Bachelor of Science Examination, 2024

(1st Year, 1st Semester)

Mathematics

Paper : MAJOR – 101

(Real Analysis)

Time : 2 *Hours* *Full Marks* : 40

Use a separate Answer-Script for each Part.

PART—I

(Marks : 20)

Answer ***any five*** questions : (4×5=20)

1. Show that the set [0, 1] is uncountable. 4

2. Let F be an Archimedean ordered field. Show that if F satisfies least upper bound property then F has Cantor's nested interval property. 4

3. Show that interior of a set is the largest open set contained in the set. Find the derived set of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{Q}^c$. 2+2

4. Prove that in $\mathbb{R}$ finite intersection of open sets is open. Give an example to show that arbitrary intersection of open sets may not be open. 2+2

5. Find the closure of the set $\{m + n\sqrt{2} : m, n \in \mathbb{Z}\}$. 4

6. Prove that every closed and bounded set in $\mathbb{R}$ is compact. 4

7. Show that an element x_0 is a limit point of a set S if and only if there exists a sequence $\{x_n\}$ of elements from $S \backslash \{x_0\}$ converging to x_0. 4

8. Prove that the set $S = \{x \in \mathbb{Q} : 2 < x^2 < 3\}$ is both closed and open in $\mathbb{Q}$. Justify whether the set S is compact or not. 2+2

PART—II

(Marks : 20)

Answer ***any four*** questions :

1. *(a)* If the subsequences $\{x_{3n-2}\}$, $\{x_{3n-1}\}$ and $\{x_{3n}\}$ of a sequence $\{x_n\}$ converge to the same limit ℓ, then prove that $\{x_n\}$ converges to ℓ.

   (b) Prove that $n^{1/n} \to 1$ as $n \to \infty$ 3+2

2. Define nests of intervals. For any nest of closed intervals $\{[a_n, b_n]\}$, prove that there exists a unique real number x such that $x \in [a_n, b_n] \forall n$. 1+4

3. Prove that $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ is convergent and

$$\cancel{2 \le \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \le 3.}$$ 5

$$2 < \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n < 3 .$$

4. *(a)* Prove that the sequence $\{u_n\}$ defined by $u_1 = \sqrt{7}$ and $u_{n+1} = \sqrt{7u_n} \forall n \ge 1$, converges to 7.

   (b) If $\{x_n\}$ is a bounded sequence and $\{y_n\}$ converges to 0, then prove that $\{x_n y_n\}$ converges to 0. 3+2

5. *(a)* Let $\sum u_n$ and $\sum v_n$ be two series of positive real numbers and $\lim_{n\to\infty}\frac{u_n}{v_n} = \ell$. If $\ell \ne 0$, then prove that $\sum u_n$ and $\sum v_n$ converge or diverge together.

   (b) If $\sum u_n$ is a convergent series of positive real numbers, then prove that $\sum \frac{u_n}{s+u_n}$ is convergent for any non-zero real number $s > 0$. 3+2

6. (a) Test the convergence of the following series :

$$\sum \frac{a^n}{n}, \ a>0$$

(b) Test the convergence of the following series :

$$\frac{1+2}{2^3}+\frac{1+2+3}{3^3}+\frac{1+2+3+4}{4^3}+\cdots$$ [3+2]

7. Let $\sum a_n$ be a convergent series of positive real numbers.

(i) Prove that the sequence of nth partial sum of the series is bounded above.

(ii) If $\{a_n\}$ is monotonically decreasing sequence, then prove that $n a_n \to 0$ as $n \to \alpha$. [2+3]

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