Homework 1

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1 Problem 1

Given equations:

$$\frac{T(x,t) - T_s}{T_i - T_s} = erf(\frac{x}{2\sqrt{\alpha t}}) \tag{1}$$

$$\alpha = \frac{k}{\rho c_p} \tag{2}$$

$$erf(t) = \frac{2}{\sqrt{\pi}} \int_0^t exp(-s^2) ds \tag{3}$$

We want to find the temperature at depth x after a time of 60 days, or 5,184,000 seconds. Therefore t can be treated as a constant and equation (1) can be rewritten as

$$\frac{T(x) - T_s}{T_i - T_s} = erf(\frac{x}{2\sqrt{\alpha t}}) \tag{4}$$

which finds the temperature at a depth X after a fixed time t. By rearranging we get:

$$T(x) = (T_i - T_s) erf(\frac{x}{2\sqrt{\alpha t}}) + T_s \tag{5}$$

Then the depth to find the depth at which the pipe freezes (T(x)=0), we get the result:

$$T(x) = (T_i - T_s) erf(\frac{x}{2\sqrt{\alpha t}}) + T_s = 0 \tag{6}$$

The derivative is then:

$$T'(x) = \frac{d}{dx}((T_i - T_s) erf(\frac{x}{2\sqrt{\alpha t}}) + T_s) \tag{7}$$

Since T_i, T_s are constants, and substituting in eqn 3 we get:

$$T'(x) = (T_i - T_s) \frac{d}{dx}(\frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{\alpha t}}} exp(-s^2) ds)) \tag{8}$$

$$T'(x) = \frac{2}{\sqrt{\pi}} (T_i - T_s) \frac{d}{dx}(\int_0^{\frac{x}{2\sqrt{\alpha t}}} exp(-s^2) ds)) \tag{9}$$

Finally,

$$T'(x) = \frac{2}{\sqrt{\pi}}(T_i - T_s)exp(-(\frac{x}{2\sqrt{\alpha t}})^2) \tag{10}$$

2 Problem 2

Given a radar reading, the US army's Counter-Rocket Artillery Mortar (C-RAM) sense and warn system is able to calculate the impact area and time of impact of an incoming threat (rocket, artillery, or mortar ordnances) is essential. In reality, the calculations take place in 3 dimensions and consider altitude, temperature, contour of the Earth, ballistic coefficient, measurement uncertainty, and more. But the problem can be simplified to a root finding problem by considering a 2-D flat Earth and simplifying drag to a constant value based on initial velocity, air pressure, and ballistic coefficient. This results in the following equations:

$$x(t) = -\frac{1}{2}F_{dx}t^2 + v_{0x}t + x_0 \tag{11}$$

$$y(t) = -\frac{1}{2}(F_{dy} + G)t^2 + v_{0y}t + y_0 \tag{12}$$

F_d can be estimated in a direction s as:

$$F_{ds} = \frac{\rho v_s^2 c}{BC} \tag{13}$$

where ρ is the air density, BC is the ballistic coefficient, and c is the drag coefficient (a constant). To find the time until impact, given a radar report giving an initial position, velocity, and ballistic coefficient: x_0, y_0, v_{x0}, v_{y0}, and BC the equation in the y direction becomes

$$y(t) = \frac{1}{2}(\frac{\rho v_{y0}^2 c}{BC} - G)t^2 + v_{0y}t + y_0 = 0 \tag{14}$$

where x_0, y_0, v_{x0}, v_{y0}, and BC (BC measured in units of kg/m^2) are given from the radar, $G = 9.81$ m/s^2 is the gravitational constant, $rho = 1.241$ kg/m^3 is the density of air, and $c = 2.00$ as the drag coefficient. The root of this function can then be plugged into the $x(t)$ function to determine the impact point.

The derivative of $y(t)$ is:

$$y'(t) = (\frac{\rho v_{y0}^2 c}{BC} - G)t + v_{0y} \tag{15}$$