Michael Goforth
CAAM 550
HW 12
12/03/2021

Problem 1
part a.

$$\begin{aligned}
&y_{k+2} = 3y_k - 2y_{k+1} + h(f(x_k, y_k) + 3f(x_{k+1}, y_{k+1})) \\
&-3y_k + 2y_{k+1} + y_{k+2} = h(f(x_k, y_k) + 3f(x_{k+1}, y_{k+1})) \\
&\alpha_0 = -3,\ \alpha_1 = 2,\ \alpha_2 = 1 \\
&\beta_0 = 1,\ \beta_1 = 3,\ \beta_2 = 0
\end{aligned}$$

Zero-stability:

$$\begin{aligned}
-3 + 2\gamma + \gamma^2 &= 0 \\
(\gamma + 3)(\gamma - 1) &= 0 \\
\gamma_1 = -3,\ \gamma_2 &= 1
\end{aligned}$$

The root γ_1 is not inside the unit disc so this multi-step method is not zero stable.
Consistent:

$$\sum_{j=0}^{2} \alpha_j = -3 + 2 + 1 = 0$$

$$\sum_{j=0}^{2} (\alpha_j j - \beta_j) = (-3(0) - 1) + (2(1) - 3) + (1(2) - 0) = 0$$

So multi-step method is consistent.
Truncation Error

$$\sum_{j=0}^{2} \frac{j^2}{2}\alpha_j - \sum_{j=0}^{2} j\beta_j = 0(-3) + \frac{1}{2}(2) + 2(1) - 0(1) - 1(3) - 2(0) = 0$$

$$\sum_{j=0}^{2} \frac{j^3}{6}\alpha_j - \sum_{j=0}^{2} \frac{j^2}{2}\beta_j = 0(-3) + \frac{1}{6}(2) + \frac{4}{3}(1) - 0(1) - \frac{1}{2}(3) - 2(0) = \frac{3}{2} \neq 0$$

So the local truncation error is order(h^2).

part b.

$$
\begin{aligned}
&y_{k+2} = \frac{1}{2}(y_k + y_{k+1}) + 2hf(x_{k+1}, y_{k+1}) \\
&-\frac{1}{2}y_k - \frac{1}{2}y_{k+1} + y_{k+2} = 2hf(x_{k+1}, y_{k+1}) \\
&\alpha_0 = -\frac{1}{2},\ \alpha_1 = -\frac{1}{2},\ \alpha_2 = 1 \\
&\beta_0 = 0,\ \beta_1 = 2,\ \beta_2 = 0
\end{aligned}
$$

Zero-stability:

$$
\begin{aligned}
-\frac{1}{2} - \frac{1}{2}\gamma + \gamma^2 = 0 \\
(\gamma - 1)(\gamma + \frac{1}{2}) = 0 \\
\gamma_1 = 1,\ \gamma_2 = -\frac{1}{2}
\end{aligned}
$$

γ_2 is in the unit disk. γ_1 is on the edge of the unit disk but is a simple root so this multi-step method is zero-stable.
Consistent:

$$
\begin{aligned}
\sum_{j=0}^{2} \alpha_j = -\frac{1}{2} - \frac{1}{2} + 1 = 0 \\
\sum_{j=0}^{2} (\alpha_j j - \beta_j) = (-\frac{1}{2}(0) - 0) + (-\frac{1}{2}(1) - 2) + (1(2) - 0) = -\frac{1}{2} \neq 0
\end{aligned}
$$

So this multi-step method is not consistent.

Problem 2
part a
See Jupyter notebook for code and results.

part b
See Jupyter notebook for code and results.

part c
part i.
See Jupyter notebook for code and results.
AB4 is accurate on the interval [-1, -.5], forward Euler is accurate over the whole interval [-1, 3].
part ii.
See Jupyter notebook for code and results.
This stiff ODE is very sensitive to changes in the h value.

part iii.
See Jupyter notebook for code and results.
part iv.
See Jupyter notebook for code and results.
These results are not surprising. Truncation error for forward Euler method is O(h), AB2 is O(h^2), AB4 is O(h^3), and RK4 is O(h^4) although RK4 error was computed using a larger h value than the rest.

part d
See Jupyter notebook for code and results.
This multistep method is not zero stable, so small errors will eventually grow unbounded.

Problem 3

part a

$$\begin{aligned} \hat{y} &= y_k + \frac{h}{4}(f(x_k, y_k) + f(\hat{x}, \hat{y})) \\ y_{k+1} &= \frac{1}{3}(4\hat{y} - y_k + hf(x_{k+1}, y_{k+1})) \\ f(x, y(x)) &= y'(x) = \lambda y(x) \end{aligned}$$

Substituting in the right-hand size of the ODE into the first equation gives

$$\begin{aligned} \hat{y} &= y_k + \frac{h}{4}(\lambda y_k + \lambda\hat{y}) \\ \hat{y} - \frac{h}{4}\lambda\hat{y} &= y_k + \frac{h}{4}\lambda y_k \\ \hat{y} &= \frac{1 + \frac{h\lambda}{4}}{1 - \frac{h\lambda}{4}} y_k \\ \hat{y} &= \frac{4 + h\lambda}{4 - h\lambda} y_k \end{aligned}$$

Substituting this result and the right hand side into the equation for y_{k+1} yields

$$\begin{aligned} y_{k+1} &= \frac{1}{3}(4(\frac{4 + h\lambda}{4 - h\lambda} y_k) - y_k + h\lambda y_{k+1}) \\ y_{k+1} - \frac{h\lambda}{3} y_{k+1} &= \frac{4(4 + h\lambda)}{3(4 - h\lambda)} y_k - \frac{1}{3} y_k \\ (1 - \frac{h\lambda}{3}) y_{k+1} &= (\frac{4(4 + h\lambda)}{3(4 - h\lambda)} - \frac{1}{3}) y_k \\ y_{k+1} &= (\frac{4(4 + h\lambda)}{(4 - h\lambda)(3 - h\lambda)} - \frac{1}{3 - h\lambda}) y_k \end{aligned}$$

part b
This method is stable if $|g(h\lambda)| < 1$, resulting in

$$\left| \frac{4(4+h\lambda)}{(4-h\lambda)(3-h\lambda)} - \frac{1}{3-h\lambda} \right| < 1 \tag{1}$$

See Jupyter notebook for code and results.

Problem 4

$$\hat{y} = y_k + \frac{h}{4}(f(x_k, y_k) + f(\hat{x}, \hat{y}))$$

Using Taylor expansion:

$$\begin{aligned}
f(\hat{x}, \hat{y}) = f(x_k + h/2, y(x_k + h/2) &= y'(x_k) + \frac{1}{2}hy''(x_k) + \frac{1}{8}hy'''(x_k) + \dots \\
\hat{y} &= y_k + \frac{h}{4}(y'(x_k) + y'(x_k) + \frac{1}{2}hy''(x_k) + \frac{1}{8}h^2y'''(x_k) + \dots) \\
\hat{y} &= y_k + \frac{h}{2}y'(x_k) + \frac{1}{8}h^2y''(x_k) + \frac{1}{32}h^3y'''(x_k) + \dots
\end{aligned}$$

Then

$$y_{k+1} = \frac{1}{3}(4\hat{y} - y_k + hf(x_{k+1}, y_{k+1}))$$

Using Taylor expansion once more

$$\begin{aligned}
f(x_{k+1}, y_{k+1}) = f(x_k + h, y(x_k + h)) &= y'(x_k) + hy''(x_k) + \frac{1}{2}h^2y'''(x_k) + \dots \\
y^*_{k+1} &= y(x_k + h) = y(x_k) + hy'(x_k) + \frac{1}{2}h^2y''(x_k) + \frac{1}{6}h^3y'''(x_k) + \dots \\
y_{k+1} &= \frac{4}{3}\hat{y} - \frac{1}{3}y_k + \frac{1}{3}hf(x_{k+1}, y_{k+1}) \\
T_k &= \frac{y^*_{k+1} - y_{k+1}}{h}
\end{aligned}$$

Order h^{-1} terms (in numerator these terms are h^1):

$$h^{-1}(y(x_k) - \frac{4}{3}(y_k) + \frac{1}{3}y_k) = 0$$

Order h^0 terms:

$$y'(x_k) - (\frac{4}{3}(\frac{1}{2}y'(x_k)) + \frac{1}{3}y'(x_k)) = 0$$

So TR-BDF2 method is consistent.
Order h^1 terms:

$$h(\frac{1}{2}y''(x_k) - (\frac{4}{3}(\frac{1}{8}y''(x_k)) + \frac{1}{3}(y''(x_k)) = 0$$

Order h^2 terms:

$$h^2(\frac{1}{6}y'''(x_k) - (\frac{4}{3}(\frac{1}{32}y'''(x_k)) + \frac{1}{3}(\frac{1}{2}y'''(x_k)) \neq 0$$

Therefore the TR-BDF2 method is accurate to the second order.