CAAM 453/550: Numerical Analysis I - Fall 2021
Homework 2 - due by 5pm on Wednesday, September 8, 2021

Instructions: You may use any of the code on the Canvas page. Turn in all modified/new MATLAB/python code (scripts and functions) and all output generated by your code. All code must be commented, and it must be clear what your output is and why you are submitting it. Additionally, all plots must be labeled.

For any problems that do not require coding, either turn in handwritten work or typeset work using LaTeX or some other typsetting software. Please do not turn in math as commented MATLAB code or math that has been typed in a word processor!

MATLAB code fragments are provided in some problems. If you program in python, you may replace them corresponding numpy or scipy code.

CAAM 453 students are to complete Problems 1, 2, 3. (75 points).
CAAM 550 students are to complete Problems 1-4. (110 points total)

CAAM 453 may complete additional problems "for fun," but you will not receive additional credit.

Problem 1 (20 points) A long conducting rod of diameter D meters and electrical resistance R per unit length is in a large enclosure whose walls (far away from the rod) are kept a temperature T_s degrees C. Air flows past the rod at temperature T_∞ degrees C. If an electrical current I passes through the rod, the temperature of the rod eventually stabilizes to T, where T satisfies

$$f(T) = \pi D h(T - T_\infty) + \pi D \varepsilon \sigma (T^4 - T_s^4) - I^2 R = 0, \tag{1}$$

where

$$\begin{aligned}
\sigma &= \text{Stefan–Boltzman constant} = 5.67 \cdot 10^{-8} \text{ Watts/meter}^2\text{Kelvin}^4, \\
\varepsilon &= \text{rod surface emissivity} = 0.8, \\
h &= \text{heat transfer coefficient of air flow} = 20 \text{ Watts/meter}^2\text{Kelvin}, \\
T_\infty &= T_s = 25\ {}^0\text{C}, \\
D &= 0.1 \text{ meter}, \\
I^2 R &= 100.
\end{aligned}$$

Be aware of the different units.[1]

i. (5 points) Plot f for T between 200K and 400K.

ii. (5 points) Compute an approximate steady state temperature using the Bisection Method with starting values $a_0 = 200$K and $b_0 = 400$K.

iii. (10 points) Compute an approximate steady state temperature using Newton's Method with starting value 200K.

[1] K = ^{0}C+273.15

Problem 2 (25 points) Let $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable. Consider the iteration

$$x_{k+1} = x_k - \frac{f(x_k)}{d} \tag{2}$$

where $d \neq 0$ is fixed.

i. (5 points) Interpret the iteration (2) as a fixed-point iteration

$$x_{k+1} = g(x_k)$$

What is g?

Use the following Theorems to answer questions ii.–iv below. Here $B_r(x_*) = (x_* - r, x_* + r)$.

Theorem 0.1 *Let $D \subset \mathbb{R}$ be open and let $g : D \to \mathbb{R}$ be continuously differentiable on D. If $x_* \in D$ is a fixed point of g and if $|g'(x_*)| < 1$, then there exists $r > 0$ such that x_* is the only fixed point in $\overline{B}_r(x_*)$ and the fixed point iteration*

$$x_{k+1} = g(x_k). \tag{3}$$

converges q–linearly to x_* for any $x_0 \in \overline{B}_r(x_*)$ with

$$\lim_{k\to\infty} \frac{|x_{k+1} - x_*|}{|x_k - x_*|} = |g'(x_*)| < 1.$$

Theorem 0.2 *Let $D \subset \mathbb{R}$ be open and let $g : D \to \mathbb{R}$ be p times continuously differentiable on D. If $x_* \in D$ is a fixed point of g, i.e.,$x_* = g(x_*)$ and if*

$$g'(x_*) = \ldots = g^{(p-1)}(x_*) = 0,$$

then there exists $r > 0$ such that x_* is the only fixed point in $B_r(x_*)$ and the fixed point iteration (3) *converges to x_* for any $x_0 \in B_r(x_*)$ with an q–order p and*

$$\lim_{k\to\infty} \frac{|x_{k+1} - x_*|}{|x_k - x_*|^p} = \frac{|g^{(p)}(x_*)|}{p!}.$$

ii. (10 points) Under what condition on d is the iteration locally convergent?

iii. (5 points) What will the convergence rate be in general?

iv. (5 points) Is there any value of d that would still yield local q–quadratic convergence?

Problem 3 (30 points) We want to reconstruct an image from a blurred version using regularized least squares. The true image is represented by a function $f:[0,1]\to[0,1]$ (think of $f(\xi)$ as the gray scale of a one-dimensional image at ξ with gray values scaled to $[0,1]$). The blurred image $g:[0,1]\to\mathbb{R}$ is given by

$$\int_0^1 k(\xi_1,\xi_2)f(\xi_2)\,d\xi_2 = g(\xi_1), \quad \xi_1\in[0,1], \tag{4}$$

where $k:[0,1]^2\to[0,\infty)$ is given by

$$k(\xi_1,\xi_2)=\frac{1}{\gamma\sqrt{2\pi}}\exp\big(-(\xi_1-\xi_2)^2/(2\gamma^2)\big), \tag{5}$$

with $\gamma>0$. The map $f\mapsto g=\int_0^1 k(\cdot,\xi_2)f(\xi_2)d\xi_2$ is called a convolution and k is called the (convolution) kernel.

Given the kernel k and the blurred image g we want to find f. The smaller the parameters $\gamma>0$ in the kernel (5) the less blurred the image is. See Figure 1.

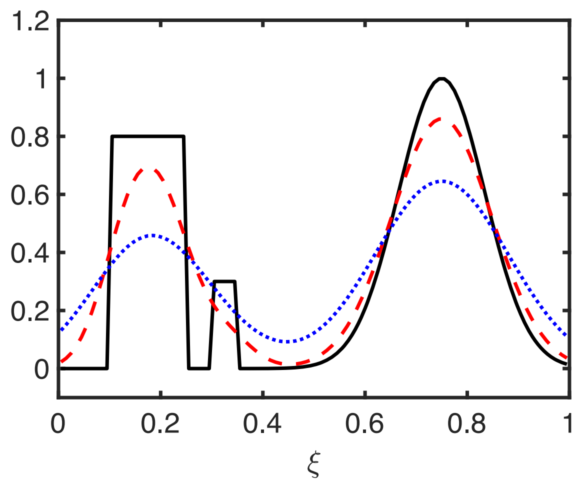

Figure 1: The image f (black solid line) and the blurred images g computed from (4) with $\gamma=0.05$ (red dashed line) and $\gamma=0.1$ (blue dotted line)

In the following computations we use

$$\gamma=0.05.$$

To discretize the problem, we divide $[0,1]$ into n equidistant intervals of length $h=1/n$. Let $\xi_i=(i-\frac{1}{2})h$ be the midpoint of the ith interval. We approximate f and g by piecewise constant functions,

$$f(\xi)\approx\sum_{i=1}^n f_i\chi_{[(i-1)h,ih]}(\xi), \qquad g(\xi)\approx\sum_{i=1}^n g_i\chi_{[(i-1)h,ih]}(\xi),$$

where χ_I is the indicator function on the interval I. We insert these approximations into (4) and approximate the integral by the midpoint rule. This leads to the $n\times n$ linear system

$$\mathbf{K}\mathbf{f}=\mathbf{g}, \tag{6}$$

where

$$\mathbf{f} = (f_1, \dots, f_n)^T, \quad \mathbf{g} = (g_1, \dots, g_n)^T,$$

and

$$\mathbf{K}_{ij} = hk(\xi_i, \xi_j), \quad i, j = 1, \dots, n.$$

Let

$$n = 100.$$

We construct the true image $\mathbf{f}^{\text{true}}$, the resulting blurred image $\mathbf{g}^{\text{true}} = \mathbf{K}\mathbf{f}^{\text{true}}$, and blurred image $\mathbf{g}$ with 0.1% noise using the MATLAB code

```
ftrue      = zeros(n,1);
ftrue      = exp( -(xi -0.75).^2 * 70 );
ind        = (0.1<=xi) & (xi<=0.25);
ftrue(ind) = 0.8;
ind        = (0.3<=xi) & (xi<=0.35);
ftrue(ind) = 0.3;
% blurred image
gtrue      = K*ftrue;
% data perturbation of 0.1%
rng('default')
g          = gtrue + 0.001*(0.5-rand(size(gtrue))).*gtrue;
```

I provide MATLAB and Python templates on the assignment page.

The true and blurred image (with noise) are shown in the left plot in Figure 2. We want to recover $\mathbf{f}^{\text{true}}$ from $\mathbf{g}$ and (6). Solving the linear system (6) using MATLAB 's backslash leads to a highly oscillatory function, indicated by the blue dashed lines in the right plot in Figure 2.

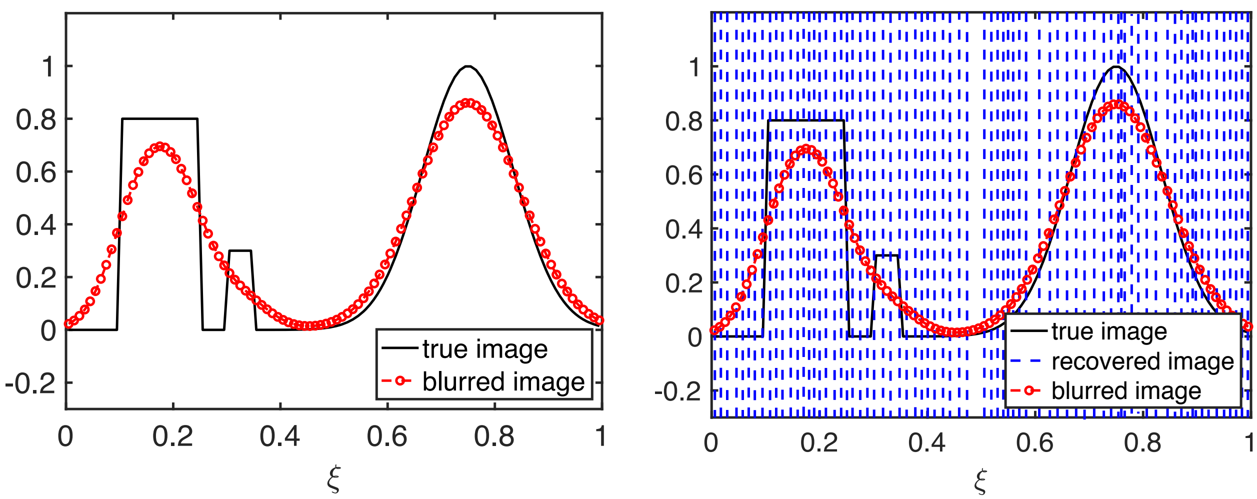

Figure 2: Left plot: The true image $\mathbf{f}^{\text{true}}$ and the blurred image $\mathbf{g}$. Right plot: The true image $\mathbf{f}^{\text{true}}$, the blurred image $\mathbf{g}$, and the recovered image $\mathbf{f} = \mathbf{K}^{-1}\mathbf{g}$ obtained by solving (6)

The provided MATLAB and Python templates create the true image $\mathbf{f}$, the kernel matrix $\mathbf{K}$, and the blurred image $\mathbf{g}$, and recreate the plots in Figure 2.

Next we try to recover the true image by solving the regularized linear least squares problem

$$\min_{\mathbf{f}\in\mathbb{R}^n} \frac{1}{2}\|\mathbf{K}\mathbf{f}-\mathbf{g}\|_2^2 + \frac{\lambda}{2}\|\mathbf{f}\|_2^2 = \min_{\mathbf{f}\in\mathbb{R}^n} \frac{1}{2}\left\| \begin{pmatrix} \mathbf{K} \\ \sqrt{\lambda} I \end{pmatrix} \mathbf{f} - \begin{pmatrix} \mathbf{g} \\ 0 \end{pmatrix} \right\|_2^2, \tag{7}$$

For every $\lambda \geq 0$ the regularized linear least squares problem (7) has a unique solution $\mathbf{f}(\lambda)$ which can be computed can be computed using MATLAB 's backslash.

```
gg      = [ g; zeros(n,1)];
KK      = [ K; sqrt(lambda)*eye(n)];
flambda = KK \ gg;
```

The Morozov discrepancy principle states that the regularization parameter $\lambda \geq 0$ should be chose so that the size of the residual $\frac{1}{2}\|\mathbf{K}\mathbf{f}(\lambda)-\mathbf{g}\|_2^2$ is approximate equal to the size of the measurement error $\mathbf{g}-\mathbf{g}^{\text{true}}$. Thus, $\lambda \geq 0$ is selected as the solution of the root finding problem

$$\phi(\lambda) = \frac{1}{2}\|\mathbf{K}\mathbf{f}(\lambda)-\mathbf{g}\|_2^2 - \frac{1}{2}\|\mathbf{g}-\mathbf{g}^{\text{true}}\|_2^2 = 0 \tag{8}$$

i. (10 points) Plot the function $\lambda \mapsto \frac{1}{2}\|\mathbf{K}\mathbf{f}(\lambda)-\mathbf{g}\|_2^2$ for $\lambda \in [10^{-10}, 10^{-2}]$ and the constant $\frac{1}{2}\|\mathbf{g}-\mathbf{g}^{\text{true}}\|_2^2$ in a loglog plot.

ii. (20 points) Apply the bisection method with $a_0 = 0$ and $b_0 = 10^{-2}$ to compute an approximate root. Stop when $b_k - a_k < 10^{-7}$.

What is the computed root λ_*? Plot the true image $\mathbf{f}^{\text{true}}$, the blurred image $\mathbf{g}$, and the recovered image $\mathbf{f}(\lambda_*)$.

Problem 4 (35 points) (For CAAM 550 students only)

Let $\mathbf{K}$, $\mathbf{f}$, $\mathbf{g}$, $\mathbf{g}^{\text{true}}$ be defined as in Problem 2. In this problem we solve the root finding problem (8) (Morozov's discrepancy principle for selecting the regularization parameter $\lambda \geq 0$) with Newton's method.

i. (20 points) Compute the derivative of the function ϕ in (8).

(Hint: Apply the chain rule. The function $[0,\infty) \ni \lambda \mapsto \mathbf{f}(\lambda) \in \mathbb{R}^n$ is defined via the normal equation

$$\left(\mathbf{K}^T\mathbf{K} + \lambda\mathbf{I}\right)\mathbf{f}(\lambda) = \mathbf{K}^T\mathbf{g} \tag{9}$$

and the derivative $\frac{d}{d\lambda}\mathbf{f}(\lambda) \in \mathbb{R}^n$ can be computed via the implicit function theorem applied to the normal equation (9).)

Describe the derivative computation on paper, and implement it in MATLAB /Python.

ii. (5 points) Compare your derivative at $\lambda = 10^{-3}$ with the (one-sided) finite difference approximation $(\phi(\lambda+\delta)-\phi(\lambda))/\delta$. Use finite difference step sizes $\delta = 10^{-i}$, $i = 1, \ldots, 10$. Plot the error between the derivative and its approximation in a log-log-scale.

iii. (10 points) Apply Newton's method with initial value $\lambda_0 = 10^{-2}$ to compute an approximate root of (8). Stop when $|\phi(\lambda_k)| < 10^{-7}$.

What is the computed root λ_*? Plot the true image $\mathbf{f}^{\text{true}}$, the blurred image $\mathbf{g}$, and the recovered image $\mathbf{f}(\lambda_*)$.

Note: The function $\phi(\lambda)$ is only defined for $\lambda \geq 0$. For some starting values (e.g., $\lambda_0 = 1$) the computed first Newton iterate can be less than zero. In this case bracketing needs to be applied.