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CAAM 550
HW 2
9/8/2021

Problem 2
part i.

$$g(x) = x - \frac{f(x)}{d} \tag{1}$$

part ii.
Let x_* be a fixed point of $g(x)$. Then from theorem 0.1, we can say that if $|g'(x)| < 1$, the fixed point iteration $x_{k+1} = g(x_k)$ converges. So

$$|g'(x_*)| = |1 - \frac{f'(x_*)}{d}| < 1 \tag{2}$$

Removing the absolute value leads us to the following 2 equations, that must both be true:

$$1 - \frac{f'(x_*)}{d} < 1 \text{ and } 1 - \frac{f'(x_*)}{d} > -1 \tag{3}$$

Then,

$$-\frac{f'(x_*)}{d} < 0 \text{ and } -\frac{f'(x_*)}{d} > -2 \tag{4}$$

$$f'(x_*) > 0 \text{ and } f'(x_*) < 2d \tag{5}$$

Combining these gives us the condition

$$0 < f'(x_*) < 2d \tag{6}$$

which if met will guarantee that this fixed point iteration is locally convergent.

part iii. From theorem 0.1, the sequence converges q-linearly to x_* with

$$\lim_{k\to\infty} \frac{|x_{k+1} - x_*|}{|x_k - x_*|} = |g'(x_*)| = |1 - \frac{f'(x_*)}{d}| < 1 \tag{7}$$

(See part iv. for special case where iteration converges quadratically.)

part iv. From theorem 0.2, since f is twice continuously differentiable, if

$$g'(x_*) = 0 \tag{8}$$

then the fixed point iteration converges to x_* with q-order 2 (quadratic convergence). So

$$g'(x_*) = 1 - \frac{f'(x_*)}{d} = 0 \tag{9}$$

$$\frac{f'(x_*)}{d} = 1 \tag{10}$$

and finally

$$f'(x_*) = d \tag{11}$$

Problem 4.
part i.
Find the derivative of function $\phi(\lambda)$ defined as:

$$\phi(\lambda) = \frac{1}{2}||Kf(\lambda) - g||_2^2 - \frac{1}{2}||g - g^{true}||_2^2 = 0 \tag{12}$$

where $f(\lambda)$ is defined by the equation

$$(K^T K + \lambda I)f(\lambda) = K^T g \tag{13}$$

Let $\phi(\lambda)$ be considered a composition of the functions $F(X)$, $G(Y)$, and $f(\lambda)$, such that

$$\phi(\lambda) = F(G(f(\lambda))) \tag{14}$$

and

$$F(X) = \frac{1}{2}x^2 + c \tag{15}$$

$$G(Y) = Ky - g \tag{16}$$

Then using the matrix chain rule,

$$J_\phi = J_F J_G J_f \tag{17}$$

$$J_F = X^T = (Kf(\lambda) - g)^T \tag{18}$$

$$J_G = K \tag{19}$$

Using the implicit function theorem, we can define $f'(\lambda)$. Let

$$g(\lambda, f(\lambda)) = (K^T K + \lambda I)f(\lambda) = K^T g \tag{20}$$

Then by the implicit function theorem,

$$\frac{\partial g}{\partial \lambda} + \frac{\partial g}{\partial f(\lambda)} \frac{df(\lambda)}{d\lambda} = 0 \tag{21}$$

$$If(\lambda) + (K^T K + \lambda I)\frac{df(\lambda)}{d\lambda} = 0 \tag{22}$$

and

$$J_f = \frac{df(\lambda)}{d\lambda} = -(K^T K + \lambda I)^{-1} f(\lambda) \tag{23}$$

Finally

$$\phi'(\lambda) = J_\phi(\lambda) = -(Kf(\lambda) - g)^T K (K^T K + \lambda I)^{-1} f(\lambda) \tag{24}$$