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A new eighth-order iterative method for solving nonlinear equations

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ABSTRACT

In this paper we present an improvement of the fourth-order Newton-type method for solving a nonlinear equation. The new Newton-type method is shown to converge of the order eight. Per iteration the new method requires three evaluations of the function and one evaluation of its first derivative and therefore the new method has the efficiency index of $\sqrt[4]{8}$, which is better than the well known Newton-type methods of lower order. We shall examine the effectiveness of the new eighth-order Newton-type method by approximating the simple root of a given nonlinear equation. Numerical comparisons are made with several other existing methods to show the performance of the presented method.

1. Introduction

In this paper, we present a new eighth-order iterative method to find a simple root α of the nonlinear equation

$$f(x) = 0, \tag{1}$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ is a scalar function on an open interval I and it is sufficiently smooth in a neighbourhood of α. It is well known that the techniques to solve nonlinear equations have many applications in science and engineering. We shall utilise two well known techniques, namely the classical Newton method for its simplicity with a second-order of convergence [4,5] and the original Newton-type method with fourth-order convergence [3].

The new Newton-type of eighth-order method requires three evaluations of the function and one evaluation of its first derivative. Hence the new Newton-type iterative method has a better efficiency index than the well known Newton-type of fourth-order method [3]. The prime motive for the development of the new Newton-type of eighth-order iterative method was to increase the order of convergence of Newton-type of fourth-order method. Furthermore, in the process we have defined minors of Newton-type methods based on order five to seven. These intermediate methods are constructed to show the gradual improvement of the fourth-order method and for the purpose of this paper establishing the eighth-order iterative method. Consequently, we have found that the new Newton-type of eighth-order is consistent, stable and convergent.

2. Development of the method and analysis of convergence

Let $f(x)$ be a real function with a simple root α and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers that converge towards α. The order of convergence m is given by

$$\lim_{x\to\infty} \frac{x_{n+1} - \alpha}{(x_n - \alpha)^m} = \beta \neq 0, \tag{2}$$

where $m \in \mathbb{R}^+$. Furthermore, the order of convergence m may be approximated by the following formula known as computational order of convergence (COC),

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$$\mathrm{COC} \approx \frac{\ln\left|(x_{n+1}-\alpha)(x_n-\alpha)^{-1}\right|}{\ln\left|(x_n-\alpha)(x_{n-1}-\alpha)^{-1}\right|}, \tag{3}$$

where $n \in \mathbb{N}$. The formula (3) was presented by Weerakoon and Fernando [6] and is widely used [1].

In order to construct the new eighth-order iterative method, we state the essentials of the established fourth-order Newton-type method [3]. Therefore in each of the sub-sections, we shall define the Newton-type methods based on order of convergence and finally the new eighth-order method. To obtain the solution of (1) by the Newton-type methods, we must evaluate the first derivative of (1) and set a particular initial approximation, ideally close to the simple root. We begin the original Newton fourth-order method and then followed by the improved Newton-type methods.

2.1. The Newton-type fourth-order method (N^4)

Since this method is well established [3], we shall state the essential expressions used in order to calculate the root of the given nonlinear equations. Hence the Newton fourth-order method is given as

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{4}$$

$$x_{n+1} = x_n - \frac{f(x_n)^2 + f(y_n)^2}{f'(x_n)(f(x_n) - f(y_n))}, \tag{5}$$

where $n \in \mathbb{N},\ x_0$ is the initial approximation and provided that the denominator of (4) and (5) are not equal to zero. Furthermore, the method given by (5) has many versions; hence we shall state few of them below

$$x_{n+1} = y_n - \left(\frac{1+\mu_n}{1-\mu_n}\right)\left(\frac{f(y_n)}{f'(x_n)}\right), \tag{6}$$

$$x_{n+1} = x_n - \left(\frac{1+\mu_n^2}{1-\mu_n}\right)\left(\frac{f(x_n)}{f'(x_n)}\right), \tag{7}$$

$$\text{where } \mu_n = \frac{f(y_n)}{f(x_n)}. \tag{8}$$

In practice, it is reasonable to choose a formula with least numerical evaluations as possible

Theorem 1. *Assume that the function $f : I \subset \mathbb{R} \to \mathbb{R}$ for an open interval I has a simple root $\alpha \in I$. Let f(x) be sufficiently smooth in the interval I, then the order of convergence of the established method defined by (5) is four and satisfies the following error equation:*

$$e_{n+1} = \left[3c_2^3 - c_2c_3\right]e_n^4 + O\left[e_n^5\right]. \tag{9}$$

2.2. The Newton-type fifth-order method (N^5)

The Newton-type fifth-order method is expressed as

$$z_n = x_n - \frac{f(x_n)^2 + f(y_n)^2}{f'(x_n)(f(x_n) - f(y_n))}, \tag{10}$$

$$x_{n+1} = z_n - (f(z_n))(f'(x_n))^{-1}, \tag{11}$$

where y_n is given by (4), $n \in \mathbb{N},\ x_0$ is the initial approximation and provided that the denominator of (10) and (11) are not equal to zero.

Theorem 2. *Assume that the function $f : I \subset \mathbb{R} \to \mathbb{R}$ for an open interval I has a simple root $\alpha \in I$. Let f(x) be sufficiently smooth in the interval I, then the order of convergence of the method defined by (11) is five and satisfies the following error equation:*

$$e_{n+1} = 2c_2^2\left[3c_2^2 - c_3\right]e_n^5 + O\left[e_n^6\right]. \tag{12}$$

2.3. The Newton-type sixth-order method (N^6)

The Newton-type sixth-order method is expressed as

$$x_{n+1} = z_n - \left(\frac{1+\mu_n^2}{1-\mu_n}\right)^2\left(\frac{f(z_n)}{f'(x_n)}\right), \tag{13}$$

where y_n, μ_n and z_n are given by (4), (8) and (10), respectively, $n \in \mathbb{N},\ x_0$ is the initial approximation and provided that the denominators of (13) are not equal to zero.

Theorem 3. *Assume that the function $f : I \subset \mathbb{R} \to \mathbb{R}$ for an open interval I has a simple root $\alpha \in I$. Let f(x) be sufficiently smooth in the interval I, then the order of convergence of the method defined by (13) is six and satisfies the following error equation:*

$$e_{n+1} = c_2\left[3c_2^4 - 4c_3c_2^2 + c_3^2\right]e_n^6 + O\left[e_n^7\right]. \tag{14}$$

2.4. The Newton-type seventh-order method (N^7)

The Newton-type seventh-order method is expressed as

$$x_{n+1} = z_n - \left[\left(\frac{1+\mu_n^2}{1-\mu_n}\right)^2 - 2(\mu_n)^2 + \frac{f(z_n)}{f(y_n)}\right]\left(\frac{f(z_n)}{f'(x_n)}\right), \tag{15}$$

where y_n, μ_n and z_n are given by (4), (8) and (10) respectively, $n \in \mathbb{N},\ x_0$ is the initial approximation and provided that the denominators of (15) are not equal to zero.

Theorem 4. *Assume that the function $f : I \subset \mathbb{R} \to \mathbb{R}$ for an open interval I has a simple root $\alpha \in I$. Let f(x) be sufficiently smooth in the interval I, then the order of convergence of the method defined by (15) is seven and satisfies the following error equation:*

$$e_{n+1} = 2c_2^2\left[9c_2^4 - 9c_3c_2^2 + 2c_3^2\right]e_n^7 + O\left[e_n^8\right]. \tag{16}$$

2.5. The new Newton-type eighth-order method (N^8)

The Newton-type eighth-order iterative method is expressed as

$$x_{n+1} = z_n - \left[\left(\frac{1+\mu_n^2}{1-\mu_n}\right)^2 - 2(\mu_n)^2 - 6(\mu_n)^3 + \frac{f(z_n)}{f(y_n)} + 4\frac{f(z_n)}{f(x_n)}\right]\left(\frac{f(z_n)}{f'(x_n)}\right), \tag{17}$$

where y_n, μ_n and z_n are given by (4), (8) and (10) respectively, $n \in \mathbb{N},\ x_0$ is the initial approximation and provided that the denominator of (17) are not equal to zero.

For the purpose of this paper and it is essential in numerical analysis to know the behaviour of an approximate method. Therefore, we shall prove the order of convergence of the new eighth-order iterative method.

Theorem 5. *Assume that the function $f : I \subset \mathbb{R} \to \mathbb{R}$ for an open interval I has a simple root $\alpha \in I$. Let f(x) be sufficiently smooth in the interval I, then the order of convergence of the new method defined by (17) is eight and satisfy the following error equation:*

$$e_{n+1} = \left[282c_2^7 + 93c_4c_2^4 + 277c_3^2c_2^3 - 43c_2c_3^3 - 52c_4c_3c_2^2 - 538c_3c_2^5 + 7c_3^2c_4\right]e_n^8 + O[e_n^9]. \tag{18}$$

Proof. Let α be a simple root of $f(x)$, i.e. $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, and the error is expressed as

$$e = x - \alpha. \tag{19}$$

Using Taylor expansion, we have

$$f(x_n) = f(\alpha) + f'(\alpha)e_n + 2^{-1}f''(\alpha)e_n^2 + 6^{-1}f'''(\alpha)e_n^3 + 24^{-1}f^{iv}(\alpha)e_n^4 + \cdots. \tag{20}$$

Taking $f(\alpha) = 0$ and simplifying, expression (20) becomes

$$f(x_n) = f'(\alpha)\left[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + \cdots\right], \tag{21}$$

where $n \in \mathbb{N}$ and

$$c_k = \frac{f^{(k)}(\alpha)}{(k!)f'(\alpha)} \quad \text{for } k = 2, 3, 4, \ldots. \tag{22}$$

Furthermore, we have

$$f'(x_n) = f'(\alpha)\left[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + \cdots\right]. \tag{23}$$

Dividing (20) by (23), we get

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 + \cdots, \tag{24}$$

and hence, we have

$$d_n = e_n - \frac{f(x_n)}{f'(x_n)} = c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 - (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 + \cdots. \tag{25}$$

The expansion of $f(y_n)$ about α is given as

$$f(y_n) = f'(\alpha)\left[d_n + c_2d_n^2 + c_3d_n^3 + c_4d_n^4 + \cdots\right]. \tag{26}$$

Substituting (25) into (26), we obtain

$$f(y_n) = f'(\alpha)\left[c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 - (7c_2c_3 - 5c_2^3 - 3c_4)e_n^4 + \cdots\right]. \tag{27}$$

Since from (10) we have

$$z_n - \alpha = x_n - \alpha - \frac{f(x_n)^2 + f(y_n)^2}{f'(x_n)(f(x_n) - f(y_n))} = c_2(3c_2^2 - c_3)e_n^4 + \cdots. \tag{28}$$

Taylor expansion of $f(z_n)$ about α is

$$f(z_n) = f'(\alpha)\left[(z_n - \alpha) + c_2(z_n - \alpha)^2 + c_3(z_n - \alpha)^3 + \cdots\right]. \tag{29}$$

In order to evaluate the essential terms of (17), we expand term by term

$$\left(\frac{1+\mu_n^2}{1-\mu_n}\right)^2 = 1 + 2c_2e_n - (c_2^2 - 4c_3)e_n^2 - 6(c_2^3 - c_4)e_n^3 + (2c_2c_4 - 38c_2^2c_3 + 25c_2^4 + 4c_3^2)e_n^4 + \cdots \tag{30}$$

$$2(\mu_n)^2 = 2c_2^2e_n^2 - 4c_2(3c_2^2 - 2c_3)e_n^3 + 2(6c_2c_4 - 32c_2^2c_3 + 25c_2^4 + 4c_3^2)e_n^4 + \cdots \tag{31}$$

$$6(\mu_n)^3 = 6c_2^3e_n^3 + (36c_2^2c_3 - 54c_2^4)e_n^4 + \cdots \tag{32}$$

$$4\frac{f(z_n)}{f(x_n)} = 4c_2(3c_2^2 - c_3)e_n^3 + 4(21c_3c_2^2 - 2c_2c_4 - 21c_2^4 - 2c_3^2)e_n^4 + \cdots, \tag{33}$$

and

$$\frac{f(z_n)}{f(y_n)} = (3c_2^2 - c_3)e_n^2 + 2(6c_3c_2 - 6c_2^3 - c_4)e_n^3 - (13c_2c_4 - 74c_2^2c_3 + 39c_2^4 + 31c_3^2 - 7c_4c_3c_2^{-1})e_n^4 + \cdots. \tag{34}$$

Substituting appropriate expressions in (17) and we obtain the error equation

$$e_{n+1} = \left[282c_2^7 + 93c_4c_2^4 + 277c_3^2c_2^3 - 43c_2c_3^3 - 52c_4c_3c_2^2 - 538c_3c_2^5 + 7c_3^2c_4\right]e_n^8 + O[e_n^9]. \tag{35}$$

The error Eq. (35) establishes the eighth-order convergence of the Newton-type method defined by (17). □

3. The Bi, Wu and Ren methods

The four particular eighth-order methods considered are given in [1]. Since these methods are well established, we shall state the essential expressions used in order to calculate the approximate solution of the given nonlinear equations and thus compare the effectiveness of the new iterative eighth-order method. Furthermore, the Bi et al. methods used a familiar divided difference scheme, known as $f[z_n,y_n] = (f(z_n) - f(y_n))/(z_n - y_n)$ and further expansion of this scheme may be found in [1,4].

3.1. The method (M1)

$$z_n = y_n - \left[\frac{2f(x_n) - f(y_n)}{2f(x_n) - 5f(y_n)}\right]\left(\frac{f(y_n)}{f'(x_n)}\right), \tag{36}$$

$$x_{n+1} = z_n - \left[\frac{f(x_n) + (\gamma+2)f(z_n)}{f(x_n) + \gamma f(z_n)}\right]\left(\frac{f(z_n)}{f[z_n,y_n] + f[z_n,x_n,x_n](z_n - y_n)}\right), \tag{37}$$

where $\gamma \in \mathbb{R}$, $f(y_n)$ is given by (4), x_0 is the initial approximation and provided that the denominators of (36) and (37) are not equal to zero.

3.2. The method (M2)

$$z_n = y_n - \left[1 + 2\frac{f(y_n)}{f(x_n)} + 5\left(\frac{f(y_n)}{f(x_n)}\right)^2 + \beta\left(\frac{f(y_n)}{f(x_n)}\right)^3\right]\left(\frac{f(y_n)}{f'(x_n)}\right), \tag{38}$$

$$x_{n+1} = z_n - \left[\frac{f(x_n) + (\gamma+2)f(z_n)}{f(x_n) + \gamma f(z_n)}\right]\left(\frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)}\right), \tag{39}$$

where $\gamma, \beta \in \mathbb{R},\ f(y_n)$ is given by (4), x_0 is the initial approximation and provided that the denominators of (38) and (39) are not equal to zero.

3.3. The method (M3)

$$z_n = y_n - \left[1 - 2\frac{f(y_n)}{f(x_n)} - \left(\frac{f(y_n)}{f(x_n)}\right)^2 + \beta\left(\frac{f(y_n)}{f(x_n)}\right)^3\right]^{-1}\left(\frac{f(y_n)}{f'(x_n)}\right), \tag{40}$$

$$x_{n+1} = z_n - \left[\frac{f(x_n) + (\gamma+2)f(z_n)}{f(x_n) + \gamma f(z_n)}\right]\left(\frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)}\right), \tag{41}$$

where $\gamma, \beta \in \mathbb{R},\ f(y_n)$ is given by (4), x_0 is the initial approximation and provided that the denominators of (40) and (41) are not equal to zero.

3.4. The method (M4)

$$z_n = y_n - \left[\frac{f(x_n) - 3f(y_n)}{f(x_n)}\right]^{\left(\frac{-2}{3}\right)}\left(\frac{f(y_n)}{f'(x_n)}\right), \tag{42}$$

$$x_{n+1} = z_n - \left[\frac{f(x_n) + (\gamma+2)f(z_n)}{f(x_n) + \gamma f(z_n)}\right]\left(\frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)}\right), \tag{43}$$

where $\gamma \in \mathbb{R},\ f(y_n)$ is given by (4), x_0 is the initial approximation and provided that the denominators of (42) and (43) are not equal to zero.

The new Newton-type eighth-order iterative method and the established methods, M1–M4, require a total of four evaluations of the function. We consider the definition of efficiency index [2] as $p^{1/w}$, where p is the order of the method and w is the number of function evaluations per iteration required by the method. Therefore, the new Newton-type eighth-order method given by (15) has the efficiency index of $\sqrt[4]{8} \approx 1.682$ which is better than $\sqrt[4]{7} \approx 1.627$ of the Newton-type seventh-order method, $\sqrt[4]{6} \approx 1.565$ of the Newton-type sixth-order method, $\sqrt[4]{5} \approx 1.495$ of the Newton-type fifth-order method and $\sqrt[3]{4} \approx 1.587$ of the Newton-type fourth-order method.

4. Application of the new eighth-order iterative method

To demonstrate the performance of the Newton eighth-order iterative method, we take four particular nonlinear equations. We shall determine the consistency and stability of results by examining the convergence of the new iterative method. The findings are generalised by illustrating the effectiveness of the Newton-type methods for determining the simple root of a nonlinear equation. The errors listed in the tables are produced by the Newton-type methods of order four to eight, given by (5), (11), (13), (15), (17), respectively and the Bi et al. eighth-order methods M1–M4 are given by (37), (39), (41), (43), respectively. In addition, we apply the formula (3) of the computational order of convergence (COC) to each of the method described and list them in the tables. The numerical computations listed in the tables were performed on an algebraic system called Maple and the errors displayed are of absolute value.

4.1. Numerical example 1

In our first example we shall demonstrate the convergence of the new eighth-order iterative method for the following nonlinear equation

$$f(x) = x^3 + 4x^2 - 10 \tag{44}$$

and the exact value of the simple root of (44) is $\alpha = 1.3652300\ldots$ In Table 1 are the errors obtained by each of the Newton-type methods described, based on the initial approximation $x_0 = 2$, $\beta = 1$ and $\gamma = 1$. We observe that the new Newton-type eighth-order iterative method is converging of the order eight.

4.2. Numerical example 2

In this subsection, we take another different type of nonlinear equation. We shall demonstrate the convergence of new eighth-order iterative method for the following nonlinear equation

Table 1
Errors occurring in the estimates of the root of (44) by the methods described.

Methods	$\lvert x_1-\alpha\rvert$	$\lvert x_2-\alpha\rvert$	$\lvert x_3-\alpha\rvert$	$\lvert x_4-\alpha\rvert$	COC
(5)	0.153e−1	0.171e−7	0.276e−31	0.187e−126	4.0000
(11)	0.620e−2	0.286e−11	0.612e−58	0.272e−291	5.0001
(13)	0.134e−2	0.343e−18	0.955e−112	0.442e−673	5.9999
(15)	0.894e−3	0.865e−22	0.692e−155	0.145e−1086	7.0000
(17)	0.500e−3	0.122e−26	0.149e−215	0.767e−1727	8.0001
(37)	0.221e−6	0.343e−55	0.114e−445	0.172e−3569	8.0000
(39)	0.210e−3	0.172e−30	0.355e−247	0.115e−1980	8.0000
(41)	0.192e−4	0.203e−39	0.320e−319	0.123e−2557	7.9999
(43)	0.998e−5	0.309e−42	0.264e−342	0.751e−2743	8.3170

Table 2
Errors occurring in the estimates of the root of (45), based on the initial approximation $x_0 = -2$.

Methods	$\lvert x_1-\alpha\rvert$	$\lvert x_2-\alpha\rvert$	$\lvert x_3-\alpha\rvert$	$\lvert x_4-\alpha\rvert$	COC
(5)	0.400	0.706e−1	0.152e−3	0.381e−14	3.974
(11)	0.365	0.374e−1	0.139e−5	0.110e−27	4.988
(13)	0.283	0.390e−2	0.776e−14	0.397e−84	6.007
(15)	0.278	0.300e−2	0.130e−15	0.368e−109	7.000
(17)	0.265	0.164e−2	0.139e−19	0.375e−156	7.999
(37)	0.434	0.338e−1	0.106e−9	0.980e−78	7.999
(39)	1.944	2.378	–	–	–
(41)	0.103	0.885e−6	0.409e−46	0.851e−369	8.000
(43)	0.486	0.140	0.322e−4	0.487e−33	7.793

Table 3
Errors occurring in the estimates of the root of (45), based on the initial approximation $x_0 = -1$.

Methods	$\lvert x_1-\alpha\rvert$	$\lvert x_2-\alpha\rvert$	$\lvert x_3-\alpha\rvert$	$\lvert x_4-\alpha\rvert$	COC
(5)	0.215e−1	0.147e−5	0.335e−22	0.901e−89	4.0000
(11)	0.175e−1	0.370e−7	0.150e−35	0.164e−177	5.0000
(13)	0.542e−2	0.316e−13	0.181e−80	0.640e−484	6.0000
(15)	0.114e−2	0.146e−18	0.851e−130	0.192e−908	7.0000
(17)	0.281e−2	0.104e−17	0.361e−141	0.771e−1129	8.0001
(37)	0.225e−3	0.407e−27	0.478e−217	0.172e−1736	8.0000
(39)	0.475e−1	0.190e−8	0.691e−67	0.215e−534	8.0000
(41)	0.160e−2	0.464e−20	0.235e−160	0.103e−1282	7.9999
(43)	0.752e−4	0.384e−31	0.176e−249	0.352e−1996	8.3170

Table 4
Errors occurring in the estimates of the root of (46) by the methods described.

Methods	$\lvert x_1-\alpha\rvert$	$\lvert x_2-\alpha\rvert$	$\lvert x_3-\alpha\rvert$	$\lvert x_4-\alpha\rvert$	COC
(5)	0.160e−1	0.251e−9	0.223e−48	0.124e−243	4.9951
(11)	0.199e−2	0.274e−19	0.257e−137	0.163e−963	6.9992
(13)	0.139e−2	0.754e−21	0.103e−148	0.909e−1044	6.9999
(15)	0.256e−3	0.122e−24	0.773e−181	0.291e−1281	7.0450
(17)	0.663e−3	0.621e−30	0.339e−273	0.119e−2346	8.5233
(37)	0.187e−3	0.385e−38	0.529e−385	0.910e−3083	7.7776
(39)	0.946e−4	0.424e−41	0.138e−414	0.193e−3319	7.7776
(41)	0.175e−3	0.199e−38	0.725e−388	0.113e−3105	7.7777
(43)	0.192e−3	0.506e−38	0.807e−384	0.268e−3073	7.7777

$$f(x) = xe^{x^2} - \sin(x)^2 + 3\cos(x) + 5 \tag{45}$$

and the exact value of the simple root of (45) is $\alpha = -1.207648\ldots$ In Tables 2 and 3 are the errors obtained by each of the methods described, $\beta = 1$ and $\gamma = 1$. Here also, we observe that the new Newton-type eighth-order method is converging of the order eight. In this particular example, we have found that M2 is diverging for an initial approximation of $x_0 = -2$, therefore inappropriate results have been omitted in Table 2.

Table 5
Errors occurring in the estimates of the root of (47) based on the initial approximation $x_0 = 1.7$.

Methods	$\|x_1 - \alpha\|$	$\|x_2 - \alpha\|$	$\|x_3 - \alpha\|$	$\|x_4 - \alpha\|$	COC
(5)	0.414e−1	0.970e−4	0.440e−14	0.186e−55	4.0000
(11)	0.286e−1	0.348e−5	0.136e−24	0.123e−121	5.0000
(13)	0.138e−1	0.156e−8	0.291e−50	0.125e−300	6.0000
(15)	0.122e−1	0.139e−9	0.410e−65	0.808e−454	7.0001
(17)	0.995e−2	0.283e−11	0.150e−87	0.914e−698	8.0003
(37)	0.520e−2	0.859e−23	0.482e−181	0.475e−1447	8.0000
(39)	0.620e−2	0.452e−13	0.203e−102	0.329e−817	8.0003
(41)	0.219e−2	0.166e−17	0.167e−138	0.176e−1106	8.0001
(43)	0.823e−2	0.173e−13	0.738e−107	0.798e−854	8.0004

Table 6
Errors occurring in the estimates of the root of (47), based on the initial approximation $x_0 = 2$.

Methods	$\|x_1 - \alpha\|$	$\|x_2 - \alpha\|$	$\|x_3 - \alpha\|$	$\|x_4 - \alpha\|$	COC
(5)	0.178	0.111e−1	0.678e−6	0.105e−22	3.9883
(11)	0.148	0.326e−2	0.946e−10	0.202e−47	4.9974
(13)	0.102	0.147e−3	0.207e−20	0.162e−121	6.0001
(15)	0.970e−1	0.736e−4	0.481e−25	0.244e−173	7.0000
(17)	0.888e−1	0.221e−4	0.203e−32	0.103e−256	7.9999
(37)	0.112	0.540e−5	0.118e−38	0.599e−308	8.0000
(39)	0.534e−1	0.331e−4	0.165e−31	0.632e−250	7.9999
(41)	0.379	2.383	–	–	–
(43)	0.192	0.146e−1	0.156e−11	0.323e−91	7.9927

4.3. Numerical example 3

In our third example we shall demonstrate the convergence of new eighth-order iterative method for a different type of nonlinear equation

$$f(x) = \sin(x)e^{-x} + \ln(1 + x^2) \tag{46}$$

and the exact value of the simple root of (46) is $\alpha = 0$. In Table 4 are the errors obtained by each of the methods described, based on the initial approximation $x_0 = 1$, $\beta = 1$ and $\gamma = 1$. Here also, we observe that the new Newton-type eighth-order method is converging of the order eight.

4.4. Numerical example 4

In the last but not least of the examples, we take another nonlinear equation. We shall demonstrate the convergence of the new eighth-order iterative method for the following nonlinear equation

$$f(x) = x^6 - x^4 - x^3 - 1 \tag{47}$$

and the exact value of the simple root of (47) is $\alpha = 1.40360212\ldots$ In Tables 5–7 are the errors obtained by each of the methods described, based on $\beta = 1$ and $\gamma = 1$. In this particular case, we observe that the new Newton-type eighth-order method is converging of the order eight and for different values of initial approximations. In Table 7, we find that the Bi et al. methods M2, M3 and M4, given by (39), (41) and (43) respectively, are diverging and hence inappropriate results have been omitted. Furthermore, this demonstrates the domain of convergence of each of the methods for different values of initial approximations.

Table 7
Errors occurring in the estimates of the root of (47), based on the initial approximation $x_0 = 3$.

Methods	$\|x_1 - \alpha\|$	$\|x_2 - \alpha\|$	$\|x_3 - \alpha\|$	$\|x_4 - \alpha\|$	$\|x_5 - \alpha\|$	COC
(5)	0.820	0.308	0.453e−1	0.134e−3	0.162e−13	3.924
(11)	0.756	0.233	0.143e−1	0.130e−6	0.991e−32	4.983
(13)	0.645	0.123	0.359e−3	0.439e−18	0.145e−107	6.000
(15)	0.633	0.112	0.157e−3	0.964e−23	0.320e−157	7.000
(17)	0.612	0.095	0.326e−4	0.455e−31	0.666e−246	8.000
(37)	0.452	0.015	0.569e−11	0.180e−86	0.177e−690	8.000
(39)	0.148	–	–	–	–	–
(41)	0.827	0.573	0.239	0.241	0.233	–
(43)	2.475	–	–	–	–	–

5. Remarks and conclusion

In this paper, we have demonstrated the performance of the new eighth-order method, namely the Newton-type eighth-order iterative method. The prime motive of the development of the new eighth-order method was to establish a higher order of convergence method than the existing fourth-order iterative method [3] and the other Newton-type methods based on order five to seven were constructed in the process. Numerical comparisons are made to show the performance of the derived method. We have examined the effectiveness of the new eighth-order iterative method by showing the accuracy of the simple root of a nonlinear equation. The main purpose of demonstrating the new Newton-type method for four types of nonlinear equations was purely to illustrate the accuracy of the approximate solution, the stability of the convergence, the consistency of the results and to determine the efficiency of the new iterative method. In addition, it should be noted that all the methods are performing differently, that is having different domain of convergence, and hence a particular choice of initial approximation is very important. Consequently, like all other iterative methods, the new method has its own domain of validity and in certain circumstances should not be used.

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