CAAM 453/550: Numerical Analysis I - Fall 2021
Homework 11 - due 5pm on Friday, November 19, 2021

Instructions: You may use any of the code on the Canvas page. Turn in all modified/new MATLAB/python code (scripts and functions) and all output generated by your code. All code must be commented, and it must be clear what your output is and why you are submitting it. Additionally, all plots must be labeled.

For any problems that do not require coding, either turn in handwritten work or typeset work using LaTeX or some other typsetting software. Please do not turn in math as commented MATLAB code or math that has been typed in a word processor!

MATLAB/Python code fragments are provided in some problems. If you program in python, you may replace them corresponding numpy or scipy code.

CAAM 453 students are to complete problems 1-3. (75 points)
CAAM 550 students are to complete problems 1 - 4. (95 points)

CAAM 453 may complete additional problems "for fun," but you will not receive additional credit.

Problem 1 (15 points) The trapezoidal method for the solution of an IVP $y'(x) = f(x, y(x))$, $y(x_0) = y_0$, $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, is given by

$$y_{k+1} = y_k + \frac{x_{k+1} - x_k}{2} \left[f(x_k, y_k) + f(x_{k+1}, y_{k+1}) \right]. \tag{1}$$

(i) (10 points) Modify `ImpEuler.m` (subfunction in `demo_ode_euler.m`) to implement the trapezoidal rule (the modified program should be called `Trap.m`) using Newton's method for the computation of y_{k+1}.

(ii) (5 points) Augment `demo_ode_euler.m` to solve each example using the explicit (backward) Euler method, the implicit (forward) Euler method, and the Trapezoidal method. Plot the approximate solution generated by the explicit Euler method, the implicit Euler method, and the Trapezoidal method.

Problem 2 (30 points) Consider the differential equation ($\lambda \in \mathbb{C}$)

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x} = \lambda y(x), \qquad y(0) = y_0.$$

(a) (5 points) Show that when applied to this equation, Heun's method results in

$$y_{k+1} = \left(1 + h\lambda + \frac{h^2\lambda^2}{2}\right) y_k.$$

(b) (10 points) Develop a formula like the one in part (a) (but different) for the fourth-order explicit Runge-Kutta method.

(c) (5 points) Compare your answers from parts (a) and (b) to the Taylor series for $y(x_{k+1})$ (the exact solution evaluated at x_{k+1}) expanded about the point x_k. Share any observations you have.

(d) (10 points) Use MATLAB/Python to plot the set of all $h\lambda \in \mathbb{C}$ for which $|y_k| \to 0$ as $k \to \infty$ for Heun's method and the fourth-order Runge Kutta method. Display the two plots separately, and then plot them in the same figure together.

Problem 3 (30 points) Consider the initial value problem

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x} = 4x - 2y(x), \quad x \in [0,1], \qquad y(0) = 0.$$

(a) (5 points) Solve this problem analytically by using an integrating factor.

(b) (10 points) Using MATLAB/Python, apply Heun's method to this problem for $h = 0.1, 0.05, 0.025$ on the interval $[0,1]$. For each value of h plot the exact solution and the numerical approximation together in the the same figure (3 figures total).

(c) (10 points) Using MATLAB, apply the 4th-order explicit Runge-Kutta (RK4) method to this problem for $h = 0.1, 0.05, 0.025$ on the interval $[0,1]$. For each value of h plot the exact solution and the numerical approximation together in the the same figure (3 figures total).

(d) (5 points) For each $h \in \{2^{-j}\}_{j=1}^{10}$, find the error in the last time step, $|y(1) - y_{x_{\text{final}}}|$, for both the approximation found by Heun's method and RK4. Plot these errors in the same loglog plot with reference lines for order 2 and order 4 convergence.

Problem 4 (20 points) CAAM 550 only

Show that the truncation error

$$T_k = \frac{y(x_{k+1}) - y(x_k)}{h} - \frac{1}{2}(f(x_k, y(x_k)) + f(x_{k+1}, y_{k+1}))$$

of the trapezoid rule is second order.