CAAM 453/550: Numerical Analysis I - Fall 2021
Homework 4 - due by 5pm on Wednesday, September 22, 2021

Instructions: You may use any of the code on the Canvas page. Turn in all modified/new MATLAB/python code (scripts and functions) and all output generated by your code. All code must be commented, and it must be clear what your output is and why you are submitting it. Additionally, all plots must be labeled.

For any problems that do not require coding, either turn in handwritten work or typeset work using LaTeX or some other typsetting software. Please do not turn in math as commented MATLAB code or math that has been typed in a word processor!

MATLAB code fragments are provided in some problems. If you program in python, you may replace them corresponding numpy or scipy code.

CAAM 453 students are to complete problems 2-4. (45 points)
CAAM 550 students are to complete problems 1-4 (75 points)

CAAM 453 may complete additional problems "for fun," but you will not receive additional credit.

Problem 1 (30 points) For a dense matrix $A \in \mathbb{R}^{m\times n}$, the standard technique for computing the QR decomposition uses Householder reflectors. An alternative approach reduces A to upper-triangular form using Givens rotations. A Givens rotation is a matrix whose entries are zero except as specified:

$$G(j,k,\theta) = \begin{bmatrix} 1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & 1 & & & & & & & & \\ & & & \cos(\theta) & & & & \sin(\theta) & & & \\ & & & & 1 & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & 1 & & & & \\ & & & -\sin(\theta) & & & & \cos(\theta) & & & \\ & & & & & & & & 1 & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 1 \end{bmatrix},$$

where the rotation matrix

$$\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix},$$

appear on rows j,k and columns j,k.

(a) (5 points) Show that for any j,k, or θ, that G is orthogonal.

(b) (15 points) Design an algorithm for computing the QR factorization $A = QR$ using Givens rotations, rather than Householder transformations. In particular, describe the order in which the lower triangular entries of A (that is, the entries in position (j,k) for $j > k$) should be eliminated, and the Givens rotations that should be used to achieve this elimination. How many total rotations are necessary? What is the matrix Q?

(c) (10 points) Write a MATLAB function `[Q,R] = givens_qr(A)` that constructs the matrices Q and R for $A \in \mathbb{R}^{m\times n}$ using the algorithm you describe in part (b). Demonstrate that your function works by reporting `norm(Q*R - A)/norm(A)` for `A = randn(10,5)` and the final matrix `R`. Please do not print out `Q` or the values of `R` at each step.

Problem 2 (5 points) In exact arithmetic $x+(y+z) = (x+y)+z$. Show that this may be no longer true if we use floating point arithmetic. Specifically, assume a floating point number system with base $\beta = 10$ and mantissa length $t = 4$. Construct floating point numbers $\bar{x}, \bar{y}, \bar{z}$ such that

$$\mathsf{fl}(\bar{x} + \mathsf{fl}(\bar{y} + \bar{z})) \neq \mathsf{fl}(\mathsf{fl}(\bar{x} + \bar{y}) + \bar{z}).$$

Problem 3 (20 points) The mean $\bar{x}$ and standard deviation σ of of the data $x_1, \ldots, x_n \in \mathbb{R}$ are defined as

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \tag{1}$$

and

$$\sigma = \left[\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2 \right]^{1/2}. \tag{2}$$

A mathematically equivalent formula for the standard deviation σ is

$$\sigma = \left[\frac{1}{n-1} \left((\sum_{i=1}^{n} x_i^2) - n\bar{x}^2 \right) \right]^{1/2}. \tag{3}$$

i. (10 points) Show that (2) and (3) are mathematically equivalent.

ii. (10 points) Generate data using

```
x = 10^k*ones(n,1) + (0.5-rand(n,1));
```

with $n = 10$ and $n = 100$ and $k = 5, \ldots, 9$.

For each case, compute the standard deviation using (2) and (3) and compare the results. (Beware of taking the square root of a negative number.)

Problem 4 (20 points) Assume the earth is a sphere of radius $r = 6370$km. Then its surface area is given by

$$A = 4\pi r^2.$$

i. (5 points) Using four-digit floating point arithmetic ($\beta = 10$ and $t = 4$) compute the surface area of the earth.

Make sue that rounding is applied after each operation. For example, compute $\mathsf{fl}\big(\mathsf{fl}(\pi)\big) * \mathsf{fl}\big(\mathsf{fl}(r) * \mathsf{fl}(r)\big)\big)$, not $\mathsf{fl}(\pi * r * r)$.

The computed result depends on the order in which the operations are performed. Use the order $fl\big(\mathsf{fl}(r) * \mathsf{fl}(r)\big) \to \mathsf{fl}\big(\mathsf{fl}(\pi)) * \mathsf{fl}\big(\mathsf{fl}(r) * \mathsf{fl}(r)\big)\big) \to \mathsf{fl}\big(4 * \mathsf{fl}\big(\mathsf{fl}(\pi)) * \mathsf{fl}\big(\mathsf{fl}(r) * \mathsf{fl}(r)\big)\big)\big)$

ii. (5 points) Using the same formula and precision, compute the difference in surface area if the value for the radius is increased by 1km.

iii. (3 points) Since

$$\frac{d}{dr}A = 8\pi r,$$

the change in surface area is approximated by $8\pi r\delta$, where δ is the change in radius. Use this approximation and the floating point arithmetic in i. to compute the difference in surface area if the value for the radius is increased by $\delta = 1$km.

iv. (2 points) Which answer, ii. or iii., is more accurate?

v. (5 points) Explain the results you obtained in i.–iii.