Independent Project - 1
Mathematical Models in Portfolio Analysis

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Abstract

This project delves into the application of traditional portfolio management techniques, specifically Mean-Variance Analysis and time series analysis, for risk assessment and the construction of optimal investment portfolios. In a landscape marked by market volatility and economic uncertainties, employing robust methodologies to manage risk and maximize returns remains paramount.

The project begins with an in-depth exploration of Mean-Variance Analysis, elucidating its foundational principles and highlighting its efficacy in portfolio optimization. Through this analysis, the project aims to demonstrate the importance of considering both return expectations and risk profiles when constructing investment portfolios.

Subsequently, the project delves into time series analysis, examining its relevance in assessing the historical performance of financial assets and forecasting future trends. By leveraging time series techniques such as autoregressive integrated moving average (ARIMA) models and exponential smoothing, investors can gain valuable insights into asset behavior and make informed decisions regarding portfolio allocation.

We will also be looking at the influence of news and other fundamental substances on stock market prices.

Contents

1 Introduction

When we look at inflation globally, we realize that inflation has risen; and it has affected the very value of the money be it in any state of any country. At the current level of inflation, that is **7.1%** in India, if we apply the rule of 72, then we realize that every ten years the value of money is decreasing by a **factor of 2**. This means that inflation will eat away all our hard-earned money if we just keep it in our bank accounts and not circulate it effectively to beat inflation.

The most effective way to beat inflation is to invest our money in **financial markets**. These financial markets beat inflation every year and; hence result in keeping the value of our money safe. The financial markets involve a lot of components and a mixture of these components can help us according to our needs whether we just want to beat inflation and protect our wealth or our goal is to make generational wealth through this opportunity. As per our choice between the two, we have to choose our **risk** and **return**.

The financial markets have various components such as **NIFTY 50** stocks that give 10% average return per year; the government **Treasury bills** that give a return of only the amount that beats inflation just by a percent or two; we also have riskier asset class of India Stock Markets, which is the **Small-cap companies** which have market cap of less than 5000 crores; There are **corporate bonds** too that have a maturity after 20 years. The major difference between corporate bonds and treasury bills is that treasury bills are **zero coupon bonds** and don't give any interest over the principle we have invested in, hence it is also a short-term asset class for debt. The corporate bonds provide us the interest of whatever the interest rate that we have signed the bond for and hence, these bonds are long-term maturity asset class. We also invest in the markets of other countries of the world to diversify our portfolio so that our risk is minimized. Our investment might also include rare and precious metals such as **gold** and **silver** to hedge our financial markets portfolio as both are negatively correlated.

1.1 Risk and Return

The Expected Return is the weighted average of the possible returns, given that the weights are the probabilities of the percentage return occurring in a particular asset class. The formula for this is as follows:

$E[R]$=Expected Return=$\sum_R p_R * R$

Two common measures of the risk of a probability distribution are its **variance** and **standard deviation**. The variance is the expected squared deviation from the mean, and the standard deviation is the square root of the variance.

Figure 1: Comparison of Nifty 100, Midcap 100 & Smallcap 100 stock returns

$Var(R) = E[(R - E[R])^2] = \sum_R p_R * (R - E[R])^2$

$SD(R) = \sqrt{Var(R)}$

In finance, we refer to the standard deviation of a return as its **volatility**.

Suppose you invest in a stock on date t for price P_t. If the stock pays a dividend, Div_{t+1}, on date t + 1, and you sell the stock at that time for price P_{t+1}, then the realized return from your investment in the stock from t to t + 1 is

$R_{t+1} = \frac{Div_{t+1}+P_{t+1}}{P_t} - 1 = \frac{Div_{t+1}}{P_t} + \frac{P_{t+1}-P_t}{P_t}$

= Dividend Yield + Capital gain rate

The realized return, R_{t+1}, is the total return we earn from dividends and capital gains, expressed as a percentage of the initial stock price.

To focus on the returns of a single security, let's assume that you reinvest all dividends immediately and use them to purchase additional shares of the same stock. For example, if a stock pays dividends at the end of each quarter, with realized returns $R_{Q1}, .., R_{Q4}$ each quarter, then its annual realized return, R_{annual}, is

$R_{annual} = (R_{Q1}+1)(R_{Q2}+1)(R_{Q3}+1)(R_{Q4}+1)-1$

The **Average Annual Return** of an investment during some historical period is simply the average of the realized returns for each year. That is, if R_t is the realized return of a security in year t, then the average annual return for years 1 through T is

$\overline{R} = \frac{R_1+R_2+R_3+...+R_T}{T} = \frac{1}{T}\sum_{i=1}^{T} R_i$

$Var(R) = \frac{1}{T-1}\sum_{i=1}^{T}(R_t - \overline{R})^2$

The **standard error** is the standard deviation of the estimated value of the mean of the actual distribution around its true value; that is, it is the standard deviation of the average return. If the distribution of a stock's return is identical each year, and each year's return is independent of prior years' returns, then we calculate the standard error of the estimate of the expected return as follows:

$SD(Average of Independent, Identical Risks) = \frac{SD(Individual Risks)}{\sqrt{Number of Observations}}$

Because the average return will be within two standard errors of the true expected return approximately 95% of the time, we can use the standard error to determine a reasonable range for the true expected value. The 95% confidence interval for the expected return is

Historical Average $\pm 2 * (Standard Error)$

Comparing historical data for **large portfolios, small stocks have had higher volatility and higher average returns** than large stocks, which have had higher volatility and higher average returns than **bonds**.

There is **no clear relationship** between the **volatility and return of individual stocks**. Larger stocks tend to have lower overall volatility, but even the **largest stocks** are typically **riskier** than a **portfolio of large stocks**. All stocks seem to have higher risk and lower returns than would be predicted based on extrapolation of data for large portfolios.

Total Risk = Idiosyncratic Risk + Systematic Risk

Variation in a stock's return due to firm-specific news is called **idiosyncratic risk**. It is risk that is independent of other shocks in the economy.

Systematic risk is risk due to market-wide news that affects all stocks simultaneously. It is risk that is common to all stocks.

Diversification eliminates **idiosyncratic risk** but does not eliminate **systematic risk**.

$\rightarrow$ Because investors **can eliminate idiosyncratic risk**, they do not require a risk premium for taking it on.

$\rightarrow$ Because investors **cannot eliminate systematic risk**, they must be compensated for holding it.

As a consequence, the risk premium for a stock depends on the **amount of its systematic risk** rather than its total risk.

An **efficient portfolio** contains only systematic risk and cannot be diversified further—that is, there is no way to reduce the risk of the portfolio without lowering its expected return. The **market portfolio** contains all shares of all stocks and securities in the market. The market portfolio is often assumed to be **efficient**. If the market portfolio is efficient, we can measure the systematic risk of a security by its beta **(β)**. The beta of a security is the sensitivity of the return of the security to the return of the overall market.The market risk premium is the expected excess return of the market portfolio:

Market Risk Premium = $E[R_{Mkt}] - r_f$

It reflects investors' overall risk tolerance and represents the market price of risk in the economy. The cost of capital for a risky investment equals the risk-free rate plus a risk premium.

$r_I = r_f + \beta_I * (E[R_{Mkt}] - r_f)$

The difference between a stock's expected return and its required return according to the security market line is the stock's alpha:

$\alpha_s = E[R_s] - r_s$

A stock's beta is equal to the covariance of the stock's returns and its benchmark index's returns over a particular time period, divided by the variance of the index's returns over that period.

$$\beta_s = \frac{Cov(Stock Returns, Indice Return)}{Variance(Indice Return)}$$

Investors should hold the market portfolio (combined with risk-free investments), and this investment advice does not depend on the quality of an investor's information or trading skills. By doing so they can avoid being taken advantage of by more sophisticated investors.

2 Portfolio Optimization

A **portfolio** is a group of financial assets held by investors.

Consider a portfolio consisting of assets $A_1, A_2, ..., A_n$. The **proportion** or **weight** of asset A_k(k=1,2,...,N) in the portfolio equals.

$$\frac{\text{\$ Value invested in asset } A_k}{\text{Total \$ value invested in the portfolio}} \quad (1)$$

We will represent a portfolio of assets A_1,A_2,...,A_n as a column $x = \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_N \end{bmatrix}$

The % return of an investment is treated as a random variable.

The **expected return** from the investment is $\mu = E(r)$

The variance Var(r) and standard deviation Σ_r are used as measures of the risk.

In one year the $ return of the portfolio x equals $\Sigma_{k=1}^{N} r_k v_k$ and the % return of the x equals:

$$\frac{\Sigma_{k=1}^{N} r_k v_k}{\Sigma_{i=1}^{N} v_i} = \Sigma_{k=1}^{N} \frac{v_k}{\Sigma_{i=1}^{N} v_i} = \Sigma_{k=1}^{N} r_k x_k \quad (2)$$

- v_k=$ value of asset A_k in portfolio x;
- r_k= % return of asset A_k in on year

Theorem-1 For any portfolio x of assets A_1,A_2,...A_N,

1. the return $x = \Sigma_{k=1}^{N} x_k r_k$;
2. $x_1 + ... + x_N$=1

Note:- A **short sale** represents the ability to sell of a security that the seller does not own. In this case, the weight of the asset in the portfolio could be negative as well.

2.1 Minimizing Risk

To measure the risk of an investment we will use the standard deviation of its return, since standard deviation expresses the variablility of the return values of the investment.

Given the covariance matrix

$$S = \begin{bmatrix} \sigma_1 & ... & \sigma_{1,n} \\ ... & ... & ... \\ \sigma_{n,1} & ... & \sigma_n \end{bmatrix} \tag{3}$$

Consider any portfolio

$$x = \begin{bmatrix} \gamma \\ 1-\gamma \end{bmatrix} \tag{4}$$

of the assets A_1 and A_2. Its return is

$$x = \gamma r_1 + (1-\gamma) r_2 \tag{5}$$

and its variance is

$$\sigma^2 = \gamma^2 \sigma_1^2 + (1-\gamma)^2 \sigma_2^2 + 2\gamma(1-\gamma)Cov(r_1, r_2). \tag{6}$$

Now, denote $f(\gamma) = \sigma^2$. We need to minimize $f(\gamma)$. The solution will be the value of γ and $1-\gamma$ which has the lowest risk.

3 Statistical Parameters for an N-Asset Portfolio

r_k denotes the return of asset A_k;$\mu_k = E(r_k)$,$\sigma_k^2 = Var(r_k)$ and $\sigma_{kj} = Cov(r_k, r_j)$
U=[1 ... 1] is the row of ones of length N.
$M = [\mu_1 ... \mu_N]$ is the row of the expected asset returns, where not all μ_k are the same. Then the vectors M and U are not proportional.
S is the covariance matrix of the asset returns.

We will assume that all expected returns $\mu_1, \mu 2 ... \mu N$ are defined and the covariance matrix S exists with det S 0. The positivity of the determinant of S is equivalent to the fact that the random variables $r_1, ..r_N$ are linearly independent. This also implies that r vectors are linearly independent as vectors in the Euclidean space H.
Theorem-2

1. $r_1, ..., r_n$ is a basis in K.
2. The dimension of K equals N.

Matrix formulas for portfolios' statistical parameters:
$\mu_x = Mx$
$\sigma_x^2 = x^T S x$
$Cov(x, y) = x^T S y$

3.1 Envelope of Financial Assets

A portfolio is called an **envelope portfolio** if it has the lowest variance among all portfolios with the same expected return
The set of all envelope portfolios is called the envelope of the assets A_1,...,A_N ans is denoted $Env(A_1, ..., A_N)$
A vector x is an envelope portfolio if it is a solution of the following minimization problem:
$Var(x) \rightarrow min$
$x_1 + ... + x_N = 1$
$E(x) = \mu$ for some real number μ.
Based on previous equations,

$$Ux = 1, Mx = \mu \tag{7}$$

This system has a solution, since M and U are not proportional. The corresponding homogeneous system is:

$$Ux = 0, Mx = 0 \tag{8}$$

Denote Q and W the sets of all solutions of the systems (6) and (7), respectively. Q is an affine subspace of K and W is the corresponding linear subspace with dimension N-2. Q can be written as $Q = \{q + w | w \in W\}$ for any solution q of the system (6).
Theorem-3 For any μ, there is a unique envelope portfolio with expected return μ. It equals:

$$x_\mu = q - Proj_w q \tag{9}$$

where q is any solution of (6).

Theorem-4 Let $v_1, \dots, v_{N-2}$ be an orthogonal system of solutions of the system (7). Then

1. for any $y \in H, (y, v_k) = Cov(y, v_k), k = 1, \dots, N-2$;
2. the envelope portfolio with expected return μ equals:

$$x_\mu = q - \Sigma_{k=1}^{N-2} \frac{(q, v_k)}{(v_k, v_k)} v_k, \tag{10}$$

   where q is a solution of (6).
3. the envelope is the set $\{x_\mu | \mu$, where x_μ is given by the previous formula.

4 Mean-Variance Analysis

Principle of Two Fund Separation. Let x and y be two envelope portfolios, x. Then the envelope equals the set $\gamma x + (1-\gamma) y | y \in R$.
Denote $\mu_1 = E(x)$ and $\mu_2 = E(y)$
Denote $q = \gamma q_1 + (1-\gamma) q_2$ and $\mu = \gamma \mu_1 + (1-\gamma)\mu_2$.

Using Theorem 4 ,eq. (6) and eq. (9), we can prove that
$Uq = 1$ and $Mq = \gamma Mq_1 + (1 - \gamma Mq_2) = \gamma\mu_1 + (1 - \gamma)\mu_2 = \mu$.
So $Uq = 1$, $Mq = \mu$ and $z = x_\mu$.
Hence, $z \in Env(A_1, \ldots, A_N)$.

4.1 Efficient Frontier

A portfolio is called an **efficient portfolio** if it has the highest expected return among all portfolios with the same variance.
The set of all efficient portfolios is called the **efficient frontier** of the assets $A_1, \ldots, A_N$ and is denoted $EF(A_1, \ldots, A_N)$.

5 Mean-Variance Relation

Each portfolio is represented by (σ, μ). The Set of all such pairs is called the **feasible set** of the assets.
Theorem-5

1. The envelope is represented on the (σ, μ)-plane by the right branch of the hyperbola:
$$\frac{\sigma^2}{A} - \frac{(\mu - \mu_0)^2}{B} = 1(\sigma > 0) \qquad (11)$$
for some constants $A > 0$,$B > 0$ and $\mu > 0$.
2. For $N \geq 3$, the feasible region is the region of the curve below, including the curve itself.
3. The portfolio x_{min} with lowest risk corresponds to the vertex of the curve, it has the mean μ_0 and the variance A.
4. The efficient frontier is represented on the (σ, μ) plane by the top half of the curve.

Theorem-6.1 Suppose N=2, $\mu_1 \neq \mu_2$ and $\rho \neq \pm 1$. The feasible set and envelope of A_1, A_2 are both represented on the (σ, μ)-plane by the right branch of the hyperbola:
$$\frac{\sigma^2}{A} - \frac{(\mu - \mu_0)^2}{B} = 1(\sigma > 0) \qquad (12)$$
for some constants A¿0,B¿0 and μ_0.

Theorem-6.2 Suppose N=2, $\mu_1 \neq \mu_2$ and $\rho = -1$. The feasible set and envelope $of A_1, A_2$ are both represented on the (σ, μ)- plane by two rays coming out of one point on the vertical μ-axis.

Theorem-6.3 Suppose N=2, $\rho = 1$, $\mu_1 \neq \mu_2$ and $\sigma_1 = \sigma_2$.

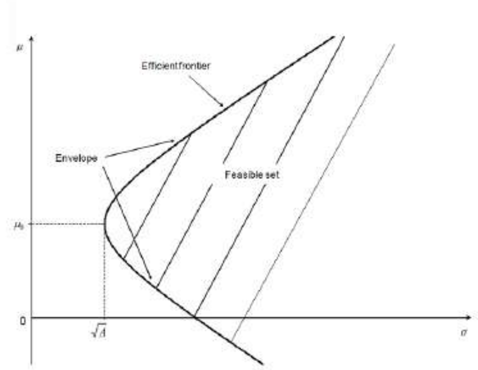
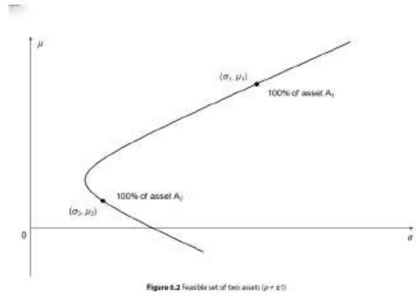


Figure 2: Figure 1



The feasible set and envelope of A_1, A_2 are both represented on the (μ, σ)-plane by the vertical straight line given by $\sigma = \sigma_1$

5.1 Implementation:-

We have taken a bunch of stocks and tried to construct an efficient frontier for those stocks.

6 Econometrics-Time Series Implementation

We have taken daily data of ICICIBANK from 2015-21.
Here is the data projection of the stock.

To do the further implementation, we will first divide the data into test and train data separately.

We will apply certain data transformations to analyze the data more efficiently and make it stationary for fitting it using ARIMA model.

After transforming the data using Log transformation and Lag values operator application, we can

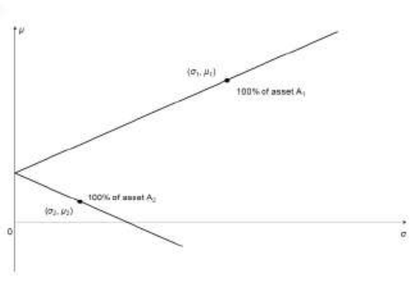


Figure 6.3 Feasible set of two assets ($\rho = -1$)

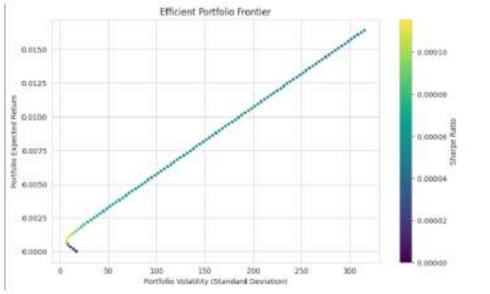


Figure 2: Stocks:-
BANKBARODA,BRITANNIA,GOODYEAR,LICHSFIN,TATASTEEL

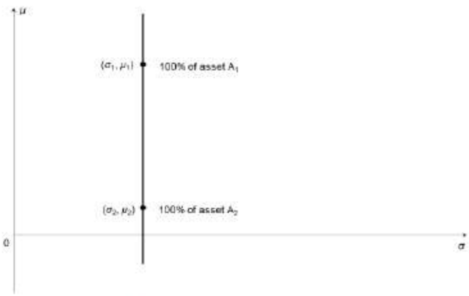
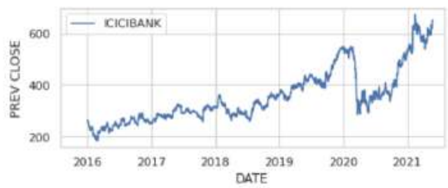


Figure 6.4 Feasible set of two assets ($\rho = 1$ & $\sigma_1 = \sigma_2$)

Data: ICICIBANK 2015-21



finally see that the data is stationary. On this data, we have fit ARIMA model after looking at its Autocorrelation and Partial Autocorrelation Function.

After applying the model, we can see the predicted trend of the data.

6.1 Exponential Smoothing

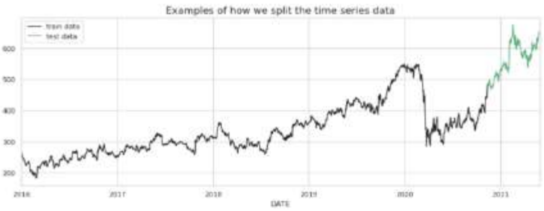
Exponential smoothing is a popular time series forecasting technique used to analyze and predict future data points based on past observations. The method assigns exponentially decreasing weights to past observations, giving more weight to recent data while gradually diminishing the influence of older data.

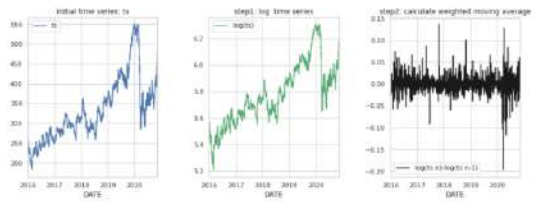
7 The Behaviour of Individual Investors and Systematic Trading Bias

Evidence shows that individual investors fail to diversify their portfolios adequately (**under diversification bias**) and favor investments in companies they are familiar with (**familiarity bias**). Investors appear to trade too much. This behavior stems, at least in part, from investor **overconfidence**: the tendency of uninformed individuals to overestimate the precision of their knowledge. Examples of behavior that could be systematic across investors include the **disposition effect** (the tendency to hang on to losers and sell winners), investor mood swings that result from common events like weather, and putting too much weight on their own experience. Investors could also **herd—actively** trying to follow each other's behavior. Stock prices appear to have more volatility than one would expect.

7.1 Effect of News Pieces on the stock prices

HDFC and HDFC Bank Merger: On **June 30**, the Boards of HDFC Limited, a leading housing finance company, and HDFC Bank, one of India's largest private sector banks, approved the effective date of the merger as **July 1, 2023**, and fixed **July 13, 2023**, as the record date for determining the shareholders of HDFC Ltd. When one company acquires another, the stock price of the acquiring company(Here, **HDCF Bank**) tends to **dip** temporarily, while the stock price of the target company(here HDFC) tends to **spike**. The acquiring company's share price drops because it often pays a premium for the target company, or incurs debt to finance





the acquisition.

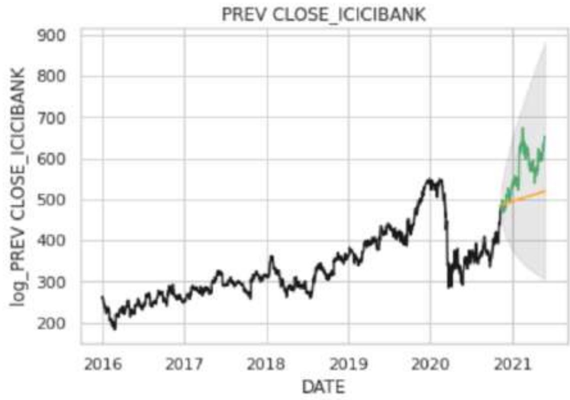
Russia-Ukraine War: Russia declared war on Ukraine on the date of 24th February 2022 and the Indian Stock market saw a dip of around 5% in NIFTY 50.

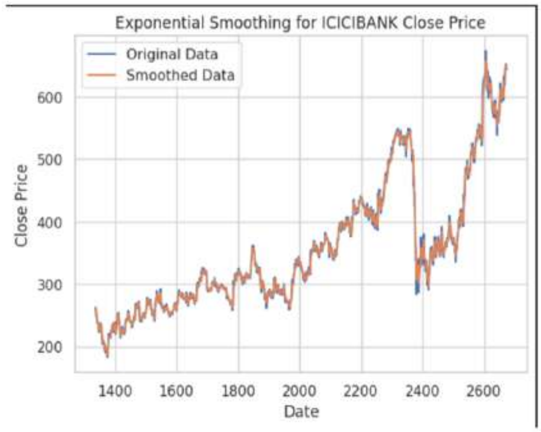
Reliance and Jio Financial Services Demerger: The record date for the Reliance-Jio Financial Services demerger is **July 20, 2023** and the ratio for demerger is set at 1:1. In the long run, demergers tend to have a positive impact on stock price, as a result of the businesses being more strategically focussed with independent management accountability. Usually, when a company demerges its business, it announces a distribution of shares from the new company to its existing investors. This also leads to a temporary **fall** in the price of the company's own stock.

The Market Correction of 20% that happened from October 2021 till June 2022: The cause of this market correction was six consecutive increases in repo rate by RBI of **250 basis points** to control inflation in the economy. The repo rate is the interest rate at which RBI lends money to commercial banks. This rate had increased from **4%** in April 2022 to **6.5%** in February 2023 and now that rate is constant since then. The rise in repo rate directly impacts the stock market and we may see a correction till then.

Dividend Announcements: The news of a dividend being announced **increases** the price of the stock. Once the dividend is distributed, the traders then sell the shares and earn good profits followed by a fall in prices after dividend distribution.

Quarterly Results announcements: In general, strong earnings generally result in the stock price **moving up** (and vice versa). There are some circumstances where although the quarterly results are good but still the stock prices fall because the results are not as much better as the public had expected.





8 Conclusion and Future Scope

In this project, we deeply dived into the aspects of the Markowitz Model of Investing that was mainly dependent on the value of expected return, μ and the variance or we may say the risk associated with

Figure 3: HDFC-HDFC Bank Merger news effect of stock price

Figure 4: Russia-Ukraine war effect on NIFTY 50(systematic risk)

Figure 5: Reliance-Jio Financial Service Demerger

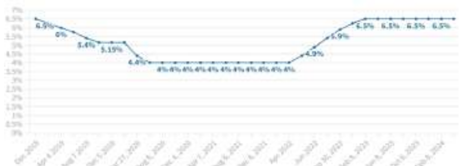


Figure 6: Repo-rate increased in order to control inflation

Figure 7: Repo-rate increased effect on NIFTY 50

Figure 8: ITC announced dividends distribution date

Figure 9: HDFC stock crashed after weak quarterly result despite demerger

the investment, ς. Then, we applied econometrics and time series to forecast the price of a stock. We were only able to take the stocks in NIFTY 50 as we observed that they were the only stocks in the financial markets that had been traded with heavy volumes, and hence they won't be operated or manipulated by any single organization or individual. We also learned about the behavior of an investor and what mistakes are commonly made by the investors. In future, we will focus further upon trading options and futures of indices as well as the individual stocks. We want to dive upon the riskier side of the market and want to learn how to optimize them and bring out marvelous returns out of this risky side. The time series model of GARCH and ARCH shall be used for the same and we will be studying about the same in the next part of project.

9 Challenges faced

The major challenge faced by us was finding the correct database for the implementation of the econometric data analysis. We couldn't apply the stock prediction on the smaller stocks for the very same reason as the data was inadequate. The other challenge faced by us was that of comparing models with different approaches. For example, we weren't able to compare the model of econometric data analysis with that of the Machine Learning model as that of to compare the time complexity, space complexity and accuracy of the model.

10 References

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