PDF and CDF of uniform and gaussian distributions

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1 Q 1.3

$$E(U) = \int_{-\infty}^{\infty} x d(F_u(x)) \tag{1}$$

$$\int_{-\infty}^{\infty} d(F_u(x)) = 1 \tag{2}$$

Given U is a uniformly distributed random variable over the interval $(0, 1)$, we have the following expression for $F_U(x)$:

$$F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0, 1) \\ 1, & x \in (1, \infty) \end{cases} \tag{3}$$

$$\implies d(F_u(x)) = 1 \times dx \tag{4}$$

$$\implies E(U) = \int_0^1 x dx \tag{5}$$

$$\therefore E(U) = 0.5 \tag{6}$$

$$E(U^2) = \int_0^1 x^2 dx \tag{7}$$

$$\implies E(U^2) = \int_{-\infty}^{\infty} x^2 d(F_u(x)) \tag{8}$$

$$\therefore E(U^2) = \frac{1}{3} \tag{9}$$

We know that

$$var(U) = E(U^2) - (E(U))^2 \tag{10}$$

$$\implies var(U) = \frac{1}{3} - \frac{1}{4} \tag{11}$$

$$\therefore var(U) = \frac{1}{12} \tag{12}$$

2 Q 1.5

We are given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{13}$$

$$\implies E[U^k] = \int_{-\infty}^{\infty} x^k p_U(x)\, dx \tag{14}$$

We know that mean μ is given by $E(U)$. Hence,

$$\mu = \int_{-\infty}^{\infty} x p_U(x)\, dx \tag{15}$$

$$\mu = \int_0^1 x\, dx \tag{16}$$

$$= \frac{x^2}{2}\Big|_0^1 \tag{17}$$

$$= \frac{1}{2} \tag{18}$$

$$var(U) = E((U - E(U))^2) \tag{19}$$

This can also be represented as

$$var(U) = E(U^2 - 2E(U)U + (E(U))^2) \tag{20}$$
$$= E(U^2) - 2(E(U))^2 + (E(U))^2 \tag{21}$$
$$= E(U^2) - (E(U))^2 \tag{22}$$

We can evaluate $E(U^2)$ using (14) as:

$$E(U^2) = \int_{-\infty}^{\infty} x^2 p_U(x)\, dx \tag{23}$$
$$= \int_0^1 x^2\, dx \tag{24}$$
$$= \frac{x^3}{3}\Big|_0^1 \tag{25}$$
$$= \frac{1}{3} \tag{26}$$

Using (18) and (22) we have

$$var(U) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \tag{27}$$

3 Q 2.5

For finding μ,

$$\mu = \int_{-\infty}^{\infty} x.p_X(x) dx \tag{28}$$
$$= \int_{-\infty}^{\infty} x.\frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \tag{29}$$

As $x.p_X(x)$ is an odd function the above integral is 0, therefore $\mu = 0$

For finding the var,

$$var = E(X^2) \tag{30}$$
$$= \int_{-\infty}^{\infty} x^2.\frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \tag{31}$$
$$\text{let } x^2 = 2t \tag{32}$$
$$\implies var = \int_{-\infty}^{\infty} \sqrt{2t}.\frac{1}{\sqrt{2\pi}} e^{-t} dt \tag{33}$$
$$= \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) \tag{34}$$
$$\implies var = 1 \tag{35}$$

4 Q 3.2

We have been given that random variable V is a function of the random variable U as follows:

$$V = -2\ln(1 - U) \tag{36}$$

Note that the obtained distribution function (CDF) for random variable U is:

$$F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0, 1) \\ 1, & x \in (1, \infty) \end{cases} \tag{37}$$

We know for any random variable X

$$F_X(x) = \Pr(X \le x) \tag{38}$$

Hence, we can write using (36) and (38)

$$\begin{aligned} F_V(x) &= \Pr(V \leq x) && (39)\\ &= \Pr(-2\ln{(1-U)} \leq x) && (40)\\ &= \Pr(\ln{(1-U)} \geq \frac{-x}{2}) && (41)\\ &= \Pr(1-U \geq \exp{\frac{-x}{2}}) && (42)\\ &= \Pr(U \leq 1-\exp{\frac{-x}{2}}) && (43)\\ &= F_U(1-\exp{\frac{-x}{2}}) && (44) \end{aligned}$$

Note that the function $f(x) = 1 - \exp{\frac{-x}{2}}$ follows:

$$f(x) \in \begin{cases} 0, & x \in (-\infty, 0) \\ (0,1) & x \in (0, \infty) \end{cases} \tag{45}$$

Hence we can write

$$F_V(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1 - \exp{\frac{-x}{2}}, & x \in (0, \infty) \end{cases} \tag{46}$$