Real Analysis 1

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December 21, 2023

Constuction of $\mathbb{R}$ and basic topology
Review of algebraic and order properties of $\mathbb{R}$, ϵ-neighborhood of a point in $\mathbb{R}$. Idea of countable sets, uncountable sets and uncountability of $\mathbb{R}$. Bounded above sets, bounded below sets, bounded sets, unbounded sets. Suprema and infima. Construction of Reals from Rationals, Cantor's nested interval Theorem, Completeness property of $\mathbb{R}$ and its equivalent properties. The Archimedean property, density of rational (and irrational) numbers in $\mathbb{R}$, intervals. Limit points of a set, isolated points, open set, closed set, derived set, Bolzano-Weierstrass theorem for sets, compact sets in $\mathbb{R}$, Heine-Borel Theorem.

Sequences
Sequences, bounded sequence, convergent sequence, limit of a sequence, $\liminf$, $\limsup$. Limit theorems. Monotone sequences, monotone convergence theorem. Subsequences, divergence criteria. Monotone subsequence theorem, Bolzano-Weierstrass theorem for sequences. Cauchy sequence, Cauchy's convergence criterion.

Series
Infinite series, convergence and divergence of infinite series, Cauchy criterion, tests for convergence: comparison test, limit comparison test, D'Alembert's test, Raabe's test, Cauchy's nth root test, Gauss test, Logarithmic test, Integral test. Alternating series, Leibniz test. Absolute and conditional convergence, Rearrangement of series, Riemann's theorem on conditionally convergent series. [50]

Contents

§1 Constuction of $\mathbb{R}$ and basic topology

§1.1 Algebraic and order properties

The first theorem we prove is a result from elementary number theory:

Theorem 1.1
There is no rational r such that $r^2 = 2$ i.e. $\pm\sqrt{2} \notin \mathbb{Q}$.

Proof. Sps for contradiction that $r = p/q,\ p, q \in \mathbb{Z} : q > 0$ and p, q are coprime.

As $r^2 = 2$ we have $p^2 = 2q^2 \implies 2|p^2 \implies 2|p$. So $p = 2k$ for some $k \in \mathbb{Z}$, $\implies 2q^2 = p^2 = 4k^2$, i.e. $q^2 = 2k^2 \implies 2|q^2 \implies 2|q$. So p, q are not coprime. Contradiction. $\square$

Remark. We will define $\mathbb{Q}$, $\mathbb{Z}$, and even $\mathbb{N}$ later in this subsection. Our first theorem shows that the rationals have *gaps* in them, not forming a perfect continuum by excluding irrationals such as $\sqrt{2}$. The irrational numbers $\mathbb{R} \setminus \mathbb{Q} = \overline{\mathbb{Q}}$ appear even in the ordinary plane Euclidean geometry — such as $\sqrt{2}$ being the length of the diagonal of a unit square. So when constructing the real line $\mathbb{R}$ we have to make sure that we include $\mathbb{Q}$ as well as $\overline{\mathbb{Q}}$.

Definition 1.2 (Functions)
A **function**, **map** or **mapping** $f : A \to B$ is an association of every element $a \in A$ with some $f(a) \in B$. Moreover f is:

1. an **injection** or **one-to-one function** iff $(\forall x, y \in A)\ f(x) = f(y) \implies x = y$;
2. a **surjection** or **onto function** iff $(\forall b \in B)\ \exists a \in A : b = f(a)$;
3. a **bijection** or **one-to-one correspondence** iff it is both of the above.

We denote the **domain** of f, A, by $D_f = A$. We define the **range** of f as $R_f = f(A) = \{f(a) : a \in A\}$. So f is surjective iff $f(A) = B$.

Remark. The set of all functions from A to B is:

$$B^A = \prod_{a \in A} B$$

where the RHS is the usual Cartesian product of sets.

Definition 1.3 (Homomorphism)
A **homomorphism** between two algebraic structures $(G, \circ)$ and $(H, \star)$ is a mapping $h : G \to H$ s.t.:

$$h(a \circ b) = h(a) \star h(b)\ \forall a, b \in G$$

with $h(e_G) = e_H$ i.e. the identity element of G is mapped to the identity element of

H.

An *injective* homomorphism is called a **monomorphism** and a *surjective* homomorphism is called an **epimorphism**.

Definition 1.4 (Isomorphism)

An **isomorphism** between two algebraic structures $(G, \circ)$ and $(H, \star)$ is a homomorphism $h : G \to H$ that is *bijective*. If such a mapping exists, we say that G is **isomorphic** to H:

$$(G, \circ) \cong (H, \star).$$

An isomorphism from a group onto itself is an **automorphism**.

Definition 1.5 (Partition)

A family of sets $\{A_\alpha\}_{\alpha \in J}$ is said to form a **partition** of the set X iff

1. $\bigcup_{\alpha \in J} A_\alpha = X$ and
2. $A_\alpha \cap A_\beta = \varnothing \; \forall \alpha, \beta \in J : \alpha \neq \beta,$ (*pairwise disjoint*)

where J is the index set of the family A_α.

Definition 1.6 (Equivalence Relations and Classes)

A relation $\sim$ on a set X is an **equivalence relation** iff:

1. $x \in X \implies x \sim x$ (*reflexive*)
2. $x, y \in X : x \sim y \implies y \sim x$ (*symmetric*)
3. $x, y, z \in X : x \sim y \,, y \sim z \implies x \sim z$ (*transitive*)

The **equivalence class** of an element $a \in X$ is defined as:

$$[a] = \{x \in X : a \sim x\}.$$

Example 1.7

A collection of mathematical objects isomorphic to each other forms an equivalence class, which we call the **isomorphism class** of those objects.

Example 1.8 (Natural numbers)

The set $\mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \dots\}$ can be defined in several ways:

1. As finite ordinals: $0 = \{\} = \varnothing$, succesor $s(n) = n \cup \{n\}$ with $s(0) := 1$.
2. As finite cardinals: isomorphism classes of finite sets.
3. As strings over a unary alphabet: “”, “a”, “aa”, “aaa”, $\dots$.

Here we consider the natural numbers to be the set $\mathbb{N}$, *excluding* 0.

Example 1.9 (Integers)

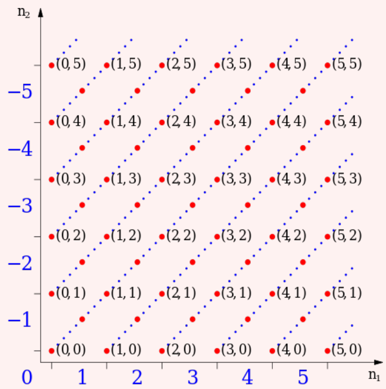
The integers $\mathbb{Z}$ can be constructed as the equivalence classes of ordered pairs of natural numbers with zero: $(a, b) \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ s.t.

$$(\forall (a,b),(c,d) \in \mathbb{N}_0) \;\; (a,b) \sim (c,d) \iff a + d = b + c.$$

We define:

1. $[(a,b)] + [(c,d)] := [(a+c, b+d)]$. (*addition*)
2. $[(a,b)] \cdot [(c,d)] := [(ac+bd, ad+bc)]$. (*multiplication*)
3. $-[(a,b)] := [(b,a)]$. (*additive inverse*)

Thus $\forall k, n \in \mathbb{N}_0$:



$$\begin{array}{ccccc}
0 = & [(0,0)] = & [(1,1)] & \dots & [(n,n)] \\
1 = & [(1,0)] = & [(2,1)] & \dots & [(n+1,n)] \\
-1 = & [(0,1)] = & [(1,2)] & \dots & [(n,n+1)] \\
2 = & [(2,0)] = & [(3,1)] & \dots & [(n+2,n)] \\
-2 = & [(0,2)] = & [(1,3)] & \dots & [(n,n+2)] \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
k = & [(k,0)] = & [(1+k,1)] & \dots & [(n+k,n)] \\
-k = & [(0,k)] = & [(1,1+k)] & \dots & [(n,n+k)] \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

and we have $\mathbb{Z} = \{[(n, n+k)], [(n+k, n)] : k, n \in \mathbb{N}_0\} = \{\cdots - 2, -1, 0, 1, 2 \dots\}$.

Example 1.10

Let $X = \mathbb{Z}$ and $n \in \mathbb{N}$. Consider the equivalence relation defined by

$$x \sim y \iff n \mid x - y,$$

then we get exactly $n - 1$ equivalence classes of the residues of integers modulo n:

$$[0], [1], \dots, [n-1]$$

and we write $\mathbb{Z}/n\mathbb{Z} = \{[0], [1], \dots, [n-1]\}$.

Example 1.11

We can construct the rationals $\mathbb{Q}$ as the equivalence classes of ordered pairs of integers $(a, b) \in \mathbb{Z} \cup \mathbb{Z} \setminus \{0\}$.

Definition 1.12 (Types of Properties of Binary Operations)
Let $G \neq \varnothing$ be a set and $\circ$ be a binary operation defined on G. Then:

1. *Closure property*: G is **closed** w.r.t. $\circ$ iff $a, b \in G \implies a \circ b \in G$.
2. *Associativity*: $a, b, c \in G \implies a \circ (b \circ c) = (a \circ b) \circ c$.
3. *Commutativity*: $a, b \in G \implies a \circ b = b \circ a$.
4. *Existence of identity element*: $\exists e \in G : a = a \circ e = e \circ a \ \forall a \in G$.
5. *Existence of inverses*: $\forall a \in G \ \exists \ a^{-1} \in G : a \circ a^{-1} = a^{-1} \circ a = e$.

Definition 1.13 (Groupoid, Semigroup, Monoid)
Let $G \neq \varnothing$ be a set and $\circ$ be a binary operation defined on G. Then:

1. The pair $(G, \circ)$ is a **groupoid** iff $\circ$ is *closed w.r.t.* G.
2. The groupoid $(G, \circ)$ is a **semigroup** iff $\circ$ is *associative*.
3. The semigroup $(G, \circ)$ is a **monoid** iff $\circ$ has *existence of identity element*.

Definition 1.14 (Group Axioms)
Let $\circ$ be a binary operation defined over a set G. Then the pair $(G, \circ)$ is called a **group** iff $\circ$

1. *is closed w.r.t. G,*
2. *is associative,*
3. *has existence of identity element and*
4. *has existence of inverses.*

Furthermore if $\circ$ is *commutative* over G then we say that $(G, \circ)$ is an **abelian group**, or **commutative group**.

Remark. An ***additive (multiplicative)* group** is one where the binary operation is *addition (multiplication)*.

Definition 1.15 (Ring)
Let $R \neq \varnothing$ be a set and $\star, \circ$ be binary operations defined on R. Then the triple $(R, \star, \circ)$ is said to be a **ring** iff the foll. hold:

1. $(R, \star)$ is an *abelian group*.
2. $(R, \circ)$ is a *semigroup*.
3. *Distributivity*:
   a) $a, b, c \in R \implies a \circ (b \star c) = (a \circ b) \star (a \circ c)$.
   b) $a, b, c \in R \implies (b \star c) \circ a = (b \circ a) \star (c \circ a)$.

Remark. In a ring we often use $+$ and $\cdot$ instead of $\star$ and $\circ$, with $(-a), 0$ and $\left(\frac{1}{a}\right), 1$

denoting the inverse and identity elements in each case. Moreover, we often write $a \cdot b$ as ab.

A **division ring** is a ring which has existence of multiplicative inverse for each nonzero element within it, i.e. division by nonzero elements is defined.

Remark (Negative times negative is positive)**.** Given a ring $(R, +, \cdot)$, we have:

1. $x, y \in R : x + y = 0 \implies x + y + (-y) = 0 + (-y) \implies x = -y$ and $x, y, z \in R : x + y = x + z \implies (-x) + x + y = (-x) + x + z \implies y = z$.
2. $(-x)$ is the additive inverse of x and x is the additive inverse of $(-x)$; replacing x with $(-x)$ we get that the additive inverse of $(-x)$ is $-(-x)$, so
$$-(-x) + (-x) = 0 = x + (-x) \implies -(-x) = x.$$
3. $x \in R \implies 0x = (0 + 0)x = 0x + 0x \implies 0 = 0x$.
4. $x, y \in R \implies x(-y) + xy = x((-y) + y) = 0 \implies x(-y) = -(xy)$ and $x, y \in R \implies (-x)y + xy = ((-x) + x)y = 0 \implies (-x)y = -(xy) = x(-y)$.
5. Thus, $(-1)1 = -(1 \cdot 1) = -1 = 1(-1)$ and $(-1)(-1) = -((-1)1) = -(-(1 \cdot 1)) = 1 \cdot 1 = 1$.

Example 1.16

The set $\mathbb{Z}/n\mathbb{Z}$ is a ring $\forall n \in \mathbb{N}$ (and if we require n to be prime it is in fact a commutative division ring).

$+$	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2]	[0]
[2]	[2]	[0]	[1]

$\times$	[0]	[1]	[2]
[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]
[2]	[0]	[2]	[1]

Definition 1.17 (Field Axioms)

Let $\mathbb{F} \neq \varnothing$ be a set with two binary additions defined on it, namely addition $(+)$ and multiplication $(\cdot)$, s.t. the foll. axioms hold:

1. $(\mathbb{F}, +)$ is an *abelian additive group.*
2. $(\mathbb{F} \setminus \{0\}, \cdot)$ is an *abelian multiplicative group.*
3. *Distributivity:* $\cdot$ *is distributive w.r.t.* $+$.
   $a \cdot (b + c) = (a \cdot b) + (a \cdot c)\ \forall a, b, c \in \mathbb{F}$.

We then say that the triple $(\mathbb{F}, +, \cdot)$ is a **field**.

Remark. One may also define a field to be a *commutative division ring.*

Example 1.18

$\mathbb{Q}, \mathbb{R}, \mathbb{C}$, $\mathbb{Z}/p\mathbb{Z}$ for prime p, $\{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ are some examples of fields.

Definition 1.19

Let $\mathbb{F}$ be a field. Then the **characteristic** of $\mathbb{F}$ is the *least positive number of copies of its multiplicative identity* 1 *that will sum to its additive identity* 0, denoted *char*($\mathbb{F}$):

$$\sum_{i=1}^{char(\mathbb{F})} 1 = 0.$$

Lemma 1.20

A field's characteristic is either 0 *or* p *(where* p *is prime).*

Definition 1.21 (Partially ordered set)

A **partially ordered set (poset)** is a set X with a relation $\prec$ s.t.:

1. If $x \in X$, then $x \not\prec x$. (*irreflexivity*)
2. If $x, y \in X$ s.t. $x \prec y$ and $y \prec x$, then $x = y$. (*antisymmetry*)
3. If $x, y, z \in X$, $x \prec y$ and $y \prec z$, then $x \prec z$. (*transitivity*)

Remark. Here we're using a strict ordering $\prec$ to define a poset. We could instead use a weak ordering $\preceq$, in which case we'd just have to replace *irreflexivity* with *reflexivity*.

Definition 1.22 (Totally ordered set)

A **totally ordered set**, **linearly ordered set**, or simply **ordered set**, is a poset X with a relation $<$ in which any two elements are comparable s.t.:

1. If $x, y, z \in X$, $x < y$ and $y < z$, then $x < z$. (*transitivity*)
2. If $x, y \in X$, exactly one of the foll. holds: $x < y$, $x = y$, $y < x$. (*trichotomy*)

A totally ordered set is also called a **chain**.

Remark. We call a set a *totally* ordered set, as opposed to simply an ordered set, when we want to emphasize the order is total, i.e. every pair of elements are related to each other, as opposed to a *partial order*, where every pair of elements need not be related to each other.

Definition 1.23 (Maximal and Minimal)

An element x in a poset $(X, \prec)$ is **maximal** iff $x \prec y \implies y \notin X$.

An element x in a poset $(X, \prec)$ is **minimal** iff $y \prec x \implies y \notin X$.

Definition 1.24 (Bounded sets)

Let $(X, \leq)$ be an poset and let $A \subseteq X$. An **upper bound** for A is an element $x \in X$ such that $a \in A \implies a \leq x$. If A has an upper bound, then we say that A is *bounded above*.

A **lower bound** for A is an element $x \in X$ such that $a \in A \implies x \leq a$. If A has a lower bound, then we say that A is *bounded below*.

A is **bounded** in X iff it is bounded above and below in X.

Definition 1.25 (Suprema and Infima)

Let $(X, \leq)$ be an poset and let $A \subseteq X$. An upper bound α is a **least upper bound** or **supremum** of A in X if whenever $\beta < \alpha$, β is not an upper bound of A. That is:

1. $(\forall a \in A)\, a \leq \alpha$
2. $(\forall \beta < \alpha)(\exists a \in A)\, \beta < a$

and we write $\alpha = \sup A \in X$.

If $\sup A \in A$, then we call it $\max A$, the **maximum** of A.

A lower bound α is a **greatest lower bound** or **infimum** of A in X if whenever $\alpha < \beta$, β is not a lower bound of A. That is:

1. $(\forall a \in A)\, \alpha \leq a$
2. $(\forall \alpha < \beta)(\exists a \in A)\, a < \beta$

and we write $\alpha = \inf A \in X$.

If $\inf A \in A$, then we call it $\min A$, the **minimum** of A.

Definition 1.26 (Order Complete)

A poset X is **order complete** iff supremum and infimum of each bounded nonempty subset $A \subseteq X$ exist and belong to A.

Remark. For a totally ordered set, its supremum and infimum (if they exist) are unique.

Definition 1.27 (Ordered Field)

An **ordered field** is a field $\mathbb{F}$ with a total order $<$ s.t.

1. if $x, y, z \in \mathbb{F}$ and $x < y$, then $x + z < y + z$
2. if $x, y, z \in \mathbb{F}$, $x < y$ and $z > 0$, then $xz < yz$

Proposition 1.28

Let $\mathbb{F}$ be an ordered field and $x \in \mathbb{F}$. Then $x^2 \geq 0$.

Proof. By trichotomy, either $x < 0$, $x = 0$ or $x > 0$. If $x = 0$, then $x^2 = 0$. So $x^2 \geq 0$. If $x > 0$, then $x^2 > 0 \cdot x = 0$. If $x < 0$, then $x - x < 0 - x$. So $0 < -x$. But then $x^2 = (x \cdot x) = (-x)(-x) = (-x)^2 > 0$. $\square$

Example 1.29

Let $X = \mathbb{Q}$. Then the supremum of $(0,1)$ is 1. The set $\{x : x^2 < 2\}$ is bounded above by 2, but has no supremum (even though $\sqrt{2}$ seems like a supremum, we are in $\mathbb{Q}$ and $\sqrt{2}$ is non-existent!).

$\max[0,1] = 1$ but $(0,1)$ has no maximum because the supremum is not in $(0,1)$.

Lemma 1.30 (Zorn's Lemma)

If every chain in a poset X is bounded above, then X has a maximal element.

Proposition 1.31 (Axiom of Choice)

If $\{A_\alpha\}_{\alpha\in J}$ is a nonempty collection of nonempty sets, then $\prod_{\alpha\in J} A_\alpha$ is nonempty.

§1.2 Construction of $\mathbb{R}$

We will define $\mathbb{R}$ as the *completion* of $\mathbb{Q}$, thus making it a *complete ordered field*. We can achieve this using Dedekind cuts or Cauchy sequences: we demonstrate using Dedekind cuts at the end of this subsection and using Cauchy sequences in the next.

Theorem 1.32

Let $\mathbb{F}$ be an ordered field and $\varnothing \neq A \subseteq \mathbb{F}$; write

$$-A = \{-x : x \in A\}$$

for the set of additive inverses of the elements of A. Let $c \in \mathbb{F}$.

1. *c is an upper bound of $A \iff -c$ is a lower bound of $-A$.*
2. *c is a lower bound of $A \iff -c$ is an upper bound of $-A$.*
3. *If A has a supremum then $-A$ has an infimum and*
$$\inf(-A) = -\sup A.$$
4. *If A has an infimum then $-A$ has a supremum and*
$$\sup(-A) = -\inf A.$$

Definition 1.33 (Completeness Property)

An ordered field $\mathbb{F}$ is said to be **complete** or said to have the **supremum property** if every nonempty subset of $\mathbb{F}$ that is bounded above has a supremum.

Theorem 1.34 (Dual of Completeness Property)

Let $\mathbb{F}$ be a complete ordered field and $\varnothing \neq A \subseteq \mathbb{F}$ be bounded below in $\mathbb{F}$. Then the *infimum of A exists and*

$$\inf A = -\sup(-A).$$

Definition 1.35 (Dedekind cuts)

A **Dedekind cut** in $\mathbb{Q}$ is a pair of nonempty subsets A, B of $\mathbb{Q}$ s.t.:

1. $A \cup B = \mathbb{Q}\,, A \cap B = \varnothing$.
2. $a \in A\,, b \in B \implies a < b$.
3. A has no maximum, i.e. $\sup A \notin A$.

We call A the left-hand part and B the right-hand part of the cut, and denote the cut itself by $x = A \mid B$.

Definition 1.36 (A real number is a Dedekind cut)

A **real number** $r \in \mathbb{R}$ is a Dedekind cut in $\mathbb{Q}$, i.e. $\mathbb{R}$ is the collection of all Dedekind cuts in $\mathbb{Q}$.

Example 1.37

1. $x = \{r \in \mathbb{Q} : r < 42\} \mid \{r \in \mathbb{Q} : r \geq 42\}$ and
2. $y = \{r \in \mathbb{Q} : r \leq 0 \text{ or } r^2 < 2\} \mid \{r \in \mathbb{Q} : r^2 \geq 2\}$

are examples of Dedekind cuts in $\mathbb{Q}$.

In the cut x the right-hand part has a minimum. We call it a **rational cut**, the cut at the rational number 42 and denote the cut itself by 42^*.

The right-hand part of the cut y has no minimum in $\mathbb{Q}$ however as proven in Theorem 1.1, and we call such a cut an **irrational cut**. This particular cut is at the irrational $\sqrt{2}$, and as before we denote the cut itself by $\sqrt{2}^*$.

In general if r is a cut at $q \in \mathbb{Q}$ or $q \in \overline{Q}$, we'll use $q^* = r$ to denote the cut. This allows us to think of $\mathbb{Q} \subset \mathbb{R}$ or $\overline{\mathbb{Q}} \subset \mathbb{R}$ by identifying q with q^*, just as we can think of $\mathbb{Z} \subset \mathbb{Q}$ by identifying n with $n/1$ or $\mathbb{N} \subset \mathbb{Z}$ by identifying n with $+n$. It is in this sense that we say:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \supset \overline{\mathbb{Q}}.$$

Lemma 1.38 (Ordering of cuts)

Let $<$ be an ordering on the collection of Dedekind cuts $\mathbb{R}$ defined by

$$p^* < q^* \iff A \subset C \;\; \forall p^* = A \mid B, q^* = C \mid D \in \mathbb{R}.$$

Then $<$ is a total order on $\mathbb{R}$.

Proof. Let $p^* = A \mid B, q^* = C \mid D, r^* = E \mid F \in \mathbb{R}$. Then clearly,

$$A \subset C,\; C \subset E \implies A \subset E$$

$$\iff p^* < q^*,\ q^* < r^* \implies p^* < r^*$$

so transitivity holds. To show that trichotomy holds, we need to show that at least one of

$$p^* < q^*, p^* = q^*, q^* < p^*$$

holds, so assuming that the first two fail we see that as A is not a subset of C there is some $a \in A : a \notin C$ and thus $c \in C \implies c < a$. As we assumed moreover that $A \neq C$, we thus have $C \subset A \iff q^* < p^*$. □

Theorem 1.39 (Real numbers)

The triple $(\mathbb{R}, +, \cdot)$ *with the total order* $<$ *is a complete ordered field and we call it the* ***field of real numbers****.*

Proof Outline. Essentially need to "just" verify that the field axioms and definition of ordered field hold for $(\mathbb{R}, +, \cdot)$ equipped with the total order $<$. □

Lemma 1.40 (Archimedean property)

Let $\mathbb{F}$ be a complete ordered field s.t. the set $\mathbb{N} \subseteq \mathbb{F}$. Then $\mathbb{N}$ is unbounded above. Equivalently,

$$\forall x, y \in \mathbb{F} : x > 0\ \exists\ n \in \mathbb{N}\ s.t.\ nx > y.$$

Proof. As $\varnothing \neq \mathbb{N} \subseteq \mathbb{F}$, if $\mathbb{N}$ is bounded above then there is some $\alpha = \sup \mathbb{N}$. Then $\alpha - 1$ is not an upper bound of $\mathbb{N}$. So we can find $n \in \mathbb{N}$ such that $n > \alpha - 1$. But then $n + 1 > \alpha$, so that α is not the least upper bound of $\mathbb{N}$. Contradiction.

Now to prove that $\forall x > 0\ \exists\ n \in \mathbb{N}$ s.t. $nx > y$, it suffices to show that $n > y/x$.

Sps for contradiction that there is no such n. Then $\forall n \in \mathbb{N}$ we must have $n \leq y/x \implies y/x$ is an upper bound of $\mathbb{N}$. Contradiction. □

Remark. There are non-complete ordered fields that are non-Archimedean i.e. in which the integers are bounded above. Consider the field of rational functions, i.e. functions in the form

$$\frac{p(x)}{q(x)}$$

with $p(x), q(x)$ being polynomials, under the usual addition and multiplication. We order two functions

$$\frac{p(x)}{q(x)}, \frac{r(x)}{s(x)}$$

as follows: these two functions intersect only finitely many times because

$$p(x)s(x) = r(x)q(x)$$

has only finitely many roots. After the last intersection, the function whose value is greater counts as the greater function. It can be checked that these form an ordered field.

In this field, the integers are the constant functions $1, 2, 3, \ldots$, but it is bounded above since the function x is greater than all of them.

§1.3 Cardinality of infinite sets

The intuition regarding the cardinalities of finite sets does not always hold up for infinite sets, so we need a general definition to cover both cases.

Definition 1.41 (Set Cardinality)

We say that two sets have the same cardinality iff there exists some bijection between them.

Definition 1.42 (Countable and Uncountable Sets)

An infinite set X is **countable** iff there is a bijection $f : \mathbb{N} \to X$. Otherwise, X is **uncountable**.

§1.4 Topology of the real line

Definition 1.43

A **metric space** is a pair (X, d) where X is a nonempty set of points and $d : X \times X \to \mathbb{R}$ (called the **distance function** or **metric**) s.t.:

1. $d(x, y) \geq 0$ with $d(x, y) = 0 \iff x = y$. (*positive definiteness*)
2. $d(x, y) = d(y, x)$. (*symmetry*)
3. $d(x, y) \leq d(x, z) + d(z, y)$. (*triangle inequality*)

Example 1.44

The most natural example of a metric space is $\mathbb{R}^n$, the Euclidean n-space. Here we consider the metric to be induced from the standard Euclidean norm $\|x\| := \sqrt{\sum_{j=1}^{n} x_j^2}$. However, other norms also exist.

$\mathbb{Q}$ is also a metric space, albeit not a complete one. $\mathbb{R}$ is a complete metric space, in fact using Cauchy sequences we can *define it to be the completion of* $\mathbb{Q}$ *w.r.t. the standard Euclidean metric.*

Definition 1.45 (Sequence)

A **sequence** in the metric space X is a mapping $f : A \to X$ where $A \subseteq \mathbb{N}$.
If $|A| = \aleph_0$ i.e. $f(n) = x_n \ \forall n \in \mathbb{N}$ then we write $(x_n)_{n=0}^{\infty}$, $(x_n)_{n\in\mathbb{N}}$ or just (x_n) to denote the sequence.
If $|A| = m \in \mathbb{N}$ i.e. the sequence is finite and starts at some index $k \in \mathbb{N}$, then we write $(x_n)_{n\in A}$, $(x_n)_{n=k}^{m-k+1}$, or equivalently $(x_n)_{n=1}^{m}$ to denote the sequence.

A sequence $(a_n)_{n\in A}$ is **(strictly) monotone increasing** iff $a_{n+1} (>) \geq a_n \ \forall n \in A$.
A sequence $(a_n)_{n\in A}$ is **(strictly) monotone decreasing** iff the sequence $(-a_n)_{n\in A}$ is (strictly) monotone increasing.

Definition 1.46 (Neighbourhoods)

Let X be a metric space and $\varepsilon > 0$. An **(ε-)neighbourhood** of $p \in X$ is defined as the set:

$$N(p; \varepsilon) := \{q \in X : d(p,q) \leq \varepsilon\}.$$

Further, a **deleted (ε-)neighbourhood** of p is defined as the ε-neighbourhood of p excluding p itself:

$$N'(p; \varepsilon) := N(p; \varepsilon) \setminus \{p\}.$$

Example 1.47

The easiest example of a neighbourhood is probably the open interval in $\mathbb{R}$,

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}.$$

Definition 1.48 (Interior points)

Let X be a metric space. A point $p \in X$ is an **interior point** of $E \subseteq X$ iff there is at least one *neighbourhood* of p that is contained in E. The **interior** $\operatorname{int} E$ of E is the set of all interior points of E. Thus:

$$p \in \operatorname{int} E \iff (\exists \varepsilon > 0 : N(p; \varepsilon) \subseteq E).$$

Example 1.49

Assume here that $X = \mathbb{R}$, then

$$\operatorname{int}[a,b] = \operatorname{int}(a,b] = \operatorname{int}[a,b) = \operatorname{int}(a,b) = (a,b).$$

Countable and discrete subsets of $\mathbb{R}$ have no interior,

$$\text{Let } S \subset \mathbb{Z}, \text{ then}$$

$$\operatorname{int} S = \operatorname{int} \mathbb{Z} = \operatorname{int} \mathbb{N} = \operatorname{int} \mathbb{Q} = \varnothing.$$

However,

$$\operatorname{int}((a,b) \cup \ S) = (a,b).$$

The irrationals are uncountable but any neighbourhood (open interval) in $\mathbb{R}$ contains both rationals and irrationals. Thus,

$$\operatorname{int} \overline{\mathbb{Q}} = \varnothing.$$

Definition 1.50 (Open sets)

Let X be a metric space and $E \subseteq X$. Then E is **open** iff $E = \operatorname{int} E$ i.e. every point in E is an interior point of E.

Equivalently, E is open iff

$$(\forall x \in E)\, \exists \varepsilon > 0 : N(p; \varepsilon) \subseteq E.$$

Theorem 1.51

Every neighbourhood is an open set.

Corollary 1.52

Every open interval (a, b) is an open set.

Remark. Negating the definition of open set, for a set G to be *not* open there must be some $x \in G$ such that $\forall \varepsilon > 0 : N(p; \varepsilon) \not\subset E$.
Thus $\varnothing$ is open (vacuously true), and $\mathbb{R} = (-\infty, \infty)$ is open.

Example 1.53 1. $(0, 1) \cup \{1, 2, 3\}$ is not open.

2. $(0, 1) \cup (1, 2)$ is open.

Theorem 1.54

Let $G_1, \ldots, G_n$ be open sets. Then the finite intersection $\displaystyle\bigcap_{i=1}^{n} G_i$ is also open.

Proof. If $\displaystyle\bigcap_{i=1}^{n} G_i = \varnothing$ then it is clearly open. Thus assume that, $\displaystyle\bigcap_{i=1}^{n} G_i \neq \varnothing$.

$$\text{Let } x \in \bigcap_{i=1}^{n} G_i$$

$$\implies x \in G_i \forall i = 1, \ldots, n$$

$$\implies \exists \delta_i > 0 : N(x; \delta_i) \subseteq G_i.$$

$$\text{Let } \delta := \min_{1 \le i \le n} \delta_i$$

$$\implies N(x; \delta) \subseteq N(x; \delta_i) \subseteq G_i$$

$$\implies N(x; \delta) \subseteq \bigcap_{i=1}^{n} G_i.$$

Thus the finite union of open sets is open. □

Theorem 1.55

Let $G = \{G_n : n \in \mathbb{N}\}$ be an infinite set of open sets. Then the infinite intersection $\bigcap G := \displaystyle\bigcap_{n \in \mathbb{N}} G_n$ need not be open.

Proof. $G_n := (-1/n,\ 1/n)$ is clearly open $\forall n \in \mathbb{N}$. Also, it is obvious that

$$\{0\} \subset \bigcap_{n\in\mathbb{N}} (-1/n, 1/n) = \bigcap G.$$

Now suppose

$$\exists \varepsilon > 0 : \{\varepsilon\} \subset \bigcap_{n\in\mathbb{N}} (-1/n, 1/n) \iff \varepsilon \in \bigcap_{n\in\mathbb{N}} (-1/n, 1/n) \quad \ldots (1)$$

$$\implies \exists n_0 \in \mathbb{N} : n_0\varepsilon > 1 \text{ (Archimedean property)}$$

$$\implies \varepsilon > 1/n_0 \implies \varepsilon \notin (-1/n_0, 1/n_0)$$

which contradicts (1). Thus $\bigcap G = \{0\}$ which is clearly not open. □

Definition 1.56 (Limit points)

Let X be a metric space. A point $p \in X$ is a **limit point** or **cluster point** or **accumulation point** of $E \subseteq X$ iff every *deleted neighbourhood* of p contains a point q s.t. $q \in E$. The **derived set** E' of E is the set of all limit points of E. Thus:

$$p \in E' \iff \left(\forall \varepsilon > 0 : N'(p; \varepsilon) \cap E \neq \varnothing\right).$$

$\operatorname{cl} E := E \cup E'$ is the **closure** of E.

Remark. Requiring a *neighbourhood* rather than a deleted neighbourhood in the definition of limit point yields the definition of an **adherent point**. Equivalently, p is adherent iff $p \in \operatorname{cl} E$.

Definition 1.57 (Isolated points)

Let X be a metric space. A point $p \in X$ is an **isolated point** of $E \subseteq X$ iff $p \in E$ and $p \notin E'$ i.e. not a limit point of E but in E.

Definition 1.58 (Closed sets)

Let X be a metric space and $E \subseteq X$. Then E is **closed** iff $E' \subseteq E$ i.e. every limit point of E is in E.

Definition 1.59 (Perfect sets)

Let X be a metric space and $E \subseteq X$. Then E is **perfect** iff E is closed and every point in E is a limit point of E.

Definition 1.60 (Bounded sets)

Let X be a metric space and $E \subseteq X$. E is **bounded** iff there is $M \in \mathbb{R}$ and $q \in X$ s.t. $d(p, q) < M\ \forall p \in E$.

Definition 1.61 (Dense sets)

Let X be a metric space and $E \subseteq X$. E is **dense in X** iff every point of X is in the closure of E i.e. $X = \operatorname{cl} E$.

Example 1.62

$\mathbb{Q}$ is dense in $\mathbb{R}$.

Theorem 1.63

E is open iff its complement $X \setminus E$ is closed.

Corollary 1.64

E is closed iff its complement $X \setminus E$ is open.

§1.5 Useful Inequalities

The following inequalities are useful in analysis.

Theorem 1.65 (Triangle Inequality)

If $x, y, z \in \mathbb{R}$, then $\|x+y\| + \|y+z\| \geq \|x+z\|$.

Theorem 1.66 (Arithmetic Mean $\geq$ Geometric $\geq$ Harmonic Mean Inequality)

If $a_1, \ldots, a_n$ are arbitrary elements of $\mathbb{R}$, then

$$\left(\frac{1}{n}\sum_{j=1}^{n} a_j\right) \geq \left(\prod_{j=1}^{n} a_j\right)^{\frac{1}{n}} \geq \left(\frac{n}{\sum_{j=1}^{n} a_j}\right).$$

Theorem 1.67 (Cauchy-Schwarz Inequality)

If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are arbitrary elements of $\mathbb{R}$, then

$$\left(\sum_{j=1}^{n} a_j^2\right)\left(\sum_{j=1}^{n} b_j^2\right) \geq \left(\sum_{j=1}^{n} a_j b_j\right)^2.$$

Moreover, if some $a_i \neq 0$ equality holds iff there is a $\lambda \in \mathbb{F}$ such that $a_j\lambda + b_j = 0$ for all $j = 1, \ldots, n$.

Theorem 1.68 (Bernoulli's Inequality)

If $x \in \mathbb{R}$ such that $x \geq -1$, then for every positive integer n

$$(1+x)^n \geq 1 + nx.$$

Moreover, if $x > -1$ and $x \neq 0$, then $(1+x)^n > 1 + nx$ for all $n \geq 2$.

Definition 1.69 (Convexity)
A function $f : D_f \to \mathbb{R}, D_f \subseteq \mathbb{R}$ *is* ***convex*** *iff* $\forall t \in (0,1)$ and $r, s \in D_f$ we have:

$$f(tr + (1-t)s) \leq tf(r) + (1-t)f(s).$$

Also, f *is* ***concave*** *iff* $-f$ *is convex.*

Theorem 1.70 (Jensen's Inequality)
Let $f : D_f \to \mathbb{R}, D_f \subseteq \mathbb{R}$ and $\{x_j\}_{j=1}^n \subseteq D_f$ with $a_1, \ldots, a_n$ being arbitrary positive reals. If

1. *f is* ***convex*** *then*

$$\frac{\sum_{j=1}^n a_j f(x_j)}{\sum_{j=1}^n a_j} \geq f\left(\frac{\sum_{j=1}^n a_j x_j}{\sum_{j=1}^n a_j}\right).$$

2. *f is* ***concave*** *then*

$$\frac{\sum_{j=1}^n a_j f(x_j)}{\sum_{j=1}^n a_j} \leq f\left(\frac{\sum_{j=1}^n a_j x_j}{\sum_{j=1}^n a_j}\right).$$

Theorem 1.71 (Minkowski's Inequality)
If $p \geq 1$ and $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are arbitrary elements of $\mathbb{R}$, then

$$\left(\sum_{k=1}^n |x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^n |y_k|^p\right)^{\frac{1}{p}} \geq \left(\sum_{k=1}^n |x_k + y_k|^p\right)^{\frac{1}{p}}.$$

Theorem 1.72 (Hölder's Inequality)
If $p, q \geq 1 : 1/p + 1/q = 1$ and $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are arbitrary elements of $\mathbb{R}$, then

$$\left(\sum_{k=1}^n |x_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^n |y_k|^q\right)^{\frac{1}{q}} \geq \left(\sum_{k=1}^n |x_k + y_k|\right).$$

Theorem 1.73 (Tschebyscheff's Inequality)
If $(a_k)_{k=1}^n, (b_k)_{k=1}^n$ are either both monotonically increasing or both monotonically decreasing sequences in $\mathbb{R}$, then

$$n\left(\sum_{k=1}^n a_k b_k\right) \geq \left(\sum_{k=1}^n a_k\right)\left(\sum_{k=1}^n b_k\right).$$

Theorem 1.74 (Rearrangement Inequality)

If $b_1, \ldots, b_n$ is any rearrangement of the positive reals $a_1, \ldots, a_n$, then:

$$\sum_{i=1}^{n} \frac{a_i}{b_i} \geq n.$$

Theorem 1.75 (Weierstrass's Inequalities)

If $\sum_{k=1}^{n} a_k < 1 : a_k \in (0,1)$ for some arbitrary positive reals $a_1, \ldots, a_n$, then:

$$\frac{1}{1 \mp \sum_{k=1}^{n} a_k} < \prod_{k=1}^{n} (1 \pm a_k) < 1 \pm \sum_{k=1}^{n} a_k.$$

§2 Sequences

Having defined real numbers, the first thing we will study is sequences. We will want to study what it means for a sequence to *converge*. Intuitively, we would like to say that $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ converges to 0, while $1, 2, 3, 4, \ldots$ diverges. However, the actual formal definition of convergence is rather hard to get right, and historically there have been failed attempts that produced spurious results.

§2.1 Definitions

Definition 2.1 (Sequence)

A *sequence* is, formally, a function $a : \mathbb{N} \to \mathbb{R}$ (or $\mathbb{C}$). Usually (i.e. always), we write a_n instead of $a(n)$. Instead of a, we usually write it as (a_n), $(a_n)_1^\infty$ or $(a_n)_{n=1}^\infty$ to indicate it is a sequence.

Definition 2.2 (Convergence of sequence)

Let (a_n) be a sequence and $\ell \in \mathbb{R}$. Then a_n *converges to* ℓ, *tends to* ℓ, or $a_n \to \ell$, if for all $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that whenever $n > N$, we have $|a_n - \ell| < \varepsilon$. In symbols, this says

$$(\forall \varepsilon > 0)(\exists N)(\forall n \geq N)\, |a_n - \ell| < \varepsilon.$$

We say ℓ is the *limit* of (a_n).

One can think of $(\exists N)(\forall n \geq N)$ as saying "eventually always", or as "from some point on". So the definition means, if $a_n \to \ell$, then given any ε, there eventually, everything in the sequence is within ε of ℓ.

Lemma 2.3

$1/n \to 0$.

Proof. Let $\varepsilon > 0$. We want to find an N such that $|1/N - 0| = 1/N < \varepsilon$. So pick N such that $N > 1/\varepsilon$. There exists such an N by the Archimedean property v1. Then for all $n \geq N$, we have $0 < 1/n \leq 1/N < \varepsilon$. So $|1/n - 0| < \varepsilon$. □

Note that the red parts correspond to the *definition* of convergence of a sequence. This is generally how we prove convergence from first principles.

Definition 2.4 (Bounded sequence)
A sequence (a_n) is *bounded* if

$$(\exists C)(\forall n)\, |a_n| \leq C.$$

A sequence is *eventually bounded* if

$$(\exists C)(\exists N)(\forall n \geq N)\, |a_n| \leq C.$$

The definition of an *eventually bounded* sequence seems a bit daft. Clearly every eventually bounded sequence is bounded! Indeed it is:

Lemma 2.5
Every eventually bounded sequence is bounded.

Proof. Let C and N be such that $(\forall n \geq N)\, |a_n| \leq C$. Then $\forall n \in \mathbb{N}$, $|a_n| \leq \max\{|a_1|, \ldots, |a_{N-1}|, C\}$. □

The proof is rather trivial. However, most of the time it is simpler to prove that a sequence is eventually bounded, and this lemma saves us from writing that long line every time.

§2.2 Sums, products and quotients

Here we prove the things that we think are obviously true, e.g. sums and products of convergent sequences are convergent.

Lemma 2.6 (Sums of sequences)
If $a_n \to a$ and $b_n \to b$, then $a_n + b_n \to a + b$.

Proof. Let $\varepsilon > 0$. We want to find a clever N such that for all $n \geq N$, $|a_n + b_n - (a + b)| < \varepsilon$. Intuitively, we know that a_n is very close to a and b_n is very close to b. So their sum must be very close to $a + b$.

Formally, since $a_n \to a$ and $b_n \to b$, we can find N_1, N_2 such that $\forall n \geq N_1$, $|a_n - a| < \varepsilon/2$ and $\forall n \geq N_2$, $|b_n - b| < \varepsilon/2$.

Now let $N = \max\{N_1, N_2\}$. Then by the triangle inequality, when $n \geq N$,

$$|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b| < \varepsilon.$$

□

We want to prove that the product of convergent sequences is convergent. However, we will not do it in one go. Instead, we separate it into many smaller parts.

Lemma 2.7 (Scalar multiplication of sequences)
Let $a_n \to a$ and $\lambda \in \mathbb{R}$. Then $\lambda a_n \to \lambda a$.

Proof. If $\lambda = 0$, then the result is trivial.

Otherwise, let $\varepsilon > 0$. Then $\exists N$ such that $\forall n \geq N$, $|a_n - a| < \varepsilon/|\lambda|$. So $|\lambda a_n - \lambda a| < \varepsilon$. □

Lemma 2.8
Let (a_n) be bounded and $b_n \to 0$. Then $a_n b_n \to 0$.

Proof. Let $C \neq 0$ be such that $(\forall n)\, |a_n| \leq C$. Let $\varepsilon > 0$. Then $\exists N$ such that $(\forall n \geq N)\, |b_n| < \varepsilon/C$. Then $|a_n b_n| < \varepsilon$. □

Lemma 2.9
Every convergent sequence is bounded.

Proof. Let $a_n \to l$. Then there is an N such that $\forall n \geq N$, $|a_n - l| \leq 1$. So $|a_n| \leq |l| + 1$. So a_n is eventually bounded, and therefore bounded. □

Lemma 2.10 (Product of sequences)
Let $a_n \to a$ and $b_n \to b$. Then $a_n b_n \to ab$.

Proof. Let $a_n = a + \varepsilon_n$. Then $a_n b_n = (a + \varepsilon_n) b_n = a b_n + \varepsilon_n b_n$.

Since $b_n \to b$, $ab_n \to ab$. Since $\varepsilon_n \to 0$ and b_n is bounded, $\varepsilon_n b_n \to 0$. So $a_n b_n \to ab$. □

Proof. (alternative) Observe that $a_n b_n - ab = (a_n - a) b_n + (b_n - b) a$. We know that $a_n - a \to 0$ and $b_n - b \to 0$. Since (b_n) is bounded, so $(a_n - a) b_n + (b_n - b) a \to 0$. So $a_n b_n \to ab$. □

Note that in this proof, we no longer write "Let $\varepsilon > 0$". In the beginning, we have no lemmas proven. So we must prove everything from first principles and use the definition. However, after we have proven the lemmas, we can simply use them instead of using first principles. This is similar to in calculus, where we use first principles to prove the product rule and chain rule, then no longer use first principles afterwards.

Lemma 2.11 (Quotient of sequences)
Let (a_n) be a sequence such that $(\forall n)\, a_n \neq 0$. Suppose that $a_n \to a$ and $a \neq 0$. Then $1/a_n \to 1/a$.

Proof. We have

$$\frac{1}{a_n} - \frac{1}{a} = \frac{a - a_n}{aa_n}.$$

We want to show that this $\to 0$. Since $a - a_n \to 0$, we have to show that $1/(aa_n)$ is bounded.

Since $a_n \to a$, $\exists N$ such that $\forall n \geq N$, $|a_n - a| \leq a/2$. Then $\forall n \geq N$, $|a_n| \geq |a|/2$. Then $|1/(a_n a)| \leq 2/|a|^2$. So $1/(a_n a)$ is bounded. So $(a - a_n)/(aa_n) \to 0$ and the result follows. □

Corollary 2.12

If $a_n \to a, b_n \to b, b_n, b \neq 0$, then $a_n/b_n = a/b$.

Proof. We know that $1/b_n \to 1/b$. So the result follows by the product rule. □

Lemma 2.13 (Sandwich rule)

Let (a_n) and (b_n) be sequences that both converge to a limit x. Suppose that $a_n \leq c_n \leq b_n$ for every n. Then $c_n \to x$.

Proof. Let $\varepsilon > 0$. We can find N such that $\forall n \geq N$, $|a_n - x| < \varepsilon$ and $|b_n - x| < \varepsilon$.

Then $\forall n \geq N$, we have $x - \varepsilon < a_n \leq c_n \leq b_n < x + \varepsilon$. So $|c_n - x| < \varepsilon$. □

Example 2.14

$1/2^n \to 0$. For every n, $n < 2^n$. So $0 < 1/2^n < 1/n$. The result follows from the sandwich rule.

Example 2.15

We want to show that

$$\frac{n^2 + 3}{(n+5)(2n-1)} \to \frac{1}{2}.$$

We can obtain this by

$$\frac{n^2 + 3}{(n+5)(2n-1)} = \frac{1 + 3/n^2}{(1 + 5/n)(2 - 1/n)} \to \frac{1}{2},$$

by sum rule, sandwich rule, Archimedean property, product rule and quotient rule.

Example 2.16

Let $k \in \mathbb{N}$ and let $\delta > 0$. Then

$$\frac{n^k}{(1+\delta)^n} \to 0.$$

This can be summarized as "exponential growth beats polynomial growth eventually".

By the binomial theorem,

$$(1+\delta)^n \geq \binom{n}{k+1}\delta^{k+1}.$$

Also, for $n \geq 2k$,

$$\binom{n}{k+1} = \frac{n(n-1)\dots(n-k)}{(k+1)!} \geq \frac{(n/2)^{k+1}}{(k+1)!}.$$

So for sufficiently large n,

$$\frac{n^k}{(1+\delta)^n} \leq \frac{n^k 2^{k+1}(k+1)!}{n^{k+1}\delta^{k+1}} = \frac{2^{k+1}(k+1)!}{\delta^{k+1}} \cdot \frac{1}{n} \to 0.$$

§2.3 Monotone-sequences property

Recall that we characterized the least upper bound property. It turns out that there is an alternative characterization of real number using sequences, known as the *monotone-sequences property*. In this section, we will show that the two characterizations are equivalent, and use the monotone-sequences property to deduce some useful results.

Definition 2.17 (Monotone sequence)

A sequence (a_n) is *increasing* if $a_n \leq a_{n+1}$ for all n.

It is *strictly increasing* if $a_n < a_{n+1}$ for all n. *(Strictly) decreasing* sequences are defined analogously.

A sequence is *(strictly) monotone* if it is (strictly) increasing or (strictly) decreasing.

Definition 2.18 (Monotone-sequences property)

An ordered field has the *monotone sequences property* if every increasing sequence that is bounded above converges.

We want to show that the monotone sequences property is equivalent to the least upper bound property.

Lemma 2.19

Least upper bound property $\Rightarrow$ monotone-sequences property.

Proof. Let (a_n) be an increasing sequence and let C an upper bound for (a_n). Then C is an upper bound for the set $\{a_n : n \in \mathbb{N}\}$. By the least upper bound property, it has a supremum s. We want to show that this is the limit of (a_n).

Let $\varepsilon > 0$. Since $s = \sup\{a_n : n \in \mathbb{N}\}$, there exists an N such that $a_N > s - \varepsilon$. Then since (a_n) is increasing, $\forall n \geq N$, we have $s - \varepsilon < a_N \leq a_n \leq s$. So $|a_n - s| < \varepsilon$. $\square$

We first prove a handy lemma.

Lemma 2.20

Let (a_n) be a sequence and suppose that $a_n \to a$. If $(\forall n)\, a_n \leq x$, then $a \leq x$.

Proof. If $a > x$, then set $\varepsilon = a - x$. Then we can find N such that $a_N > x$. Contradiction. $\square$

Before showing the other way implication, we will need the following:

Lemma 2.21

Monotone-sequences property $\Rightarrow$ Archimedean property.

Proof. We prove version 2, i.e. that $1/n \to 0$.

Since $1/n > 0$ and is decreasing, by MSP, it converges. Let δ be the limit. By the previous lemma, we must have $\delta \geq 0$.

If $\delta > 0$, then we can find N such that $1/N < 2\delta$. But then for all $n \geq 4N$, we have $1/n \leq 1/(4N) < \delta/2$. Contradiction. Therefore $\delta = 0$. $\square$

Lemma 2.22

Monotone-sequences property $\Rightarrow$ least upper bound property.

Proof. Let A be a non-empty set that's bounded above. Pick u_0, v_0 such that u_0 is not an upper bound for A and v_0 is an upper bound. Now do a repeated bisection: having chosen u_n and v_n such that u_n is not an upper bound and v_n is, if $(u_n + v_n)/2$ is an upper bound, then let $u_{n+1} = u_n$, $v_{n+1} = (u_n + v_n)/2$. Otherwise, let $u_{n+1} = (u_n + v_n)/2$, $v_{n+1} = v_n$.

Then $u_0 \leq u_1 \leq u_2 \leq \dots$ and $v_0 \geq v_1 \geq v_2 \geq \dots$. We also have

$$v_n - u_n = \frac{v_0 - u_0}{2^n} \to 0.$$

By the monotone sequences property, $u_n \to s$ (since (u_n) is bounded above by v_0). Since $v_n - u_n \to 0$, $v_n \to s$. We now show that $s = \sup A$.

If s is not an upper bound, then there exists $a \in A$ such that $a > s$. Since $v_n \to s$, then there exists m such that $v_m < a$, contradicting the fact that v_m is an upper bound.

To show it is the *least* upper bound, let $t < s$. Then since $u_n \to s$, we can find m such that $u_m > t$. So t is not an upper bound. Therefore s is the least upper bound. $\square$

Why do we need to prove the Archimedean property first? In the proof above, we secretly used the it. When showing that $v_n - u_n \to 0$, we required the fact that $\frac{1}{2^n} \to 0$. To prove this, we sandwiched it with $\frac{1}{n}$. But to show $\frac{1}{n} \to 0$, we need the Archimedean property.

Lemma 2.23

A sequence can have at most 1 limit.

Proof. Let (a_n) be a sequence, and suppose $a_n \to x$ and $a_n \to y$. Let $\varepsilon > 0$ and pick N such that $\forall n \geq N$, $|a_n - x| < \varepsilon/2$ and $|a_n - y| < \varepsilon/2$. Then $|x - y| \leq |x - a_N| + |a_N - y| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Since ε was arbitrary, x must equal y. $\square$

Lemma 2.24 (Nested intervals property)

Let $\mathbb{F}$ be an ordered field with the monotone sequences property. Let $I_1 \supseteq I_2 \supseteq \dots$ be closed bounded non-empty intervals. Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. Let $T_n = [a_n, b_n]$ for each n. Then $a_1 \leq a_2 \leq \dots$ and $b_1 \geq b_2 \geq \dots$. For each n, $a_n \leq b_n \leq b_1$. So the sequence a_n is bounded above. So by the monotone sequences property, it has a limit a. For each n, we must have $a_n \leq a$. Otherwise, say $a_n > a$. Then for all $m \geq n$, we have $a_m \geq a_n > a$. This implies that $a > a$, which is nonsense.

Also, for each fixed n, we have that $\forall m \geq n$, $a_m \leq b_m \leq b_n$. So $a \leq b_n$. Thus, for all n, $a_n \leq a \leq b_n$. So $a \in I_n$. So $a \in \bigcap_{n=1}^{\infty} I_n$. $\square$

We can use this to prove that the reals are uncountable:

Proposition 2.25

$\mathbb{R}$ is uncountable.

Proof. Suppose the contrary. Let $x_1, x_2, \dots$ be a list of all real numbers. Find an interval that does not contain x_1. Within that interval, find an interval that does not contain x_2. Continue *ad infinitum*. Then the intersection of all these intervals is non-empty, but the elements in the intersection are not in the list. Contradiction. $\square$

A powerful consequence of this is the *Bolzano-Weierstrass theorem*. This is formulated in terms of subsequences:

Definition 2.26 (Subsequence)

Let (a_n) be a sequence. A *subsequence* of (a_n) is a sequence of the form $a_{n_1}, a_{n_2}, \dots$, where $n_1 < n_2 < \dots$.

Example 2.27

$1, 1/4, 1/9, 1/16, \dots$ is a subsequence of $1, 1/2, 1/3, \dots$.

Theorem 2.28 (Bolzano-Weierstrass theorem)

Let $\mathbb{F}$ be an ordered field with the monotone sequences property (i.e. $\mathbb{F} = \mathbb{R}$).

Then every bounded sequence has a convergent subsequence.

Proof. Let u_0 and v_0 be a lower and upper bound, respectively, for a sequence (a_n). By repeated bisection, we can find a sequence of intervals $[u_0, v_0] \supseteq [u_1, v_1] \supseteq [u_2, v_2] \supseteq \dots$ such that $v_n - u_n = (v_0 - u_0)/2^n$, and such that each $[u_n, v_n]$ contains infinitely many terms of (a_n).

By the nested intervals property, $\bigcap_{n=1}^{\infty}[u_n, v_n] \neq \emptyset$. Let x belong to the intersection. Now pick a subsequence $a_{n_1}, a_{n_2}, \dots$ such that $a_{n_k} \in [u_k, v_k]$. We can do this because $[u_k, v_k]$ contains infinitely many a_n, and we have only picked finitely many of them. We will show that $a_{n_k} \to x$.

Let $\varepsilon > 0$. By the Archimedean property, we can find K such that $v_K - u_K = (v_0 - u_0)/2^K \leq \varepsilon$. This implies that $[u_K, v_K] \subseteq (x - \varepsilon, x + \varepsilon)$, since $x \in [u_K, v_K]$.

Then $\forall k \geq K$, $a_{n_k} \in [u_k, v_k] \subseteq [u_K, v_K] \subseteq (x - \varepsilon, x + \varepsilon)$. So $|a_{n_k} - x| < \varepsilon$. □

§2.4 Cauchy sequences

The third characterization of real numbers is in terms of Cauchy sequences. Cauchy convergence is an alternative way of defining convergent sequences without needing to mention the actual limit of the sequence. This allows us to say $\{3, 3.1, 3.14, 3.141, 3.1415, \dots\}$ is *Cauchy convergent* in $\mathbb{Q}$ even though the limit π is not in $\mathbb{Q}$.

Definition 2.29 (Cauchy sequence)

A sequence (a_n) is *Cauchy* if for all ε, there is some $N \in \mathbb{N}$ such that whenever $p, q \geq N$, we have $|a_p - a_q| < \varepsilon$. In symbols, we have

$$(\forall \varepsilon > 0)(\exists N)(\forall p, q \geq N)\, |a_p - a_q| < \varepsilon.$$

Roughly, a sequence is Cauchy if all terms are eventually close to each other (as opposed to close to a limit).

Lemma 2.30

Every convergent sequence is Cauchy.

Proof. Let $a_n \to a$. Let $\varepsilon > 0$. Then $\exists N$ such that $\forall n \geq N$, $|a_n - a| < \varepsilon/2$. Then $\forall p, q \geq N$, $|a_p - a_q| \leq |a_p - a| + |a - a_q| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. □

Lemma 2.31

Let (a_n) be a Cauchy sequence with a subsequence (a_{n_k}) that converges to a. Then $a_n \to a$.

Proof. Let $\varepsilon > 0$. Pick N such that $\forall p, q \geq N$, $|a_p - a_q| < \varepsilon/2$. Then pick K such that $n_K \geq N$ and $|a_{n_K} - a| < \varepsilon/2$.

Then $\forall n \geq N$, we have

$$|a_n - a| \leq |a_n - a_{n_K}| + |a_{n_K} - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

An important result we have is that in $\mathbb{R}$, Cauchy convergence and regular convergence are equivalent.

Theorem 2.32 (The general principle of convergence)

Let $\mathbb{F}$ be an ordered field with the monotone-sequence property. Then every Cauchy sequence of $\mathbb{F}$ converges.

Proof. Let (a_n) be a Cauchy sequence. Then it is eventually bounded, since $\exists N$, $\forall n \geq N$, $|a_n - a_N| \leq 1$ by the Cauchy condition. So it is bounded. Hence by Bolzano-Weierstrass, it has a convergent subsequence. Then (a_n) converges to the same limit. □

Definition 2.33 (Complete ordered field)

An ordered field in which every Cauchy sequence converges is called *complete*.

Hence we say that $\mathbb{R}$ is a complete ordered field.

However, not every complete ordered field is (isomorphic to) $\mathbb{R}$. For example, we can take the rational functions as before, then take the Cauchy completion of it (i.e. add all the limits we need). Then it is already too large to be the reals (it still doesn't have the Archimedean property) but is a complete ordered field.

To show that completeness implies the monotone-sequences property, we need an additional condition: the Archimedean property.

Lemma 2.34

Let $\mathbb{F}$ be an ordered field with the Archimedean property such that every Cauchy sequence converges. The $\mathbb{F}$ satisfies the monotone-sequences property.

Proof. Instead of showing that every bounded monotone sequence converges, and is hence Cauchy, We will show the equivalent statement that every increasing non-Cauchy sequence is not bounded above.

Let (a_n) be an increasing sequence. If (a_n) is not Cauchy, then

$$(\exists \varepsilon > 0)(\forall N)(\exists p, q > N)\, |a_p - a_q| \geq \varepsilon.$$

wlog let $p > q$. Then

$$a_p \geq a_q + \varepsilon \geq a_N + \varepsilon.$$

So for any N, we can find a $p > N$ such that

$$a_p \geq a_N + \varepsilon.$$

Then we can construct a subsequence $a_{n_1}, a_{n_2}, \ldots$ such that

$$a_{n_{k+1}} \geq a_{n_k} + \varepsilon.$$

Therefore

$$a_{n_k} \geq a_{n_1} + (k-1)\varepsilon.$$

So by the Archimedean property, (a_{n_k}), and hence (a_n), is unbounded. $\square$

Note that the definition of a convergent sequence is

$$(\exists l)(\forall \varepsilon > 0)(\exists N)(\forall n \geq N)\, |a_n - l| < \varepsilon,$$

while that of Cauchy convergence is

$$(\forall \varepsilon > 0)(\exists N)(\forall p, q \geq N)\, |a_p - a_q| < \varepsilon.$$

In the first definition, l quantifies over all real numbers, which is uncountable. However, in the second definition, we only have to quantify over natural numbers, which is countable (by the Archimedean property, we only have to consider the cases $\varepsilon = 1/n$).

Since they are equivalent in $\mathbb{R}$, the second definition is sometimes preferred when we care about logical simplicity.

§2.5 Limit supremum and infimum

Here we will define the limit supremum and infimum. While these are technically not part of the course, eventually some lecturers will magically assume students know this definition. So we might as well learn it here.

Definition 2.35 (Limit supremum/infimum)

Let (a_n) be a bounded sequence. We define the *limit supremum* as

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty}\left(\sup_{m\geq n} a_m\right).$$

To see that this exists, set $b_n = \sup_{m\geq n} a_m$. Then (b_n) is decreasing since we are taking the supremum of fewer and fewer things, and is bounded below by any lower bound for (a_n) since $b_n \geq a_n$. So it converges.

Similarly, we define the *limit infimum* as

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty}\left(\inf_{m\geq n} a_m\right).$$

Example 2.36

Take the sequence

$$2, -1, \frac{3}{2}, -\frac{1}{2}, \frac{4}{3}, -\frac{1}{3}, \dots$$

Then the limit supremum is 1 and the limit infimum is 0.

Lemma 2.37

Let (a_n) be a sequence. The following two statements are equivalent:

- $a_n \to a$

- $\limsup a_n = \liminf a_n = a$.

Proof. If $a_n \to a$, then let $\varepsilon > 0$. Then we can find an n such that

$$a - \varepsilon \leq a_m \leq a + \varepsilon \text{ for all } m \geq n$$

It follows that

$$a - \varepsilon \leq \inf_{m \geq n} a_m \leq \sup_{m \geq n} a_m \leq a + \varepsilon.$$

Since ε was arbitrary, it follows that

$$\liminf a_n = \limsup a_n = a.$$

Conversely, if $\liminf a_n = \limsup a_n = a$, then let $\varepsilon > 0$. Then we can find n such that

$$\inf_{m \geq n} a_m > a - \varepsilon \text{ and } \sup_{m \geq n} a_m < a + \varepsilon.$$

It follows that $\forall m \geq n$, we have $|a_m - a| < \varepsilon$. $\square$

§3 Series

In this section, we investigate which infinite *sums*, as opposed to sequences, converge. We would like to say $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$, while $1 + 1 + 1 + 1 + \ldots$ does not converge. The majority of the section is coming up with different tests to figure out if infinite sums converge.

§3.1 Infinite sums

Definition 3.1 (Convergence of infinite sums and partial sums)
Let (a_n) be a real sequence. For each N, define

$$S_N = \sum_{n=1}^{N} a_n.$$

If the sequence (S_N) converges to some limit s, then we say that

$$\sum_{n=1}^{\infty} a_n = s,$$

and we say that the series $\sum_{n=1}^{\infty} a_n$ *converges*.

We call S_N the Nth *partial sum*.

There is an immediate necessary condition for a series to converge.

Lemma 3.2

If $\sum_{n=1}^{\infty} a_n$ converges. Then $a_n \to 0$.

Proof. Let $\sum_{n=1}^{\infty} a_n = s$. Then $S_n \to s$ and $S_{n-1} \to s$. Then $a_n = S_n - S_{n-1} \to 0$. □

However, the converse is false!

Example 3.3 (Harmonic series)
If $a_n = 1/n$, then $a_n \to 0$ but $\sum a_n = \infty$.

We can prove this as follows:

$$S_{2^n} - S_{2^{n-1}} = \frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n} \geq \frac{2^{n-1}}{2^n} = \frac{1}{2}.$$

Therefore $S_{2^n} \geq S_1 + n/2$. So the partial sums are unbounded.

Example 3.4 (Geometric series)
Let $|\rho| < 1$. Then

$$\sum_{n=0}^{\infty} \rho^n = \frac{1}{1-\rho}.$$

We can prove this by considering the partial sums:

$$\sum_{n=0}^{N} \rho^n = \frac{1-\rho^{N+1}}{1-\rho}.$$

Since $\rho^{N+1} \to 0$, this tends to $1/(1-\rho)$.

Example 3.5
$\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ converges. This is since

$$\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}.$$

So

$$\sum_{n=2}^{N} \frac{1}{n(n-1)} = 1 - \frac{1}{N} \to 1.$$

Lemma 3.6
Suppose that $a_n \geq 0$ for every n and the partial sums S_n are bounded above. Then $\sum_{n=1}^{\infty} a_n$ converges.

Proof. The sequence (S_n) is increasing and bounded above. So the result follows form the monotone sequences property. □

The simplest convergence test we have is the *comparison test.* Roughly speaking, it says that if $0 \le a_n \le b_n$ for all n and $\sum b_n$ converges, then $\sum a_n$ converges. However, we will prove a much more general form here for convenience.

Lemma 3.7 (Comparison test)

Let (a_n) and (b_n) be non-negative sequences, and suppose that $\exists C, N$ such that $\forall n \ge N$, $a_n \le C b_n$. Then if $\sum b_n$ converges, then so does $\sum a_n$.

Proof. Let $M > N$. Also for each R, let $S_R = \sum_{n=1}^{R} a_n$ and $T_R = \sum_{n=1}^{R} b_n$. We want S_R to be bounded above.

$$S_M - S_N = \sum_{n=N+1}^{M} a_n \le C \sum_{n=N+1}^{M} b_n \le C \sum_{n=N+1}^{\infty} b_n.$$

So $\forall M \ge N$, $S_M \le S_n + C\sum_{n=N+1}^{\infty} b_n$. Since the S_M are increasing and bounded, it must converge. □

Example 3.8

1. $\sum \frac{1}{n2^n}$ converges, since $\sum \frac{1}{2^n}$ converges.
2. $\sum \frac{n}{2^n}$ converges.

   If $n \ge 4$, then $n \le 2^{n/2}$. That's because $4 = 2^{4/2}$ and for $n \ge 4$, $(n+1)/n < \sqrt{2}$, so when we increase n, we multiply the right side by a greater number by the left. Hence by the comparison test, it is sufficient to show that $\sum 2^{n/2}/2^n = \sum 2^{-n/2}$ converges, which it does (geometric series).
3. $\sum \frac{1}{\sqrt{n}}$ diverges, since $\frac{1}{\sqrt{n}} \ge \frac{1}{n}$. So if it converged, then so would $\sum \frac{1}{n}$, but $\sum \frac{1}{n}$ diverges.
4. $\sum \frac{1}{n^2}$ converges, since for $n \ge 2$, $\frac{1}{n^2} \le \frac{1}{n(n-1)}$, and we have proven that the latter converges.
5. Consider $\displaystyle\sum_{n=1}^{\infty} \frac{n+5}{n^3 - 7n^2/2}$. We show this converges by noting

   $$n^3 - \frac{7n^2}{2} = n^2\left(n - \frac{7}{2}\right).$$

   So if $n \ge 8$, then

   $$n^3 - \frac{7n^2}{2} \ge \frac{n^3}{2}.$$

   Also, $n + 5 \le 2n$. So

   $$\frac{n+5}{n^3 - 7n^2/2} \le 4/n^2.$$

So it converges by the comparison test.

6. If $\alpha > 1$, then $\sum 1/n^\alpha$ converges.

   Let $S_N = \sum_{n=1}^N 1/n^\alpha$. Then

$$\begin{aligned} S_{2^n} - S_{2^{n-1}} &= \frac{1}{(2^{n-1}+1)^\alpha} + \cdots + \frac{1}{(2^n)^\alpha} \\ &\le \frac{2^{n-1}}{(2^{n-1})^\alpha} \\ &= (2^{n-1})^{1-\alpha} \\ &= (2^{1-\alpha})^{n-1}. \end{aligned}$$

   But $2^{1-\alpha} < 1$. So

$$S_{2^n} = (S_{2^n} - S_{2^{n-1}}) + (S_{2^{n-1}} - S_{2^{n-2}}) + \ldots (S_2 - S_1) + S_1$$

   and is bounded above by comparison with the geometric series $1 + 2^{1-\alpha} + (2^{1-\alpha})^2 + \ldots$

§3.2 Absolute convergence

Here we'll consider two stronger conditions for convergence — absolute convergence and unconditional convergence. We'll prove that these two conditions are in fact equivalent.

Definition 3.9 (Absolute convergence)
A series $\sum a_n$ *converges absolutely* if the series $\sum |a_n|$ converges.

Example 3.10
The series $\sum \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ converges, but not absolutely.

To see the convergence, note that

$$a_{2n-1} + a_{2n} = \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{2n(2n-1)}.$$

It is easy to compare with $1/n^2$ to get that the partial sums S_{2n} converges. But $S_{2n+1} - S_{2n} = 1/(2n+1) \to 0$, so the S_{2n+1} converges to the same limit.

It does not converge absolutely, because the sum of the absolute values is the harmonic series.

Lemma 3.11
Let $\sum a_n$ converge absolutely. Then $\sum a_n$ converges.

Proof. We know that $\sum |a_n|$ converges. Let $S_N = \sum_{n=1}^N a_n$ and $T_N = \sum_{n=1}^N |a_n|$.

We know two ways to show random sequences converge, without knowing what they converge to, namely monotone-sequences and Cauchy sequences. Since S_N is not monotone, we shall try Cauchy sequences.

If $p > q$, then

$$|S_p - S_q| = \left| \sum_{n=q+1}^{p} a_n \right| \leq \sum_{n=q+1}^{p} |a_n| = T_p - T_q.$$

But the sequence T_p converges. So $\forall \varepsilon > 0$, we can find N such that for all $p > q \geq N$, we have $T_p - T_q < \varepsilon$, which implies $|S_p - S_q| < \varepsilon$. □

Definition 3.12 (Unconditional convergence)

A series $\sum a_n$ *converges unconditionally* if the series $\sum_{n=1}^{\infty} a_{\pi(n)}$ converges for every bijection $\pi : \mathbb{N} \to \mathbb{N}$, i.e. no matter how we re-order the elements of a_n, the sum still converges.

Theorem 3.13

If $\sum a_n$ converges absolutely, then it converges unconditionally.

Proof. Let $S_n = \sum_{n=1}^{N} a_{\pi(n)}$. Then if $p > q$,

$$|S_p - S_q| = \left| \sum_{n=q+1}^{p} a_{\pi(n)} \right| \leq \sum_{n=q+1}^{\infty} |a_{\pi(n)}|.$$

Let $\varepsilon > 0$. Since $\sum |a_n|$ converges, pick M such that $\sum_{n=M+1}^{\infty} |a_n| < \varepsilon$.

Pick N large enough that $\{1, \ldots, M\} \subseteq \{\pi(1), \ldots, \pi(N)\}$.

Then if $n > N$, we have $\pi(n) > M$. Therefore if $p > q \geq N$, then

$$|S_p - S_q| \leq \sum_{n=q+1}^{p} |a_{\pi(n)}| \leq \sum_{n=M+1}^{\infty} |a_n| < \varepsilon.$$

Therefore the sequence of partial sums is Cauchy. □

The converse is also true.

Theorem 3.14

If $\sum a_n$ converges unconditionally, then it converges absolutely.

Proof. We will prove the contrapositive: if it doesn't converge absolutely, it doesn't converge unconditionally.

Suppose that $\sum |a_n| = \infty$. Let (b_n) be the subsequence of non-negative terms of a_n, and (c_n) be the subsequence of negative terms. Then $\sum b_n$ and $\sum c_n$ cannot both converge, or else $\sum |a_n|$ converges.

wlog, $\sum b_n = \infty$. Now construct a sequence $0 = n_0 < n_1 < n_2 < \dots$ such that $\forall k$,

$$b_{n_{k-1}+1} + b_{n_{k-1}+2} + \cdots + b_{n_k} + c_k \geq 1,$$

This is possible because the b_n are unbounded and we can get it as large as we want.

Let π be the rearrangement

$$b_1, b_2, \dots b_{n_1}, c_1, b_{n_1+1}, \dots b_{n_2}, c_2, b_{n_2+1}, \dots b_{n_3}, c_3, \dots$$

So the sum up to c_k is at least k. So the partial sums tend to infinity. □

We can prove an even stronger result:

Lemma 3.15

Let $\sum a_n$ be a series that converges absolutely. Then for any bijection $\pi : \mathbb{N} \to \mathbb{N}$,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\pi(n)}.$$

Proof. Let $\varepsilon > 0$. We know that both $\sum |a_n|$ and $\sum |a_{\pi(n)}|$ converge. So let M be such that $\sum_{n>M} |a_n| < \frac{\varepsilon}{2}$ and $\sum_{n>M} |a_{\pi(n)}| < \frac{\varepsilon}{2}$.

Now N be large enough such that

$$\{1, \dots, M\} \subseteq \{\pi(1), \dots, \pi(N)\},$$

and

$$\{\pi(1), \dots, \pi(M)\} \subseteq \{1, \dots, N\}.$$

Then for every $K \geq N$,

$$\left| \sum_{n=1}^{K} a_n - \sum_{n=1}^{K} a_{\pi(n)} \right| \leq \sum_{n=M+1}^{K} |a_n| + \sum_{n=M+1}^{K} |a_{\pi(n)}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We have the first inequality since given our choice of M and N, the first M terms of the $\sum a_n$ and $\sum a_{\pi(n)}$ sums are cancelled by some term in the huge sum.

So $\forall K \geq N$, the partial sums up to K differ by at most ε. So $|\sum a_n - \sum a_{\pi(n)}| \leq \varepsilon$.

Since this is true for all ε, we must have $\sum a_n = \sum a_{\pi(n)}$. □

§3.3 Convergence tests

We'll now come up with a *lot* of convergence tests.

Lemma 3.16 (Alternating sequence test)

Let (a_n) be a decreasing sequence of non-negative reals, and suppose that $a_n \to 0$. Then $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ converges, i.e. $a_1 - a_2 + a_3 - a_4 + \dots$ converges.

Proof. Let $S_N = \sum_{n=1}^{N} (-1)^{n+1} a_n$. Then

$$S_{2n} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-1} - a_{2n}) \geq 0,$$

and (S_{2n}) is an increasing sequence.

Also,

$$S_{2n+1} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n} - a_{2n+1}),$$

and (S_{2n+1}) is a decreasing sequence. Also $S_{2n+1} - S_{2n} = a_{2n+1} \geq 0$.

Hence we obtain the bounds $0 \leq S_{2n} \leq S_{2n+1} \leq a_1$. It follows from the monotone sequences property that (S_{2n}) and (S_{2n+1}) converge.

Since $S_{2n+1} - S_{2n} = a_{2n+1} \to 0$, they converge to the same limit. □

Example 3.17

$$1 - \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} - \frac{1}{\sqrt[3]{4}} + \ldots \text{converges.}$$

Lemma 3.18 (Ratio test)

We have three versions:

1. If $\exists c < 1$ such that
$$\frac{|a_{n+1}|}{|a_n|} \leq c,$$
for all n, then $\sum a_n$ converges.
2. If $\exists c < 1$ and $\exists N$ such that
$$(\forall n \geq N) \, \frac{|a_{n+1}|}{|a_n|} \leq c,$$
then $\sum a_n$ converges. Note that just because the ratio is always less than 1, it doesn't necessarily converge. It has to be always less than a fixed number c. Otherwise the test will say that $\sum 1/n$ converges.
3. If $\exists \rho \in (-1, 1)$ such that
$$\frac{a_{n+1}}{a_n} \to \rho,$$
then $\sum a_n$ converges. Note that we have the *open* interval $(-1, 1)$. If $\frac{|a_{n+1}|}{|a_n|} \to 1$, then the test is inconclusive!

Proof.

1. $|a_n| \leq c^{n-1}|a_1|$. Since $\sum c^n$ converges, so does $\sum |a_n|$ by comparison test. So $\sum a_n$ converges absolutely, so it converges.
2. For all $k \geq 0$, we have $|a_{N+k}| \leq c^k |a_N|$. So the series $\sum |a_{N+k}|$ converges, and

therefore so does $\sum |a_k|$.

3. If $\frac{a_{n+1}}{a_n} \to \rho$, then $\frac{|a_{n+1}|}{|a_n|} \to |\rho|$. So (setting $\varepsilon = (1 - |\rho|)/2$) there exists N such that $\forall n \geq N$, $\frac{|a_{n+1}|}{|a_n|} \leq \frac{1+|\rho|}{2} < 1$. So the result follows from (ii).

□

Example 3.19

If $|b| < 1$, then $\sum nb^n$ converges, since

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)b^{n+1}}{nb^n} = \left(1 + \frac{1}{n}\right) b \to b < 1.$$

So it converges.

We can also evaluate this directly by considering $\displaystyle\sum_{i=1}^{\infty} \sum_{n=i}^{\infty} b^n$.

The following two tests were taught at the end of the course, but are included here for the sake of completeness.

Theorem 3.20 (Condensation test)

Let (a_n) be a decreasing non-negative sequence. Then $\sum_{n=1}^{\infty} a_n < \infty$ if and only if

$$\sum_{k=1}^{\infty} 2^k a_{2^k} < \infty.$$

Proof. This is basically the proof that $\sum \frac{1}{n}$ diverges and $\sum \frac{1}{n^\alpha}$ converges for $\alpha < 1$ but written in a more general way.

We have

$$\begin{aligned} & a_1 + a_2 + (a_3 + a_4) + (a_5 + \dots + a_8) + (a_9 + \dots + a_{16}) + \dots \\ \geq & a_1 + a_2 + 2a_4 + 4a_8 + 8a_{16} + \dots \end{aligned}$$

So if $\sum 2^k a_{2^k}$ diverges, $\sum a_n$ diverges.

To prove the other way round, simply group as

$$\begin{aligned} & a_1 + (a_2 + a_3) + (a_4 + \dots + a_7) + \dots \\ \leq & a_1 + 2a_2 + 4a_4 + \dots . \end{aligned}$$

□

Example 3.21

If $a_n = \frac{1}{n}$, then $2^k a_{2^k} = 1$. So $\sum_{k=1}^{\infty} 2^k a_{2^k} = \infty$. So $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

After we formally define integrals, we will prove the integral test:

Theorem 3.22 (Integral test)

Let $f : [1, \infty] \to \mathbb{R}$ be a decreasing non-negative function. Then $\sum_{n=1}^{\infty} f(n)$ converges iff $\int_1^{\infty} f(x) \ \mathrm{d}x < \infty$.

§3.4 Complex versions

Most definitions in the course so far carry over unchanged to the complex numbers. e.g. $z_n \to z$ iff $(\forall \varepsilon > 0)(\exists N)(\forall n \geq N)\, |z_n - z| < \varepsilon$.

Two exceptions are least upper bound and monotone sequences, because the complex numbers do not have an ordering! (It cannot be made into an ordered field because the square of every number in an ordered field has to be positive) Fortunately, Cauchy sequences still work.

We can prove the complex versions of most theorems so far by looking at the real and imaginary parts.

Example 3.23

Let (z_n) be a Cauchy sequence in $\mathbb{C}$. Let $z_n = x_n + iy_n$. Then (x_n) and (y_n) are Cauchy. So they converge, from which it follows that $z_n = x_n + iy_n$ converges.

Also, the Bolzano-Weierstrass theorem still holds: If (z_n) is bounded, let $z_n = x_n + y_n$, then (x_n) and (y_n) are bounded. Then find a subsequence (x_{n_k}) that converges. Then find a subsequence of (y_{n_k}) that converges.

Then nested-intervals property has a "nested-box" property as a complex analogue.

Finally, the proof that absolutely convergent sequences converge still works. It follows that the ratio test still works.

Example 3.24

If $|z| < 1$, then $\sum nz^n$ converges. Proof is the same as above.

However, we do have an extra test for complex sums.

Lemma 3.25 (Abel's test)

Let $a_1 \geq a_2 \geq \cdots \geq 0$, and suppose that $a_n \to 0$. Let $z \in \mathbb{C}$ such that $|z| = 1$ and $z \neq 1$. Then $\sum a_n z^n$ converges.

Proof. We prove that it is Cauchy. We have

$$\begin{aligned}
\sum_{n=M}^{N} a_n z^n &= \sum_{n=M}^{N} a_n \frac{z^{n+1} - z^n}{z - 1} \\
&= \frac{1}{z-1} \sum_{n=M}^{N} a_n (z^{n+1} - z^n)
\end{aligned}$$

$$= \frac{1}{z-1}\left(\sum_{n=M}^{N} a_n z^{n+1} - \sum_{n=M}^{N} a_n z^n\right)$$
$$= \frac{1}{z-1}\left(\sum_{n=M}^{N} a_n z^{n+1} - \sum_{n=M-1}^{N-1} a_{n+1} z^{n+1}\right)$$
$$= \frac{1}{z-1}\left(a_N z^{N+1} - a_M z^M + \sum_{n=M}^{N-1} (a_n - a_{n+1}) z^{n+1}\right)$$

We now take the absolute value of everything to obtain

$$\begin{aligned}\left|\sum_{n=M}^{N} a_n z^n\right| &\le \frac{1}{|z-1|}\left(a_N + a_M + \sum_{n=M}^{N-1} (a_n - a_{n+1})\right) \\ &= \frac{1}{|z-1|}\left(a_N + a_M + (a_M - a_{M+1}) + \cdots + (a_{N-1} - a_N)\right) \\ &= \frac{2a_M}{|z-1|} \to 0.\end{aligned}$$

So it is Cauchy. So it converges □

Note that here we transformed the sum $\sum a_n(z^{n+1} - z^n)$ into $a_N z^{N+1} - a_M z^M + \sum (a_n - a_{n+1}) z^{n+1}$. What we have effectively done is a discrete analogue of integrating by parts.

Example 3.26

The series $\sum z^n/n$ converges if $|z| < 1$ or if $|z| = 1$ and $z \neq 1$, and it diverges if $z = 1$ or $|z| > 1$.

The cases $|z| < 1$ and $|z| > 1$ are trivial from the ratio test, and Abel's test is required for the $|z| = 1$ cases.