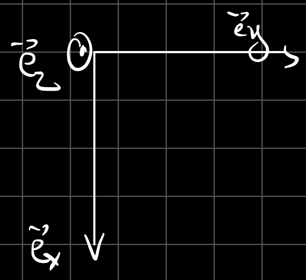
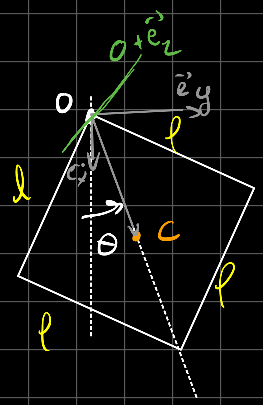




Masse M
carré plein homogène
coté l

Bilan des Forces:
Gravité: $\vec{F_G} = Mg\,\vec{e_x}$
→ Par symétrie, C = centre de masse. $C = \left(\frac{l}{\sqrt{2}}\cos(\theta)\vec{e_x} + \frac{l}{\sqrt{2}}\sin(\theta)\vec{e_y}\right)$

Moment cinétique

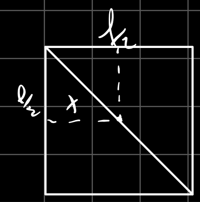
$\frac{l}{\sqrt{2}}\cos(\theta)\cancel{\vec{e_x}} + \frac{l}{\sqrt{2}}\sin(\theta)\vec{e_y}$

$$\frac{d\vec{J_O}}{dt} = \sum \vec{OP_\alpha}\wedge\vec{F_\alpha} = \vec{OC}\wedge Mg\,\vec{e_x} = \frac{l}{\sqrt{2}}\sin(\theta)Mg\ -\vec{e_z}$$

$$\frac{d\vec{J_O}}{dt} = M_O = I_O\,\ddot{\theta}\ \vec{e_z}$$

On a alors $\frac{dJ_O}{dt} = M_O \iff \frac{l}{\sqrt{2}}\sin(\theta)Mg + I_O\ddot{\theta} = 0$

$$\Rightarrow \ddot{\theta} + \frac{l\sin(\theta)Mg}{\sqrt{2}\,I_O} = 0 \quad (1)$$



$x^2 = \frac{l^2}{4}+\frac{l^2}{4}$
$x^2 = \frac{l^2}{2} \Rightarrow x = l\frac{1}{\sqrt{2}}$

Moment d'inertie d'un carré plein

Voyons I_O passant par son sommet O (quelconque)
- Masse M
- Epaisseur négligeable
- Longueur / largeur = l.

Par def: $I_O = \sum d^2 m = \int_S d^2\,dm$

$= \rho\int_V d^2\,dV$ avec $\rho = \frac{M}{l^2}$ pour un carré
ρ = densité surfacique.

$= \rho\int_0^l\int_0^l d^2\,dx\,dy$

$= \rho\int_0^l\int_0^l (x^2+y^2)\,dx\,dy = \rho\int_0^l \left[\frac{x^3}{3}+xy^2\right]_{x=0}^{l}$

$= \rho\left[y\frac{l^3}{3}+l\frac{y^3}{3}\right]_{y=0}^{l}$

$= \rho\left[\frac{l^4}{3}+\frac{l^4}{3}\right] = \rho\frac{2l^4}{3}$

$= \frac{M}{l^2}\cdot\frac{2l^4}{3} = \frac{2}{3}Ml^2 \quad (2)$

(1) devient:

$\ddot{\theta} + \frac{Mgl\,3}{\sqrt{2}\ 2Ml^2}\sin(\theta) = 0$

petit θ
$\sin(\theta)\simeq\theta$

$\iff \ddot{\theta} + \frac{3}{2\sqrt{2}}\frac{g}{l}\,\theta = 0 \quad (3)$

$\frac{3}{2\sqrt{2}} = \frac{3\sqrt{2}}{4}$

On pose $\omega_0^2 = \sqrt{\frac{g}{l}}\sqrt{\frac{3\sqrt{2}}{4}}$

$\omega = 2\pi f$

$\Rightarrow f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{l}}\cdot\frac{\sqrt{3}}{2}\cdot\sqrt[4]{2}$

→ pendule équivalent de longueur $l' = \frac{2\sqrt{2}}{3} l$.

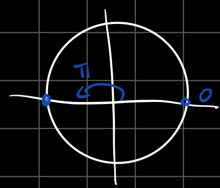
Position d'équilibre :

$\ddot{\theta} = 0$

On a en général : $\omega_0^2 \sin(\theta) = 0$

Annulation si : $\sin(\theta) = 0$

$\iff \underset{\theta_1}{\theta = 0}$ ou $\underset{\theta_2}{\pi}$



Stabilité

$\theta(t) = \theta_i + \varepsilon_i(t)$

$\overset{(3)}{\Rightarrow} \left\{ \begin{array}{l} \ddot{\varepsilon}_1 + \omega_0^2 \sin(\varepsilon_1) = 0 \\ \ddot{\varepsilon}_2 + \omega_0^2 \underbrace{\sin(\pi + \varepsilon_2)}_{= -\sin(\varepsilon_2)} = 0 \end{array} \right]$ si petites oscillations $\sin(\theta_i) \simeq \theta_i$

Stable $\ddot{\varepsilon}_1 + \omega_0^2 \varepsilon_1 = 0 \longrightarrow p^2 + \omega_0^2 = 0 \iff p^2 = \pm i \omega_0^2$

Instable $\ddot{\varepsilon}_2 - \omega_0^2 \varepsilon_2 = 0$

$\Rightarrow p(t) = A e^{i\omega_0 t} + B e^{-i\omega_0 t}$

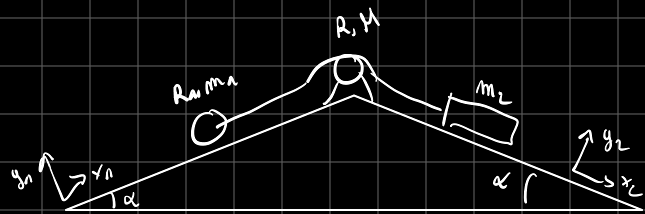
$= A \cos(\omega_0 t) + B \sin(\omega_0 t)$

$\rightarrow p^2 - \omega_0^2 = 0 \iff p^2 = \omega_0^2$

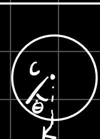
$p = \pm \omega_0$

$p(t) = A e^{\omega_0 t} + B e^{-\omega_0 t}$

$= A \cosh(\omega_0 t) + B \sinh(\omega_0 t)$



cylindre roule sans glissement



$\vec{v}_K = \vec{v}_C + \vec{\omega} \wedge \vec{CK}$

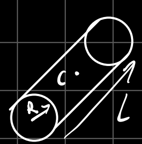
$0 = \dot{x}_c - R\dot{\theta}$

$\Rightarrow \dot{x}_c = R\dot{\theta}$

Moments d'inertie :

Poulie : cylindre plein :

$\rho =$



$I_C = \rho \int_V d^2 \, dV = \rho \int_0^L \int_0^{2\pi} \int_0^R r^3 \, dr \, d\varphi \, dz$

On a $\rho = \frac{M}{\pi R^2 L}$

$= \frac{R^4}{4} 2\pi \cdot L \cdot \rho$

$= \frac{R^{\cancel{4}}}{\cancel{4}\,2} \cancel{2\pi} \cancel{L} \cdot \frac{M}{\cancel{\pi R^2 L}}$

$= \frac{1}{2} M R^2$

- Pour la poulie : $I_{poulie} = \frac{1}{2} M R^2$
- Pour le cylindre : $I_{cyl} = \frac{1}{2} M_1 R_1^2$

$\rightarrow \left[\begin{array}{l} \dot{x}_{C,1} = R_1 \dot{\theta} \\ \dot{x}_{C,poulie} = R\dot{\theta} \end{array} \right] \quad \frac{\dot{x}_{C,1}}{R_1} = \frac{\dot{x}_{C,poulie}}{R}$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} M \dot{x}_C^2 + \frac{1}{2} I_p \dot{\theta}^2 + \frac{1}{2} m_2 \dot{x}_2$$

$$= \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_1 R_1^2 \dot{\theta} + \frac{1}{2} M \dot{x}_1^2 \left(\frac{R}{R_1}\right)^2 + \frac{1}{2} M R^2 \dot{\theta}^2 + \frac{1}{2} m_2 \dot{x}_2 \qquad \text{et } \dot{x}_1 = \dot{x}_2$$

$$= \dot{x}_1^2 \cdot \left(\frac{1}{2} m_1 + \frac{1}{2} m_2\right) + \frac{1}{2} m_1 \frac{\dot{x}_1}{R_1} \cdot R_1^2 + \frac{1}{2} M \dot{x}_1^2 \left(\frac{R}{R_1}\right)^2 + \frac{1}{2} M$$

$$T = \frac{1}{2} \left(m_1 \dot{x}_1^2 + I_1 \dot{\theta}_1^2 + I_p \dot{\theta}_p^2 + m_2 \dot{x}_2^2 \right)$$

$\dot{x}_1 = R\dot{\theta} \qquad \dot{\theta}_1 = \dot{\theta}_p = \dot{\theta}$

$\dot{x}_1 = \dot{x}_2$

$I_i = \frac{1}{2} M_i R_i^2$

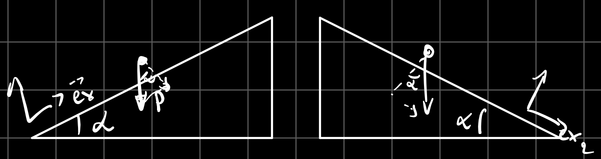
$$\frac{1}{2} \left(m_1 \dot{x}_1^2 + \frac{1}{2} m_1 R_1^2 \cdot \dot{x}_1^2 \cdot \frac{1}{R_1^2} + \frac{1}{2} M R^2 \dot{x}_1^2 \cdot \frac{1}{R^2} + m_2 \dot{x}_1^2 \right)$$

$$= \frac{1}{2} \dot{x}_1^2 \cdot \left(m_1 + m_2 + \frac{1}{2} m_1 + \frac{1}{2} M \right)$$

$$= \frac{1}{2} \dot{x}_1^2 \left(\underbrace{\frac{3}{2} m_1 + m_2 + \frac{1}{2} M}_{= M_T} \right)$$

$$\begin{cases} \dot{x}_{1,C} = R\dot{\theta} \\ x_2 = x_{1,C} \end{cases}$$

On a $\frac{dT}{dt} = \sum_\alpha \vec{F}_\alpha \cdot \vec{v}_\alpha = -m_1 g_1 \sin(\alpha)\, \dot{x}_1 + m_2 g_2 \sin(\alpha)\, \dot{x}_2 + \underbrace{\vec{T}_2 \cdot \dot{\vec{x}}_2}_{-h \dot{\vec{x}}_C \cdot \dot{\vec{x}}_2 \,=\, -h \dot{x}_1^2}$



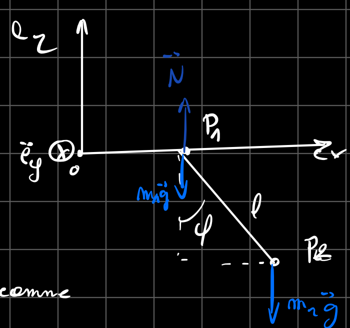
$-P.\sin(\alpha)$

Donc on a : $-m_1 g_1 \sin(\alpha)\, \dot{x}_1 + m_2 g_2 \sin(\alpha)\, \dot{x}_1 - h \dot{x}_1^2$
$= \dot{x}_1 \ddot{x}_1 M_T$

$\iff \ddot{x}_1 M_T = (m_2 - m_1) \sin(\alpha) - h \dot{x}_1$

$$\ddot{x}_1 = \frac{(m_2 - m_1)}{M_T} \sin(\alpha) - h_1 \dot{x}_1$$

Le pendule d'Euler :



2 ddl : mvt dans le plan (x,z)
- m_1, m_2
- l const

$S = (O, e_x, e_y, e_z)$: ref fixe
- Pas de force interne car syst. considéré comme un solide (dist. (e) P_1 et P_2 fixe).

p.9

$\vec{p}_1 = m_1 \vec{v}_1 \rightarrow \vec{p}_1 = m_1 \dot{x}_1 \vec{e}_x$

- $\vec{v}_1 = \dot{\vec{x}}_1 = \dot{x}_1 \vec{e}_x$

$\vec{p}_2 = m_2 \vec{v}_2$

- $\vec{v}_2 = \dot{\vec{x}}_2 = \frac{d}{dt}(\vec{OP}_2) = \frac{d}{dt}\left((x_1 + l\sin(\varphi))\vec{e}_x - l\cos(\varphi)\,\vec{e}_y\right)$

$$= \dot{x}_1 + \dot{\varphi}\, l \cos(\varphi)\,\vec{e}_x + l\dot{\varphi}\sin(\varphi)\vec{e}_y$$

$\vec{p}_2 = m_2\dot{x}_1 + m_2\dot{\varphi}\, l\cos(\varphi)\vec{e}_x + m_2\dot{\varphi}\, l\sin(\varphi)\vec{e}_y$

$\vec{p}_{tot} = \vec{p}_1 + \vec{p}_2 = \left[m_1\dot{x}_1 + m_2\cdot(\dot{x}_1 + \dot{\varphi}\, l\cos(\varphi))\right]\vec{e}_x + m_2\dot{\varphi}\, l\sin(\varphi)\vec{e}_y$

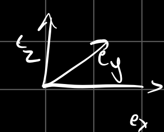
$$\frac{d\vec{p}_{tot}}{dt} = m_1\ddot{x}_1 + m_2\ddot{x}_1 + m_2 l\cdot(\ddot{\varphi}\cos(\varphi) - \dot{\varphi}^2\sin(\varphi))\vec{e}_x + \left[m_2\cdot(\ddot{\varphi}\sin(\varphi) + \dot{\varphi}^2\cos(\varphi))\right]\vec{e}_y$$

$$= \sum_\alpha \vec{F}_{\alpha\, ext} = -m_1 g\,\vec{e}_y - m_2 g\vec{e}_y + N\vec{e}_y$$
$$= N - (m_1+m_2)g\,\vec{e}_y$$

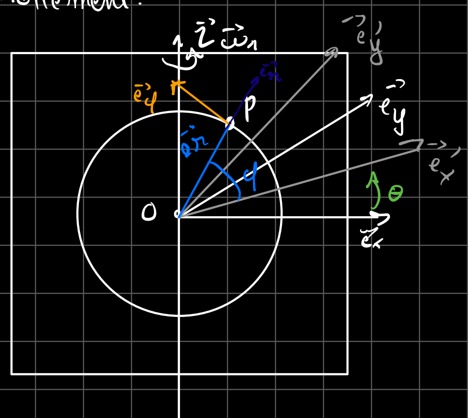
projection

$\Rightarrow$
$$\begin{cases} \ddot{x}_1\cdot(m_1+m_2) + m_2 l\,(\ddot{\varphi}\cos(\varphi) - \dot{\varphi}^2\sin(\varphi)) = 0 & (1) \\ m_2\cdot(\ddot{\varphi}\sin(\varphi) + \dot{\varphi}^2\cos(\varphi)) = N - (m_1+m_2) & (2) \end{cases}$$

Bille oscillant sur un cercle tournant



- Anneau circulaire de rayon R.
- $\vec{\omega}_1$ const $\rightarrow$
- P glisse sans frottement.



$S(O, \vec{e}_x, \vec{e}_y, \vec{e}_y)$: ref fixe

$S'(O, \vec{e}_x', \vec{e}_y', \vec{e}_z')$: ref lié au plan

$S''(P, \vec{e}_r, \vec{e}_y', \vec{e}_\varphi')$: ref lié à la part. P

avec $\vec{e}_r \parallel \vec{OP}$

$\rightarrow \vec{\omega}_{S'/S} = \omega_1\vec{e}_z = \dot{\theta}\vec{e}_z$

$\rightarrow \vec{\omega}_{S''/S'} = -\dot{\varphi}\,\vec{e}_y'$

Donc $\vec{\omega}_{S''/S} = \dot{\theta}\,\vec{e}_z - \dot{\varphi}\,\vec{e}_y'$

- $\vec{e}_r = \cos(\varphi)\vec{e}_x' + \sin(\varphi)\vec{e}_z$
- $\vec{e}_\varphi = -\sin(\varphi)\vec{e}_x' + \cos(\varphi)\vec{e}_z$

Calcul de $\vec{r}$, $\vec{v}$ et $\vec{a}$

$\vec{r} = R\vec{e}_r = R(\cos(\varphi)\,\vec{e}_x' + \sin(\varphi)\vec{e}_z)$

$$\vec{v} = \frac{d\vec{r}}{dt} \stackrel{\text{en } \in S''}{=} \underbrace{\dot{R}\vec{e}_r}_{=0 \text{ car } R \text{ const sur disque}} + R\,\vec{\omega}_{S''/S}\wedge\vec{e}_r = R(\dot{\theta}\vec{e}_z - \dot{\varphi}\,\vec{e}_y')\wedge\vec{e}_r)$$

$$= R\dot{\theta}\,(\vec{e}_z\wedge\vec{e}_r) - R\dot{\varphi}\,(\vec{e}_y'\wedge\vec{e}_r)$$

$$= R\dot{\theta}\,(\vec{e}_z \wedge \overbrace{\cos(\varphi)\vec{e}_x'}^{\vec{e}_y'} + \cancel{\sin(\varphi)\vec{e}_z})$$
$$- R\dot{\varphi}\,(\vec{e}_y' \wedge \overbrace{\cos(\varphi)\vec{e}_x'}^{-\vec{e}_z} + \overbrace{\sin(\varphi)\vec{e}_z}^{\vec{e}_x'})$$

$$= R\dot\theta\cos(\varphi)\vec{e}_{y'} + R\dot\varphi\cos(\varphi)\vec{e}_{z'} - R\dot\varphi\sin(\varphi)\vec{e}_{x'}$$

$\vec{\omega}_{S'/S}$ car $\vec{e}_{x'}, \vec{e}_{y'}, \vec{e}_{z'} \in S'$

$\downarrow \dot\theta = \text{const}$
$[\ddot\theta = 0]$

$$\vec{a} = \frac{d\vec{v}}{dt} = -R\dot\theta\dot\varphi\sin(\varphi)\vec{e}_{y'} + R\dot\theta\cos(\varphi)\dot\theta\underbrace{(\vec{e}_{z'}\wedge\vec{e}_{y'})}_{-\vec{e}_{x'}}$$
$$+ R\ddot\varphi\cos(\varphi)\vec{e}_{z'} - R\dot\varphi^2\sin(\varphi)\vec{e}_{z'}$$
$$- R\ddot\varphi\sin(\varphi)\vec{e}_{x'} - R\dot\varphi^2\cos(\varphi)\vec{e}_{x'} - R\dot\varphi\sin(\varphi)\dot\theta\overbrace{(\vec{e}_{z'}\wedge\vec{e}_{x'})}^{\vec{e}_{y'}}$$

$$= \vec{e}_{x'}\cdot\left(-R\cos(\varphi)(\dot\theta^2+\dot\varphi^2) - R\ddot\varphi\sin(\varphi)\right)$$
$$- \vec{e}_{y'}\cdot 2R\dot\varphi\dot\theta\sin(\varphi)$$
$$+ \vec{e}_{z'}\left(R\ddot\varphi\cos(\varphi) - R\dot\varphi^2\sin(\varphi)\right)$$

- Cas 1 : plan fixe $\Rightarrow \dot\theta = 0$

$$\vec{v} = -R\dot\varphi\sin(\varphi)\vec{e}_{x'} + R\dot\varphi\cos(\varphi)\vec{e}_{z}$$

$$= R\dot\varphi\cdot\underbrace{(-\sin(\varphi)\vec{e}_{x'} + \cos(\varphi)\vec{e}_{z})}_{\vec{e}_{\varphi}} = R\dot\varphi\vec{e}_{\varphi}$$

$$\vec{a} = \vec{e}_{x}\cdot\left(-R\cos(\varphi)\dot\varphi^2 - R\ddot\varphi\sin(\varphi)\right)$$
$$+ \vec{e}_{z'}\cdot\left(R\ddot\varphi\cos(\varphi) - R\dot\varphi^2\sin(\varphi)\right)$$

$$= -R\dot\varphi^2\cdot(\cos(\varphi)\vec{e}_{x'} + \sin(\varphi)\vec{e}_{z}) + R\ddot\varphi\,(-\sin(\varphi)\vec{e}_{x'} + \cos(\varphi)\vec{e}_{z})$$
$$= -R\dot\varphi^2\vec{e}_{r} + R\ddot\varphi\vec{e}_{\varphi}$$