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CAAM 550
HW 5
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Problem 1
part i

$$I_n = \int_0^1 x^n e^x dx$$

then

$$I_{n+1} = \int_0^1 x^{n+1} e^x dx$$

and using integration by parts gives

$$\begin{aligned} I_{n+1} &= x^{n+1}e^x|_0^1 - \int_0^1 (n+1)x^n e^x dx \\ I_{n+1} &= e - (n+1)I_n \end{aligned}$$

Because the functions x^n and e^x are greater than 0 on the interval (0, 1],

$$I_n = \int_0^1 x^n e^x dx > 0$$

for all n. Similarly, on the interval (0, 1], $x^n > x^{n+1}$, so

$$I_n = \int_0^1 x^n e^x dx > \int_0^1 x^{n+1} e^x dx = I_{n+1}$$

for all $n \geq 0$. Combining the two results above yields

$$I_1 > I_2 > \ldots > 0$$

part ii
See Jupyter notebook for code and output.

part iii
Let the perturbed data be denoted by I_n^*, then

$$\begin{aligned} &I_1^* = I_1 + \epsilon \\ &I_2^* = e - 2I_1^* = e - 2I_1 - 2\epsilon = I_2 - 2\epsilon \\ &I_3^* = e - 3I_2^* = e - 3I_2 - 6\epsilon = I_3 - 6\epsilon \\ &\ldots \\ &I_n^* = e - nI_{n-1}^* = e - nI_{n-1} - n!\epsilon = I_n - n!\epsilon \end{aligned}$$

So the perturbation grows at a factorial growth rate as n increases, and is therefore ill-suited to solve the integral for large n.

Problem 2
part i

$$A^{-1}A = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \ddots & \\ 1 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \ddots & \\ -1 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \ddots & \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = I$$

part ii
Because the 1-norm of a matrix is the maximum of the absolute column sums, and $||A||_1 = ||A^{-1}||_1 = n$.

$$\kappa_1(A) = ||A||_1||A^{-1}||_1 = n^2$$

Because the infinity norm of a matrix is the maximum absolute row sum, $||A||_\infty = ||A^{-1}||_\infty = 2$.

$$\kappa_\infty(A) = ||A||_\infty||A^{-1}||_\infty = 4$$

Problem 3
See Jupyter notebook for code and numerical results.
As n increases, the problem becomes more ill-conditioned, as seen in the increasing κ_2 value. As a result, the difference between the actual solution and computed solution increases as n increases.

Problem 4
part i.
See Jupyter notebook for code and results.

part ii.
See Jupyter notebook for code and results.

part iii.
For part i, $\gamma^2 - 4\delta < 0$ so eqn 10 is used.

$$y(x) = e^{\mu x}\sin(\theta x)\frac{s - \mu\alpha}{\theta} + e^{\mu x}\cos(\theta x)\alpha$$
$$\frac{\partial y}{\partial s} = \frac{1}{\theta}e^{\mu x}\sin(\theta x)$$

At the boundary condition $y(10)$, and with $\alpha = 0, \beta = 1, \gamma = 1, \delta = 1$

$$\frac{\partial y(10)}{\partial s} = 5.39e - 3$$

(See Jupyter notebook for calculation.) For part ii, $\gamma^2 - 4\delta \geq 0$ so eqn 9 is used.

$$y(x) = e^{\lambda_1 x}\frac{\lambda_2\alpha - s}{\lambda_2 - \lambda_1} + e^{\lambda_2 x}\frac{s - \lambda_1\alpha}{\lambda_2 - \lambda_1}$$
$$\frac{\partial y}{\partial s} = \frac{-e^{\lambda_1 x} + e^{\lambda_2 x}}{\lambda_2 - \lambda_1}$$

Plugging in the values from part ii, $\alpha = 0, \beta = 1, \gamma = -2, \delta = -2$, the partial derivative at the boundary condition $y(10)$ is

$$\frac{\partial y(10)}{\partial s} = 2.12e11$$

(See Jupyter notebook for calculation.)

Problem 5
part i
Since $x_1 = x_{j+1} - x_j = 1$, eqn 18a can be rewritten as:

$$\exp\begin{pmatrix} 0 & 1 \\ -\delta & -\gamma \end{pmatrix}\begin{pmatrix} \alpha \\ s_{2,0} \end{pmatrix} = \begin{pmatrix} s_{1,1} \\ s_{2,1} \end{pmatrix}$$

Let

$$E = \exp\begin{pmatrix} 0 & 1 \\ -\delta & -\gamma \end{pmatrix} = \begin{pmatrix} e_{1,1} & e_{1,2} \\ e_{2,1} & e_{2,2} \end{pmatrix}$$

then eqn 18a can be rewritten as the series

$$e_{1,2}s_{2,0} - s_{1,1} = e_{1,1}\alpha$$
$$e_{2,2}s_{2,0} - s_{2,1} = e_{2,1}\alpha$$

Similarly, eqn 18b can be rewritten as the series

$$e_{1,1}s_{1,j} + e_{1,2}s_{2,j} - s_{1,j+1} = 0$$
$$e_{2,1}s_{1,j} + e_{2,2}s_{2,j} - s_{2,j+1} = 0$$

Finally eqn 18c can be rewritten as

$$e_{1,1}s_{1,9} + e_{1,2}s_{2,9} = \beta$$

With these equations, the problem can be written in the form $\boldsymbol{A}x = b$ where

$$x = (s_{2,0}, s_{1,1}, s_{2,1}, s_{1,2}, s_{2,2}, s_{1,3}, s_{2,3}, s_{1,4}, s_{2,4}, s_{1,5}, s_{2,5}, s_{1,6}, s_{2,6}, s_{1,7}, s_{2,7}, s_{1,8}, s_{2,8}, s_{1,9}, s_{2,9})^T$$

$$b = (-e_{1,1}\alpha, -e_{2,1}\alpha, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \beta)^T$$

$$\boldsymbol{A} = \begin{bmatrix}
e_{1,2} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e_{2,2} & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & e_{1,1} & e_{1,2} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & e_{2,1} & e_{2,2} & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & e_{1,1} & e_{1,2} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & e_{2,1} & e_{2,2} & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & e_{1,1} & e_{1,2} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & e_{2,1} & e_{2,2} & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & e_{1,1} & e_{1,2} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & e_{2,1} & e_{2,2} & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_{1,1} & e_{1,2} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_{2,1} & e_{2,2} & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_{1,1} & e_{1,2} & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_{2,1} & e_{2,2} & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_{1,1} & e_{1,2} & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_{2,1} & e_{2,2} & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_{1,1} & e_{1,2} & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_{2,1} & e_{2,2} & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e_{1,1} & e_{1,2}
\end{bmatrix}$$

See Python code for solution and plots.

part ii

Same calculations as part i. See Jupyter notebook for code and results.

part iii

The condition number of matrix A is

$$\kappa_2 = ||A||_p||A^{-1}||_2 \approx 47.3$$

Then the perturbation of the data is $1e-8$, so the maximum error in the data is

$$\kappa_2 \frac{||\Delta s||}{||s||} \approx 7.28e-6$$

So because the condition number is relatively small, this solution is not very sensitive to perturbations (especially perturbations as small as 1e-8).