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Linear Algebra

→ Defination of vector space :- A non-empty set 'V' together ~~with~~ with the operation of addition and scalar multiplication is called a vector space if it satisfies following properties. For all $u, v, w \in V$ and for all $\alpha, \beta \in R$

(i) $u+v \in V$ } [closure property for addition & scalar multiplication
(ii) $\alpha \cdot u \in V$ }

(iii) $u+v = v+u$ [commutative property for addition]

(iv) $(u+v)+w = u+(v+w)$ [associative property for addition]

(v) There exist an element $0 \in V$ such that for every $u \in V$,
$u+0 = u = 0+u$ [existence of identity element for addition]

(vi) There exist an element $(-u) \in V$ such that for every u
$u+(-u) = 0 = (-u)+u$ [Existence of ~~identity~~ Inverse element for addition]

(vii) $\alpha(u+v) = \alpha u + \alpha v$, [Distributive law of scalar multiplication for addition].

(viii) $(\alpha+\beta)u = \alpha u + \beta u$, [Distributive law for scalar multiplication]

(ix) $(\alpha\beta)u = \alpha(\beta u)$ [associative law for scalar multiplication]

(x) $1 \cdot u = u$ [identity property for scalar multiplication

Ex. Show that R^3 the set of 3 couples of real no. (u_1, u_2, u_3) with usual vector addition & scalar multiplication forms a vector space over R

- let $V = R^3$, let $u, v, w \in V = R^3$. Then $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ & $w = (w_1, w_2, w_3)$

let $\alpha, \beta \in R$

(1) $u+v = (u_1, u_2, u_3) + (v_1, v_2, v_3)$
$= (u_1+v_1, u_2+v_2, u_3+v_3)$
$\in R^3$

$u+v \in V = R^3$

(3) $u+v = (u_1,u_2,u_3)+(v_1,v_2,v_3)$

$= (u_1+v_1, u_2+v_2, u_3+v_3)$

$= (v_1+u_1, v_2+u_2, v_3+u_3)$

$= (v_1,v_2,v_3)+(u_1,u_2,u_3)$

$u+v = v+u$

(4) $u+(v+w) = (u_1,u_2,u_3)+(v_1+w_1, v_2+w_2, v_3+w_3)$

$= (u_1+v_1+w_1, u_2+v_2+w_2, u_3+v_3+w_3)$

$=$

$= (u_1+v_1, u_2+v_2, u_3+v_3)+(w_1,w_2,w_3)$

$= (u+v)+w$.

5) let $(0,0,0) \in R^3$ than for every $u \in R^3$ we have (u_1,u_2,u_3) $+(0,0,0) = (u_1,u_2,u_3)$

∴ i.e. $u+0 = u = 0+u$

6) let $-u \in V = R^3$ than $(-u_1,-u_2,-u_3) \in R^3$.

∴ for every $u \in R^3$ we have

$(u_1,u_2,u_3)+(-u_1,-u_2,-u_3) = (0,0,0)$

$u+(-u) = 0 = (-u)+u$

7) $\alpha.(u+v) = \cancel{\alpha u+\alpha v}\ \alpha(u_1+v_1, u_2+v_2, u_3+v_3)$

$= (\alpha u_1+\alpha v_1, \alpha u_2+\alpha v_2, \alpha u_3+\alpha v_3)$

$= (\alpha u_1, \alpha u_2, \alpha u_3)+(\alpha v_1, \alpha v_2, \alpha v_3)$

$= \alpha(u_1,u_2,u_3)+\alpha(v_1,v_2,v_3)$

$= \alpha u+\alpha v.$

8) $(\alpha+\beta)u = (\alpha+\beta)(u_1,u_2,u_3)$

$= ((\alpha+\beta)u_1, (\alpha+\beta)u_2, (\alpha+\beta)u_3)$

$= (\alpha u_1+\beta u_1, \alpha u_2+\beta u_2, \alpha u_3+\beta u_3)$

$= (\alpha u_1, \alpha u_2, \alpha u_3)+(\beta u_1, \beta u_2, \beta u_3)$

$= \alpha(u_1,u_2,u_3)+\beta(u_1,u_2,u_3)$

$= \alpha u+\beta u$

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9) $(\alpha\beta)u = (\alpha\beta)(u_1, u_2, u_3)$

$= (\alpha\beta u_1, \alpha\beta u_2, \alpha\beta u_3)$

$= \alpha[\beta(u_1, u_2, u_3)]$

$= \alpha(\beta u)$

10) for any $1 \in R$ we have

$1.u = 1(u_1, u_2, u_3)$

$= (u_1, u_2, u_3)$

$= u$

hence R^3 is vector space

Note:- 1) The set R^n of n-tuples of real no.'s $(u_1, u_2, \cdots, u_n)$ is a vector space with usual vector addition & scalar multiplication over R.

2) The set P_n of polynomials of degree $\leq n$ is a vector space with usual addition & scalar multiplication of polynomials.

e.g. P.T. P_4 is a vector space

i.e. $P_n = \{ a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \mid a_0, a_1, \cdots, a_n \text{ in } R \}$

→ check that wether the set M_{33}, the set of all Matrices of 3x3 is a vector space with Matrix addition & scalar multiplication of Matrix.

We are given $V = M_{33}$.

Let $A, B, C \in M_{33}$ than

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

and $\alpha, \beta \in R$

(i) $$A + B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix} \in M_{33}$$

$\therefore A + B \in M_{33}$.

(ii) $$\alpha \cdot A = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \alpha a_{13} \\ \alpha a_{21} & \alpha a_{22} & \alpha a_{23} \\ \alpha a_{31} & \alpha a_{32} & \alpha a_{33} \end{bmatrix}$$

$\therefore \alpha A \in M_{33}$

(iii) $$A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} b_{11}+a_{11} & b_{12}+a_{12} & b_{13}+a_{13} \\ b_{21}+a_{21} & b_{22}+a_{22} & b_{23}+a_{23} \\ b_{31}+a_{31} & b_{32}+a_{32} & b_{33}+a_{33} \end{bmatrix}$$

$A + B = B + A$

(iv) $A + (B + C)$ $$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} b_{11}+c_{11} & b_{12}+c_{12} & b_{13}+c_{13} \\ b_{21}+c_{21} & b_{22}+c_{22} & b_{23}+c_{23} \\ b_{31}+c_{31} & b_{32}+c_{32} & b_{33}+c_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{21} \\ a_{31}+b_{31} & a_{32}+b_{12} & a_{33}+b_{33} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{11} & c_{23} \\ c_{31} & c_{31} & c_{33} \end{bmatrix}$$

$A+(B+C) = (A+B)+C$

(v) let $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in M_{33}$

such that $A+O = A = O+A$

hence O_3 is ~~additive~~ identity element for M_{33}

(vi) let $-A \in M_{33}$

i.e, $-A = \begin{bmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \\ -a_{31} & -a_{32} & -a_{33} \end{bmatrix}$

such that $A+(-A) = O_3 = (-A)+A$

hence $(-A)$ is an inverse element of M_{33}

hence M_{33} is a vector space over R

Note:- The set $M_{m \times n}$ of all matrices of order $m \times n$ is a vector space with usual matrix addition and scalar multiplication

→ Check weather the set R^+ the set of all positive real no's with usual vector addition & scalar multi. form a vector space.

- We know that R^+ is a set of all positive Real no.

∴ $0 \notin R^+$ hence the set R^+ has no any identity for addition. Hence R^+ is not a vector space

→ determine weather the set R^+ of all +ve real no. with the operation $x+y = xy$ and $kx = x^k$, where k is any scalar is a vector space or not.

Q- We are given $V = R^+$ with $x+y = xy$ & $kx = x^k$

let $x, y, z \in R^+$ & $\alpha, \beta \in R$

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(i) $x+y = xy \in R^+$ has x, y both are +ve real no.,

$\therefore x+y \in R^+$

(ii) $ax = x^{a} \in R^+$

$\therefore a x \in R^+$

(iii) $x+y = xy = yx = y+x$

$\therefore x+y = y+x$

(iv) $x+(y+z) = x(y+z)$

$= x(yz)$

$= (xy)z$

$= (x+y)z$

$= (x+y)+z$

(v) let $a \in R^+$ such that $x+a = x$

$x \cdot a = x$

$a = 1$

$a + x = x$

hence '1' is the identity element for R^+ with given operations.

(vi) let $a \in R^+$ such that $x+a = 1$

$xa = 1$

$a = 1/x$

$a + x = 1$

hence '$1/x$' is the inverse for R^+ with given operation

(vii) $\alpha(x+y) = \alpha(x \cdot y)$

$= (xy)^{\alpha}$

$= x^{\alpha} y^{\alpha}$

$= (\alpha x)(\alpha y)$ $= (\alpha x + \alpha y)$

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(viii) $(\alpha+\beta)x = x^{(\alpha+\beta)}$

$= x^{\alpha} x^{\beta}$

$= (\alpha x)(\beta x)$

$= (\alpha x + \beta x)$

(ix) $(\alpha\beta)x = x^{\alpha\beta}$

$= x^{\beta\alpha}$

$= (x^{\beta})^{\alpha}$

$= (\beta x)^{\alpha}$

$= \alpha(\beta x)$

(x) $(1)x = x^1 = x$

hence $V = R^+$ is a vector space with given operations

→ $P_4 = \{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \mid a_i \in R, 0 \le i \le 4\}$

$= \{\sum_{k=1}^{4} a_k x^k \mid a_i \in R, 0 \le i \le 4\}$

= The set of all polynomial of degree less than or equal to 4 is a vector space, with the operation defined by

for $p(x) = \sum_{k=0}^{4} a_k x^k$ & $q(x) = \sum_{k=0}^{4} b_k x^k$

① $p(x) + q(x) = \sum_{k=0}^{4} (a_k + b_k) x^k$

② $\alpha\, p(x) = \sum_{k=0}^{4} (\alpha\, a_k) x^k$

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→ ① $p(x) + q(x) = \sum_{k=0}^{4} (a_k + b_k) x^k$ (

$\in P_4$ ($\because a_i + b_i \in R$ as $a_i, b_i \in R$
$0 \le i \le 4$)

② $\alpha \cdot p(x) = \sum_{k=0}^{4} (\alpha a_k) x^k$

$\in P_4$ ($\because a_k, \alpha \in R \Rightarrow \alpha a_k \in R$
$0 \le k \le 4$)

③ $p(x) + q(x) = \sum_{k=0}^{4} (a_k + b_k) x^k$

$= \sum_{k=0}^{4} (b_k + a_k) x^k$ ($\because a_k, b_k \in R$ & $+$ is R is commute)

$= q(x) + p(x)$

'+' is commutative in P_4

④ $p(x) + (q(x) + r(x)) = p(x) + \sum_{k=0}^{4} (b_k + c_k) x^k$

$= \sum_{k=0}^{4} a_k x^k + \left(\sum_{k=0}^{4} (b_k + c_k) x^k \right)$ ($\because b_k, c_k \in R$)

$= \sum_{k=0}^{4} (a_k + (b_k + c_k)) x^k$ ($\because a_k, b_k, c_k \in R$)

$= (p(x) + q(x)) + r(x)$. ~~...~~

∴ '+' is associative in P_4

$= \sum_{k=0}^{4} [(a_k + b_k) + c_k] x^k$

$= \sum_{k=0}^{k} \cancel{[a_k + (b_k + c_k)] x^k}$

5) for $p(x) = \sum_{k=0}^{3}$

* Sub-space

- A non-empty subset 'S' of a vector space 'V' is said to be a subspace of V, if the set S itself is a vector space under the operation defined on V.

Note:- Every vector space has atleast two sub-spaces, vector space itself and $\{0\}$.

here the subspace $\{0\}$ is called the zero subspace containing only the zero as the element

* Condition to check subspace:-

Thm:- A non-empty subset 'S' of a vector space V is a sub-space of V, if an only if

(i) $u+v \in S$, for all $u, v \in S$

(ii) $\alpha u \in S$, for all $u \in S$ & $\alpha \in R$

i.e, closure property for addition & scalar multiplication in S

Ex Show that $\{(x,y) / x = 3y\}$ is a subspace of R^2.

- We are given that $S = \{(x,y) / x = 3y\}$

let $u = (x_1, y_1)$ & $v = (x_2, y_2)$ be in S such that $x_1 = 3y_1$ & $x_2 = 3y_2$

Then $u+v = (x_1, y_1) + (x_2, y_2)$

$= (x_1 + x_2, y_1 + y_2)$

But here $x_1 = 3y_1$, $x_2 = 3y_2$

$x_1 + x_2 = 3y_1 + 3y_2$

$x_1 + x_2 = 3(y_1 + y_2)$

$\therefore$ $u+v = \{x_1 + x_2, y_1 + y_2 / x_1 + x_2 = 3(y_1 + y_2)\} \in S$

- for any $\alpha \in R$, we have

$\alpha u = \alpha(x_1, y_1)$

$= (\alpha x_1, \alpha y_1)$

But here $x_1 = 3y_1$

$\alpha x_1 = \alpha 3 y_1$

$= 3(\alpha y_1)$

$\therefore \alpha . u = \{\alpha x_1, \alpha y_1 / \alpha x_1 = 3\alpha y_1\} \in S$, $\alpha u \in S$.

hence S is a subspace of R^2

* Check whether the following sets are subspaces of the respective vector space under the standard operation of ...

- $S = \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 / a_0 = 0\}$, $V = P_3$.

we are given $S = \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 / a_0 = 0\}$ &

$g(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$. be in S. such that

$a_0 = 0$ & $b_0 = 0$

- Then $f(x) + g(x) = (a_0 + a_1 x + a_2 x^2 + a_3 x^3) + (b_0 + b_1 x + b_2 x^2 + b_3 x^3)$

$= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3$

but here $a_0 = 0$ & $b_0 = 0$

$\therefore$ addition of both of them is $a_0 + b_0 = 0$

$\therefore f(x) + g(x) = \{(a_0 + b_0) + (a_1 + b_1)x, (a_2 + b_2)x^2, (a_3 + b_3)x^3 / a_0 + b_0 = 0\}$

$f(x) + g(x) \in S$

→ for any $\alpha \in R$, we have

$\alpha f(x) = \alpha a_0 + \alpha a_1 x + \alpha a_2 x^2 + \alpha a_3 x^3$.

but $a_0 = 0$

$\therefore \alpha a_0 = 0$.

$\therefore \alpha . f(x) = \{\alpha a_0 + \alpha a_1 x + \alpha a_2 x^2 + \alpha a_3 x^3 / \alpha a_0 = 0\}$

$\alpha f(x) \in S$.

hence S is a subspace of P_3

→ $S = \{(x,y) / x^2 = y^2\}$, $V = R^2$.

— let $S = \{(x,y) / x^2 = y^2\}$

let $u = (x_1, y_1)$ & $v = (x_2, y_2)$ be in S such that

$x_1^2 = y_1^2$ & $x_2^2 = y_2^2$

Then $u+v = (x_1, y_1) + (x_2, y_2)$

$= (x_1 + x_2, y_1 + y_2)$

~~but here~~ $x_1^2 = y_1^2$ & $x_2^2 = y_2^2$

& $x_1^2 + x_2^2 = y_1^2 + y_2^2$

∴ here we can never get $(x_1+x_2)^2 = (y_1+y_2)^2$, i.e. here $(x_1+x_2)^2 \neq (y_1+y_2)^2$

∴ $u+v \notin S$. hence S is not a subspace of R^2

e.g. $u = (-1, 1)$ & $v = (2, 2)$

$u+v = (1, 3)$

$1^2 \neq 3^2$

$u+v \notin S$

→ $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} / a+b+c+d = 0 \right\}$, $V = M_{22}$.

let $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ & $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ in S, E

$a_1 + b_1 + c_1 + d_1 = 0$ & $a_2 + b_2 + c_2 + d_2 = 0$

$A+B = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$

~~but here~~ $a_1+b_1+c_1+d_1 = 0$ & $a_2+b_2+c_2+d_2 = 0$

& $a_1+b_1+c_1+d_1 + a_2+b_2+c_2+d_2 = 0$

$(a_1+a_2) + (b_1+b_2) + (c_1+c_2) + (d_1+d_2) = 0$

∴ $a+b = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}$, E

$(a_1+a_2) + (b_1+b_2)$ ——— $= 0$

∴ $A+B \in S$

for any $\alpha \in R$,

$$\alpha A = \begin{bmatrix} \alpha a_1 & \alpha b_1 \\ \alpha c_1 & \alpha d_1 \end{bmatrix}$$

$\therefore$ $a_1 + b_1 + c_1 + d_1 = 0$

$\alpha a_1 + \alpha b_1 + \alpha c_1 + \alpha d_1 = 0$

$\therefore$ $\alpha A \in S$

hence S is a subspace of M_{22}.

→ $S = \{(x, y, z) \,/\, x^2 + y^2 + z^2 \leq 1\}$, $V = R^3$.

let $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$

such that $x_1^2 + y_1^2 + z_1^2 \leq 1$ & $x_2^2 + y_2^2 + z_2^2 \leq 1$

$\therefore$ $u + v = (x_1^2 + x_2^2 + y_1^2 + y_2^2 + z_1^2 + z_2^2)$

$u + v \notin S$

e.g. $u = (1, 0, 0)$, $v = (0, 1, 0)$

$\therefore$ $u + v = 2 \neq 1$

→ $S = \{A_{nn} \,/\, AB = BA$ for fixed $B_{nn}\}$ $V = M_{nn}$

let A_1 & A_2 be in S such that

$A_1 B = B A_1$ & $A_2 B = B A_2$

then, $(A_1 + A_2) B = A_1 B + A_2 B$

$= B A_1 + B A_2$

$= B(A_1 + A_2)$

hence, $A_1 + A_2 \in S$

$\therefore$ for any $\alpha \in R$, we have

$(\alpha A_1) B = ~~B \alpha A_1~~ \alpha(A_1 B)$

$= \alpha(B A_1)$

$(\alpha A_1) B = B(\alpha A_1)$

$\alpha A_1 \in S$

hence S is a subspace of M_{nn}

$\rightarrow$ $S = \{(x,y,z) \mid y = x+z+1\}$, $V = \mathbb{R}^3$

$u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$ in S.

~~then~~ $y_1 = x_1 + z_1 + 1$ & $y_2 = x_2 + z_2 + 1$

Then $u + v = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

But here $y_1 = x_1 + z_1 + 1$ & $y_2 = x_2 + z_2 + 1$

$\therefore y_1 + y_2 = (x_1 + x_2 + z_1 + z_2 + 2)$

$\therefore u + v \notin S$.

* Linear Combination:-

Let S be any non-empty subset of a vector space V. Than a vector v in V is said to be linear combination of the vectors in S if there exist a finite subset $\{v_1, v_2, \ldots, v_n\}$ in S such that

$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$ for any scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$

Note :- The method to check a vector v is a linear combination of the given vectors $v_1, v_2, \ldots, v_n$ is as follows:-

(i) express a vector v as a linear combination of $v_1, v_2, \ldots, v_n$ and than form the system of eq^n.

i.e. $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$

(ii) If the above system of eq^n is consistent than v is a linear combination of $v_1, v_2, \ldots, v_n$. Otherwise it is not a linear combination of $v_1, v_2, \ldots, v_n$

Q1. Express a vector $(2,-2,3)$ as a linear combination of a set $\{(0,1,-1), (2,0,1), (-3,2,5)\}$ of R^3

let $S = \{(0,1,-1), (2,0,1), (-3,2,5)\}$

than, $v_1 = (0,1,-1)$, $v_2 = (2,0,1)$, $v_3 = (-3,2,5)$

let $v = (2,-2,3)$

than, for any scalars $\alpha_1, \alpha_2, \alpha_3$ we have

$v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$.

$(2,-2,3) = \alpha_1 (0,1,-1) + \alpha_2 (2,0,1) + \alpha_3 (-3,2,5)$

$(2,-2,3) = (0,\alpha_1,-\alpha_1) + (2\alpha_2, 0, \alpha_2) + (-3\alpha_3, 2\alpha_3, 5\alpha_3)$

$(2,-2,3) = [(2\alpha_2 - 3\alpha_3), (\alpha_1 + 2\alpha_3), (-\alpha_1 + \alpha_2 + 5\alpha_3)]$

Now by equating the components on both sides we get

$2\alpha_2 - 3\alpha_3 = 2$

$\alpha_1 + 2\alpha_3 = -2$

$-\alpha_1 + \alpha_2 + 5\alpha_3 = 3$.

Than from above system augmented matrix is

$$[A|B] = \left[\begin{array}{ccc|c} 0 & 2 & -3 & 2 \\ 1 & 0 & 2 & -2 \\ -1 & 1 & 5 & 3 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & -2 \\ 0 & 2 & -3 & 2 \\ -1 & 1 & 5 & 3 \end{array}\right]$$

$R_3 + R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & -2 \\ 0 & 2 & -3 & 2 \\ 0 & 1 & 7 & 1 \end{array}\right]$$

$R_2 - R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & -2 \\ 0 & 1 & -10 & 1 \\ 0 & 1 & 7 & 1 \end{array}\right]$$

$R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & -2 \\ 0 & 1 & -10 & 1 \\ 0 & 0 & -17 & 0 \end{array}\right]$$

$R_3 / -17$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & -2 \\ 0 & 1 & 7 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \rightarrow (i)$$

Here Rank of $A = r(A|B) = 3$ = no. of unknown

$\therefore$ given system is consistent

$\therefore$ $v = (2,-2,3)$ is a linear combination of (v_1, v_2, v_3)

from (i) we have the system

$\alpha_1 + 2\alpha_3 = -2$ $\qquad\qquad$ $\alpha_1 = -2$

$\alpha_2 + 7\alpha_3 = 1$ $\qquad\qquad$ $\alpha_2 = 1$

$\alpha_3 = 0$ $\qquad\qquad$ $\alpha_3 = 0$

$\therefore$ Given vector v can be express as

$(2,-2,3) = -2(0,1,-1) + 1(2,0,1), 0(-3,2,5)$

$= (0,-2,2) + (2,0,1)$

$= (2,-2,3)$

* **Span of a set**: The span of a non-empty subset S of a vector space V is the set of all linear combination of any finite no. of elements of S.

It is denoted by 'Span(S)' or $L[S]$

i.e. $\text{Span}(S) = \{\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \mid v_1, v_2, \ldots v_n \in S,$
$\alpha_1, \alpha_2 \ldots \alpha_n \in R\}$.

eg. Determine the span of a set $\{(4,2)\}$ of R^2.

- let $S = \{(4,2)\}$, then $v_1 = (4,2)$.
let $v \in R^2$,
Then for any $\alpha_1 \in R$ we have vector $= v_1$
$v = \alpha_1 v_1$
$v = \alpha_1 (4,2) = (4\alpha_1, 2\alpha_1)$.

$\therefore$ ~~Span of S~~:
$\text{Span}(S) = \{ v \in R^2 \mid v = (4\alpha_1, 2\alpha_1), \alpha_1 \in R\}$.

4 Determine the span of $\{(1,0,0), (0,1,0), (0,0,1)\}$ in R^3.

- let $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ let $v \in R^3$
$v_1 = (1,0,0)$, $v_2 = (0,1,0)$, $v_3 = (0,0,1)$
Then for any scalars $\alpha_1, \alpha_2, \alpha_3 \in R$ we have
$v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$
$= \alpha_1 (1,0,0) + \alpha_2 (0,1,0) + \alpha_3 (0,0,1)$
$v = (\alpha_1, \alpha_2, \alpha_3)$

$\therefore$ $\text{Span}(S) = \{ v \in R^3 \mid v = (\alpha_1, \alpha_2, \alpha_3), \alpha_1, \alpha_2, \alpha_3 \in R\}$

- Determine the span of $\{1 + 3x, x + x^2\}$ in P_2

- let $S = \{(1+3x), x + x^2\}$
let polynomial $p_x \in P_2$ & $p_1(x) = (1+3x)$, $p_2 = (x + x^2)$
Then for any scalars α_1, α_2 we have
$p_1(x) = \alpha_1 \cdot p(x) + \alpha_2 p_2(x)$
$= \alpha_1 (1+3x) + \alpha_2 (x + x^2)$
$= \alpha_1 + 3\alpha_1 x + \alpha_2 x + \alpha_2 x^2$
$= \alpha_1 + 3\alpha_1 x + \alpha_2 x + \alpha_2 x^2\}$

$\therefore$ $\text{Span}(S) = \{ v \in R^2, v = (\alpha_1 + 3\alpha_1 x + \alpha_2 x + \alpha_2 x^2) \mid \alpha_1, \alpha_2 \in R\}$

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- Set $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix} \right\}$ in M_{22}

let $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix} \right\}$

let $A \in M_{22}$

$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}$

Then for any $\alpha_1, \alpha_2, \alpha_3$ we have

$$A = \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3$$

$$= \alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2\alpha_2 \\ 3\alpha_2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2\alpha_3 \\ -\alpha_3 & \alpha_3 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 & 2\alpha_2 + 2\alpha_3 \\ 3\alpha_2 - \alpha_3 & \alpha_3 \end{bmatrix}$$

* Linear - dependence & independence of a set

Def:- Linearly dependent set (LD) :- a finite set $\{v_1, v_2, \dots, v_n\}$ of a vector space V is said to be linearly dependent set if there exist scalar $\alpha_1, \alpha_2, \dots, \alpha_n$, not all zero (i.e. atleast one of them must be non-zero), such that $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$.

→ Linearly independent set (LI) :- A ~~set~~ finite set $\{v_1, v_2, \dots, v_n\}$ of a vector space V is said to be linearly independent set if there exist scalars ~~such~~ $\alpha_1, \alpha_2, \dots, \alpha_n$ such that $\alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_n v_n = 0$

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Note:- A method to check a set LI or LD
↳ If the system of equations $\alpha_1 V_1 + \alpha_2 V_2 + \dots + \alpha_n V_n = 0$ (i.e Homogenous system) has trivial solution then set is LI and if non trivial solution then the set is LD.

Example:- Check wheater the following set of vectors are LI or LD.

1.) $\{(1,2,3), (0,2,1), (0,1,3)\}$ in R^3

Soln:- We have given the set

$$\{(1,2,3), (0,2,1), (0,1,3)\}$$

Then take $V_1 = (1,2,3)$
$V_2 = (0,2,1)$
$V_3 = (0,1,3)$

Then for any scalars $\alpha_1, \alpha_2, \alpha_3$ we consider a linear equation.

$$\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 = 0.$$

$$\therefore \alpha_1(1,2,3) + \alpha_2(0,2,1) + \alpha_3(0,1,3) = (0,0,0)$$

$$\therefore (\alpha_1, 2\alpha_1, 3\alpha_1) + (0, 2\alpha_2, \alpha_2) + (0, \alpha_3, 3\alpha_3) = (0,0,0).$$

$$\therefore (\alpha_1, 2\alpha_1 + 2\alpha_2 + \alpha_3, 3\alpha_1 + \alpha_2 + 3\alpha_3) = (0,0,0).$$

Now equating components on both side we

$\alpha_1 = 0$

$2\alpha_1 + 2\alpha_2 + \alpha_3 = 0$

$3\alpha_1 + \alpha_2 + 3\alpha_3 = 0.$

Then from above system

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ 3 & 1 & 3 \end{bmatrix}$$

$R_2 \to R_2 + (-2)R_1$
$R_3 \to R_3 + (-3)R_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 2 & 1 \end{bmatrix}$$

$R_3 \to R_3 + (-2)R_2.$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & -5 \end{bmatrix}$$

$R_3 \to R_3 / -5$

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$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore r(A) = 3 =$ no. of unknowns

$\therefore$ given system has trivial solution

$\therefore \alpha_1 = \alpha_2 = \alpha_3 = 0$

$\therefore$ given set is LI.

2.) $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 6 \\ 4 & 6 \end{bmatrix} \right\}$ in M_{22}.

Soln:- We have given the set

$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$

$A_4 = \begin{bmatrix} 2 & 6 \\ 4 & 6 \end{bmatrix}$

Then for any sclar

$\alpha_1, \alpha_2, \alpha_3, \alpha_4$ we consider a linear combination.

$$\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \alpha_4 A_4 = 0.$$

$\cancel{\alpha_1 A_1 A_1}$

$$\alpha_1 \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$$

$$+ \alpha_4 \begin{bmatrix} 2 & 6 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 & \alpha_1 \\ \alpha_1 & 2\alpha_1 \end{bmatrix} + \begin{bmatrix} \alpha_2 & 0 \\ 0 & 2\alpha_2 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 3\alpha_3 \\ \alpha_3 & 2\alpha_3 \end{bmatrix} + \begin{bmatrix} 2\alpha_4 & 6\alpha_4 \\ 4\alpha_4 & 6\alpha_4 \end{bmatrix} = \begin{bmatrix} 0 & \\ 0 & \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1+\alpha_2+2\alpha_4 & \alpha_1+3\alpha_3+6\alpha_4 \\ \alpha_1+\alpha_3+4\alpha_4 & 2\alpha_1+2\alpha_2+2\alpha_3+6\alpha_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Now by equationg components on both side

$$\alpha_1+\alpha_2+2\alpha_4 = 0 \quad \text{——} (i.)$$
$$\alpha_1+3\alpha_3+6\alpha_4 = 0 \quad \text{——} (ii)$$
$$\alpha_1+\alpha_3+4\alpha_4 = 0 \quad \text{——} (iii)$$
$$2\alpha_1+2\alpha_2+2\alpha_3+6\alpha_4 = 0 \quad \text{——} (iv)$$

$$\therefore A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 0 & 3 & 6 \\ 1 & 0 & 1 & 4 \\ 2 & 2 & 2 & 6 \end{bmatrix}$$

$$R_2 \to R_2 + (-1)R_1$$
$$R_3 \to R_3 - R_1$$
$$R_4 \to R_4 + (-2)R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & -1 & 3 & 4 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$$R_2 \to R_2(-1)$$

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$$\sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & -3 & -4 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$$R_3 \to R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$$R_4 \to R_4 + R_3$$
$$R_3 \to R_3 / -2$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$r(A) = 3 < 4 <$ no. of unknowns

∴ The set is non trivial solution.

∴ It is LD.

H.W $\left\{ \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \right\}$

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* Basis of a set

→ A subset S of vectors of a vector space V is said to be basis for a subset of V if

(i) Set S is L.I

(ii) Set S spans V.

(i) Basis for a vector space is not unique

(ii) Standard Basis ~~for~~ of different vector spaces.

- $\{(1,0),(0,1)\}$ is basis for R^2.

- $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ for M_2

- $\{1, x, x^2\}$ for P_2.

Ex. P.T. the set $\{(1,0,0),(2,2,0),(3,3,3)\}$ is a basis for R^3

- let set $S = \{(1,0,0),(2,2,0),(3,3,3)\}$

Then take $v_1 = (1,0,0)$, $v_2 = (2,2,0)$, $v_3 = (3,3,3)$

let $x = (x_1, x_2, x_3)$ be in R^3.

Then for any scalars ~~then~~ $\alpha_1, \alpha_2, \alpha_3$ the vector x is a linear combination as $x = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$

$(x_1, x_2, x_3) = \alpha_1(1,0,0) + \alpha_2(2,2,0) + \alpha_3(3,3,3)$

$(x_1, x_2, x_3) = (\alpha_1, 0, 0) + (2\alpha_2, 2\alpha_2, 0) + (3\alpha_3, 3\alpha_3, 3\alpha_3)$

$(x_1, x_2, x_3) = (\alpha_1 + 2\alpha_2 + 3\alpha_3) + (2\alpha_2 + 3\alpha_3) + (3\alpha_3)$

Now by equating components on both side we get.

$\therefore$ $\alpha_1 + 2\alpha_2 + 3\alpha_3 = x_1$

$2\alpha_2 + 3\alpha_3 = x_2$

$3\alpha_3 = x_3$

→ from above the augmented matrix is

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & x_1 \\ 0 & 2 & 3 & x_2 \\ 0 & 0 & 3 & x_3 \end{array}\right]$$

$R_2 \to R_2/2$, $R_3/3$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 3 & x_1 \\ 0 & 1 & 3/2 & x_2/2 \\ 0 & 0 & 1 & x_3/3 \end{array}\right] \to (1)$$

here we can observe that for any value of (x_1, x_2, x_3)
(i.e in R^3) rank of $A = r[A|B] = 3 =$ no. of unknowns
$\therefore$ given system is consistent
$\therefore$ S spans R^3.

→ To prove set S is L.I or L.D we consider.
$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$ (i.e $(x_1, x_2, x_3) = (0, 0, 0)$)
from (1) we can observe that $r(A) = 3 =$ no. of unknown's
$\therefore$ the system has trivial solution. $(\alpha_1 = \alpha_2 = \alpha_3 = 0)$
$\therefore$ S is L.I.
$\therefore$ Set S span's R^3 and S is L.I, hence S forms a basis for R^3.

→ Determine weather the set $\{(1,0), (0,1), (2,-1)\}$ forms a basis for R^2 or not.
let $S = \{(1,0), (0,1), (2,-1)\}$
Then take $v_1 = (1,0)$, $v_2 = (0,1)$, $v_3 = (2,-1)$
let $x = (x_1, x_2)$ be in R^2
Then for any scalars $\alpha_1, \alpha_2, \alpha_3$ the vector x is a L.C as $x = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$
$(x_1, x_2) = \alpha_1(1,0) + \alpha_2(0,1) + \alpha_3(2,-1)$
$= (\alpha_1, 0) + (0, \alpha_2) + (2\alpha_3, -\alpha_3)$
$(x_1, x_2) = ((\alpha_1 + 2\alpha_3), (\alpha_2 - \alpha_3))$
Now by equating components on both sides we have
$\alpha_1 + 2\alpha_3 = x_1$
$\alpha_2 - \alpha_3 = x_2$
from above the augment matrix
$[A|B] = \left[\begin{array}{ccc|c} 1 & 0 & 2 & x_1 \\ 0 & 1 & -1 & x_2 \end{array}\right] \rightarrow (1)$

$\therefore$ for any value of (x_1, x_2) we have $r(A) = r(A|B)$
$= 2 < 3$ (no. of variables)

$\therefore$ given system is consistent

$\therefore$ S spans R

To prove set S is L.I or L.D, we consider
$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$

from (i) we can observe that $r(A) = 2 < 3 =$ no. of variables

$\therefore$ The system has non trivial sol^n

$\therefore$ S is L.D.

$\therefore$ Set S spans R^2 but S is not L.I

$\therefore$ Set S not form a basis for R^2.

→ Determine whether $(1-3x+2x^2, 1+x+4x^2, 1-7x)$ forms a basis for P_2 or not.

→ we have given set $S = \{1-3x+2x^2, 1+x+4x^2, 1-7x\}$ then take

$p_1(x) = 1-3x+2x^2$

$p_2(x) = 1+x+4x^2$

$p_3(x) = 1-7x$

let $p(x) = a_0 + a_1 x + a_2 x^2$ be in P_2

then for any scalars $\alpha_1, \alpha_2, \alpha_3$, $p(x)$ is a L.C a

$p(x) = \alpha_1 p_1(x) + \alpha_2 p_2(x) + \alpha_3 p_3(x)$

$a_0 + a_1 x + a_2 x^2 = \alpha_1 (1+3x+4x^2) + \alpha_2 (1+x+4x^2) + \alpha_3 (1-7x)$

$= (\alpha_1 + 3\alpha_1 x + 4\alpha_1 x^2) + (\alpha_2 + \alpha_2 x + 3\alpha_2 x^2)$
$+ \alpha_3 - 7\alpha_3 x$

$(a_0 + a_1 x + a_2 x^2) = (\alpha_1 + \alpha_2 + \alpha_3) + (-3\alpha_1 + \alpha_2 - 7\alpha_3) x +$
$(2\alpha_1 + 4\alpha_2) x^2$

Now comparing

$\cancel{(A|B)} = \alpha_1 + \alpha_2 + \alpha_3 = a_0$

$-3\alpha_1 + \alpha_2 - 7\alpha_3 = a_1$

$2\alpha_1 + 4\alpha_2 = a_2$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & a_0 \\ -3 & 1 & -7 & a_1 \\ 2 & 4 & 0 & a_2 \end{array}\right]$$

$R_2 + 3R_1$; $R_3 \to R_3 + (-2)R_1$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & a_0 \\ 0 & 4 & -4 & a_1 + 3a_0 \\ 0 & 2 & -2 & a_2 - 2a_0 \end{array}\right]$$

$R_2 \to R_2/4$, $R_3 \to R_3/2$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & a_0 \\ 0 & 1 & -1 & a_1 + 3a_0/4 \\ 0 & 1 & -1 & a_2 - 2a_0/2 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & a_0 \\ 0 & 1 & -1 & a_1 + 3a_0/4 \\ 0 & 0 & 0 & \frac{a_2 - 2a_0}{2} - a_1 + 3a_0/4 \end{array}\right]$$

here we can observe that for any value of a_0, a_1 & a_2 the system is inconsistent

$\therefore$ S cannot spans P_2

$\therefore$ S not forms a basis.

→ Determine weather the set $\left\{ \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix} \right\}$ forms a basis in M_{22} or not

→ Definition :-

<u>Dimension</u> :- The number of vectors in a basis of a non-zero vector space 'V' is known as dimension of 'V' & it is denoted by 'dim (V)"

Note:- Dimension of some standard vector space can be obtained directly from there standard basis

e.g. (i) dimension of ~~R^n~~ R^n = n

(ii) ~~...~~ dim (M_{mn}) = $m \times n$

(iii) dim (P_n) = $n+1$

(iv) dim $(\{0\})$ = 0 (as singelton 0 is L.D set and hence it not form a basis).

* Linear Transformations :-

let V & W be two vector spaces than a linear Transformation is a function from V to W. i.e., $T : V \to W$. Such that

Linearity property $\begin{cases} \text{(i) } T(u+v) = T(u) + T(v). \\ \text{(ii) } T(\alpha u) = \alpha T(u) \end{cases}$

for all u, v in V. & for all scalars α.

e.g. determine weather the following functions are Linear transformation or not.

Q. (i) $T : R^2 \to R^2$, where $T(x,y) = (x+2y, 3x-y)$

We are given that $T: R^2 \to R^2$ where $T(x,y) = (x+2y, 3x-y)$.

Let $u=(x_1, y_1)$ & $v=(x_2, y_2)$ be in R^2

Then $T(u) = T(x_1, y_1)$
$= (x_1 + 2y_1, 3x_1 - y_1)$

& $T(v) = T(x_2, y_2)$
$= (x_2 + 2y_2, 3x_2 - y_2)$

here $u+v = (x_1, y_1) + (x_2, y_2)$
$= (x_1 + x_2, y_1 + y_2)$

Now, $T(u+v) = T(x_1 + x_2, y_1 + y_2)$
$= (x_1 + x_2 + 2y_1 + 2y_2, 3x_1 + 3x_2 - y_1 - y_2)$.
$= (x_1 + 2y_1 + x_2 + 2y_2, 3x_1 - y_1 + 3x_2 - y_2)$
$= (x_1 + 2y_1, 3x_1 - y_1) + (x_2 + 2y_2, 3x_2 - y_2)$

$\therefore T(u+v) = T(u) + T(v)$

Now, $\alpha u = \alpha(x_1, y_1) = (\alpha x_1, \alpha y_1)$

$T(\alpha u) = T(\alpha x_1, \alpha y_1)$
$= (\alpha x_1 + 2\alpha y_1, 3\alpha x_1 - \alpha y_1)$
$= \alpha(x_1 + 2y_1, 3x_1 - y_1)$

$T(\alpha u) = \alpha T(u)$

has T is a linear Transformation

→ $T: P_2 \to P_2$, where $T(a_0 + a_1 x + a_2 x^2) = (a_0 + 1) + (a_1 + 1)x + (a_2 + 1)x^2$.

~~others are~~

We are given ~~P from~~ ~~$R_2 \to R_2$~~ $T: P_2 \to P_2$

~~others~~ let $p_1(x) = a_0 + a_1 x + a_2 x^2$ & $p_2(x) = b_0 + b_1 x + b_2 x^2$

be in P_2.

Then $T(P_1(x)) = T(a_0 + a_1 x + a_2 x^2)$
$= (a_0 + 1) + (a_1 + 1)x + (a_2 + 1)x^2$

& $T(P_2(x)) = T(b_0 + b_1x + b_2x^2)$
$= (b_0+1) + (b_1+1)x + (b_2+1)x^2$;

here

$P_1(x) + P_2(x) = (a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2)$
$= (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2$

Now,

$T(P_1(x) + P_2(x)) = T[(a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2]$
$= [(a_0+b_0)+1) + ((a_1+b_1)+1)x + ((a_2+b_2)+1)$
$= [(a_0+1) + (a_1+1)x + (a_2+1)x^2] + (b_0+b_1x + b_2x^2)$
$= T(P_1(x) + P_2(x))$.

$T(P_1(x) + P_2(x)) \neq T(P_1x) + T(P_2(x))$

has T is not a Linear Transform

→ $T: M_{mn} \to R$, when $T(A) = \det(A)$

– let A_1 & A_2 be in M_{mn}, then

$T(A_1) = \det(A_1)$

$T(A_2) = \det(A_2)$

Now $T(A_1 + A_2) = \det(A_1 + A_2)$
$\neq \det(A_1) + \det(A_2)$
$\neq T(A_1) + T(A_2)$.

has T is not a linear transformation.

→ $T: R^3 \to R^2$, when $T(x,y,z) = (2x-y+z, y-4z)$

– let $U = (x_1, y_1, z_1)$ & $V = (x_2, y_2, z_2)$ in R_3

$T(U) = T(x_1, y_1, z_1)$
$= (2x_1 - y_1 + z_1, y_1 - 4z_1)$

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$P(v) = T(x_2, y_2, z_2)$
$= (2x_2 - y_2 + z_2, y_2 - 4z_2)$

$\cancel{T(u)+} \; T(u+v) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$
$= (2x_1 + 2x_2 - y_1 - y_2 + z_1 + z_2,$
$y_1 + y_2 - 4z_1 - 4z_2)$
$= (2x_1 - y_1 + z_1 + 2x_2 - y_2 + z_2,$
$y_1 - 4z_1 + y_2 - 4z_2)$
$= \cancel{T(u) + T(v)}$
$= (2x_1 - y_1 + z_1, y_1 - 4z_1) + (2x_2 - y_2 + z_2, y_2 - 4z_2)$
$T(u+v) = T(u) + T(v)$
$\cancel{T(\alpha u + \alpha v) = T(2\alpha x_1 - \alpha y_1 + \alpha z_1)}, \cancel{(2\alpha x_2 - \alpha y_2 + \alpha z_2)}$
$= \cancel{\alpha T}($

$T(\alpha u) = T(2\alpha x_1 - \alpha y_1 + \alpha z, \alpha y_1 - 4\alpha z_1)$
$= \alpha T(2x_1 - y_1 + z, y_1 - 4z_1)$
$= \alpha T(u)$

→ Matrix representation of L.T.

- Suppose $T : R^n \to R^m$ is a linear Transformation. Then, there exist a mat (A) of order $m \times n$, such that
$T(X) = AX$
$T(x_1, x_2, \dots, x_n) = \begin{bmatrix} & \end{bmatrix}_{m \times n} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$

- e.g. Suppose $T : R^3 \to R^3$ is a linear Transformation where
$T(x, y, z) = (x - y + 2z, 2x + y - z, -x - 2y)$
$T(x, y, z) = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

i.e. $T(X) = AX$ where $A \begin{bmatrix} & \end{bmatrix}_{3 \times 3}$

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* Range and Kernel of a L.T.

- Let V & W be two vector spaces & $T : V \to W$ be a linear Transformation. Then the set $\{T(x) / x \in V\}$ is called range of T.

- It is denoted by '$R(T)$'.
The dimension of '$R(T)$' is known as rank of T., it is denoted by $\text{rank}(T) = \rho(T) = \dim(R(T))$

- The set ~~for ex~~ $\{x \in V / T(x) = 0\}$ is called Kernel of T (or Null space of T) & it is denoted by '$\text{Ker}(T)$' or N
The dimension of ~~Kernel of T or Null~~ $\text{Ker}(T)$ or $N(T)$ is known as Nullity of T, It is denoted by
$\text{Nullity}(T) = n(T) = \dim(N(T))$

Note: If $T : R^n \to R^n$ is a L.T such that $T(x) = AX$, where A is matrix of L.T., than ~~$\rho(A) =$~~
$\rho(T) = \rho(A)$ &
$n(T) = n(A)$

→ Continue from middle of other part →

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