**Michael Goforth**
**CAAM 550**
**HW 11**
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**Problem 1**
**part i.**
See Jupyter notebook for code and results.

**part ii.**
See Jupyter notebook for code and results.

**Problem 2**
**part a**
Heun's method states

$$y_{k+1} = y_k + \frac{h}{2}(f(x_k, y_k) + f(x_{k+1}, y_k + hf(x_k, y_k)))$$

Since $f(x, y(x)) = \lambda y(x)$,

$$\begin{aligned}
f(x_k, y_k) &= \lambda y_k \\
f(x_{k+1}, y_k + hf(x_k, y_k)) &= f(x_{k+1}, y_k + h\lambda y_k)) \\
f(x_{k+1}, y_k + hf(x_k, y_k)) &= \lambda(y_k + h\lambda y_k) \\
y_{k+1} &= y_k + \frac{h}{2}(\lambda y_k + \lambda(y_k + h\lambda y_k)) \\
y_{k+1} &= y_k + \frac{h\lambda}{2}y_k + \frac{h\lambda}{2}y_k + \frac{h^2\lambda^2}{2}\lambda y_k \\
y_{k+1} &= y_k(1 + h\lambda + \frac{h^2\lambda^2}{2}\lambda)
\end{aligned}$$

**part b**
The fourth order Runge-Kutta method is

$$\begin{aligned}
Y_1 &= y_k \\
Y_2 &= y_k + \frac{h}{2}f(x_k, Y_1) \\
Y_3 &= y_k + \frac{h}{2}f(x_i + \frac{h}{2}, Y_2) \\
Y_4 &= y_k + hf(x_k + \frac{h}{2}, Y_3) \\
y_{k+1} &= y_k + \frac{h}{6}\left(f(x_k, Y_1) + 2f(x_i + \frac{h}{2}, Y_2), +2f(x_k + \frac{h}{2}, Y_3) + f(x_{k+1}, Y_4)\right)
\end{aligned}$$

Using the fact that $f(x, y(x)) = \lambda y(x)$,

$$
\begin{aligned}
Y_1 &= y_k \\
Y_2 &= y_k + \frac{h}{2}\lambda y_k \\
Y_3 &= y_k + \frac{h}{2} f(x_i + \frac{h}{2}, y_k + \frac{h}{2}\lambda y_k) \\
Y_3 &= y_k + \frac{h}{2}\lambda y_k + \frac{h^2\lambda^2}{4} y_k \\
Y_4 &= y_k + hf(x_k + \frac{h}{2}, y_k + \frac{h}{2}\lambda y_k + \frac{h^2\lambda^2}{4} y_k) \\
Y_4 &= y_k + h\lambda y_k + \frac{h^2\lambda^2}{2} y_k + \frac{h^3\lambda^3}{4} y_k \\
y_{k+1} &= y_k + \frac{h}{6}\left(\lambda Y_1 + 2\lambda Y_2 + 2\lambda Y_3 + \lambda Y_4\right) \\
y_{k+1} &= y_k \left(h\lambda + \frac{1}{2}h^2\lambda^2 + \frac{1}{6}h^3\lambda^3 + \frac{1}{24}h^4\lambda^4\right)
\end{aligned}
$$

**part c**

Taylor series of the point $x_{k+1}$ expanded around $x_k$:

$$y(x_{k+1}) = y(x_k + h) = y(x_k) + hy'(x_k) + \frac{h^2}{2}y''(x_k) + \frac{h^3}{6}y'''(x_k) + \frac{h^4}{24}y''''(x_k) + \ldots$$

For the case $y'(x, y(x)) = \lambda y(x)$,

$$y(x_{k+1}) = y(x_k + h) = y(x_k) + h\lambda y(x_k) + \frac{\lambda^2 h^2}{2}y(x_k) + \frac{\lambda^3 h^3}{6}y(x_k) + \frac{\lambda^4 h^4}{24}y(x_k) + \ldots$$

Thus the Heun's method results in the second order Taylor series of $x_{k+1}$ expanded around $x_k$, and the fourth order Runge-Kutta method results in the fourth order Taylor series around the same point.

**part d**

See Jupyter notebook for code and results.

**Problem 3**

**part a**

$$
\begin{aligned}
\frac{dy(x)}{dx} &= 4x - 2y(x) \\
x &\in [0, 1] \\
y(0) &= 0
\end{aligned}
$$

Using the integrating factor $\phi(x) = e^{2x}$,

$$\begin{aligned}
e^{2x}\frac{dy(x)}{dx} + 2e^{2x}y(x) = e^{2x}4x \\
\frac{d}{dx}(e^{2x}y(x)) = e^{2x}4x \\
e^{2x}y(x) = \int e^{2x}4x dx \\
e^{2x}y(x) = 2x\, e^{2x} - e^{2x} + C \\
0 = y(0) = -1 + C \\
C = 1 \\
e^{2x}y(x) = 2x\, e^{2x} - e^{2x} + 1 \\
y(x) = 2x - 1 + e^{-2x}
\end{aligned}$$

**part b**
See Jupyter notebook for code and results.

**part c**
See Jupyter notebook for code and results.

**part d**
See Jupyter notebook for code and results.

**Problem 4**

$$T_k = \frac{y(x_{k+1}) - y(x_k)}{h} - \frac{1}{2}(f(x_k, y(x_k)) + y(x_{k+1}, y_{k+1}))$$

Using Taylor series expansion,

$$\begin{aligned}
y(x_{k+1}) = y(x_k + h) = y(x_k) + hy'(x_k) + \frac{h^2}{2}y''(x_k) + \frac{h^3}{6}y'''(\zeta_1) \\
y(x_{k+1}) - y(x_k) = hy'(x_k) + \frac{h^2}{2}y''(x_k) + \frac{h^3}{6}y'''(\zeta_1)
\end{aligned}$$

and also using Taylor expansion

$$\begin{aligned}
f(x_{k+1}, y_{k+1}) = y'(x_{k+1}) = y'(x_k + h) \\
f(x_{k+1}, y_{k+1}) = y'(x_k) + hy''(x_k) + \frac{h^2}{2}y'''(\zeta_2)
\end{aligned}$$

Combining these gives

$$T_k = \frac{hy'(x_k) + \frac{h^2}{2}y''(x_k) + \frac{h^3}{6}y'''(\zeta_1)}{h} - \frac{1}{2}(f(x_k, y(x_k)) + y'(x_k) + hy''(x_k) + \frac{h^2}{2}y'''(\zeta_2))$$

$$T_k = y'(x_k) + \frac{h}{2}y''(x_k) + \frac{h^2}{6}y'''(\zeta_1) - y'(x_k) - \frac{h}{2}y''(x_k) - \frac{h^2}{4}y'''(\zeta_2)$$

$$T_k = \frac{h^2}{6}y'''(\zeta_1) - \frac{h^2}{4}y'''(\zeta_2)$$

Thus the truncation error of the trapezoid rule is order 2.