

PDF and CDF of uniform and gaussian distributions

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I. Q 1.3

$$E(U) = \int_{-\infty}^{\infty} x d(F_u(x)) \quad (1)$$

$$\int_{-\infty}^{\infty} d(F_u(x)) = 1 \quad (2)$$

Given U is a uniformly distributed random variable over the interval $(0, 1)$, we have the following expression for $F_U(x)$:

$$F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0, 1) \\ 1, & x \in (1, \infty) \end{cases} \quad (3)$$

$$\implies d(F_u(x)) = 1 \times dx \quad (4)$$

$$\implies E(U) = \int_0^1 x dx \quad (5)$$

$$\therefore E(U) = 0.5 \quad (6)$$

$$E(U^2) = \int_0^1 x^2 dx \quad (7)$$

$$\implies E(U^2) = \int_{-\infty}^{\infty} x^2 d(F_u(x)) \quad (8)$$

$$\therefore E(U^2) = \frac{1}{3} \quad (9)$$

We know that

$$var(U) = E(U^2) - (E(U))^2 \quad (10)$$

$$\implies var(U) = \frac{1}{3} - \frac{1}{4} \quad (11)$$

$$\therefore var(U) = \frac{1}{12} = 0.0825 \quad (12)$$

using ./code/exrand.c, we get the variance for the uniform distribution as 0.083301 and mean as 0.500007}

II. Q 1.5

We are given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (13)$$

$$\implies E[U^k] = \int_{-\infty}^{\infty} x^k p_U(x) dx \quad (14)$$

We know that mean μ is given by $E(U)$. Hence,

$$\mu = \int_{-\infty}^{\infty} x p_U(x) dx \quad (15)$$

$$\mu = \int_0^1 x dx \quad (16)$$

$$= \frac{x^2}{2} \Big|_0^1 \quad (17)$$

$$= \frac{1}{2} \quad (18)$$

$$var(U) = E((U - E(U))^2) \quad (19)$$

This can also be represented as

$$var(U) = E(U^2 - 2E(U)U + (E(U))^2) \quad (20)$$

$$= E(U^2) - 2(E(U))^2 + (E(U))^2 \quad (21)$$

$$= E(U^2) - (E(U))^2 \quad (22)$$

We can evaluate $E(U^2)$ using (14) as:

$$E(U^2) = \int_{-\infty}^{\infty} x^2 p_U(x) dx \quad (23)$$

$$= \int_0^1 x^2 dx \quad (24)$$

$$= \frac{x^3}{3} \Big|_0^1 \quad (25)$$

$$= \frac{1}{3} \quad (26)$$

Using (18) and (22) we have

$$var(U) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \quad (27)$$

III. Q 2.5

For finding μ ,

$$\mu = \int_{-\infty}^{\infty} x.p_X(x)dx \quad (28)$$

$$= \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \quad (29)$$

As $x.p_X(x)$ is an odd function the above integral is 0, therefore $\mu = 0$

For finding the var ,

$$var = E(X^2) \quad (30)$$

$$= \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \quad (31)$$

$$\text{let } x^2 = 2t \quad (32)$$

$$\Rightarrow var = \int_{-\infty}^{\infty} \sqrt{2t} \cdot \frac{1}{\sqrt{2\pi}} e^{-t} dt \quad (33)$$

$$= \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) \quad (34)$$

$$\Rightarrow var = 1 \quad (35)$$

using code `./code/exrand.c`, we get the mean as 0.000326 and variance as 0.000906.

Note that the function $f(x) = 1 - \exp \frac{-x}{2}$ follows:

$$f(x) \in \begin{cases} 0, & x \in (-\infty, 0) \\ (0, 1) & x \in (0, \infty) \end{cases} \quad (45)$$

Hence we can write

$$F_V(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1 - \exp \frac{-x}{2}, & x \in (0, \infty) \end{cases} \quad (46)$$

IV. Q 3.2

We have been given that random variable V is a function of the random variable U as follows:

$$V = -2 \ln(1 - U) \quad (36)$$

Note that the obtained distribution function (CDF) for random variable U is:

$$F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0, 1) \\ 1, & x \in (1, \infty) \end{cases} \quad (37)$$

We know for any random variable X

$$F_X(x) = \Pr(X \leq x) \quad (38)$$

Hence, we can write using (36) and (38)

$$F_V(x) = \Pr(V \leq x) \quad (39)$$

$$= \Pr(-2 \ln(1 - U) \leq x) \quad (40)$$

$$= \Pr(\ln(1 - U) \geq \frac{-x}{2}) \quad (41)$$

$$= \Pr(1 - U \geq \exp \frac{-x}{2}) \quad (42)$$

$$= \Pr(U \leq 1 - \exp \frac{-x}{2}) \quad (43)$$

$$= F_U(1 - \exp \frac{-x}{2}) \quad (44)$$