# CAAM 453/550: Numerical Analysis I - Fall 2021
## Homework 12 - due 5pm on Friday, December 3, 2021 - No extensions

*Instructions: You may use any of the code on the Canvas page. Turn in all modified/new MATLAB/python code (scripts and functions) and all output generated by your code. All code must be commented, and it must be clear what your output is and why you are submitting it. Additionally, all plots must be labeled.*

*For any problems that do not require coding, either turn in handwritten work or typeset work using LaTeX or some other typsetting software. Please do not turn in math as commented MATLAB code or math that has been typed in a word processor!*

*MATLAB/Python code fragments are provided in some problems. If you program in python, you may replace them corresponding numpy or scipy code.*

*CAAM 453 students are to complete problems 1-3. (90 points)*
*CAAM 550 students are to complete problems 1 - 4. (110 points)*

*CAAM 453 may complete additional problems "for fun," but you will not receive additional credit.*

**Problem 1 (20 points)** For each method below decide if it is i) zero-stable, ii) consistent, and if it is consistent, iii) find the order of accuracy of the truncation error.

(a) (10 points) $$y_{k+2} = 3y_k - 2y_{k+1} + h(f(x_k, y_k) + 3f(x_{k+1}, y_{k+1}))$$

(b) (10 points) $$y_{k+2} = \frac{1}{2}(y_k + y_{k+1}) + 2hf(x_{k+1}, y_{k+1})$$

**Problem 2 (50 points) Pledged! Complete this problem on your own.**
Consider the initial value problem

$$y'(x) = f(x, y(x)), \qquad y(x_0) = y_0.$$

The goal of this problem is to create a "black box" function that will solve the above ODE for any explicit linear multistep method.

In order to do this, you need to input the initial condition $y_0$, the initial timestep $x_0$, the final timestep $x_{\text{final}}$, the stepsize $h$, a function handle representing the right-hand side of the ODE, $f(x, y)$, and two sets of coefficients $\{\alpha_j\}_{j=0}^m$ and $\{\beta_j\}_{j=0}^{m-1}$ that define the multistep method.

The function should return a set of gridpoints $\{x_k\}_{k=0}^N$ and a solution vector with components $\{y_k\}_{k=0}^N$ with $y_k \approx y(x_k)$. Here $N = (x_{\text{final}} - x_0)/h$. Since you are only inputting 1 initial condition $y_0$, for an $m$-step method you will need to compute $m-1$ additional initial conditions. Use the 4th-order explicit Runge-Kutta method (RK4) to compute these initial conditions. In other words, if you use an $m$-step method, then $y_1$ through $y_{m-1}$ should be found using RK4.

(a) (10 points) To check your code, apply the 2-step Adams-Bashforth method to the problem

$$y'(x) = 2y(x) \qquad \text{on } [0, 1], \qquad y(0) = 1,$$

for $h = 0.2, 0.1$, and $0.05$. Plot the exact solution together with each of your approximations (3 plots total).

(b) (10 points) Use your code to solve the problem

$$y'(x) = y(x) - x \qquad \text{on } [0, 2], \qquad y(0) = \frac{1}{2},$$

using the leapfrog method

$$y_{k+2} = y_k + 2hf(x_{k+1}, y_{k+1}),$$

with time steps $h = 0.4, 0.2, 0.1$. The exact solution to this problem is $y(x) = x + 1 - \frac{1}{2}e^x$. Create plots with your approximations and the exact solution (3 plots total).

(c) (20 points) Consider the problem

$$y'(x) = 5y(x)(1 - y(x)) \qquad \text{on } [-1, 3], \qquad y(-1) = 0.3.$$

The exact solution is $y(x) = 1 - \frac{1}{1+Ke^{5x}}$, with $K = \frac{3}{7}e^5$.

(i) (5 points) Apply forward Euler, and the 4-step Adams-Bashforth (AB4) methods to this problem with $h = 0.1$ and plot both solutions together with the exact solution (1 plot). In which portions of the interval $[-1, 3]$ is each method most accurate?

(ii) (5 points) This is a stiff ODE for AB4. Apply AB4 again with $h = 0.04, 0.08, 0.09$, and $0.1$. Plot all four solutions in the same plot. How sensitive are the numerical solutions to changes in $h$?

(iii) (5 points) Apply AB2 and RK4 to this problem with $h = 0.1$ and plot the results in the same plot.

(iv) (5 points) Using $h = 0.05$ calculate the error $|y(3) - y_{\text{final}}|$ (the error at $x = 3$), for Euler's method, AB2 and AB4. Are the results surprising?
Also compute the same error using RK4 with $h = 0.2$, and compare to the previous three errors. Your table should look like

```
method          error

-------    --------------------
'Euler'    4.46591619240877e-09
'AB2'      2.72634526066184e-09
'AB4'      1.36243349935228e-10
'RK4'      1.82796155989706e-09
```

(d) (10 points) Apply the 2-step explicit method

$$y_{k+2} = 5y_k - 4y_{k+1} + 2h(f(x_k, y_k) + 2f(x_{k+1}, y_{k+1})),$$

to the problem

$$y'(x) = 0 \qquad \text{on } [0, 2], \qquad y(0) = 1.$$

Modify your code so that $y_1 = 1 + 10 * \varepsilon_{\text{machine}}$, where $\varepsilon_{\text{machine}}$ denotes machine precision, instead of computing $y_1$ using RK4. Plot the results using $h = 0.02$ and $h = 0.05$ in separate plots. What is happening? Why?

**Problem 3 (20 points)** This problem involves a method for solving ODE that combines the implicit trapezoidal rule with the implicit BDF2 method. This method, known as TR-BDF2, involves two stages: take a trapezoidal step of length $h/2$

$$\widehat{y} = y_k + \frac{h}{4}(f(x_k, y_k) + f(\widehat{x}, \widehat{y})),$$

followed by a step similar to BDF2

$$y_{k+1} = \frac{1}{3}(4\widehat{y} - y_k + hf(x_{k+1}, y_{k+1})).$$

We are going to apply this method to

$$y'(x) = \lambda y(x) \qquad \lambda \in \mathbb{C}, \qquad y(x_0) = y_0.$$

(a) (10 points) Eliminate $\widehat{y}$ from the method by plugging the right-hand side of the ODE into the above expressions and thereby reduce the update to a single step

$$y_{k+1} = g(h\lambda)\, y_k,$$

where $g$ depends only on $h\lambda$.

(b) (10 points) Plot the region of absolute stability of TR-BDF2 in the complex plane.

**Problem 4 (20 points) CAAM550 Only.**

Show that the TR-BDF2 method in Problem 3 above is consistent and therefore convergent. Find the global order of accuracy for this method.