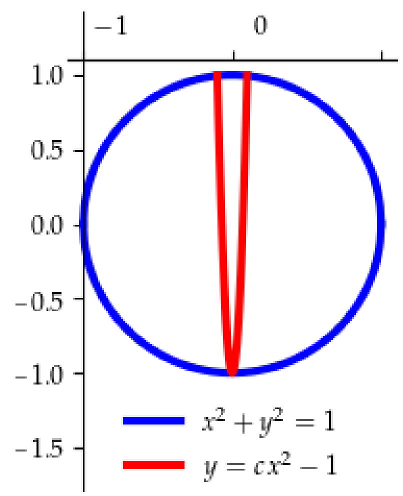
Theorem. *The part of a parabola inside the unit circle can have arc length > 4.*

We consider arcs cut by the unit circle of parabolas $y = cx^2 - 1$ for $c \to \infty$. At an intersection point between such a parabola and the unit circle we have



$$\begin{aligned} x^2 + y^2 = 1 = x^2 + (cx^2 - 1)^2 = \\ c^2x^4 - (2c-1)x^2 + 1, \text{ so} \\ x^2(c^2x^2 - (2c-1)) = 0, \text{ and so} \\ x = 0, \pm f(c), \text{ where } f(c) = \frac{1}{c}\sqrt{2c-1} \end{aligned}$$

The right half-arc of the parabola for c, then, has length $A(c)$ given by

$$A(c) = \int_0^{f(c)} \sqrt{1 + 4c^2x^2}\, dx$$

Changing variables to $t = 2cx$ and setting $g(c) = 2\sqrt{2c-1}$, we get

$$A(c) = \frac{1}{2c}\int_0^{g(c)} \sqrt{1+t^2}\, dt,$$

We have

$$\sqrt{1+t^2} = t + \left(\sqrt{1+t^2} - t\right) = t + \frac{1}{\sqrt{1+t^2}+t}$$

and so

$$A(c) = \frac{1}{2c}\int_0^{g(c)} \left[t + \frac{1}{\sqrt{1+t^2}+t}\right] dt =$$

$$\frac{1}{2c}\left[\frac{t^2}{2}\bigg|_0^{g(c)} + \int_0^{g(c)} \frac{t}{\sqrt{1+t^2}+t} dt\right] = \frac{2c-1}{c} + \frac{1}{2c}\int_0^{g(c)} \frac{1}{\sqrt{1+t^2}+t} dt$$

Because $\dfrac{1}{\sqrt{1+t^2}+t} > \dfrac{1}{3t}$ for $t >1$, we now have that

$$A(c) > 2 - \frac{1}{c} + \frac{1}{2c}\int_1^{g(c)} \frac{dt}{3t} = 2 - \frac{1}{c} + \frac{1}{6c}\ln(g(c)) > 2$$

when $\ln(g(c)) > 6$. This last relationship is true for all sufficently large c. □