УНИВЕРСИТЕТ ИТМО

Methods & Models for Multivariate Data Analysis

Lecture 1. Univariate random variable analysis

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Random Events Reminder

Backgrounds of random events theory

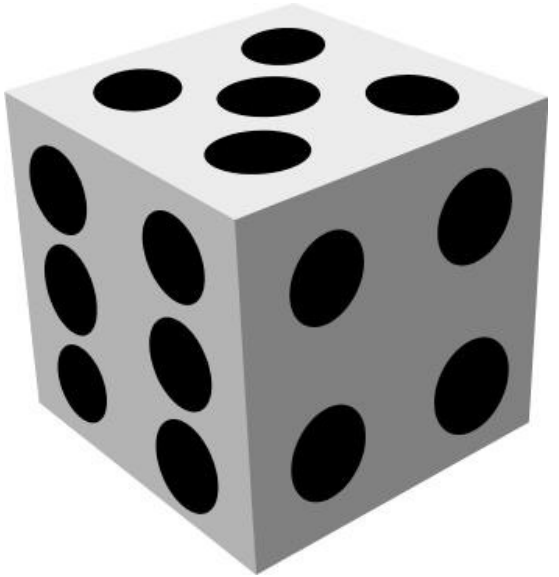
УНИВЕРСИТЕТ ИТМО

Виды событий: достоверное, невозможное, случайное.
Events are: reliable, impossible, random.

Reliable event – I
Impossible event - $\varnothing$

Random event – event that occurs with certain probability
Inverse (обратное) event – $\overline{A}$

Operations on events:
1. Sum - $\bigcup A_k$ **2. Production - $\bigcap A_k$**



Probability of events

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Elementary event **is an outcome of experiment**

Random event **is a sum of set of elementary events** ω_i **:**

$$A = \cup\, \omega_i$$

All random events in a scheme form set of random events F – ***field or algebra of events*****.**

Axiomatic definition of probability:

Probability is a numerical function P(A) defined on F algebra, with properties:

1) **P(I)=1**
2) **P(Ø)=0**
3) **0 < P(A) < 1**

Probability of events

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Classic definition of probability:

Probability of event is a ratio of number of outcomes (m) with event A to number of all outcomes (n) of experiments

$$P(A) = \frac{m}{n}$$

Statistical definition* of probability:

Probability of event is frequency of occurrence of this event that becomes stable in a case of large number of experiments

* Statistical definition doesn't give us formula for calculation of probability itself, but gives us opportunity for calculation an estimate of probability

$$P(A) \approx P^*(A) = \frac{\mu}{n}$$

Probability of events

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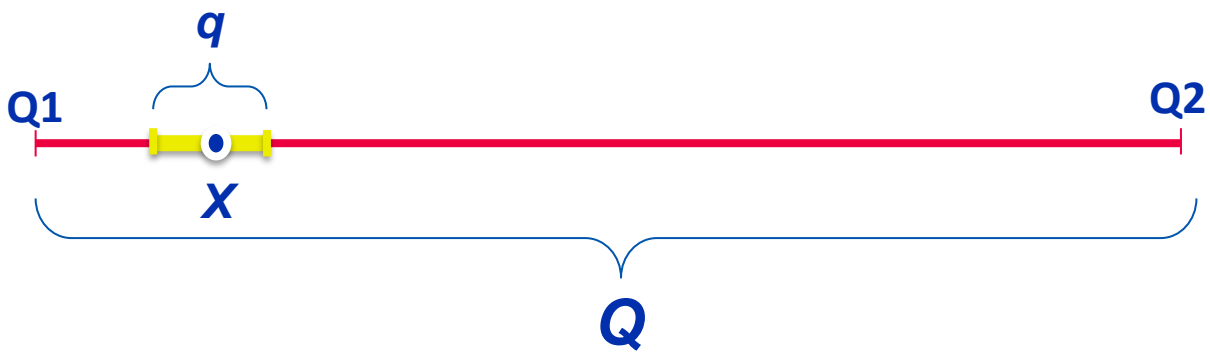
Geometric definition of probability:

Occurrence of X on the interval Q is a reliable event

Occurrence of X on the interval q is a random event



$$P(A) = \frac{length\ \ q}{length\ \ Q}$$

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What are compatible and incompatible events?

Backgrounds of random events theory

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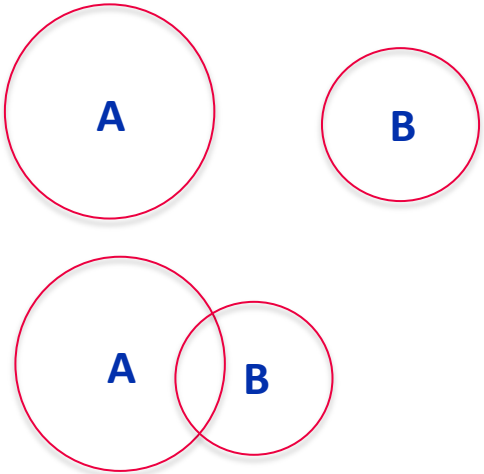
Types of random events: совместные и несовместные (compatible - incompatible)

Incompatible events $A_1 A_2 \ldots A_k = \varnothing$

Compatible events $A_1 + A_2 \ldots + A_k = I$

$A\overline{A} = \varnothing$ $A + \overline{A} = I$



Probability of events

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Addition of probabilities for incompatible events:

$$P(\overline{A} \cup A) = P(\overline{A}) + P(A) = 1$$

$$P(\overline{A}) = 1 - P(A)$$

Addition of probabilities for compatible events:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Conditional (условная) probability $P(B \mid A)$ **and multiplication of probabilities:**

$$P(A \cap B) = P(A)P(B \mid A)$$

Probabilistic independence (вероятностная независимость):

$$P(A \cap B) = P(A)P(B)$$

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Random variables

Random variable

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Discrete random variable **– a variable that can take on only specific values or countable set of values**

Probability law **– it is relation between certain value of random variable and it's probability**

$$p_k = P(\xi = \alpha_k), k = 1,2,...$$

Examples of discrete probability distributions:

- **Binomial probability distribution**
- **Hypergeometric probability distribution**
- **Multinomial probability distribution**
- **Negative binomial distribution**
- **Poisson probability distribution**

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Random variable

УНИВЕРСИТЕТ ИТМО

Continuous random variable **- a variable that can take on any value in a specified range**

Probability distribution function (cumulative) **– probability that random outcome ξ takes on a value less than x**

$$F_\xi(x) = P(\xi < x),\ x \in R*$$

* Strictly speaking $F_\xi(x)$ is defined for discrete random variables, BUT $\xi, x \in N$

Distribution function properties (main):

1. $0 < F_\xi(x) \le 1$
2. $P(a \le \xi < b) = F(b) - F(a)$

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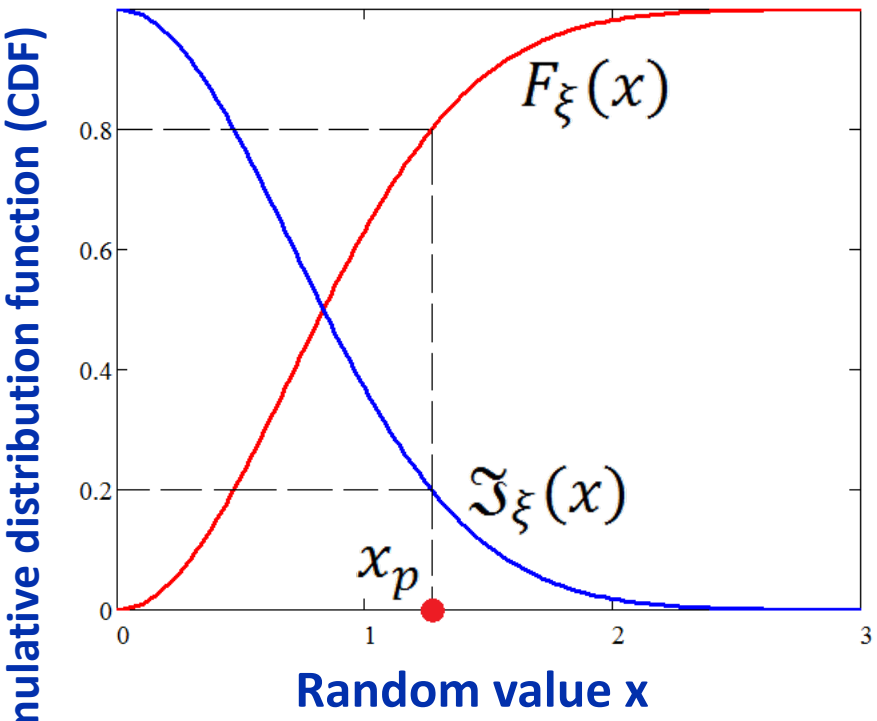
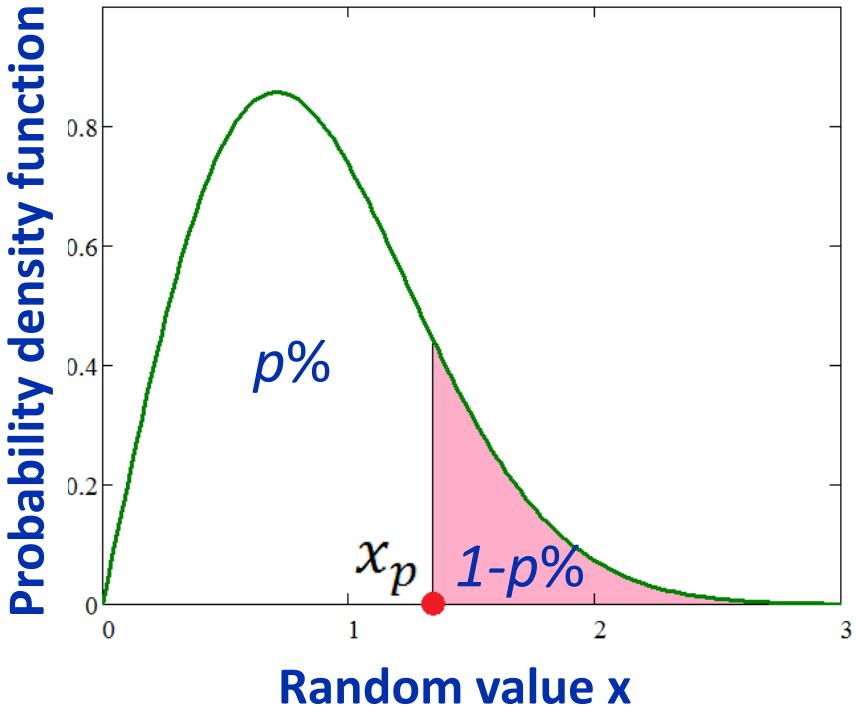
What is relation
between *probability distribution*
***function* and *probability density function*?**

CDF and PDF





p%-quantile :

$$x_p: \quad F_\xi(x_p) = p$$

Distribution density:

$$f_\xi(x) = \frac{dF_\xi(x)}{dx}$$

Random variable

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What is relation between *probability distribution function* and *survival function*?

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Useful model: categorical distribution

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A categorical (multi-nomial) distribution is a generalized probability distribution of categories (k > 2) or discrete random value.

$$f(x = i \mid \boldsymbol{p}) = p_i, \quad \sum_{i=1}^{k} p_i = 1.$$

If there are only two categories k, the multi-nomial distribution - > Bernoulli distribution:

$$f(k; p) = \begin{cases} p & \text{if } k = 1, \\ q = 1 - p & \text{if } k = 0. \end{cases}$$

Why it is so important?

Plays an important role in the topic modeling in ML and NLP.

Useful model: Gaussian distribution

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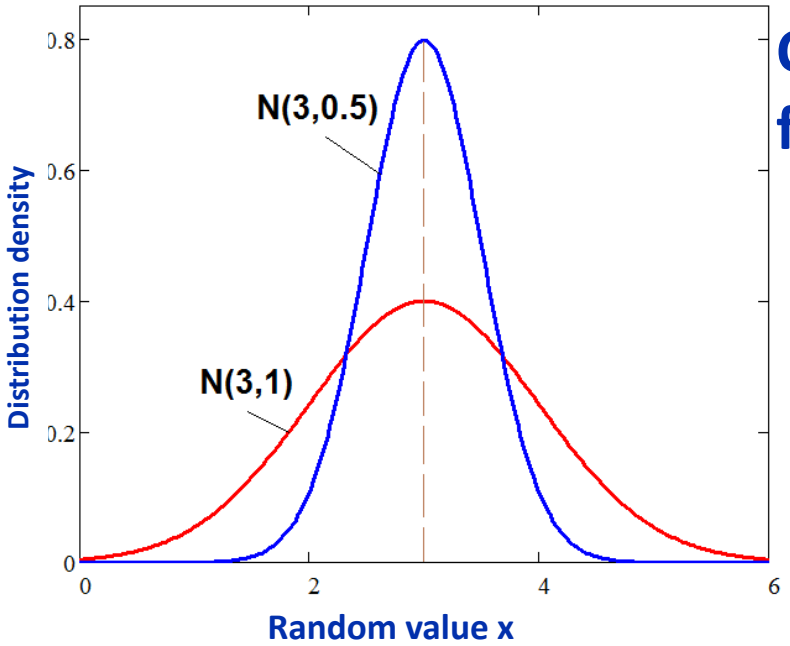
Distribution density: $$f_\xi(x) = \frac{1}{\sqrt{2\pi}\sigma_\xi} exp\left[-\frac{(x - M_\xi)^2}{2\sigma_\xi^2}\right]$$

Nomenaclature: $\boldsymbol{N}(M_\xi, \sigma_\xi)$



Calculation of p%-quantile for Gaussian distribution:

$$x_p = M_\xi + \sigma_\xi u_p$$

Why it is so important?

The mixture of Gaussian distributions is a universal approximation for any smooth density function.

Random variable

What is the probability that random outcome *continuous* RV will have a certain value?

$$P(\xi = b) = ???$$

Nonparametric distribution estimators (1/2)

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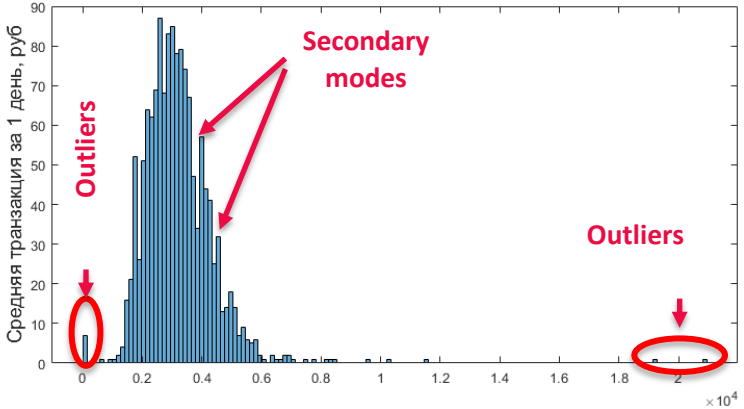
Histogram as way to display frequency characteristics of random values:

$$f^*(x) = \frac{n}{\Delta x n}, x - \frac{\Delta x}{2} < x_i < x + \frac{\Delta x}{2}$$

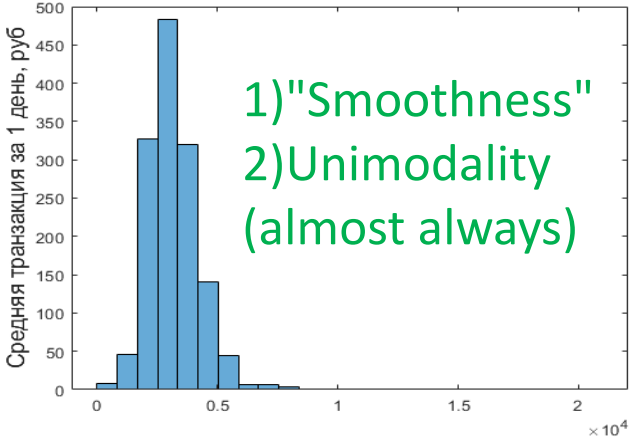


Assessment of number (m) of histogram columns:

1) $m = 1 + 3.32 \lg n$

2) $m = 5 \lg n$

3) **Own choice**



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Nonparamteric distribution estimators (2/2)

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The kernel smoothing method is often applied for "smooth" estimates of distribution density.

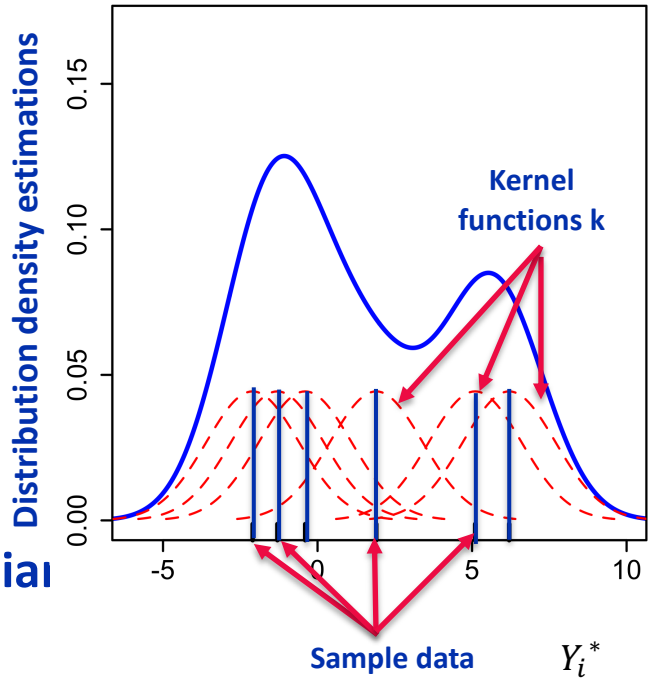
Essence: replacement of the rectangular not crossed histogram columns with the sum of curves (functions) with the centers in sample data:

$$f_{KDE}(y) = \frac{1}{Nh}\sum_i K\left(\frac{y - Y_i^*}{h}\right)$$

Sample data

Kenrel function

Smoothing parameter



There is a lot of kernel functions, but usually Gaussian function is enough:

$$f_\xi(x) = \frac{1}{\sqrt{2\pi}\sigma_\xi} exp\left[-\frac{(x - M_\xi)^2}{2\sigma_\xi^2}\right]$$

Width of a window (h) is selected empirically, but it is possible to estimate on a formula: $h = \sigma^* N^{-0.2}$

Estimation of probability model

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Probabilistic definition

1. **Probability distribution function (cumulative) for continuous variables:**

$$F_\xi(x) = P(\xi < x),\ x \in R$$

2. **Survival probability distribution function for continuous variables:**

$$\widetilde{F}_\xi(x) = P(\xi \geq x) = 1 - F_\xi(x)$$

Statistical definition

1. **Sample estimation of CDF:**

$$F^*(x) = \begin{cases} 0, & x \leq x_1 \\ \dfrac{k}{n}, & x_k < x \leq x_{k+1} \\ 1, & x > x_n \end{cases}$$

2. **Sample estimation of Survival CDF:**

$$1\text{-}F^*(x) = \begin{cases} 1, & x \leq x_1 \\ 1 - \dfrac{k}{n}, & x_k < x \leq x_{k+1} \\ 0, & x > x_n \end{cases}$$

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Estimation of probability model

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Probabilistic definition

3. Probability (density) function:

$$f_\xi(x) = F'_\xi(x)$$

Probability density function properties (main):

1. $\int_{-\infty}^{\infty} f(x)dx = 1$
2. $P(a \le \xi < b) = \int_a^b f(x)dx$

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Statistical definition

3. Sample estimation of PDF:

$$f^*(x) = \frac{n(x)}{\Delta x n}, x - \frac{\Delta x}{2} < x_i < x + \frac{\Delta x}{2}$$

Kernel density estimation is another way for PDF estimation:

$$fk(y) := \frac{1}{N \cdot h} \cdot \sum_i K\left(\frac{y - A_i}{h}\right)$$

$$K(x) := \frac{1}{\sqrt{2 \cdot \pi}} \cdot \exp\left(\frac{-x^2}{2}\right)$$

Random variable

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What are *moments* of probability distribution?

What is *mathematical expectation* from probability theory point of view? How it could be estimated in statistics?

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Estimation of probability model

УНИВЕРСИТЕТ ИТМО

Probabilistic definition

4. High-order moments of probability distribution

$$m_k = \int_R (x - c)^k f_\xi(x) dx$$

If $c = 0$ → m_k is a **raw** moment

If $c = M[X]$ → m_k is a **central** moment

First raw moment – *mathematical expectation*

$$M[x] = m_1 = \int_R x f_\xi(x) dx$$

Statistical definition

4. High-order moments estimation

First raw moment point unbiased estimation – *mean value*

$$\bar{x} = \frac{\sum_n x}{n}$$

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Estimation of probability model

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Probabilistic definition

4. **High-order moments of probability distribution**

Second central moment – *variance*

$$V[x] = \int_{-\infty}^{\infty} (x - M[x])^2 f_{\xi}(x) dx$$

Statistical definition

4. **High-order moments estimation**

Second central moment point *biased* estimation

$$V^* = \frac{\sum_n (x - \bar{x})^2}{n}$$

Second central moment point *unbiased* estimation

$$V^{**} = \frac{\sum_n (x - \bar{x})^2}{n - 1}$$

For big samples *V*=V***

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Estimation of probability model

УНИВЕРСИТЕТ ИТМО

What is *quantile (percentile)*
of probability distribution?
Median?
Mode?

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Estimation of probability model

УНИВЕРСИТЕТ ИТМО

Probabilistic definition

5. Quintiles (percentiles) of probability function:

$$x_p: \quad F_\xi(x) = p$$

$x_{0,25}, x_{0,75}$ - Quartiles

$Me = x_{0,5}$ - Median

$Q = x_{0,75} - x_{0,25}$ - Inter quartile range

6. Mode of probability function:

$$x_m: \quad f_\xi(x_m) \to \max$$

Statistical definition

5. Quintiles (percentiles) sample estimation using *order statistics*

Sample in ascending order:

$$x_1 < x_2 < x_i < \ldots < x_n$$

$$x_p = x_i, i = np + 1$$

6. Mode of probability function:

$$Mode^* = \max(f^*(x))$$

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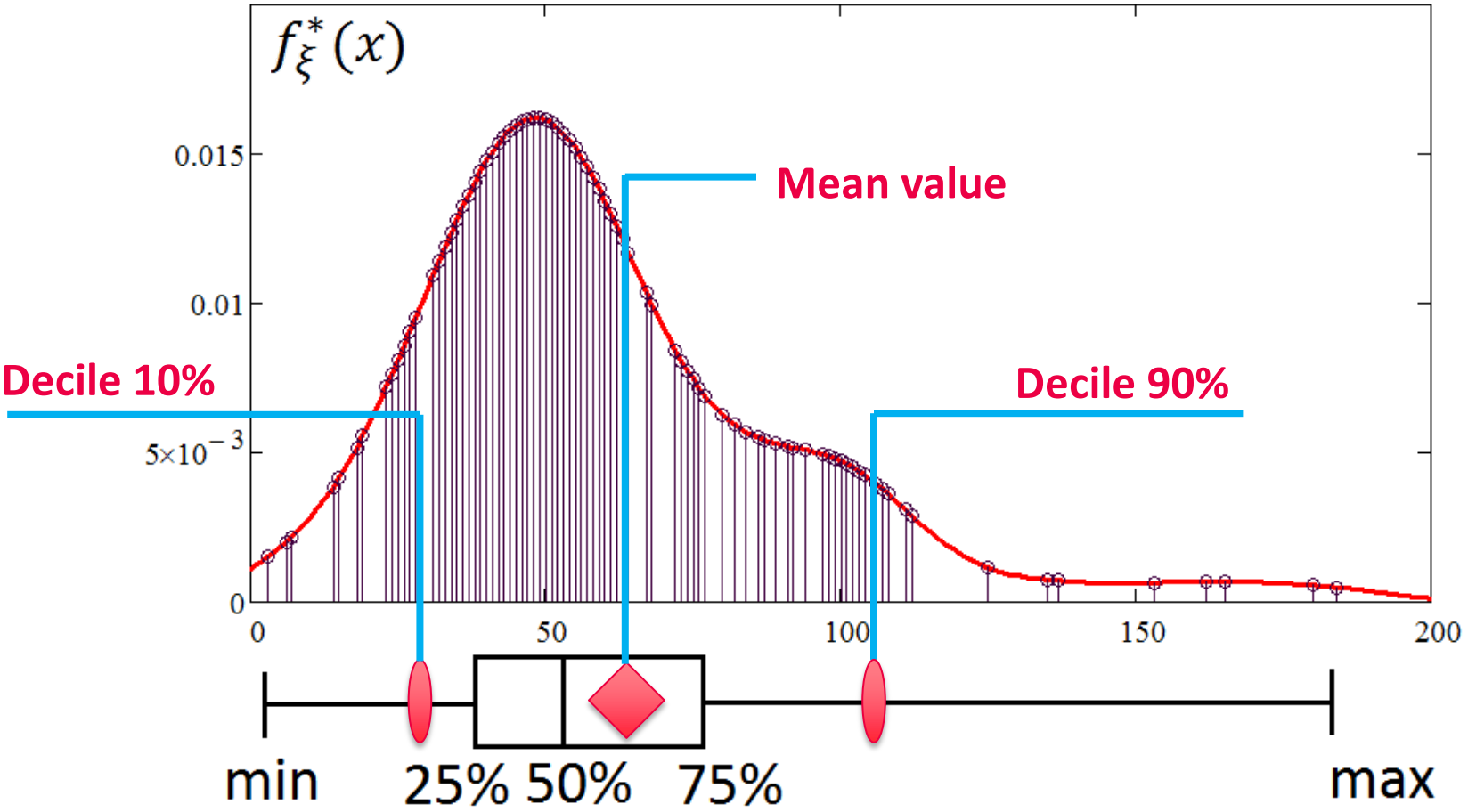
Mapping of quantiles: boxes with moustaches



Is sample mean value always enough? Why we need mode and median?

Measures of central tendency

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Is sample mean value always enough?

Sample mean value: $M_\xi^* \stackrel{\text{def}}{=} \bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$

(is suitable only for real numbers)

For example: average clients per day ~~=~~ **10,42**

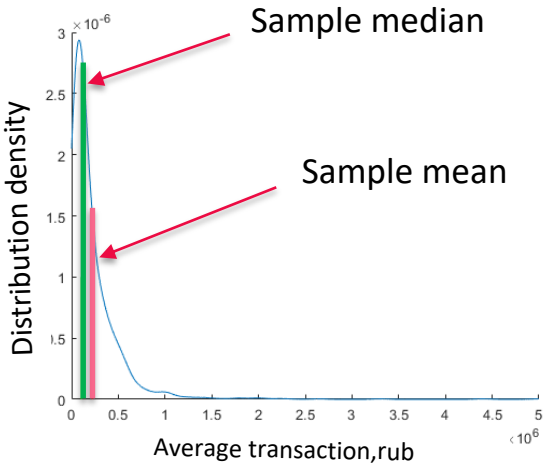
Sample median: $Me = x_{p=0.5}$

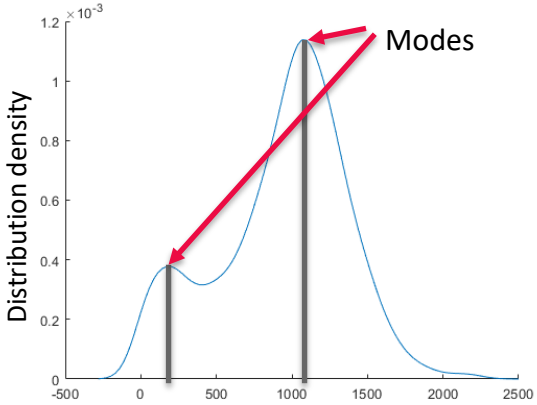
(is suitable for integers, real and ordinal numbers)

Sample mode: $Md = argmax(f^*)$

(is suitable for all numbers)

For all scales, except continuous, the mode is defined by the value having bigger repeatability than the next values.





Measures of deviation

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The measure of spread characterizes "density" of data around the central trend

Sample variance:

$$D_\xi^* \stackrel{\text{def}}{=} s^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2$$

Sample root-mean-square deviation :

$$s = \left(D_\xi^*\right)^{1/2}$$

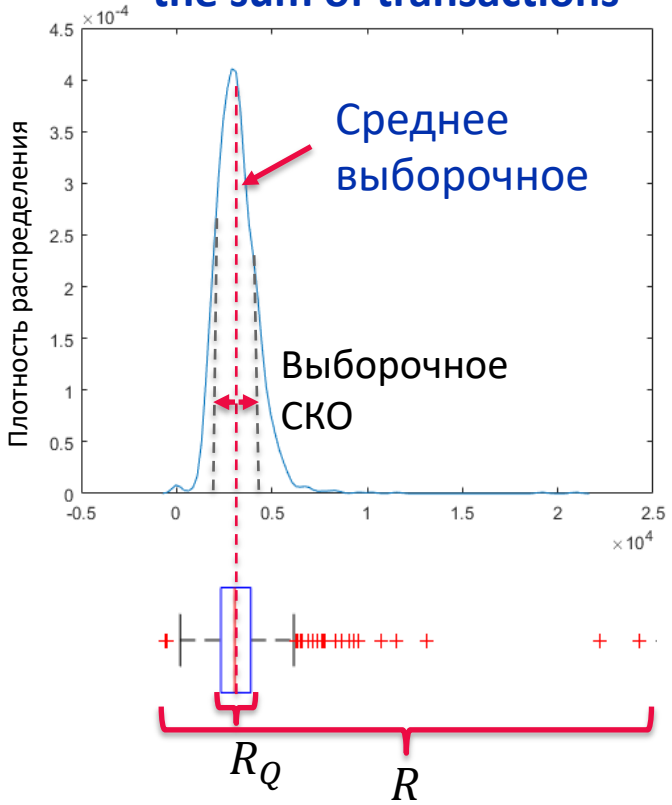
Sample range:

$$R = X_{max} - X_{min}$$

Interquartile distance:

$$R_Q = X_{0.75} - X_{0.25}$$

Example: Estimate of outliers for the sum of transactions



What brilliant feature has ordered statistics (quantiles) vs. parametric estimation of distributions?

Variety of distributions

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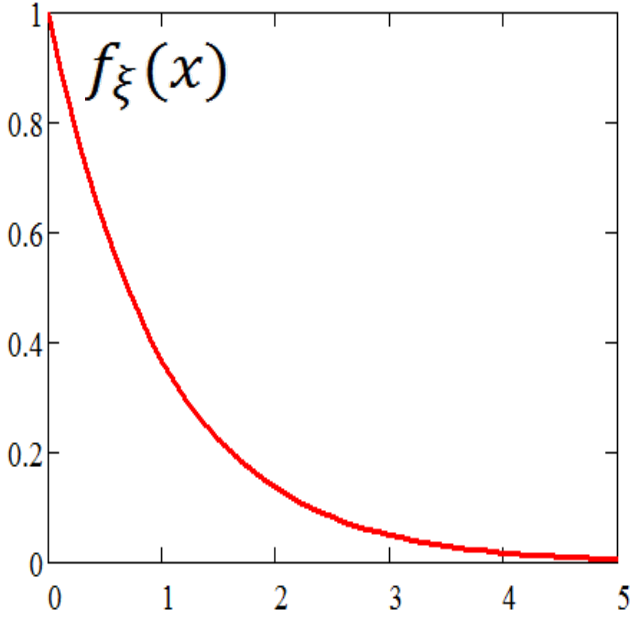
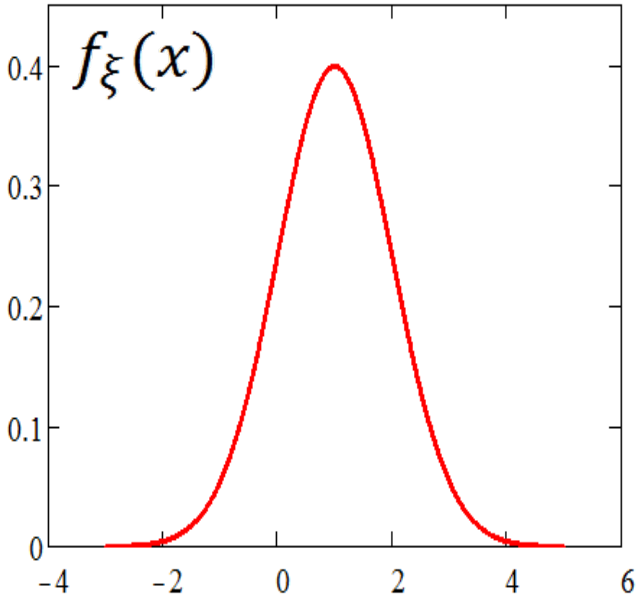
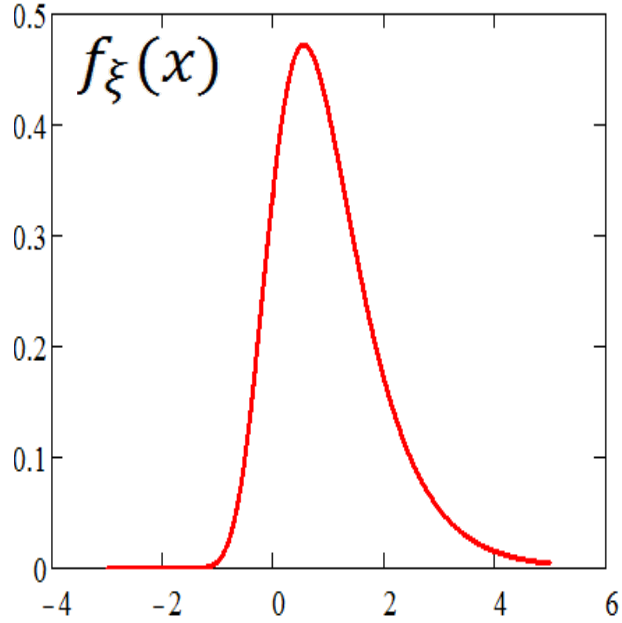
Exponential distribution



Gaussian distribution



Gumbel distribution



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Distribution characteristics: $\bar{x} = 1, \quad s^2 = 1$

Characteristic quantiles and their complexes

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Selective median:

$$Me^* = \begin{cases} x_{(N/2+1)}, & N - \text{odd} \\ (x_{(N/2+1)} + x_{(N/2)})/2, & N - \text{even} \end{cases}$$

Interquartile distance :

$$Q = x_{(\lfloor 3N/4 \rfloor + 1)} - x_{(\lfloor N/4 \rfloor + 1)}$$

Interdecile distance:

$$D = x_{(\lfloor 9N/10 \rfloor + 1)} - x_{(\lfloor N/10 \rfloor + 1)}$$

Sample range:

$$R = x_{(N)} - x_{(1)}$$

k-average amplified range:

$$R_{(k)} = \frac{1}{k}\left(\sum_{j=0}^{k-1} x_{(N-j)} - \sum_{j=1}^{k} x_{(j)}\right)$$

Interval estimates

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What *is interval estimate*?

In what cases we use *interval estimates?*

What is difference between *probability interval* and *confidence interval?* (3 points)

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Interval estimates

Probability interval should not be confused with *Confidence interval*

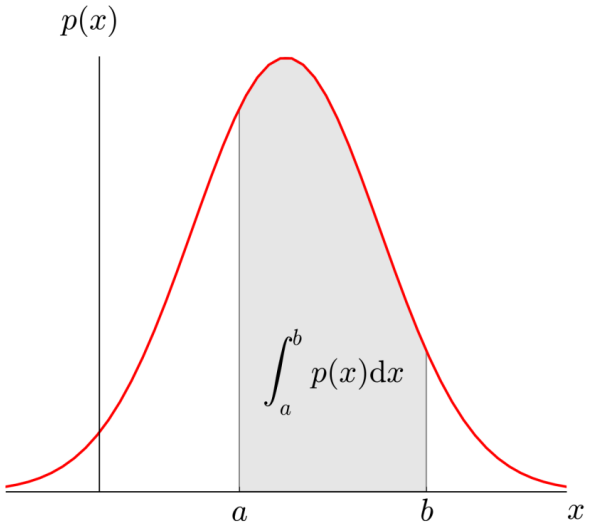
More general definition:

Probability interval - is an estimate of an interval in which random outcome of random variable will fall, with a certain probability, given by known or observed probability distribution function

Probability interval definition is followed from properties of probability distribution functions:

$$P(a \le \xi < b) = F(b) - F(a)$$

$$P(a \le \xi < b) = \int_a^b f(x)dx$$



Interval estimates

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Confidence interval is a particular case for probability interval for distribution function parameter estimates:

- Confidence interval is based on known distribution function for estimate of parameter.
- Confidence interval range can vary from sample to sample due to its size.
- Confidence interval can be built using known and unknown parameters

$$P[\widetilde{\Theta} - u_{1-\alpha/2}\sigma_{\widetilde{\Theta}} < \Theta < \widetilde{\Theta} + u_{1-\alpha/2}\sigma_{\widetilde{\Theta}}] \approx 1 - \alpha$$

where α is significance level, (1-α) is a confidence level.

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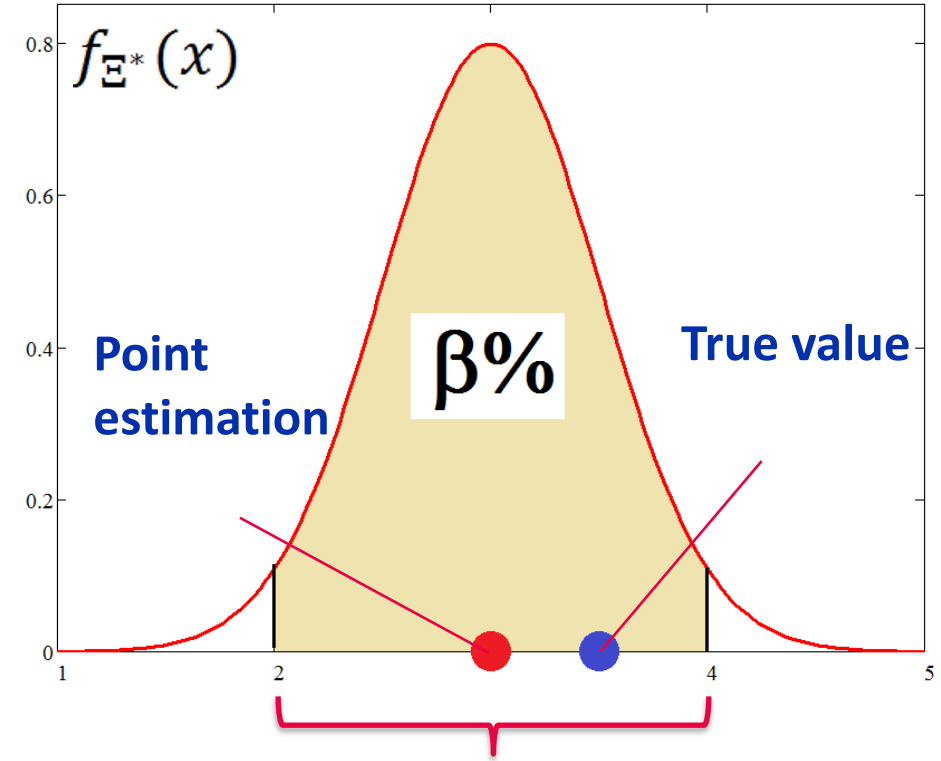
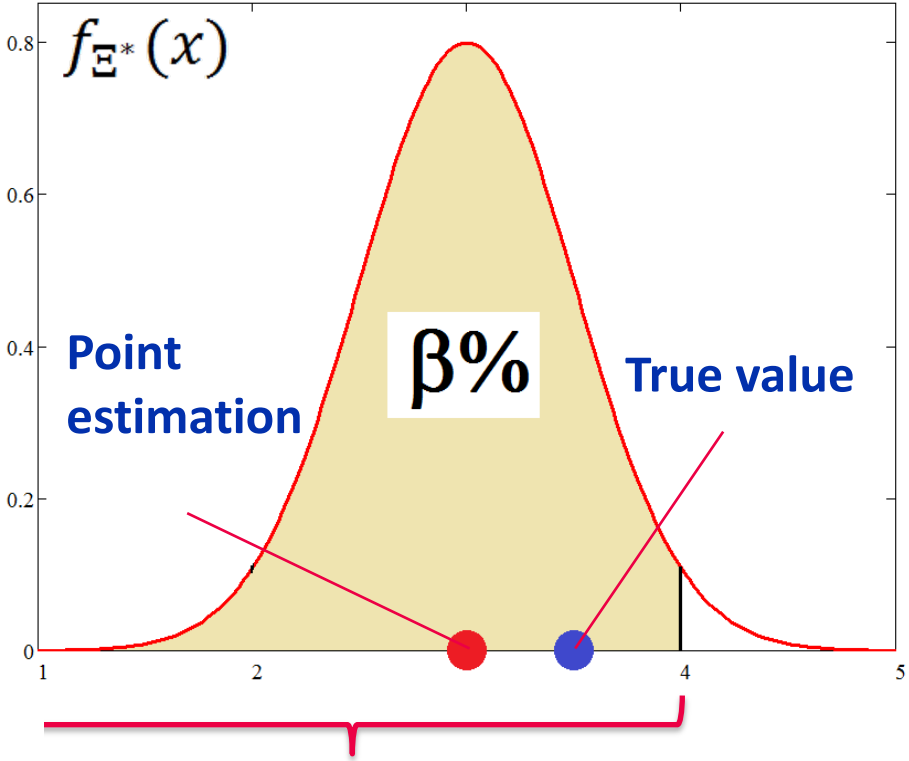
The higher the confidence level, the wider bounds of confidence interval.

Interval estimates



β% *two-sided confidence interval*



β% one-sided confidence interval

Interval estimates

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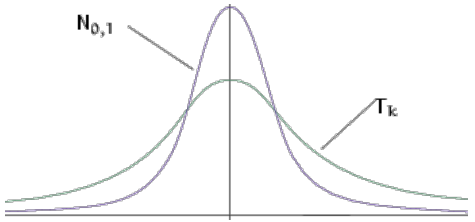
Confidence interval for mean (σ is a sample estimate):

$$P[\mu^* - t_{1-\alpha/2}\frac{\sigma_\mu}{\sqrt{n}} < \mu < \mu^* + t_{1-\alpha/2}\frac{\sigma_\mu}{\sqrt{n}}] \approx 1-\alpha$$

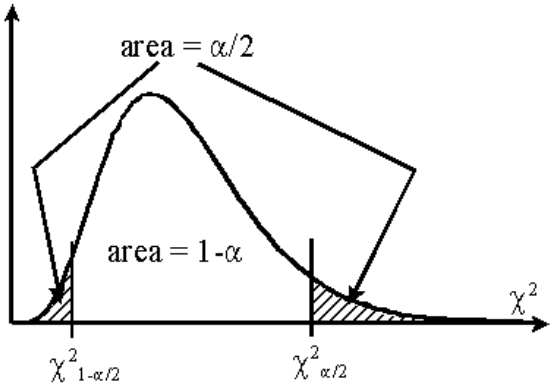
t -Student distribution



Confidence interval for variance:

$$P[\frac{D^*(n-1)}{\chi^2_{\alpha/2}} < D < \frac{D^*(n-1)}{\chi^2_{1-\alpha/2}}] \approx 1-\alpha$$

χ^2 probability density:



Confidence interval for standard deviation:

$$P[\frac{\sigma^*\sqrt{n-1}}{\chi_{\alpha/2}} < \sigma < \frac{\sigma^*\sqrt{n-1}}{\chi_{1+(1-\alpha)/2}}] \approx 1-\alpha$$

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Interval estimates

УНИВЕРСИТЕТ ИТМО

Prediction interval **- is an estimate of an interval in which future observations will fall, with a certain probability, given what has already been observed**

From "Wikipedia"

Prediction interval = Probability interval

In other word *prediction interval* shows that you expect next predicted value will be within boundaries of calculated distribution of observed population.

Prediction interval **tells you about the distribution of values, and about its uncertainty.**

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Interval estimates

УНИВЕРСИТЕТ ИТМО

What is tolerance interval? (3 points)

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Tolerance interval

УНИВЕРСИТЕТ ИТМО

Tolerance interval **is an interval within which, with some confidence level, a specified proportion of a sampled population falls.**

Tolerance interval may be described as confidence interval for estimate of probability interval.

$$P\left[\int_{x_{(a)}}^{x_{(b)}} f(x)dx \geq \gamma\right] = \beta$$

$$P\left[F(x_{(b)}) - F(x_{(a)}) \geq \gamma\right] = \beta$$

where $x_{(a)}$ - quintiles of distribution, γ - width of probability interval,

β - confidence level

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Estimation of distribution parameters

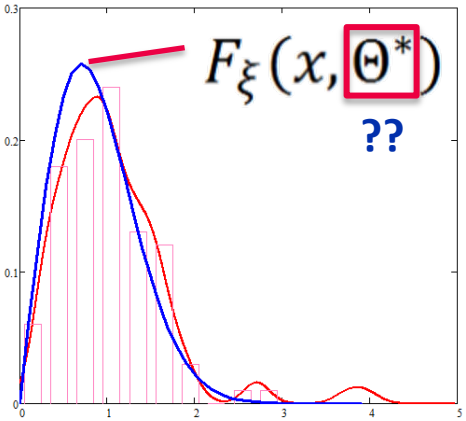
УНИВЕРСИТЕТ ИТМО

Purpose: to understand under what law the analysed random value lives (the income of the client, turnover of the bank branch, etc.)

Means: parametrical distributions

$$F_\xi^*(x) = F_\xi(x, \Theta^*)$$

Essence of estimation of distribution parameters: search of parameters which minimize a divergence betwee theoretical and empirical distributior



Known distribution

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Estimation of distribution parameters

УНИВЕРСИТЕТ ИТМО

There are several methods of parameter estimation in statistics:

1. Method of moments
2. Method of quantiles (inverse distribution method)
3. Maximum likelihood method
4. Least square method

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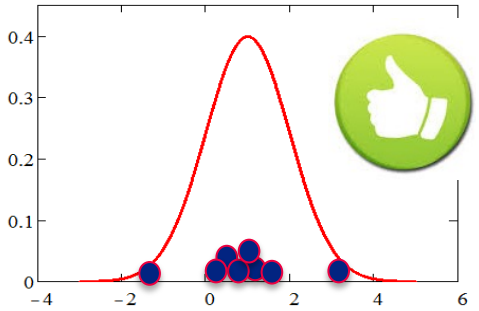
Maximum likelihood method

УНИВЕРСИТЕТ ИТМО

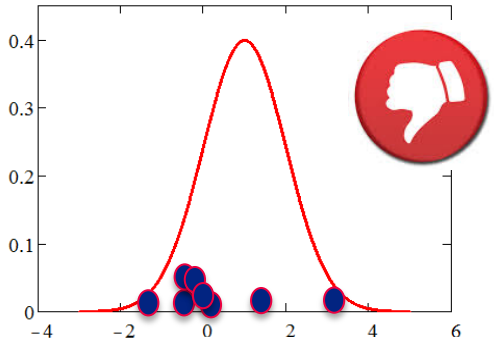
Likelyhood function:

$$L(x_1, \dots, x_N | \Theta) = \prod_{i=1}^{N} f_\xi(x_i, \Theta)$$



Idea:

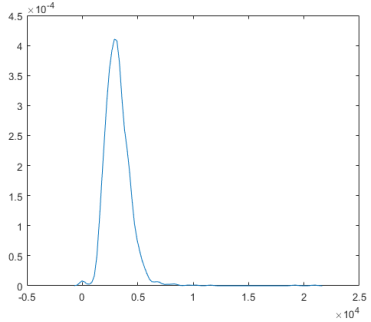
$$L(x_1, \dots, x_N | \Theta) \underset{\Theta}{\rightarrow} max$$

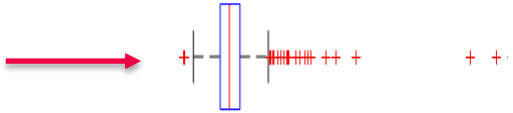


The transformed task:

$$\ln(L) \underset{\Theta}{\rightarrow} max$$

Example: distribution
of transaction sum
and estimation of likelyhood of
Gaussian distribution





Least Squares Method

УНИВЕРСИТЕТ ИТМО

Residuals as Euclidean distance:

$$d(x_p) = (x_p^* - x_p(\Theta))^2$$

The idea (squares sum minimization of residuals by L characteristic quantiles):

$$\sum_{k=1}^{L} (x_{p_k}^* - x_{p_k}(\Theta))^2 \underset{\Theta}{\rightarrow} min$$

Method implementation (equation system):

$$\sum_{k=1}^{L} (x_{p_k}^* - x_{p_k}(\Theta)) \frac{\partial x_{p_k}(\Theta)}{\partial \theta_j} = 0, \qquad j = 1, \dots, l$$

LS method for linaerly scalable variables

Quintile of linearly scalable variable:

$$x_{p_k}(\Theta) = \theta_1 + \theta_2 U_{p_k}$$

$$U_{p_k} = F^{-1}_{\xi(0)}(p_k)$$ - **quantile of normilized model distribution**

LS implementation:
System with 2 linear algebraic equations

$$\begin{cases} L\theta_1 + \theta_2 \sum_{k=1}^{L} U_{p_k} = \sum_{k=1}^{L} x^*_{p_k} \\ \theta_1 \sum_{k=1}^{L} U_{p_k} + \theta_2 \sum_{k=1}^{L} U^2_{p_k} = \sum_{k=1}^{L} U_{p_k} x^*_{p_k} \end{cases}$$

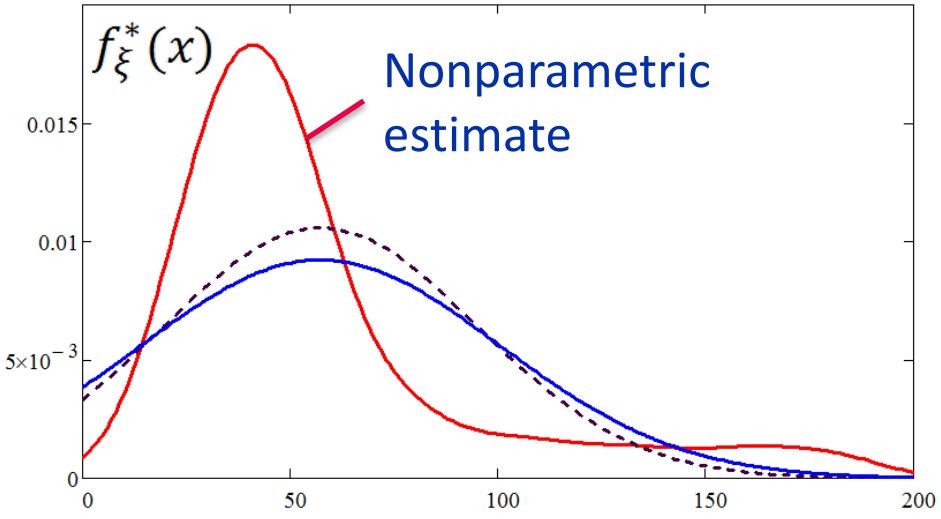
The simplest approximation: throughout all sample

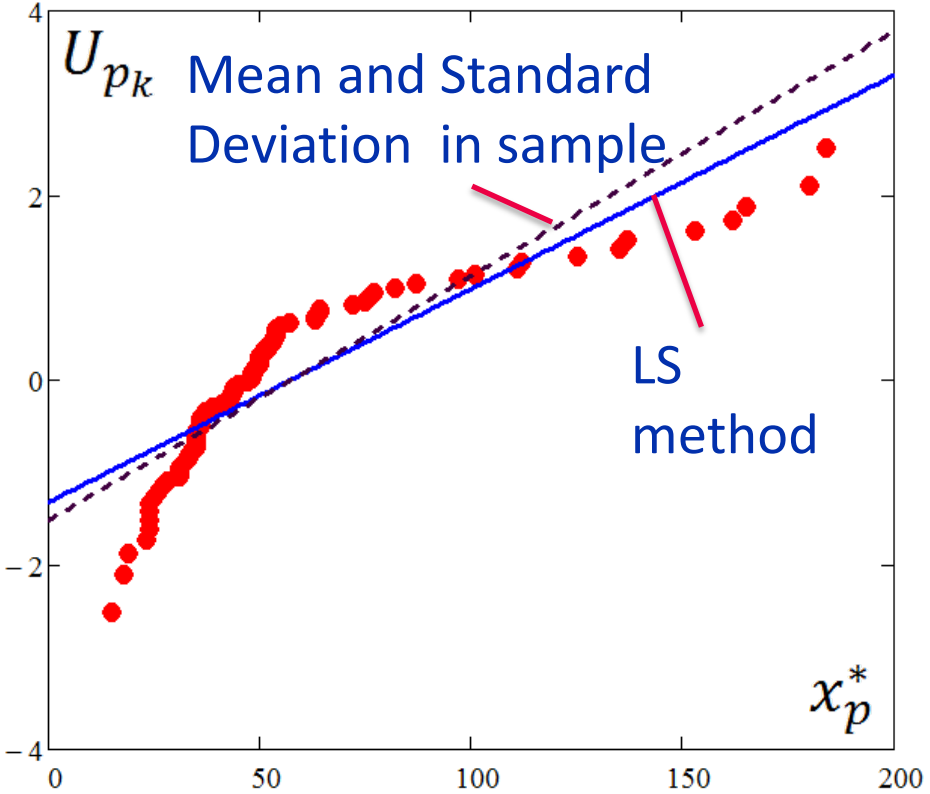
УНИВЕРСИТЕТ ИТМО

Task: approximate model distribution to sample, which not quite corresponds to it

All sample quantiles are eqvivalent

$$L = N$$





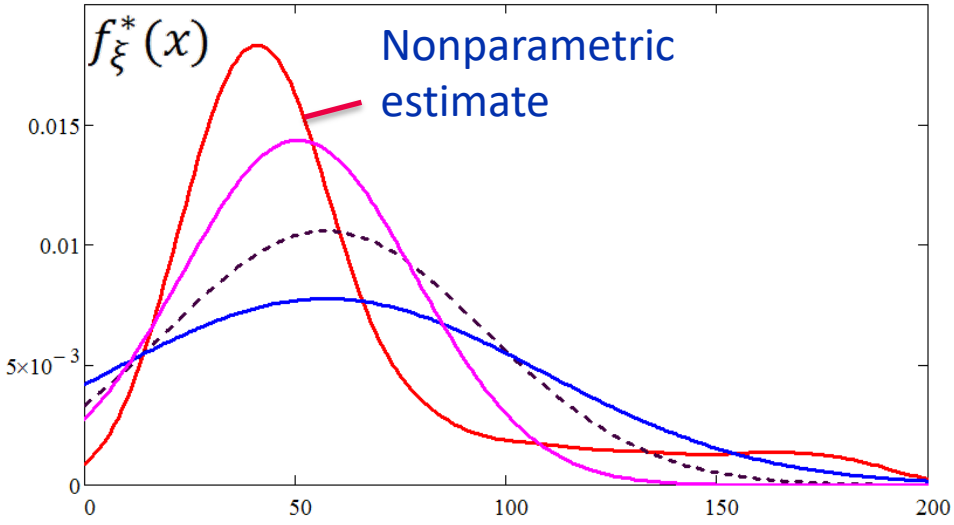
Weighted LS: consider aspects

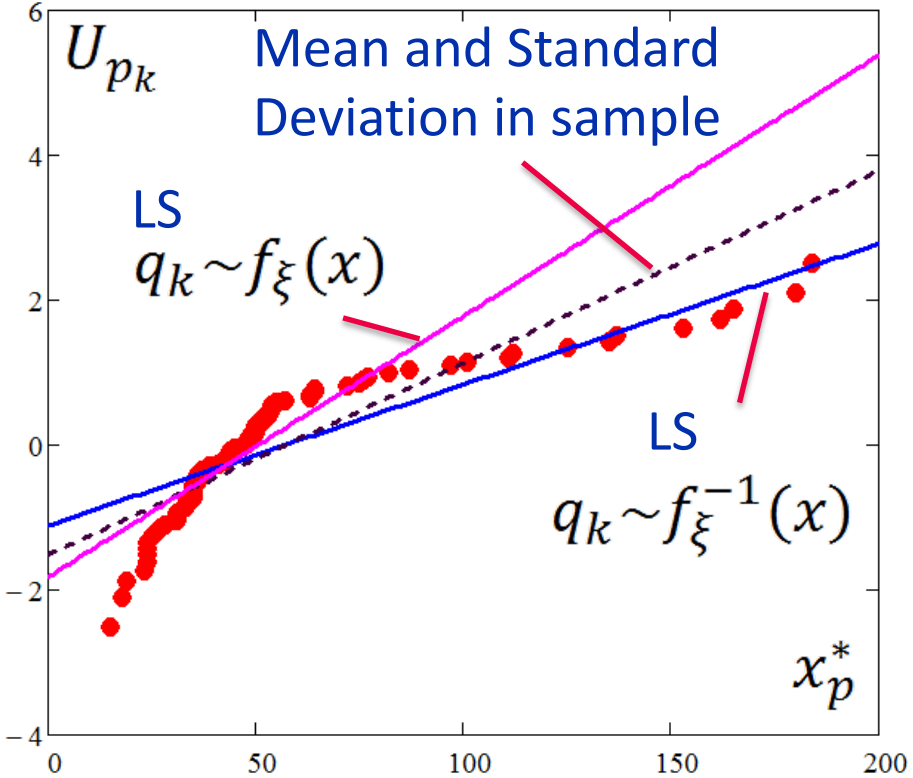
$$\sum_{k=1}^{L} q_k \left(x^*_{p_k} - x_{p_k}(\Theta)\right)^2 \underset{\Theta}{\rightarrow} min$$

Objective function with weights q_k

All sample quantiles are not eqvivalent

$$L = N$$





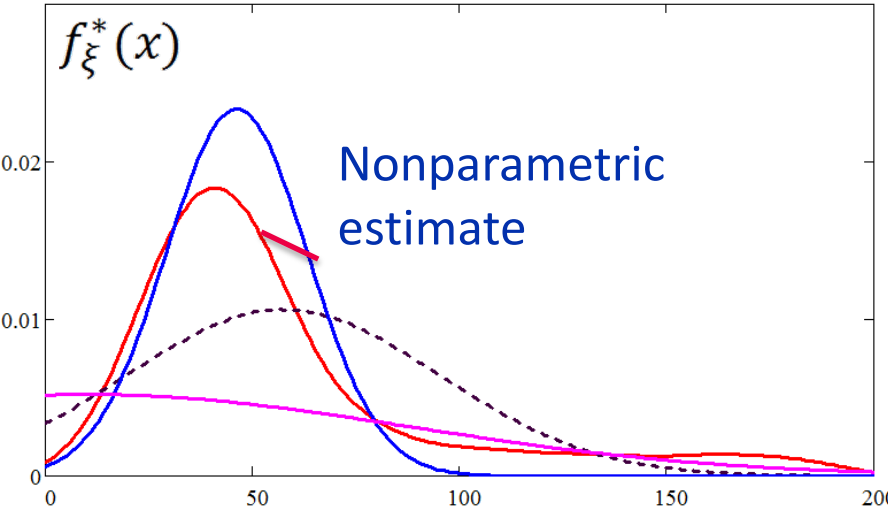
LS method with selected quantiles

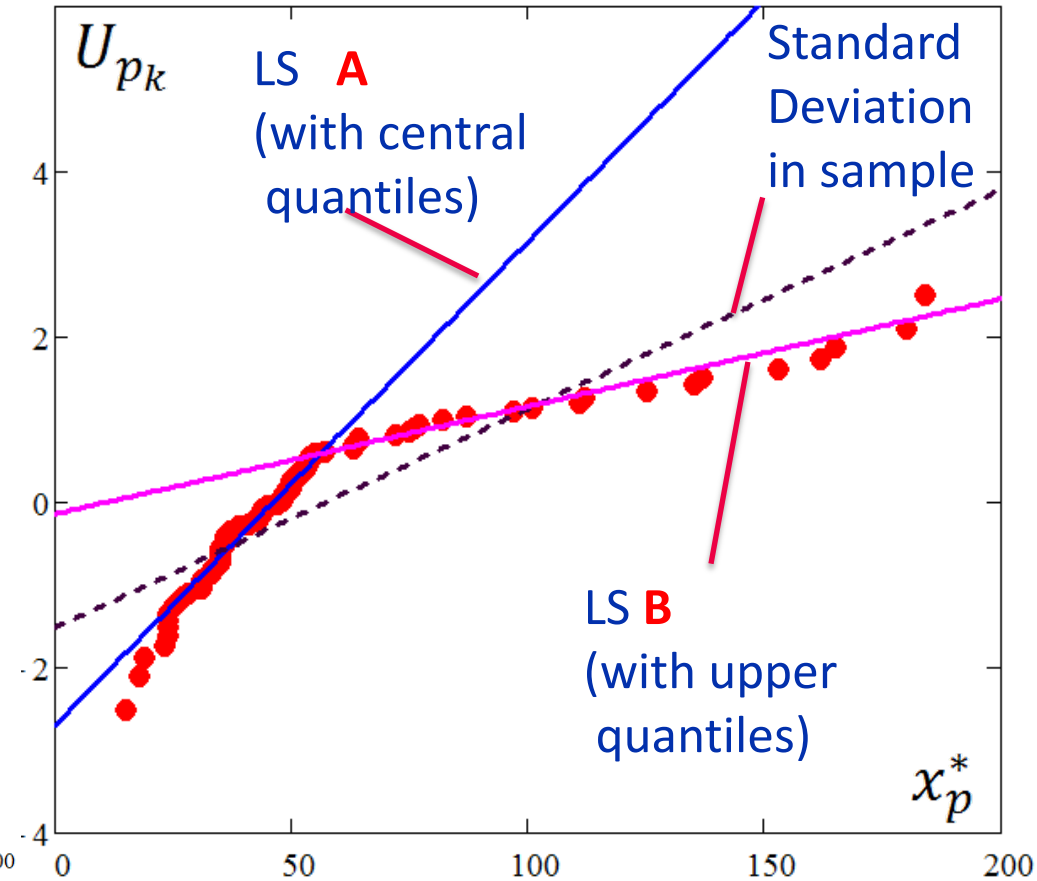
УНИВЕРСИТЕТ ИТМО

Presumtion: selected quantiles are eqvivalent $L < N$

A: quantiles (10%,25%,35%, 50%, 65%, 75%)

B: quantiles (75%,85%,90%,95%, 97%,99%)

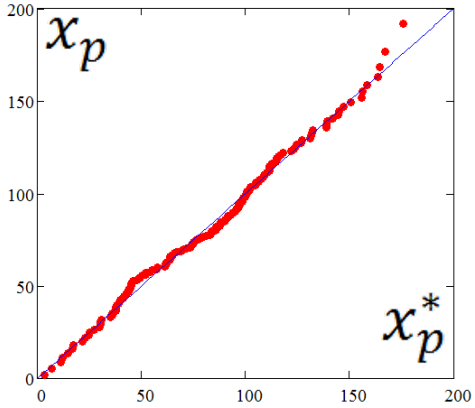


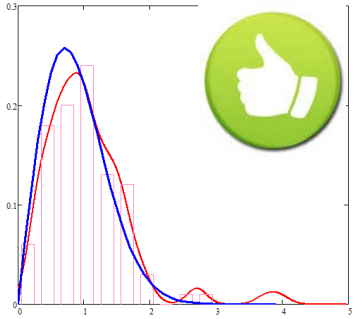


Quality analysis of the estimated parameters

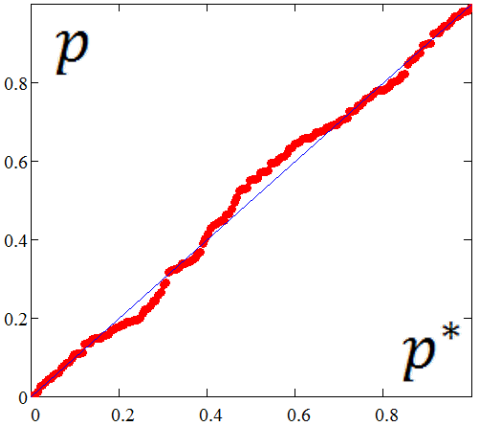
УНИВЕРСИТЕТ ИТМО

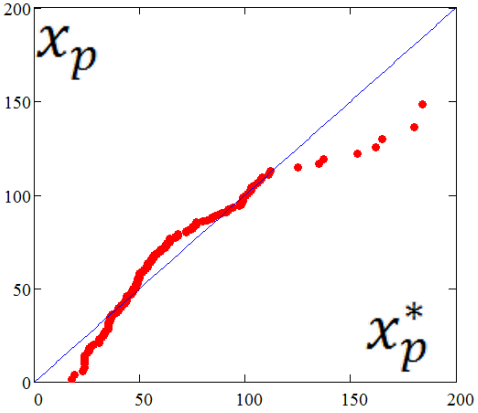
Quantile biplot (*q-q plot*)

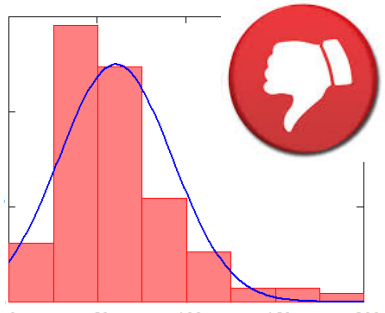


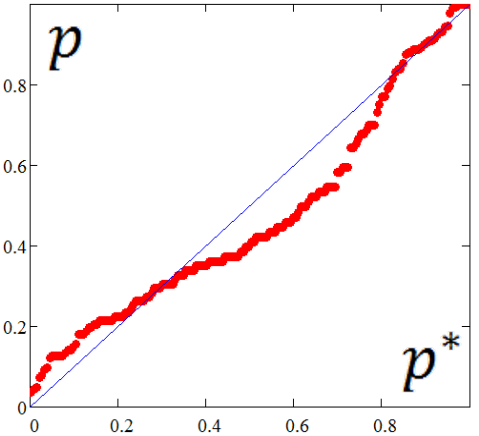


Probability biplot (*p-p plot*)









Goodness-of-fit criteria

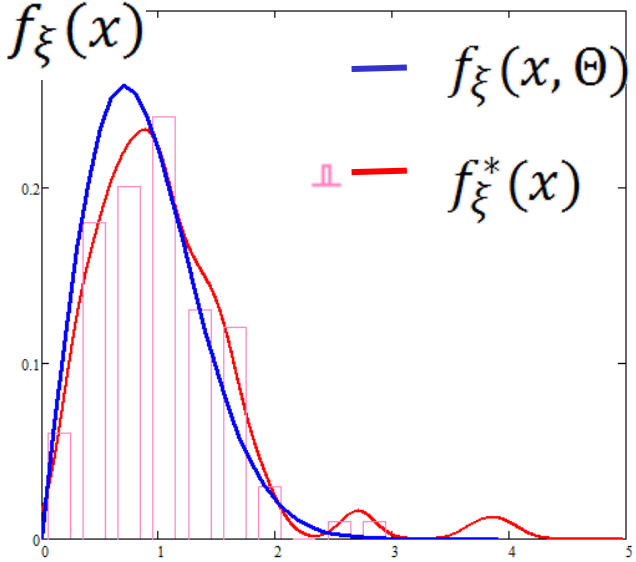
УНИВЕРСИТЕТ ИТМО

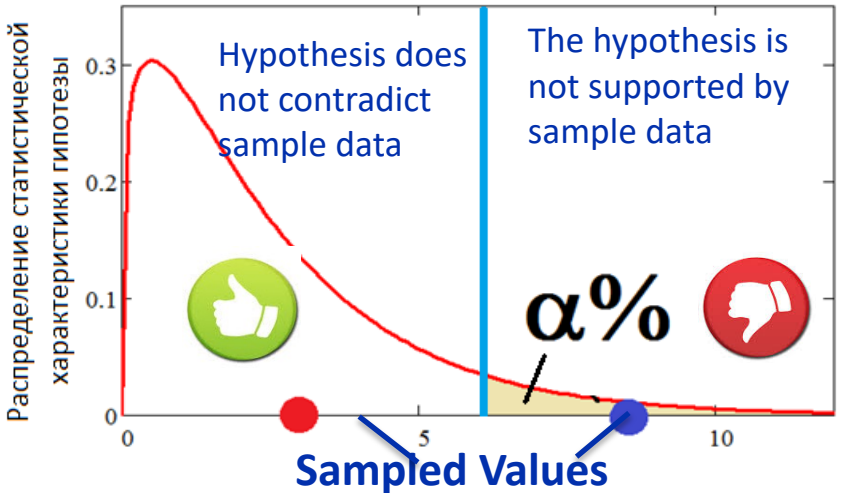
Goodness-of-fit criterion: a mechanism for testing the null hypothesis by comparing selected data with a theoretical “standard”

The idea: the characteristic of the residual is selected to compare reality and the “standard”, and a parametric distribution model is created for this characteristic.

Null hypothesis - H_0 **:** $f_\xi(x,\Theta) = f_\xi^*(x)$

Alternative H_1 **:** $f_\xi(x,\Theta) \neq f_\xi^*(x)$



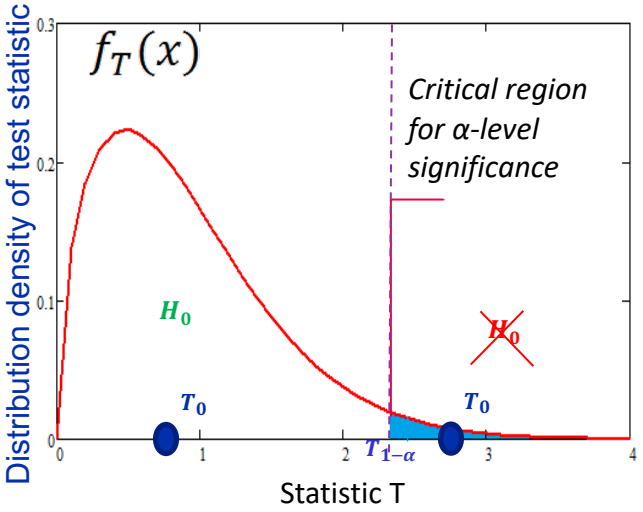


When we need to compare probability distributions?

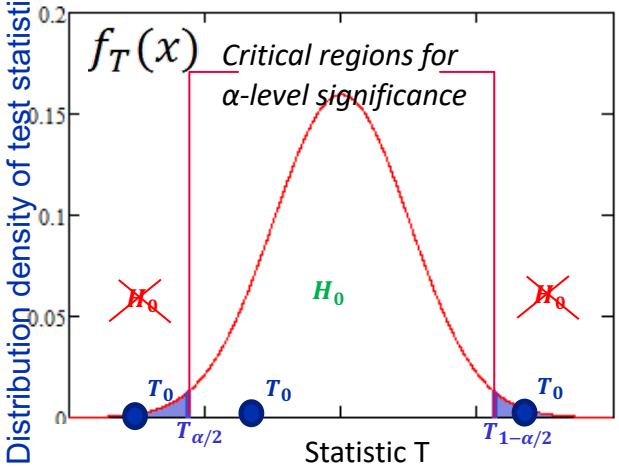
Tests: comparison of statistics

УНИВЕРСИТЕТ ИТМО

One-sided test



Both-sided test



Error of non-parametric testing (*I kind error*) – possibility to reject correct hypothesis with α probability

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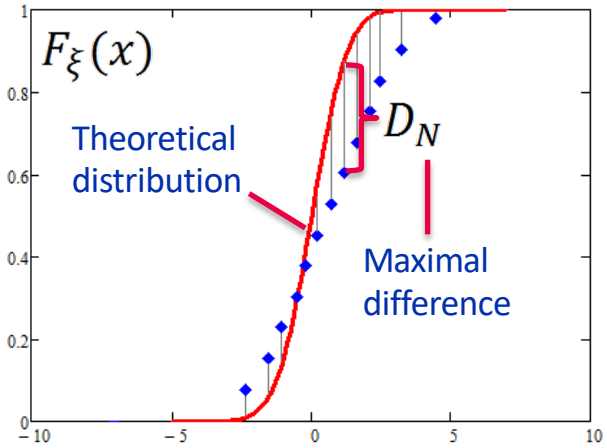
Kolmogorov test

УНИВЕРСИТЕТ ИТМО

Goodness of fit testing



Estimated statistic:

$$D_N = max|F^*(x) - F(x)|$$

Test (*rejection of* H_0):

$$\sqrt{N} D_N \geq T_{1-\alpha}$$

$T_{1-\alpha}$ **- quantile of Kolmogorov distribution**

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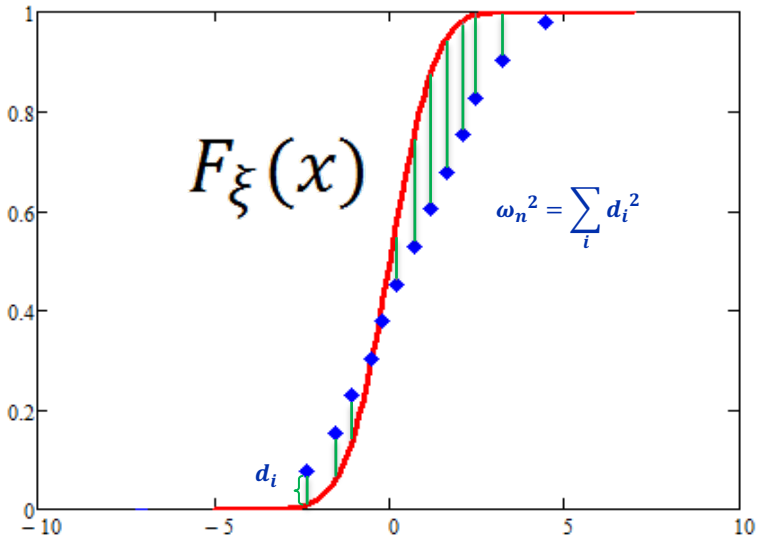
Omega squared test (Cramér–von Mises)

УНИВЕРСИТЕТ ИТМО

Goodness of fit testing



Estimated statistic:

$$N\omega_N^2 = \frac{1}{12N} + \sum_{i=1}^{N}\left[F\left(x_{(i)}\right) - \frac{2i-1}{2N}\right]^2$$

Test (*rejection of* H_0):

$$N\omega_N^2 \geq T_{1-\alpha}$$

$T_{1-\alpha}$ **- quantile of omega distribution**

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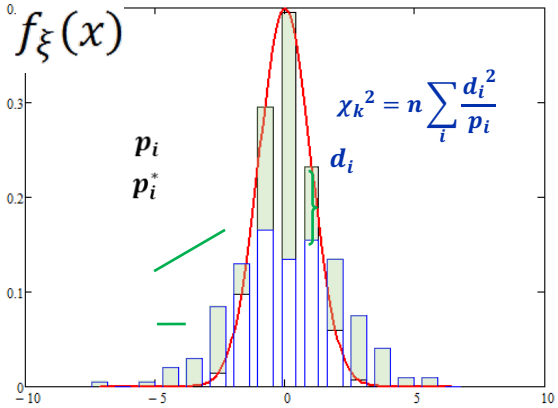
Pearson's test

УНИВЕРСИТЕТ ИТМО

Goodness of fit testing for grouped data



Random variable *x*

Estimated statistic:

$$\chi_N^2 = N\sum_{j=1}^{M}\frac{(p_j^* - p_j)^2}{p_j}$$

Test (*rejection of* H_0):

$$\chi_N^2 \geq T_{1-\alpha}$$

$T_{1-\alpha}$ **- quantile of** χ_{N-1}^2 **distribution**

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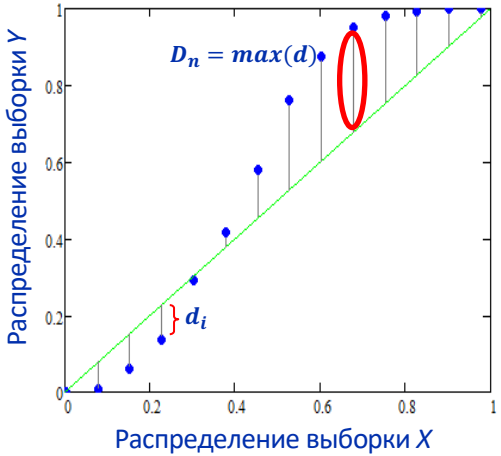
Homogeneity tests



Null hypothesis $H_0: f_\xi^*(x) = g_\xi^*(x)$

Kolmogorov test ⟹ **Smirnov**

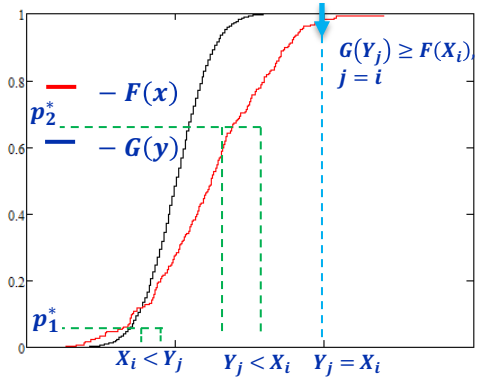
$$D_{NM} = \max|F^*(x) - G^*(x)|,$$

Test ω^2 ⟹ **Rosenblatt**

$$\frac{NM}{N+M}\omega_{NM}^2 = \frac{1}{N+M}\left(\frac{1}{6} + \frac{1}{M}\sum_{i=1}^{N}(R_i - i)^2 + \frac{1}{N}\sum_{j=1}^{M}(S_j - j)^2\right) - \frac{2}{3}$$

Rules for testing the same with goodness of fit criterions

Wilcoxon rank-sum test



Null hypothesis H_0: $F(x) = G(x)$

Alternative hypothesis H_2: $F(x) < G(x)$

Samples $(x_1, \dots, x_N)$ **and** $(y_1, \dots, y_M)$

Estimated statistic:

$$U_{st} = \frac{\min\{U_X, U_Y\} - \frac{1}{2}MN}{\sqrt{\frac{1}{12}MN(M+N+1)}}$$

$$U_Y = \sum_{i=1}^{N}\sum_{j=1}^{M}[x_i > y_j] \qquad U_X = \sum_{i=1}^{N}\sum_{j=1}^{M}[x_i < y_j]$$

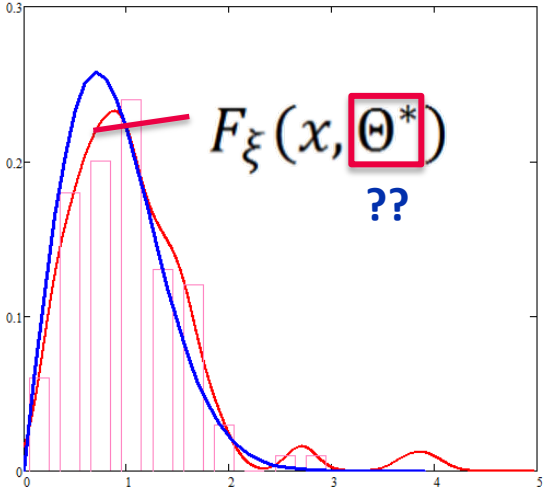
Test (*rejection of* H_0):

H_0

$$U_{st} \notin [Z_{\alpha/2}, Z_{1-\alpha/2}]$$

$$Z \sim \mathcal{N}(0,1)$$

Essence of parametric criterions

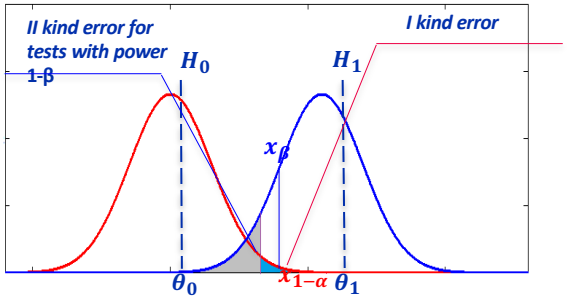


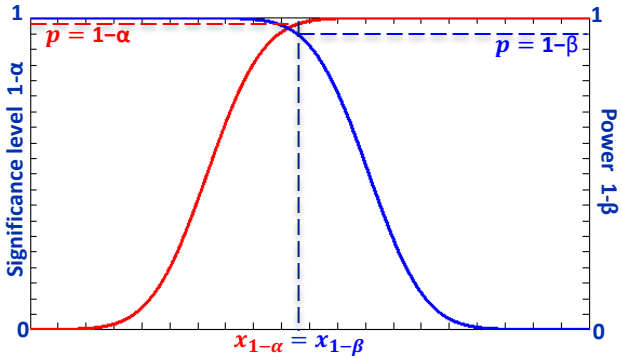
We investigate Θ sample of parameters, that describe distribution of statistical population

$$F(x) = F(x, \Theta)$$

Is it true that $\Theta^* = \Theta_0$ (null hypothesis);
or $\Theta^* = \Theta_1$ (alternative hypothesis)?

Errors of I and II kind





Errors for interval estimation (probability 1-β): *miss* true value from β%- confidence interval

Errors for hypothesis testing:

А) *I kind* (probability α): reject true hypothesis

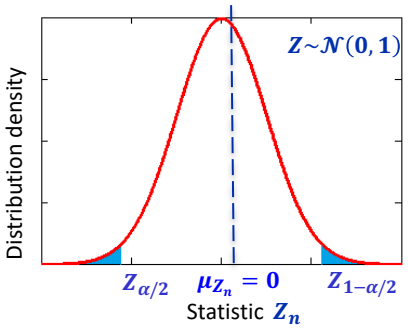
Б) *II kind* (probability γ): accept wrong hypothesis

Building the most powerful tests :

$$1 - \gamma \rightarrow max$$

for fixed significance α

Test for the mean value

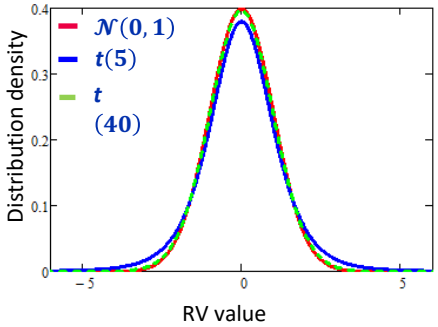


Null hypothesis: $\bar{x} = \mu$

For large samples:

Statistic: $Z_N = \left| \frac{\bar{x} - \mu}{s/\sqrt{N}} \right|$

Test: $Z_N > |Z_{1-\alpha}| \quad Z \sim \mathcal{N}(0,1)$



For small samples (N<30):

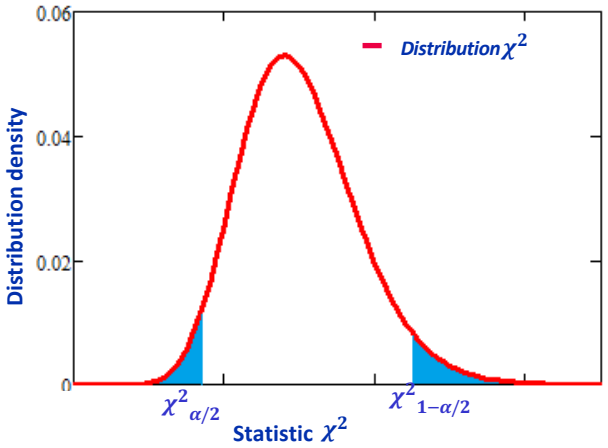
Statistic: $t_N = \sqrt{N-1}\,\frac{|\bar{x} - \mu|}{s}$

Test: $t_N > |t|_{1-\alpha,N-1}$

Test for variance

УНИВЕРСИТЕТ ИТМО

Null hypothesis: $s^2 = \sigma^2$



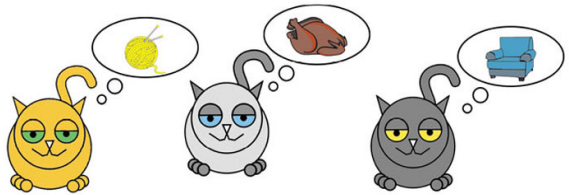
Statistic:

$$\chi_N{}^2 = \frac{(N-1)s^2}{\sigma^2}$$

Test:

$$\chi_N{}^2 \in \left[0, \chi^2{}_{\frac{\alpha}{2},N-1}\right] \cup \left[\chi^2{}_{1-\frac{\alpha}{2},N-1}, \infty\right]$$

$\chi^2{}_{\frac{\alpha}{2},N-1}$ - quantile of χ^2 distribution



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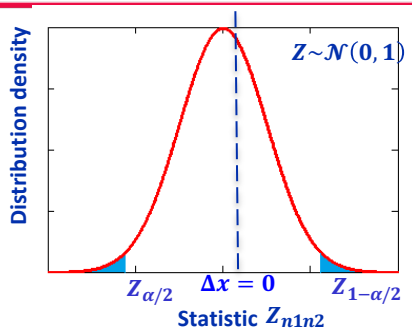
Test for mean and variance comparison



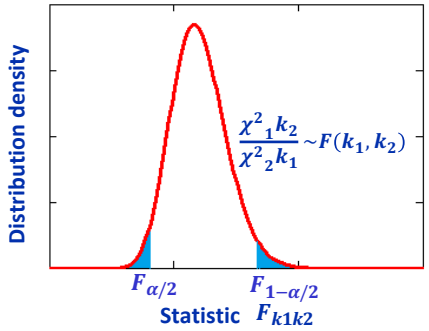
Comparison of mean values:

Statistic:

$$Z_{NM} = \frac{|\bar{x}_1 - \bar{x}_2|}{\sqrt{\frac{{s_1}^2}{N} + \frac{{s_2}^2}{M}}}$$

Test:

$$Z_{NM} > |Z|_{1-\alpha}$$



Comparison of variances:

Statistic:

$$F_{NM} = \frac{{s_1}^2}{{s_2}^2}$$

Test:

$$F_{NM} \in \left[0, F_{\frac{\alpha}{2},N-1,M-1}\right] \cup \left[F_{1-\frac{\alpha}{2},N-1,M-1}, \infty\right]$$

F - **Fisher distribution**

$$F_{NM} \in \left[0, F_{\frac{\alpha}{2},N-1,M-1}\right] \cup \left[F_{1-\frac{\alpha}{2},N-1,M-1}, \infty\right]$$

Thanks!

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