Selberg's Elementary Proof of the Prime Number Theorem

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Abstract

This report provides an exposition of Selberg's proof of the Prime Number Theorem (PNT), $\pi(x) \sim \frac{x}{\log x}$, or rather the equivalent statement $\psi(x) \sim x$. Selberg's proof is remarkable due to it being the first proof of the PNT not requiring any complex analytic techniques. The key is Selberg's asymptotic formula $\psi(x)\log x + \sum_{n \leqslant x} \Lambda(n)\psi\left(\frac{x}{n}\right) = 2x\log x + \mathrm{O}(x)$. A generalisation to prime ideals in a number field, the Prime Ideal Theorem, is also discussed briefly.

1. Introduction

The Prime Number Theorem (PNT) is the pinnacle of classical analytic number theory, and a fundamental result about the distribution of primes. The original proof of the PNT, due to Hadamard and de la Valée Poussin, uses complex analytic techniques and properties of Riemann's ζ-function. It was not known whether an elementary proof was possible (Hardy even thought it to be impossible), until Selberg's proof [Sel49] in 1949; Erdős also published a proof around the same time.

[Apo89, Chapter 4] sketches an outline of Selberg's proof. My aim here is to fill in the technical details and present a complete proof, following very closely the proof given in [HW08]. I also briefly discuss a generalisation to prime ideals in a number field, the so-called Prime Ideal Theorem.

1.1. Acknowledgments

This report is a culmination of a month of working through Apostol's *Introduction to Analytic Number Theory* [Apo89] under the supervision of Professor Satadal Ganguly as part of a summer internship. Thus, I am extremely grateful to Professor Satadal Ganguly for his kind mentorship and guidance throughout the entire program.

2. Preliminaries

2.1. Notation

I use the $\mathrm{O}(\cdot)$, $\mathrm{o}(\cdot)$ and $\sim$ notations. For functions $f\colon \mathbb{R} \to \mathbb{C}$, $g : \mathbb{R} \to [0, \infty)$, $f(x) = \mathrm{O}\big(g(x)\big)$ means $|f| \leqslant Cg$ for some constant C, $f(x) = \mathrm{o}\big(g(x)\big)$ means $|f/g| \to 0$ as $x \to \infty$, and $f(x) \sim g(x)$ means $|f/g| \to 1$ as $x \to \infty$. The notation $\lfloor x \rfloor$ refers to the floor function, the largest integer n such that $n \leqslant x$. Note that I *never* use $\{x\}$ to refer to the fractional part of x.

Finally, I always use the letter p is to denote a prime, and $\mathbb{N}$ to denote the positive integers starting from 1.

2.2. Basic Notions

The PNT is usually stated in terms of the prime counting function $\pi(x)$,

Definition 2.1. The *prime counting function* $\pi(x) : (0, \infty) \to \mathbb{C}$ is given by

$$\pi(x) = \#\left\{p : p \leqslant x\right\}.$$

Theorem 2.2 (Prime Number Theorem)**.**

$$\pi(x) \sim \frac{x}{\log x}.$$

This is the main result that we will prove, and for this we need a few more preliminary definitions and results.

Definition 2.3. The *Euler-Mascheroni* constant is defined as

$$\gamma = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right).$$

Definition 2.4. The *Riemann ζ-function* is defined by

$$\zeta(s) = \begin{cases} \sum_{n=1}^{\infty} \dfrac{1}{n^s}, & \text{if } s > 1 \\ \lim_{x\to\infty} \left(\sum_{n\leqslant x} \dfrac{1}{n^s} - \dfrac{x^{1-s}}{1-s} \right), & \text{if } 0 < s < 1. \end{cases}$$

Definition 2.5. The *von Mangoldt function* $\Lambda : \mathbb{N} \to \mathbb{C}$ is given by

$$\Lambda(n) = \begin{cases} \log(p), & \text{if } n = p^k, \text{ where } k \geqslant 1 \text{ is an integer} \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.6. The *Chebyshev ϑ-function* $\vartheta : (0,\infty) \to \mathbb{C}$ is given by

$$\vartheta(x) = \sum_{p\leqslant x} \log(p).$$

Definition 2.7. The *Chebyshev ψ-function* $\psi : (0,\infty) \to \mathbb{C}$ is given by

$$\psi(x) = \sum_{n\leqslant x} \Lambda(n).$$

We will prove the PNT using Selberg's method, by proving the equivalent PNT in Chebyshev form.

Theorem 2.8 (PNT in Chebyshev form [Apo89, Theorem 4.4])**.** *We have*

$$\pi(x) \sim \frac{x}{\log x} \iff \vartheta(x) \sim x \iff \psi(x) \sim x.$$

The following theorem due to Tatuzawa and Iseki [TI51] will be used to prove Selberg's asymptotic formula ((3.1)).

Theorem 2.9 ([Apo89, Theorem 4.17])**.** *Let* $F : (0,\infty) \to \mathbb{C}$ *and* $G(x) = \log x \sum_{n\leqslant x} F\left(\frac{x}{n}\right)$. *Then*

$$F(x)\log x + \sum_{n\leqslant x} F\left(\frac{x}{n}\right)\Lambda(n) = \sum_{d\leqslant x} \mu(d) G\left(\frac{x}{d}\right).$$

Proof. First write $F(x)\log x$ as a sum

$$F(x)\log x = \sum_{n\leqslant x} \left[\frac{1}{n}\right] F\left(\frac{x}{n}\right) \log\frac{x}{n} = \sum_{n\leqslant x} F\left(\frac{x}{n}\right) \log\frac{x}{n} \sum_{d\mid n} \mu(d)$$

and using the identity $\Lambda(n) = \sum_{d\mid n} \mu(d) \log\frac{n}{d}$ we can write

$$\sum_{n\leqslant x} F\left(\frac{x}{n}\right)\Lambda(n) = \sum_{n\leqslant x} F\left(\frac{x}{n}\right) \sum_{d\mid n} \mu(d)\log\frac{n}{d}.$$

Now adding these two equations we get

$$
\begin{aligned}
F(x)\log x + \sum_{n\leqslant x} F\left(\frac{x}{n}\right)\Lambda(n) &= \sum_{n\leqslant x} F\left(\frac{x}{n}\right)\sum_{d|n}\mu(d)\left\{\log\frac{x}{n} + \log\frac{n}{d}\right\} \\
&= \sum_{n\leqslant x} F\left(\frac{x}{n}\right)\sum_{d|n}\mu(d)\log\frac{x}{d} \\
&= \sum_{n\leqslant x}\sum_{d|n} F\left(\frac{x}{n}\right)\mu(d)\log\frac{x}{d} \\
&= \sum_{q\leqslant x/d}\sum_{d\leqslant x} F\left(\frac{x}{qd}\right)\mu(d)\log\frac{x}{d} \qquad \text{write } n = qd \\
&= \sum_{d\leqslant x}\mu(d)\log\frac{x}{d}\sum_{q\leqslant x/d} F\left(\frac{x}{qd}\right) \\
&= \sum_{d\leqslant x}\mu(d)G\left(\frac{x}{d}\right).
\end{aligned}
$$

□

The following technique called *Abel summmation* will be used several times.

Theorem 2.10 (Abel summmation [Apo89, Theorem 4.2]). *For any arithmetic function $a : \mathbb{N} \to \mathbb{C}$ let*

$$A(x) = \sum_{n\leqslant x} a(n),$$

where $A(x) = 0$ for all $x < 1$. Assume f has a continuous derivative on the interval $[y, x]$, where $0 < y < x$. Then

$$\sum_{y<n\leqslant x} a(n)f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)\mathrm{d}t.$$

Proof. As $A(x)$ is a step function with jump $f(n)$ at each integer n, we can express the sum on the left as a Riemann-Stieltjes integral

$$\sum_{y\leqslant n\leqslant x} a(n)f(n) = \int_y^x f(t)\mathrm{d}A(t).$$

Integrating by parts, we get

$$
\begin{aligned}
\sum_{y\leqslant n\leqslant x} a(n)f(n) &= f(x)A(x) - f(y)A(y) - \int_y^x A(t)\mathrm{d}f(t) \\
&= f(x)A(x) - f(y)A(y) - \int_y^x A(t)f'(t)\mathrm{d}t.
\end{aligned}
$$

In particular, for $y < 1$, we have $\sum_{n\leqslant x} a(n)f(n) = A(x)f(x) - \int_1^x A(t)f'(t)\mathrm{d}t.$ □

The following estimates will also be useful.

Theorem 2.11 ([Apo89, Theorem 3.2(a)]). *If $x \geqslant 1$, then*

$$\sum_{n \leqslant x} \frac{1}{n} = \log x + \gamma + \mathrm{O}\left(\frac{1}{x}\right).$$

Theorem 2.12 ([Apo89, Theorem 3.2(b)]). *If $x \geqslant 1,\ s > 0$ and $s \neq 1$ then*

$$\sum_{n \leqslant x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + \mathrm{O}(x^{-s}).$$

Theorem 2.13 ([Apo89, Theorem 4.9]).

$$\sum_{n \leqslant x} \frac{\Lambda(n)}{n} = \log x + \mathrm{O}(1).$$

Theorem 2.14 ([Apo89, Theorem 4.11]). *For all $x \geqslant 1$ we have*

$$\sum_{n \leqslant x} \psi\left(\frac{x}{n}\right) = x \log x - x + \mathrm{O}(\log x).$$

Theorem 2.15 ([HW08, Theorem 414]).

$$\psi(x) = \mathrm{O}(x).$$

Proof. Using Theorem (2.14), and the fact that $\log x < x$, we get

$$\psi(x) - \psi\left(\frac{x}{2}\right) = x \log x + \mathrm{O}(x) - 2\left(\frac{x}{2} \log \frac{x}{2} + \mathrm{O}(x)\right) = \mathrm{O}(x).$$

So there exists a constant $K > 0$ such that

$$\psi(x) - \psi\left(\frac{x}{2}\right) \leqslant Kx \quad \forall x \geqslant 1.$$

Replacing x successively by $x/2$, $x/4$, $\ldots$ we obtain

$$\psi\left(\frac{x}{2}\right) - \psi\left(\frac{x}{4}\right) \leqslant K\frac{x}{2}$$

$$\psi\left(\frac{x}{4}\right) - \psi\left(\frac{x}{8}\right) \leqslant K\frac{x}{4}$$

and so forth. Note that $\psi(x/2^n) = 0$ when $2^n > x$. Adding these inequalities yields

$$\psi(x) \leqslant Kx\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right) = 2Kx.$$

Hence, $\psi(x) \leqslant Bx$ with $B = 2K$, so $\psi(x) = \mathrm{O}(x)$. $\square$

2.3. Outline of the argument

The key lemma is Selberg's asymptotic formula

$$\psi(x)\log x + \sum_{n\leqslant x}\Lambda(n)\psi\left(\frac{x}{n}\right) = 2x\log x + \mathrm{O}(x).$$

It is natural (it will become clear later why) to define a function

$$\sigma(x) = e^{-x}\psi(e^x) - 1,$$

then Selberg's formula implies the inequality

$$x^2\big|\sigma(x)\big| \leqslant 2\int_0^x\int_0^y\big|\sigma(u)\big|\mathrm{d}u\ \mathrm{d}y + \mathrm{O}(x). \tag{2.1}$$

The PNT is then equivalent to the statement: $\sigma(x)\to 0$ as $x\to\infty$. Hence, if we let

$$C = \limsup_{x\to\infty}\big|\sigma(x)\big|,\ \ K = \limsup_{x\to\infty}\frac{1}{x}\int_0^x\big|\sigma(u)\big|\mathrm{d}u$$

then the PNT is equivalent to showing that $C = 0$. This is proved as follows by assuming towards a contradiction that $C > 0$. From the definition of C and K,

$$\big|\sigma(x)\big| \leqslant C + \mathrm{o}(1),\ \ \big|\sigma(x)\big| \leqslant K + \mathrm{o}(1) \tag{2.2}$$

with $C \leqslant K$. If $C > 0$, then this inequality along with (2.1) yields $K < C$, which is absurd. So $C = 0$.

3. Proof of the main result

3.1. Selberg's asymptotic formula

The following asymptotic formula of Selberg is the key lemma in this proof

Theorem 3.1 (Selberg's theorem). *For $x > 0$ we have*

$$\psi(x)\log x + \sum_{n\leqslant x}\Lambda(n)\psi\left(\frac{x}{n}\right) = 2x\log x + \mathrm{O}(x)$$

and

$$\sum_{n\leqslant x}\Lambda(n)\log n + \sum_{mn\leqslant x}\Lambda(m)\Lambda(n) = 2x\log x + \mathrm{O}(x).$$

Proof. The two statements above are equivalent as

$$\sum_{n\leqslant x}\Lambda(n)\psi\left(\frac{x}{n}\right) = \sum_{n\leqslant x}\Lambda(n)\sum_{m\leqslant x/n}\Lambda(m) = \sum_{mn\leqslant x}\Lambda(m)\Lambda(n)$$

and using Abel summmation with $a(n) = \Lambda(n)$ and $f(t) = \log t$ we get

$$\sum_{n\leqslant x}\Lambda(n)\log n = \psi(x)\log x - \int_2^x\frac{\psi(t)}{t}\mathrm{d}t = \psi(x)\log x + \mathrm{O}(x)$$

using Theorem (2.15). Now, we apply Theorem (2.9) to the functions $F_1 = \psi(x)$ as well as $F_2(x) = x - \gamma - 1$, where γ is the Euler-Mascheroni constant. For F_1, using Theorem (2.14), we have

$$G_1(x) = \log x \sum_{n \leqslant x} \psi\left(\frac{x}{n}\right) = x\log^2 x - x\log x + \mathrm{O}(\log^2 x).$$

For F_2, using Theorem (2.11) we have

$$\begin{aligned}
G_2(x) &= \log x \sum_{n \leqslant x} \left(\frac{x}{n} - \gamma - 1\right) \\
&= x \log x \sum_{n \leqslant x} \frac{1}{n} - (\gamma + 1) \log x \sum_{n \leqslant x} 1 \\
&= x \log x \left(\log x + \gamma + \mathrm{O}\left(\frac{1}{x}\right)\right) - (\gamma + 1)\big(x + \mathrm{O}(1)\big) \log x \\
&= x \log^2 x - x \log x + \mathrm{O}(\log x).
\end{aligned}$$

Hence, $G_1(x) - G_2(x) = \mathrm{O}(\log^2 x)$. Only the weaker estimate $G_1(x) - G_2(x) = \mathrm{O}(\sqrt{x})$ is needed in fact. Now applying Theorem (2.9) to F_1 and F_2 yields

$$\begin{aligned}
&\{F_1(x) - F_2(x)\} \log x + \sum_{n \leqslant x} \left\{F_1\left(\frac{x}{n}\right) - F_2\left(\frac{x}{n}\right)\right\} \Lambda(n) \\
&= \sum_{d \leqslant x} \mu(d) \left\{G_1\left(\frac{x}{d}\right) - G_2\left(\frac{x}{d}\right)\right\} = \mathrm{O}\left(\sum_{d \leqslant x} \sqrt{\frac{x}{d}}\right).
\end{aligned}$$

Then applying Theorem (2.12) to the above yields

$$\{\psi(x) - (x - \gamma - 1)\} \log x + \sum_{n \leqslant x} \left\{\psi\left(\frac{x}{n}\right) - \left(\frac{x}{n} - \gamma - 1\right)\right\} \Lambda(n) = \mathrm{O}\left(\sqrt{x} \sum_{d \leqslant x} \frac{1}{\sqrt{d}}\right) = \mathrm{O}(x).$$

Hence, using Theorem (2.13), we get

$$\begin{aligned}
\psi(x) \log x + \sum_{n \leqslant x} \Lambda(n) \psi\left(\frac{x}{n}\right) &= (x - \gamma - 1) \log x + \sum_{n \leqslant x} \left(\frac{x}{n} - \gamma - 1\right) \Lambda(n) + \mathrm{O}(x) \\
&= x \log x + x \sum_{n \leqslant x} \frac{\Lambda(n)}{n} - (\gamma + 1) \left\{\log x + \sum_{n \leqslant x} \Lambda(n)\right\} + \mathrm{O}(x) \\
&= 2x \log x + \mathrm{O}(1) - 2(\gamma + 1) \log x + \mathrm{O}(x) \\
&= 2x \log x + \mathrm{O}(x)
\end{aligned}$$

where the last step is due to the fact that $\log x < x$. $\square$

3.2. Proof of the PNT in Chebyshev form

We now prove the PNT in Chebyshev form (Theorem (2.8)). Set $\psi(x) = x + R(x)$; the aim is to show that $R(x) = \mathrm{o}(x)$. From Theorem (3.1) we get

$$x \log x + R(x) \log x + \sum_{n \leqslant x} \Lambda(n) \left(\frac{x}{n}\right) + \sum_{n \leqslant x} \Lambda(n) R\left(\frac{x}{n}\right) = 2x \log x + \mathrm{O}(x).$$

Then using Theorem (2.13) we get

$$R(x) \log x + \sum_{n \leqslant x} \Lambda(n) R\left(\frac{x}{n}\right) = \mathrm{O}(x).$$

Replacing n by m, x by x/n

$$R\left(\frac{x}{n}\right) \log\left(\frac{x}{n}\right) + \sum_{m \leqslant x/n} \Lambda(m) R\left(\frac{x}{mn}\right) = \mathrm{O}\left(\frac{x}{n}\right).$$

Hence, using Theorem (2.13) again

$$\begin{aligned}
&\log x \left\{ R(x) \log x + \sum_{n \leqslant x} \Lambda(n) R\left(\frac{x}{n}\right) \right\} \\
&- \sum_{n \leqslant x} \Lambda(n) \left\{ R\left(\frac{x}{n}\right) \log\left(\frac{x}{n}\right) + \sum_{m \leqslant x/n} \Lambda(m) R\left(\frac{x}{mn}\right) \right\} \\
&= \mathrm{O}(x \log x) + \mathrm{O}\left(x \sum_{n \leqslant x} \frac{\Lambda(n)}{n} \right) = \mathrm{O}(x \log x).
\end{aligned}$$

Distributing the first and second terms, and using $\log(x/n) = \log x - \log n$

$$\begin{aligned}
R(x) \log^2 x &= -\sum_{n \leqslant x} \Lambda(n) R\left(\frac{x}{n}\right) \log n \\
&\quad + \sum_{mn \leqslant x} \Lambda(m)\Lambda(n) R\left(\frac{x}{mn}\right) + \mathrm{O}(x \log x) \\
&= -\sum_{n \leqslant x} \Lambda(n) R\left(\frac{x}{n}\right) \log n \\
&\quad + \sum_{n \leqslant x} \sum_{hk=n} \Lambda(h)\Lambda(k) R\left(\frac{x}{hk}\right) + \mathrm{O}(x \log x) \\
\implies |R(x)| \log^2 x &\leqslant \sum_{n \leqslant x} \Lambda(n) \left| R\left(\frac{x}{n}\right) \right| \log n \\
&\quad + \sum_{n \leqslant x} \sum_{hk=n} \Lambda(h)\Lambda(k) \left| R\left(\frac{x}{hk}\right) \right| + \mathrm{O}(x \log x) \\
&\leqslant \sum_{n \leqslant x} \left\{ \Lambda(n) \log n + \sum_{hk=n} \Lambda(h)\Lambda(k) \right\} \left| R\left(\frac{x}{n}\right) \right| + \mathrm{O}(x \log x)
\end{aligned}$$

from where, noting that $\sum\limits_{mn\leqslant x}\Lambda(m)\Lambda(n)R\left|\left(\frac{x}{mn}\right)\right| = \sum\limits_{\ell\leqslant x}\Lambda(m)\Lambda\left(\frac{\ell}{m}\right)R\left|\left(\frac{x}{\ell}\right)\right|$, we get

$$|R(x)|\log^2 x \leqslant \sum_{n\leqslant x} a_n \left|R\left(\frac{x}{n}\right)\right| + \mathrm{O}(x\log x) \tag{3.1}$$

where $a_n = \Lambda(n)\log n + \sum\limits_{hk=n}\Lambda(h)\Lambda(k)$ and $\sum\limits_{n\leqslant x} a_n = 2x\log x + \mathrm{O}(x)$. Now we replace the sum with an integral.

Lemma 3.2.

$$|R(x)|\log^2 x \leqslant 2\int_1^x \left|R\left(\frac{x}{t}\right)\right|\log t\mathrm{d}t + \mathrm{O}(x\log x). \tag{3.2}$$

Proof. If $t > t' \geqslant 0$, and $F(t) := \psi(t) + t = \mathrm{O}(t)$ be an increasing function, then

$$\begin{aligned} ||R(t)| - |R(t')|| &\leqslant |R(t) - R(t')| = |\psi(t) - \psi(t') - (t - t')| \\ &\leqslant \psi(t) - \psi(t') + t - t' = F(t) - F(t') \end{aligned}$$

$$\begin{aligned} \Longrightarrow \sum_{n\leqslant x-1} n\left\{F\left(\frac{x}{n}\right) - F\left(\frac{x}{n+1}\right)\right\} &= \sum_{n\leqslant x} F\left(\frac{x}{n}\right) - [x]F\left(\frac{x}{[x]}\right) \\ &= \mathrm{O}\left(x\sum_{n\leqslant x}\frac{1}{n}\right) = \mathrm{O}(x\log x). \end{aligned}$$

Let $c(1) = 0,\ c(n) = a_n - 2\int_{n-1}^n \log t\mathrm{d}t,\ f(n) = \left|R(\frac{x}{n})\right|,\ C(x) = \sum\limits_{n\leqslant x} c(n),$

then $C(x) = \sum\limits_{n\leqslant x} a_n - 2\int_1^{[x]}\log t\mathrm{d}t = \mathrm{O}(x)$ and using

$$\sum_{n\leqslant x} c(n)f(n) = \sum_{n\leqslant x-1} C(n)\left\{f(n) - f(n+1)\right\} + C(x)f\left([x]\right)$$

we have

$$\begin{aligned} &\sum_{n\leqslant x} a_n\left|R\left(\frac{x}{n}\right)\right| - 2\sum_{2\leqslant n\leqslant x}\left|R\left(\frac{x}{n}\right)\right|\int_{n-1}^n \log t\mathrm{d}t \\ &\qquad = \sum_{n\leqslant x-1} C(n)\left\{\left|R\left(\frac{x}{n}\right)\right| - \left|R\left(\frac{x}{n+1}\right)\right|\right\} + C(x)\left|R\left(\frac{x}{[x]}\right)\right| \\ &\qquad = \mathrm{O}\left(\sum_{n\leqslant x-1} n\left\{F\left(\frac{x}{n}\right) - F\left(\frac{x}{n+1}\right)\right\}\right) + \mathrm{O}(x) \\ &\qquad = \mathrm{O}(x\log x). \end{aligned} \tag{3.3}$$

Now

$$\begin{aligned} &\left|\left|R\left(\frac{x}{n}\right)\right|\int_{n-1}^n \log t\mathrm{d}t - \int_{n-1}^n\left|R\left(\frac{x}{t}\right)\right|\log t\mathrm{d}t\right| \leqslant \int_{n-1}^n\left|\left|R\left(\frac{x}{n}\right)\right| - \left|R\left(\frac{x}{t}\right)\right|\right|\log t\mathrm{d}t \\ &\qquad \leqslant \int_{n-1}^n\left\{F\left(\frac{x}{t}\right) - F\left(\frac{x}{n}\right)\right\}\log t\mathrm{d}t \leqslant (n-1)\left\{F\left(\frac{x}{n-1}\right) - F\left(\frac{x}{n}\right)\right\} \end{aligned}$$

so that

$$\sum_{2\leqslant n\leqslant x}\left|R\left(\frac{x}{n}\right)\right|\int_{n-1}^{n}\log t\mathrm{d}t-\int_{1}^{x}\left|R\left(\frac{x}{t}\right)\right|\log t\mathrm{d}t$$

$$\begin{aligned}&=\mathrm{O}\left(\sum_{n\leqslant x-1}n\left\{F\left(\frac{x}{n}\right)-F\left(\frac{x}{n+1}\right)\right\}\right)+\mathrm{O}(x\log x)\\&=\mathrm{O}(x\log x).\end{aligned}\tag{3.4}$$

Adding the equations (3.3) and 2×(3.4) yields (3.2). □

Hence, we can rewrite (3.1) as

$$\log^2(z)\,|R(z)|\leqslant 2\int_1^z\left|R\left(\frac{z}{t}\right)\right|\log t\mathrm{d}t+\mathrm{O}(z\log z).\tag{3.5}$$

Showing $R(x)=\mathrm{o}(x)$ directly using (3.5) is hard because the behaviour of the von Mangoldt function $\Lambda(n)$ depends on the location of primes which is exactly what we're trying to find. Hence it is natural to define a smoother function $\sigma(x)=e^{-x}R(e^x)=e^{-x}\psi(e^x)-1$. Substitute $z=e^x,\ t=ze^{-u}\implies \mathrm{d}t=-ze^{-u}\mathrm{d}u$. Then for the integral's limits in (3.5), $t=z$ when $u=0$ and $t=1$ when $u=x$. Also $|R(z/t)|=|R(e^u)|=e^u|\sigma(u)|$ and $\log t=x-u$. So the integral becomes simplified as

$$\begin{aligned}\int_1^z\left|R\left(\frac{z}{t}\right)\right|\log t\mathrm{d}t&=-z\int_x^0 e^u|\sigma(u)|(x-u)e^{-u}\mathrm{d}u\\=z\int_0^x|\sigma(u)|(x-u)\mathrm{d}u&=e^x\int_0^x|\sigma(u)|\int_u^x\mathrm{d}y\mathrm{d}u\\\text{(change the order of integration)}\ &=e^x\int_0^x\int_0^y|\sigma(u)|\mathrm{d}u\mathrm{d}y.\end{aligned}$$

Hence we may rewrite (3.5) as the simpler

$$x^2|\sigma(x)|\leqslant 2\int_0^x\int_0^y|\sigma(u)|\mathrm{d}u\mathrm{d}y+\mathrm{O}(x).\tag{3.6}$$

As $\psi(x)=\mathrm{O}(x)$ by Theorem (2.15), by definition $\sigma(x)$ is bounded for large x. So the upper limits

$$C=\limsup_{x\to\infty}|\sigma(x)|\quad\text{and}\quad K=\limsup_{x\to\infty}\frac{1}{x}\int_0^x|\sigma(u)|\mathrm{d}u\tag{3.7}$$

exist. Then

$$\sigma(x)\leqslant C+\mathrm{o}(1)\quad\text{and}\quad\int_0^x|\sigma(u)|\mathrm{d}u\leqslant Kx+\mathrm{o}(x)\tag{3.8}$$

so using (3.6) we get

$$\sigma(x)\leqslant K+\mathrm{o}(1).\tag{3.9}$$

Hence $C\leqslant K$. Now $R(x)=\mathrm{o}(x)\iff\sigma(x)=\mathrm{o}(1)$ so our aim is to show $C=0$. So we assume for contradiction that $C>0$, then show that $K<C$ which is absurd. We need the

following two lemmas.

Lemma 3.3. *There is a fixed* $A_1 > 0$ *such that for all* $x_1, x_2 > 0$ *we have* $\left|\int_{x_1}^{x_2} \sigma(u)\mathrm{d}u\right| < A_1$.

Proof. From Theorem (2.13), and using Abel summmation with $a(n) = \Lambda(n), f(t) = \frac{1}{t}$ we get the estimate $\int_2^z \frac{\psi(t)}{t^2}\mathrm{d}t = \log z + \mathrm{O}(1)$. Substituting $z = e^x, t = e^u$ then yields

$$\int_0^x \sigma(u)\mathrm{d}u = \int_0^x \left\{e^{-u}\psi(e^u) - 1\right\}\mathrm{d}u = \int_1^z \left\{\frac{\psi(t)}{t^2} - \frac{1}{t}\right\}\mathrm{d}t = \mathrm{O}(1).$$

To prove the lemma it suffices to show that the integral is $\mathrm{O}(1)$, so we note that

$$\int_{x_1}^{x_2} \sigma(u)\mathrm{d}u = \int_0^{x_2} - \int_0^{x_1} \sigma(u)\mathrm{d}u = \mathrm{O}(1). \qquad \square$$

Lemma 3.4. *If* $\sigma(u_0) = 0$ *for some* $u_0 > 0$ *then* $\int_0^C |\sigma(u_0 + t)|\,\mathrm{d}t \leqslant \frac{C^2}{2} + \mathrm{O}(u_0^{-1})$.

Proof. We rewrite Selberg's formula (Theorem (3.1)) as

$$\psi(x)\log x + \sum_{mn \leqslant x} \Lambda(m)\Lambda(n) = 2x\log x + \mathrm{O}(x).$$

If $x > x_0 \geqslant 1$, the same holds for x_0 in place of x. Subtracting the two yields

$$\psi(x)\log x - \psi(x_0)\log x_0 + \sum_{x_0 < mn \leqslant x} \Lambda(m)\Lambda(n) = 2\left(x\log x - x_0\log x_0\right) + \mathrm{O}(x).$$

Since $\Lambda(n) \geqslant 0$, we have $0 \leqslant \psi(x)\log x - \psi(x_0)\log x_0 \leqslant 2\left(x\log x - x_0\log x_0\right) + \mathrm{O}(x)$. This implies $|R(x)\log x - R(x_0)\log x_0| \leqslant x\log x - x_0\log x_0 + \mathrm{O}(x)$. Put $x = e^{u_0+t}$, $x_0 = u_0$ so that $R(x_0) = 0$. Then for $0 \leqslant t \leqslant C$, we have

$$\begin{aligned} |\sigma(u_0 + t)| &\leqslant 1 - \left(\frac{u_0}{u_0 + t}\right)e^{-t} + \mathrm{O}\left(\frac{1}{u_0}\right) \\ &= 1 - e^{-t} + \mathrm{O}\left(\frac{1}{u_0}\right) \leqslant t + \mathrm{O}\left(\frac{1}{u_0}\right). \end{aligned}$$

Hence, $\displaystyle\int_0^C |\sigma(u_0 + t)|\mathrm{d}t \leqslant \int_0^C t\mathrm{d}t + \mathrm{O}(u_0^{-1}) = \frac{C^2}{2} + \mathrm{O}(u_0^{-1}).$ $\square$

Now let $\delta = \dfrac{3C^2 + 4A_1}{2C} > C > 0$ and let $y > 0$ be arbitrary. We study the behaviour of $\sigma(u)$ on the interval $[y, y + \delta - C]$. By its definition, $\sigma(u) = e^{-u}\psi(e^u) - 1$ is monotone increasing only at the jump discontinuities $u = \log p^k$ where it increases by $\log p$ and between any two jump discontinuities $\sigma(u)$ decreases monotonically as $\psi(e^u)$ remains constant whilst e^{-u} decreases. This means that either $\sigma(u)$ vanishes at some point $u = u_0$ or $\sigma(u)$ changes sign at most once.

Case I: As $\sigma(u_0) = 0$ for some $u_0 \in [y, y + \delta - C]$, we use (3.8) and Lemma (3.4) to obtain

$$\int_y^{y+\delta} |\sigma(u)|\mathrm{d}u = \int_y^{u_0} + \int_{u_0}^{u_0+C} + \int_{u_0+C}^{y+\delta} |\sigma(u)|\mathrm{d}u$$

$$
\begin{aligned}
&\leqslant C(u_0 - y) + \frac{C^2}{2} + C\big(y + \delta - u_0 - C\big) + \mathrm{o}(1) \\
&= C\left(\delta - \frac{C}{2}\right) = \mathrm{o}(1) = C'\delta + \mathrm{o}(1)
\end{aligned}
$$

for all y sufficiently large, where we took $C' = C\left(1 - \dfrac{C}{2\delta}\right) < C$.

<u>Case II</u>: If $\sigma(u)$ changes sign exactly once at some point $u = u_1 \in [y, y+\delta - C]$, then by Lemma (3.3)

$$
\int_y^{y+\delta-C} |\sigma(u)|\mathrm{d}u = \left|\int_y^{u_1} \sigma(u)\mathrm{d}u\right| + \left|\int_{u_1}^{y+\delta-C} \sigma(u)\mathrm{d}u\right| < 2A_1.
$$

If $\sigma(u)$ does not change sign at all in the interval, then by Lemma (3.3) again

$$
\int_y^{y+\delta-C} |\sigma(u)|\mathrm{d}u = \left|\int_y^{y+\delta-C} \sigma(u)\mathrm{d}u\right| < A_1 < 2A_1.
$$

Hence,

$$
\begin{aligned}
\int_y^{y+\delta} |\sigma(u)|\mathrm{d}u &= \int_y^{y+\delta-C} + \int_{y+\delta-C}^{y+\delta} |\sigma(u)|\mathrm{d}u \\
&< 2A_1 + \int_{y+\delta-C}^{y+\delta} |C + \mathrm{o}(1)|\mathrm{d}u \\
&= 2A_1 + C^2 + \mathrm{o}(1) = C''\delta + \mathrm{o}(1)
\end{aligned}
$$

where we took $C'' = \dfrac{2A_1 + C^2}{\delta} = C\left(\dfrac{4A_1 + 2C^2}{4A_1 + 3C^2}\right) = C\left(1 - \dfrac{C}{2\delta}\right) = C'$.

In both cases we always have

$$
\int_y^{y+\delta} |\sigma(u)|\mathrm{d}u \leqslant C'\delta + \mathrm{o}(1).
$$

If $M = [x/\delta]$, then

$$
\begin{aligned}
\int_0^x |\sigma(u)|\mathrm{d}u &= \sum_{m=0}^{M-1} \int_{m\delta}^{(m+1)\delta} |\sigma(u)|\mathrm{d}u + \int_{M\delta}^{x} |\sigma(u)|\mathrm{d}u \\
&\leqslant C'M\delta + \mathrm{o}(M) + \mathrm{O}(1) = C'x + \mathrm{o}(x).
\end{aligned}
$$

Hence,

$$
K = \limsup_{x\to\infty} \frac{1}{x}\int_0^x |\sigma(u)|\mathrm{d}u \leqslant C' < C.
$$

An absurdity. Hence, we must have $C = 0$. This proves the PNT in Chebyshev form.

4. Further Generalisations

It is possible to generalise Theorem (2.2) to prime ideals in a number field. Let K be a number field and $\pi_K(X)$ denote the number of prime ideals in K of norm at most X,

$$\pi_K(X) = \#\{\mathfrak{p} : \mathrm{N}_K(\mathfrak{p}) \leqslant X\}.$$

In 1903, Landau [Lan03] proved the following with an asymptotic analogous to the PNT

Theorem 4.1 (Prime Ideal Theorem)**.**

$$\pi_K(X) \sim \frac{X}{\log X}.$$

Landau's original proof involves complex analysis and properties of the Riemann ζ-function, but an elementary proof in the spirit of Selberg's proof of the PNT is also possible as shown by Shapiro [Sha49] in 1949.

References

[Apo89] Tom M. Apostol. *Introduction to Analytic Number Theory*. Springer International Student Edition. Springer-Verlag, 1989.

[HW08] G.H. Hardy and E.M. Wright. *An Introduction to the Theory of Numbers*. Oxford Mathematics. Oxford University Press, 2008.

[Lan03] Edmund Landau. "Neuer Beweis des Primzahlsatzes und Beweis des Primidealsatzes". In: *Mathematische Annalen* 56.4 (1903).

[Sel49] Atle Selberg. "An Elementary Proof of the Prime-Number Theorem". In: *Annals of Mathematics* 50.2 (1949).

[Sha49] Harold Shapiro. "An elementary proof of the prime ideal theorem". In: *Communications on Pure and Applied Mathematics* 2.4 (1949).

[TI51] Tikao Tatuzawa and Kanesiroo Iseki. "On Selberg's elementary proof of the prime-number theorem". In: *Proceedings of the Japan Academy, Series A, Mathematical Sciences* 27.7 (1951).

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