CAAM 453/550: Numerical Analysis I - Fall 2021

Homework 3 - due by 5pm on Wednesday, September 15, 2021

Instructions: You may use any of the code on the Canvas page. Turn in all modified/new MATLAB/python code (scripts and functions) and all output generated by your code. All code must be commented, and it must be clear what your output is and why you are submitting it. Additionally, all plots must be labeled.

For any problems that do not require coding, either turn in handwritten work or typeset work using LaTeX or some other typsetting software. Please do not turn in math as commented MATLAB code or math that has been typed in a word processor!

MATLAB code fragments are provided in some problems. If you program in python, you may replace them corresponding numpy or scipy code.

CAAM 453 students are to complete problems 1 - 4. (60 points)
CAAM 550 students are to complete problems 1 - 5. (90 points)

CAAM 453 may complete additional problems "for fun," but you will not receive additional credit.

Problem 1 (10 points) Let E be the $n \times n$ matrix that extracts the "even part" of an n-vector: $E\mathbf{x} = \frac{1}{2}(\mathbf{x} + F\mathbf{x})$, where F is the $n \times n$ matrix that reorders $(x_1, \ldots, x_n)^T$ to $(x_n, \ldots, x_1)^T$. Is E an orthogonal projector, an oblique projector, or not a projector at all? Why? What are its entries?

Problem 2 (20 points) Consider the vector $\mathbf{x} = (5, 12)^\top$.

(a) (5 points) Find a vector $\mathbf{v}$ such that the Householder reflector $H(\mathbf{v})$ yields

$$H(\mathbf{v})\mathbf{x} = \begin{bmatrix} \|\mathbf{x}\|_2 \\ 0 \end{bmatrix}.$$

(b) (5 points) Compute $H(\mathbf{v})^T H(\mathbf{v})$ for this particular $\mathbf{v}$ to verify that $H(\mathbf{v})$ is unitary.

(c) (2 points) Compute the orthogonal projector P onto $\text{span}\{\mathbf{v}\}$.

(d) (8 points) Produce a precise drawing (or plot) showing $\text{span}\{\mathbf{v}\}$, $\text{span}\{\mathbf{v}\}^\perp$, $\mathbf{x}$, $P\mathbf{x}$, $(I-P)\mathbf{x}$, and $H(\mathbf{v})\mathbf{x}$. Be sure to label your illustration (or plot) clearly.

Problem 3 (20 points) (**Pledged! Complete this problem on your own**)
Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

(a) (15 points) Perform a Householder based QR factorization of A by hand (i.e. without computational assistance).

(b) (1 point) Confirm your result with MATLAB.

(c) (2 points) Deduce the rank of A.

(d) (2 points) Use the R matrix to calculate the determinant of A.

Problem 4 (10 points) (Pledged! Complete this problem on your own)
For this problem you need to use a full QR factorization. you can use your own code or MATLAB's own `qr`. For a square matrix A, we do the following: start with $A_0 = A$, and then for $j \geq 0$

- decompose $A_j = Q_j R_j$,
- build $A_{j+1} = R_j Q_j$.

Note that

$$A_{j+1} = Q_j^T Q_j R_j Q_j = Q_j^T A_j Q_j = \cdots = (Q_0 \ldots Q_j)^T A_0 (Q_0 \ldots Q_j),$$

so all the matrices in the sequence A_j are similar. In many cases the sequence A_j converges to an upper triangular matrix (thus revealing the eigenvalues of A. If A is symmetric, all matrices A_j are symmetric and therefore, if the method converges, it converges to a diagonal matrix. (As given this is not a great method. There are ways to improve it). Code this method and test it with several matrices. It is a requirement for convergence that all eigenvalues are real. Why?

Problem 5 (30 points - CAAM 550 only) Before we introduce the problem, we need a definition

> Suppose $\|\mathbf{x}\|_a$ and $\|\mathbf{x}\|_b$ are two different norms acting on vectors $\mathbf{x} \in \mathbb{R}^k$. These two norms are **equivalent** if there are two constants $c, C > 0$ such that for every $\mathbf{x}$,
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> $$c\|\mathbf{x}\|_b \leq \|\mathbf{x}\|_a \leq C\|\mathbf{x}\|_b.$$
>
> In other words, two equivalent norms $\|\mathbf{x}\|_a$ and $\|\mathbf{x}\|_b$ are not independent; they are always within a constant factor of each other, although C and $1/c$ may be large. This concept of norm equivalence also applies to matrix norms.
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> *Fun fact:* any two norms (vector or matrix) in a finite-dimensional space are equivalent, however, you will not be asked to show this.

i. (10 points) Show that the $2-$norm and the $\infty-$norm on $\mathbb{R}^k$ are equivalent. Specifically, show that for every $\mathbf{x} \in \mathbb{R}^k$,
$$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \leq \sqrt{k}\|\mathbf{x}\|_\infty.$$

ii. (5 points) Show that if two vector norms are equivalent, their induced matrix norms are also equivalent.

iii. (10 points) Following the previous two parts, show that for any $A \in \mathbb{R}^{m \times n}$,
$$\frac{1}{\sqrt{n}}\|A\|_\infty \leq \|A\|_2 \leq \sqrt{m}\|A\|_\infty.$$

iv. (5 points) Show that $\|A\|_1 = \|A^\top\|_\infty$ for any matrix $A \in \mathbb{R}^{m \times n}$.