2\theta° \sin(-\theta)° \\ &= -2\cos 2\theta° \sin\theta°.\end{aligned}$$

Hence $\qquad \cos 2\theta° = 0 \text{ or } \sin\theta° = 0.$

The solutions of cos $2\theta°$ = 0 are −135°, −45°, 45°, 135° and the solutions of sin $\theta°$ = 0 are −180°, 0°, 180°.

Therefore the solutions of the original equation are
$-180°, -135°, -45°, 0°, 45°, 135°, 180°.$ ∎

You could also have expanded sin 3θ in the form 3 sin θ − 4 $\sin^3\theta$ and then solved the equation 4 $\sin^3\theta$ − 2 sin θ = 0 to get the same result.

Example 11.7

Prove that if $A + B + C = 180$, then

$$\frac{\sin A + \sin B - \sin C}{\sin A + \sin B + \sin C} = \tan A \tan B.$$

$$\begin{aligned}
\text{LHS} &= \frac{(\sin A + \sin B) - \sin C}{(\sin A + \sin B) + \sin C} \\
&= \frac{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B) - 2\sin\frac{1}{2}C\cos\frac{1}{2}C}{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B) + 2\sin\frac{1}{2}C\cos\frac{1}{2}C} \\
&= \frac{2\cos\frac{1}{2}C\cos\frac{1}{2}(A-B) - 2\sin\frac{1}{2}C\cos\frac{1}{2}C}{2\cos\frac{1}{2}C\cos\frac{1}{2}(A-B) + 2\sin\frac{1}{2}C\cos\frac{1}{2}C} \\
&= \frac{2\cos\frac{1}{2}C\left(\cos\frac{1}{2}(A-B) - \sin\frac{1}{2}C\right)}{2\cos\frac{1}{2}C\left(\cos\frac{1}{2}(A-B) + \sin\frac{1}{2}C\right)} \\
&= \frac{\cos\frac{1}{2}(A-B) - \sin\frac{1}{2}C}{\cos\frac{1}{2}(A-B) + \sin\frac{1}{2}C} \\
&= \frac{\cos\frac{1}{2}(A-B) - \cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B) + \cos\frac{1}{2}(A+B)}
\end{aligned}$$

$$= \frac{2\sin A \sin B}{2\cos A \cos B} = \tan A \tan B = \text{RHS}.$$

As the LHS = RHS, the identity is true. ∎

Notice how in Example 11.7, the fact that $\cos(90 - \theta)° = \sin \theta°$ and $\sin(90 - \theta)° = \cos \theta°$ was used in the forms $\cos\frac{1}{2}(A+B) = \sin\frac{1}{2}C$ and $\sin\frac{1}{2}(A+B) = \cos\frac{1}{2}C$.

Exercise 11.2

In questions 1 to 6, express the given sum or difference as the product of two trigonometric ratios.

1 $\sin 4A + \sin 2A$

2 $\sin 5A - \sin A$

3 $\cos 4\theta - \cos 2\theta$

4 $\cos A - \cos 5A$

5 $\cos 47° + \cos 35°$

6 $\sin 49° - \sin 23°$

In questions 7 to 10, use the factor formulae to simplify the given expressions.

7 $\dfrac{\sin 30° + \sin 60°}{\cos 30° - \cos 60°}$

8 $\dfrac{\sin\alpha + \sin\beta}{\cos\alpha - \cos\beta}$

9 $\dfrac{\cos\theta - \cos 3\theta}{\sin\theta + \sin 3\theta}$

10 $\dfrac{\sin\theta + 2\sin 2\theta + \sin 3\theta}{\cos\theta + 2\cos 2\theta + \cos 3\theta}$

In questions 11 to 14, use the factor formulae to solve the following equations, giving all solutions in the interval 0 to 360.

11 $\cos \theta° - \cos 2\theta° = 0$

12 $\sin \theta° + \sin 2\theta° = 0$

13 $\sin\theta° + \sin 2\theta° + \sin 3\theta° = 0$

14 $\cos\theta° + 2\cos 2\theta° + \cos 3\theta° = 0$

In questions 15 to 17, you are given that $A + B + C = 180°$. Prove that each of the following results is true.

15 $\sin A + \sin B + \sin C = 4\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$

16 $\sin A + \sin B - \sin C = 4\sin\frac{1}{2}A\sin\frac{1}{2}B\cos\frac{1}{2}C$

17 $\cos A + \cos B + \cos C - 1 = 4\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C$

Key ideas

- You can use the factor formulae to express the sum or difference of two sines or two cosines as a product of sines and cosines and vice versa.
- To write a product as the sum or difference of two sines or two cosines use the relationships:

$$2\sin A\cos B=\sin(A+B)+\sin(A-B)$$

$$2\cos A\sin B=\sin(A+B)-\sin(A-B)$$

$$2\cos A\cos B=\cos(A+B)+\cos(A-B)$$

$$2\sin A\sin B=\cos(A-B)-\cos(A+B).$$

- To write the sum or difference of two sines or two cosines as a product use the relationships:

$$\sin C+\sin D=2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$\sin C-\sin D=2\sin\frac{C-D}{2}\cos\frac{C+D}{2}$$

$$\cos C+\cos D=2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$\cos C-\cos D=2\sin\frac{C+D}{2}\sin\frac{D-C}{2}.$$

- $\cos C-\cos D=2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$ is the same as

$\cos C-\cos D=-2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$.

12
Circles related to triangles

In this chapter you will learn:

- *that the circumcircle is the circle which passes through the vertices of a triangle*
- *that the incircle and the three ecircles all touch the three sides of a triangle*
- *how to calculate the radii of these circles from information about the triangle.*

12.1 The circumcircle

The circumcircle of a triangle *ABC* (see Figure 12.1) is the circle that passes through each of the vertices. The centre of the circumcircle will be denoted by *O* and its radius will be denoted by *R*.

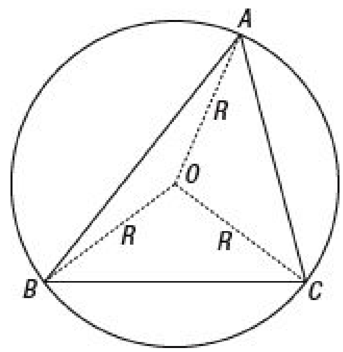


Figure 12.1

To calculate the radius *R* of the circumcircle, drop the perpendicular from *O* on to the side *BC* to meet *BC* at *N* (see below).

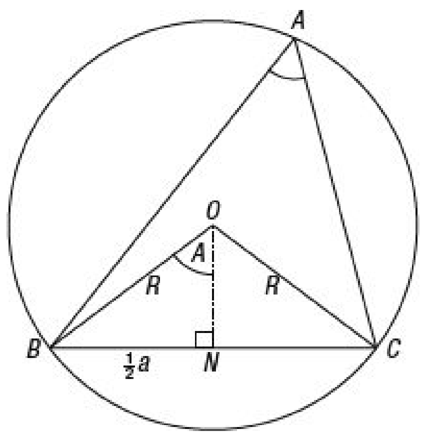


Figure 12.2

As *BOC* is an isosceles triangle, because two of its sides are *R*, the line *ON* bisects the base. Therefore $BN = NC = \frac{1}{2}a$.

In addition, angle *BOC*, the angle at the centre of the circle standing on the arc *BC*, is twice the angle *BAC*, which stands on the same arc. Thus angle *BOC* = 2*A*, so angle *BON* = *A*.

Therefore, in triangle *BON*,

$$\sin A = \frac{\frac{1}{2}a}{R}$$

so

$$2R = \frac{a}{\sin A}.$$

You will recognize that the right-hand side of this formula for *R* also occurs in the sine formula.

Therefore

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \qquad \text{Equation 1}$$

Nugget

Equation 1, which enables you to calculate the radius *R* of the circumcircle, is what some people understand by the sine formula for a triangle, rather than the shorter version in Section 6.3.

Note that the equation $2R = \frac{a}{\sin A}$ would itself give you a proof of the sine formula for the triangle, for, by symmetry, the radius of the circumcircle must be a property of the triangle as a whole, and not 'biased' towards one particular vertex. Therefore, if you had started with another vertex, you would obtain the formulae $2R = \frac{b}{\sin B}$ or $2R = \frac{c}{\sin C}$.

Thus $$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Example 12.1

Find the radius of the circumcircle of the triangle with sides of length 4 cm, 5 cm and 6 cm.

Let $a = 4$, $b = 5$ and $c = 6$. Then calculate an angle of the triangle, say the largest angle, C, using the cosine formula. From the value of cos C, work out the value of sin C, and then use Equation 1 to find R.

Using the cosine formula, $c^2 = a^2 + b^2 - 2ab\cos C$, $6^2 = 4^2 + 5^2 - 2 \times 4 \times 5 \times \cos C$

which reduces to $40\cos C = 5$

or $\cos C = \frac{1}{8}$.

Using $\sin^2 C = 1 - \cos^2 C$, so

$$\sin^2 C = 1 - \frac{1}{64} = \frac{63}{64}$$

$$\sin C = \sqrt{\frac{63}{64}} = \frac{3\sqrt{7}}{8}.$$

Finally, using the full version of the sine formula, Equation 1, $2R = \frac{c}{\sin C}$, you obtain

$$2R = \frac{6}{\frac{3\sqrt{7}}{8}} = \frac{16}{\sqrt{7}}.$$

Thus the radius R of the circumcircle is $\frac{8}{\sqrt{7}}$ cm. ∎

Example 12.2

The angles A, B and C of a triangle are 50°, 60° and 70°, and the radius R of its circumcircle is 10 cm. Calculate the area of the triangle.

The standard formula for the area Δ of a triangle is $\Delta = \frac{1}{2}ab\sin C$.

You can find a and b from the full version of the sine formula.

Thus, from $2R = \dfrac{a}{\sin A} = \dfrac{b}{\sin B}$,

$$a = 2R\sin A = 20\sin 50°$$

and

$$b = 2R\sin B = 20\sin 60°.$$

Using these in the formula $\Delta = \frac{1}{2}ab\sin C$,

$$\begin{aligned}\Delta &= \tfrac{1}{2} \times 20\sin 50° \times 20\sin 60° \times \sin 70° \\ &= 200\sin 50° \sin 60° \sin 70°.\end{aligned}$$

The area of the triangle is approximately 125 cm^2. ∎

The full version of the sine formula, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, can often be used, together with the fact that the sum of the angles of a triangle is 180°, to prove other formulae concerned with a triangle.

As an example, here is a proof of the cosine formula using this method. This proof is certainly not the recommended proof of the cosine formula, but it does have the advantage that you do not need to consider the acute-angled and obtuse-angled triangles separately.

Example 12.3

Use the sine formula to prove the cosine formula

$$a^2 = b^2 + c^2 - 2bc\cos A°.$$

Starting with the more complicated right-hand side, first substitute $b = 2R\sin B°$ and $c = 2R\sin C°$. Then manipulate the right-hand side, keeping symmetry as much as possible and using $A + B + C = 180°$ judiciously, into a form that you can recognize as the left-hand side.

You use $A + B + C = 180°$ by substituting $B + C = 180° - A$ and then recognizing that $\sin(B + C)° = \sin(180 - A)° = \sin A°$ and that $\cos(B + C)° = \cos(180 - A)° = -\cos A°$.

$$\begin{aligned}
\text{RHS} &= b^2 + c^2 - 2bc\cos A° \\
&= 4R^2\sin^2 B° + 4R^2\sin^2 C° - 8R^2\sin B°\sin C°\cos A° \\
&= 2R^2(2\sin^2 B° + 2\sin^2 C° - 4\sin B°\sin C°\cos A°) \\
&= 2R^2((1-\cos 2B°) + (1-\cos 2C°) - 2\cos A°(2\sin B°\sin C°)) \\
&= 2R^2(2 - (\cos 2B° + \cos 2C°) - 2\cos A°(2\sin B°\sin C°)) \\
&= 2R^2(2 - (2\cos(B+C)°\cos(B-C)°) - 2\cos A°(2\sin B°\sin C°))
\end{aligned}$$

$$\begin{aligned}
&= 2R^2(2 + 2\cos A°\cos(B-C)° - 2\cos A°(\cos(B-C)° + \cos A°)) \\
&= 2R^2(2 + 2\cos A°\cos(B-C)° - 2\cos A°\cos(B-C)° - 2\cos^2 A°) \\
&= 2R^2(2\sin^2 A°) \\
&= (2R\sin A°)^2 \\
&= a^2 = \text{LHS}.
\end{aligned}$$

Since RHS = LHS, the identity is true. ▮

Exercise 12.1

1 Calculate the radius of the circumcircle of the triangle *ABC* given that a = 10 cm and that angle A = 30°.

2 Find the exact radius of the circumcircle of a triangle with sides 2 cm, 3 cm and 4 cm, leaving square roots in your answer.

3 The area of a triangle *ABC* is 40 cm^2 and angles B and C are 50° and 70° respectively. Find the radius of the circumcircle.

4 Prove that $a = b \cos C + c \cos B$.

5 Let $s = \frac{1}{2}(a+b+c)$. Prove that $s = 4R \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C$.

12.2 The incircle

The incircle of a triangle *ABC* (see Figure 12.3) is the circle that touches each of the sides and lies inside the triangle. The centre of the incircle will be denoted by I and its radius will be denoted by r.

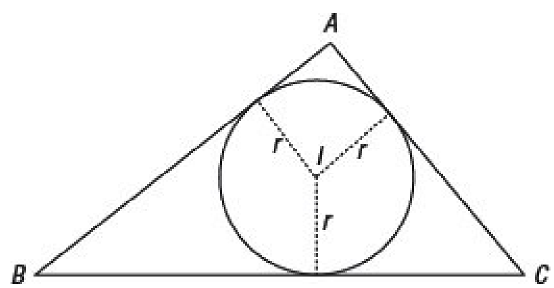


Figure 12.3

Nugget

Note each side of the triangle is a tangent to the circle.

To calculate the radius *r* of the incircle, let the perpendiculars from *I* drop on to the sides of the triangle *ABC* to meet the sides at *L*, *M* and *N*. Join *IA*, *IB* and *IC* (see below).

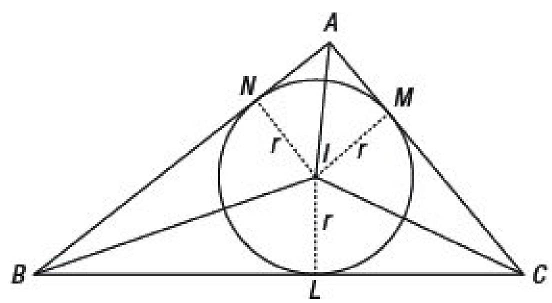


Figure 12.4

Using the formula $\frac{1}{2}$ base × height, the area of the triangle *BIC* is $\frac{1}{2}ar$.

Similarly, the areas of the triangles *CIA* and *AIB* are $\frac{1}{2}br$ and $\frac{1}{2}cr$.

Adding these you get Δ, the area of the triangle *ABC*. Therefore

$$\begin{aligned}\Delta &= \tfrac{1}{2}ar + \tfrac{1}{2}br + \tfrac{1}{2}cr \\ &= r \times \frac{a+b+c}{2}.\end{aligned}$$

Denoting the expression $\frac{a+b+c}{2}$ by *s* (for semi-perimeter), you find that

$$rs = \Delta. \qquad \text{Equation 2}$$

Equation 2 is used to derive results concerning the incircle of a triangle.

Example 12.4

The sides of a triangle are 4 cm, 5 cm and 6 cm. Calculate the radius of the incircle.

Let $a = 4$, $b = 5$ and $c = 6$. Use the cosine formula to calculate the cosine of the largest angle, C. Then find $\sin C$, and use this in the formula $\frac{1}{2}ab\sin C$ to find the area of the triangle. Then use Equation 2.

Using the cosine formula, $c^2 = a^2 + b^2 - 2ab\cos C$, gives $36 = 16 + 25 - 2\times 4\times 5\cos C$

which leads to $$\cos C = \frac{1}{8}.$$

Using $\sin^2 C = 1 - \cos^2 C$ shows that $\sin C = \sqrt{1-\frac{1}{64}} = \sqrt{\frac{63}{64}} = \frac{3\sqrt{7}}{8}.$

Using the formula $\frac{1}{2}ab\sin C$ for area, you obtain $\Delta = \frac{1}{2}\times 4\times 5\times\frac{3\sqrt{7}}{8} = \frac{15\sqrt{7}}{4}.$

For this triangle $s = \frac{a+b+c}{2} = \frac{4+5+6}{2} = \frac{15}{2}.$

Therefore, using Equation 2, $rs = \Delta$

$$r\times\frac{15}{2} = \frac{15\sqrt{7}}{4}$$

so $$r = \frac{\sqrt{7}}{2}.$$

The radius of the incircle is $\frac{1}{2}\sqrt{7}$ cm. ∎

12.3 The ecircles

The ecircles of a triangle *ABC* are the circles that touch each of the sides and lie outside the circle. There are three such circles. Figure

12.5 shows part of the ecircle opposite the vertex *A*.

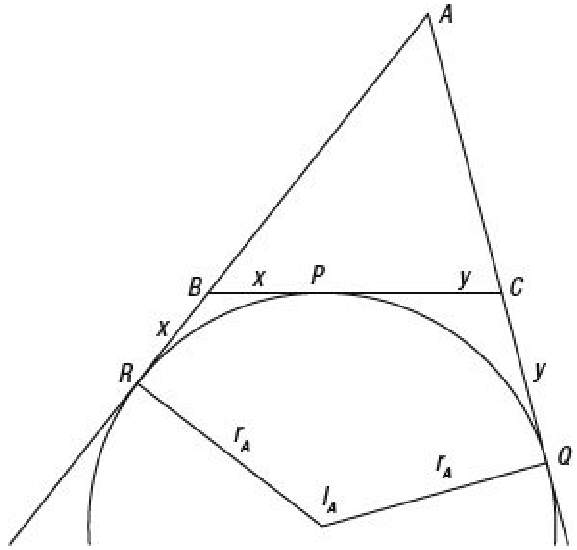


Figure 12.5

The centre of the ecircle opposite *A* will be denoted by I_A and its radius will be denoted by r_A.

To calculate r_A, let the points of contact of the ecircle with *BC* and the lines that extend from triangle *ABC* be *P*, *Q* and *R*.

Let $PB = x$ and $PC = y$. Then $RB = x$ and $QC = y$.

Then, from Figure 12.5, $x + y = a$ and as tangents from an external point, in this case *A*, are equal $c + x = b + y$.

Solving the equations

$$x + y = a$$
$$x - y = b - c$$

simultaneously gives

$$x = \frac{a+b-c}{2} = s - c \text{ and } y = \frac{a-b+c}{2} = s - b.$$

Notice also that $AR = c + x = c + (s - c) = s$ and that $AQ = s$.

The area Δ of triangle *ABC* is

$$\begin{aligned}
\Delta &= \text{area } ARI_AQ - \text{area } BRI_AP - \text{area } CQI_AP \\
&= 2\times\text{area } ARI_A - 2\times\text{area } BRI_A - 2\times\text{area } CQI_A \\
&= 2\times\tfrac{1}{2}r_As - 2\times\tfrac{1}{2}r_A(s-c) - 2\times\tfrac{1}{2}r_A(s-b) \\
&= r_A(s-(s-c)-(s-b)) \\
&= r_A(b+c-s) \\
&= r_A(s-a).
\end{aligned}$$

Therefore $\Delta = r_A (s - a)$. Equation 3

There are related formulae for r_B and r_C.

12.4 Heron's formula: the area of a triangle A number of the formulae in the previous section involved the semi-perimeter of a triangle, that is *s*. There is a very famous formula, called Heron's formula, involving *s* for the area of a triangle. It is $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$.

The method of establishing Heron's formula comes from first using the cosine formula to calculate cos *A*. The value of sin *A* is then calculated using $\sin^2 A = 1 - \cos^2 A$. Finally this value is substituted in the formula for area, $\Delta = \frac{1}{2} bc \sin A$.

Rearranging the cosine formula $a^2 = b^2 + c^2 - 2bc \cos A$ gives

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

Then using $\sin^2 A = 1 - \cos^2 A = (1 + \cos A)(1 - \cos A)$, you obtain

$$\sin^2 A = (1 + \cos A)(1 - \cos A)$$

$$= \left(1 + \frac{b^2 + c^2 - a^2}{2bc}\right)\left(1 - \frac{b^2 + c^2 - a^2}{2bc}\right)$$

$$= \frac{(2bc + b^2 + c^2 - a^2)\,(2bc - b^2 - c^2 + a^2)}{4b^2c^2}$$

$$= \frac{((b + c)^2 - a^2)\,(a^2 - (b - c)^2)}{4b^2c^2}$$

$$= \frac{(b + c + a)\,(b + c - a)\,(a + b - c)\,(a - b + c)}{4b^2c^2}$$

$$= \frac{2s\,(2(s - a))\,(2(s - c))\,(2(s - b))}{4b^2c^2}$$

$$= \frac{4s\,(s - a\,)\,(s - b)\,(s - c)}{b^2c^2}$$

Therefore

$$\Delta = \tfrac{1}{2}\,bc\,\sin A$$

$$= \tfrac{1}{2}\,bc \times \frac{\sqrt{4s(s - a)\,(s - b)\,(s - c)}}{bc}$$

$$= \sqrt{s(s - a)\,(s - b)\,(s - c)}.$$

Therefore $\quad \Delta = \sqrt{s(s - a)\,(s - b)\,(s - c)}. \qquad$ Equation 4

Equation 4 is called Heron’s formula.

You can use the four equations established in this chapter, together with the technique used in Example 12.4, to establish most of the formulae you need.

Example 12.5

Find the radii of the three ecircles of the triangle with sides of 4 cm, 5 cm and 6 cm.

Let $a = 4$, $b = 5$ and $c = 6$.

Then $s = \frac{1}{2}(a+b+c) = \frac{15}{2}$, $s-a = \frac{7}{2}$, $s-b = \frac{5}{2}$ and $s-c = \frac{3}{2}$.

In Equation 3, $\boldsymbol{\Delta} = r_{\mathrm{A}}(s - a)$, use Heron's formula, Equation 4, to find the area of the triangle.

Then $$\Delta = \sqrt{\frac{15}{2}\times\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}} = \frac{15}{4}\sqrt{7}.$$

Therefore, from $\Delta = r_A(s - a)$

$$\frac{15}{4}\sqrt{7} = \frac{7}{2}r_A$$

so $$r_A = \frac{15\sqrt{7}}{4}\times\frac{2}{7} = \frac{15\sqrt{7}}{14}.$$

Similarly $$r_B = \frac{15\sqrt{7}}{4}\times\frac{2}{5} = \frac{3\sqrt{7}}{2}$$

and $$r_C = \frac{15\sqrt{7}}{4}\times\frac{2}{3} = \frac{5\sqrt{7}}{2}.$$

The radii of the ecircles are $\frac{15\sqrt{7}}{14}$ cm, $\frac{3\sqrt{7}}{2}$ cm and $\frac{5\sqrt{7}}{2}$ cm.∎

Notice that the area of this triangle was found by an alternative method in Example 12.4.

Example 12.6

Prove that $\sin\frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}}$.

Use the cosine formula to find cos *A*, and then use Equation 16 in Chapter 9, $\cos A = 1 - 2\sin^2\frac{1}{2}A$, to find $\sin\frac{1}{2}A$.

Rearranging the cosine formula $a^2 = b^2 + c^2 - 2bc\cos A$ gives

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

Therefore $1 - 2\sin^2\frac{1}{2}A = \frac{b^2 + c^2 - a^2}{2bc}$

so $2\sin^2\frac{1}{2}A = 1 - \frac{b^2 + c^2 - a^2}{2bc}$.

Therefore

$$\begin{aligned}
\sin^2\tfrac{1}{2}A &= \tfrac{1}{2}\left(1 - \frac{b^2 + c^2 - a^2}{2bc}\right) \\
&= \frac{2bc - b^2 - c^2 + a^2}{4bc} \\
&= \frac{a^2 - (b-c)^2}{4bc} \\
&= \frac{(a + (b-c))(a - (b-c))}{4bc} \\
&= \frac{(2(s-c))(2(s-b))}{4bc} \\
&= \frac{(s-b)(s-c)}{bc}.
\end{aligned}$$

Therefore $\sin\frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}}$. ∎

Exercise 12.2

1 For the triangle with sides of length 2 cm, 3 cm and 4 cm, find the area and the radii of the incircle and the three ecircles.

2 Find the radius of the incircle of the triangle with sides 3 cm and 4 cm with an angle of 60° between them.

3 Prove that $rr_A r_B r_C = \Delta^2$.

4 Prove that $r = 4R \sin\frac{1}{2}A \sin\frac{1}{2}B \sin\frac{1}{2}C$.

5 Prove that $r_C = 4R \cos\frac{1}{2}A \cos\frac{1}{2}B \sin\frac{1}{2}C$.

6 In Example 12.6 a formula for $\sin\frac{1}{2}A$ is derived. Use a similar method to derive a formula for $\cos\frac{1}{2}A$, and then use both of them to find a formula for $\tan\frac{1}{2}A$.

7 Prove the formula $\tan\frac{1}{2}(B - C) = \frac{b - c}{b + c}\cot\frac{1}{2}A$.

8 Prove that $r = (s - a)\tan\frac{1}{2}A$.

Key ideas

- The circumcircle of a triangle is the circle which passes through each of the vertices of the triangle.
- For a triangle *ABC* the radius, *R*, of the circumcircle is given by the formula: $2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.
- The area, **Δ**, of a triangle *ABC* is given by the formula: $\Delta = \frac{1}{2}ab\sin C$.
- The incircle of a triangle lies inside the triangle and touches each of the sides of the triangle.
- The semi-perimeter, *s*, of a triangle *ABC* is the distance equal to half of the perimeter.

$$s = \frac{a+b+c}{2}$$

- The area, **Δ**, of a triangle is given by the formula: $\Delta = rs$, where *r*, is the radius of the incircle and *s* is the semi-perimeter of the triangle.
- The ecircles of a triangle are circles which touch each side and lie outside the triangle. A triangle has three ecircles.
- The area, **Δ**, of a triangle *ABC* is given by the formula: $\Delta = r_A(s - a)$, where r_A is the radius of the ecircle opposite *A* and *s*, is the semi-perimeter of the triangle.
- Heron's formula for the area of a triangle *ABC* is: $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, where *s* is the semi-perimeter of the triangle.

13
General solutions of equations

In this chapter you will learn:

- *how to find all the solutions of simple trigonometric equations*
- *general formulae for these solutions.*

13.1 The equation $\sin \theta = \sin \alpha$

In earlier chapters you have always, when asked to solve an equation, been given an interval such as −180 to 180 or 0 to 360 in which to find the solutions. In this chapter, the task will be to find a way of giving all solutions of a trigonometric equation, not just those solutions confined to a given interval.

For example, if you are given the equation $\sin \theta° = 0.6427\ldots$ you can immediately look up the corresponding principal angle, in this case 40°. You can therefore replace the equation $\sin \theta° = 0.6427\ldots$ by the equation $\sin \theta° = \sin 40°$ and the problem of finding all solutions of $\sin \theta° = 0.6427\ldots$ by finding all solutions of $\sin \theta° = \sin 40°$.

In Section 5.2 you saw that, if 40 is a solution, then $180 - 40 = 140$ is also a solution, and that you can add any whole number multiple of 360 to give all solutions.

Thus all solutions of $\sin \theta° = \sin 40°$ are

$$\ldots, -320, 40, 400, 760, \ldots$$

and

$$\ldots, -220, 140, 500, 860, \ldots .$$

You can write these solutions in the form

$40 + 360n$ and $140 + 360n$ (n is an integer).

Notice that these two formulae follow a pattern. Starting from 40 you can write them, in ascending order, as

$$\begin{array}{l} 0 \times 180 + 40 \\ 1 \times 180 - 40 \\ 2 \times 180 + 40 \\ 3 \times 180 - 40 \\ \vdots \end{array}$$

The pattern works backwards also, so all solutions fall into the pattern

$$
\begin{array}{r}
\vdots \\
-2\times 180+40 \\
-1\times 180-40 \\
0\times 180+40 \\
1\times 180-40 \\
2\times 180+40 \\
3\times 180-40 \\
\vdots
\end{array}
$$

You can use this pattern to write all the solutions in one formula as

$$180n + (-1)^n\, 40$$

where $(-1)^n$ takes the value -1 when n is odd and $+1$ when n is even.

You can generalize this result further. If you solve for $\theta°$ the equation $\sin\theta° = \sin\alpha°$ to give the solutions in terms of α you would obtain

$$\theta° = 180n + (-1)^n\alpha° \quad (n \text{ is an integer}). \qquad \textbf{Equation 1}$$

Nugget

n can be positive or negative. When $n = 0$ the formula gives the principal angle (the answer from your calculator).

Notice that in the formula $180n + (-1)^n\alpha°$ there is no reason why $\alpha°$ should be an acute angle.

If the original angle had been 140, the formula $\theta = 180n + (-1)^n 140$

where n is an integer gives the solutions

$\ldots, -360 + 140, -180 - 140, 0 + 140, 180 - 140, 360 + 140, \ldots$

or $\ldots, -220, -320, 140, 40, 500, \ldots$.

Therefore the formula $180n + (-1)^n\alpha°$ where n is an integer gives you all solutions of the equation $\sin\theta° = \sin\alpha°$ provided you have one solution $\alpha°$.

In radians, the same formula gives

$$\theta = n\pi + (-1)^n\alpha \quad (n \text{ is an integer}) \qquad \text{Equation 2}$$

as the solution of the equation $\sin\theta = \sin\alpha$.

Example 13.1

Find all the solutions in degrees of the equation $\sin\theta° = -\frac{1}{2}$.

The principal solution of the equation $\sin\theta° = -\frac{1}{2}$ is −30

Using Equation 1, all solutions are $180n + (-1)^n(-30)$ or alternatively, $180n - (-1)^n 30$ for integer n. ∎

Example 13.2

Find all solutions in radians of the equation $\sin\left(2\theta+\frac{1}{6}\pi\right)=\frac{1}{2}\sqrt{3}$..

Let $y=2\theta+\frac{1}{6}\pi$. Then the equation becomes $\sin y=\frac{1}{2}\sqrt{3}$.

The principal angle is $\frac{1}{3}\pi$. So the solution of the equation is $y=n\pi+(-1)^n\frac{1}{3}\pi$.

Therefore $\quad 2\theta+\frac{1}{6}\pi=n\pi+(-1)^n\frac{1}{3}\pi,$

so $\quad \theta=\frac{1}{2}n\pi+(-1)^n\frac{1}{6}\pi-\frac{1}{12}\pi$ (n is an integer). ∎

13.2 The equation cos *θ* = cos *α*

It is convenient to consider the equation cos $\theta°$ = 0.7660…, that is the equation cos $\theta°$ = cos 40°.

Using the method of Section 5.3, the solutions of this equation are

…, −320, 40, 400, 760,…

and …, −400, −40, 320, 680,… .

You can write these solutions in the form

$40+360n$ and $-40+360n$ (n is an integer).

The pattern for these solutions is easier to follow than that for the sine function. It is $\theta=360n\pm40$ (n is an integer).

You can generalize this further, as in the case of the sine function. The solution of the equation cos $\theta°$ = cos $\alpha°$ is

$\theta°=360n\pm\alpha°$ (n is an integer). **Equation 3**

Nugget

When $n = 0$ the formula gives ± the principal angle (the answer from your calculator). Once you have these two angles you just need to add multiples of 360° in order to find all the other solutions.

In radians, the same formula gives

$$\theta = 2n\pi \pm \alpha \quad (n \text{ is an integer}) \qquad \text{Equation 4}$$

as the solution of the equation $\cos\theta = \cos\alpha$.

13.3 The equation $\tan\theta = \tan\alpha$

Consider the equation $\tan\theta° = \tan 40°$.

Using the method of Section 5.4, the solutions of this equation are
$\ldots, -320, -140, 40, 220, 400, 580, 760, \ldots$.

An alternative form is

$$40 + 180n \quad (n \text{ is an integer}).$$

The solution of the equation $\tan\theta° = \tan\alpha°$ is
$\theta° = 180n + \alpha° \quad (n \text{ is an integer}).$ Equation 5

Nugget

Remember the graph of $y = \tan \theta°$ has a period of 180° and so once you have found the principal angle from your calculator you need to add or subtract multiples of 180° in order to find all the other solutions.

In radians, the same formula gives

$$n\pi + \alpha \quad (n \text{ is an integer}) \qquad \text{Equation 6}$$

as the solution of the equation $\tan \theta = \tan \alpha$.

Example 13.3

Find the general solution in degrees of the equation $\cos\theta° = -\frac{1}{2}\sqrt{3}$.

The principal angle is 150°, so the general solution is
$\theta° = 360n \pm 150°$ (n is an integer). $\blacksquare$

Example 13.4

Find the general solution in radians of the equation $\tan 2x = -\sqrt{3}$.

Let $2x = y$. The principal angle for equation $\tan y = -\sqrt{3}$ is $-\frac{1}{3}\pi$, so the general solution for y is $y = n\pi + \left(-\frac{1}{3}\pi\right)$ (n is an integer).

This is the same as $y = n\pi + \frac{1}{3}\pi$ (n is an integer).

Therefore, as $x = \frac{1}{2}y, x = \frac{1}{2}n\pi + \frac{1}{6}\pi$ (n is an integer). ∎

Example 13.5

Solve in radians the equation cos 2θ = cos θ.

Using Equation 4,

$$2\theta = 2n\pi \pm \theta \ (n \text{ is an integer}).$$

Taking the positive sign gives

$$2\theta = 2n\pi \ (n \text{ is an integer}),$$

and the negative sign

$$3\theta = 2n\pi \ (n \text{ is an integer}).$$

Therefore the complete solution is $\theta = \frac{2}{3}n\pi$ (n is an integer). ∎

Notice that the solution given, $\theta = \frac{2}{3}n\pi$ (n is an integer), includes, when n is a multiple of three, the solution obtained by taking the positive sign.

Example 13.6

Solve in radians the equation $\sin 3\theta = \sin \theta$.

Using Equation 1,

$$3\theta = n\pi + (-1)^n \theta \text{ (} n \text{ is an integer).}$$

Taking n even, and writing $n = 2m$ gives $3\theta = 2m\pi + \theta$, that is $\theta = m\pi$.

Taking n odd, and writing $n = 2m + 1$ gives $3\theta = (2m + 1)\pi - \theta$, that is $4\theta = (2m + 1)\pi$.

Thus the complete solution is

$$\text{either } \theta = m\pi \text{ or } \theta = \tfrac{1}{4}(2m + 1)\pi \text{ where } m \text{ is an integer.} \blacksquare$$

Example 13.7

Solve the equation $\cos 2\theta° = \sin \theta°$ giving all solutions in degrees.

Use the fact that $\sin \theta° = \cos(90 - \theta)°$ to rewrite the equation as
$\cos 2\theta° = \cos(90 - \theta)°$.

Using Equation 3,

$$2\theta = 360n \pm (90 - \theta) \quad (n \text{ is an integer}).$$

Taking the positive sign gives

$$3\theta = 360n + 90 \text{ or } \theta = 120n + 30,$$

and the negative sign

$$\theta = 360n - 90.$$

Thus the complete solution is

$$\text{either } \theta = 120n + 30 \text{ or } \theta = 360n - 90 \text{ where } n \text{ is an integer.} \blacksquare$$

Exercise 13.1

In questions 1 to 10 find the general solution in degrees of the given equation.

1 $\sin\theta° = \frac{1}{2}\sqrt{3}$

2 $\cos\theta° = \frac{1}{2}$

3 $\tan\theta° = -1$

4 $\sin(2\theta - 30)° = -\frac{1}{2}$

5 $\cos 3\theta° = -1$

6 $\tan(3\theta + 60)° = \frac{1}{3}\sqrt{3}$

7 $\tan 2\theta° = \tan\theta°$

8 $\cos 3\theta° = \cos 2\theta°$

9 $\sin 3\theta° = \sin\theta°$

10 $\sin 3\theta° = \cos\theta°$

In questions 11 to 20 give the general solution in radians of the given equation.

11 $\sin 2\theta = -\frac{1}{2}$

12 $\cos 3\theta = 0$

13 $\tan\left(2\theta + \frac{1}{2}\pi\right) = 0$

14 $\sin\left(3\theta + \frac{1}{6}\pi\right) = \frac{1}{2}$

15 $\cos\left(2\theta + \frac{1}{2}\pi\right) = 0$

16 $\tan\left(\frac{1}{2}\pi - 2\theta\right) = -1$

17 $\tan 3\theta = \cot(-\theta)$ **18** $\cos 3\theta = \sin 2\theta$

19 $\cos 3\theta = \sin\left(\frac{1}{2}\pi - \theta\right)$

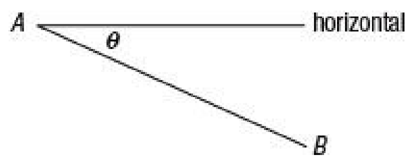
20 $\sin 2\theta = -\sin\theta$

Key ideas

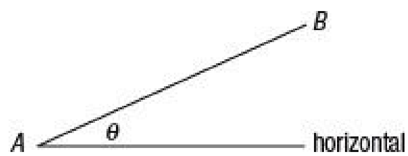
- The general solution in degrees of $\sin \theta° = \sin \alpha°$ is $\theta° = 180n + (-1)^n \alpha°$ (n is an integer).
- The general solution in radians of $\sin \theta = \sin \alpha$ is $\theta = n\pi + (-1)^n \alpha$ (n is an integer).
- The general solution in degrees of $\cos \theta° = \cos \alpha°$ is $\theta° = 360n \pm \alpha°$ (n is an integer).
- The general solution in radians of $\cos \theta = \cos \alpha$ is $\theta = 2n\pi \pm \alpha$ (n is an integer).
- The general solution in degrees of $\tan \theta° = \tan \alpha°$ is $\theta° = 180n + \alpha°$ (n is an integer).
- The general solution in radians of $\tan \theta = \tan \alpha$ is $\theta = n\pi + \alpha$ (n is an integer).

Glossary

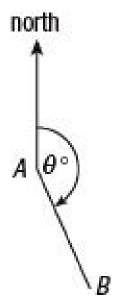
angle of depression The angle of depression of *B* from *A*, where *B* is below *A*, is the angle θ that the line *AB* makes with the horizontal.



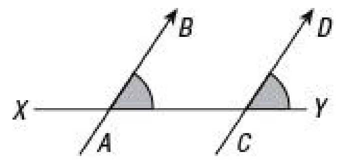
angle of elevation The angle of elevation of *B* from *A*, where *B* is above *A*, is the angle θ that the line *AB* makes with the horizontal.



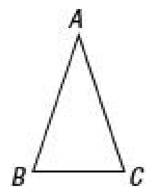
bearing The bearing θ of *B* from *A* is the angle in degrees between north and the line *AB*, measured clockwise from the north. See figure below.



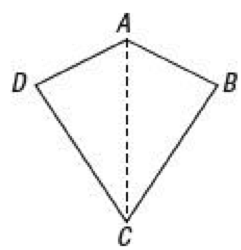
corresponding angles When two parallel lines, *AB* and *CD*, are traversed by a line *XY*, the marked angles in the figure below are called corresponding angles, and are equal.



isosceles triangle An isosceles triangle is a triangle with two equal sides. In the figure below, $AB = AC$.

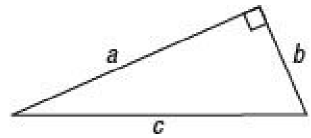


kite A kite is a quadrilateral which has one line of reflective symmetry. In the figure below, *ABCD* is a kite with *AC* as its axis of symmetry.

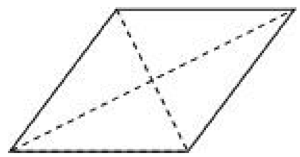


periodic, period A periodic function $f(x)$ is a function with the property that there exists a number c such that $f(x) = f(x + c)$ for all values of x. The smallest such value of c is called the period of the function.

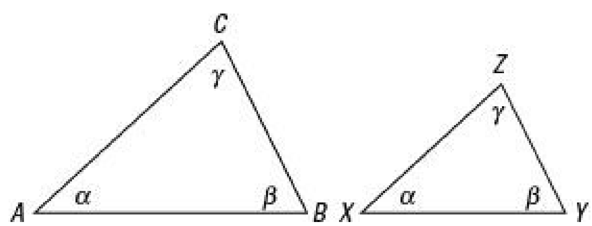
Pythagoras's theorem Pythagoras's theorem states that in a right-angled triangle with sides a, b and c, where c is the hypotenuse, $c^2 = a^2 + b^2$.



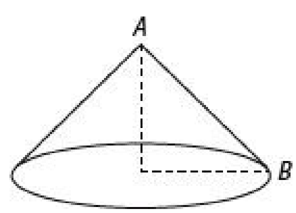
rhombus A rhombus is a quadrilateral which has all four sides equal in length. In a rhombus, the diagonals cut at right angles, and bisect each other.



similar triangles Two triangles which have equal angles are similar. The sides of similar triangles are proportional to each other. Triangles *ABC* and *XYZ* are similar, and $\frac{BC}{YZ}=\frac{CA}{ZX}=\frac{AB}{XY}$.



slant height, of a cone The slant height of the cone is the length of the sloping edge *AB*.



Summary of trigonometric formulae

Relations between the ratios

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Pythagoras's equation

$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \mathrm{cosec}^2\theta$$

Relations for solving equations

$$\sin\theta^\circ = \sin(180-\theta)^\circ$$
$$\cos\theta = \cos(-\theta)$$
$$\tan\theta^\circ = \tan(180+\theta)^\circ$$

Solution of triangles

$$\Delta = \tfrac{1}{2}bc\sin A = \tfrac{1}{2}ca\sin B = \tfrac{1}{2}ab\sin C$$
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$
$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = c^2 + a^2 - 2ca\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Compound angles

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$
$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

Multiple angles

$$\sin 2A = 2\sin A\cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A$$
$$\cos 2A = 2\cos^2 A - 1$$
$$\cos 2A = 1 - 2\sin^2 A$$
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Factor formulae

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$
$$2\cos A\sin B = \sin(A+B) - \sin(A-B)$$
$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$
$$2\sin A\sin B = \cos(A-B) - \cos(A+B)$$
$$\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$$
$$\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$$
$$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$$
$$\cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$$

General solutions of equations

The general solution in degrees of the equation sin $\theta°$ = sin$\alpha°$ is
$\theta° = 180n + (-1)^n\alpha°$ (n is an integer).

The general solution in radians of the equation sin θ = sinα is
$\theta = n\pi + (-1)^n\alpha$ (n is an integer).

The general solution in degrees of the equation cos $\theta°$ = cos $\alpha°$ is
$\theta° = 360n \pm \alpha°$ (n is an integer).

The general solution in radians of the equation cos θ = cos α is
$\theta = 2n\pi \pm \alpha$ (n is an integer).

The general solution in degrees of the equation tan $\theta°$ = tan $\alpha°$ is
$\theta° = 180n + \alpha°$ (n is an integer).

The general solution in radians of the equation tan θ = tan α is
$\theta = n\pi + \alpha$ (n is an integer).

Radians

$$\pi \text{ rad} = 180 \text{ degrees}$$

Length of arc

$$s = r\theta$$

Area of circular sector

$$A = \frac{1}{2} r^2 \theta$$

Radius of circumcircle

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Radius of incircle

$$rs = \Delta$$

where $s = \frac{1}{2}(a + b + c)$.

Radius of ecircle

$$\Delta = r_A(s-a) = r_B(s-b) = r_C(s-c)$$

where $s = \frac{1}{2}(a+b+c)$.

Heron's formula for area of triangle

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a+b+c)$.

Answers

Answers involving lengths are given either correct to three significant figures or to three decimal places, whichever seems more appropriate. Answers for angles are usually given correct to two decimal places.

Exercise 1.1 (Section 1.4)

1 0.364

2 0.577

3 5729.578

4 0.852

5 1.881

6 0.009

7 18.88°

8 63.43°

9 80.72°

10 0.01°

11 45.00°

12 60.00°

Exercise 1.2 (Section 1.6)

1 8.36 m

2 038.66°

3 4.69 miles

4 19.54 m

5 126 m

6 21.3 m

7 53.01°, 36.99°

8 144 m

9 75.21°

Exercise 1.3 (Section 1.7)

1 68.20°

2 3.73 cm

3 2.51 cm

4 48.81°

5 11.59 cm

6 3.46 cm

7 8.06 cm

8 24.78°

9 63.86°

10 3.60 cm

Exercise 2.1 (Section 2.3)

1 66.42°

2 5.47 cm

3 2.25 cm

4 28.96°

5 3.37 cm

6 1.73 cm

7 6.85 cm

8 27.49°

9 60.61°

10 2.68 cm

11 0.994, 0.111

12 5.14 cm, 3.06 cm

13 36.87°, 53.13°

14 38.43°

15 9.32 km

16 4.82 cm

17 2.87 cm

18 7.19 km

19 38.42°, 51.58°, 7.40 cm

Exercise 2.2 (Section 2.7)

1 73.30°

2 6.66 cm

3 5.79 cm

4 48.81°

5 9.44 cm

6 2.1 cm

7 10.03 cm

8 67.38°

9 19.02°

10 5.38 cm

11 35.02°, 2.86 cm

12 44.20°

13 55.5 mm, 72.7 mm

14 30.50°, 59.50°

15 2.66 cm, 1.86 cm, 3.88 cm

16 44.12°, 389.8 mm

17 69.51°

18 14.7 km, 10.3 km

19 0.681 cm

20 $\frac{1}{2}x\sqrt{3}$ m, $\frac{1}{2}\sqrt{3}$, $\frac{1}{2}$

21 2.60 cm, 2.34 cm

22 3.59°

23 10.17 km, 11.70 km

24 328.17°, 17.1 km

Exercise 2.3 (Section 2.8)

1 10.05 cm

2 4.61 cm

3 145.42°

4 4.33 cm

5 10.91 km, 053.19°

6 85.06°

7 44.80 cm

8 8.58 cm

9 25.24 cm

10 13.20 cm

Exercise 3.1 (Section 3.4)

1 68.20°, 60.50°

2 73.57°, 78.22°

3 3.68 m

4 97.08°

5 35.26°

6 12.72°, 15.76°

7 54.74°

8 5.20 cm, 50.77°

9 56.31°

10 35.26°

Exercise 4.1 (Section 4.3)

1 0.766

2 −0.766

3 −0.940

4 0.985

5 −0.342

6 0.174

7 0

8 0

9 3

10 1

11 1

12 1

13 2

14 4

15 1

16 0

17 0

18 −1

19 −1

20 1

Exercise 4.2 (Section 4.6)

1 −1.732

2 −0.577

3 0.364

4 −5.671

6 0.213

7 −0.176

Exercise 5.1 (Section 5.2)

1 17.46°, 162.54°

2 26.74°, 153.26°

3 0°, 180°, 360°

4 90°

5 270°

6 185.74°, 354.26°

7 206.74°, 333.26°

8 210°, 330°

9 −171.37°, −8.63°

10 −150°, −30°

11 −180°, 0°, 180°

12 90°

13 −90°

14 64.16°, 115.84°

15 −115.84°, −64.16°

16 −130.00°, −50.00°

17 15°, 75°, 195°, 255°

18 13.37°, 76.63°, 193.37°, 256.63°

19 0°, 60°, 120°, 180°, 240°, 300°, 360°

20 135°, 315°

21 60°, 300°

22 180°

23 70°, 110°, 190°, 230°, 310°, 350°

24 20.91°, 69.09°, 200.91°, 249.09°

25 1.62 hours

26 About 122 days

Exercise 5.2 (Section 5.4)

1 109.47°, 250.53°

2 63.43°, 243.43°

3 41.41°, 318.59°

4 153.43°, 333.43°

5 30°, 150°, 210°, 330°

6 22.5°, 112.5°, 202.5°, 292.5°

7 203.07°

8 143.18°

9 70.00°, 110.00°, 250.00°, 290.00°

10 87.14°, 177.14°, 267.14°, 357.14°

11 −126.27°, −53.73°, 53.73°, 126.27°

12 −103.28°, −13.28°, 76.72°, 166.72°

13 −168.21°, −101.79°, 11.79°, 78.21°

14 −90°, 90°

15 −90°, 30°, 150°

16 −45°

17 4.14 pm and 7.46 pm

Exercise 6.1 (Section 6.4)

These answers are given in alphabetical order.

1 15.82 cm, 14.73 cm, 94.22 cm^2

2 20.29 cm, 30.36 cm, 152.22 cm^2

3 7.18 mm, 6.50 mm, 18.32 mm^2

4 5.59 cm, 7.88 cm, 22.00 cm^2

5 23.06 cm, 17.32 cm, 192.88 cm^2

6 C = 28.93°, A = 126.07°, a = 58.15 cm or C = 151.07°, A = 3.93°, a = 4.93 cm

7 C = 51.31°, A = 88.69°, a = 109.26 cm or C = 128.69°, A = 11.31°, a = 21.43 cm

8 A = 61.28°, B = 52.72°, b = 87.10 cm

9 A = 35.00°, B = 115.55°, b = 143.13 cm or A = 145.00°, B = 5.55°, b = 15.35 cm

Exercise 6.2 (Section 6.5)

1 37.77 cm

2 5.30 cm

3 54.73 cm

4 25.9 cm

5 2.11 cm

6 7.97 cm

7 28.96°, 46.57°, 104.48°

8 40.11°, 57.90°, 81.99°

9 62.19°, 44.44°, 73.37°

10 28.91°, 31.99°, 119.10°

11 106.23°

12 43.84°

13 The cosine of the largest angle is −1 and the largest angle is 180° – this triangle is impossible to draw; the longest side is longer than the sum of the other two sides.

14 5.93 km

15 52.01°, 88.05°, 39.93°

16 45.17°, 59.60°, 7.25 cm

17 56.09°

18 16.35 m, 13.62 m

19 41.04°

20 Two triangles. 66.82°, 63.18° and 29.13 cm 16.82°, 113.18° and 9.44° cm

21 98.34 m

22 5.71 m, 6.08 m

23 3.09 mm

24 7.98 cm, 26.32° and 29.93°

25 3.81 cm, 4.20 cm, 7.81 cm^2

26 4.51 hours

27 4.41 km

28 0.305 m^2

29 49.46°, 58.75°

Exercise 6.3 (Section 6.10)

1 15.2 m

2 546 m

3 276 m

4 192 m

5 889 m

6 1.26 km

7 3700 m

8 2.23 km

9 2.88 km

10 2.17 km

11 500 m

12 3.64 km, 315°, 5.15 km

13 72.9 m, 51.1 m

14 1.246 km

15 189 m

16 63.7 m

17 3470 m, 7270 m

Exercise 7.1 (Section 7.5)

1 60°

2 15°

3 270°

4 120°

5 135°

6 720°

7 0.588

8 0.924

9 0.309

10 0.383

11 0.966

12 0.5

13 13.41°

14 $\frac{1}{12}\pi$ rad

15 $\frac{2}{5}\pi$ rad

16 $\frac{11}{30}\pi$ rad

17 $\frac{7}{12}\pi$ rad

18 4.75 rad

19 1.12 cm

20 6.72 cm^2

21 1.6 rad

22 0.611 rad, 35.01°

23 $\frac{1}{4}\pi$ rad, $\frac{1}{3}\pi$ rad and $\frac{5}{12}\pi$ rad

24 23.18 cm^2 and 55.36 cm^2

25 2.90 cm^2

26 3.03 cm^2

Exercise 8.1 (Section 8.2)

1 −0.5735

2 ±3.180

3 ±1.077

4 $\dfrac{k}{\sqrt{k^2-1}}$

5 $-\sqrt{1+t^2}, -\dfrac{1}{\sqrt{1+t^2}}, \dfrac{t}{\sqrt{1+t^2}}$

6 $\dfrac{1}{\sqrt{s^2-1}}, \dfrac{\sqrt{s^2-1}}{s}$

7 $-120°, -60°, 60°, 120°$

8 $-135°, -45°, 45°, 135°$

9 $-180°, 0°, 30°, 150°, 180°$

10 $-180°, 0°, 180°$

11 $38.17°, 141.83°$

12 $-116.57°, 63.43°$

13 $-153.43°, -26.57°, 26.57°, 153.43°$

14 $-75.52°, 0°, 75.52°$

15 $-60°, 30°, 60°, 150°$

Exercise 9.1 (Section 9.5)

1 $0.663, -0.749$

2 $\dfrac{\sqrt{6}-\sqrt{2}}{4}, \dfrac{\sqrt{6}-\sqrt{2}}{4}$

3 Expanding gives $\sin(90 - \theta)° = \sin 90° \cos \theta° - \cos 90° \sin \theta° = 1 \times \cos \theta° - 0 \times \sin \theta° = \cos \theta°$

4 -0.997

5 $0.894, 1.999$

6 $\dfrac{3+\sqrt{3}}{3-\sqrt{3}}$ or $\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$ or $2+\sqrt{3}$

7 $\frac{40}{13} \approx 3.077, \frac{20}{37} \approx 0.541$

8 Using Equation 6,
$$\tan(180-\theta)° = \frac{\tan 180° - \tan\theta°}{1+\tan 180° \tan\theta°} = \frac{0-\tan\theta°}{1+0\times\tan\theta°} = -\tan\theta°$$

9 $\sin 34° = 0.56$

10 $\cos 61° = 0.48$

11 tan 68° = 2.48

12 tan 39° = 0.81

Exercise 9.2 (Section 9.6)

1 $\frac{24}{25}, \frac{7}{25}, \frac{24}{7}$

2 $-\frac{24}{25}, \frac{7}{25}, -\frac{24}{7}$

3 0.484, 0.875, 0.553

4 1, 0

5 0.992, −0.129

6 −0.992, −0.129

7 sin 72° = 0.951, cos 72° = 0.309

8 0.5, −0.5

9 Use Equation 16, $\cos\theta = 1 - 2\sin^2\frac{1}{2}\theta$, to get $\sin^2\frac{1}{2}\theta = \dfrac{1-\cos\theta}{2}$ and the result follows. Use Equation 15, $\cos\theta = 2\cos^2\frac{1}{2}\theta - 1$, to get $\cos^2\frac{1}{2}\theta = \dfrac{1+\cos\theta}{2}$ and the result follows.

10 $\pm\frac{1}{2}, \pm\frac{1}{2}\sqrt{3}$

11 ± 0.6

12 tan 20° = 0.364

Exercise 9.3 (Section 9.7)

In the solutions to the identities, the reason for each step is given by placing an Equation number from Chapter 9 in bold-faced type in brackets. Where no reason is given, either an algebraic simplification or the use of $\sin^2 A + \cos^2 A = 1$ is involved.

1 $$\begin{aligned} \text{LHS} &= \sin(A+B) + \sin(A-B) \\ &= \sin A\cos B + \cos A\sin B + \sin A\cos B - \cos A\sin B(\mathbf{1}\ \&\ \mathbf{2}) \\ &= 2\sin A\cos B = \text{RHS} \end{aligned}$$

2 $\text{LHS} = \cos(A+B) + \cos(A-B)$

$= \cos A\cos B - \sin A\sin B + \cos A\cos B + \sin A\sin B(3\ \&\ 4)$

$= 2\cos A\cos B = \text{RHS}$

3 $\text{LHS} = \dfrac{\cos 2A}{\cos A + \sin A}$

$= \dfrac{\cos^2 A - \sin^2 A}{\cos A + \sin A}(9)$

$= \dfrac{(\cos A + \sin A)(\cos A - \sin A)}{\cos A + \sin A}$

$= \cos A - \sin A = \text{RHS}$

4 $\text{LHS} = \sin 3A = \sin(2A + A)$

$= \sin 2A\cos A + \cos 2A\sin A(1)$

$= 2\sin A\cos A \times \cos A + (1 - 2\sin^2 A) \times \sin A(7\ \&\ 11)$

$= 2\sin A\cos^2 A + \sin A - 2\sin^3 A$

$= 2\sin A(1 - \sin^2 A) + \sin A - 2\sin^3 A$

$= 3\sin A - 4\sin^3 A = \text{RHS}$

5 $\text{LHS} = \dfrac{\sin A}{\cos A} + \dfrac{\cos A}{\sin A}$

$= \dfrac{\sin^2 A + \cos^2 A}{\sin A\cos A}$

$= \dfrac{1}{\sin A\cos A} = \dfrac{2}{2\sin A\cos A}$

$= \dfrac{2}{\sin 2A}(7) = \text{RHS}$

6 $\text{LHS} = \dfrac{1}{\sin\theta} + \dfrac{\cos\theta}{\sin\theta}$

$= \dfrac{1 + \cos\theta}{\sin\theta}$

$= \dfrac{2\cos^2 \frac{1}{2}\theta}{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}(12\ \&\ 8)$

$= \dfrac{\cos\frac{1}{2}\theta}{\sin\frac{1}{2}\theta} = \text{RHS}$

7 $$\text{LHS} = \frac{\sin\theta}{\sin\phi} + \frac{\cos\theta}{\cos\phi}$$
$$= \frac{\sin\theta\cos\phi + \sin\phi\cos\theta}{\sin\phi\cos\phi}$$
$$= \frac{\sin(\theta+\phi)}{\frac{1}{2}\times 2\sin\phi\cos\phi}(1)$$
$$= \frac{2\sin(\theta+\phi)}{\sin 2\phi}(7) = \text{RHS}$$

Exercise 9.4 (Section 9.8)

1 −90°, 30°, 90°, 150°

2 −150°, −30°, 30°, 150°

3 −148.28°, −58.28°, 31.72°, 121.72°

4 −165°, −105°, 15°, 75°

5 −157.5°, −67.5°, 22.5°, 112.5°

6 −150°, −30°, 30°, 150°

7 −120°, 0°, 120°

8 0°, ±60°, ±120°, ±180°

9 ±30°, ±90°, ±150°

10 −157.5°, −67.5°, 22.5°, 112.5°

Exercise 10.1 (Section 10.2)

1 $\sqrt{2}\sin(x+45)°$

2 13 sin(x + 67.38)°

3 $\sqrt{29}\sin(x+21.80)°$

4 $\sqrt{2}\sin(x+135)°$

5 $\sqrt{10}\sin(x+288.43)°$

6 $\sqrt{10}\sin(x+108.43)°$

7 $\sqrt{2}\cos(x+\frac{7}{4}\pi)$

8 $\sqrt{2}\cos(x-\frac{1}{4}\pi)$

9 $\sqrt{2}\sin(x-\frac{7}{4}\pi)$

10 Both functions have the form $R\sin(x + a)$ with $R=\sqrt{50}$. This means that in both cases the maximum value is $\sqrt{50}$.

Exercise 10.2 (Section 10.3)

1 0°, 90°, 360°

2 90°, 330°

3 40.21°, 252.41°

4 45.24°, 180°

5 159.24°, 283.13°

6 45°, 90°, 225°, 270°

7 −180°, −90°, 180°

8 −180°, −60°, 180°

9 −69.20°, 32.33°

10 −157.38°, −90°

11 289.59°, 349.20°

12 −135°, −75°, 45°, 105°

13 $\sqrt{5}$ at $x=116.57°$, $-\sqrt{5}$ at $x=-63.43°$

14 5 at x = −53.13°, −5 at x = 126.87°

15 2 at x = −15° and 165°, −2 at x = −105° and 75°

16 $\sqrt{2}$ at $x=-22.5°$ and $157.5°$, $-\sqrt{2}$ at $x=-112.5°$ and $67.5°$

17 7 at $x = 36.87°$, -3 at $x = -143.13°$

18 4.732 at $x = -17.63°$ and $162.37°$, 1.268 at $x = -107.63°$ and $72.37°$

Exercise 11.1 (Section 11.1)

1 $\frac{1}{2}(\sin 4\theta + \sin 2\theta)$

2 $\frac{1}{2}(\sin 80° - \sin 10°)$

3 $\frac{1}{2}(\cos 80° + \cos 20°)$

4 $\frac{1}{2}(\sin 8\theta - \sin 2\theta)$

5 $\frac{1}{2}(\cos 3(C + D) + \cos(C - D))$

6 $\frac{1}{2}(\sin 90° - \sin 30°) = \frac{1}{4}$

7 $\cos 2A - \cos 4A$

8 $\frac{1}{2}(\sin 6C - \sin 10D)$

9 $2\cos C \sin D = \sin(C + D) - \sin(C - D)$

10 $2\cos C \cos B = \cos(C + B) + \cos(C - B)$

Exercise 11.2 (Section 11.2)

1 $2\sin 3A \cos A$

2 $2\cos 3A \sin 2A$

3 $-2\sin 3\theta \sin\theta$

4 $2\sin 3A \sin 2A$

5 $2\cos 41° \cos 6°$

6 $2\cos 36° \sin 13°$

7 $\cot 15°$

8 $-\cot\left(\frac{\alpha-\beta}{2}\right)$

9 $\tan\theta$

10 $\tan 2\theta$

11 0°, 120°, 240°, 360°

12 0°, 120°, 180°, 240°, 360°

13 0°, 90°, 120°, 180°, 240°, 270°, 360°

14 45°, 135°, 180°, 225°, 315°

15
$$\begin{aligned}
\text{LHS} &= \sin A + \sin B + \sin C \\
&= (\sin A + \sin B) + \sin(A+B) \\
&= 2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) + 2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A+B) \\
&= 2\sin\tfrac{1}{2}(A+B)(\cos\tfrac{1}{2}(A-B) + \cos\tfrac{1}{2}(A+B)) \\
&= 2\cos\tfrac{1}{2}C(2\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B) \\
&= 4\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C = \text{RHS}
\end{aligned}$$

16
$$\begin{aligned}
\text{LHS} &= \sin A + \sin B - \sin C \\
&= (\sin A + \sin B) - \sin(A+B) \\
&= 2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) - 2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A+B) \\
&= 2\sin\tfrac{1}{2}(A+B)\left(\cos\tfrac{1}{2}(A-B) - \cos\tfrac{1}{2}(A+B)\right) \\
&= 2\cos\tfrac{1}{2}C\left(2\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\right) \\
&= 4\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\cos\tfrac{1}{2}C = \text{RHS}
\end{aligned}$$

17
$$\begin{aligned}
\text{LHS} &= \cos A + \cos B + \cos C - 1 \\
&= (\cos A + \cos B) + (\cos C - 1) \\
&= 2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) - 2\sin^2\tfrac{1}{2}C \\
&= 2\sin\tfrac{1}{2}C(\cos\tfrac{1}{2}(A-B) - \sin\tfrac{1}{2}C) \\
&= 2\sin\tfrac{1}{2}C(\cos\tfrac{1}{2}(A-B) - \cos\tfrac{1}{2}(A+B)) \\
&= 2\sin\tfrac{1}{2}C(2\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B) \\
&= 4\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C = \text{RHS}
\end{aligned}$$

Exercise 12.1 (Section 12.1)

1 10 cm

2 $\dfrac{8}{\sqrt{15}}$ cm

3 $\sqrt{\dfrac{20}{\sin 50° \sin 60° \sin 70°}}$ cm

4
$$\begin{aligned} \text{RHS} &= b\cos C + c\cos B \\ &= 2R\sin B\cos C + 2R\sin C\cos B \\ &= 2R(\sin B\cos C + \sin C\cos B) \\ &= 2R(\sin(B+C)) \\ &= 2R\sin A = a = \text{LHS} \end{aligned}$$

5
$$\begin{aligned} \text{LHS} &= s \\ &= \tfrac{1}{2}(a+b+c) \\ &= R(\sin A + \sin B + \sin C) \\ &= R\left(2\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A + (\sin B + \sin C)\right) \\ &= R\left(2\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A + 2\sin\tfrac{1}{2}(B+C)\cos\tfrac{1}{2}(B-C)\right) \\ &= 2R\left(\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A + \cos\tfrac{1}{2}A\cos\tfrac{1}{2}(B-C)\right) \\ &= 2R\cos\tfrac{1}{2}A\left(\sin\tfrac{1}{2}A + \cos\tfrac{1}{2}(B-C)\right) \\ &= 2R\cos\tfrac{1}{2}A\left(\cos\tfrac{1}{2}(B+C) + \cos\tfrac{1}{2}(B-C)\right) \\ &= 2R\cos\tfrac{1}{2}A\left(2\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C\right) \\ &= 4R\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C = \text{RHS} \end{aligned}$$

Exercise 12.2 (Section 12.4)

1 $\tfrac{3}{4}\sqrt{15}$ cm^2, $\tfrac{1}{6}\sqrt{15}$ cm, $\tfrac{3}{10}\sqrt{15}$ cm, $\tfrac{1}{2}\sqrt{15}$ cm, $\tfrac{3}{2}\sqrt{15}$ cm

2 $\dfrac{6\sqrt{3}}{7+\sqrt{13}} = \dfrac{\sqrt{3}(7-\sqrt{13})}{6}$ cm

3 Using Equations 3 and 4:

$$\begin{aligned} \text{LHS} &= rr_A r_B r_C \\ &= \frac{\Delta}{s}\times\frac{\Delta}{s-a}\times\frac{\Delta}{s-b}\times\frac{\Delta}{s-c} \\ &= \frac{\Delta^4}{\Delta^2} = \Delta^2 = \text{RHS} \end{aligned}$$

4 $\text{LHS} = \dfrac{\Delta}{s}$

$$= \frac{\frac{1}{2}ab\sin C}{\frac{1}{2}(a+b+c)}$$

$$= \frac{2R\sin A \times 2R\sin B \times \sin C}{2R\sin A + 2R\sin B + 2R\sin C}$$

$$= 2R\frac{\sin A\sin B\sin C}{\sin A + \sin B + \sin C}$$

$$= 2R\frac{2\sin\frac{1}{2}A\cos\frac{1}{2}A \times 2\sin\frac{1}{2}B\cos\frac{1}{2}B \times 2\sin\frac{1}{2}C\cos\frac{1}{2}C}{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B) + 2\sin\frac{1}{2}C\cos\frac{1}{2}C}$$

$$= 2R\frac{8\sin\frac{1}{2}A\cos\frac{1}{2}A\sin\frac{1}{2}B\cos\frac{1}{2}B\sin\frac{1}{2}C\cos\frac{1}{2}C}{2\cos\frac{1}{2}C(\cos\frac{1}{2}(A-B) + \sin\frac{1}{2}C)}$$

$$= 8R\frac{\sin\frac{1}{2}A\cos\frac{1}{2}A\sin\frac{1}{2}B\cos\frac{1}{2}B\sin\frac{1}{2}C}{\cos\frac{1}{2}(A-B) + \cos\frac{1}{2}(A+B)}$$

$$= 8R\frac{\sin\frac{1}{2}A\cos\frac{1}{2}A\sin\frac{1}{2}B\cos\frac{1}{2}B\sin\frac{1}{2}C}{2\cos\frac{1}{2}A\cos\frac{1}{2}B}$$

$$= 4R\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C = \text{RHS}$$

5 $\text{LHS} = \dfrac{\Delta}{s-c}$

$$= \frac{\frac{1}{2}ab\sin C}{\frac{1}{2}(a+b-c)}$$

$$= \frac{2R\sin A \times 2R\sin B \times \sin C}{2R\sin A + 2R\sin B - 2R\sin C}$$

$$= 2R\frac{\sin A\sin B\sin C}{\sin A + \sin B - \sin C}$$

$$= 2R\frac{2\sin\frac{1}{2}A\cos\frac{1}{2}A \times 2\sin\frac{1}{2}B\cos\frac{1}{2}B \times 2\sin\frac{1}{2}C\cos\frac{1}{2}C}{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B) - 2\sin\frac{1}{2}C\cos\frac{1}{2}C}$$

$$= 2R\frac{8\sin\frac{1}{2}A\cos\frac{1}{2}A\sin\frac{1}{2}B\cos\frac{1}{2}B\sin\frac{1}{2}C\cos\frac{1}{2}C}{2\cos\frac{1}{2}C(\cos\frac{1}{2}(A-B) - \sin\frac{1}{2}C)}$$

$$= 8R\frac{\sin\frac{1}{2}A\cos\frac{1}{2}A\sin\frac{1}{2}B\cos\frac{1}{2}B\sin\frac{1}{2}C}{\cos\frac{1}{2}(A-B) - \cos\frac{1}{2}(A+B)}$$

$$= 8R\frac{\sin\frac{1}{2}A\cos\frac{1}{2}A\sin\frac{1}{2}B\cos\frac{1}{2}B\sin\frac{1}{2}C}{2\sin\frac{1}{2}A\sin\frac{1}{2}B}$$

$$= 4R\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\sin\tfrac{1}{2}C = \text{RHS}$$

6 $\cos\frac{1}{2}A=\sqrt{\dfrac{s(s-a)}{bc}}$; $\tan\frac{1}{2}A=\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$

7 $$\begin{aligned}\text{RHS}&=\frac{b-c}{b+c}\cot\tfrac{1}{2}A\\&=\frac{2R\sin B-2R\sin C}{2R\sin B+2R\sin C}\times\frac{\cos\frac{1}{2}A}{\sin\frac{1}{2}A}\\&=\frac{\sin B-\sin C}{\sin B+\sin C}\times\frac{\cos\frac{1}{2}A}{\sin\frac{1}{2}A}\\&=\frac{2\cos\frac{1}{2}(B+C)\sin\frac{1}{2}(B-C)}{2\sin\frac{1}{2}(B+C)\cos\frac{1}{2}(B-C)}\times\frac{\cos\frac{1}{2}A}{\sin\frac{1}{2}A}\\&=\frac{\sin\frac{1}{2}A\sin\frac{1}{2}(B-C)}{\cos\frac{1}{2}A\cos\frac{1}{2}(B-C)}\times\frac{\cos\frac{1}{2}A}{\sin\frac{1}{2}A}\\&=\frac{\sin\frac{1}{2}(B-C)}{\cos\frac{1}{2}(B-C)}\\&=\tan\tfrac{1}{2}(B-C)=\text{LHS}\end{aligned}$$

8 $$\begin{aligned}\text{RHS}&=(s-a)\times\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\\&=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}\\&=\frac{1}{s}\sqrt{s(s-a)(s-b)(s-c)}\\&=\frac{\Delta}{s}\\&=r=\text{LHS}\end{aligned}$$

Exercise 13 (Section 13.3)

In all the answers to this exercise, *n* and *m* are integers. It is possible that your answer might take a different form from the one given and still be correct.

1 $180n+(-1)^n 60$

2 $360n\pm 60$

3 $180n-45$

4 $180m$ or $180m+120$

5 $120n \pm 60$

6 $60n - 10$

7 $180n$

8 $72n$

9 $180m$ or $90m + 45$

10 $(4m+1)45$ or $(4m+1)22\frac{1}{2}$

11 $\frac{1}{2}n\pi - \frac{1}{12}(-1)^n\pi$

12 $\frac{2}{3}n\pi \pm \frac{1}{6}\pi$

13 $\frac{1}{2}n\pi - \frac{1}{4}\pi$

14 $\frac{2}{3}m\pi$ or $\frac{1}{3}(2m+1)\pi - \frac{1}{9}\pi$

15 $n\pi$ or $n\pi - \pi\frac{1}{2}$

16 $-\frac{1}{2}n\pi + \frac{3}{8}\pi$

17 $\frac{1}{2}n\pi + \frac{1}{4}\pi$

18 $\frac{1}{10}(4n+1)\pi$ or $\frac{1}{2}(4n-1)\pi$

19 $\frac{1}{2}n\pi$

20 $\frac{2}{3}m\pi$ or $(2m+1)\pi$