CAAM 453/550: Numerical Analysis I - Fall 2021
Homework 10 - due at 5pm on Friday, November 12, 2021

Instructions: You may use any of the code on the Canvas page. Turn in all modified/new MATLAB/python code (scripts and functions) and all output generated by your code. All code must be commented, and it must be clear what your output is and why you are submitting it. Additionally, all plots must be labeled.

For any problems that do not require coding, either turn in handwritten work or typeset work using LaTeX or some other typsetting software. Please do not turn in math as commented MATLAB code or math that has been typed in a word processor!

MATLAB code fragments are provided in some problems. If you program in python, you may replace them corresponding numpy or scipy code.

CAAM 453 students are to complete problems 1-3. (60 points)
CAAM 550 students are to complete problems 1 - 4. (70 points)

CAAM 453 may complete additional problems "for fun," but you will not receive additional credit.

Problem 1 (15 points) Let $x_i = a + ih$, $i = 0, \ldots, n$, $h = (b-a)/n$, and

$$\begin{aligned} T(h) &= \frac{h}{2} f(a) + h \sum_{i=1}^{n-1} f(a+ih) + \frac{h}{2} f(b), \\ S(h) &= \frac{h}{6} f(a) + \frac{2h}{3} \sum_{i=0}^{n-1} f(a + (i + \frac{1}{2})h) + \frac{h}{3} \sum_{i=1}^{n-1} f(a+ih) + \frac{h}{6} f(b). \end{aligned}$$

In Homework 9 you have wrriten a programs that approximates $\int_a^b f(x)dx$ by the composite trapezoidal rule $T(h)$ and the composite Simpson rule $S(h)$ that starting with $h = (b-a)$ compute $T(h)$ and $S(h)$ and reduce h by a factor of 2 until

$$|T(h) - T(h/2)|/|T(h/2)| < \texttt{tol}, \qquad |S(h) - S(h/2)|/|S(h/2)| < \texttt{tol}.$$

i. (10 points) Write a program that approximates $\int_a^b f(x)dx$ using Romberg integration. Your program should compute the rows in the Romberg successively, until $|T_{n,n-1} - T_{n,n}|/|T_{n,n}| < \texttt{tol}$. Use `tol = 1.e-6` . Your program should return the Romberg table and it should return the total number of function evaluations $f(x)$. Your program should reuse computed function values as much as possible.

ii. (5 points) Apply your programs in i.–iii. to approximate the five integrals

$$\begin{aligned} \int_0^3 \frac{x}{1+x^2} dx &= \tfrac{1}{2} \log(10), \\ \int_0^{0.95} \frac{1}{1-x} dx &= \log(20), \\ \int_0^{\pi/2} \frac{1}{\sqrt{1 - m \sin^2(x)}} dx & \qquad m = 0.5, 0.8, 0.95. \end{aligned}$$

Print the computed integral, as well as the number of function values used by each method to compute the approximate integral.

Problem 2 (15 points) Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ cannot be evaluated exactly, but instead of $f(x)$ a value $\widehat{f}(x) = f(x) + \delta(x)$ is computed, where $|\delta(x)| \le \delta$ for all $x \in \mathbb{R}$.

Suppose one is interested in the integral

$$\int_a^b f(x)\,dx$$

and one wants to approximate the integral by the composite trapezoidal rule $T(h)$ with step size $h = (b-a)/N$.

i. (5 points) Derive an estimate for the error $|\int_a^b f(x)\,dx - T(h)|$ when inexact function evaluations are used.

ii. (10 points) Given δ, what would the range of 'reasonable' step sizes h be? Justify your answer.

Demonstrate your findings by approximating $\int_0^1 x^{5/2} dx$, where instead of $f(x) = x^{5/2}$ the function $\widehat{f}(x) = x^{5/2} + 0.01 r(x)$ is used in the composite trapezoidal formula, where $r(x)$ is a normally distributed random variable. Generate the $\widehat{f}$ function values using

```
h = (b-a)/N;
x =  (a:h:b)';
fhat = x.^(5/2) + 0.01*randn(N+1,1);
```

Problem 3 (30 points) Pledged! Complete this problem on your own.
Let $\Omega \subset \mathbb{R}^2$. We want to compute the integral of a continuous function $f : \Omega \to \mathbb{R}$ over Ω. If we can write

$$\Omega = \left\{ (x,y) \in \mathbb{R}^2 \ : \ g(x) \le y \le h(x),\ x \in [a,b] \right\}.$$

where $g, h : [a,b] \to \mathbb{R}$ are functions with $g(x) \le h(x)$ for $x \in [a,b]$, then

$$\int\int_\Omega f(x,y) d(x,y) = \int_a^b \int_{g(x)}^{h(x)} f(x,y) dy dx.$$

We define $F(x) = \int_{g(x)}^{h(x)} f(x,y) dy$ and apply a quadrature formula

$$\int\int_\Omega f(x,y) d(x,y) = \int_a^b F(x) dx \approx \sum_{i=0}^{n_x} w_i^x F(x_i)$$

in which each function value $F(x_i)$ is approximated using another quadrature formula

$$F(x_i) = \int_{g(x_i)}^{h(x_i)} f(x_i,y) dy \approx \sum_{j=0}^{n_y} w_j^y f(x_i, y_j)$$

for the inner integral.

Write a program that uses the composite trapezoidal rule or the composite Simpson rule to implement the reduction approach outlined above to approximate

$$\int\int_\Omega \exp(-xy) d(x,y)$$

where

i. $\Omega = \left\{(x,y) \in \mathbb{R}^2 \ : \ x,y \in [0,1]\right\}$, the unit square,

ii. $\Omega = \left\{(x,y) \in \mathbb{R}^2 \ : \ x^2+y^2 \leq 1,\ x,y \geq 0\right\}$, the quarter of the unit disk lying in the first quadrant.

Experiment with the number of subintervals in the composite quadrature rules. In each experiment document the approximate value of the integral obtained by your approach and the number of function values required.

Problem 4 (10 points) CAAM 550 only
Show that applying one step of Romberg integration to the composite trapezoid rule results in the composite Simpson's rule (remember that the convergence rate of the trapezoid rule is order 2).