Unit -2 Matrices

⇒ **Definition of Matrix:** A rectangular array of numbers is called a matrix.

- It is a rectangular array of *m* rows and *n* columns enclosed by a pair of brackets is called matrix of order *m* X *n.*
- The matrix is denoted by capital letters like A, B, C, D, etc.

A matrix A of ORDER $m \times n$ can be represented in the following form:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

where a_{ij} is the entry at the intersection of the i^{th} row and j^{th} column.

- Example :-

Here is a matrix of size 2×3 ("2 by 3"), because it has 2 rows and 3 columns:

Let $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 5 & 6 \end{bmatrix}$. Then $a_{11} = 1,\ a_{12} = 3,\ a_{13} = 7,\ a_{21} = 4,\ a_{22} = 5,$ and $a_{23} = 6.$

⇒ **Types of Matrix:**

1. **Column Matrix:** A matrix with only one column is called a column matrix or column vector.
   - Example: A = $\begin{bmatrix} 4 \\ 6 \\ -3 \end{bmatrix}$
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2. **Row Matrix:** A matrix with only one row is called a row matrix or row vector.
   - Example : A = $\begin{bmatrix} 3 & 5 & -6 \end{bmatrix}$

3. **Zero or Null Matrix:** A matrix whose all elements are zero is called as Zero Matrix and order $n \times m$ Zero matrix denoted by $0_{n \times m}$.

- Example :

$$\mathbf{0}_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \mathbf{0}_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

4. **Square Matrix:** A matrix in which the number of rows equal to the number of columns is called a square matrix.

- Example : A = $\begin{bmatrix} 2 & 4 & 7 \\ -5 & 3 & 4 \\ 2 & -4 & 9 \end{bmatrix}$

5. **Diagonal Matrix:** If all elements except diagonal elements of a square matrix are zero, the matrix is called a diagonal matrix.

- Example :

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

6. **Unit Matrix:** If all diagonal elements are unity or one is called a unit or an identity matrix. It is written as I_n.

- Example :

$$I_1 = [1],\ I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

⇒ **Transpose of Matrix:** The matrix obtained from a given matrix A by interchanging rows and columns is called transpose of A and it is denoted by A' or A^T.

- Example:

$$If \quad [A]_{m\times n} \quad then \quad [A^T]_{n\times m}$$

$$[A]=\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\\1 & 0 & 1\end{bmatrix}_{4x3} \; ; \; [A^T]=\begin{bmatrix}1 & 4 & 7 & 1\\2 & 5 & 8 & 0\\3 & 6 & 9 & 1\end{bmatrix}_{3x4}$$

$$Vector \quad [b]=\begin{bmatrix}1 & 2 & 4\end{bmatrix}_{1x3} \; ; \; [b^T]=\begin{bmatrix}1\\2\\3\end{bmatrix}_{3x1}$$

⇒ **Basic Operations on Matrix:**

1. **Addition of two Matrices:** The matrix C = A + B is a matrix each of whose element is the sum of elements at the matching positions of A and B.
    - Example:

$$\begin{bmatrix}A11 & A12\\A21 & A22\end{bmatrix} + \begin{bmatrix}B11 & B12\\B21 & B22\end{bmatrix} = \begin{bmatrix}A11+B11 & A12+B12\\A21+B21 & A22+B22\end{bmatrix}$$

For Example:

$$A = \begin{bmatrix}0 & 1\\4 & 2\end{bmatrix}, \quad B = \begin{bmatrix}2 & 5\\3 & 6\end{bmatrix}$$

$$A + B = \begin{bmatrix}0 & 1\\4 & 2\end{bmatrix} + \begin{bmatrix}2 & 5\\3 & 6\end{bmatrix} = \begin{bmatrix}0+2 & 1+5\\4+3 & 2+6\end{bmatrix}$$

$$A + B = \begin{bmatrix}2 & 6\\7 & 8\end{bmatrix}$$

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2. **<u>Subtraction of two Matrices:</u>** Subtraction of two matrices of the same order can be obtained by subtracting the corresponding elements of these matrices.
    - <u>Example :</u>

$$\begin{bmatrix} A11 & A12 \\ A21 & A22 \end{bmatrix} - \begin{bmatrix} B11 & B12 \\ B21 & B22 \end{bmatrix} = \begin{bmatrix} A11-B11 & A12-B12 \\ A21-B21 & A22-B22 \end{bmatrix}$$

For Example:

$$A = \begin{bmatrix} 0 & 1 \\ 4 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 0 & 1 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 5 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 0-2 & 1-5 \\ 4-3 & 2-6 \end{bmatrix}$$

$$A - B = \begin{bmatrix} -2 & -4 \\ 1 & -4 \end{bmatrix}$$

3. **<u>Multiplication of two Matrices:</u>** We work **across** the 1st row of the first matrix, multiplying **down** the 1st column of the second matrix, element by element. We **add** the resulting products.
    - <u>Example :</u>

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 2 \\ 2 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 2 & 1 & 3 \\ 4 & 5 & 2 \\ 6 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} (1x2 + 3x4+5x6) & (1x1+ 3x5+5x2) & (1x3+3x2+5x3) \\ (2x2 + 4x4+2x6) & (2x1+ 4x5+2x2) & (2x3+4x2+2x3) \\ (2x2 + 5x4+6x6) & (2x1+ 5x5+6x2) & (2x3+5x2+6x3) \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 26 & 24 \\ 32 & 26 & 20 \\ 58 & 39 & 34 \end{bmatrix}$$

⇒ Minor :

Minor of an element a_{ij} = the determinant formed by removing the rows 'i' and column 'j'

Eg: If A =

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 7 \\ 3 & 8 & 0 \end{bmatrix}$$

Minor of 1 =

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 7 \\ 3 & 8 & 0 \end{bmatrix} = \begin{vmatrix} 5 & 7 \\ 8 & 0 \end{vmatrix}$$

$$\text{Minor of } 5 = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 7 \\ 3 & 8 & 0 \end{bmatrix} = \begin{vmatrix} 1 & 4 \\ 3 & 0 \end{vmatrix}$$

⇒ **Adjoint of a Square Matrix:** Adjoint of a square matrix A is the transpose of the matrix of the cofactors of a given matrix A. It is denoted by adj. A.

1. Find the Matrix of Minors
2. Find the Matrix of Co-factors by applying the following signs:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \text{ derived from } (-1)^{(i+j)}$$

3. Transpose the matrix of Cofactors
4. This is the adjoint!

$$A^{-1} = \frac{1}{|A|} \times \text{Adjoint } A$$

- <u>Example : 2 X 2 Matrix</u>

Adjoint of a 2x2 matrix can be found by following the steps given below:

$$B = \begin{bmatrix} 2 & 4 \\ -6 & 8 \end{bmatrix}$$

First, we have to calculate the cofactors of each element. So,

$c_{11}=(-1)^{1+1}8 = 8$

$c_{12}=(-1)^{1+2} (-6)= 6$

$c_{21}=(-1)^{2+1} (4) = -4$

$c_{22}=(-1)^{2+2} (2) = 2$

Now, form a cofactor matrix of B.

$$C = \begin{bmatrix} 8 & 6 \\ -4 & 2 \end{bmatrix}$$

$$\text{adj } B = C^T = \begin{bmatrix} 8 & 6 \\ -4 & 2 \end{bmatrix}^T$$

$$\text{adj } B = \begin{bmatrix} 8 & -4 \\ 6 & 2 \end{bmatrix}$$

- Example : 3 X 3 Matrix

$$A = \begin{bmatrix} 6 & 2 & 1 \\ 1 & 3 & 5 \\ 4 & 4 & 1 \end{bmatrix}$$

For adj A, we follow some steps:

Calculate cofactors of each element

Cofactor of 6 (element of 1st row and 1st column) = $(-1)^{1+1} \begin{vmatrix} 3 & 5 \\ 4 & 1 \end{vmatrix}$

$= 1(3 - 20) = -17$

Cofactor of 2 (element of 1^{st} row and 2^{nd} column) = $(-1)^{1+2} \begin{vmatrix} 1 & 5 \\ 4 & 1 \end{vmatrix}$

$= (-1)(1- 20) =19$

Cofactor of 1 (element of 1^{st} row and 3^{rd} column) = $(-1)^{4} \begin{vmatrix} 1 & 3 \\ 4 & 41 \end{vmatrix}$

$= 1(4 -12) = -8$

cofactor of 1 (element of 2^{nd} row and 1^{st} column) = $(-1)^{3} \begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix}$

$= (-1) (2-4) = 2$

Cofactor of 3 (element of 2^{nd} row and 2^{nd}column) = $(-1)^{4} \begin{vmatrix} 6 & 1 \\ 4 & 1 \end{vmatrix}$

$= 1(6 - 4) = 2$

Cofactor of 5 (element of 2^{nd} row and 3^{rd}column) = $(1-)^5 \begin{vmatrix} 6 & 2 \\ 4 & 4 \end{vmatrix}$

= (-1) (24 - 8) = 16

Cofactor of 4 (element of 3^{rd} row and 1^{st}column) = $(-1)^4 \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix}$

= 1(10 - 3) = 7

Cofactor of 4 (element of 3^{rd} row and 2^{nd} column) = $(-1)^5 \begin{vmatrix} 6 & 1 \\ 1 & 5 \end{vmatrix}$

= (-1) (30 - 1) = -29

Cofactor of 4 (element of 3^{rd} row and 3^{rd}column) = $(-1)^6 \begin{vmatrix} 6 & 2 \\ 1 & 3 \end{vmatrix}$

= 1(18 - 2) = 16

Now, form a cofactor matrix C using these values.

$$C = \begin{bmatrix} -17 & 19 & -8 \\ 2 & 2 & 16 \\ 7 & -29 & 16 \end{bmatrix}$$

Then, **adj B** = $C^T = \begin{bmatrix} -17 & 2 & 7 \\ 19 & 2 & -29 \\ -8 & 16 & 16 \end{bmatrix}$

⇒ **<u>Inverse of a Matrix:</u>** Adjoint of a square matrix A is the transpose of the matrix of the cofactors of a given matrix A. It is denoted by adj. A.

- **<u>Formula :</u>**

$$A^{-1} = \frac{1}{|A|} \cdot \left(Adj\,(A)^{t} \right)$$

- <u>Example : 2 x 2 Matrix</u>

For example, if

$$\text{original matrix} = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}$$

$$\text{determinant value} = 1(8) - 3(2) = 2$$

then

$$\text{inverse} = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}^{-1} = \begin{pmatrix} 8/2 & -2/2 \\ -3/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ -1.5 & 0.5 \end{pmatrix}$$

- <u>Example : 3 x 3 Matrix</u>

$$\begin{pmatrix} 3 & 7 & 5 \\ 2 & 0 & 3 \\ 4 & 1 & 8 \end{pmatrix} = \text{original matrix}$$

Determining the value of the determinant

Value of determinant:

$$\left|\begin{array}{ccc|cc} 3 & 7 & 5 & 3 & 7 \\ 2 & 0 & 3 & 2 & 0 \\ 4 & 1 & 8 & 4 & 1 \end{array}\right.$$

$$\text{value} = 0 + 84 + 10 - 0 - 9 - 112 = -27$$

The inverse is found by dividing each element in the adjoint by -27:

$$\text{inverse} = \begin{pmatrix} {}^{-3}\!/_{-27} & {}^{-51}\!/_{-27} & {}^{21}\!/_{-27} \\ {}^{-4}\!/_{-27} & {}^{4}\!/_{-27} & {}^{1}\!/_{-27} \\ {}^{2}\!/_{-27} & {}^{25}\!/_{-27} & {}^{-14}\!/_{-27} \end{pmatrix}$$

$$= \begin{pmatrix} {}^{3}\!/_{27} & {}^{51}\!/_{27} & {}^{-21}\!/_{27} \\ {}^{4}\!/_{27} & {}^{-4}\!/_{27} & {}^{-1}\!/_{27} \\ {}^{-2}\!/_{27} & {}^{-25}\!/_{27} & {}^{14}\!/_{27} \end{pmatrix}$$

We can verify that this is indeed the correct inverse of the original matrix by multiplying the original matrix times the inverse:

$$\begin{array}{ccccc} \text{original matrix} & \times & \text{inverse} & = & \text{identity matrix} \end{array}$$

$$\begin{pmatrix} 3 & 7 & 5 \\ 2 & 0 & 3 \\ 4 & 1 & 8 \end{pmatrix} \times \begin{pmatrix} {}^{3}\!/_{27} & {}^{51}\!/_{27} & {}^{-21}\!/_{27} \\ {}^{4}\!/_{27} & {}^{-4}\!/_{27} & {}^{-1}\!/_{27} \\ {}^{-2}\!/_{27} & {}^{-25}\!/_{27} & {}^{14}\!/_{27} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
