Michael Goforth
CAAM 550
HW 10
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Problem 1
See Jupyter notebook for code and results.

Problem 2

part i

$$\left|\int_a^b f(x)dx - \hat{T}(h)\right| \leq \left|\int_a^b f(x)dx - T(h) + T(h) - \hat{T}(h)\right|$$

$$
\begin{aligned}
&T(h) = \frac{h}{2}(f(a) + f(b)) + h\sum_{i=1}^{n-1} f(a+ih) \\
&\hat{T}(h) = \frac{h}{2}(f(a) + \delta(a) + f(b) + \delta(b)) + h\sum_{i=1}^{n-1}(f(a+ih) + \delta(a+ih)) \\
&T(h) - \hat{T}(h) = \frac{h}{2}(\delta(a) + \delta(b)) + h\sum_{i=1}^{n-1} \delta(a+ih) \\
&T(h) - \hat{T}(h) \leq \frac{h}{2}(\delta + \delta) + h\sum_{i=1}^{n-1} \delta \\
&T(h) - \hat{T}(h) \leq (b-a)\delta \\
&\left|\int_a^b f(x)dx - T(h)\right| \leq \frac{b-a}{12}h^2 \max_{[a,b]} |f''(x)| \\
&\left|\int_a^b f(x)dx - \hat{T}(h)\right| \leq \frac{b-a}{12}h^2 \max_{[a,b]} |f''(x)| + (b-a)\delta
\end{aligned}
$$

part b

$$T(h) - \hat{T}(h) = \frac{h}{2}(\delta(a) + \delta(b)) + h\sum_{i=1}^{n-1} \delta(a+ih)$$

Because the δ function is random, on average this will equal 0. However for small sample sizes (N), there could be significant variance here. The standard error of the mean is $\sigma/\sqrt{n}$, so to reduce the effects of the variance a value of n should be chosen so that it is large enough to reduce the variance. For example, if the goal is less than 1% error 95% (2 sigma) of the time, then $\sigma/\sqrt{n} = .005$.

If sigma = 1 (as it does in the matlab randn function), then n would need to be 40,000.
See Jupyter notebook for code and results.

Problem 3

part i
See Jupyter notebook for code and results.

part ii
See Jupyter notebook for code and results.

Problem 4
part i

$$T_{0,0} = T(h) = \frac{h}{2}f(a) + \frac{h}{2}f(b)$$
$$T_{1,0} = T(h/2) = \frac{h}{4}f(a) + \frac{h}{2}f(a + \frac{h}{2}) + \frac{h}{4}f(b)$$
$$T_{1,1} = T_{1,0} + \frac{1}{3}(T_{1,0} - T_{0,0})$$
$$T_{1,1} = \frac{h}{4}f(a) + \frac{h}{2}f(a + \frac{h}{2}) + \frac{h}{4}f(b) + \frac{h}{12}f(a) + \frac{h}{6}f(a + \frac{h}{2}) + \frac{h}{12}f(b) - \frac{h}{6}f(a) - \frac{h}{6}f(b)$$
$$T_{1,1} = \frac{1}{6}f(a) + \frac{2}{3}f(a + \frac{h}{2}) + \frac{1}{6}f(b)$$

Also known as Simpson's rule.