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CAAM 550
HW 3
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Problem 1

$$Ex = \frac{1}{2}(x + Fx) \tag{1}$$

For a vector $x = (x_1, x_2, ..., x_n)^T$,

$$Ex = \frac{1}{2}(I + F)x \tag{2}$$

$$E = \frac{1}{2}(I + F) \tag{3}$$

$$E^2 = \frac{1}{4}(I + F)(I + F) \tag{4}$$

$$E^2 = \frac{1}{4}(I + F)(I + F) \tag{5}$$

$$E^2 = \frac{1}{4}(I + 2F + F^2) \tag{6}$$

Because F simply reverses the order of the elements in x, F^2 reverses the already reversed vector and $F^2 = I$. So

$$E^2 = \frac{1}{4}(I + 2F + I) = \frac{1}{2}(I + F) = E \tag{7}$$

Since $E^2 = E$, the matrix E is a projector.

$$E^T = \frac{1}{2}(I + F)^T \tag{8}$$

$$E^T = \frac{1}{2}(I^T + F^T) \tag{9}$$

F is a matrix that reverses the order of the elements of a vector, x, F is an anti-diagonal exchange matrix:

$$F = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix} = F^T \tag{10}$$

Therefore,

$$E^T = \frac{1}{2}(I + F) = E \tag{11}$$

and E is an orthogonal projection.

$$E = \frac{1}{2}(I + F) \tag{12}$$

Using F as shown above, then E is half the sum of I and F, or a matrix with .5 on both diagonals, 0 elsewhere, and if n is odd the center term will be 1:

$$E_{i,j} = \begin{cases} 1, & i = j = n/2 \\ .5, & j = n - i + 1 \\ .5, & i = j \\ 0, & \text{elsewhere} \end{cases} \tag{13}$$

Problem 2
part a)

$$\boldsymbol{x} = \begin{pmatrix} 5 \\ 12 \end{pmatrix} \tag{14}$$

$$\boldsymbol{v} = \boldsymbol{x} \pm ||x||_2 \boldsymbol{e_1} \tag{15}$$

$$\boldsymbol{v} = \begin{pmatrix} 5 \\ 12 \end{pmatrix} - 13 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{16}$$

$$\boldsymbol{v} = \begin{pmatrix} -8 \\ 12 \end{pmatrix} \tag{17}$$

$$\boldsymbol{H} = \boldsymbol{I} - 2\frac{\boldsymbol{v}\boldsymbol{v}^T}{||\boldsymbol{v}||_2^2} \tag{18}$$

$$\boldsymbol{H} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{104} \begin{bmatrix} 64 & -96 \\ -96 & 144 \end{bmatrix} \tag{19}$$

$$\boldsymbol{H} = \begin{bmatrix} 5/13 & 12/13 \\ 12/13 & -5/13 \end{bmatrix} \tag{20}$$

$$\boldsymbol{H}\boldsymbol{x} = \begin{bmatrix} 5/13 & 12/13 \\ 12/13 & -5/13 \end{bmatrix} \begin{pmatrix} 5 \\ 12 \end{pmatrix} = \begin{pmatrix} 13 \\ 0 \end{pmatrix} \tag{21}$$

part b)

$$\boldsymbol{H}^T\boldsymbol{H} = \begin{bmatrix} 5/13 & 12/13 \\ 12/13 & -5/13 \end{bmatrix} \begin{bmatrix} 5/13 & 12/13 \\ 12/13 & -5/13 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{22}$$

so $\boldsymbol{H}(\boldsymbol{v})$ is unitary.

part c)

$$\boldsymbol{P} = \boldsymbol{I} - \frac{\boldsymbol{v}\boldsymbol{v}^T}{||\boldsymbol{v}||_2^2} \tag{23}$$

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{208} \begin{bmatrix} 64 & -96 \\ -96 & 144 \end{bmatrix} \tag{24}$$

$$\boldsymbol{P} = \begin{bmatrix} 9/13 & 6/13 \\ 6/13 & 4/13 \end{bmatrix} \tag{25}$$

part d
See Jupyter notebook for plot and code to create plot.

Problem 3
part a)

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \tag{26}$$

Step 1

$$\boldsymbol{v_1} = \boldsymbol{a_1} + \text{sign}(a_{11})||\boldsymbol{a_1}||_2 \boldsymbol{e_1} \tag{27}$$

$$\boldsymbol{v_1} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + || \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} ||_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{28}$$

$$\boldsymbol{v_1} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \sqrt{2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{29}$$

$$\boldsymbol{v_1} = \begin{pmatrix} 1 + \sqrt{2} \\ -1 \\ 0 \end{pmatrix} \tag{30}$$

$$\boldsymbol{H_1} = I - 2\frac{\boldsymbol{v_1 v_1^T}}{||\boldsymbol{v_1}||_2^2} \tag{31}$$

$$\boldsymbol{v_1 v_1^T} = \begin{pmatrix} 1 + \sqrt{2} \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 + \sqrt{2} & -1 & 0 \end{pmatrix} \tag{32}$$

$$\boldsymbol{v_1 v_1^T} = \begin{pmatrix} 3 + 2\sqrt{2} & -1 - \sqrt{2} & 0 \\ -1 - \sqrt{2} & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{33}$$

$$||\boldsymbol{v_1}||_2^2 = (1 + \sqrt{2})^2 + 1 = 4 + 2\sqrt{2} \tag{34}$$

$$\boldsymbol{H_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{2}{4 + 2\sqrt{2}} \begin{pmatrix} 3 + 2\sqrt{2} & -1 - \sqrt{2} & 0 \\ -1 - \sqrt{2} & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{35}$$

$$\boldsymbol{H_1} = \frac{1}{2 + \sqrt{2}} \begin{pmatrix} -1 - \sqrt{2} & 1 + \sqrt{2} & 0 \\ 1 + \sqrt{2} & 1 + \sqrt{2} & 0 \\ 0 & 0 & 2 + \sqrt{2} \end{pmatrix} \tag{36}$$

$$\frac{-1-\sqrt{2}}{2+\sqrt{2}} = \frac{(-1-\sqrt{2})(2-\sqrt{2})}{(2+\sqrt{2})(2-\sqrt{2})} = \frac{-\sqrt{2}}{2} \tag{37}$$

$$\boldsymbol{H_1} = \begin{pmatrix} -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{38}$$

Step 1

$$\boldsymbol{H_1 A} = \begin{pmatrix} -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \tag{39}$$

$$\boldsymbol{H_1 A} = \begin{pmatrix} -\sqrt{2}/2-\sqrt{2}/2 & -\sqrt{2}+\sqrt{2} & -3\sqrt{2}/2+\sqrt{2}/2 \\ \sqrt{2}/2-\sqrt{2}/2 & \sqrt{2}+\sqrt{2} & 3\sqrt{2}/2+\sqrt{2}/2 \\ 0 & 1 & 1 \end{pmatrix} \tag{40}$$

$$\boldsymbol{H_1 A} = \begin{pmatrix} -\sqrt{2} & 0 & -\sqrt{2} \\ 0 & 2\sqrt{2} & 2\sqrt{2} \\ 0 & 1 & 1 \end{pmatrix} \tag{41}$$

$$(A^{(2)}) = \begin{pmatrix} 2\sqrt{2} & 2\sqrt{2} \\ 1 & 1 \end{pmatrix} \tag{42}$$

$$\boldsymbol{v_2} = \boldsymbol{a_1^{(2)}} + \operatorname{sign}(a_{11}^{(2)})||\boldsymbol{a_1^{(2)}}||_2 \boldsymbol{e_1} \tag{43}$$

$$\boldsymbol{v_2} = \begin{pmatrix} 2\sqrt{2} \\ 1 \end{pmatrix} + || \begin{pmatrix} 2\sqrt{2} \\ 1 \end{pmatrix} ||_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{44}$$

$$\boldsymbol{v_2} = \begin{pmatrix} 3+2\sqrt{2} \\ 1 \end{pmatrix} \tag{45}$$

$$\boldsymbol{H_2} = I - 2\frac{\boldsymbol{v_2 v_2^T}}{||\boldsymbol{v_2}||_2^2} \tag{46}$$

$$\boldsymbol{v_2 v_2^T} = \begin{pmatrix} 3+2\sqrt{2} \\ 1 \end{pmatrix} \begin{pmatrix} 3+2\sqrt{2} & 1 \end{pmatrix} = \begin{pmatrix} 17+12\sqrt{2} & 3+2\sqrt{2} \\ 3+2\sqrt{2} & 1 \end{pmatrix} \tag{47}$$

$$||\boldsymbol{v_2}||_2^2 = 9+12\sqrt{2}+8+1 = 18+12\sqrt{2} \tag{48}$$

$$\boldsymbol{H_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{2}{18+12\sqrt{2}} \begin{pmatrix} 17+12\sqrt{2} & 3+2\sqrt{2} \\ 3+2\sqrt{2} & 1 \end{pmatrix} \tag{49}$$

$$\boldsymbol{H_2} = \frac{1}{9+6\sqrt{2}} \begin{pmatrix} -8-6\sqrt{2} & -3-2\sqrt{2} \\ -3-2\sqrt{2} & 8+6\sqrt{2} \end{pmatrix} \tag{50}$$

$$\frac{8+6\sqrt{2}}{9+6\sqrt{2}} = \frac{8+6\sqrt{2}}{9+6\sqrt{2}}\frac{9-6\sqrt{2}}{9-6\sqrt{2}} = \frac{72+54\sqrt{2}-48\sqrt{2}-72}{81-72} = \frac{2\sqrt{2}}{3} \tag{51}$$

$$\frac{-3-2\sqrt{2}}{9+6\sqrt{2}} = -\frac{1}{3} \tag{52}$$

$$\boldsymbol{H_2} = \begin{pmatrix} -2\sqrt{2}/3 & -1/3 \\ -1/3 & 2\sqrt{2}/3 \end{pmatrix} \tag{53}$$

$$\tilde{\boldsymbol{H_2}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2\sqrt{2}/3 & -1/3 \\ 0 & -1/3 & 2\sqrt{2}/3 \end{pmatrix} \tag{54}$$

Step 3

$$\boldsymbol{R} = \tilde{\boldsymbol{H_2}}\boldsymbol{H_1}\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2\sqrt{2}/3 & -1/3 \\ 0 & -1/3 & 2\sqrt{2}/3 \end{pmatrix} \begin{pmatrix} -\sqrt{2} & 0 & -\sqrt{2} \\ 0 & 2\sqrt{2} & 2\sqrt{2} \\ 0 & 1 & 1 \end{pmatrix} \tag{55}$$

$$\boldsymbol{R} = \begin{pmatrix} -\sqrt{2} & 0 & -\sqrt{2} \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{pmatrix} \tag{56}$$

$$\boldsymbol{Q^T} = \tilde{\boldsymbol{H_2}}\boldsymbol{H_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2\sqrt{2}/3 & -1/3 \\ 0 & -1/3 & 2\sqrt{2}/3 \end{pmatrix} \begin{pmatrix} -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{57}$$

$$\boldsymbol{Q^T} = \begin{pmatrix} -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -2/3 & -2/3 & -1/3 \\ -\sqrt{2}/6 & -\sqrt{2}/6 & 2\sqrt{2}/3 \end{pmatrix} \tag{58}$$

$$\boldsymbol{Q} = \begin{pmatrix} -\sqrt{2}/2 & -2/3 & -\sqrt{2}/6 \\ \sqrt{2}/2 & -2/3 & -\sqrt{2}/6 \\ 0 & -1/3 & 2\sqrt{2}/3 \end{pmatrix} \tag{59}$$

part b
Confirmed. See Jupyter notebook for code and code output.

part c
Since $\boldsymbol{Q}$ is orthogonal it is full rank. Therefore $\text{rank}(\boldsymbol{A}) = \text{rank}(\boldsymbol{QR}) = \text{rank}(\boldsymbol{R})$. $\text{rank}(\boldsymbol{R}) = 2$ since the rank of an upper triangular matrix is the number of non-zero rows.

part d
Since $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$ for any matrices $\boldsymbol{A}, \boldsymbol{B}$. So then

$$\det(\boldsymbol{A}) = \det(\boldsymbol{Q})\det(\boldsymbol{R}) = 0 \tag{60}$$

since the determinant of a triangular matrix is the product of its diagonal elements and $\det(\boldsymbol{R}) = 0$.

Problem 4
Let $A \in \mathbb{R}^{nxn}$, have complex eigenvalues $\lambda_1, \dots \lambda_n$. Suppose that by following the method outlined in the problem, A_j converged to an upper triangular

matrix A_∞. Since a triangular matrix must have its diagonal elements equal to its eigenvalues, A_∞ must have non-real diagonal elements. However, a QR factorization of a real matrix will only ever yield real matrices, so all elements of all matrices A_j must also be real. Therefore, by contradiction, it is impossible for a real matrix with complex eigenvalues to converge to a triangular matrix by this method.

Problem 5
part i

$$||x||_\infty = \max_{1\le i\le n} |x_i| \text{ for a vector } \boldsymbol{x} \in \mathbb{R}^n \tag{61}$$

Without loss of generality, let x_j be the element of x whose absolute value is the greatest, so

$$||x||_\infty = |x_j| \tag{62}$$

Then

$$||\boldsymbol{x}||_2 = (\sum_{i=1}^{n}(|x_i|^2))^{1/2} \text{ for a vector } \boldsymbol{x} \in \mathbb{R}^n \tag{63}$$

$$||\boldsymbol{x}||_2 = (x_1^2 + x_2^2 + \ldots + x_j^2 + \ldots + x_i^2)^{1/2} \tag{64}$$

and since $x \in \mathbb{R}$ then $x_i^2 \ge 0$ and

$$||\boldsymbol{x}||_\infty = |x_j| = ((x_j)^2)^{1/2} \le (x_1^2 + x_2^2 + \ldots + x_j^2 + \ldots + x_i^2)^{1/2} \tag{65}$$

and

$$||\boldsymbol{x}||_\infty \le ||\boldsymbol{x}||_2 \tag{66}$$

Further since $|x_{i\neq j}| \le |x_j|$,

$$||\boldsymbol{x}||_2 = (x_1^2 + x_2^2 + \ldots + x_j^2 + \ldots + x_i^2)^{1/2} \le (nx_j^2)^{1/2} \tag{67}$$

$$||\boldsymbol{x}||_2 = (x_1^2 + x_2^2 + \ldots + x_j^2 + \ldots + x_i^2)^{1/2} \le (nx_j^2)^{1/2} \tag{68}$$

and

$$||\boldsymbol{x}||_2 \le \sqrt{n}||\boldsymbol{x}||_\infty \tag{69}$$

Thus combining equations 19 and 22 gives:

$$||\boldsymbol{x}||_\infty \le ||\boldsymbol{x}||_2 \le \sqrt{n}||\boldsymbol{x}||_\infty \tag{70}$$

part ii.
Given equivalent vector norms a and b, such that

$$c_1||\boldsymbol{x}||_a \le ||\boldsymbol{x}||_b \le c_2||\boldsymbol{x}||_a \tag{71}$$

The induced matrix norms defined by norms a and b are

$$||A||_i := \max_{x \in \mathbb{R}^n} \frac{||Ax||_i}{||x||_i} \tag{72}$$

for $i = a, b$. Let $A \in \mathbb{R}^{nxn}$. Then without loss of generality let $x_0 \in \mathbb{R}^n$ be the vector that maximizes $\frac{||Ax_0||_a}{||x_0||_a}$. Then

$$c_1 ||Ax_0||_a \leq ||Ax_0||_b \tag{73}$$

$$\frac{c_1}{||x_0||_a ||x_0||_b} ||Ax_0||_a \leq \frac{1}{||x_0||_a ||x_0||_b} ||Ax_0||_b \tag{74}$$

$$\frac{c_1}{||x_0||_b} ||A||_a \leq \frac{1}{||x_0||_a} \frac{||Ax_0||_b}{||x_0||_b} \tag{75}$$

$$\frac{c_1 ||x_0||_a}{||x_0||_b} ||A||_a \leq \frac{||Ax_0||_b}{||x_0||_b} \leq ||A||_b \tag{76}$$

Similarly, if $x_1 \in \mathbb{R}^n$ is the vector that maximizes $\frac{||Ax_1||_b}{||x_1||_b}$, then

$$||Ax_1||_b \leq c_2 ||Ax_1||_a \tag{77}$$

$$\frac{1}{||x_1||_a ||x_1||_b} ||Ax_1||_b \leq \frac{c_2}{||x_1||_a ||x_1||_b} ||Ax_1||_a \tag{78}$$

$$\frac{1}{||x_1||_a} ||A||_b \leq \frac{c_2}{||x_1||_b} \frac{||Ax_1||_a}{||x_1||_a} \tag{79}$$

$$||A||_b \leq \frac{c_2 ||x_1||_a}{||x_1||_b} \frac{||Ax_1||_a}{||x_1||_a} \leq \frac{c_2 ||x_1||_a}{||x_1||_b} ||A||_a \tag{80}$$

Therefore the induced matrix norms a and b are equivalent.

part iii.
Combining the results of part i and ii above,

$$\frac{||x_0||_\infty}{||x_0||_2} ||A||_\infty \leq ||A||_2 \tag{81}$$

where x_0 is the vector that maximizes the equation

$$||A||_\infty := \max_{x \in \mathbb{R}^n} \frac{||Ax||_\infty}{||x||_\infty} \tag{82}$$

Because the ∞ norm for a vector is simply the absolute value of the largest element in the matrix, x_0 is always a vector in which all elements are ± 1. Therefore $||x_0||_\infty = 1$ and $||x_0||_2 = \sqrt{n}$, giving the result

$$\frac{1}{\sqrt{n}} ||A||_\infty \leq ||A||_2 \tag{83}$$

Similarly,

$$||A||_2 \leq \frac{\sqrt{m}||x_1||_\infty}{||x_1||_2}||A||_\infty \tag{84}$$

In general $||A||_2$ is the largest singular value of A, so therefore x_1 must be a normalized eigenvalue, meaning $||x_1||_2 = 1$ and $||x_1||_\infty <= 1$. This gives the result

$$||A||_2 \leq \sqrt{m}||x_1||_\infty||A||_2 \leq \sqrt{m}||A||_\infty \tag{85}$$

Combining the above results yields

$$\frac{1}{\sqrt{n}}||A||_\infty \leq ||A||_2 \leq \sqrt{m}||A||_\infty \tag{86}$$

part iv.

$$||A||_1 := \max_{x \in \mathbb{R}^n} \frac{||Ax||_1}{||x||_1} = \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}| \tag{87}$$

which is the maximum absolute sum of the columns of A.

$$||A||_\infty := \max_{x \in \mathbb{R}^n} \frac{||Ax||_\infty}{||x||_\infty} = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}| \tag{88}$$

which is the maximum absolute sum of the rows of A. Because the columns of A are the rows of A^T,

$$||A||_1 = ||A^T||_\infty \tag{89}$$

for any matrix $A \in \mathbb{R}^{nxn}$.