PDF and CDF of uniform and gaussian distributions

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I. Q 1.3

Given U is a uniformly distributed random variable over the interval $(0, 1)$, we have the density function $p_U(x)$:

$$p_U(x) = \begin{cases} 1, & x \in (0,1) \\ 0, & otherwise \end{cases} \tag{1}$$

We know that,

$$F_U(x) = \int_{-\infty}^{x} p_U(x)\, dx \tag{2}$$

$\therefore$ We have the following expression for $F_U(x)$:

$$F_U(x) = \begin{cases} 0, & x \in (-\infty,0) \\ x, & x \in (0,1) \\ 1, & x \in (1,\infty) \end{cases} \tag{3}$$

$$\therefore d(F_U(x)) = 1 \times dx \tag{4}$$

$$\tag{5}$$

$$\text{Now,} E(U) = \int_{-\infty}^{\infty} x d(F_U(x)) \tag{6}$$

$$\implies E(U) = \int_{0}^{1} x dx \tag{7}$$

$$\implies E(U) = 0.5 \tag{8}$$

$$E(U^1) = \int_{-\infty}^{\infty} x^2 d(F_u(x)) \tag{9}$$

$$\implies E(U^2) = \int_{0}^{1} x^2 dx \tag{10}$$

$$\therefore E(U^2) = \frac{1}{3} \tag{11}$$

We know that

$$var(U) = E(U^2) - (E(U))^2 \tag{12}$$

$$\implies var(U) = \frac{1}{3} - \frac{1}{4} \tag{13}$$

$$\therefore var(U) = \frac{1}{12} = 0.0825 \tag{14}$$

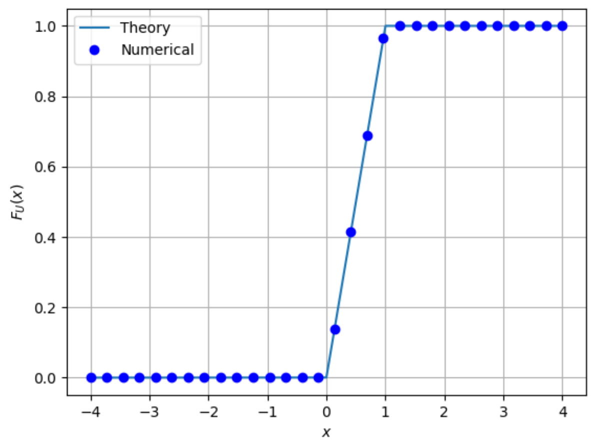

Fig. 1. The CDF of U

using ./code/exrand.c, we get the variance for the uniform distribution as 0.083301 and mean as 0.500007}

II. Q 1.5

We are given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{15}$$

$$\implies E[U^k] = \int_{-\infty}^{\infty} x^k p_U(x)\, dx \tag{16}$$

We know that mean μ is given by $E(U)$. Hence,

$$\mu = \int_{-\infty}^{\infty} x p_U(x)\, dx \tag{17}$$

$$\mu = \int_{0}^{1} x\, dx \tag{18}$$

$$= \frac{x^2}{2}\Big|_0^1 \tag{19}$$

$$= \frac{1}{2} \tag{20}$$

$$var(U) = E((U - E(U))^2) \tag{21}$$

This can also be represented as

$$var(U) = E(U^2 - 2E(U)U + (E(U))^2) \tag{22}$$
$$= E(U^2) - 2(E(U))^2 + (E(U))^2 \tag{23}$$
$$= E(U^2) - (E(U))^2 \tag{24}$$

We can evaluate $E(U^2)$ using (16) as:

$$E(U^2) = \int_{-\infty}^{\infty} x^2 p_U(x)\, dx \tag{25}$$
$$= \int_{0}^{1} x^2\, dx \tag{26}$$
$$= \frac{x^3}{3}\Big|_0^1 \tag{27}$$
$$= \frac{1}{3} \tag{28}$$

Using (20) and (24) we have

$$var(U) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \tag{29}$$

III. Q 2.2

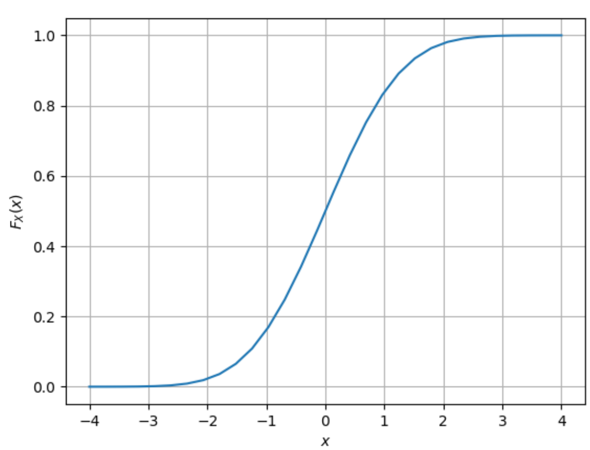

Fig. 2. The CDF of X

IV. Q 2.4

using code ./code/exrand.c, we get the mean as 0.000326 and variance as 0.000906.

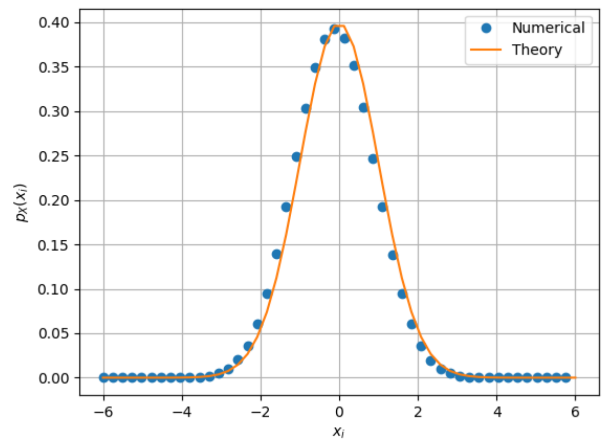

Fig. 3. The PDF of X

V. Q 2.5

For random variable X, we have been given the density function as follows

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp \frac{-x^2}{2} \tag{30}$$
$$\tag{31}$$

For finding μ,

$$\mu = \int_{-\infty}^{\infty} x.p_X(x)dx \tag{32}$$
$$= \int_{-\infty}^{\infty} x.\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \tag{33}$$

As $x.p_X(x)$ is an odd function the above integral is 0, therefore $\mu = 0$

For finding the variance var,

$$var = E(X^2) \tag{34}$$
$$= \int_{-\infty}^{\infty} x^2.\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \tag{35}$$
$$\text{let } x^2 = 2t \tag{36}$$
$$\implies var = \int_{-\infty}^{\infty} \sqrt{2t}.\frac{1}{\sqrt{2\pi}}e^{-t}dt \tag{37}$$
$$= \frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}\right) \tag{38}$$
$$\implies var = 1 \tag{39}$$

VI. Q 3.2

We have been given that random variable V is a function of the random variable U as follows:

$$V = -2\ln(1 - U) \tag{40}$$

Note that the obtained distribution function (CDF) for random variable U is:

$$F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0, 1) \\ 1, & x \in (1, \infty) \end{cases} \tag{41}$$

We know for any random variable X

$$F_X(x) = \Pr(X \leq x) \tag{42}$$

Hence, we can write:

$$\begin{aligned} F_V(x) &= \Pr(V \leq x) && (43) \\ &= \Pr(-2\ln(1-U) \leq x) && (44) \\ &= \Pr(U \leq 1 - \exp\frac{-x}{2}) && (45) \\ &= F_U(1 - \exp\frac{-x}{2}) && (46) \end{aligned}$$

Note that the function $f(x) = 1 - \exp\frac{-x}{2}$ follows:

$$f(x) \in \begin{cases} 0, & x \in (-\infty, 0) \\ (0, 1) & x \in (0, \infty) \end{cases} \tag{47}$$

Hence we can write

$$F_V(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1 - \exp\frac{-x}{2}, & x \in (0, \infty) \end{cases} \tag{48}$$

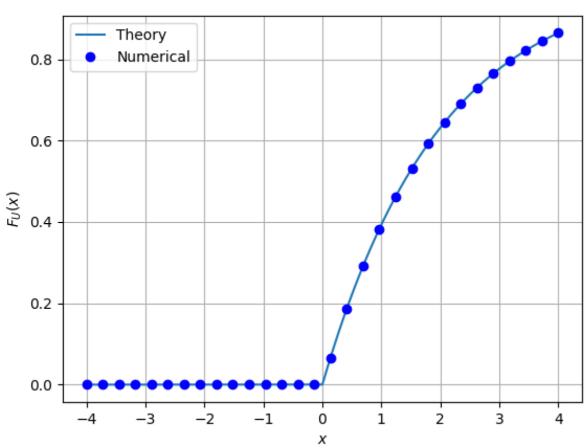

Fig. 4. The CDF of V