Theory of Real Functions

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Unit - 1
Limits of functions (ε - δ approach), sequential criterion for limits, divergence criteria. Limit theorems, one sided limits. Infinite limits and limits at infinity. Continuous functions, sequential criterion for continuity and discontinuity. Algebra of continuous functions. Continuous functions on closed and bounded interval, intermediate value theorem, location of roots theorem, preservation of intervals theorem. Classification of discontinuity, discontinuity of monotonic functions. Uniform continuity, non-uniform continuity criteria, uniform continuity theorem on compact sets.

Unit - 2

Differentiability of a function at a point and in an interval, Caratheodory's theorem, algebra of differentiable functions. Relative extrema, interior extremum theorem. Rolle's theorem. Mean value theorem: Lagrange's mean value theorem, Cauchy's mean value theorem, Darboux's theorem on derivatives. Applications of mean value theorem to inequalities and approximation of polynomials.

Unit - 3

Taylor's theorem with Lagrange's form of remainder, Taylor's theorem with Cauchy's form of remainder and Young's form of remainder, application of Taylor's theorem to convex functions, Jensen's inequality, relative extrema. Taylor's series and Maclaurin's series expansions of exponential and trigonometric functions, $\ln(1+x)$, $\frac{1}{(ax+b)}$ and $(x+1)^n$. Application of Taylor's theorem to inequalities. L'Hospital's rule.

Unit - 4

Higher order derivatives, Leibnitz rule, concavity and inflection points, envelopes, asymptotes, curvature, curve tracing in cartesian coordinates. Reduction formulae, derivations and illustrations of reduction formulae, parametric equations, parameterizing a curve, arc length of a curve, arc length of parametric curves, area under a curve, area and volume of surface of revolution. [50]

Contents

§1 Limits and Continuity

§1.1 Limit of a function

Definition 1.1 (ε-adherent points)

Let $X \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is said to be ε-*adherent to* X iff there exists a $y \in X$ which is ε-close to x (i.e. $|x - y| \leq \varepsilon$).

Definition 1.2 (Adherent point)

Let $X \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is called an **adherent point** of X iff it is ε-adherent to X for every $\varepsilon > 0$.

Definition 1.3 (Limit point)

Let $X \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is called a **limit point** (or **cluster point**) of X if it is an adherent point of $X \setminus \{x\}$.

Definition 1.4 (ε-closeness and local ε-closeness)

Let $X \subseteq \mathbb{R}$, $f : X \to \mathbb{R}$, $L \in \mathbb{R}$ and $\varepsilon > 0$. We say the function f is ε-*close to* L iff $f(x)$ is ε-close to L for every $x \in X$. Let x_0 be an adherent point of X. We say that f is ε-*close to* L *near* x_0 iff there exists a $\delta > 0$ such that f becomes ε-close to L when restricted to the set $\{x \in X : |x - x_0| < \delta\}$.

Theorem 1.5

A point $c \in \mathbb{R}$ is a limit point of $A \subseteq \mathbb{R}$ if and only if there exists a sequence (a_n) in A such that $a_n \neq c$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} a_n = c$.

Definition 1.6 (Limit of a function at a point)

Let $\varnothing \neq A \subseteq \mathbb{R}$ and c be a limit pt. of A. Then a function $f : A \to \mathbb{R}$, is said to have a limit at c, if there exists a fixed real L such that for every real $\varepsilon > 0$ there exists a real $\delta > 0$ (depending on both ε and c) such that

$$x \in A, |x - c| < \delta \implies |f(x) - L| < \varepsilon$$

and we thus write

$$\lim_{x\to c} f(x) = L.$$

§2 Differentiability

§2.1 Differentiability of a real valued function

Remark. Recall that a linear operator is a linear map that maps a vector space to itself. So a linear operator over a field $\mathbb{F}$ is a map $T : V \to V$ such that

$$T(\alpha v + w) = \alpha T(v) + T(w).$$

The only linear operator on $\mathbb{R}$ is the function $f(x) = ax$ for any fixed real constant a. Also, the $\dim(\mathbb{R}) = 1$ over $\mathbb{R}$, as the dimension of any field over itself is 1. Any nonzero real can act as a basis in the real vector space $\mathbb{R}$.

The derivative of a real valued function is, in fact, a linear operator on the real vector space $\mathbb{R}$.

§2.2 Algebra of derivatives

§2.3 Mean Value Theorems

§2.4 Taylor's Theorem

§3 Advanced Calculus in One Real Variable

§3.1 Higher order derivatives

§3.1.1 Leibnitz rule

Theorem 3.1 (Leibnitz rule)

If f and g are n-times differentiable functions, then the product fg is also n-times differentiable and its n^{th} derivative is given by

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)},$$

where $f^{(j)}$ is the j^{th} derivative of f with $f^{(0)} = f$.

Proof. We proceed by induction on n.

For $n = 1$, $(fg)' = f'g + fg'$. This proves the base case.

Assume for our induction hypothesis that the theorem holds for a fixed $n \in \mathbb{Z}^+$, i.e.

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}.$$

Then,

$$\begin{aligned}
(fg)^{(n+1)} &= \left[\sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}\right]' \\
&= \sum_{k=0}^{n} \binom{n}{k} f^{(n+1-k)} g^{(k)} + \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k+1)} \\
&= \sum_{k=0}^{n} \binom{n}{k} f^{(n+1-k)} g^{(k)} + \sum_{k=1}^{n+1} \binom{n}{k-1} f^{(n+1-k)} g^{(k)} \\
&= \binom{n}{0} f^{(n+1)} g^{(0)} + \sum_{k=1}^{n} \binom{n}{k} f^{(n+1-k)} g^{(k)} + \sum_{k=1}^{n} \binom{n}{k-1} f^{(n+1-k)} g^{(k)} + \binom{n}{n} f^{(0)} g^{(n+1)} \\
&= \binom{n+1}{0} f^{(n+1)} g^{(0)} + \left(\sum_{k=1}^{n} \left[\binom{n}{k-1} + \binom{n}{k}\right] f^{(n+1-k)} g^{(k)}\right) + \binom{n+1}{n+1} f^{(0)} g^{(n+1)} \\
&= \binom{n+1}{0} f^{(n+1)} g^{(0)} + \sum_{k=1}^{n} \binom{n+1}{k} f^{(n+1-k)} g^{(k)} + \binom{n+1}{n+1} f^{(0)} g^{(n+1)} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(n+1-k)} g^{(k)}.
\end{aligned}$$

□

§3.2 Concavity and inflection points

§3.3 Envelopes

§3.4 Asymptotes

§3.5 Curvature

§3.6 Curve tracing in cartesian coordinates

§3.7 Integration by reduction formulas

Theorem 3.2 (Integration by reduction)

Let $m, n \in \mathbb{Z}$.

1. If $I_n = \int \sin^n(x)\,dx$ then
$$I_n = \frac{-\sin^{n-1}(x)\cos(x)}{n} + \frac{(n-1)I_{n-2}}{n}.$$

2. If $I_n = \int \cos^n(x)\,dx$ then
$$I_n = \frac{\cos^{n-1}(x)\sin(x)}{n} + \frac{(n-1)I_{n-2}}{n}.$$

3. If $I_n = \int \tan^n(x)\,dx$ then
$$I_n = \frac{\tan^{n-1}(x)}{n-1} - I_{n-2}.$$

4. If $I_n = \int \sec^n(x)\,dx$ then
$$I_n = \frac{\sec^{n-2}(x)\tan(x)}{n-1} + \frac{(n-2)I_{n-2}}{n-1}.$$

5. If $I_{m,n} = \int \sin^m(x)\cos^n(x)\,dx$ then
$$\begin{aligned} I_{m,n} &= \frac{\sin^{m+1}(x)\cos^{n-1}(x)}{m+n} + \frac{(m-1)I_{m,n-2}}{m+n} \\ &= \frac{-\sin^{m-1}(x)\cos^{n+1}(x)}{m+n} + \frac{(m-1)I_{m-2,n}}{m+n} \end{aligned}$$

Proof. 1. $I_n = \displaystyle\int \sin^n(x)\,dx = \int \sin^{n-1}(x)\sin(x)\,dx$

$$\begin{aligned} &= \sin^{n-1}(x)\int \sin(x)\,dx - \int \frac{d}{dx}\left(\sin^{n-1}(x)\right)\left(\int \sin(x)\,dx\right)dx \\ &= -\sin^{n-1}(x)\cos(x) - \int \cos(x)\left((n-1)\sin^{n-2}(x)\right)(-\cos(x))\,dx \\ &= -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\cos^2(x)\,dx \end{aligned}$$

$$= -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\left(1-\sin^2(x)\right)\,\mathrm{d}x$$

$$= -\sin^{n-1}(x)\cos(x) + (n-1)\int \left(\sin^{n-2}(x) - \sin^n(x)\right)\,\mathrm{d}x$$

$$= -\sin^{n-1}(x)\cos(x) + (n-1)(I_{n-2} - I_n).$$

Thus, $nI_n = I_n + (n-1)I_n = -\sin^{n-1}(x)\cos(x) + (n-1)I_{n-2}$.

2. $I_n = \displaystyle\int \cos^n(x)\,\mathrm{d}x = \int \cos^{n-1}(x)\cos(x)\,\mathrm{d}x$

$$= \cos^{n-1}(x)\int \cos(x)\,\mathrm{d}x - \int \frac{d}{\mathrm{d}x}\left(\cos^{n-1}(x)\right)\left(\int \cos(x)\,\mathrm{d}x\right)\,\mathrm{d}x$$

$$= \cos^{n-1}(x)\sin(x) + \int \sin(x)\left((n-1)\cos^{n-2}(x)\right)\sin(x)\,\mathrm{d}x$$

$$= \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)\sin^2(x)\,\mathrm{d}x$$

$$= \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)\left(1-\cos^2(x)\right)\,\mathrm{d}x$$

$$= \cos^{n-1}(x)\sin(x) + (n-1)\int \left(\cos^{n-2}(x) - \cos^n(x)\right)\,\mathrm{d}x$$

$$= \cos^{n-1}(x)\sin(x) + (n-1)(I_{n-2} - I_n).$$

Thus, $nI_n = I_n + (n-1)I_n = \cos^{n-1}(x)\sin(x) + (n-1)I_{n-2}$.

3. $I_n = \displaystyle\int \tan^n(x)\,\mathrm{d}x = \int \tan^{n-2}(x)\tan^2(x)\,\mathrm{d}x$

$$= \int \overbrace{\tan^{n-2}(x)}^{t^{n-2}}\underbrace{\sec^2(x)\,\mathrm{d}x}_{\mathrm{d}t} - \int \tan^{n-2}(x)\,\mathrm{d}x$$

$$= \int t^{n-2}\,\mathrm{d}t - I_{n-2}$$

$$= \frac{t^{n-1}}{n-1} - I_{n-2}$$

$$= \frac{\tan^{n-1}(x)}{n-1} - I_{n-2}.$$

4. $I_n = \displaystyle\int \sec^n(x)\,\mathrm{d}x = \int \sec^{n-2}(x)\sec^2(x)\,\mathrm{d}x$

$$= \sec^{n-1}(x)\int \sec^2(x)\,\mathrm{d}x - \int \frac{d}{\mathrm{d}x}(\sec^{n-1}(x))\left(\int \sec^2(x)\,\mathrm{d}x\right)\,\mathrm{d}x$$

$$= \sec^{n-1}(x)\tan(x) - (n-2)\int \sec^{n-3}(x)\sec(x)\tan(x)\tan(x)\,\mathrm{d}x$$

$$= \sec^{n-1}(x)\tan(x) - (n-2)\int \sec^{n-2}(x)\tan^2(x)\,\mathrm{d}x$$

$$= \sec^{n-1}(x)\tan(x) - (n-2)\int \sec^{n-2}(x)(\sec^2(x)-1)\,\mathrm{d}x$$

$$= \sec^{n-1}(x)\tan(x) - (n-2)\int \left(\sec^{n}(x) - \sec^{n-2}(x)\right)\,\mathrm{d}x$$

$$= \sec^{n-1}(x)\tan(x) - (n-2)\int \left(I_n - I_{n-2}\right).$$

Thus,

$$(n-1)I_n = I_n + (n-2)I_n = \sec^{n-1}(x)\tan(x) + (n-2)I_{n-2}.$$

$\square$

§3.8 Parametric equations

§3.9 Parameterizing a curve

§3.10 Arc length of a curve

§3.11 Arc length of parametric curves

§3.12 Area under a curve

§3.13 Area and volume of surface of revolution.