# CAAM 453/550: Numerical Analysis I - Fall 2021
## Homework 9 - due by 5pm on Friday, November 5, 2021

*Instructions: You may use any of the code on the Canvas page. Turn in all modified/new MATLAB/Python code (scripts and functions) and all output generated by your code. All code must be commented, and it must be clear what your output is and why you are submitting it. Additionally, all plots must be labeled.*

*For any problems that do not require coding, either turn in handwritten work or typeset work using LaTeX or some other typsetting software. Please do not turn in math as commented MATLAB/Python code or math that has been typed in a word processor!*

*MATLAB/Python code fragments are provided in some problems. If you program in python, you may replace them corresponding numpy or scipy code.*

*CAAM 453 students are to complete problems 1, 2, 4, 5. (90 points)*
*CAAM 550 students are to complete problems 1 - 5. (120 points)*

*CAAM 453 may complete additional problems "for fun," but you will not receive additional credit.*

**Problem 1 (10 points)** This problem uses splines to fit a smooth curve through points in $\mathbb{R}^2$. We are given values $(x_i, y_i)$, $i = 0, \ldots, n$, and we want to fit a piecewise smooth curve through these point. Specifically, we want to find piecewise cubic functions $x(t), y(t)$ such that $x(t_i) = x_i$, $y(t_i) = y_i$, $i = 0, \ldots, n$. The $t_i$'s are computed recursively using

$$\begin{aligned} t_0 &= 0, \\ t_i &= t_{i-1} + \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}, \quad i = 1, \ldots, n. \end{aligned}$$

Consider the points

| $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_i$ | 25.0 | 19.0 | 13 | 9.0 | 5.0 | 2.2 | 1.0 | 3.0 | 8.0 | 13.0 | 18.0 | 25.0 |
| $y_i$ | 5.0 | 7.5 | 9.1 | 9.4 | 9.0 | 7.5 | 5.0 | 2.1 | 2.0 | 3.5 | 4.5 | 5.0 |

Use (free) cubic splines to compute smooth piecewise cubic polynomials $x(t), y(t)$ that fit the data above.

Return a plot that shows the data $(x_i, y_i)$, $i = 0, \ldots, n$, as well as the computed $(x(t), y(t))$ (use a fine partition of $[t_0, t_n]$ to obtain a smooth plot).

**Problem 2 (25 points) Pledged! Complete this problem on your own.** This problem is about piecewise cubic *Hermite interpolation*. The cubic polynomials

$$\begin{aligned} H_0(x) &= (1-x)^2(1+2x), \\ H_1(x) &= x^2(3-2x), \\ h_0(x) &= x(1-x)^2, \\ h_1(x) &= x^2(x-1) \end{aligned}$$

satisfy (here $'$ denotes differentiation with respect to $x$)

$$\begin{array}{llll} H_0(0) = 1 & H_0(1) = 0 & H_0'(0) = 0 & H_0'(1) = 0, \\ H_1(0) = 0 & H_1(1) = 1 & H_1'(0) = 0 & H_1'(1) = 0, \\ h_0(0) = 0 & h_0(1) = 0 & h_0'(0) = 1 & h_0'(1) = 0, \\ h_1(0) = 0 & h_1(1) = 0 & h_1'(0) = 0 & h_1'(1) = 1. \end{array}$$

Given a subdivision

$$a = x_0 < x_1 < \ldots < x_n = b$$

of the interval $[a,b]$ and data

$$\begin{array}{c|c|c|c} x_0 & x_1 & \cdots & x_n \\ \hline f_0 & f_1 & \cdots & f_n \\ f_0' & f_1' & \cdots & f_n' \end{array}.$$

we want to find a function $P : [a,b] \to \mathbb{R}$ such that

- On each $[x_i, x_{i+1}]$, $i = 0, \ldots, n-1$, $P$ is a polynomial of degree at most 3.
- $P(x_i) = f_i$ and $P'(x_i) = f_i'$, $i = 0, \ldots, n$.

a. (10 points) Describe how the function $P$ can be constructed using the polynomials $H_0, H_1, h_0, h_1$.

b. (15 points) Write a MATLAB program that implements your approach and apply it to approximate Runge's function $f(x) = 1/(1+x^2)$ on $[-5,5]$ using equally spaced points.

Plot the computed function for $n = 5$ and for $n = 10$.

**Problem 3 (30 points) CAAM 550 only.** This problem is about Bernstein polynomials, yet another type of polynomial that can be used for polynomial interpolation.

(a) (10 points) The Bernstein polynomials $B_i^n \in \mathcal{P}_n([0,1])$ (here we are only concerned with polynomials on the interval $[0,1]$), $0 \le i \le n$ are given by

$$B_i^n(x) := \binom{n}{i}(1-x)^{n-i}x^i,$$

where

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}.$$

Show that the Bernstein polynomials are nonnegative on $[0,1]$ and represent a partition of unity, i.e.,

$$B_i^n(x) \ge 0, \qquad \sum_{i=0}^{n} B_i^n(x) = 1, \qquad x \in [0,1].$$

(b) (10 points) Show that the Bernstein polynomials satisfy the recursion

$$B_i^n(x) = xB_{i-1}^{n-1}(x) + (1-x)B_i^{n-1}(x), \quad x \in [0,1], \quad 1 \le i \le n.$$

(c) (10 points) Suppose you have a polynomial written in the Bernstein basis. Describe how to transform the coefficients to write your polynomial in a monomial basis. Write the transformation matrix for cubic polynomials.

**Problem 4 (30 points)** Let $x_i = a + ih$, $i = 0, \ldots, n$, $h = (b-a)/n$, and

$$\begin{aligned} T(h) &= \frac{h}{2}f(a) + h\sum_{i=1}^{n-1} f(a+ih) + \frac{h}{2}f(b), \\ S(h) &= \frac{h}{6}f(a) + \frac{2h}{3}\sum_{i=0}^{n-1} f(a+(i+\frac{1}{2})h) + \frac{h}{3}\sum_{i=1}^{n-1} f(a+ih) + \frac{h}{6}f(b). \end{aligned}$$

i. (12 points) Write a program that approximates $\int_a^b f(x)dx$ by the composite trapezoidal rule $T(h)$. Starting with $h = (b-a)$ compute $|T(h) - T(h/2)|/|T(h/2)|$ and reduce $h$ by a factor of 2 until

$$|T(h) - T(h/2)|/|T(h/2)| < \texttt{tol}.$$

Your program should return a table with $h/2$, $T(h/2)$, $|T(h) - T(h/2)|/|T(h/2)|$ and it should return the total number of function evaluations $f(x)$. Use `tol = 1.e-6` . Your program should reuse computed function values as much as possible!

ii. (13 points) Repeat i. using the composite Simpson rule.

iv. (5 points) Apply your programs in i.–iii. to approximate the five integrals

$$\begin{aligned}
\int_0^3 \frac{x}{1+x^2}dx &= \tfrac{1}{2}\log(10), \\
\int_0^{0.95} \frac{1}{1-x}dx &= \log(20), \\
\int_0^{\pi/2} \frac{1}{\sqrt{1-m\sin^2(x)}}dx & \qquad m = 0.5, 0.8, 0.95.
\end{aligned}$$

Print the computed integral, (when exact value of integral is available) the absolute and relative errors between computed and exact integral, and the number of function values used by each method to compute the approximate integral. Your output should look like the following.

```
f(x) = x/(1+x^2), [a,b] = [0,3]
Composite trapezoidal rule
Approximate value of the integral = 1.151292e+00
Absolute error                    = 1.931191e-07
Relative error                    = 1.677411e-07
Number of f evaluations required  =   2049

Composite Simpson rule
Approximate value of the integral = 1.151293e+00
Absolute error                    = 1.004330e-08
Relative error                    = 8.723502e-09
Number of f evaluations required  =    129
```

**Problem 5 (25 points)** In class we have determined the quadrature weights $w_i$, $i = 0, \dots, n$, for a Newton-Cotes quadrature formula using Lagrange polynomials. Another way to compute weights $w_i$, $i = 0, \dots, n$, for given nodes $x_i$, $i = 0, \dots, n$, is as follows. A Newton-Cotes quadrature formula with nodes $x_i$, $i = 0, \dots, n$, is exact for all polynomials of degree $n$. In particular for the polynomials $1, x, x^2, \dots, x^n$ it must hold that

$$\begin{aligned}
\sum_{i=0}^{n} w_i &= \int_a^b 1 dx = b - a, \\
\sum_{i=0}^{n} w_i x_i &= \int_a^b x dx = \frac{1}{2}(b^2 - a^2), \\
\sum_{i=0}^{n} w_i x_i^2 &= \int_a^b x^2 dx = \frac{1}{3}(b^3 - a^3), \\
&\dots \\
\sum_{i=0}^{n} w_i x_i^n &= \int_a^b x^n dx = \frac{1}{n+1}(b^{n+1} - a^{n+1}).
\end{aligned}$$

This is a system of $n+1$ linear equations in the unknowns $w_0, \dots, w_n$.

i. (3 points) Would you recommend the approach outlined above, in general, to compute quadrature weights? Justify your answer! (Hint: The matrix should look familiar.)

ii. (7 points) Choose equidistant nodes $x_i = -5 + i\frac{10}{n}$, $i = 0, \dots, n$. For $n = 5, 10, 15$, compute the corresponding weights using the approach outlined above, and compute the approximation

$$\sum_{i=0}^{n} w_i f(x_i)$$

$f(x) = 1/(1+x^2)$ of the integral

$$\int_{-5}^{5} \frac{1}{1+x^2} dx.$$

For each $n$, list the nodes, the corresponding weights, and the computed approximation of the integral. What do you observe?

iii. (10 points) Based on our discussion of polynomial interpolation, select better quadrature nodes $x_i$, $i = 0, \dots, n$. Specify the nodes. (Note that in our discussion of polynomial interpolation we labeled the nodes $x_1, \dots, x_n$. Here we label them $x_0, \dots, x_n$!) For $n = 5, 10, 15$, and your selection of nodes compute the corresponding weights and compute the approximation

$$\sum_{i=0}^{n} w_i f(x_i)$$

of the integral. For each $n$, list the nodes, the corresponding weights, and the computed approximation of the integral. What do you observe?

iv. (5 points) Use your codes for the computation of the composite trapezoidal rule and the composite Simpson rule to compute approximations of $\int_{-5}^{5} \frac{1}{1+x^2} dx$ with `tol = 1.e-4` . Print the computed integral, as well as the number of function values used by each method to compute the approximate integral.