Discrete Mathematics

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December 17, 2023

Finite Structures
Partial and linear orderings. Chains and antichains. Lattices. Distributive lattices. Complementation. Boolean algebras, Duality, Atoms, Boolean functions. Normal forms.

Combinatorics
Pigeon hole principle. Finite combinatorics. Generating functions. Partitions. Recurrence relations. Linear difference equations with constant coefficients.

Graph Theory
Graphs: Subgraph, Complete graph, Bipartite graph, Connected graph, Tree, Eulerian graph, Hamiltonian graph, planar graph. [50]

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§0 Introduction

> "The advanced reader who skips parts that appear too elementary may miss more than the less advanced reader who skips parts that appear too complex."
> — George Pólya

Here we rigorously develop the foundations for further study of discrete and algebraic structures.

§1 Finite Structures

§1.1 Relations

Definition 1.1

A **binary relation** from set A to set B is any subset $\rho \subseteq A \times B$.
An element $a \in A$ *is related to* $b \in B$ in the relation ρ iff $(a, b) \in \rho$, which we denote by $a \, \rho \, b$. If $(a, b) \notin \rho$ then we write $a \not\rho \, b$.
Let A be a set. Then any subset $\rho \subseteq A \times A$ is a *binary relation* on A.

Definition 1.2

Let A be a set and let ρ be relation on A. Then ρ is said to be

1. **reflexive** iff $a \, \rho \, a \; \forall a \in A$,
2. **symmetric** iff $a \, \rho \, b \implies b \, \rho \, a \; \forall a, b \in A$,
3. **transitive** iff $a \, \rho \, b$ and $b \, \rho \, c \implies a \, \rho \, c \; \forall a, b, c \in A$,
4. **irreflexive** iff $a \not\rho \, a \; \forall a \in A$,
5. **asymmetric** iff $a \, \rho \, b \implies b \not\rho \, a \; \forall a, b \in A$,
6. **intransitive** iff $a \, \rho \, b$ and $b \, \rho \, c \implies a \not\rho \, c \; \forall a, b, c \in A$,
7. **antisymmetric** iff $a \, \rho \, b$ and $b \, \rho \, a \implies a = b \; \forall a, b \in A$,

and if ρ is all of (i), (ii) and (iii) then ρ is said to be an **equivalence relation** on A.
If ρ is all of (i), (ii) and (vii) then ρ is said to be a **partial order relation** on A.

In the following six definitions ρ and σ are binary relations from A to B.

Definition 1.3

The **complement** of ρ is the relation $\overline{\rho} \subseteq A \times B$ such that $(a \overline{\rho} b) \iff (a \not\rho \, b) = \neg(a \, \rho \, b)$.

Definition 1.4

A relation on a set is an **equivalence relation** iff it is reflexive, symmetric and transitive.

Definition 1.5

A relation $\preceq$ on a set A is said to be a **partial order relation**, or simply a *partial order*, iff $\preceq$ is

1. reflexive $\iff (a \preceq a \; \forall a \in A)$,
2. antisymmetric $\iff ((a \preceq b) \land (b \preceq a) \implies (a = b) \; \forall a, b \in A)$,
3. transitive $\iff ((a \preceq b) \land (b \preceq c) \implies (a \preceq c) \; \forall a, b, c \in A)$

on A.

§1.2 Partial and linear orderings

§1.3 Chains and antichains

§1.4 Lattices

§2 Combinatorics

§2.1 Pigeon hole principle

Theorem 2.1 (Pigeonhole Principle)

If n pigeonholes are occupied by $n+1$ or more pigeons, then at least one pigeonhole is occupied by 2 or more pigeons.

Theorem 2.2 (Generalised Pigeonhole Principle)

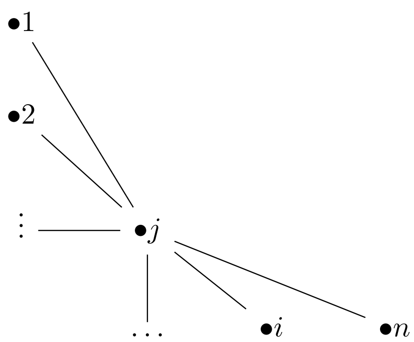
If n pigeonholes are occupied by $kn+1$ or more pigeons for $k \in \mathbb{N}$, then at least one pigeonhole is occupied by $k+1$ or more pigeons.

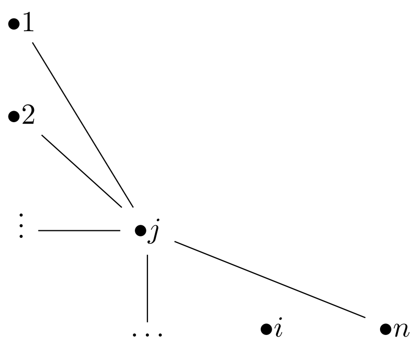
§3 Graph Theory

§3.1 Graphs

Problem 3.1. *In any gathering of $n > 1$ people, there are at least two people with exactly the same number of (mutual) friends. (Nobody can be his/her own friend.)*

Answer. We can represent every person as a *vertex* or node, and each pair of mutual friendships with an *edge* - thus turning it into a problem on graphs. For each of the n people, the number of friends can only be some number in $\{0, \dots, n-1\}$. Define f_i to be the number of friends of the i^{th} person, $i = 1, \dots, n$. Then $f_i \in \{0, \dots, n-1\}$ for all i.





Now sps $f_i = 0$ for some i and $f_j = n - 1$ for some $j \neq i$.
That would mean the j^{th} person is a friend of everyone else which includes the i^{th} person ($\because i \neq j$), but the i^{th} person is a friend of no one. Contradiction.
So either $\{f_{\sigma(1)}, \dots, f_{\sigma(n)}\} = \{1, \dots, n-1\}$ or $\{f_{\sigma(1)}, \dots, f_{\sigma(n)}\} = \{0, \dots, n-2\}$, for some permutation σ of the indices. Thus,

$$|\{f_1, \dots, f_n\}| = n - 1 < n.$$

Applying pigeon hole principle yields the required result.

Definition 3.2 ((Undirected) Graph)

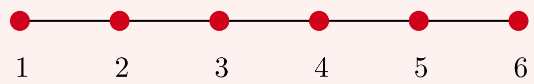
A **graph** G is a triple $G = (V(G), E(G), g)$ where $V(G) = V \neq \varnothing$ is its *vertex set*, $E(G) = E$ (possibly empty) is its *edge set* and

$$\forall e \in E \; e \mapsto g(e) = \{v, w\}, \; v, w \in V(\text{not necessarily distinct})$$

is its *incidence function* assigning to each edge its endpoints.

Example 3.3 (Example of a Graph)

The ordered pair (V, E) where $V = \{1, 2, \dots, 6\}$ and $E = \{\{1,2\}, \{2,3\}, \dots, \{5,6\}\}$ is a graph.



This graph is known as P_6, a path on 6 vertices.