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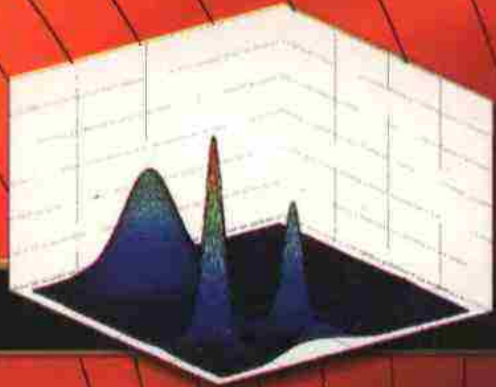

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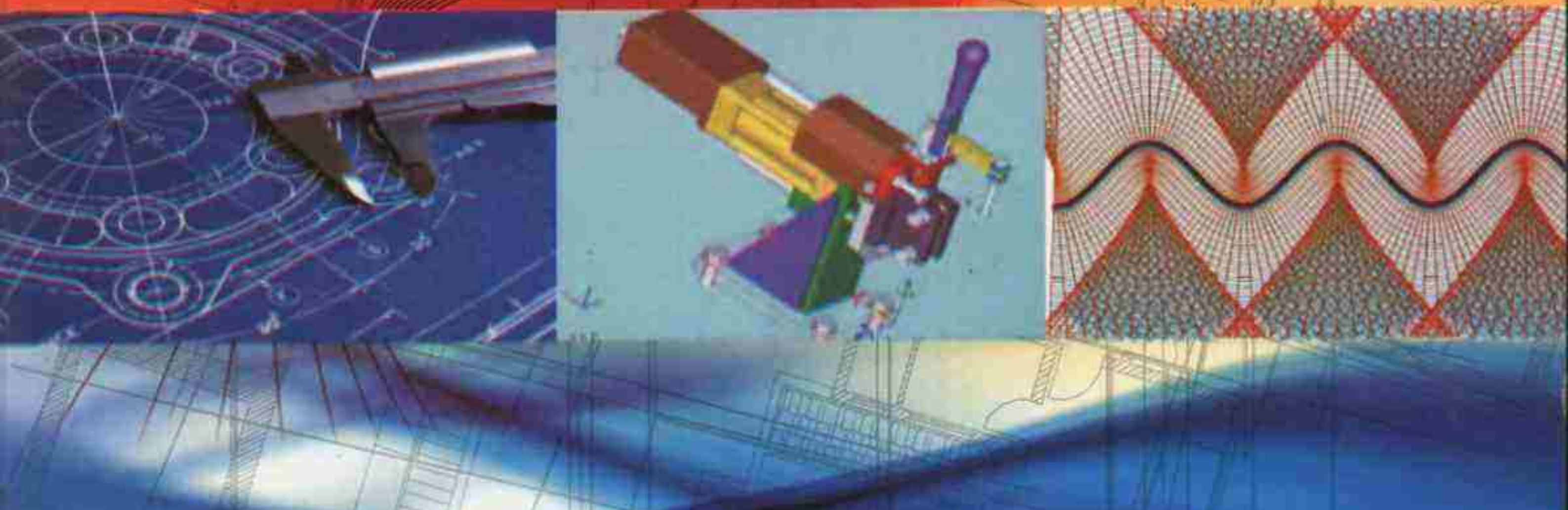

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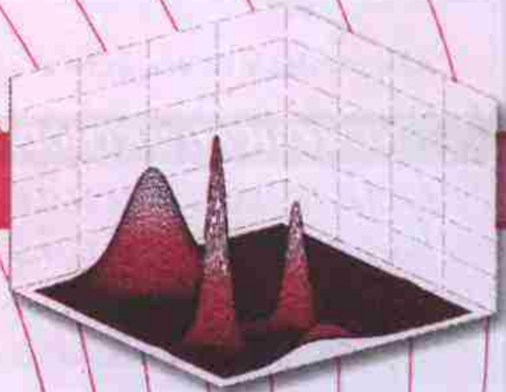

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42nd Edition

Higher Engineering Mathematics

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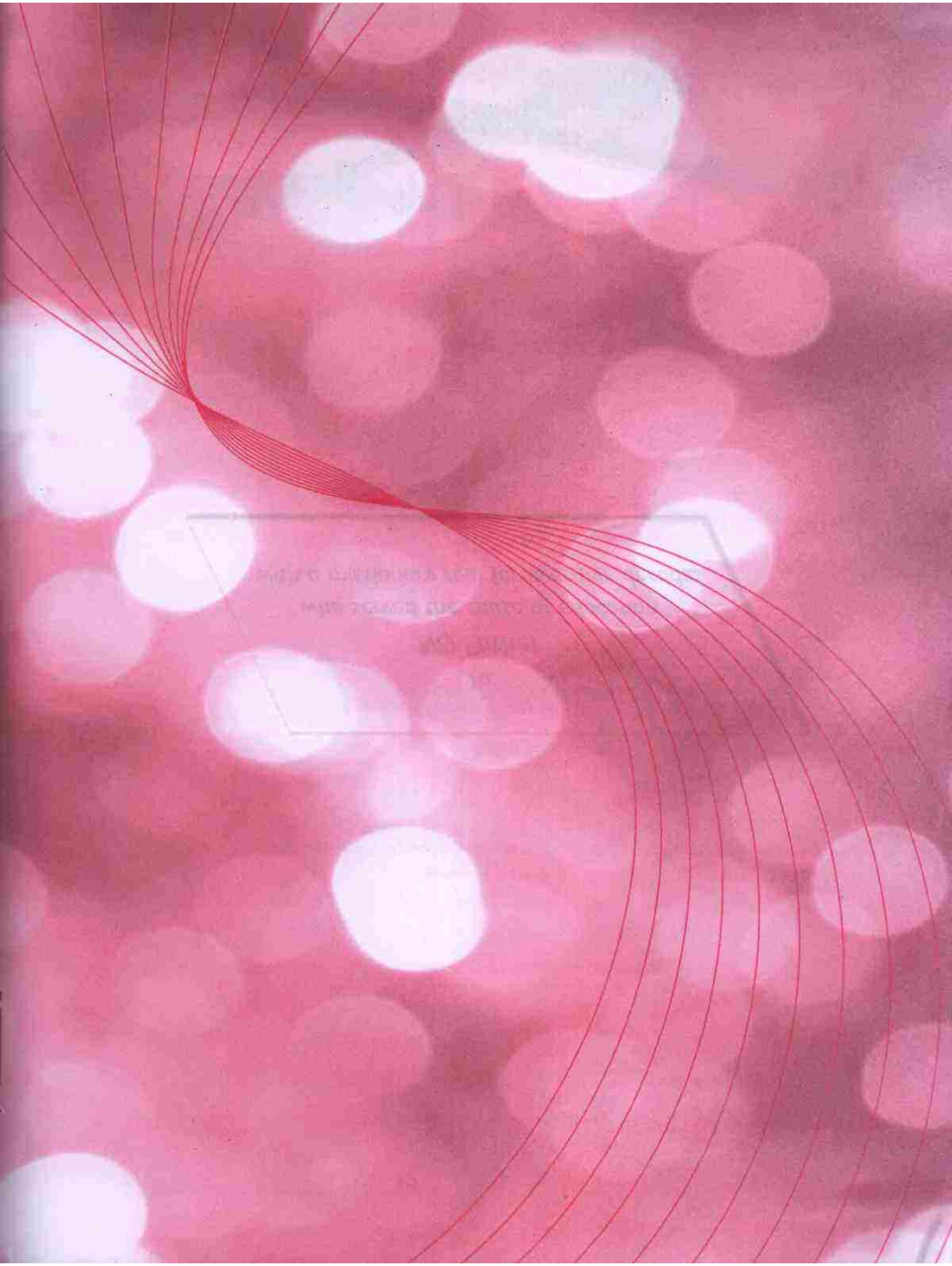
To

My Father

who served the cause of education
with a missionary zeal for over five decades



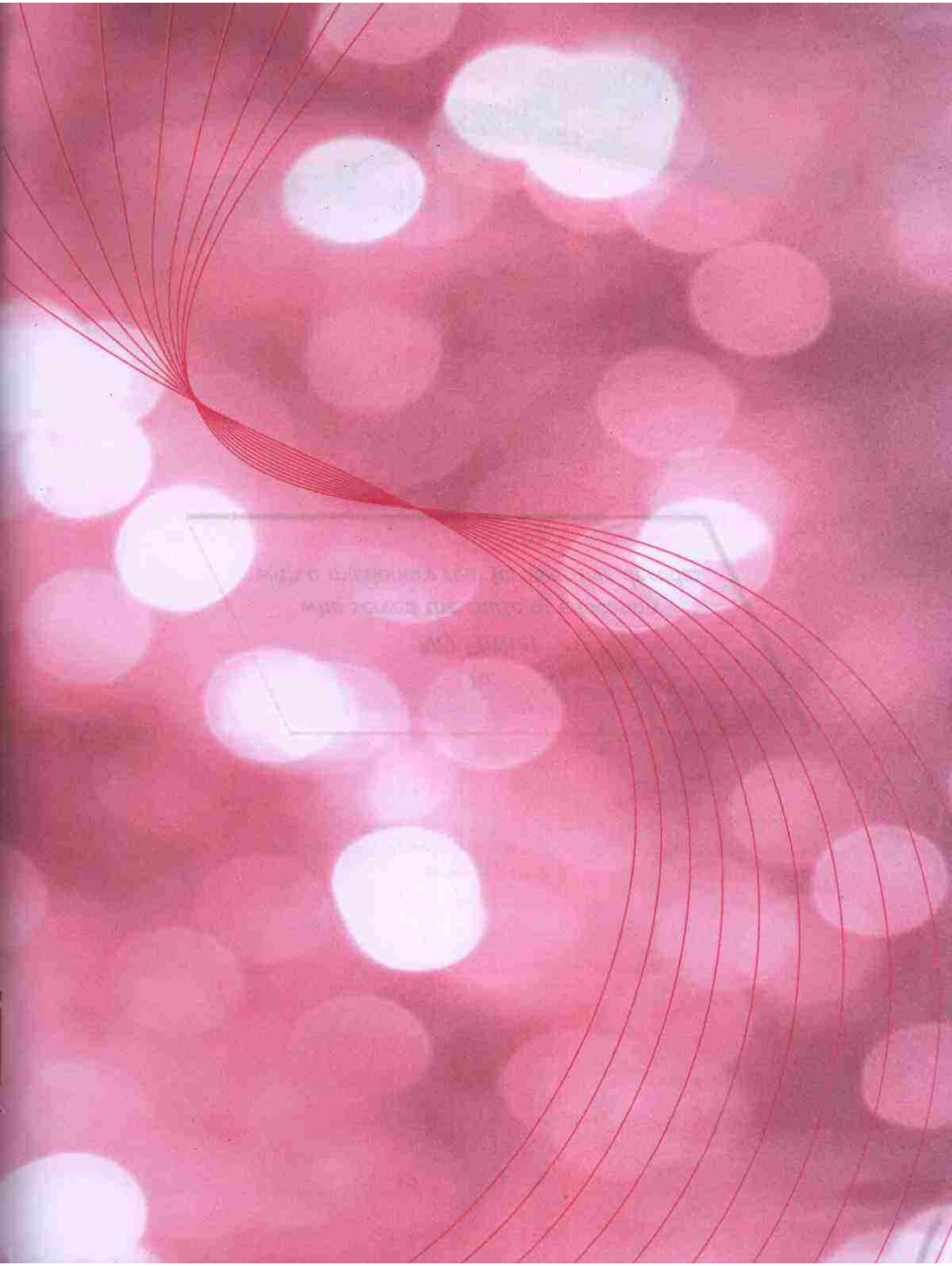
Preface to the 42nd Edition

The book has now been recast in an attractive new format, retaining its main features which have made it so popular. The text has been carefully revised, the number of illustrative examples has been increased and problems from the latest university question papers have been added. The 'Objective Type of Questions' have been updated and given at the end of each chapter. It is hoped that the book in its new form will enjoy its ever increasing popularity.

The author takes this opportunity to thank the numerous readers in India and abroad for their letters of appreciation and fellow professors for their suggestions and patronage of the book. In particular, he is grateful to Prof. Jeevargi Phakirappa, V.N. Engg. College, Bellary (Kar.); Prof. P. Annapurna, N.B.K.R. Inst. of Technology, Vidyanagar (A.P.); Dr. A.P. Burnwal, R.I.T., Koderma (Jh. Kh.); Prof. M. Vasudeva Reddy, Vaishnavi Inst. of Technology, Tarapalli (Tirupati); Dr. K.P. Ghadle, B.A.M. University, Aurangabad (Mah.); Prof. B.K. Yadav, Chauksey Engg. College, Bilaspur (C.G.); Prof. D. Ravi Kumar, Vignan University, Guntur (A.P.); Dr. J.C. Prajapati, Charotara University of Sc. & Technology, Changa (Guj.); Prof. Ramesh Chandra, S.R. Technology Institute, Nalgonda (A.P.); Dr. Latika Bhandari, R.V.S. College of Engg. & Technology, Bhillai; Prof. R. Saraswathi, Sri Padmavati Engg. College, Kavalli (A.P.) and Prof. Vikas Goyal, J.M. Inst. of Technology, Radur (Haryana).

Suggestions for improvement of the text and intimation of misprints will be thankfully acknowledged.

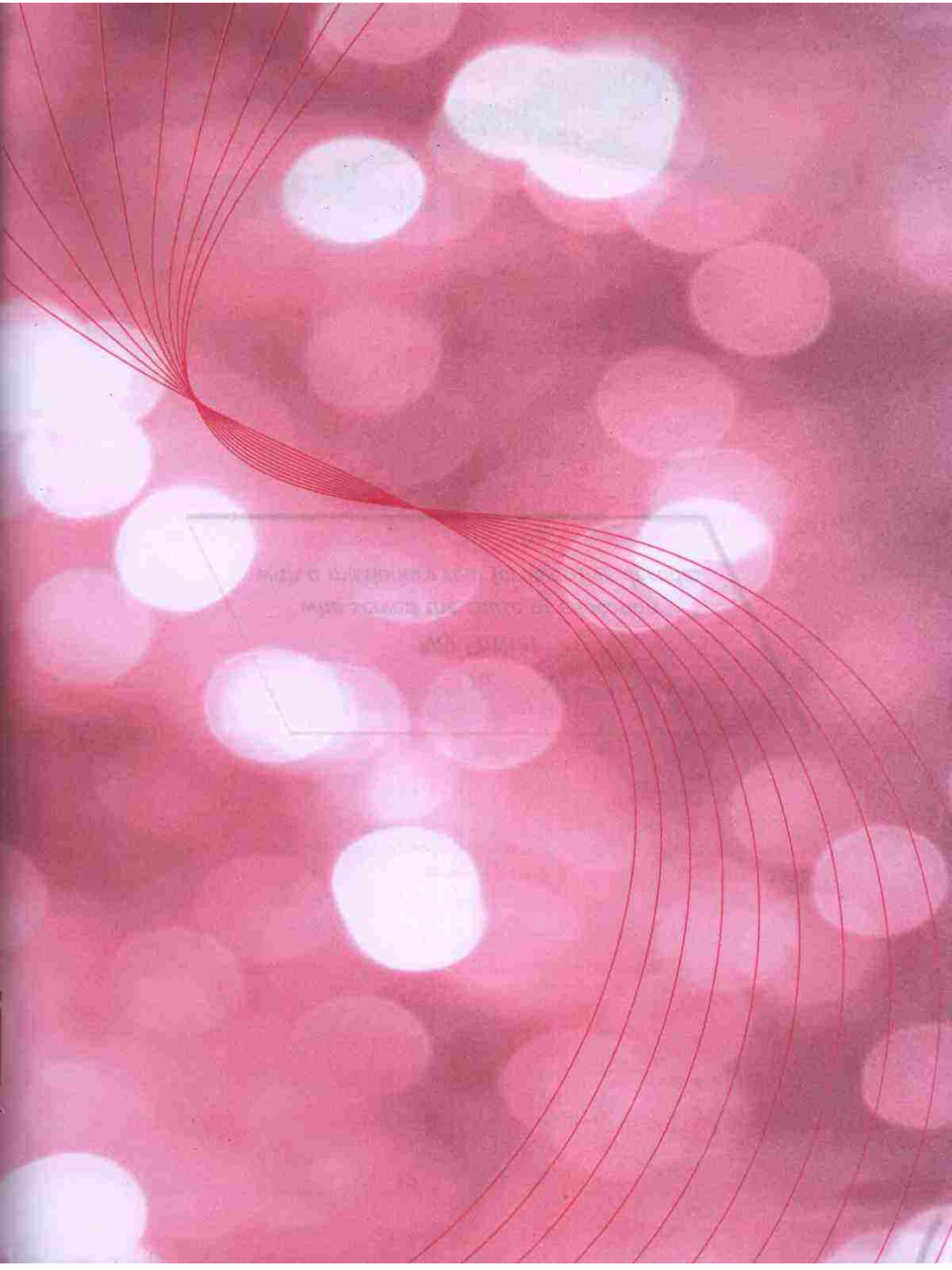
New Delhi **B.S. GREWAL**



Contents

Note : *The references given alongside the problems pertain to the* **Degree Engineering Examinations** *of the various universities and professional bodies. The abbreviations used for some of these are given below :*

Agra	*stands for*	Dr. B.R. Ambedkar University, Agra
Andhra	"	Andhra University, Waltair
Anna	"	Anna University, Chennai
Bhopal	"	Rajiv Gandhi Technical University, Bhopal
B.P.T.U.	"	Biju Patnaik Technical University, Rourkela
Coimbatore	"	Bharathiyar University, Coimbatore
CUSAT	"	Cochin University of Science and Technology, Kochi
Calicut	"	Calicut University, Cochin
Hazaribag	"	Vinoba Bhave University, Hazaribag
Hissar	"	Guru Jambeshwar University, Hissar
I.E.T.E.	"	Graduateship Examination of the Institute of Electronics and Telecommunication Engineers (India)
I.I.T.	"	Degree Engineering Examination of Indian Institute of Technology
I.S.M.	"	Indian School of Mines, Dhanbad
Kottayam	"	Mahatama Gandhi Memorial University, Kottayam
Kurukshetra	"	National Institute of Technology, Kurukshetra
Madurai	"	Madurai Kamaraj University, Madurai
Marathwada	"	B.A.M. University, Aurangabad
Nagarjuna	"	Acharya Nagarjuna University
P.T.U.	"	Punjab Technical University, Jalandhar
Raipur	"	Pt. Ravi Shankar Shukla University, Raipur
R.T.U.	"	Rajasthan Technical University, Kota
Rohtak	"	Maharishi Dayanand University, Rohtak
S. Patel	"	Sardar Patel University, Vallabh Vidyanagar
S.V.T.U.	"	Swami Vivekanand Technical University, Chhatisgarh
Tirupati	"	Sri Venkateswara University, Tirupati
Tiruchirapalli	"	Bharathidasan University, Tiruchirapalli
U.P.T.U.	"	UP Technical University, Lucknow
U.K.T.U.	"	Uttarakhand Technical University, Dehradun
V.T.U.	"	Visveswaraiah Technological University, Belgaum
Warangal	"	Warangal University of Technology
W.B.T.U.	"	West Bengal University of Technology, Kolkata



CHAPTER 1

Solution of Equations

1.1 INTRODUCTION

The expression $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$

where a's are constants ($a_0 \neq 0$) and n is a positive integer, is called a *polynomial in x* of degree n. The polynomial $f(x) = 0$ is called an *algebraic equation of degree n*. If $f(x)$ contains some other functions such as trigonometric, logarithmic, exponential etc. ; then $f(x) = 0$ is called a *transcendental equation*.

The value of x which satisfies $f(x) = 0$, ...(1)

is called its root. Geometrically, a root of (1) is that value of x where the graph of $y = f(x)$ crosses the x-axis. The process of finding the roots of an equation is known as *solution* of that equation. This is a problem of basic importance in applied mathematics. We often come across problems in deflection of beams, electrical circuits and mechanical vibrations which depend upon the solution of equations. As such, a brief account of solution of equations is given in this chapter.

1.2 GENERAL PROPERTIES

I. *If α is a root of the equation $f(x) = 0$, then the polynomial $f(x)$ is exactly divisible by $x - \alpha$ and conversely.*

For instance, 3 is a root of the equation $x^4 - 6x^2 - 8x - 3 = 0$, because $x = 3$ satisfies this equation.

$\therefore$ $x - 3$ divides $x^4 - 6x^2 - 8x - 3$ completely, *i.e.*, $x - 3$ is its factor.

II. *Every equation of the nth degree has n roots (real or imaginary).*

Conversely if $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the nth degree equation $f(x) = 0$, then

$$f(x) = A(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n) \text{ where } A \text{ is a constant.}$$

Obs. If a polynomial of degree n vanishes for more than n value of x, it must be identically zero.

Example 1.1. *Solve the equation $2x^3 + x^2 - 13x + 6 = 0$.*

Solution. By inspection, we find $x = 2$ satisfies the given equation.

$\therefore$ 2 is its root, *i.e.* $x - 2$ is a factor of $2x^3 + x^2 - 13x + 6$. Dividing this polynomial by $x - 2$, we get the quotient $2x^2 + 5x - 3$ and remainder 0.

Equating the quotient to zero, we get $2x^2 + 5x - 3 = 0$.

Solving this quadratic, we get $x = \dfrac{-5 \pm \sqrt{[5^2 - 4 \,.\, (2) \,.\, (-3)]}}{2 \times 2} = \dfrac{-5 \pm 7}{4} = -3, \dfrac{1}{2}$.

Hence, the roots of the given equation are 2, – 3, 1/2.

Note. *The labour of dividing the polynomial by $x - 2$ can be saved considerably by the following simple device called* **synthetic division.**

2	1	–13	6	2
	4	10	–6	
2	5	–3	0	

[*Explanation* : (*i*) Write down the coefficient of the powers of x in order (supplying the missing powers of x by zero co-efficients and write 2 on extreme right.

(*ii*) Put 2 as the first term of 3rd row and multiply it by 2, write 4 under 1 and add, giving 5.

(*iii*) Multiply 5 by 2, write 10 under – 13 and add, giving – 3.

(*iv*) Multiply – 3 by 2, write – 6 under 6 and add given zero].

Thus the quotient is $2x^2 + 5x - 3$ and remainder is zero.

Obs. To divide a polynomial by $x + h$, we write $- h$ on the extreme right.

III. Intermediate value property. If $f(a)$ and $f(b)$ *have different signs, then the equation $f(x) = 0$ has atleast one root between $x = a$ and $x = b$.*

The polynomial $f(x)$ is a continuous function of x (Fig. 1.1). So while x changes from a to b, $f(x)$ must pass through all the values from $f(a)$ to $f(b)$. But since one of these quantities $f(a)$ or $f(b)$ is positive and the other negative, it follows that at least for one value of x(say α) lying between a and b, $f(x)$ must be zero. Then α is the required root.

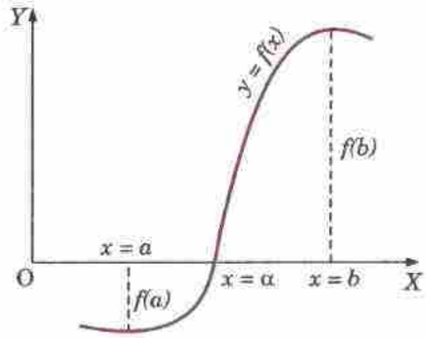


Fig. 1.1

IV. *In an equation with real coefficients, imaginary roots occur in conjugate pairs, i.e.,* if $\alpha + i\beta$ is a root of the equation $f(x) = 0$, then $\alpha - i\beta$ must also be its root. (See p. 534)

Similarly if $a + \sqrt{b}$ is an irrational root of an equation, then $a - \sqrt{b}$ must also be its root.

Obs. *Every equation of the odd degree has at least one real root.*

This follows from the fact that imaginary roots occur in conjugate pairs.

Example 1.2. *Solve the equation $3x^3 - 4x^2 + x + 88 = 0$, one root being $2 + \sqrt{7}i$.*

Solution. Since one root is $2 + \sqrt{7}i$, the other root must be $2 - \sqrt{7}i$.

$\therefore$ The factors corresponding to these roots are

$$(x - 2 - \sqrt{7}i) \text{ and } (x - 2 + \sqrt{7}i)$$

or $$(x - 2 - \sqrt{7}i)(x - 2 + \sqrt{7}i) = (x - 2)^2 + 7 = x^2 - 4x + 11,$$

which is a divisor of $3x^3 - 4x^2 + x + 88$ ...(*i*)

$\therefore$ Division of (*i*) by $x^2 - 4x + 11$ gives $3x + 8$ as the quotient.

Thus the depressed equation is $3x + 8 = 0$. Its root is $- 8/3$. Hence the roots of the given equation are $2 \pm \sqrt{7}i, - 8/3$.

V. Descarte's rule of signs. **The equation $f(x) = 0$ cannot have more positive roots than the changes of signs in $f(x)$; and more negative roots than the changes of signs in $f(-x)$.*

For instance, consider the equation $f(x) = 2x^7 - x^5 + 4x^3 - 5 = 0$ ...(1)

Sign of $f(x)$ are + – + –

Clearly, $f(x)$ has 3 changes of signs (from + to – or – to +).

Thus (*i*) cannot have more than 3 positive roots.

*After the French mathematician and philosopher *Rene Descartes* (1596–1650), who invented Analytic geometry in 1637.

Also $f(-x) = 2(-x)^7 - (-x)^5 + 4(-x)^3 - 5$

$= -2x^7 + x^5 - 4x^3 - 5$

This shows that $f(x)$ has 2 changes of signs. Thus (i) cannot have more than 2 negative roots.

Obs. Existence of imaginary roots. *If an equation of the nth degree has at the most p positive roots and at the most q negative roots, then it follows that the equation has at least n – (p + q) imaginary roots.*

Evidently (1) above is an equation of the 7th degree and has at the most 3 positive roots and 2 negative roots. Thus (1) has at least 2 imaginary roots.

VI. Relations between roots and coefficients, *If* $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$ *be the roots of the equation*

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n = 0 \quad ...(1)$$

then

$$\Sigma\alpha_1 = -\frac{a_1}{a_0}, \quad \Sigma\alpha_1\alpha_2 = \frac{a_2}{a_0}, \quad \Sigma\alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}$$

....

$$\alpha_1\alpha_2\alpha_3 \ldots\ldots \alpha_n = (-1)^n \frac{a_n}{a_0}.$$

Example 1.3. *Solve the equation $x^3 - 7x^2 + 36 = 0$, given that one root is double of another.*

Solution. Let the roots be α, β, γ such that $\beta = 2\alpha$.

Also $\alpha + \beta + \gamma = 7, \alpha\beta + \beta\gamma + \gamma\alpha = 0, \alpha\beta\gamma = -36$

$\therefore \quad 3\alpha + \gamma = 7 \quad ...(i)$

$2\alpha^2 + 3\alpha\gamma = 0 \quad ...(ii)$

$2\alpha^2\gamma = -36 \quad ...(iii)$

Solving (i) and (ii), we get $\alpha = 3, \gamma = -2$.

[The values $\alpha = 0, \gamma = 7$ are inadmissible, as they do not satisfy (iii)].

Hence the required roots are 3, 6 and – 2.

Example 1.4. *Solve the equation $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$, given that the sum of two of its roots is zero.* *(Cochin, 2005 ; Madras, 2003)*

Solution. Let the roots be $\alpha, \beta, \gamma, \delta$ such that $\alpha + \beta = 0$.

Also $\alpha + \beta + \gamma + \delta = 2 \quad \therefore \quad \gamma + \delta = 2$

Thus the quadratic factor corresponding to α, β is of the form $x^2 - 0x + p$, and that corresponding to γ, δ is of the form of $x^2 - 2x + q$.

$\therefore \quad x^4 - 2x^3 + 4x^2 + 6x - 21 = (x^2 + p)(x^2 - 2x + q) \quad ...(i)$

Equating the coefficients of x^2 and x from both sides of (i), we get

$4 = p + q, \quad 6 = -2p.$

$\therefore \quad p = -3, \quad q = 7.$

Hence the given equation is equivalent to $(x^2 - 3)(x^2 - 2x + 7) = 0$

$\therefore$ The roots are $x = \pm\sqrt{3}, 1 \pm i\sqrt{6}$.

Example 1.5. *Find the condition that the cubic $x^3 - lx^2 + mx - n = 0$ should have its roots in*

(a) arithmetical progression. *(Madras, 2000 S)*

(b) geometrical progression.

Solution. (a) Let the roots be $a - d, a, a + d$ so that the sum of the roots $= 3a = l$ *i.e.*, $a = l/3$.

Since a is the root of the given equation

$\therefore \quad a^3 - la^2 + ma - n = 0 \quad ...(i)$

Substituting $a = l/3$, we get $(l/3)^3 - l(l/3)^2 + m(l/3) - n = 0$.

or $\quad 2l^3 - 9lm + 27n = 0$, which is the required condition.

(*b*) Let the roots be a/r, a, ar, so that the product of the roots $= a^3 = n$.

Putting $a = (n)^{1/3}$, in (*i*), we get $n - ln^{2/3} + mn^{1/3} - n = 0$ or $m = ln^{1/3}$

Cubing both sides, we get $m^3 = l^3 n$, which is the required condition.

Example 1.6. *Solve the equation $x^4 - 2x^3 - 21x^2 + 22x + 40 = 0$ whose roots are in A.P.*

Solution. Let the roots be $a - 3d$, $a - d$, $a + d$, $a + 3d$, so that the sum of the roots $= 4a = 2$.

$$\therefore \qquad a = 1/2$$

Also product of the roots $= (a^2 - 9d^2)(a^2 - d^2) = 40$

or
$$\left(\frac{1}{4} - 9d^2\right)\left(\frac{1}{4} - d^2\right) = 40 \quad \text{or} \quad 144d^4 - 40d^2 - 639 = 0$$

$$\therefore \qquad d^2 = 9/4 \quad \text{or} \quad -7/36$$

Thus, $d = \pm 3/2$, the other value is not admissible.

Hence the required roots are $-4, -1, 2, 5$.

Example 1.7. *Solve the equation $2x^4 - 15x^3 + 35x^2 - 30x + 8 = 0$, whose roots are in G.P.*

Solution. Let the roots be a/r^3, a/r, ar, ar^3, so that product of the roots $= a^4 = 4$.

Also the product of a/r^3, ar^3 and a/r, ar are each $= a^2 = 2$.

$\therefore$ The factors corresponding to a/r^3, ar^3 and a/r, ar are $x^2 + px + 2$, $x^2 + qx + 2$.

Thus, $2x^4 - 15x^3 + 35x^2 - 30x + 8 = 2(x^2 + px + 2)(x^2 + qx + 2)$

Equating the coefficients of x^3 and x^2

$$-15 = 2p + 2q \quad \text{and} \quad 35 = 8 + 2pq$$

$$\therefore \qquad p = -9/2, q = -3.$$

Thus the given equation is $2\left(x^2 - \frac{9}{2}x + 2\right)(x^2 - 3x + 2) = 0$

Hence the required roots are 1/2, 4 and 1, 2 *i.e.*, $\frac{1}{2}$, 1, 2, 4.

Example 1.8. *If α, β, γ be the roots of the equation $x^3 + px + q = 0$, find the value of*
(a) $\Sigma\alpha^2\beta$, (b) $\Sigma\alpha^4$ (c) $\Sigma\alpha^2\beta$.

Solution. We have
$$\alpha + \beta + \gamma = 0 \qquad ...(i)$$
$$\alpha\beta + \beta\gamma + \gamma\alpha = p \qquad ...(ii)$$
$$\alpha\beta\gamma = -q \qquad ...(iii)$$

(*a*) Multiplying (*i*) and (*ii*), we get

$$\alpha^2\beta + \alpha^2\gamma + \beta^2\gamma + \beta^2\alpha + \gamma^2\alpha + \gamma^2\beta + 3\alpha\beta\gamma = 0$$

or
$$\Sigma\alpha^2\beta = -3\alpha\beta\gamma = 3q \qquad \text{[By } (iii)]$$

(*b*) Multiplying the given equation by x, we get $x^4 + px^2 + qx = 0$

Putting $x = \alpha, \beta, \gamma$ successively and adding, we get $\Sigma\alpha^4 + p\Sigma\alpha^2 + q\Sigma\alpha = 0$

or
$$\Sigma\alpha^4 = -p\Sigma\alpha^2 - q(0) \qquad ...(iv)$$

Now squaring (*i*), we get $\alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha) = 0$

or
$$\Sigma\alpha^2 = -2p \qquad \text{[By } (ii)]$$

Hence, substituting the value of $\Sigma\alpha^2$ in (*iv*), we obtain

$$\Sigma\alpha^4 = -p(-2p) = 2p^2.$$

(*c*) $\Sigma\alpha^3\beta = \Sigma\alpha^2\,\Sigma\alpha\beta - \alpha\beta\gamma\Sigma\alpha = -2p(p) - (-q)(0) = -2p^2$.

PROBLEMS 1.1

1. Form the equation of the fourth degree whose roots are $3 + i$ and $\sqrt{7}$. *(Madras, 2000 S)*
2. Solve the equation (*i*) $x^3 + 6x + 20 = 0$, one root being $1 + 3i$.
   (*ii*) $x^4 - 2x^3 - 22x^2 + 62x - 15 = 0$, given that $2 + \sqrt{3}$ is a root.
3. Show that $x^7 - 3x^4 + 2x^3 - 1 = 0$ has at least four imaginary roots. *(Cochin, 2005)*
4. Show that the equation $x^4 + 15x^2 + 7x - 11 = 0$ has one positive, one negative and two imaginary roots.
5. Find the number and position of real roots of $x^4 + 4x^3 - 4x - 13 = 0$.
6. Solve the equation $3x^3 - 11x^2 + 8x + 4 = 0$, given that two of its roots are equal.
7. If r_1, r_2, r_3 are the roots of the equation $2x^3 - 3x^2 + kx - 1 = 0$, find constant k if sum of two roots is 1. *(S.V.T.U., 2009)*
8. The equation $x^4 - 4x^3 + ax^2 + 4x + b = 0$ has two pairs of equal roots. Find the values of a and b.
   Solve the following equations 9–14 :
9. $x^3 - 9x^2 + 14x + 24 = 0$, given that two of its roots are in the ratio 3 : 2.
10. $x^3 - 4x^2 - 20x + 48 = 0$ given that the roots α and β are connected by the relation $\alpha + 2\beta = 0$. *(S.V.T.U., 2007)*
11. $x^4 - 6x^3 + 13x^2 - 12x + 4 = 0$, given that it has two parts of equal roots. *(Madras, 2003)*
12. $x^4 - 8x^3 + 21x^2 - 20x + 5 = 0$ given that the sum of two of the roots is equal to the sum of the other two.
13. $x^3 - 12x^2 + 39x - 28 = 0$, roots being in arithmetical progression. *(Madras, 2001 S)*
14. $8x^3 - 14x^2 + 7x - 1 = 0$, roots being in geometrical progression. *(Osmania, 1999)*
15. O, A, B, C are the four points on a straight line such that the distances of A, B, C from O are the roots of equation $ax^3 + 3bx^2 + 3cx + d = 0$. If B is the middle point of AC, show that $a^2d - 3abc + 2b^3 = 0$. *(S.V.T.U., 2006)*
16. Solve the equations (*i*) $x^4 + 2x^3 - 21x^2 - 22x + 40 = 0$ whose roots are in A.P.
    (*ii*) $x^4 + 5x^3 - 30x^2 + 40x + 64 = 0$ whose roots are in G.P.
17. If α, β, γ be the roots of the equation $x^3 - lx^2 + mx - n = 0$, find the value of
    (*i*) $\Sigma\alpha^2\beta^2$, (*ii*) $(\beta + \gamma)(\gamma + \alpha)(\alpha + \beta)$
18. Find the sum of the cubes of the roots of the equation $x^3 - 6x^2 + 11x - 6 = 0$.
19. If α, β, γ are the roots of $x^3 + 4x - 3 = 0$, find the value of (*i*) $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$ (*ii*) $\alpha^{-2} + \beta^{-2} + \gamma^{-2}$.
20. If α, β, γ be the roots of $x^3 + px + q = 0$, show that
    (*i*) $\alpha^5 + \beta^5 + \gamma^5 = 5\alpha\beta\gamma(\beta\gamma + \gamma\alpha + \alpha\beta)$, (*ii*) $3\Sigma\alpha^2\,\Sigma\alpha^5 = 5\Sigma\alpha^3\,\Sigma\alpha^4$.

1.3 TRANSFORMATION OF EQUATIONS

(1) To find an equation whose roots are *m* times the roots of the given equation, *multiply the second term by m, third term by m^2 and so on* (all missing terms supplied with zero coefficients).

For instance, let the given equation be

$$3x^4 + 6x^3 + 4x^2 - 8x + 11 = 0 \qquad ...(i)$$

To multiply its roots by m, put $y = mx$ (or $x = y/m$) in (*i*).

Then $$3(y/m)^4 + 6(y/m)^3 + 4(y/m)^2 + 8(y/m) + 11 = 0$$

Multiplying by m^4, we get $$3y^4 + m(6y^3) + m^2(4y^2) - m^3(8y) + m^4(11) = 0$$

This is same as multiplying the second term by m, third term by m^2 and so on in (*i*).

Cor. To find an equation whose roots are with opposite signs to those of the given equation, *change the signs of the every alternative term of the given equation beginning with the second.*

Changing the signs of the roots of (*i*) is same as multiplying its roots by – 1.

∴ The required equation will be

$$3x^4 + (-1)6x^3 + (-1)^2\,4x^2 - (-1)^3\,8x + (-1)^4\,11 = 0$$

or $$3x^4 - 6x^3 + 4x^3 + 8x + 11 = 0$$

which is (*i*) with signs of every alternate term changed beginning with the second.

(2) To find an equation whose roots are reciprocal of the root of the given equation, *change x to 1/x.*

Example 1.9. *Solve $6x^3 - 11x^2 - 3x + 2 = 0$, given that its roots are in harmonic progression.*

Solution. Since the roots of the given equation are in H.P., the roots of the equation having reciprocal roots will be in A.P.

The equation with reciprocal roots is $6(1/x)^3 - 11(1/x)^2 - 3(1/x) + 2 = 0$

or $$2x^3 - 3x^2 - 11x + 6 = 0 \qquad ...(i)$$

Since the roots of the given equation are in H.P., therefore, the roots of (i) are in A.P. Let the root be $a - d$, a, $a + d$. Then

$$3a = 3/2 \text{ and } a(a^2 - d^2) = -3.$$

Solving these equations, we get $a = 1/2$, $d = 5/2$.

Thus the roots of (i) are -2, $1/2$, 3.

Hence the required roots of the given equation are $-1/2$, 2, $1/3$.

Example 1.10. *If α, β, γ be the roots of the cubic equation $x^3 - px^2 + qx - r = 0$, form the equation whose roots are $\beta\gamma + 1/\alpha$, $\gamma\alpha + 1/\beta$, $\alpha\beta + 1/\gamma$.*

Hence evaluate $\Sigma(\alpha\beta + 1/\gamma)(\beta\gamma + 1/\alpha)$. *(S.V.T.U., 2008)*

Solution. If x is a root of the given equation and y a root of the required equation, then

$$y = \beta\gamma + 1/\alpha = \frac{\alpha\beta\gamma + 1}{\alpha} = \frac{r+1}{\alpha} \qquad [\because \alpha\beta\gamma = r]$$

or $$y = \frac{r+1}{x} \Rightarrow x = \frac{r+1}{y}$$

Thus substituting $x = (r + 1)/y$ in the given equation, we get

$$\left(\frac{r+1}{y}\right)^3 - p\left(\frac{r+1}{y}\right)^2 + q\left(\frac{r+1}{y}\right) - r = 0$$

or $ry^3 - q(r + 1)y^2 + p(r + 1)^2 y - (r + 1)^3 = 0$, which is the required equation.

Hence $\Sigma(\alpha\beta + 1/\gamma)(\beta\gamma + 1/\alpha) = p(r + 1)^2/r$.

Example 1.11. *Form an equation whose roots are cubes of the roots of $x^3 - 3x^2 + 1 = 0$.* ...(i)

Solution. If y be a root of the required equation, then $y = x^3$ ...(ii)

Now we have to eliminate x from (i) and (ii)

$\therefore$ Rewriting (i) as $x^3 + 1 = 3x^2$

Cubing both sides, $x^9 + 3x^6 + 3x^3 + 1 = 27x^6$

Substituting $x^3 = y$, we get $y^3 - 24y^2 + 3y + 1 = 0$, which is the required equation.

(3) To diminish the roots of an equation $f(x) = 0$ by h, *divide $f(x)$ by $x - h$ successively. Then the successive remainders determine the coefficients of the required equation.*

Let the given equation be

$$a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = 0 \qquad ...(i)$$

To diminish its roots by h, put $y = x - h$ (or $x = y + h$) in (i) so that

$$a_0(y + h)^n + a_1(y + h)^{n-1} + ... + a_n = 0 \qquad ...(ii)$$

On simplification, it takes the form

$$A_0 y^n + A_1 y^{n-1} + ... + A_n = 0 \qquad ...(iii)$$

Its coefficient $A_0, A_1, ... A_n$ can easily be found with the help of *synthetic division* (p. 2). For this, we put $y = x - h$ in (iii) so that

$$A_0(x - h)^n + A_1(x - h)^{n-1} + ... + A_n = 0 \qquad ...(iv)$$

Clearly, (i) and (iv) are identical. If we divide L.H.S. of (iv) by $x - h$, the remainder is A_n and the quotient $Q = A_0(x - h)^{n-1} + A_1(x - h)^{n-2} + ... + A_{n-1}$. Similarly, if we divide Q by $x - h$, the remainder is A_{n-1} and the quotient is Q_1(say). Again dividing Q_1 by $x - h$, A_{n-2} will be obtained as remainder and so on.

Obs. To increase the roots by h, we take h negative.

Example 1.12. *Transform the equation $x^3 - 6x^2 + 5x + 8 = 0$ into another in which the second term is missing. Hence find the equation of its squared differences.* (Cochin, 2005)

Solution. Sum of the roots of the given equation = 6.

In order that the second term in the transformed equation is missing, the sum of the roots is to be zero.

Since the equation has 3 roots, if we decrease each root by 2, the sum of the roots of the new equation will become zero.

$\therefore$ Dividing $x^3 - 6x^2 + 5x + 8$ by $x - 2$ successively, we have

$$\begin{array}{ccccc}
1 & -6 & 5 & 8 & (2 \\
 & 2 & -8 & -6 & \\
\hline
 & -4 & -3 & \mathbf{2} & \\
 & 2 & -4 & & \\
\hline
 & -2 & \mathbf{-7} & & \\
 & 2 & & & \\
\hline
\mathbf{1} & \mathbf{0} & & &
\end{array}$$

Thus the transformed equation is $x^3 - 7x + 2 = 0$. ...(*i*)

If α, β, γ be the roots of the given equation, then the roots of (*i*) are $\alpha - 2, \beta - 2, \gamma - 2$.

Let these roots be denoted by a, b, c.

Then $b - c = \beta - \gamma$. Also $a + b + c = 0, abc = -2$.

Now $(b - c)^2 = (b + c)^2 - 2bc = (a + b + c - a)^2 - \dfrac{2abc}{a} = a^2 + 4/a$

$\therefore$ The equation of squared differences of (*i*) is given by the transformation $y = x^2 + 4/x$

or $$x^3 - xy + 4 = 0 \qquad ...(ii)$$

Subtracting (*ii*) from (*i*), we get $-7x + xy - 2 = 0$ or $x = 2/(y - 7)$

Substituting for x in (*i*), the equation becomes

$$[2/(y - 7)]^3 - 7[2/(y - 7)] + 2 = 0 \quad \text{or} \quad y^3 - 28y^2 + 245y - 682 = 0 \qquad ...(iii)$$

Roots of this equation are $(b - c)^2, (c - a)^2, (a - b)^2$ *i.e.*, $(\beta - \gamma)^2, (\gamma - \alpha)^2, (\alpha - \beta)^2$.

Hence (*iii*) is the required equation.

1.4 RECIPROCAL EQUATIONS

If an equation remains unaltered on changing x to $1/x$, it is called a **reciprocal equation.**

Such equations are of the following standard types :

I. *A reciprocal equation of an odd degree having coefficients of terms equidistant from the beginning and end equal.* It has a root $= -1$.

II. *A reciprocal equation of an odd degree having coefficients of terms equidistant from the beginning and end equal but opposite in sign.* It has root $= 1$.

III. *A reciprocal equation of an even degree having coefficients of terms equidistant from the beginning and end equal but opposite in sign.* Such an equation has two roots $= 1$ and -1.

The substitution $x + 1/x = y$ reduces the degree of the equation of half its former degree.

Example 1.13. *Solve $6x^5 - 41x^4 + 97x^3 - 97x^2 + 41x - 6 = 0$.* (Coimbatore, 2001 S)

Solution. This is a reciprocal equation of odd degree with opposite signs. $\therefore$ $x = 1$ is a root.

Dividing L.H.S. by $x - 1$, the given equation reduces to

$$6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$$

Dividing by x^2, we have

$$6(x^2 + 1/x^2) - 35(x + 1/x) + 62 = 0$$

Putting $x + 1/x = y$ and $x^2 + 1/x^2 = y^2 - 2$, we get

$$6(y^2 - 2) - 35y + 62 = 0 \quad \text{or} \quad 6y^2 - 35y + 50 = 0 \quad \text{or} \quad (3y - 1)(2y - 5) = 0$$

$\therefore$ $$x + 1/x = y = 1/3 \quad \text{or} \quad 5/2$$

i.e., $3x^2 - 10x + 3 = 0$ or $2x^2 - 5x + 2 = 0$

i.e., $(3x - 1)(x - 3) = 0$ or $(2x - 1)(x - 2) = 0$

$\therefore$ $x = 1/3, 3$ or $1/2, 2$

Hence the required roots are 1, 1/3, 3, 1/2, 2.

Example 1.14. *Solve* $6x^6 - 25x^5 + 31x^4 - 31x^2 + 25x - 6 = 0.$ *(Madras, 2003)*

Solution. This is a reciprocal equation of even degree with opposite signs. $\therefore$ $x = 1, -1$ are its roots.

Dividing L.H.S. by $x - 1$ and $x + 1$, the given equation reduces to

$$6x^4 - 25x^3 + 37x^2 - 25x + 6 = 0$$

Dividing by x^2, we get

$$6(x^2 + 1/x^2) - 25(x + 1/x) + 37 = 0.$$

Putting $x + 1/x = y$ and $x^2 + 1/x^2 = y^2 - 2$, it becomes

$$6(y^2 - 2) - 25y + 37 = 0 \quad \text{or} \quad 6y^2 - 25y + 25 = 0$$

or $(2y - 5)(3y - 5) = 0$

$\therefore$ $x + 1/x = y = 5/2$ or $5/3$.

i.e., $2x^2 - 5x + 2 = 0$ or $3x^2 - 5x + 3 = 0$

$\therefore$ $x = 2, 1/2$ or $x = \dfrac{5 \pm i\sqrt{11}}{6}$

Hence the required roots of the given equation are $1, -1, 2, 1/2, \dfrac{5 \pm i\sqrt{11}}{6}$.

PROBLEMS 1.2

1. Find the equation whose roots are 3 times the roots of $x^3 + 2x^2 - 4x + 1 = 0$.
2. Form the equation whose roots are the reciprocals of the roots of $2x^5 + 4x^3 - 13x^2 + 7x - 6 = 0$. *(S.V.T.U., 2009)*
3. Find the equation whose roots are the negative reciprocals of the roots of
$$x^4 + 7x^3 + 8x^2 - 9x + 10 = 0.$$
4. Solve the equation $6x^3 - 11x^2 - 3x + 2 = 0$, given that its roots are in H.P.
5. Find the equation whose roots are the roots of
   (*i*) $x^3 - 6x^2 + 11x - 6 = 0$ each increased by 1. *(S.V.T.U., 2009)*
   (*ii*) $x^4 + x^3 - 3x^2 - x + 2 = 0$ each diminished by 3.
   (*iii*) $x^5 - 5x^4 + 10x^3 - 10x^2 + 5x + 6 = 0$ each diminished by 1.
6. Find the equation whose roots are the squares of the roots of $x^3 - x^2 + 8x - 6 = 0$.
7. Find the equation whose roots are the cubes of the roots of $x^3 + px^2 + q = 0$.
8. If α, β, γ are the roots of the equation $2x^3 + 3x^2 - x - 1 = 0$, form the equation whose roots are $(1 - \alpha)^{-1}$, $(1 - \beta)^{-1}$ and $(1 - \gamma)^{-1}$.
9. If a, b, c are the roots of the equation $x^3 + px^2 + qx + r = 0$, find the equation whose roots are ab, bc and ca. *(Madras, 2003)*
10. If α, β, γ be the roots of $x^3 + mx + n = 0$, form the equation whose roots are
    (*a*) $\alpha + \beta - \gamma, \beta + \gamma - \alpha, \gamma + \alpha - \beta$, (*b*) $\beta\gamma/\alpha, \gamma\alpha/\beta, \alpha\beta/\gamma$ (*c*) $\dfrac{1}{\beta} + \dfrac{1}{\gamma}, \dfrac{1}{\gamma} + \dfrac{1}{\alpha}, \dfrac{1}{\alpha} + \dfrac{1}{\beta}$.
11. Find the equation of squared differences of the roots of the cubic $x^3 + 6x^2 + 7x + 2 = 0$.
12. Solve the equations :
    (*i*) $6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0$ (*ii*) $4x^4 - 20x^3 + 33x^2 - 20x + 4 = 0$. *(Madras, 2003)*
    (*iii*) $8x^5 - 22x^4 - 55x^3 + 55x^2 + 22x - 8 = 0$. (*iv*) $6x^5 + x^4 - 43x^3 - 43x^2 + x + 6 = 0$ *(S.V.T.U., 2006)*
    (*v*) $3x^6 + x^5 - 27x^4 + 27x^2 - x - 3 = 0$.
13. Show that the equation $x^4 - 10x^3 + 23x^2 - 6x - 15 = 0$ can be transformed into reciprocal equation by diminishing the roots by 2. Hence solve the equation.
14. By suitable transformation, reduce the equation $x^4 + 16x^3 + 83x^2 + 152x + 84 = 0$ to an equation in which term in x^3 is absent and hence solve it. *(Madras, 2002)*

1.5 SOLUTION OF CUBIC EQUATIONS–CARDAN'S METHOD*

Consider the equation $ax^3 + bx^2 + cx + d = 0$ ...(1)

Dividing by a, we get an equation of the form $x^3 + lx^2 + mx + n = 0$.

To remove the x^2 term, put $y = x - (-l/3)$ or $x = y - l/3$ so that the resulting equation is of the form

$$y^3 + py + q = 0 \quad ...(2)$$

To solve (2), put $y = u + v$

so that $y^3 = u^3 + v^3 + 3uv(u + v) = u^3 + v^3 + 3uvy$

or $y^3 - 3uvy - (u^3 + v^3) = 0$ ...(3)

Comparing (2) and (3), we get

$$uv = -p/3,\ u^3 + v^3 = -q \text{ or } u^3 + v^3 = -q \text{ and } u^3 v^3 = -p^3/27$$

$\therefore$ u^3, v^3 are the roots of the equation $t^2 + qt - p^3/27 = 0$

which gives $u^3 = \frac{1}{2}(-q + \sqrt{q^2 + 4p^3/27}) = \lambda^3$ (say)

and $v^3 = \frac{1}{2}(-q - \sqrt{q^2 + 4p^3/27})$

$\therefore$ The three values of u are $\lambda, \lambda\omega, \lambda\omega^2$, where ω is one of the imaginary cube roots of unity.

From $uv = -p/3$, we have $v = -p/3u$

$\therefore$ When $u = \lambda, \lambda\omega$ and $\lambda\omega^2$,

$$v = -\frac{p}{3\lambda}, -\frac{p\omega^2}{3\lambda} \text{ and } -\frac{p\omega}{3\lambda}. \quad [\because \omega^3 = 1]$$

Hence the three roots of (2) are $\lambda - \frac{p}{3\lambda}, \lambda\omega - \frac{p\omega^2}{3\lambda}, \lambda\omega^2 - \frac{p\omega}{3\lambda}$ (Being $= u + v$)

Having known y, the corresponding values of x can be found from the relation $x = y - l/3$.

Obs. 1. If one value of u is found to be a rational number, find the corresponding value of v giving one root $y = u + v$. Then find the corresponding root $x = \alpha$ (say). Finally, divide the left hand side of (1) by $x - \alpha$, giving the remaining quadratic equation from which the other two roots can be found readily.

Obs. 2. If u^3 and v^3 turn out to be conjugate complex numbers, the roots of the given cubic can be obtained in neat forms by employing De Moivre's theorem. (§ 19.5)

Example 1.15. *Solve by Cardan's method $x^3 - 3x^2 + 12x + 16 = 0$.* *(U.P.T.U., 2008)*

Solution. Given equation is $x^3 - 3x^2 + 12x + 16 = 0$ ...(*i*)

To remove the second term from (i), diminish each root of (*i*) by 3/3 = 1, *i.e.*, put $y = x - 1$ or $x = y + 1$ [$\therefore$ Sum of roots = 3]. Then (*i*) becomes

$$(y + 1)^3 - 3(y + 1) + 12(y + 1) + 16 = 0 \quad \text{or} \quad y^3 + 9y^2 + 26 = 0 \quad ...(ii)$$

To solve (*ii*), put $y = u + v$ so that $y^3 - 3uvy - (u^3 + v^3) = 0$ ...(*iii*)

Comparing (*ii*) and (*iii*), we get $uv = -3$ and $u^3 + v^3 = -26$

$\therefore$ u^3, v^3 are the roots of the equation $t^2 + 26t - 27 = 0$

or $(t + 27)(t - 1) = 0$ whence $t = -27, t = 1$.

or $u^3 = -27$ *i.e.*, $u = -3$ and $v^3 = 1$ *i.e.*, $v = 1$

$\therefore$ $y = u + v = -3 + 1 = -2$ and $x = y + 1 = -1$

Dividing L.H.S. of (*i*) by $x + 1$, we obtain $x^2 - 4x + 16 = 0$

or $$x = \frac{4 \pm \sqrt{(16 - 64)}}{2} = 2 \pm i\ 2\sqrt{3}$$

Hence the required roots of the given equation are $-1, 2 \pm i\ 2\sqrt{3}$.

*Named after an Italian mathematician *Girolamo Cardan* (1501–1576) who was the first to use complex number as roots of an equation.

Example 1.16. *Solve the cubic equation $28x^3 - 9x^2 + 1 = 0$ by Cardan's method.*

Solution. Since the term in x is missing, let us put $x = 1/y$ in the given equation so that the transformed equation is $y^3 - 9y + 28 = 0$ ...(*i*)

To solve (*i*), put $y = u + v$ so that $y^3 - 3uvy - (u^3 + v^3) = 0$ ...(*ii*)

Comparing (*ii*) and (*iii*), we get $uv = 3$ and $u^3 + v^3 = -28$.

$\therefore$ u^3, v^3 are the roots of $t^2 + 28t + 27 = 0$

or $(t + 1)(t + 27) = 0$ or $t = -1, -27$ or $u = -1, v = -3$

$\therefore$ $y = u + v = -4$. Dividing L.H.S. of (*i*) by $y + 4$, we obtain $y^2 - 4y + 7 = 0$ whence $y = 2 \pm i\sqrt{3}$.

$\therefore$ Roots of (*i*) are $-4, 2 \pm i\sqrt{3}$.

Hence the roots of the given cubic equation are $-\dfrac{1}{4}, \dfrac{1}{2 \pm i\sqrt{3}}$ or $-\dfrac{1}{4}, (2 - i\sqrt{3})/7, (2 + i\sqrt{3})/7$.

Example 1.17. *Solve the equation $x^3 + x^2 - 16x + 20 = 0$.*

Solution. Instead of diminishing the roots of the given equation by $-1/3$, we first multiply its roots by 3, so that the equation becomes

$$x^3 + 3x^2 - 144x + 540 = 0 \qquad ...(i)$$

To remove the x^2 term, put $y = x - (-3/3)$ or $x = y - 1$ in (*i*)

so that $$(y - 1)^3 + 3(y - 1)^2 - 144(y - 1) + 540 = 0$$

or $$y^3 - 147y + 686 = 0 \qquad ...(ii)$$

To solve (iii), let $y = u + v$, so that

$$y^3 - 3uvy - (u^3 + v^3) = 0 \qquad ...(iii)$$

Comparing (*ii*) and (*iii*), we get

$$uv = 49, u^3 + v^3 = -686, \text{ so that } u^3 v^3 = (343)^2.$$

$\therefore$ u^3, v^3 are the roots of the quadratic

$$t^2 + 686t + (343)^2 = 0 \quad \text{or} \quad (t + 343)^2 = 0$$

$\therefore$ $t = -343$ *i.e.*, $u^3 = v^3 = -343$ or $u = v = -7$.

Thus $y = u + v = -14$ and $x = y - 1 = -15$.

Dividing L.H.S. of (*i*) by $x + 15$, we get

$$(x - 6)^2 = 0 \quad \text{or} \quad x = 6, 6.$$

$\therefore$ The root of (*i*) are $-15, 6, 6$.

Hence the roots of the given equation are $-5, 2, 2$.

Example 1.18. *Solve $x^3 - 3x^2 + 3 = 0$.* *(S.V.T.U., 2006)*

Solution. Given equation is $x^3 - 3x^2 + 3 = 0$ ...(*i*)

To remove the x^2 term, put $y = x - 3/3$ or $x = y + 1$,

so that (*i*) becomes $(y + 1)^3 - 3(y + 1)^2 + 3 = 0$

or $$y^3 - 3y + 1 = 0 \qquad ...(ii)$$

To solve it, put $y = u + v$

so that $$y^3 - 3uvy - (u^3 + v^3) = 0 \qquad ...(iii)$$

Comparing (*ii*) and (*iii*), we get $uv = 1, u^3 + v^3 = -1$

$\therefore$ u^3, v^3 are the roots of the equation $t^2 + t + 1 = 0$

Hence $u^3 = \dfrac{-1 + i\sqrt{3}}{2}$ and $v^3 = \dfrac{-1 - i\sqrt{3}}{2}$

$\therefore$ $u = \left(\dfrac{-1 + i\sqrt{3}}{2}\right)^{1/3}$ put $-\dfrac{1}{2} = r\cos\theta$ and $\sqrt{3}/2 = r\sin\theta$

$= [r(\cos\theta + i\sin\theta)]^{1/3}$ so that $r = 1, \theta = 2\pi/3$

$= [\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi)]^{1/3}$,

where n is any integer or zero. Using De Moivre's theorem (p. 647).

$$u = \cos\left(\frac{\theta + 2n\pi}{3}\right) + i \sin\left(\frac{\theta + 2n\pi}{3}\right)$$

Giving n the value 0, 1, 2 successively we get the three values of u to be

$$\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}, \cos\frac{\theta + 2\pi}{3} + i\sin\frac{\theta + 2\pi}{3}, \cos\frac{\theta + 4\pi}{3} + i\sin\frac{\theta + 4\pi}{3}$$

i.e.,
$$\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}, \cos\frac{8\pi}{9} + i\sin\frac{8\pi}{9}, \cos\frac{14\pi}{9} + i\sin\frac{14\pi}{9}.$$

The corresponding values of v are

$$\cos\frac{2\pi}{9} - i\sin\frac{2\pi}{9}, \cos\frac{8\pi}{9} - i\sin\frac{8\pi}{9}, \cos\frac{14\pi}{9} - i\sin\frac{14\pi}{9}.$$

$\therefore$ The three values of $y = u + v$ are $2\cos 2\pi/9$, $2\cos 8\pi/9$, $2\cos 14\pi/9$.

Hence the roots of (i) are found from $x = 1 + y$ to be

$$1 + 2\cos 2\pi/9,\ 1 + 2\cos 8\pi/9,\ 1 + 2\cos 14\pi/9.$$

PROBLEMS 1.3

Solve the following equations by Cardan's method :

1. $x^3 - 27x + 54 = 0$. *(U.P.T.U., 2003)*
2. $x^3 - 18x + 35 = 0$ *(Osmania, 2003)*
3. $x^3 - 15x = 126$ *(S.V.T.U., 2009)*
4. $2x^3 + 5x^2 + x - 2 = 0$ *(U.P.T.U., 2003)*
5. $9x^3 + 6x^2 - 1 = 0$ *(S.V.T.U., 2008)*
6. $x^3 - 6x^2 + 6x - 5 = 0$
7. $x^3 - 3x + 1 = 0$
8. $27x^3 + 54x^2 + 198x - 73 = 0$

1.6 SOLUTION OF BIQUADRATIC EQUATIONS

(1) Ferrari's method

This method of solving a biquadratic equation is illustrated by the following examples :

Example 1.19. *Solve the equation $x^4 - 12x^3 + 41x^2 - 18x - 72 = 0$ by Ferrari's method.* *(S.V.T.U., 2007)*

Solution. Combining x^4 and x^3 terms into a perfect square, the given equation can be written as

$$(x^2 - 6x + \lambda)^2 + (5 - 2\lambda)x^2 + (12\lambda - 18)x - (\lambda^2 + 72) = 0$$

or
$$(x^2 - 6x + \lambda)^2 = \{(2\lambda - 5)x^2 + (18 - 12\lambda)x + (\lambda^2 + 72)\} \qquad ...(i)$$

This equation can be factorised if R.H.S. is a perfect square

i.e., if $(18 - 12\lambda)^2 = 4(2\lambda - 5)(\lambda^2 + 72)$ $[b^2 = 4ac]$

i.e., if $2\lambda^3 - 41\lambda^2 + 252\lambda - 441 = 0$ which gives $\lambda = 3$.

$\therefore$ (i) reduces to $(x^2 - 6x + 3)^2 = (x - 9)^2$

i.e., $(x^2 - 5x - 6)(x^2 - 7x + 12) = 0$.

Hence the roots of the given equation are -1, 3, 4 and 6.

Example 1.20. *Solve the equation $x^4 - 2x^3 - 5x^2 + 10x - 3 = 0$ by Ferrari's method.*

Solution. Combining x^4 and x^3 terms into a perfect square, the given equation can be written as $(x^2 - x + \lambda)^2 = (2\lambda + 6)x^2 - (2\lambda + 10)x + (\lambda^2 + 3)$. This equation can be factorised, if R.H.S. is a perfect square *i.e.*, if $(2\lambda + 10)^2 = 4(2\lambda + 6)(\lambda^2 + 3)$ $[b^2 = 4ac]$

or
$$2\lambda^3 + 5\lambda^2 - 4\lambda - 7 = 0, \text{ which gives } \lambda = -1.$$

$\therefore$ (i) reduces to $(x^2 - x - 1)^2 = 4x^2 - 8x + 4$

or
$$(x^2 - x - 1)^2 - (2x - 2)^2 = 0 \quad \text{or} \quad (x^2 + x - 3)(x^2 - 3x + 1) = 0$$

$\therefore$
$$x = \frac{-1 \pm \sqrt{1 + 12}}{2} \quad \text{or} \quad \frac{3 \pm \sqrt{9 - 4}}{2}$$

Hence the roots are $\dfrac{-1 \pm \sqrt{13}}{2}, \dfrac{3 \pm \sqrt{5}}{2}$.

(2) Descarte's method

This method of solving a biquadratic equations consists in removing the term in x^3 and then expressing the new equation as product of two quadratics. It has been best illustrated by the following examples :

Example 1.21. *Solve the equation $x^4 - 8x^2 - 24x + 7 = 0$ by Descarte's method.* *(U.P.T.U., 2001)*

Solution. In the given equation, the term in x^3 is already absent so we assume that

$$x^4 - 8x^2 - 24x + 7 = (x^2 + px + q)(x^2 - px + q') \quad ...(i)$$

Equating coefficients of the like powers of x in (i), we get

$$-8 = q + q' - p^2, -24 = p(q' - q) ; 7 = qq'$$

$\therefore \quad q + q' = p^2 - 8, q - q' = 24/p$

$\therefore \quad (p^2 - 8)^2 - (24/p)^2 = 4 \times 7$

$$p^2 - 16p^4 + 36p^2 - 576 = 0 \quad \text{or} \quad t^3 - 16t^2 + 36t - 576 = 0 \text{ where } t = p^2$$

Now $t = 16$ satisfies this cubic so that $p = 4$.

$\therefore \quad q + q' = 8, q - q' = 6 \quad \therefore \quad q = 7, q' = 1$

Thus (i) takes the form $(x^2 + 4x + 7)(x^2 - 4x + 1) = 0$

whence $$x = \frac{-4 \pm \sqrt{(16 - 28)}}{2} \text{ or } x = \frac{4 \pm \sqrt{(16 - 4)}}{2}$$

Hence $x = -2 \pm \sqrt{3}i, 2 \pm \sqrt{3}$.

Example 1.22. *Solve the equation $x^4 - 6x^3 - 3x^2 + 22x - 6 = 0$ by Descarte's method.*

Solution. Here sum of roots = 6 and number of roots = 4

$\therefore$ To remove the second term, we have to diminish the roots by 6/4 (= 3/2) which will be a problem. Therefore, we first multiply the roots by 2. $\therefore$ $y^4 - 12y^3 + 12y^2 + 176y - 96 = 0$ where $y = 2x$. Now diminishing the roots by 3, we obtain $z^4 - 42z^2 + 32z + 297 = 0$ where $z = y - 3$.

Assuming that $$z^4 - 42z^2 + 32z + 297 = (z^2 + pz + q)(z^2 - pz + q') \quad ...(i)$$

and comparing coefficients, we get

$$-42 = q + q' - p^2 ; 32 = p(q' - q) ; 297 = q q'$$

$\therefore \quad q + q' = p^2 - 42 ; q - q' = -32/p, q q' = 297$

$\therefore \quad (p^2 - 42)^2 - (-32/p)^2 = 4 \times 297$

or $\quad t^3 - 84t^2 + 576t - 1024 = 0 \quad$ where $t = p^2$

Now $t = 4$ satisfies this cubic so that $p = 2$.

$\therefore \quad q + q' = -38, q - q' = -16, \quad \therefore \quad q = -27, q' = -11.$

Thus (i) takes the form $(z^2 + 2z - 27)(z^2 - 2z - 11) = 0$

Whence $$z = \frac{-2 \pm \sqrt{(4 + 108)}}{2} \quad \text{or} \quad z = \frac{2 \pm \sqrt{(4 + 44)}}{2}$$

or $$x = \frac{1}{2}y = \frac{1}{2}(z + 3) = \frac{1}{2}(2 \pm \sqrt{28}) = \frac{1}{2}(4 \pm \sqrt{12})$$

Hence $\quad x = 1 \pm \sqrt{7}, 2 \pm \sqrt{3}$.

PROBLEMS 1.4

Solve by Ferrari's method, the equations :

1. $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$ *(U.P.T.U., 2003)*
2. $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0$ *(U.P.T.U., 2002)*
3. $x^4 - 10x^2 - 20x - 16 = 0$
4. $x^4 - 8x^3 - 12x^2 + 60x + 63 = 0$ *(U.P.T.U., 2005)*

Solve the following equations by Descartes method :

5. $x^4 - 6x^3 + 3x^2 + 22x - 6 = 0$
6. $x^4 + 12x - 5 = 0$
7. $x^4 - 8x^3 - 24x + 7 = 0$ *(U.P.T.U., 2001)*
8. $x^4 - 10x^3 + 44x^2 - 104x + 96 = 0$

Obs. We have obtained algebraic solutions of cubic and biquadratic equations. But the need often arises to solve higher degree or transcendental equations for which no algebraic methods are available in general. Such equations can be best solved by *graphical method* (explained below) or by *numerical methods* (§28.2).

1.7 GRAPHICAL SOLUTION OF EQUATIONS

Let the equation be $f(x) = 0$.

(i) *Find the interval (a, b) in which a root of f(x) = 0 lies.*

[At least one root of $f(x) = 0$ lies in (a, b) if $f(a)$ and $f(b)$ are of opposite signs—§1.2(III) p. 2].

(ii) *Write the equation $f(x) = 0$ as $\phi(x) = \psi(x)$ where $\psi(x)$ contains only terms in x* and the constants.

(iii) *Draw the graphs of $y = \phi(x)$ and $y = \psi(x)$ on the same scale and with respect to the same axes.*

(iv) *Read the abscissae of the points of intersection of the curves $y = \phi(x)$ and $y = \psi(x)$. These are required real roots of $f(x) = 0$.*

Sometimes it may not be convenient to write the given equation $f(x) = 0$ in the form $\phi(x) = \psi(x)$. In such cases, we proceed as follows :

(i) Form a table for the value of x and $y = f(x)$ directly.

(ii) Plot these points and pass a smooth curve through them.

(iii) Read the abscissae of the points where this curve cuts the x-axis. These are the required roots of $f(x) = 0$.

Obs. The roots, thus located graphically are approximate and to improve their accuracy, the curves are replotted on the larger scale in the immediate vicinity of each point of intersection. This gives a better approximation to the value of desired root. The above graphical operation may be repeated until the root is obtained correct upto required number of decimal places. But this method of repeatedly drawing graphs is very tedious. It is, therefore, advisable to improve upon the accuracy of an approximate root by numerical method of §28.2.

Example 1.23. *Find graphically an approximate value of the root of the equation.*

$$3 - x = e^{x-1}.$$

Solution. Let $\quad f(x) = e^{x-1} + x - 3 = 0 \qquad \ldots(i)$

$f(1) = 1 + 1 - 3 = -\text{ ve}$

and $\quad f(2) = e + 2 - 3 = 2.718 - 1 = +\text{ ve}$

$\therefore$ A root of (i), lies between $x = 1$ and $x = 2$.

Let us rewrite (i) as $\quad e^{x-1} = 3 - x$.

The abscissa of the point of intersection of the curves

$$y = e^{x-1} \qquad \ldots(ii)$$

and
$$y = 3 - x \qquad \ldots(iii)$$

will give the required root.

To plot (ii), we form the following table of values :

$x =$	$y = e^{x-1}$
1.1	1.11
1.2	1.22
1.3	1.35
1.4	1.49
1.5	1.65
1.6	1.82
1.7	2.01
1.8	2.23
1.9	2.46
2.0	2.72

Taking the origin at (1, 1) and 1 small unit along either axis = 0.02, we plot these points and pass a smooth curve through them as shown in Fig. 1.2.

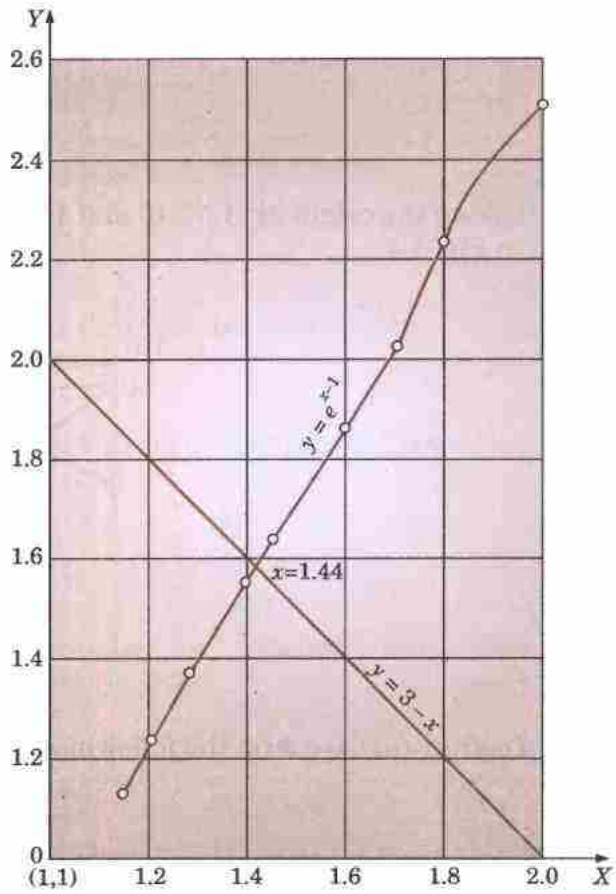


Fig. 1.2

To draw the line (iii), we join the points (1, 2) and (2, 1) on the same scale and with the same axes.

From the figure, we get the required root to be $x = 1.44$ nearly.

Example 1.24. *Obtain graphically an approximate value of the root of $x = \sin x + \pi/2$.*

Solution. Let us write the given equation as $\sin x = x - \pi/2$

The abscissa of the point of intersection of the curve $y = \sin x$ and the line $y = x - \pi/2$ will give a rough estimate of the root.

To draw a curve $y = \sin x$, we form the following table :

x	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
y	0	0.71	1	0.71	0

Taking 1 unit along either axis = $\pi/4 = 0.8$ nearly, we plot the curve as shown in Fig. 1.3.

Also we draw the line $y = x - \pi/2$ to the same scale and with the same axis.

From the graph, we get $x = 2.3$ radians approximately.

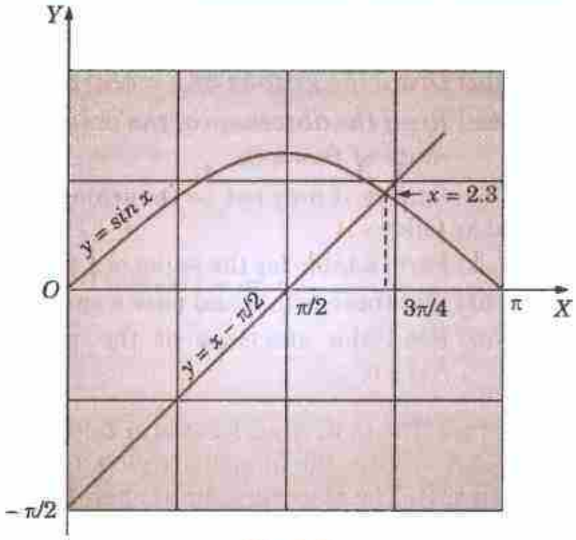


Fig. 1.3

Example 1.25. *Obtain graphically the lowest root of $\cos x \cosh x = -1$.*

Solution. Let $f(x) = \cos x \cosh x + 1 = 0$ ...(*i*)

$\because$ $f(0) = +\text{ ve}, f(\pi/2) = +\text{ ve and } f(\pi) = -\text{ ve}.$

$\therefore$ The lowest root of (*i*) lies between $x = \pi/2$ and $x = \pi$.

Let us write (*i*) as $\cos x = -\operatorname{sech} x$.

The abscissa of the point of intersection of the curves

$y = \cos x$ ...(*ii*) and $y = -\operatorname{sech} x$ ...(*iii*)

will give the required root. To draw (*ii*), we form the following table of values :

$x =$	$\pi/2 = 1.57$	$3\pi/4 = 2.36$	$\pi = 3.14$
$y = \cos x$	0	-0.71	-1

Taking the origin at (1.57, 0) and 1 unit along either axes = $\pi/8 = 0.4$ nearly, we plot the cosine curve as shown in Fig. 1.4.

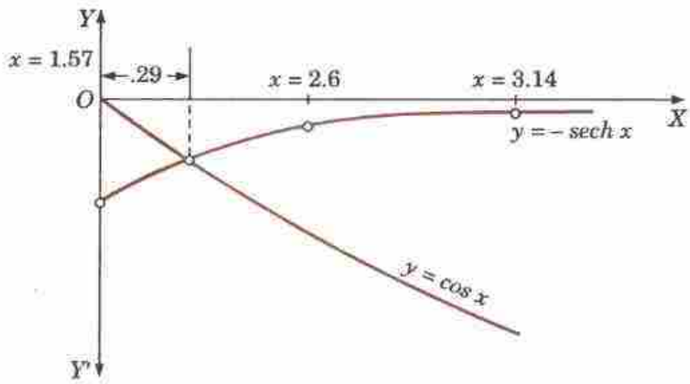


Fig. 1.4

To draw (iii), we form the following table :

$x =$	1.57	2.36	3.14
$\cosh x =$	2.58	5.56	11.12
$y = -\operatorname{sech} x$	-0.39	-0.18	-0.09

Then we plot the curve (*iii*) to the same scale with the same axes.

From the figure we get the lowest root to be approximately $x = 1.57 + 0.29 = 1.86$.

PROBLEMS 1.5

Solve the following equations graphically :

1. $x^3 - x - 1 = 0$ (*Madras, 2000 S*)
2. $x^3 - 3x - 5 = 0$
3. $x^3 - 6x^2 + 9x - 3 = 0$.
4. $\tan x = 1.2\,x$
5. $x = 3\cos(x - \pi/4)$
6. $e^x = 5x$ which is near $x = 0.2$.

1.8 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 1.6

Choose the correct answer or fill up the blanks in the following problems :

1. If for the equation $x^3 - 3x^2 + kx + 3 = 0$, one root is the negative of another, then the value of k is
(*a*) 3 (*b*) – 3 (*c*) 1 (*d*) – 1.
2. If the roots of the equation $x^n - 1 = 0$ are $1, \alpha_1, \alpha_2, ..., \alpha_{n-1}$, then $(1 - \alpha_1)(1 - \alpha_2) ...... (1 - \alpha_{n-1})$ is equal to
(*a*) 0 (*b*) 1 (*c*) n (*d*) $n + 1$.
3. If α, β, γ are the roots of $2x^3 - 3x^2 + 6x + 1 = 0$, then $\alpha^2 + \beta^2 + \gamma^2$ is
(*a*) 15/4 (*b*) – 3 (*c*) – 15/4 (*d*) 33/4.
4. $x + 2$ is a factor of
(*a*) $x^4 + 2$ (*b*) $x^4 - x^2 + 12$
(*c*) $x^4 - 2x^3 - x + 2$ (*d*) $x^4 + 2x^3 - x - 2$
5. If $\alpha + \beta + \gamma = 5$; $\alpha\beta + \beta\gamma + \gamma\alpha = 7$; $\alpha\beta\gamma = 3$, then the equation whose roots are α, β and γ is
(*a*) $x^3 - 7 = 0$ (*b*) $x^3 - 7x^2 + 3 = 0$
(*c*) $x^3 - 5x^2 + 7x - 3 = 0$ (*d*) $x^3 + 7x^2 - 3 = 0$.
6. If one of the roots of the equation $x^3 - 6x^2 + 11x - 6 = 0$ is 2, then the other two roots are
(*a*) 1 and 3 (*b*) 0 and 4
(*c*) – 1 and 5 (*d*) – 2 and 6.
7. The equation whose roots are the reciprocals of the roots of $x^3 + px^2 + r = 0$ is
(*a*) $x^3 + 1/p.x^2 + 1/r = 0$ (*b*) $1/r \,.\, x^3 + 1/p.x + 1 = 0$
(*c*) $rx^3 + px^2 + 1 = 0$ (*d*) $rx^3 + px + 1 = 0$.
8. If 1 and 2 are two roots of the equation $x^4 - x^3 - 19x^2 + 49x - 30 = 0$, then the remaining two roots are
(*a*) – 3 and 5 (*b*) 3 and – 5
(*c*) – 6 and 5 (*d*) 6 and –5.
9. If the roots of $x^3 - 3x^2 + px + 1 = 0$, are in arithmetic progression, then the sum of squares of the largest and the smallest roots is
(*a*) 3 (*b*) 5 (*c*) 6 (*d*) 10.
10. A root of $x^3 - 8x^2 + px + q = 0$ where p and q are real numbers is $3 + i\sqrt{3}$. The real root is
(*a*) 2 (*b*) 6 (*c*) 9 (*d*) 12.
11. One of the roots of the equation $f(x) = x^n + a_{n-1}x^{n-1} + ... + a_1x + a_0 = 0$ where $a_0, a_1, ... a_{n-1}$ are real, is given to be $2 - 3i$. Of the remaining, the next $n - 2$ roots are given to be 1, 2, 3, ..., $n - 2$. The nth root is
(*a*) n (*b*) $n - 1$ (*c*) $2 + 3i$ (*d*) $-2 + 3i$.
12. If a real root of $f(x) = 0$ lies in $[a, b]$, then the sign of $f(a) \,.\, f(b)$ is ......
13. Descartes rule of signs states that ......
14. If α, β, γ are the roots of the equation $x^3 - px + q = 0$, then $\Sigma\, 1/\alpha =$ ......
15. If α, β, γ are the roots of $x^3 = 7$, then $\Sigma\alpha^3$ is ......
16. One real root of the equation $x^3 + 2x^2 + 5 = 0$ lies between ......

17. In an equation with real coefficients, imaginary roots must occur in ......
18. If $f(\alpha)$ and $f(\beta)$ are of opposite signs, then $f(x) = 0$ has at least one root between α and β provided ......
19. If α, β, γ are the roots of the equation $x^3 + 2x + 3 = 0$, then $\alpha + 3, \beta + 3, \gamma + 3$ are the roots of the equation ......
20. If one root is double of another in $x^3 - 7x^2 + 36 = 0$, then its roots are ......
21. The equation whose roots are 10 times those $x^3 - 2x - 7 = 0$, is ......
22. If α, β, γ are the roots of $x^3 + px^2 + qx + r = 0$, then $\Sigma (1/\alpha\beta) =$ ......
23. $\sqrt{3}$ and $-1 + i$ are the roots of the biquadratic equation ......
24. If α, β, γ are the roots of $x^3 - 3x + 2 = 0$, then the value of $\alpha^2 + \beta^2 + \gamma^2$ is ......
25. If there is a root of $f(x) = 0$ in the interval $[a, b]$, then sign of $f(a)/f(b)$ is ......
26. If α, β, γ are the roots of $x^3 + px^2 + qx + r = 0$, then the condition for $\alpha + \beta = 0$ is ......
27. The three roots of $x^3 = 1$ are ......
28. One real root of the equation $x^3 + x - 5 = 0$ lies in the interval
    (*i*) (2, 3), (*ii*) (3, 4), (*iii*) (1, 2), (*iv*) (– 3, – 2)
29. If two roots of $x^3 - 3x^2 + 2 = 0$ are equal, then its roots are ......
30. The cubic equation whose two roots are 5 and $1 - i$ is ......
31. The sum and product of the roots of the equation $x^5 = 2$ are ...... and .....
32. If the roots of the equation $x^4 + 2x^3 - ax^2 - 22x + 40 = 0$ are $-5, -2, 1$ and 4, then $a =$ ......
33. A root of $x^3 - 3x^2 + 2.5 = 0$ lies between 1.1 and 1.2. (True or False)
34. The equation $x^6 - x^5 - 10x + 7 = 0$ has four imaginary roots. (True or False)

CHAPTER 2

Linear Algebra : Determinants, Matrices

1. Introduction. 2. Determinants, Cofactors, Laplace's expansion. 3. Properties of determinants. 4. Matrices, Special matrices. 5. Matrix operations. 6. Related matrices. 7. Rank of a matrix, Elementary transformations, Elementary matrices, Inverse from elementary matrices, Normal form of a matrix. 8. Partition method. 9. Solution of linear system of equations. 10. Consistency of linear system of equations. 11. Linear and orthogonal transformations. 12. Vectors ; Linear dependence. 13. Eigen values and eigen vectors. 14. Properties of eigen values. 15. Cayley-Hamilton theorem. 16. Reduction to diagonal form. 17. Reduction of quadratic form to canonical form. 18. Nature of quadratic form. 19. Complex matrices. 20. Objective Types of Questions.

2.1 INTRODUCTION

Linear algebra comprises of the theory and applications of linear system of equation, linear transformations and eigen value problems. In linear algebra, we make a systematic use of matrices and to a lesser extent determinants and their properties.

Determinants were first introduced for solving linear systems and have important engineering applications in systems of differential equations, electrical networks, eigen-value problems and so on. Many complicated expressions occurring in electrical and mechanical systems can be elegently simplified by expressing them in the form of determinants.

Cayley* discovered matrices in the year 1860. But it was not until the twentieth century was well-advanced that engineers heard of them. These days, however, matrices have been found to be of great utility in many branches of applied mathematics such as algebraic and differential equations, mechanics theory of electrical circuits, nuclear physics, aerodynamics and astronomy. With the advent of computers, the usage of matrix methods has been greatly facilitated.

2.2 DETERMINANTS

(1) **Definition.** The expression $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is called a *determinant of the second order* and stands for '$a_1b_2 - a_2b_1$'. It contains 4 numbers a_1, b_1, a_2, b_2 (called *elements*) which are arranged along two horizontal lines (called *rows*) and two vertical lines (called *columns*).

Similarly, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is called a *determinant of the third order*. It consists of 9 *elements* which are arranged in 3 *rows* and 3 *columns*.

**Arthur Cayley* (1821–1895) was a professor at Cambridge and is known for his important contributions to algebra, matrices and differential equations.

In general, *a determinant of the nth order* is denoted by $\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \ldots l_1 \\ a_2 & b_2 & c_2 & d_2 \ldots l_2 \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_n & b_n & c_n & d_n \ldots l_n \end{vmatrix}$

which is a block of n^2 *elements* arranged in the form of a square along *n-rows* and *n-columns.* The diagonal through the left hand top corner which contains the elements $a_1, b_2, c_3, \ldots l_n$ is called the *leading or principal diagonal.*

(2) Cofactors

The cofactor *of any element in a determinant is obtained by deleting the row and column which intersect in that element with the proper sign. The sign of an element in the ith row and jth column is* $(-1)^{i+j}$. The cofactor of an element is usually denoted by the corresponding capital letter.

For instance, in $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, the cofactor of b_3 *i.e.*, $B_3 = (-1)^{3+2} \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$ and $C_2 = -\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$.

(3) Laplace's expansion.* *A determinant can be expanded in terms of any row (or column) as follows :*

Multiply each element of the row (or column) in terms of which we intend expanding the determinant, by its cofactor and then add up all these terms.

∴ Expanding by R_1 (*i.e.*, 1st row),

$$\Delta = a_1A_1 + b_1B_1 + c_1C_1 = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

Similarly, expanding by C_2 (*i.e.*, 2nd column)

$$\Delta = b_1B_1 + b_2B_2 + b_3B_3 = -b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

$$= -b_1(a_2c_3 - a_3c_2) + b_2(a_1c_3 - a_3c_1) - b_3(a_1c_2 - a_2c_1)$$

and expanding by R_3 (*i.e.*, 3rd row), $\Delta = a_3A_3 + b_3B_3 + c_3C_3$.

Thus Δ is the sum of the products of the elements of any row (or column) by the corresponding cofactors.

If, however, the sum of the products of the elements of any row (or column) by the cofactors of another row (or column) be taken, the result is zero.

e.g., in Δ, $$a_3A_2 + b_3B_2 + c_3C_2 = -a_3 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_3 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_3 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

$$= -a_3(b_1c_3 - b_3c_1) + b_3(a_1c_3 - a_3c_1) - c_3(a_1b_3 - a_3b_1) = 0$$

In general, $a_iA_j + b_iB_j + c_iC_j = \Delta$ when $i = j$

$= 0$ when $i \neq j$

Example 2.1. *Expand* $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$.

Solution. Expanding by R_1, $\Delta = a\begin{vmatrix} b & f \\ f & c \end{vmatrix} - h\begin{vmatrix} h & f \\ g & c \end{vmatrix} + g\begin{vmatrix} h & b \\ g & f \end{vmatrix}$

$$= a(bc - f^2) - h(hc - gf) + g(hf - gb) = abc + 2fgh - af^2 - bg^2 - ch^2.$$

*Named after a great French mathematician *Pierre Simon Marquis De Laplace* (1749–1827). He made important contributions to probability theory, special functions, potential theory and astronomy. While a professor in Paris, he taught Napolean Bonapart for a year.

Example 2.2. *Find the value of* $\Delta = \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{vmatrix}$.

Solution. Since there are two zeros in the second row, therefore, expanding by R_2, we get

$$\Delta = -\begin{vmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} + 0 - 3\begin{vmatrix} 0 & 1 & 3 \\ 2 & 3 & 1 \\ 3 & 0 & 2 \end{vmatrix} + 0$$

(Expand by C_1) (Expand by R_1)

$$= -[1(0 \times 2 - 1 \times 1) - 3(2 \times 2 - 1 \times 3) + 0] - 3[0 - (2 \times 2 - 3 \times 1) + 3(2 \times 0 - 3 \times 3)]$$
$$= -(-1-3) - 3(-1-27) = 4 + 84 = 88.$$

2.3 PROPERTIES OF DETERMINANTS

The following properties, are proved for determinants of the third order, but these hold good for determinants of any order. These properties enable us to simplify a given determinant and evaluate it without expanding the given determinant.

I. *A determinant remains unaltered by changing its rows into columns and columns into rows.*

Let $$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{[Expand by } R_1]$$
$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

Then $$\Delta' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \text{[Expand by } R_1]$$
$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$
$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) = \Delta.$$

Obs. 1. Any theorem concerning the rows of a determinant, therefore, applies equally to its columns and *vice-versa*.
2. When a *row* or a *column* is referred to in a general manner, it is called a *line*.

II. *If two parallel lines of a determinant are interchanged, the determinant retains its numerical value but changes in sign.*

Let $$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{[Expand by } R_1]$$
$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

Interchanging C_2 and C_3, we have

$$\Delta' = \begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix} \quad \text{[Expand by } R_1]$$
$$= a_1(c_2b_3 - c_3b_2) - c_1(a_2b_3 - a_3b_2) + b_1(a_2c_3 - a_3c_2)$$
$$= -[a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)] = -\Delta.$$

Cor. *If a line of Δ be passed over two parallel lines, i.e., if the resulting determinant is like*

$$\Delta' = \begin{vmatrix} b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \\ b_3 & c_3 & a_3 \end{vmatrix}, \qquad \text{then } \Delta' = (-1)^2 \Delta.$$

In general, if any line of a determinant be passed over m parallel lines, the resulting determinant

$$\Delta' = (-1)^m \Delta.$$

III. *A determinant vanishes if two parallel lines are identical.*

Consider a determinant Δ in which two parallel lines are identical.

Interchange of the identical lines leaves the determinant unaltered yet by the previous property, the interchanges of two parallel lines changes the sign of the determinant.

Hence $\Delta = \Delta' = -\Delta$ or $2\Delta = 0$, or $\Delta = 0$.

IV. *If each element of a line be multiplied by the same factor, the whole determinant is multiplied by that factor.*

i.e.,
$$\begin{vmatrix} a_1 & pb_1 & c_1 \\ a_2 & pb_2 & c_2 \\ a_3 & pb_3 & c_3 \end{vmatrix} = p \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

For on expanding by C_2,

$$\text{L.H.S.} = -pb_1(a_2c_3 - a_3c_2) + pb_2(a_1c_3 - a_3c_1) - pb_3(a_1c_2 - a_2c_1)$$
$$= p\{-b_1B_1 + b_2B_2 - b_3B_3\} = \text{R.H.S.}$$

Similarly,
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ ka_1 & kb_2 & kc_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Cor. *If two parallel lines be such that the elements of one are equi-multiples of the elements of the other, the determinant vanishes.*

i.e.,
$$\begin{vmatrix} a_1 & b_1 & pb_1 \\ a_2 & b_2 & pb_2 \\ a_3 & b_3 & pb_3 \end{vmatrix} = p \begin{vmatrix} a_1 & b_1 & b_1 \\ a_2 & b_2 & b_2 \\ a_3 & b_3 & b_3 \end{vmatrix} = p(0) = 0$$

V. *If each element of a line consists of m terms, the determinant can be expressed as the sum of m determinants.*

Consider the determinant $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 + d_1 - e_1 \\ a_2 & b_2 & c_2 + d_2 - e_2 \\ a_3 & b_3 & c_3 + d_3 - e_3 \end{vmatrix}$

end of whose third column elements consists of three terms.

Expanding Δ by C_3, we have

$$\Delta = (c_1 + d_1 - e_1)(a_2b_3 - a_3b_2) - (c_2 + d_2 - e_2)(a_1b_3 - a_3b_1) + (c_3 + d_3 - e_3)(a_1b_2 - a_2b_1)$$
$$= [c_1(a_2b_3 - a_3b_2) - c_2(a_1b_3 - a_3b_1) + c_3(a_1b_2 - a_2b_1)] + [d_1(a_2b_3 - a_3b_2) - d_2(a_1b_3 - a_3b_1)$$
$$+ d_3(a_1b_2 - a_2b_1)] - [e_1(a_2b_3 - a_3b_2) - e_2(a_1b_3 - a_3b_1) + e_3(a_1b_2 - a_2b_1)]$$
$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} - \begin{vmatrix} a_1 & b_1 & e_1 \\ a_2 & b_2 & e_2 \\ a_3 & b_3 & e_3 \end{vmatrix}$$

Further, *if the elements of three parallel lines consist of m, n and p terms respectively, the determinants can be expressed as the sum of m × n × p determinants.*

Example 2.3. *If* $\begin{vmatrix} a & a^2 & a^3 - 1 \\ b & b^2 & b^3 - 1 \\ c & c^2 & c^3 - 1 \end{vmatrix} = 0$ *in which a, b, c are different, show that abc = 1.*

Solution. As each term of C_3 in the given determinant consists of two terms, we express it as a sum of two determinants.

$$\begin{vmatrix} a & a^2 & a^3 - 1 \\ b & b^2 & b^3 - 1 \\ c & c^2 & c^3 - 1 \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} + \begin{vmatrix} a & a^2 & -1 \\ b & b^2 & -1 \\ c & c^2 & -1 \end{vmatrix} = abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

[Taking common a, b, c from R_1, R_2, R_3 respectively of the first determinant and -1 from C_3 of the second determinant.]

$$= abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

[Passing C_3 over C_2 and C_1 in the second determinant]

$$\therefore \quad \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} (abc - 1) = 0. \text{ Hence } abc = 1, \text{ since } \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \neq 0 \text{ as } a, b, c \text{ are all different.}$$

VI. *If to each elements of a line be added equi-multiples of the corresponding elements of one or more parallel lines, the determinants remains unaltered.*

Let
$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Then
$$\Delta' = \begin{vmatrix} a_1 + pb_1 - qc_1 & b_1 & c_1 \\ a_2 + pb_2 - qc_2 & b_2 & c_2 \\ a_3 + pb_3 - qc_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} pb_1 & b_1 & c_1 \\ pb_2 & b_2 & c_2 \\ pb_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} -qc_1 & b_1 & c_1 \\ -qc_2 & b_2 & c_2 \\ -qc_3 & b_3 & c_3 \end{vmatrix}$$

$$= \Delta + 0 + 0 = \Delta. \qquad \text{[by } IV\text{-}Cor.\text{]}$$

Obs. *This property is very useful for simplifying determinants.* To add equi-multiples of parallel lines, we shall employ the following **notation :**

Suppose to the elements of the second row, we add p times the elements of the first row and q times the element of the third row ; then we say :

Operate $R_2 + pR_1 + qR_3$.

Similarly Operate '$C_3 + mC_1 - nC_2$'

means that to the elements of the third column add m times the elements of the first column and $-n$ times the elements of the second column.

Example 2.4. *Evaluate* $\begin{vmatrix} 21 & 17 & 7 & 10 \\ 24 & 22 & 6 & 10 \\ 6 & 8 & 2 & 3 \\ 6 & 7 & 1 & 2 \end{vmatrix}$.

Solution. Operating $R_1 - R_2 - R_4$, $R_2 - 3R_3$, $R_3 - 2R_4$, the given determinant becomes

$$\Delta = \begin{vmatrix} -8 & -12 & 0 & -2 \\ 6 & -2 & 0 & 1 \\ -4 & -6 & 0 & -1 \\ 5 & 7 & 1 & 2 \end{vmatrix} \qquad \text{[Expand by } C_1\text{]}$$

$$= -\begin{vmatrix} -8 & -12 & -2 \\ 6 & -2 & 1 \\ -4 & -6 & -1 \end{vmatrix} = 0 \qquad [\because \quad R_1 = 2R_2]$$

Example 2.5. *Solve the equation* $\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix} = 0.$

Solution. Operating $R_3 - (R_1 + R_2)$, we get

$$\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 0 & 1 & 3x+8 \end{vmatrix} = 0 \qquad \text{(Operate } R_2 - R_1 \text{ and } R_1 + R_3\text{)}$$

or
$$\begin{vmatrix} x+2 & 2x+4 & 6x+12 \\ x+1 & x+1 & x+1 \\ 0 & 1 & 3x+8 \end{vmatrix} = 0 \quad \text{or} \quad (x+1)(x+2)\begin{vmatrix} 1 & 2 & 6 \\ 1 & 1 & 1 \\ 0 & 1 & 3x+8 \end{vmatrix} = 0$$

To bring one more zero in C_1, operate $R_1 - R_2$.

$$\therefore \qquad (x+1)(x+2)\begin{vmatrix} 0 & 1 & 5 \\ 1 & 1 & 1 \\ 0 & 1 & 3x+8 \end{vmatrix} = 0$$

Now expand by C_1. $\therefore$ $-(x+1)(x+2)(3x+8-5) = 0$ or $-3(x+1)(x+2)(x+1) = 0$

Thus, $x = -1, -1, -2$.

Example 2.6. *Prove that* $\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right).$

Solution. Let Δ be the given determinant. Taking a, b, c, d common from R_1, R_2, R_3, R_4 respectively, we get

$$\Delta = abcd\begin{vmatrix} a^{-1}+1 & a^{-1} & a^{-1} & a^{-1} \\ b^{-1} & b^{-1}+1 & b^{-1} & b^{-1} \\ c^{-1} & c^{-1} & c^{-1}+1 & c^{-1} \\ d^{-1} & d^{-1} & d^{-1} & d^{-1}+1 \end{vmatrix}$$

[Operate $R_1 + (R_2 + R_3 + R_4)$ and take out the common factor from R_1]

$$= abcd\,(1 + a^{-1} + b^{-1} + c^{-1} + d^{-1})\begin{vmatrix} 1 & 1 & 1 & 1 \\ b^{-1} & b^{-1}+1 & b^{-1} & b^{-1} \\ c^{-1} & c^{-1} & c^{-1}+1 & c^{-1} \\ d^{-1} & d^{-1} & d^{-1} & d^{-1}+1 \end{vmatrix}$$

[Operate $C_2 - C_1, C_3 - C_1, C_4 - C_1$]

$$= abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\begin{vmatrix} 1 & 0 & 0 & 0 \\ b^{-1} & 1 & 0 & 0 \\ c^{-1} & 0 & 1 & 0 \\ d^{-1} & 0 & 0 & 1 \end{vmatrix} = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

Obs. If all elements on one side of the leading diagonal are zero, then the determinant is equal to the product of leading diagonal elements and such a determinants is called a *triangular determinant.*

VII. Factor Theorem. *If the elements of a determinant Δ are functions of x and two parallel lines become identical when $x = a$, then $x - a$ is a factor of Δ.*

Let $\Delta = f(x)$

Since $\Delta = 0$ when $x = a$, $\therefore$ $f(a) = 0$.

i.e., $(x - a)$ is a factor of $f(x)$.

Hence $x - a$ is a factor of Δ.

Obs. *If k parallel lines of a determinant Δ become identical when $x = a$, then $(x - a)^{k-1}$ is a factor of Δ.*

Example 2.7. *Factorize* $\Delta = \begin{vmatrix} a^3 & a^2 & a & 1 \\ b^3 & b^2 & b & 1 \\ c^3 & c^2 & c & 1 \\ d^3 & d^2 & d & 1 \end{vmatrix}$.

Solution. Putting $a = b$, $R_1 \equiv R_2$ and hence $\Delta = 0$. $\therefore$ $a - b$ is a factor of Δ.

Similarly, $a - c$ and $a - d$ are also factors of Δ.

Again putting $b = c$, $R_2 \equiv R_3$ and hence $\Delta = 0$. $\therefore$ $b - c$ is a factor of Δ.

Similarly $b - d$ and $c - d$ are also factors of Δ.

Also Δ is of the sixth degree in a, b, c, d and therefore, there cannot be any other algebraic factor of Δ.

$\therefore$ Suppose $\Delta = k(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$, where k is a numerical constant.

The leading term in $\Delta = a^3b^2c$. The corresponding term on R.H.S. $= ka^3b^2c$.

$\therefore$ $k = 1$.

Hence, $\Delta = (a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$.

Example 2.8. *Prove that* $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc\,(a+b+c)^3.$ *(J.N.T.U., 1998)*

Solution. Let the given determinant be Δ. If we put $a = 0$,

$$\Delta = \begin{vmatrix} (b+c)^2 & 0 & 0 \\ 0 & c^2 & b^2 \\ c^2 & c^2 & b^2 \end{vmatrix} = 0$$

$\therefore$ a is a factor of Δ. Similarly b and c are its factors.

Again if we put $a + b + c = 0$.

$$\Delta = \begin{vmatrix} (-a)^2 & a^2 & a^2 \\ b^2 & (-b)^2 & b^2 \\ c^2 & c^2 & (-c)^2 \end{vmatrix} = 0$$

In this, three columns being identical, $(a + b + c)^2$ is a factor of Δ.

As Δ is of the sixth degree and is symmetrical in a, b, c the remaining factor must therefore, be of the first degree and of the form $k\,(a + b + c)$.

Thus $\Delta = kabc\,(a + b + c)^3$

To determine k, put $a = b = c = 1$, then

$$\begin{vmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{vmatrix} = 27k \quad \text{or} \quad 54 = 27k \quad i.e., k = 2$$

Hence $\Delta = 2abc\,(a + b + c)^3$.

Otherwise : Operating $C_1 - C_3$ and $C_2 - C_3$, we have

$$\Delta = \begin{vmatrix} (b+c)^2 - a^2 & 0 & a^2 \\ 0 & (c+a)^2 - b^2 & b^2 \\ c^2 - (a+b)^2 & c^2 - (a+b)^2 & (a+b)^2 \end{vmatrix} \qquad [\text{Take } (a+b+c) \text{ common from } C_1 \text{ and } C_2]$$

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix} \qquad [\text{Operate } R_3 - R_1 - R_2]$$

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix} \qquad \left[\text{Operate } C_1 + \frac{1}{a} C_3, C_2 + \frac{1}{b} C_3\right]$$

$$= (a+b+c)^2 \begin{vmatrix} b+c & a^2/b & a^2 \\ b^2/a & c+a & b^2 \\ 0 & 0 & 2ab \end{vmatrix} \qquad [\text{Expand by } R_3]$$

$$= 2ab(a+b+c)^2\,[(b+c)(c+a) - ab] = 2abc\,(a+b+c)^3.$$

VIII. Multiplication of Determinants. *The product of two determinants of the same order is itself a determinant of that order.*

Let $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$

then their product is defined as

$$\Delta_1\Delta_2 = \begin{vmatrix} a_1l_1 + b_1m_1 + c_1n_1, & a_1l_2 + b_1m_2 + c_1n_2, & a_1l_3 + b_1m_3 + c_1n_3 \\ a_2l_1 + b_2m_1 + c_2n_1, & a_2l_2 + b_2m_2 + c_2n_2, & a_2l_3 + b_2m_3 + c_2n_3 \\ a_3l_1 + b_3m_1 + c_3n_1, & a_3l_2 + b_3m_2 + c_3n_2, & a_3l_3 + b_3m_3 + c_3n_3 \end{vmatrix}$$

Similarly, the product of two determinants of the nth order is a determinant of the nth order.

Example 2.9. *Evaluate* $\begin{vmatrix} a^2+\lambda^2 & ab+c\lambda & ca-b\lambda \\ ab-c\lambda & b^2+\lambda^2 & bc+a\lambda \\ ca+b\lambda & bc-a\lambda & c^2+\lambda^2 \end{vmatrix} \times \begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix}$

Solution. By the rule of multiplication of determinants, the resulting determinant

$$\Delta = \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix}$$

where $d_{11} = (a^2+\lambda^2)\,\lambda + (ab+c\lambda)c + (ca-b\lambda)(-b) = \lambda(a^2+b^2+c^2+\lambda^2)$

$d_{12} = (a^2+\lambda^2)(-c) + (ab+c\lambda)\lambda + (ca-b\lambda)a = 0$

$d_{13} = 0,$

$d_{21} = 0, d_{22} = \lambda(a^2+b^2+c^2+\lambda^2), d_{23} = 0.$

$d_{31} = 0, d_{32} = 0, d_{33} = \lambda(a^2+b^2+c^2+\lambda^2).$

Hence $$\Delta = \begin{vmatrix} \lambda(a^2+b^2+c^2+\lambda^2) & 0 & 0 \\ 0 & \lambda(a^2+b^2+c^2+\lambda^2) & 0 \\ 0 & 0 & \lambda(a^2+b^2+c^2+\lambda^2) \end{vmatrix}$$

$$= \lambda^3(a^2+b^2+c^2+\lambda^2)^3.$$

Example 2.10. *Show that* $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$ *where A, B etc. are the co-factors of a, b, etc. respectively in the determinant* $(a_1b_2c_3)$.

Solution. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta' = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$

Then $$\Delta\Delta' = \begin{vmatrix} a_1A_1+b_1B_1+c_1C_1, & a_1A_2+b_1B_2+c_1C_2, & a_1A_3+b_1B_3+c_1C_3 \\ a_2A_1+b_2B_1+c_2C_1, & a_2A_2+b_2B_2+c_2C_2, & a_2A_3+b_2B_3+c_2C_3 \\ a_3A_1+b_3B_1+c_3C_1, & a_3A_2+b_3B_2+c_3C_2, & a_3A_3+b_3B_3+c_3C_3, \end{vmatrix} = \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3$$

Hence $\Delta' = \Delta^2$.

Obs. Δ' is called the **reciprocal** *or* **adjugate determinant** *of* Δ.

Example 2.11. *Express* $\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$ *as the square of a determinant, and hence find its value.*

Solution. Given determinant

$$= \begin{vmatrix} a.(-a)+b.c+c.b, & a.(-b)+b.a+c.c, & a.(-c)+b.b+c.a \\ b.(-a)+c.c+a.b, & b.(-b)+c.a+a.c, & b.(-c)+c.b+a.a \\ c.(-a)+a.c+b.b, & c.(-b)+a.a+b.c, & c.(-c)+a.b+b.a \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix}$$

[Taking out (–1) common from C_1 and interchange C_2, C_3.]

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times (-1)^2 \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \Delta^2$$

where $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3+b^3+c^3-3abc)$

Hence the given determinant $= \Delta^2 = (a^3+b^3+c^3-3abc)^2$.

PROBLEMS 2.1

1. Prove, without expanding, that $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix}$ vanishes.

2. If $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$, then prove, without expansion, that $xyz = -1$ where x, y, z are unequal.

(Andhra, 1999 ; Assam, 1999)

3. Show that (*i*) $\begin{vmatrix} x & l & m & 1 \\ \alpha & x & n & 1 \\ \alpha & \beta & x & 1 \\ \alpha & \beta & \gamma & 1 \end{vmatrix} = (x-\alpha)(x-\beta)(x-\gamma)$.

(*ii*) $\begin{vmatrix} a & b & c \\ b+c & c+a & a+b \\ a^2 & b^2 & c^2 \end{vmatrix} = -(a-b)(b-c)(c-a)(a+b+c)$.

4. If a, b, c are all different and $\begin{vmatrix} a & a^3 & a^4-1 \\ b & b^3 & b^4-1 \\ c & c^3 & c^4-1 \end{vmatrix} = 0$, then show that $abc\,(bc+ca+ab) = a+b+c$.

5. Evaluate (*i*) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 3 & 4 \\ 1 & 2 & 4 & 4 \\ 1 & 2 & 3 & 5 \end{vmatrix}$ (*ii*) $\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$

Prove the following results : (6 to 12)

6. $\begin{vmatrix} a+b & b+c & c+a \\ l+m & m+n & n+l \\ p+q & q+r & r+p \end{vmatrix} \div \begin{vmatrix} a & b & c \\ l & m & n \\ p & q & r \end{vmatrix} = 2$

7. $\begin{vmatrix} a-b-c & 2b & 2c \\ 2a & b-c-a & 2c \\ 2a & 2b & c-a-b \end{vmatrix} = (a+b+c)^3$

8. $\begin{vmatrix} 1+a^2-b^2 & 2b & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a^2 & 1-a^2-b^2 \end{vmatrix}$ is a perfect cube.

9. $\begin{vmatrix} 1 & \cos A & \sin A \\ 1 & \cos B & \sin B \\ 1 & \cos C & \sin C \end{vmatrix} = 4\sin\dfrac{B-C}{2}\sin\dfrac{C-A}{2}\sin\dfrac{A-B}{2}$.

10. $\begin{vmatrix} 4 & 5 & 6 & x \\ 5 & 6 & 7 & y \\ 6 & 7 & 8 & z \\ x & y & z & 0 \end{vmatrix}$ is a perfect square.

11. $\begin{vmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & b & b^2 & b^3+cda \\ 1 & c & c^2 & c^3+dab \\ 1 & d & d^2 & d^3+abc \end{vmatrix}$ vanishes.

12. $\begin{vmatrix} a^2+\lambda & ab & ac & ad \\ ab & b^2+\lambda & bc & bd \\ ac & bc & c^2+\lambda & cd \\ ad & bd & cd & d^2+\lambda \end{vmatrix} = \lambda^3(a^2+b^2+c^2+d^2+\lambda)$

Factorize each of the following determinants : (13 to 15)

13. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$ *(Andhra, 1998)*

14. $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$.

15. $\begin{vmatrix} b^2c^2+a^2 & bc+a & 1 \\ c^2a^2+b^2 & ca+b & 1 \\ a^2b^2+c^2 & ab+c & 1 \end{vmatrix}$

16. $\begin{vmatrix} a^2 & b^2 & c^2 & d^2 \\ a & b & c & d \\ 1 & 1 & 1 & 1 \\ bcd & cda & dab & abc \end{vmatrix}$

17. If $a+b+c=0$, solve $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$ *(Andhra, 1999)*

18. Solve the equation $\begin{vmatrix} x+1 & 2x+1 & 3x+1 \\ 2x & 4x+3 & 6x+3 \\ 4x+1 & 6x+4 & 8x+4 \end{vmatrix} = 0.$

19. Show that $\begin{vmatrix} b^2+c^2 & ab & ac \\ ba & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix} = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = 4a^2b^2c^2.$

2.4 MATRICES

(1) Definition. *A system of mn numbers arranged in a rectangular formation along m rows and n columns and bounded by the brackets [ ] is called an m by n* **matrix** ; *which is written as m × n matrix.* A matrix is also denoted by a single capital letter.

Thus
$$A = \begin{bmatrix} a_{11} & a_{12} & ...a_{1j} & ...a_{1n} \\ a_{21} & a_{22} & ...a_{2j} & ...a_{2n} \\ ... & ... & ... & ... \\ a_{i1} & a_{i2} & ...a_{ij} & ...a_{in} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & a_{mj} & ...a_{mn} \end{bmatrix}$$
is a matrix of *order mn*. It has *m rows* and *n columns*. Each of the *mn* numbers is called an element of the matrix.

To locate any particular element of a matrix, the elements are denoted by a letter followed by two suffixes which respectively specify the rows and columns. Thus a_{ij} is the element in the i-th row and j-th column of A. In this notation, the matrix A is denoted by $[a_{ij}]$.

A matrix should be treated as a single entity with a number of components, rather than a collection of numbers. For example, the coordinates of a point in solid geometry, are given by a set of three numbers which can be represented by the matrix $[x, y, z]$. Unlike a determinant, a matrix cannot reduce to a single number and the question of finding the value of a matrix never arises. The difference between a determinant and a matrix is brought out by the fact that an interchange of rows and columns does not alter the determinant but gives an entirely different matrix.

(2) Special matrices

Row and column matrices. *A matrix having a single row is called a row matrix, e.g.,* $[1 \quad 3 \quad 5 \quad 7]$.

A matrix having a single column is called a column matrix, e.g., $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$

Row and column matrices are sometimes called *row vectors* and *column vectors*.

Square matrix. *A matrix having n rows and n columns is called a square matrix of order n.*

The determinant having the same elements as the square matrix A is called the *determinant of the matrix* and is denoted by the symbol $|\,A\,|$. For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}, \text{ then } |\,A\,| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

The diagonal of this matrix containing the elements 1, 3, 5 is called the *leading or principal diagonal. The sum of the diagonal elements of a square matrix A is called the* **trace** *of A.*

A square matrix is said to be **singular** *if its determinant is zero otherwise* **non-singular.**

Diagonal matrix. *A square matrix all of whose elements except those in the leading diagonal, are zero is called a diagonal matrix.*

A diagonal matrix whose all the leading diagonal elements are equal is called a *scalar matrix.* For example,

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \text{ and } \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

are the diagonal and scalar matrices respectively.

Unit matrix. *A diagonal matrix of order n which has unity for all its diagonal elements, is called a unit matrix or an identity matrix of order n* and is denoted by I_n. For example, unit matrix of order 3 is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Null matrix. *If all the elements of a matrix are zero, it is called a null or zero matrix* and is denoted by '**0**' ; *e.g.*,

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ is a null matrix.}$$

Symmetric and skew-symmetric matrices. *A square matrix* $A = [a_{ij}]$ *is said to be symmetric when* $a_{ij} = a_{ji}$ *for all i and j.*

If $a_{ij} = -a_{ji}$ *for all i and j so that all the leading diagonal elements are zero, then the matrix is called a skew-symmetric matrix.* Examples of symmetric and skew-symmetric matrices are

$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \text{ and } \begin{bmatrix} 0 & h & -g \\ -h & 0 & f \\ g & -f & 0 \end{bmatrix} \text{ respectively.}$$

Triangular matrix. A square matrix all of whose elements below the leading diagonal are zero, is called an *upper triangular matrix.* A square matrix all of whose elements above the leading diagonal are zero, is called a *lower triangular matrix.* Thus

$$\begin{bmatrix} a & h & g \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & -5 & 4 \end{bmatrix}$$

are upper and lower triangular matrices respectively.

2.5 MATRICES OPERATIONS

(1) Equality of Matrices

Two matrices A and B are said to equal if and only if

(i) they are of the same order

and (ii) each element of A is equal to the corresponding element of B.

(2) Addition and subtraction of matrices. *If A, B be two matrices of the same order, then their sum A + B is defined as the matrix each element of which is the sum of the corresponding elements of A and B.*

Thus, $$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} + \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \\ c_3 & d_3 \end{bmatrix} = \begin{bmatrix} a_1 + c_1 & b_1 + d_1 \\ a_2 + c_2 & b_2 + d_2 \\ a_3 + c_3 & b_3 + d_3 \end{bmatrix}$$

Similarly, *A – B is defined as a matrix whose elements are obtained by subtracting the elements of B from the corresponding elements of A.*

Thus, $$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} - \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 - c_1 & b_1 - d_1 \\ a_2 - c_2 & b_2 - d_2 \end{bmatrix}$$

Obs. 1. Only matrices of the same order can be added or subtracted.

2. Addition of matrices is *commutative*,

i.e., $A + B = B + A$.

3. Addition and subtraction of matrices is *associative*.

i.e. $(A + B) - C = A + (B - C) = B + (A - C)$.

(3) Multiplication of matrix by a scalar. *The product of a matrix A by a scalar k is a matrix whose each element is k times the corresponding elements of A.*

Thus,
$$k\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} = \begin{bmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \end{bmatrix}$$

The distributive law holds for such products, *i.e.*, $k(A + B) = kA + kB$.

Obs. All the laws of ordinary algebra hold for the addition or subtraction of matrices and their multiplication by scalars.

Example 2.12. *Find x, y, z and w given that*
$$3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 5 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 6 & x+y \\ z+w & 5 \end{bmatrix}$$

Solution. We have $\begin{bmatrix} 3x & 3y \\ 3z & 3w \end{bmatrix} = \begin{bmatrix} x+6 & 5+x+y \\ -1+z+w & 2w+5 \end{bmatrix}$

Equating the corresponding elements, we get
$$3x = x + 6,\ 3y = 5 + x + y,\ 3z = -1 + z + w,\ 3w = 2w + 5.$$
or
$$2x = 6,\ 2y = 5 + x,\ 2z = w - 1,\ w = 5$$
Hence $x = 3, y = 4, z = 2, w = 5$.

Example 2.13. *Express* $\begin{bmatrix} 3 & 5 & -7 \\ -8 & 11 & 4 \\ 13 & -14 & 6 \end{bmatrix}$ *as the sum of a lower triangular matrix with zero leading diagonal and an upper triangular matrix.*

Solution. Let $L = \begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{bmatrix}$ be the lower triangular matrix with zero leading diagonal.

and $U = \begin{bmatrix} l & m & n \\ 0 & p & q \\ 0 & 0 & r \end{bmatrix}$ be the upper triangular matrix.

Then
$$\begin{bmatrix} 3 & 5 & -7 \\ -8 & 11 & 4 \\ 13 & -14 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{bmatrix} + \begin{bmatrix} l & m & n \\ 0 & p & q \\ 0 & 0 & r \end{bmatrix}$$

Equating corresponding elements from both sides, we obtain $3 = l, 5 = m, -7 = n, -8 = a, 11 = p, 4 = q, 13 = b, -14 = c, 6 = r$.

Hence
$$L = \begin{bmatrix} 0 & 0 & 0 \\ -8 & 0 & 0 \\ 13 & -14 & 0 \end{bmatrix} \text{ and } U = \begin{bmatrix} 3 & 5 & -7 \\ 0 & 11 & 4 \\ 0 & 0 & 6 \end{bmatrix}$$

(4) Multiplication of matrices. *Two matrices can be multiplied only when the number of columns in the first is equal to the number of rows in the second. Such matrices are said to be* **conformable.**

For instance, the product $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix} \times \begin{bmatrix} l_1 & l_2 \\ m_1 & m_2 \\ n_1 & n_2 \end{bmatrix}$

is defined as the matrix $\begin{bmatrix} a_1l_1 + b_1m_1 + c_1n_1 & a_1l_2 + b_1m_2 + c_1n_2 \\ a_2l_1 + b_2m_1 + c_2n_1 & a_2l_2 + b_2m_2 + c_2n_2 \\ a_3l_1 + b_3m_1 + c_3n_1 & a_3l_2 + b_3m_2 + c_3n_2 \\ a_4l_1 + b_4m_1 + c_4n_1 & a_4l_2 + b_4m_2 + c_4n_2 \end{bmatrix}$

In general, if $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}$

be two $m \times n$ and $n \times p$ conformable matrices, then their product is defined as the $m \times p$ matrix

$$AB = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{bmatrix}$$

where $c_{ij} = a_{i1} b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \dots + a_{in}b_{nj}$, *i.e.*, the element in the ith row and the jth column of the matrix AB is obtained by weaving the ith row of A with jth column of B. The expression for c_{ij} is known as the *inner product* of the ith row with the jth column.

Post-multiplication and Pre-multiplication. In the product AB, the matrix A is said to be *post-multiplied* by the matrix B. Whereas in BA, the matrix A is said to be *pre-multiplied* by B. In one case the product may exist and in the other case it may not. Also the product in both cases may exist yet may or may not be equal.

Obs. 1. *Multiplication of matrices is associative. i.e., $(AB)C = A(BC)$*
provided A, B are conformable for the product AB and B, C are conformable for the product BC. (Ex. 2.16).

Obs. 2. *Multiplication of matrices is distributive. i.e., $A(B + C) = AB + AC$.*
provided A, B are conformable for the product AB and A, C are conformable for the product AC.

Obs. 3. Power of a matrix. If A be a square matrix, then the product AA is defined as A^2. Similarly, we define higher powers of A. *i.e.*, $A \cdot A^2 = A^3$, $A^2 \cdot A^2 = A^4$ etc.

If $A^2 = A$, then the matrix A is called idempotent.

Example 2.14. *If $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}$, form the product of AB. Is BA defined?*

Solution. Since the number of columns of A = the number of rows of B (each being = 3).

$\therefore$ The product AB is defined and

$$= \begin{bmatrix} 0.1 + 1.-1 + 2.2, & 0.-2 + 1.0 + 2.-1 \\ 1.1 + 2.-1 + 3.2, & 1.-2 + 2.0 + 3.-1 \\ 2.1 + 3.-1 + 4.2, & 2.-2 + 3.0 + 4.-1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 5 & -5 \\ 7 & -8 \end{bmatrix}$$

Again since the number of columns of $B \neq$ the number of rows of A.

$\therefore$ The product BA is not possible.

Example 2.15. *If $A = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$, compute AB and BA and show that $AB \neq BA$.*

Solution. Considering rows of A and columns of B, we have

$$AB = \begin{bmatrix} 1.2 + 3.1 + 0.-1, & 1.3 + 3.2 + 0.1, & 1.4 + 3.3 + 0.2 \\ -1.2 + 2.1 + 1.-1, & -1.3 + 2.1 + 1.1, & -1.4 + 2.3 + 1.2 \\ 0.2 + 0.1 + 2.-1, & 0.3 + 0.2 + 2.1, & 0.4 + 0.3 + 2.2 \end{bmatrix} = \begin{bmatrix} 5 & 9 & 13 \\ -1 & 2 & 4 \\ -2 & 2 & 4 \end{bmatrix}$$

Again considering the rows of B and columns of A, we have

$$BA = \begin{bmatrix} 2.1 + 3.-1 + 4.0, & 2.3 + 3.2 + 4.0 & 2.0 + 3.1 + 4.2 \\ 1.1 + 2.-1 + 3.0, & 1.3 + 2.2 + 3.0 & 1.0 + 2.1 + 3.2 \\ -1.1 + 1.-1 + 2.0, & -1.3 + 1.2 + 2.0 & -1.0 + 1.1 + 2.2 \end{bmatrix} = \begin{bmatrix} -1 & 12 & 11 \\ -1 & 7 & 8 \\ -2 & -1 & 5 \end{bmatrix}$$

Evidently $AB \neq BA$.

Example 2.16. *If* $A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 3 & 1 \\ 5 & 3 & 4 \end{bmatrix}$, *find the matrix B such that* $AB = \begin{bmatrix} 3 & 4 & 2 \\ 1 & 6 & 1 \\ 5 & 6 & 4 \end{bmatrix}$. (Mumbai, 2005)

Solution. Let $AB = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 3 & 1 \\ 5 & 3 & 4 \end{bmatrix} \begin{bmatrix} l & m & n \\ p & q & r \\ u & v & w \end{bmatrix}$

$$= \begin{bmatrix} 3l+2p+2u & 3m+2q+2v & 3n+2r+2w \\ l+3p+u & m+3q+v & n+3r+w \\ 5l+3p+4u & 5m+3q+4v & 5n+3r+4w \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 & 2 \\ 1 & 6 & 1 \\ 5 & 6 & 4 \end{bmatrix} \quad \text{(given)}$$

Equating corresponding elements, we get

$$3l + 2p + 2u = 3, \quad l + 3p + u = 1, \quad 5l + 3p + 4u = 5 \qquad ...(i)$$

$$3m + 2q + 2v = 4, \quad m + 3q + v = 6, \quad 5m + 3q + 4v = 6 \qquad ...(ii)$$

$$3n + 2r + 2w = 2, \quad n + 3r + w = 1, \quad 5n + 3r + 4w = 4 \qquad ...(iii)$$

Solving the equations (*i*), we get $l = 1, p = 0, u = 0$

Similarly equations (*ii*) give $m = 0, q = 2, v = 0$

and equations (*iii*) give $n = 0, r = 0, w = 1$

Thus, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Example 2.17. *Prove that* $A^3 - 4A^2 - 3A + 11I = 0$, *where* $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$.

Solution. $A^2 = A \times A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1+6+2 & 3+0+4 & 2-3+6 \\ 2+0-1 & 6+0-2 & 4+0-3 \\ 1+4+3 & 3+0+6 & 2-2+9 \end{bmatrix} = \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix}$

$$A^3 = A^2 \times A = \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 9+14+5 & 27+0+10 & 18-7+15 \\ 1+8+1 & 3+0+2 & 2-4+3 \\ 8+18+9 & 24+0+18 & 16-9+27 \end{bmatrix} = \begin{bmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix}$$

$A^3 - 4A^2 - 3A + 11I$

$$= \begin{bmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix} - 4\begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix} - 3\begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 28-36-3+11 & 37-28-9+0 & 26-20-6+0 \\ 10-4-6-0 & 5-16+0+11 & 1-4+3+0 \\ 35-32-3+0 & 42-36-6+0 & 34-36-9+11 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

Example 2.18. *By mathematical induction, prove that if*

$$A = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix}, \text{ then } A^n = \begin{bmatrix} 1+10n & -25n \\ 4n & 1-10n \end{bmatrix}.$$

Solution. When $n = 1$, A^n gives $A^1 = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix}$ ...(*i*)

Let us assume that the result is true for any positive integer k, so that

$$A^k = \begin{bmatrix} 1+10k & -25k \\ 4k & 1-10k \end{bmatrix}$$

$\therefore$
$$A^{k+1} = A^k \cdot A^1 = \begin{bmatrix} 1+10k & -25k \\ 4k & 1-10k \end{bmatrix} \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix}$$
$$= \begin{bmatrix} 11(1+10k)-100k & -25(1+10k)+225k \\ 44k+4(1-10k) & -100k-9(1-10k) \end{bmatrix}$$
$$= \begin{bmatrix} 1+10(k+1) & -25(k+1) \\ 4(k+1) & 1-10(k+1) \end{bmatrix}$$

This is true for $n = k + 1$ ...(*ii*)

We have seen in (*i*) that the result is true for $n = 1$.

$\therefore$ It is true for $n = 1 + 1 = 2$ [by (*ii*)]

Similarly, it is true for $n = 2 + 1 = 3$ and so on.

Hence by mathematical induction, the result is true for all positive integers n.

Example 2.19. *Prove that (AB)C = A(BC), where A, B, C are matrices conformable for the products.* (*J.N.T.U., 2002 S*)

Solution. Let $A = [a_{ij}]$ be of order $m \times n$, $B = [b_{ij}]$ be of order $n \times p$ and $C = [c_{ij}]$ be of order of $p \times q$.

Then $\quad AB = [a_{ik}]\,[b_{kj}] = \sum_{k=1}^{n} a_{ik}\, b_{kj}$

$\therefore$
$$(AB)C = \left[\sum_{k=1}^{n} a_{ik}\, b_{kl}\right] \cdot [c_{lj}] = \left[\sum_{l=1}^{p} \left(\sum_{k=1}^{n} a_{ik}\, b_{kl}\right) c_{lj}\right] = \left[\sum_{k=1}^{n} \sum_{l=1}^{p} a_{ik}\, b_{kl}\, c_{lj}\right]$$

Similarly, $\quad BC = [b_{kl}] \cdot [c_{lj}] = \sum_{l=1}^{p} b_{kl}\, c_{lj}$

$\therefore$
$$A(BC) = [a_{ik}] \left[\sum_{l=1}^{p} b_{kl}\, c_{lj}\right] = \left[\sum_{k=1}^{n} a_{ik} \left(\sum_{l=1}^{p} b_{kl}\, c_{lj}\right)\right] = \left[\sum_{k=1}^{n} \left(\sum_{l=1}^{p} a_{ik}\, b_{kl}\, c_{lj}\right)\right]$$

Hence $\quad (AB)C = A(BC)$.

PROBLEMS 2.2

1. For what values of x, the matrix $\begin{bmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{bmatrix}$ is singular ?

2. Find the values of x, y, z and a which satisfy the matrix equation $\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$.

3. Matrix A has x rows and $x + 5$ columns. Matrix B has y rows and $11 - y$ columns. Both AB and BA exist. Find x and y.

4. If $A + B = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix}$ and $A - B = \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}$, calculate the product AB.

5. If $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, find AB or BA, whichever exists.

6. If $A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$, verify that $(AB)C = A(BC)$ and $A(B + C) = AB + AC$.

7. Evaluate (*i*) $[x, y, z] \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$; (*ii*) $\begin{bmatrix} 2 & 1 & -1 \\ 4 & -5 & 6 \\ -3 & 7 & 3 \end{bmatrix} \times \begin{bmatrix} 3 & 1 \\ -6 & 4 \\ -2 & 5 \end{bmatrix} \times \begin{bmatrix} 5 & 3 \\ -2 & 1 \end{bmatrix}$; (*iii*) $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \times [4 \;\; 5 \;\; 2] \times \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} \times [3 \;\; 2]$

8. Prove that the product of two matrices

$$\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \text{ and } \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$$

is a null matrix when θ and ϕ differ by an odd multiple of $\pi/2$.

9. If $A = \begin{bmatrix} 0 & -\tan\alpha/2 \\ \tan\alpha/2 & 0 \end{bmatrix}$, show that $I + A = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$.

10. If $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$, find the value of $A^2 - 6A + 8I$, where I is a unit matrix of second order. *(B.P.T.U., 2006)*

11. If $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$, and I is the unit matrix of order 3, evaluate $A^2 - 3A + 9I$.

12. If $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 0 & 2 \\ 4 & -3 & 2 \end{bmatrix}$, verify the result $(A + B)^2 = A^2 + BA + AB + B^2$.

13. If $E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $F = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$,

calculate the products EF and FE and show that $E^2F + FE^2 = E$.

14. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, show that $A^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$, when n is a positive integer.

15. Factorize the matrix $A = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$ into the form LU, where L is lower triangular and U is upper triangular matrix.

2.6 RELATED MATRICES

(1) Transpose of a matrix. *The matrix obtained from any given matrix A, by interchanging rows and columns is called the* **transpose** *of A and is denoted by A′.*

Thus the transposed matrix of $A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}$ is $A' = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix}$

Clearly, the transpose of an $m \times n$ matrix is an $n \times m$ matrix. Also the transpose of the transpose of a matrix coincides with itself, *i.e.*, $(A')' = A$.

For a symmetric matrix, $A' = A$ and for a skew-symmetric matrix, $A' = -A$.

Obs. 1. *The transpose of the product of the two matrices is the product of their transposes taken in the reverse order i.e.*, $(\mathbf{AB})' = \mathbf{B'A'}$.

For, the element in the ith row and jth col. of $(AB)'$

= element in the jth row and ith col. of AB = inner product of jth row of A with ith col. of B

= inner product of jth col. of A' with ith row of B' = element in the ith row and jth col. of $B'A'$

Hence $(AB)' = B'A'$.

Obs. 2. *Every square matrix can be uniquely expressed as a sum of a symmetric and a skew-symmetric matrix.* *(J.N.T.U., 2001)*

Let A be the given square matrix, then $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$.

Let $B = \frac{1}{2}(A + A')$ and $C = \frac{1}{2}(A - A')$

$\therefore$ $B' = \left[\frac{1}{2}(A + A')\right]' = \frac{1}{2}[A' + (A')'] = \frac{1}{2}(A' + A) = B$, *i.e.*, $B = \frac{1}{2}(A + A')$ is a symmetric matrix.

Again, $C' = \left[\frac{1}{2}(A - A')\right]' = \frac{1}{2}[A' - (A')'] = \frac{1}{2}(A' - A) = -C$, *i.e.*, $C = \frac{1}{2}(A - A')$ is a skew-symmetric matrix.

Hence A can be expressed as the sum of a symmetric and a skew-symmetric matrix.

To prove the uniqueness, assume that P is a symmetric matrix and Q is a skew-symmetric matrix such that $A = P + Q$.

Then $A' = (P + Q)' = P' + Q' = P - Q$

Thus, $P = \frac{1}{2}(A + A')$ and $Q = \frac{1}{2}(A - A')$

which shows that there is one and only one way of expressing A as the sum of a symmetric and skew-symmetric matrix.

Example 2.20. *Express the matrix A as the sum of a symmetric and a skew-symmetric matrix where*

$$A = \begin{bmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{bmatrix}$$

Solution. We have $A' = \begin{bmatrix} 4 & 1 & -5 \\ 2 & 3 & 0 \\ -3 & -6 & -7 \end{bmatrix}$

Then $A + A' = \begin{bmatrix} 8 & 3 & -8 \\ 3 & 6 & -6 \\ -8 & -6 & -14 \end{bmatrix}$ and $A - A' = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -6 \\ -2 & 6 & 0 \end{bmatrix}$

$$\therefore \quad A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = \begin{bmatrix} 4 & 1.5 & -4 \\ 1.5 & 3 & -3 \\ -4 & -3 & -7 \end{bmatrix} + \begin{bmatrix} 0 & 0.5 & 1 \\ -0.5 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}.$$

(2) Adjoint of a square matrix. The determinant of the square matrix

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \text{ is } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

The matrix formed by the cofactors of the elements in Δ is

$\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$. Then the transpose of this matrix, *i.e.*, $\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$

is called the *adjoint of the matrix A* and is written as *Adj. A*.

Thus the **adjoint** *of A is the transposed matrix of cofactors of A.*

(3) Inverse of a matrix. *If A be any matrix, then a matrix B if it exists, such that AB = BA = I, is called the* **Inverse** *of A* which is denoted by A^{-1} so that $AA^{-1} = I$.

Also $$\mathbf{A^{-1} = \frac{Adj.A}{|A|}}$$

For $$A\,(Adj\,A) = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or $A \cdot \dfrac{Adj.A}{|A|} = I \quad [\because \; |A| \neq 0]$ or $\dfrac{Adj.A}{|A|}$ is the inverse of A.

Obs. 1. *Inverse of a matrix, is unique.*

If possible, let the two inverses of the matrix A be B and C,

then $AB = BA = I$ and $AC = CA = I$

$\therefore$ $CAB = (CA)B = IB = B$ and $CAB = C(AB) = CI = C$

Thus, $B = C$.

Obs. 2. *The reciprocal of the product of two matrices is the product of their reciprocals taken in the reverse order i.e.,* $\mathbf{(AB)^{-1} = B^{-1} A^{-1}}$ *(Assam, 1999)*

If A, B be two matrices, then the reciprocal of their product is $(AB)^{-1}$.

Clearly, $(AB) . (B^{-1} A^{-1}) = A(BB^{-1}) A^{-1}$ [by Associative law]

$$= AIA^{-1} = AA^{-1} = I.$$

Similarly, $(B^{-1}A^{-1}) . (AB) = I$

Hence $B^{-1} A^{-1}$ is the reciprocal of AB.

Obs. 3. Multiplication by an inverse matrix plays the same role in matrix algebra that division plays in ordinary algebra.

i.e., if $[A][B] = [C][D]$, then $[A]^{-1} [A][B] = [A^{-1}][C][D]$

or $B = A^{-1}[C][D]$, *i.e.*, $\dfrac{[C][D]}{[A]} = A^{-1} [C][D]$.

Example 2.21. *Find the inverse of* $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

Solution. The determinant of the given matrix A is

$$\Delta = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \text{ (say)}$$

If $A_1, A_2, ...$ be the cofactors of $a_1, a_2...$ in Δ, then $A_1 = -24, A_2 = -8, A_3 = -12$; $B_1 = 10, B_2 = 2, B_3 = 6$; $C_1 = 2, C_2 = 2, C_3 = 2$.

Thus $\Delta = a_1A_1 + a_2A_2 + a_3A_3 = -8.$

and $$adj\, A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix}.$$

Hence the inverse of the given matrix A

$$= \frac{adj\, A}{\Delta} = \frac{1}{-8}\begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

Note. For other methods see Examples 2.25 ; 2.28 and 2.46.

Example 2.22. *Find the matrix A if* $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}$ *(Mumbai, 2008)*

Solution. If $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = B, \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = C$ and $\begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix} = D$, then

$$BAC = D \quad \text{or} \quad AC = B^{-1}D$$

$$\therefore \quad A = B^{-1} DC^{-1}$$

Now, $$B^{-1} = \frac{adj\, B}{|B|} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

Similarly, $$C^{-1} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

Hence, $$A = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 14 & 8 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 24 & 13 \\ -34 & -18 \end{bmatrix}.$$

PROBLEMS 2.3

1. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, verify that $AA' = I = A'A$, where I is the unit matrix.

2. Express each of the following matrices as the sum of a symmetric and a skew-symmetric matrix :

(i) $\begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} a & a & b \\ c & b & b \\ c & a & c \end{bmatrix}$

3. If A is a non-singular matrix of order n, prove that $A \; adj\, A = |A| I$. *(Mumbai, 2006)*

Verify that $A\,(adj\, A) = (adj\, A)\, A = |A|\, I$, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$.

4. Find the inverse of the matrix (i) $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ *(Mumbai, 2009)* (ii) $\begin{bmatrix} 5 & -2 & 4 \\ -2 & 1 & 1 \\ 4 & 1 & 0 \end{bmatrix}$ *(B.P.T.U., 2005)*

5. If $A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 3 & 1 \\ 5 & 3 & 4 \end{bmatrix}$, compute $adj\, A$ and A^{-1}. Also find B such that $AB = \begin{bmatrix} 3 & 4 & 2 \\ 1 & 6 & 1 \\ 5 & 6 & 4 \end{bmatrix}$. *(Mumbai, 2008)*

6. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, (i) find A^{-1} ; (ii) show that $A^3 = A^{-1}$.

7. Find the inverse of the matrix

$$S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ and if } A = \frac{1}{2}\begin{bmatrix} 4 & -1 & 1 \\ -2 & 3 & -1 \\ 2 & 1 & 5 \end{bmatrix},$$

show that SAS^{-1} is a diagonal matrix dig (2, 3, 1). *(Mumbai, 2007)*

8. If $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$, prove that $A^{-1} = A'$.

9. Show that $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & \tan\theta/2 \\ -\tan\theta/2 & 1 \end{bmatrix}^{-1}$.

10. If $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$, verify that $(AB)' = B'A'$, where A' is the transpose of A.

11. $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 9 & 3 \\ 1 & 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$.

12. If A is a square matrix, show that (i) $A + A'$ is symmetric, and (ii) $A - A'$ is skew-symmetric. *(P.T.U., 1999)*

13. If $D = \text{diag}\,[d_1, d_2, d_3]$, $d_1, d_2, d_3 \neq 0$, prove that $D^{-1} = \text{diag}\,[d_1^{-1}, d_2^{-1}, d_3^{-1}]$.

14. If A and B are square matrices of the same order and A is symmetrical, show that $B'AB$ is also symmetrical. [**Hint.** Show that $(B'AB)' = B'AB$]

15. If a non-singular matrix A is symmetric, show that A^{-1} is also symmetric.

2.7 (1) RANK OF A MATRIX

If we select any r rows and r columns from any matrix A, deleting all the other rows and columns, then the determinant formed by these $r \times r$ elements is called the *minor of A of order r*. Clearly, there will be a number of different minors of the same order, got by deleting different rows and columns from the same matrix.

Def. *A matrix is said to be of rank r when*

(i) *it has at least one non-zero minor of order r,*

and (ii) *every minor of order higher than r vanishes.*

Briefly, the rank of a matrix is the largest order of any non-vanishing minor of the matrix.

If a matrix has a non-zero minor of order r, its rank is $\geq r$.

If all minors of a matrix of order $r + 1$ are zero, its rank is $\leq r$.

The rank of a matrix A shall be denoted by $\rho(A)$.

(2) Elementary transformation of a matrix. The following operations, three of which refer to rows and three to columns are known as *elementary transformations :*

I. The interchange of any two rows (columns).

II. The multiplication of any row (column) by a non-zero number.

III. The addition of a constant multiple of the elements of any row (column) to the corresponding elements of any other row (column).

Notation. The *elementary row transformations* will be denoted by the following symbols :

(*i*) R_{ij} for the interchange of the ith and jth rows.

(*ii*) kR_i for multiplication of the ith row by k.

(*iii*) $R_i + pR_j$ for addition to the ith row, p times the jth row.

The corresponding column transformation will be denoted by writing C in place of R.

Elementary transformations do not change either the order or rank of a matrix. While the value of the minors may get changed by the transformation I and II, their zero or non-zero character remains unaffected.

(3) Equivalent matrix. *Two matrices A and B are said to be equivalent if one can be obtained from the other by a sequence of elementary transformations.* Two equivalent matrices have the same order and *the same rank.* The symbol ~ is used for equivalence.

Example 2.23. *Determine the rank of the following matrices :*

(*i*) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ (V.T.U., 2011)

Solution. (*i*) Operate $R_2 - R_1$ and $R_3 - 2R_1$ so that the given matrix

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix} = A \text{ (say)}$$

Obviously, the 3rd order minor of A vanishes. Also its 2nd order minors formed by its 2nd and 3rd rows are all zero. But another 2nd order minor is $\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} = -1 \neq 0$.

$\therefore$ $\rho(A) = 2$. Hence the rank of the given matrix is 2.

(*ii*) Given matrix

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 1 & -3 & -1 \\ 1 & 1 & -3 & -1 \end{bmatrix} \qquad \sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

[Operating $C_3 - C_1, C_4 - C_1$] [Operating $R_3 - R_1, R_4 - R_1$]

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A \text{ (say)}$$

[Operating $R_3 - 3R_2, R_4 - R_2$] [Operating $C_3 + 3C_2, C_4 + C_2$]

Obviously, the 4th order minor of A is zero. Also every 3rd order minor of A is zero. But, of all the 2nd order minors, only $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1 \neq 0$. $\therefore$ $\rho(A) = 2$.

Hence the rank of the given matrix is 2.

(4) Elementary matrices. *An elementary matrix is that, which is obtained from a unit matrix, by subjecting it to any of the elementary transformations.*

Examples of elementary matrices obtained from

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ are } R_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = C_{23} \,;\, kR_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix} \,;\, R_1 + pR_2 = \begin{bmatrix} 1 & p & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(5) Theorem. *Elementary row (column) transformations of a matrix A can be obtained by pre-multiplying (post-multiplying) A by the corresponding elementary matrices.*

Consider the matrix $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

Then $R_{23} \times A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{bmatrix}$

So a pre-multiplication by R_{23} has interchanged the 2nd and 3rd rows of A. Similarly, pre-multiplication by kR_2 will multiply the 2nd row of A by k and pre-multiplication by $R_1 + pR_2$ will result in the addition of p times the 2nd row of A to its 1st row.

Thus the pre-multiplication of A by elementary matrices results in the corresponding elementary row transformation of A. It can easily be seen that post mutiplication will perform the elementary column transformations.

(6) Gauss-Jordan method of finding the inverse*. *Those elementary row transformations which reduce a given square matrix A to the unit matrix, when applied to unit matrix I give the inverse of A.*

Let the successive row transformations which reduce A to I result from pre-multiplication by the elementary matrices $R_1, R_2, \ldots R_i$ so that

$$R_i R_{i-1} \ldots R_2 R_1 A = I$$

$\therefore$ $$R_i R_{i-1} \ldots R_2 R_1 A A^{-1} = I A^{-1}$$

or $$R_i R_{i-1} \ldots R_2 R_1 I = A^{-1} \qquad [\because \; AA^{-1} = I]$$

Hence the result.

Working rule to evaluate A^{-1}. *Write the two matrices A and I side by side. Then perform the same row transformations on both. As soon as A is reduced to I, the other matrix represents A^{-1}.*

Example 2.24. *Using the Gauss-Jordan method, find the inverse of the matrix*

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}. \qquad \text{(Kurukshetra, 2006)}$$

Solution. Writing the same matrix side by side with the unit matrix of order 3, we have

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right] \qquad \text{(Operate } R_2 - R_1 \text{ and } R_3 + 2R_1)$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right] \qquad \text{(Operate } \tfrac{1}{2}R_2 \text{ and } \tfrac{1}{2}R_3)$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 1 & 1 & 0 & \frac{1}{2} \end{array}\right] \qquad \text{(Operate } R_1 - R_2 \text{ and } R_3 + R_2)$$

*Named after the great German mathematician *Carl Friedrich Gauss* (1777–1855) who made his first great discovery as a student at Gottingen. His important contributions are to algebra, number theory, mechanics, complex analysis, differential equations, differential geometry, non-Euclidean geometry, numerical analysis, astronomy and electromagnetism. He became director of the observatory at Gottingen in 1807.

Name after another German mathematician and geodesist *Wilhelm Jordan* (1842–1899).

$$\sim\begin{bmatrix} 1 & 0 & 6: & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -3: & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -2: & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad \left[\text{Operate } R_1 + 3R_3,\ R_2 - \frac{3}{2}R_3 \text{ and } \left(-\frac{1}{2}\right)R_2\right]$$

$$\sim\begin{bmatrix} 1 & 0 & 0: & 3 & 1 & \frac{3}{2} \\ 0 & 1 & 0: & -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1: & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

Hence the inverse of the given matrix is $\begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$ [cf. Example 2.21]

(7) Normal form of a matrix. *Every non-zero matrix A of rank r, can be reduced by a sequence of elementary transformations, to the form*

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \text{ called the } \textbf{normal form} \text{ of } A. \qquad \ldots(i)$$

Cor. 1. *The rank of a matrix A is r if and only if it can be reduced to the normal form (i).*

Cor. 2. *Since each elementary transformation can be affected by pre-multiplication or post-multiplication with a suitable elementary matrix and each elementary matrix is non-singular, therefore, we have the following result :*

Corresponding to every matrix A of rank r, there exist non-singular matrices P and Q such that PAQ equals (i).

If A be a $m \times n$ matrix, then P and Q are square matrices of orders m and n respectively.

Example 2.25. *Reduce the following matrix into its normal form and hence find its rank.*

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}. \qquad \textit{(U.P.T.U., 2005)}$$

Solution.

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} \qquad [\text{By } R_{12}]$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} \qquad [\text{By } R_2 - 2R_1, R_3 - 3R_1, R_4 - 6R_1]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} \qquad [\text{By } C_2 + C_1, C_3 + 2C_1, C_4 + 4C_1]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad [\text{By } R_4 - R_2 - R_3]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{[By } R_2 - R_3]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{[By } R_3 - 4R_2]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{[By } C_3 + 6C_2, C_4 + 3C_2]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \left[\text{By } \frac{1}{33} C_3\right]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{[By } C_4 - 22C_3]$$

$$\sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence $\rho(A) = 3$.

Example 2.26. *For the matrix* $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$, *find non-singular matrices P and Q such that PAQ is in the normal form. Hence find the rank of A.*

(Kurukshetra, 2005)

Solution. We write $A = IAI$, *i.e.*, $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

We shall affect every elementary row (column) transformation of the product by subjecting the pre-factor (post-factor) of A to the same.

Operate $C_2 - C_1, C_3 - 2C_1$, $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Operate $R_2 - R_1$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Operate $C_3 - C_2$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

Operate $R_3 + R_2$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

which is of the normal form $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$

Hence, $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ and $\rho(A) = 2$.

PROBLEMS 2.4

Determine the rank of the following matrices (1–4) :

1. $\begin{bmatrix} 1 & 4 & 5 \\ 2 & 6 & 8 \\ 3 & 7 & 22 \end{bmatrix}$ (*P.T.U., 2005*)

2. $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$ (*W.B.T.U., 2005*)

3. $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ (*Kottayam, 2005*)

4. $\begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{bmatrix}$ (*Rohtak, 2004*)

5. $\begin{bmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{bmatrix}$ (*Bhopal, 2008*)

6. Determine the values of p such that the rank of $A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ p & 2 & 2 & 2 \\ 9 & 9 & p & 3 \end{bmatrix}$ is 3. (*Mumbai, 2007*)

7. Use Gauss-Jordan method to find the inverse of the following matrices :

(*i*) $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ (*Mumbai, 2008*)

(*iii*) $\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ (*B.P.T.U., 2006*) (*iv*) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ (*Kurukshetra, 2006*)

8. Find the non-singular matrices P and Q such that $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$ is reduced to normal form. Also find its rank. (*S.V.T.U., 2009 ; Mumbai, 2007*)

9. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, find A^{-1}. Also find two non-signular matrices P and Q such that $PAQ = I$, where I is the unit matrix and verify that $A^{-1} = QP$.

10. Find non-singular matrices P and Q such that PAQ is in the normal form for the matrices :

(*i*) $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ (*Rohtak, 2004*) (*ii*) $A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$

11. Reduce each of the following matrices to normal form and hence find their ranks :

(*i*) $\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$ (*Kurukshetra, 2005*) (*ii*) $A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ (*Bhopal 2009*)

(*iii*) $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ (*Mumbai, 2008*) (*iv*) $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$ (*U.T.U., 2010*)

2.8 PARTITION METHOD OF FINDING THE INVERSE

According to this method of finding the inverse, if the inverse of a matrix A_n of order n is known, then the inverse of the matrix A_{n+1} can easily be obtained by adding $(n+1)$th row and $(n+1)$th column to A_n.

Let $$A = \begin{bmatrix} A_1 & : & A_2 \\ \dots & : & \dots \\ A_3' & : & \alpha \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} X_1 & : & X_2 \\ \dots & : & \dots \\ X_3' & : & x \end{bmatrix}$$

where A_2, X_2 are column vectors and A_3', X_3' are row vectors (being transposes of column vectors A_3, X_3) and α, x are ordinary numbers. We also assume that A_1^{-1} is known.

Then, $AA^{-1} = I_{n+1}$, *i.e.*, $$\begin{bmatrix} A_1 & : & A_2 \\ \dots & : & \dots \\ A_3' & : & \alpha \end{bmatrix}\begin{bmatrix} X_1 & : & X_2 \\ \dots & : & \dots \\ X_3' & : & x \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & 1 \end{bmatrix}$$

gives
$$A_1X_1 + A_2X_3' = I_n \quad \text{...(i)}$$
$$A_1X_2 + A_2x = 0 \quad \text{...(ii)}$$
$$A_3'X_1 + \alpha X_3' = 0 \quad \text{...(iii)}$$
$$A_3'X_2 + \alpha x = 1 \quad \text{...(iv)}$$

From (*ii*), $X_2 = -A_1^{-1}A_2x$ and using this, (*iv*) gives $x = (\alpha - A_3'A_1^{-1}A_2)^{-1}$

Hence x and then X_2 are given.

Also from (*i*), $X_1 = A_1^{-1}(I_n - A_2X_3')$

and using this, (*iii*) gives $X_3' = -A_3'A_1^{-1}(\alpha - A_3'A_1^{-1}A_2)^{-1} = -A_3'A_1^{-1}x$

Then X_1 is determined and hence A^{-1} is computed.

Obs. This is also known as the *'Escalator method'*. For evaluation of A^{-1} we only need to determine two inverse matrices A_1^{-1} and $(\alpha - A_3'A_1^{-1}A_2)^{-1}$.

Example 2.27. *Using the partition method, find the inverse of* $\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$.

Solution. Let $$A = \begin{bmatrix} 1 & 1 & : & 1 \\ 4 & 3 & : & -1 \\ \dots & \dots & : & \dots \\ 3 & 5 & : & 3 \end{bmatrix} = \begin{bmatrix} A_1 & : & A_2 \\ \dots & : & \dots \\ A_3' & : & \alpha \end{bmatrix}$$

so that $$A_1^{-1} = \begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix}^{-1} = -\begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix}$$

Let $$A^{-1} = \begin{bmatrix} X_1 & : & X_2 \\ \dots & : & \dots \\ X_3' & : & x \end{bmatrix} \text{ so that } AA^{-1} = I.$$

$$\alpha - A_3'A_1^{-1}A_2 = 3 + [3 \quad 5] = \begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -10$$

$\therefore$ $$x = (\alpha - A_3'A_1^{-1}A_2)^{-1} = -\frac{1}{10}$$

Also, $$X_2 = -A_1^{-1}A_2x = \begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\left(-\frac{1}{10}\right) = -\frac{1}{10}\begin{bmatrix} 4 \\ -5 \end{bmatrix}$$

Then $$X_3' = -A_3'A_1^{-1}x = [3 \quad 5]\begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix}\left(-\frac{1}{10}\right) = -\frac{1}{10}[-11 \quad 2]$$

Finally, $$X_1 = A_1^{-1}(I - A_2X_3') = -\begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} - \frac{1}{10}\begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}[-11 \quad 2]$$

$$= \begin{bmatrix} -3 & 1 \\ 4 & -1 \end{bmatrix} - \frac{1}{10}\begin{bmatrix} -44 & 8 \\ 55 & -10 \end{bmatrix} = \begin{bmatrix} 1.4 & 0.2 \\ -1.5 & 0 \end{bmatrix}$$

Hence $$A^{-1} = \begin{bmatrix} 1.4 & 0.2 & -0.4 \\ -1.5 & 0 & 0.5 \\ 1.1 & -0.2 & -0.1 \end{bmatrix}.$$

Example 2.28. *If A and C are non-singular matrices, then show that* $\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ -C^{-1}BA^{-1} & C^{-1} \end{bmatrix}$

Hence find inverse of $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 4 & 0 \\ 0 & 1 & 0 & 3 \end{bmatrix}$. (*Mumbai, 2005*)

Solution. Let the given matrix be $M = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$ and its inverse be $M^{-1} = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ both in the partioned form where A, B, C, P, Q, R, S are all matrices.

$\therefore$ $$MM^{-1} = \begin{bmatrix} A & O \\ B & C \end{bmatrix}\begin{bmatrix} P & Q \\ R & S \end{bmatrix} = I$$

or $$\begin{bmatrix} AP + OR & AQ + OS \\ BP + CR & BQ + CS \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix}$$

$\therefore$ *Equating corresponding elements, we have*

$$AP + OR = I, AQ + OS = 0, BP + CR = 0, BQ + CS = I.$$

Second relation gives $AQ = 0$, *i.e.*, $Q = 0$ as A is non-singular.

First relation gives $AP = I$, *i.e.*, $P = A^{-1}$.

From third equation, $BP + CR = 0$, *i.e.*, $CR = -BP = -BA^{-1}$

$\therefore$ $C^{-1}CR = -C^{-1}BA^{-1}$ or $IR = -C^{-1}BA^{-1}$ or $R = -C^{-1}BA^{-1}$

From fourth equation, $BQ + CS = I$, or $CS = I$ or $S = C^{-1}$

Hence $$M^{-1} = \begin{bmatrix} A^{-1} & 0 \\ -C^{-1}BA^{-1} & C^{-1} \end{bmatrix}.$$

(*ii*) Let $$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 4 & 0 \\ 0 & 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} A & O \\ B & C \end{bmatrix}$$

Whence $$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$$

$\therefore$ $$A^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, C^{-1} = \frac{1}{12}\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$$

$\therefore$ $$-C^{-1}(BA^{-1}) = -\frac{1}{12}\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}\left\{\frac{1}{2}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

$$= -\frac{1}{24}\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix} = -\frac{1}{24}\begin{bmatrix} 18 & 0 \\ 0 & 4 \end{bmatrix}$$

Hence, $$M^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ -3/4 & 0 & 1/4 & 0 \\ 0 & -1/6 & 0 & 1/3 \end{bmatrix}.$$

PROBLEMS 2.5

Find the inverse of each of the following matrices using the partition method :

1. $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ *(Nagpur, 1997)*

2. $\begin{bmatrix} 3 & 2 & 4 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 3 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 3 & -1 & 10 & 2 \\ 5 & 1 & 20 & 3 \\ 9 & 7 & 39 & 4 \\ 1 & -2 & 2 & 1 \end{bmatrix}$.

2.9 SOLUTION OF LINEAR SYSTEM OF EQUATIONS

(1) Method of determinants—Cramer's* rule

Consider the equations $\left\{\begin{matrix} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{matrix}\right\}$ ...(*i*)

If the determinant of coefficient be $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

then $$x\Delta = \begin{vmatrix} xa_1 & b_1 & c_1 \\ xa_2 & b_2 & c_2 \\ xa_3 & b_3 & c_3 \end{vmatrix} \quad [\text{Operate } C_1 + yC_2 + zC_3]$$

$$= \begin{vmatrix} a_1x + b_1y + c_1z & b_1 & c_1 \\ a_2x + b_2y + c_2z & b_2 & c_2 \\ a_3x + b_3y + c_3z & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \quad [\text{By } (i)]$$

Thus $$x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ provided } \Delta \neq 0. \quad ...(ii)$$

Similarly, $$y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad ...(iii)$$

and $$z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad ...(iv)$$

Equation (*ii*), (*iii*) and (*iv*) *giving the values of x, y, z constitute the* **Cramer's rule,** which reduces the solution of the linear equations (*i*) to a problem in evaluation of determinants.

(2) Matrix inversion method

If $$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

then the equations (*i*) are equivalent to the matrix equation $AX = D$ ...(*v*)

where A is the *coefficient matrix.*

Multiplying both sides of (*v*) by the reciprocal matrix A^{-1}, we get

$$A^{-1}AX = A^{-1}D \quad \text{or} \quad IX = A^{-1}D \qquad [\because \ A^{-1}A = I]$$

**Gabriel Cramer* (1704–1752), a Swiss mathematician.

or $$X = A^{-1}D \quad i.e., \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \times \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \qquad ...(vi)$$

where A_1, B_1 etc. are the cofactors of a_1, b_1 etc. in the determinant $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ $(\Delta \neq 0)$

Hence equating the values of x, y, z to the corresponding elements in the product on the right side of (vi), we get the desired solutions.

Obs. *When A is a singular matrix, i.e., $\Delta = 0$, the above methods fail. These also fail when the number of equations and the number of unknowns are unequal. Matrices can, however, be usefully applied to deal with such equations as will be seen in § 2.10(2).*

Example 2.29. *Solve the equations $3x + y + 2z = 3$, $2x - 3y - z = -3$, $x + 2y + z = 4$ by (i) determinants (ii) matrices.*

Solution. (*i*) *By determinants :*

Here $$\Delta = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3(-3+2) - 2(1-4) + (-1+6) = 8 \quad \text{[Expanding by } C_1\text{]}$$

$\therefore$ $$x = \frac{1}{\Delta}\begin{vmatrix} 3 & 1 & 2 \\ -3 & -3 & -1 \\ 4 & 2 & 1 \end{vmatrix} \quad \text{[Expand by } C_1\text{]}$$

$$= \frac{1}{8}[3(-3+2) + 3(1-4) + 4(-1+6)] = 1$$

Similarly, $$y = \frac{1}{\Delta}\begin{vmatrix} 3 & 3 & 2 \\ 2 & -3 & -1 \\ 1 & 4 & 1 \end{vmatrix} = 2 \quad \text{and} \quad z = \frac{1}{\Delta}\begin{vmatrix} 3 & 1 & 3 \\ 2 & -3 & -3 \\ 1 & 2 & 4 \end{vmatrix} = -1$$

Hence $x = 1, y = 2, z = -1.$

Note. The use of Cramer's rule involves a lot of labour when the number of equations exceeds four. In such and other cases, the numerical methods given in § 28.4 to 28.6 are perferable.

(*ii*) *By matrices :*

Here $$\Delta = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{(say).}$$

Then $A_1 = -1, A_2 = 3, A_3 = 5$; $B_1 = -3, B_2 = 1, B_3 = 7$; $C_1 = 7, C_2 = -5, C_3 = -11.$

Also $\Delta = a_1A_1 + a_2A_2 + a_3A_3 = 8.$

$\therefore$ $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{\Delta}\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \times \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix} \times \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -3-9+20 \\ -9-3+28 \\ 21+15-44 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Hence $x = 1, y = 2, z = -1.$

Example 2.30. *Solve the equations $x_1 - x_2 + x_3 + x_4 = 2$; $x_1 + x_2 - x_3 + x_4 = -4$; $x_1 + x_2 + x_3 - x_4 = 4$; $x_1 + x_2 + x_3 + x_4 = 0$, by finding the inverse by elementary row operations.*

Solution. Given system can be written as $AX = B$, where

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, B = \begin{bmatrix} 2 \\ -4 \\ 4 \\ 0 \end{bmatrix}$$

To find A^{-1}, we write

$$[A : I] = \left[\begin{array}{rrrr:rrrr} 1 & -1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} R_2 - R_1 \\ R_3 + R_1 \\ R_4 + R_1 \end{bmatrix}$$

$$= \left[\begin{array}{rrrr:rrrr} 1 & -1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 & -1 & 1 & 0 & 0 \\ 2 & 0 & 2 & 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} \frac{1}{2}R_2 \\ \frac{1}{2}R_3 \\ \frac{1}{2}R_4 \end{bmatrix}$$

$$= \left[\begin{array}{rrrr:rrrr} 1 & -1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1/2 & 1/2 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1/2 & 0 & 1/2 & 0 \\ 1 & 0 & 1 & 1 & 1/2 & 0 & 0 & 1/2 \end{array}\right] \begin{bmatrix} R_3 - R_2 \\ R_4 - R_3 \end{bmatrix}$$

$$= \left[\begin{array}{rrrr:rrrr} 1 & 0 & 0 & 1 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1/2 & 1/2 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1/2 & 1/2 \end{array}\right] \begin{bmatrix} R_1 - R_4 \\ R_2 + R_3 \end{bmatrix}$$

$$= \left[\begin{array}{rrrr:rrrr} 1 & 0 & 0 & 0 & 1/2 & 1/2 & +1/2 & -1/2 \\ 1 & 1 & 0 & 0 & 0 & 1/2 & 1/2 & 0 \\ 1 & 0 & 1 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1/2 & 1/2 \end{array}\right] \begin{bmatrix} R_2 - R_1 \\ R_3 - R_1 \end{bmatrix}$$

$$= \left[\begin{array}{rrrr:rrrr} 1 & 0 & 0 & 0 & 1/2 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 0 & 0 & -1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 0 & -1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1/2 & 1/2 \end{array}\right]$$

Thus,
$$A^{-1} = \begin{bmatrix} 1/2 & 1/2 & 1/2 & -1/2 \\ -1/2 & 0 & 0 & 1/2 \\ 0 & -1/2 & 0 & 1/2 \\ 0 & 0 & -1/2 & 1/2 \end{bmatrix}$$

Hence,
$$X = A^{-1}B = \begin{bmatrix} 1/2 & 1/2 & 1/2 & -1/2 \\ -1/2 & 0 & 0 & 1/2 \\ 0 & -1/2 & 0 & 1/2 \\ 0 & 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}$$

i.e., $x_1 = 1, x_2 = -1, x_3 = 2, x_4 = -2$.

PROBLEMS 2.6

Solve the following equations with the help of determinants (1 to 4) :

1. $x + y + z = 4$; $x - y + z = 0$; $2x + y + z = 5$. *(Osmania, 2003)*
2. $x + 3y + 6z = 2$; $3x - y + 4z = 9$; $x - 4y + 2z = 7$.
3. $x + y + z = 6.6$; $x - y + z = 2.2$; $x + 2y + 3z = 15.2$.
4. $x^2 z^3/y = e^8$; $y^2 z/x = e^4$; $x^3 y/z^4 = 1$.
5. $2vw - wu + uv = 3uvw$; $3vw + 2wu + 4uv = 19uvw$; $6vw + 7wu - uv = 17uvw$.

Solve the following system of equations by matrix method (6 to 8) :

6. $x_1 + x_2 + x_3 = 1, x_1 + 2x_2 + 3x_3 = 6, x_1 + 3x_2 + 4x_3 = 6$. *(P.T.U., 2006)*
7. $x + y + z = 3$; $x + 2y + 3z = 4$; $x + 4y + 9z = 6$. *(Bhopal, 2003)*
8. $2x - 3y + 4z = -4, x + z = 0, -y + 4z = 2$. *(W.B.T.U., 2005)*
9. $2x - y + 3z = 8$; $x - 2y - z = -4$; $3x + y - 4z = 0$. *(Mumbai, 2005)*
10. $2x_1 + x_2 + 2x_3 + x_4 = 6, 4x_1 + 3x_2 + 3x_3 - 3x_4 = -1, 6x_1 - 6x_2 + 6x_3 + 12x_4 = 36, 2x_1 + 2x_2 - x_3 + x_4 = 10$. *(U.P.T.U., 2001)*

11. By finding A^{-1}, solve the linear equation $AX = B$, where $A = \begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 0 \\ 5 & 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}$.
12. In a given electrical network, the equations for the currents i_1, i_2, i_3 are
$3i_1 + i_2 + i_3 = 8$; $2i_1 - 3i_2 - 2i_3 = -5$; $7i_1 + 2i_2 - 5i_3 = 0$.
Calculate i_1 and i_3 by Cramer's rule.
13. Using the loop current method on a circuit, the follwoing equations are obtained :
$7i_1 - 4i_2 = 12, -4i_1 + 12i_2 - 6i_3 = 0, -6i_2 + 14i_3 = 0$.
By matrix method, solve for i_1, i_2 and i_3.
14. Solve the following equations by calculating the inverse by elementary row operations :
$2x_1 + 2x_2 + 2x_3 - 3x_4 = 2$; $3x_1 + 6x_2 - 2x_3 + x_4 = 8$; $x_1 + x_2 - 3x_3 - 4x_4 = -1$; $2x_1 + x_2 + 5x_3 + x_4 = 5$.

2.10 (1) CONSISTENCY OF LINEAR SYSTEM OF EQUATIONS

Consider the system of m linear equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= k_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= k_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= k_m \end{aligned}\right\} \quad \dots(i)$$

containing the n unknowns $x_1, x_2, \dots, x_n$. To determine whether the equations (i) are consistent (*i.e.*, possess a solution) or not, we consider the ranks of the matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & k_1 \\ a_{21} & a_{22} & \dots & a_{2n} & k_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & k_m \end{bmatrix}$$

A is the *co-efficient matrix* and K is called the *augmented matrix* of the equations (i).

(2) Rouche's theorem. *The system of equations (i) is consistent if and only if the coefficient matrix A and the augmented matrix K are of the same rank otherwise the system is inconsistent.*

Proof. We consider the following two possible cases :

I. Rank of A = rank of K = r ($r \le$ the smaller of the numbers m and n). The equations (i) can, be suitable row operations, be reduced to

$$\left.\begin{aligned} b_{11}x_1 + b_{12}x_2 + \dots + b_{1n}x_n &= l_1 \\ 0.x_1 + b_{22}x_2 + \dots + b_{2n}x_n &= l_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ 0.x_1 + 0.x_2 + \dots + b_{rn}x_n &= l_r \end{aligned}\right\} \quad \dots(ii)$$

and the remaining $m - r$ equations being all of the form $0.x_1 + 0.x_2 + \dots\dots + 0.x_n = 0$.

The equations (ii) will have a solution, though $n - r$ of the unknowns may be chosen arbitrarily. The solution, will be unique only when $r = n$. Hence the equations (i) are consistent.

II. Rank of A (i.e., r) < rank of K. In particular, let the rank of K be $r + 1$. In this case, the equations (i) will reduce, by suitable row operations, to

$$\begin{aligned} b_{11}x_1 + b_{12}x_2 + \dots + b_{1n}x_n &= l_1, \\ 0.x_1 + b_{22}x_2 + \dots + b_{2n}x_n &= l_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots&, \\ 0.x_1 + 0.x_2 + \dots + b_{rn}x_n &= l_r, \\ 0.x_1 + 0.x_2 + \dots + 0.x_n &= l_{r+1}, \end{aligned}$$

and the remaining $m - (r + 1)$ equations are of the form $0.x_1 + 0.x_2 + \dots + 0.x_n = 0$.

Clearly, the $(r + 1)$th equation cannot be satisfied by any set of values for the unknowns. Hence the equations (i) are inconsistent.

[Procedure to test the consistency of a system of equations in n unknowns :

Find the ranks of the coefficient matrix A and the augmented matrix K, by reducing A to the triangular form by elementary row operations. Let the rank of A be r and that of K be r'.

(i) If $r \neq r'$, the equations are inconsistent, i.e., there is no solution.

(ii) If $r = r' = n$, the equations are consitent and there is a unique solution.

(iii) If $r = r' < n$, the equations are consistent and there are infinite number of solutions. Giving arbitrary values to $n - r$ of the unknowns, we may express the other r unknowns in terms of these.]

Example 2.31. *Test for consistency and solve*

$$5x + 3y + 7z = 4,\ 3x + 26y + 2z = 9,\ 7x + 2y + 10z = 5.$$

(Bhopal, 2008 ; J.N.T.U., 2005 ; P.T.U., 2005)

Solution. We have
$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

Operate $3R_1, 5R_2$,
$$\begin{bmatrix} 15 & 9 & 21 \\ 15 & 130 & 10 \\ 7 & 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 45 \\ 5 \end{bmatrix}$$

Operate $R_2 - R_1$,
$$\begin{bmatrix} 15 & 9 & 21 \\ 0 & 121 & -11 \\ 7 & 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 33 \\ 5 \end{bmatrix}$$

Operate $\frac{7}{3}R_1, 5R_3, \frac{1}{11}R_2$,
$$\begin{bmatrix} 35 & 21 & 49 \\ 0 & 11 & -1 \\ 35 & 10 & 50 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 28 \\ 3 \\ 25 \end{bmatrix}$$

Operate $R_3 - R_1 + R_2, \frac{1}{7}R_1$,
$$\begin{bmatrix} 5 & 3 & 7 \\ 0 & 11 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$$

The ranks of coefficient matrix and augmented matrix for the last set of equations, are both 2. Hence the equations are consistent. Also the given system is equivalent to

$$5x + 3y + 7z = 4,\ 11y - z = 3, \quad \therefore\ y = \frac{3}{11} + \frac{z}{11} \quad \text{and} \quad x = \frac{7}{11} - \frac{16}{11}z$$

where z is a parameter.

Hence $\quad x = \frac{7}{11}, y = \frac{3}{11}$ and $z = 0$, is a particular solution.

Obs. In the above solution, the coefficient matrix is reduced to an upper triangular matrix by row-transformations.

Example 2.32. *Investigate the values of λ and μ so that the equations*

$$2x + 3y + 5z = 9,\ 7x + 3y - 2z = 8,\ 2x + 3y + \lambda z = \mu,$$

have (i) no solution, (ii) a unique solution and (iii) an infinite number of solutions.

(Mumbai, 2007 ; V.T.U., 2007)

Solution. We have
$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

The system admits of unique solution if, and only if, the coefficient matrix is of rank 3. This requires that

$$\begin{vmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{vmatrix} = 15(5 - \lambda) \neq 0$$

Thus for a unique solution $\lambda \neq 5$ and μ may have any value. If $\lambda = 5$, the system will have no solution for those values of μ for which the matrices

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & 5 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & 5 & \mu \end{bmatrix}$$

are not of the same rank. But A is of rank 2 and K is not of rank 2 unless $\mu = 9$. Thus if $\lambda = 5$ and $\mu \neq 9$, the system will have no solution.

If $\lambda = 5$ and $\mu = 9$, the system will have an infinite number of solutions.

Example 2.33. *Test for consistency the following equations and solve them if consistent :* $x - 2y + 3t = 2$, $2x + y + z + t = -4$; $4x - 3y + z + 7t = 8$. *(Mumbai, 2008)*

Solution. Given equation can be written as

$$\begin{bmatrix} 1 & -2 & 0 & 3 \\ 2 & 1 & 1 & 1 \\ 4 & -3 & 1 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 8 \end{bmatrix}$$

Operate $R_2 - 2R_1$, $R_3 - 4R_1$,

$$\begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 5 & 1 & -5 \\ 0 & 5 & 1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Operate $R_3 - R_2$,
$$\begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 5 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Clearly, rank of the coefficient matrix is 2 and the rank of augmented matrix is also 2. Hence the given equations are consistent. But the rank 2 < 4, the number of unknowns.

$\therefore$ The number of parameters is $4 - 2 = 2$

Thus the equations have doubly infinite solutions. Now putting $t = k_1$ and $z = k_2$ in

$$x - 2y + 3t = 2 \quad \text{and} \quad 5y + z - 5t = 0,$$

we get $x - 2y + 3k_1 = 2$ and $5y + k_2 - 5k_1 = 0$

Hence $y = k_1 - k_2/5$

and
$$\begin{aligned} x &= 2 + 2y - 3k_1 \\ &= 2 + 2(k_1 - k_2/5) - 3k_1 \\ &= 2 - k_1 - \frac{2}{5}k_2 \end{aligned}$$

(3) System of linear homogeneous equations. Consider the homogeneous linear equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= 0 \end{aligned}\right\} \qquad \dots(iii)$$

Find the rank r of the coefficient matrix A by reducing it to the triangular form by elementary row operations.

I. *If $r = n$, the equations (iii) have only a trivial zero solution*

$$x_1 = x_2 = \dots = x_n = 0$$

If $r < n$, the equation (iii) have $(n - r)$ linearly independent solutions.

The number of linearly independent solutions is $(n - r)$ means, if arbitrary values are assigned to $(n - r)$ of the variables, the values of the remaining variables can be uniquely found.

Thus the equations (*iii*) will have an infinite number of solutions.

II. *When $m < n$* (*i.e.*, the number of equations is less than the number of variables), *the solution is always other than $x_1 = x_2 = \dots = x_n = 0$.* The number of solutions is infinite.

III. *When $m = n$* (*i.e.*, the number of equations = the number of variables), *the necessary and sufficient condition for solutions other than $x_1 = x_2 = \dots = x_n = 0$, is that the determinant of the coefficient matrix is zero.* In this case the equations are said to be *consistent* and such a solution is called *non-trivial solution. The determinant is called the* **eliminant** *of the equations.*

Example 2.34. *Solve the equations*
(i) $x + 2y + 3z = 0, 3x + 4y + 4z = 0, 7x + 10y + 12z = 0$
(ii) $4x + 2y + z + 3w = 0, 6x + 3y + 4z + 7w = 0, 2x + y + w = 0.$

Solution. *(i)* Rank of the coefficient matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 7 & 10 & 12 \end{bmatrix} \qquad [\text{Operating } R_3 - 3R_1]$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix} \qquad [\text{Operating } R_3 - 7R_1 - 2R_2]$$

is 3 which = the number of variables (*i.e.*, $r = n$)

$\therefore$ The equations have only a trivial solution : $x = y = z = 0$.

(ii) Rank of the coefficient matrix

$$\begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5/2 & 5/2 \\ 0 & 0 & -1/2 & -1/2 \end{bmatrix} \qquad \left[\text{Operating } R_2 - \frac{3}{2} R_1, R_3 - \frac{1}{2} R_1\right.$$

$$\sim \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5/2 & 5/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \left[\text{Operating } R_3 + \frac{1}{5} R_2\right.$$

is 2 which < the number of variable (*i.e.*, $r < n$)

$\therefore$ Number of independent solutions $= 4 - 2 = 2$. Given system is equivalent to

$$4x + 2y + z + 3w = 0, z + w = 0.$$

$\therefore$ We have $z = -w$ and $y = -2x - w$

which give an infinite number of non-trivial solutions, x and w being the parameters.

Example 2.35. *Find the values of k for which the system of equations* $(3k - 8)x + 3y + 3z = 0, 3x + (3k - 8)y + 3z = 0, 3x + 3y + (3k - 8) z = 0$ *has a non-trivial solution.* *(U.P.T.U., 2006)*

Solution. For the given system of equations to have a non-trivial solution, the determinant of the coefficient matrix should be zero.

i.e., $$\begin{vmatrix} 3k-8 & 3 & 3 \\ 3 & 3k-8 & 3 \\ 3 & 3 & 3k-8 \end{vmatrix} = 0 \text{ or } \begin{vmatrix} 3k-2 & 3 & 3 \\ 3k-2 & 3k-8 & 3 \\ 3k-2 & 3 & 3k-8 \end{vmatrix} = 0 \qquad [\text{Operating } C_1 + (C_2 + C_3)]$$

or $$(3k-2)\begin{vmatrix} 1 & 3 & 3 \\ 1 & 3k-8 & 3 \\ 1 & 3 & 3k-8 \end{vmatrix} = 0 \text{ or } (3k-2)\begin{vmatrix} 1 & 3 & 3 \\ 0 & 3k-11 & 0 \\ 0 & 0 & 3k-11 \end{vmatrix} = 0 \qquad [\text{Operating } R_2 - R_1, R_3 - R_1]$$

or $$(3k - 2)(3k - 11)^2 = 0 \text{ whence } k = 2/3, 11/3, 11/3.$$

Example 2.36. *If the following system has non-trivial solution, prove that* $a + b + c = 0$ *or* $a = b = c$ *:* $ax + by + cz = 0, bx + cy + az = 0, cx + ay + bz = 0.$ *(Mumbai, 2006)*

Solution. For the given system of equations to have non-trivial solution, the determinant of the coefficient matrix is zero.

i.e., $$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0 \text{ or } \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix} = 0 \qquad [\text{Operating } R_1 + R_2 + R_3]$$

or $$(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix} = 0 \text{ or } (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ b & c-b & a-b \\ c & a-c & b-c \end{vmatrix} = 0 \qquad [\text{Operating } C_2 - C_1, C_3 - C_1]$$

or $(a+b+c)[(c-b)(b-c)-(a-c)(a-b)]=0$

or $(a+b+c)(-a^2-b^2-c^2+ab+bc+ca)=0$

i.e., $a+b+c=0$ or $a^2+b^2+c^2-ab-bc-ca=0$

or $a+b+c=0$ or $\frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]=0$

or $a+b+c=0\ ;\ a=b,\ b=c,\ c=a.$

Hence the given system has a non-trivial solution if $a+b+c=0$ or $a=b=c$.

Example 2.37. *Find the values of λ for which the equations*

$$(\lambda-1)x+(3\lambda+1)y+2\lambda z=0$$
$$(\lambda-1)x+(4\lambda-2)y+(\lambda+3)z=0$$
$$2x+(3\lambda+1)y+3(\lambda-1)z=0$$

are consistent, and find the ratios of $x:y:z$ when λ has the smallest of these values. What happens when λ has the greatest of these values. (*Kurukshetra, 2006 ; Delhi, 2002*)

Solution. The given equations will be consistent, if

$$\begin{vmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ \lambda-1 & 4\lambda-2 & \lambda+3 \\ 2 & 3\lambda+1 & 3(\lambda-1) \end{vmatrix}=0 \qquad [\text{Operate } R_2-R_1]$$

or if, $$\begin{vmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ 0 & \lambda-3 & 3-\lambda \\ 2 & 3\lambda+1 & 3(\lambda-1) \end{vmatrix}=0 \qquad [\text{Operate } C_3+C_2]$$

or if, $$\begin{vmatrix} \lambda-1 & 3\lambda+1 & 5\lambda+1 \\ 0 & \lambda-3 & 0 \\ 2 & 3\lambda+1 & 6\lambda-2 \end{vmatrix}=0 \qquad [\text{Expand by } R_2]$$

or if, $(\lambda-3)\begin{vmatrix} \lambda-1 & 5\lambda+1 \\ 2 & 2(3\lambda+1) \end{vmatrix}=0$ or if, $2(\lambda-3)[(\lambda-1)(3\lambda-1)-(5\lambda+1)]=0$

or if, $6\lambda(\lambda-3)^2=0$ or if, $\lambda=0$ or 3.

(*a*) *When* $\lambda=0$, the equations become $-x+y=0$ ...(*i*)

$-x-2y+3z=0$ ...(*ii*)

$2x+y-3z=0$ ...(*iii*)

Solving (*ii*) and (*iii*), we get $\dfrac{x}{6-3}=\dfrac{y}{6-3}=\dfrac{z}{-1+4}$. Hence $x=y=z$.

(*b*) *When* $\lambda=3$, equations becomes identical.

PROBLEMS 2.7

1. Investigate for consistency of the following equations and if possible find the solutions :
$4x-2y+6z=8,\ x+y-3z=-1,\ 15x-3y+9z=21.$
2. For what values of k the equations $x+y+z=1,\ 2x+y+4z=k,\ 4x+y+10z=k^2$ have a solution and solve them completely in each case. (*Bhopal, 2008 ; Mumbai, 2008 ; V.T.U., 2006*)
3. Investigate for what values of λ and μ the simultaneous equations
$x+y+z=6,\ x+2y+3z=10,\ x+2y+\lambda z=\mu,$
have (*i*) no solution, (*ii*) a unique solution, (*iii*) an infinite number of solutions.
(*Mumbai, 2007 ; U.P.T.U., 2006 ; Rohtak, 2004*)
4. Test for consistency and solve,
(*i*) $2x-3y+7z=5,\ 3x+y-3z=13,\ 2x+19y-47z=32.$ (*Bhopal, 2009 ; Kurukshetra, 2005 ; Raipur, 2005*)
(*ii*) $x+2y+z=3,\ 2x+3y+2z=5,\ 3x-5y+5z=2,\ 3x+9y-z=4.$ (*Bhilai, 2005 ; Madras, 2002*)
(*iii*) $2x+6y+11=0,\ 6x+20y-6z+3=0,\ 6y-18z+1=0.$ (*Rajasthan, 2005*)
(*iv*) $3x+3y+2z=1,\ x+2y=4,\ 10y+3z=-2,\ 2x-3y-z=5.$ (*U.T.U., 2010 ; Nagarjuna, 2008*)

5. Find the values of a and b for which the equations

$$x + ay + z = 3, x + 2y + 2z = b, x + 5y + 3z = 9$$

are consistent. When will these equations have a unique solution ? (*Kurukshetra, 2005 ; Madras, 2003*)

6. Show that if $\lambda \neq -5$, the system of equations

$$3x - y + 4z = 3, x + 2y - 3z = -2, 6x + 5y + \lambda z = -3,$$

have a unique solution. If $\lambda = -5$, show that the equations are consistent. Determine the solutions in each case.

7. Show that the equations

$$3x + 4y + 5z = a, 4x + 5y + 6z = b, 5x + 6y + 7z = c$$

do not have a solution unless $a + c = 2b$. (*Raipur, 2004 ; Nagpur, 2001*)

8. Prove that the equations $5x + 3y + 2z = 12, 2x + 4y + 5z = 2, 39x + 43y + 45z = c$ are incompatible unless $c = 74$; and in that case the equations are satisfied by $x = 2 + t, y = 2 - 3t, z = -2 + 2t$, where t is any arbitrary quantity.

9. Find the values of λ for which the equations $(2 - \lambda)x + 2y + 3 = 0, 2x + (4 - \lambda)y + 7 = 0, 2x + 5y + (6 - \lambda) = 0$ are consistent and find the values of x and y corresponding to each of these values of λ.

10. Show that there are three real values of λ for which the equations $(a - \lambda) x + by + cz = 0, bx + (c - \lambda)y + az = 0, cx + ay + (b - \lambda)z = 0$ are simultaneously true and that the product of these values of λ is

$$D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$$

11. Determine the values of k for which the following system of equations has non-trivial solutions and find them :

$$(k - 1) x + (4k - 2) y + (k + 3) z = 0, (k - 1) x + (3k + 1) y + 2kz = 0, 2x + (3k + 1) y + 3(k - 1) z = 0.$$

(*Mumbai, 2005*)

12. Show that the system of equations $2x_1 - 2x_2 + x_3 = \lambda x_1, 2x_1 - 3x_2 + 2x_3 = \lambda x_2, -x_1 + 2x_2 = \lambda x_3$ can possess a non-trivial solution only if $\lambda = 1, \lambda = -3$. Obtain the general solution in each case.

13. Determine the values of λ for which the following set of equations may possess non-trivial solution :

$$3x_1 + x_2 - \lambda x_3 = 0, 4x_1 - 2x_2 - 3x_3 = 0, 2\lambda x_1 + 4x_2 + \lambda x_3 = 0.$$

For each permissible value of λ, determine the general solution. (*Kurukshetra, 2006*)

14. Solve completely the system of equations

(*i*) $x + y - 2z + 3w = 0 ; x - 2y + z - w = 0 ; 4x + y - 5z + 8w = 0 ; 5x - 7y + 2z - w = 0.$

(*ii*) $3x + 4y - z - 6w = 0 ; 2x + 3y + 2z - 3w = 0 ; 2x + y - 14z - 9w = 0 ; x + 3y + 13z + 3w = 0.$ (*J.N.T.U., 2002 S*)

2.11 (1) LINEAR TRANSFORMATIONS

Let (x, y) be the co-ordinates of a point P referred to set of rectangular axes OX, OY. Then its co-ordinates (x', y') referred to OX', OY', obtained by rotating the former axes through an angle θ given by

$$\left.\begin{aligned} x' &= x \cos \theta + y \sin \theta, \\ y' &= -x \sin \theta + y \cos \theta \end{aligned}\right\} \quad ...(i)$$

A more general transformation than (*i*) is

$$\left.\begin{aligned} x' &= a_1 x + b_1 y \\ y' &= a_2 x + b_2 y \end{aligned}\right\} \quad ...(ii)$$

which in matrix notation is $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Such transformations as (*i*) and (*ii*), are called *linear transformations* in two dimensions.

Similarly, the relations of the type $\left.\begin{aligned} x' &= l_1 x + m_1 y + n_1 z \\ y' &= l_2 x + m_2 y + n_2 z \\ z' &= l_3 x + m_3 y + n_3 z \end{aligned}\right\}$ ...(*iii*)

give a *linear transformation* from (x, y, z) to (x', y', z') in three dimensional problems.

In general, the relation $Y = AX$ where $Y = \begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{bmatrix}, A = \begin{bmatrix} a_1 & b_1 & c_1 & \cdots & k_1 \\ a_2 & b_2 & c_2 & \cdots & k_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n & b_n & c_n & \cdots & k_n \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$ ...(*iv*) give *linear transformation* from n variables $x_1, x_2, ..., x_n$ to the variables $y_1, y_2, ..., y_n$ *i.e.*, the transformation of the vector X to the vector Y.

This transformation is called linear because the linear relations $A(X_1 + X_2) = AX_1 + AX_2$ and $A(bX) = bAX$, hold for this transformation.

If the transformation matrix A is singular, the transformation also is said to be singular otherwise non-singular. For a non-singular transformation $Y = AX$, we can also write the inverse transformation $X = A^{-1} Y$. A non-singular transformation is also called a *regular* transformation.

Cor. If a transformation from (x_1, x_2, x_3) to (y_1, y_2, y_3) is given by $Y = AX$ and another transformation of (y_1, y_2, y_3) to (z_1, z_2, z_3) is given by $Z = BY$, then the transformation from (x_1, x_2, x_3) to (z_1, z_2, z_3) is given by

$$Z = BY = B(AX) = (BA)X.$$

(2) Orthogonal transformation. *The linear transformation (iv), i.e., $Y = AX$, is said to be* ***orthogonal*** *if, it transforms*

$$y_1^2 + y_2^2 + ... + y_n^2 \textit{ into } x_1^2 + x_2^2 + ... + x_n^2$$

The matrix of an orthogonal transformation is called an **orthogonal matrix.**

We have
$$X'X = [x_1 x_2 ...... x_n] \times \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{bmatrix} = x_1^2 + x_2^2 + ... + x_n^2$$

and similarly, $Y'Y = y_1^2 + y_2^2 + ... + y_n^2.$

$\therefore$ If $Y = AX$ is an orthogonal transformation, then

$$X'X = Y'Y = (AX)'(AX) = X'A'AX \text{ which is possible only if } A'A = I.$$

But $A^{-1}A = I$, therefore, $A' = A^{-1}$ for an orthogonal transformation.

Hence a *square matrix A is said to be orthogonal if $AA' = A'A = I$.*

Obs. 1. *If A is orthogonal, A′ and A^{-1} are also orthogonal.*

Since A is orthogonal, $A' = A^{-1}$.

$\therefore$ $(A')' = (A^{-1})' = (A')^{-1}$, *i.e.*, $B' = B^{-1}$ where $B = A'$

Hence B (*i.e.*, A') is orthogonal. As $A' = A^{-1}$, A^{-1} is also orthogonal.

Obs. 2. *If A is orthogonal, then* $|A| = \pm 1$.

Since $AA' = A'A = I$ $\therefore$ $|A||A'| = |I|$ *(Mumbai, 2006)*

But $|A'| = |A|$, $\therefore$ $|A||A| = |1|$

or $|A|^2 = 1$ *i.e.*, $|A| = \pm 1$.

Example 2.38. *Show that the transformation*

$$y_1 = 2x_1 + x_2 + x_3,\ y_2 = x_1 + x_2 + 2x_3,\ y_3 = x_1 - 2x_3$$

is regular. Write down the inverse transformation.

Solution. The given transformation may be written as

$$Y = AX$$

where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}$

Now
$$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{vmatrix} = -1$$

Thus the matrix A is non-singular and hence the transformatiuon is regular.

$\therefore$ The inverse transformation is given by

$$X = A^{-1} Y$$

where $A^{-1} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix}$

Thus $x_1 = 2y_1 - 2y_2 - y_3$; $x_2 = -4y_1 + 5y_2 + 3y_3$; $x_3 = y_1 - y_2 - y_3$
is the inverse transformation.

Example 2.39. *Prove that the following matrix is orthogonal :*

$$\begin{bmatrix} -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}$$ (Kurukshetra, 2005)

Solution. We have $AA' = \begin{bmatrix} -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix} \times \begin{bmatrix} -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix}$

$$= \begin{bmatrix} 4/9 + 1/9 + 4/9 & -4/9 + 2/9 + 2/9 & -2/9 - 2/9 + 4/9 \\ -4/9 + 2/9 + 2/9 & 4/9 + 4/9 + 1/9 & 2/9 - 4/9 + 2/9 \\ -2/9 - 2/9 + 4/9 & 2/9 - 4/9 + 2/9 & 1/9 + 4/9 + 4/9 \end{bmatrix} = I.$$

Hence the matrix is orthogonal.

Example 2.40. *If* $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{bmatrix}$ *is orthogonal, find a, b, c and* A^{-1}. (Mumbai, 2006)

Solution. As A is orthogonal, $AA' = I$

$$\therefore \quad \frac{1}{3}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{bmatrix} \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or
$$\begin{bmatrix} 1+4+a^2 & 2+2+ab & 2-4+ac \\ 2+2+ab & 4+1+b^2 & 4-2+bc \\ 2-4+ac & 4-2+bc & 4+4+c^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$\therefore$ $5 + a^2 = 9, 5 + b^2 = 9, 8 + c^2 = 9,$ *i.e.*, $a^2 = 4, b^2 = 4, c^2 = 1$

Thus $a = 2, b = 2, c = 1$.

Since A is orthogonal, $A^{-1} = A' = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$.

2.12 (1) VECTORS

Any quantity having n-components is called a *vector of order n.* Therefore, the coefficients in a linear equation or the elements in a row or column matrix will form a vector. Thus any n numbers $x_1, x_2, \ldots, x_n$ written in a particular order, constitute a vector $\mathbf{x}$.

(2) Linear dependence. *The vectors* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ *are said to be* **linearly dependent,** *if there exist numbers* $\lambda_1, \lambda_2, \ldots, \lambda_r$ *not all zero, such that*

$$\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \ldots + \lambda_r\mathbf{x}_r = \mathbf{0}. \qquad \ldots(i)$$

If no such numbers, other than zero, exist, the vectors are said to be **linearly independent.** If $\lambda_1 \neq 0$, transposing $\lambda_1\mathbf{x}_1$ to the other side and dividing by $-\lambda_1$, we write (i) in the form

$\mathbf{x}_1 = \mu_2\mathbf{x}_2 + \mu_3\mathbf{x}_3 + \ldots + \mu_r\mathbf{x}_r$.

Then the vector $\mathbf{x}_1$ is said to be a linear combination of the vectors $\mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_r$.

Example 2.41. *Are the vectors* $\mathbf{x}_1 = (1, 3, 4, 2)$, $\mathbf{x}_2 = (3, -5, 2, 2)$ *and* $\mathbf{x}_3 = (2, -1, 3, 2)$ *linearly dependent ? If so express one of these as a linear combination of the others.*

Solution. The relation $\quad \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \lambda_3\mathbf{x}_3 = \mathbf{0}.$

i.e., $\quad \lambda_1(1, 3, 4, 2) + \lambda_2(3, -5, 2, 2) + \lambda_3(2, -1, 3, 2) = \mathbf{0}$

is equivalent to $\lambda_1 + 3\lambda_2 + 2\lambda_3 = 0,\ 3\lambda_1 - 5\lambda_2 - \lambda_3 = 0,$
$4\lambda_1 + 2\lambda_2 + 3\lambda_3 = 0,\ 2\lambda_1 + 2\lambda_2 + 2\lambda_3 = 0$

As these are satisfied by the values $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = -2$ which are not zero, the given vectors are linearly dependent. Also we have the relation,

$$\mathbf{x}_1 + \mathbf{x}_2 - 2\mathbf{x}_3 = \mathbf{0}$$

by means of which any of the given vectors can be expressed as a linear combination of the others.

Obs. Applying elementary row operations to the vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, we see that the matrices

$$A = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_1 + \mathbf{x}_2 - 2\mathbf{x}_3 \end{bmatrix}$$

have the same rank. The rank of B being 2, the rank of A is also 2. Moreover $\mathbf{x}_1, \mathbf{x}_2$ are linearly independent and $\mathbf{x}_3$ can be expressed as a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$ $\left[\because\ \mathbf{x}_3 = \frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2)\right]$. Similar results will hold for column operations and for any matrix. In general, we have the following results :

If a given matrix has r linearly independent vectors (rows or columns) and the remaining vectors are linear combinations of these r vectors, then rank of the matrix is r. Conversely, if a matrix is of rank r, it contains r linearly independent vectors are remaining vectors (if any) can be expressed as a linear combination of these vectors.

PROBLEMS 2.8

1. Represent each of the transformations
$$x_1 = 3y_1 + 2y_2, y_1 = z_1 + 2z_2 \text{ and } x_2 = -y_1 + 4y_2, y_2 = 3z_1$$
by the use of matrices and find the composite transformation which express x_1, x_2 in terms of z_1, z_2.
2. If $\xi = x\cos\alpha - y\sin\alpha$, $\eta = x\sin\alpha + y\cos\alpha$, write the matrix A of transformation and prove that $A^{-1} = A'$. Hence write the inverse transformation.
3. A transformation from the variables x_1, x_2, x_3 to y_1, y_2, y_3 is given by $Y = AX$, and another transformation from y_1, y_2, y_3 to z_1, z_2, z_3 is given by $Z = BY$, where
$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}.$$ Obtain the transformation from x_1, x_2, x_3 to z_1, z_2, z_3.
4. Find the inverse transformation of $y_1 = x_1 + 2x_2 + 5x_3$; $y_2 = 2x_1 + 4x_2 + 11x_3$; $y_3 = -x_2 + 2x_3$.
5. Verify that the following matrix is orthogonal :

   (*i*) $\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$ *(Hissar, 2005 S ; P.T.U., 2003)*

   (*ii*) $\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$ *(Kurukshetra, 2005)*
6. Find the values of a, b, c if $A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ is orthogonal ? *(Mumbai, 2005 S)*
7. Prove that $\begin{bmatrix} l & m & n & 0 \\ 0 & 0 & 0 & -1 \\ n & l & -m & 0 \\ -m & n & -l & 0 \end{bmatrix}$ is orthogonal when $l = 2/7, m = 3/7, n = 6/7$.
8. If A and B are orthogonal matrices, prove that AB is also orthogonal. *(Anna, 2005)*
9. Are the following vectors linearly dependent. If so, find the relation between them :

   (*i*) (2, 1, 1), (2, 0, – 1), (4, 2, 1). *(Mumbai, 2009)*

   (*ii*) (1, 1, 1, 3), (1, 2, 3, 4), (2, 3, 4, 9).

   (*iii*) $\mathbf{x}_1 = (1, 2, 4), \mathbf{x}_2 = (2, -1, 3), \mathbf{x}_3 = (0, 1, 2), \mathbf{x}_4 = (-3, 7, 2)$. *(U.P.T.U., 2003 ; Nagpur, 2001)*

2.13 (1) EIGEN VALUES

If A is any square matrix of order n, we can form the matrix $A - \lambda I$, where I is the nth order unit matrix. The determinant of this matrix equated to zero,

i.e.,
$$| A - \lambda I | = \begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix} = 0$$

is called the *characteristic equation of A*. On expanding the determinant, the characteristic equation takes the form

$$(-1)^n \lambda^n + k_1\lambda^{n-1} + k_2\lambda^{n-2} + \dots + k_n = 0,$$

where k's are expressible in terms of the elements a_{ij}. The roots of this equation are called the *eigenvalues* or *latent roots* or *characteristic roots* of the matrix A.

(2) Eigen vectors

If $X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$ and $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$, then the linear transformation $Y = AX$ ...(*i*)

carries the column vector X into the column vector Y by means of the square-matrix A. In practice, it is often required to find such vectors which transform into themselves or to a scalar multiple of themselves.

Let X be such a vector which transforms into λX by means of the transformation (*i*).

Then $\lambda X = AX$ or $AX - \lambda IX = 0$ or $[A - \lambda I]X = 0$ ...(*ii*)

This matrix equation represents n homogeneous linear equations

$$\left.\begin{array}{l} (a_{11}-\lambda)x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + (a_{22}-\lambda)x_2 + \dots + a_{2n}x_n = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + (a_{nn}-\lambda)x_n = 0 \end{array}\right\} \quad \text{...(iii)}$$

which will have a non-trivial solution only if the coefficient matrix is singular, *i.e.*, if $| A - \lambda I | = 0$.

This is called the characteristic equation of the transformation and is same as the characteristic equation of the matrix A. It has n roots and corresponding to each root, the equation (*ii*) [or (*iii*)] will have a non-zero solution.

$X = [x_1, x_2, \dots\dots, x_n]'$, which is known as the *eigen vector* or *latent vector*.

Obs. 1. Corresponding to n distinct eigen values, we get n independent eigen vectors. But when two or more eigen values are equal, it may or may not be possible to get linearly independent eigen vectors corresponding to the repeated roots.

Obs. 2. If X_i is a solution for a eigen value λ_i then it follows from (*ii*) that cX_i is also a solution, where c is arbitrary constant. Thus *the eigen vector corresponding to a eigen value is not unique* but may be any one of the vectors cX_i.

Example 2.42. *Find the eigen values and eigen vectors of the matrix* $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$. (*Bhopal, 2008*)

Solution. The characteristic equation is $|A - \lambda I| = 0$

i.e., $\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$ or $\lambda^2 - 7\lambda + 6 = 0$

or $(\lambda - 6)(\lambda - 1) = 0 \quad \therefore \lambda = 6, 1.$

Thus the eigen values are 6 and 1.

If x, y be the components of an eigen vector corresponding to the eigen value λ, then

$$[A - \lambda I]\, X = \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

Corresponding to $\lambda = 6$, we have $\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

which gives only one independent equation $-x + 4y = 0$

$\therefore \quad \dfrac{x}{4} = \dfrac{y}{1}$ giving the eigen vector (4, 1).

Corresponding to $\lambda = 1$, we have $\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

which gives only one independent equation $x + y = 0$.

$\therefore \quad \dfrac{x}{1} = \dfrac{y}{-1}$ giving the eigen vector $(1, -1)$.

Example 2.43. *Find the eigen values and eigen vectors of the matrix* $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

(*Bhopal, 2009 ; Raipur, 2005*)

Solution. The characteristic equation is $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix}$, *i.e.*, $\lambda^3 - 7\lambda^2 + 36 = 0$

Since $\lambda = -2$ satifies it, we can write this equation as

$$(\lambda + 2)(\lambda^2 - 9\lambda + 18) = 0 \quad \text{or} \quad (\lambda + 2)(\lambda - 3)(\lambda - 6) = 0.$$

Thus the eigen values of A are $\lambda = -2, 3, 6$.

If $x, y\ z$ be the components of an eigen vector corresponding to the eigen value λ, we have

$$[A - \lambda I]\, X = \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \qquad \text{...(}i\text{)}$$

Putting $\lambda = -2$, we have $3x + y + 3z = 0$, $x + 7y + z = 0$, $3x + y + 3z = 0$.

The first and third equations being the same, we have from the first two

$$\frac{x}{-20} = \frac{y}{0} = \frac{z}{20} \quad \text{or} \quad \frac{x}{-1} = \frac{y}{0} = \frac{z}{1}$$

Hence the eigen vector is $(-1, 0, 1)$. Also every non-zero multiple of this vector is an eigen vector corresponding to $\lambda = -2$.

Similarly, the eigen vectors corresponding to $\lambda = 3$ and $\lambda = 6$ are the arbitrary non-zero multiples of the vectors $(1, -1, 1)$ and $(1, 2, 1)$ which are obtained from (*i*).

Hence the three eigen vectors may be taken as $(-1, 0, 1)$, $(1, -1, 1)$, $(1, 2, 1)$.

Example 2.44. *Find the eigen values and eigen vectors of the matrix* $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ (*U.P.T.U., 2005*)

Solution. The characteristic equation is

$$[A - \lambda I] = 0, \quad i.e., \quad \begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix} = 0$$

or
$$(3 - \lambda)(2 - \lambda)(5 - \lambda) = 0$$

Thus the eigen values of A are 2, 3, 5.

If x, y, z be the components of an eigen vector corresponding to the eigen value λ, we have

$$[A - \lambda I]\, X = \begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Putting $\lambda = 2$, we have $x + y + 4z = 0$, $6z = 0$, $3z = 0$, *i.e.*, $x + y = 0$ and $z = 0$.

$$\therefore \quad \frac{x}{1} = \frac{y}{-1} = \frac{z}{0} = k_1 \text{ (say)}$$

Hence the eigen vector corresponding to $\lambda = 2$ is $k_1(1, -1, 0)$.

Putting $\lambda = 3$, we have $y + 4z = 0$, $-y + 6z = 0$, $2z = 0$, *i.e.*, $y = 0$, $z = 0$.

$$\therefore \quad \frac{x}{1} = \frac{y}{0} = \frac{z}{0} = k_2$$

Hence the eigen vector corresponding to $\lambda = 3$ is k_2 (1, 0, 0).

Similarly, the eigen vector corresponding to $\lambda = 5$ is k_3 (3, 2, 1).

2.14 PROPERTIES OF EIGEN VALUES

I. Any square matrix A and its transpose A′ have the same eigen values.

We have $$(A - \lambda I)' = A' - \lambda I' = A' - \lambda I$$

$$| (A - \lambda I)' | = | A' - \lambda I |$$

$$| A - \lambda I | = | A' - \lambda I | \qquad [\because | B' | = | B |$$

$\therefore$ $| A - \lambda I | = 0$ if and only if $| A' - \lambda I | = 0$

i.e., λ is an eigen value of A if and only if it is an eigen value of A'.

II. The eigen values of a triangular matrix are just the diagonal elements of the matrix.

Let $$A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{vmatrix}$$ be a triangular matrix of order n.

Then $| A - \lambda I | = (a_{11} - \lambda)(a_{22} - \lambda) \dots\dots (a_{nn} - \lambda)$.

$\therefore$ Roots of $| A - \lambda I | = 0$ are $\lambda = a_{11}, a_{22}, \dots, a_{nn}$.

Hence the eigen values of A are the diagonal elements of A, *i.e.*, $a_{11}, a_{22}, \dots, a_{nn}$.

Cor. *The eigen values of a diagonal matrix are just the diagonal elements of the matrix.*

III. The eigen values of an idempotent matrix are either zero or unity.

Let A be an idempotent matrix so that $A^2 = A$. If λ be an eigen value of A, then there exists a non-zero vector X such that

$$AX = \lambda X \qquad \dots(1)$$

$\therefore$ $A(AX) = A(\lambda X)$, *i.e.*, $A^2X = \lambda(AX)$

i.e. $$AX = \lambda(\lambda X) \qquad [\because A^2 = A \text{ and } AX = \lambda X$$

$\therefore$ $$AX = \lambda^2 X \qquad \dots(2)$$

From (1) and (2), we get $\lambda^2 X = \lambda X$ or $(\lambda^2 - \lambda) X = 0$

or $$\lambda^2 - \lambda = 0 \text{ whence } \lambda = 0 \text{ or } 1.$$

Hence the result.

IV. The sum of the eigen values of a matrix is the sum of the elements of the principal diagonal.

[This property will be proved for a matrix of order 3, but the method will be capable of easy extension to matrices of any order.]

Consider the square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \dots(i)$$

so that $$| A - \lambda I | = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} \qquad \text{(On expanding)}$$

$$= -\lambda^3 + \lambda^2(a_{11} + a_{22} + a_{33}) - \dots \qquad \dots(ii)$$

If $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A, then $| A - \lambda I | = (-1)^3 (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$

$$= -\lambda^3 + \lambda^2 (\lambda_1 + \lambda_2 + \lambda_3) - \dots \qquad \dots(iii)$$

Equating the right hand sides of (*ii*) and (*iii*) and comparing coefficients of λ^2, we get

$$\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}.$$ Hence the result.

V. The product of the eigen values of a matrix A is equal to its determinant.

Putting $\lambda = 0$ in (*iii*), we get the result.

VI. If λ is an eigen value of a matrix A, then $1/\lambda$ is the eigen value of A^{-1}.

If X be the eigen vector corresponding to λ, then $AX = \lambda X$ $\qquad \dots(i)$

Premultiplying both sides by A^{-1}, we get $A^{-1} AX = A^{-1}\lambda X$

i.e., $\quad IX = \lambda A^{-1} X \quad$ or $\quad X = \lambda(A^{-1} X), \quad$ *i.e.*, $\quad A^{-1}X = (1/\lambda) X$

This being of the same form as (*i*), shows that $1/\lambda$ is an eigen value of the inverse matrix A^{-1}.

VII. If λ is an eigen value of an orthogonal matrix, then $1/\lambda$ is also its eigen value.

We know that if λ is an eigen value of a matrix A, then $1/\lambda$ is an eigen value of A^{-1}. [Property *V*]. Since A is an orthogonal matrix, A^{-1} is same as its transpose A'.

$\therefore$ $1/\lambda$ is an eigen value of A'.

But the matrices A and A' have the same eigen values, since the determinants $| A - \lambda I |$ and $| A' - \lambda I |$ are the same.

Hence $1/\lambda$ is also an eigen value of A

VIII. If $\lambda_1, \lambda_2 ..., \lambda_n$ are the eigen values of a matrix A, then A^m has the eigen values $\lambda_1^m, \lambda_2^m, ..., \lambda_n^m$ (m being a positive integer). (*Mumbai, 2006*)

Let A_i be the eigen value of A and X_i the corresponding eigen vector. Then

$$AX_i = \lambda_i X_i \qquad ...(i)$$

We have $\quad A^2X_i = A(AX_i) = A(\lambda_i X_i) = \lambda_i(AX_i) = \lambda(\lambda_i X_i) = \lambda_i^2 X_i$

Similarly, $\quad A^3 X_i = \lambda_i^3 X_i$. In general, $A^m X_i = \lambda_i^m X_i$ which is of the same form as (*i*).

Hence λ_i^m is an eigen value of A^m.

The corresponding eigen vector is the same X_i.

2.15 CAYLEY-HAMILTON THEOREM*

Every square matrix satisfies its own characteristic equation ; *i.e.*, if the characteristic equation for the nth order square matrix A is

$$| A - \lambda I | = (-1)^n \lambda^n + k_1\lambda^{n-1} + ... + k_n = 0 \qquad ...(i)$$

then
$$(-1)^n A^n + k_1 A^{n-1} + ... + k_n = 0.$$

Let the adjoint of the matrix $A - \lambda I$ be P. Clearly, the elements of P will be polynomials of the $(n-1)$th degree in λ, for the cofactors of the elements in $| A - \lambda I |$ will be such polynomials.

$\therefore$ P can be split up into a number of matrices, containing terms with the same powers of λ, such that

$$P = P_1\lambda^{n-1} + P_2\lambda^{n-2} + ... + P_{n-1}\lambda + P_n \qquad ...(ii)$$

where $P_1, P_2, ..., P_n$ are all the square matrices of order n whose elements are functions of the elements of A.

Since the product of a matrix by its adjoint = determinant of the matrix × unit matrix.

$\therefore$ $\quad [A - \lambda I]P = | A - \lambda I | \times I$

$\therefore$ by (*i*) and (*ii*),
$$[A - \lambda I]\,[P_1 \lambda^{n-1} + P_2 \lambda^{n-2} + ... + P_{n-1}\lambda + P_n]$$
$$= [(-1)^n \lambda^n + k_1 \lambda^{n-1} + ... + k_{n-1}\lambda + k_n]\, I.$$

Equating the coefficients of various powers of λ, we get

$$-P_1 = (-1)^n I \qquad [\because \; IP_1 = P_1]$$
$$AP_1 - P_2 = k_1 I,$$
$$AP_2 - P_3 = k_2 I,$$
$$.........................$$
$$AP_{n-1} - P_n = k_{n-1} I,$$
$$AP_n = k_n I.$$

Now pre-multiplying the equations by $A^n, A^{n-1}, ..., A, I$ respectively and adding, we get

$$(-1)^n A^n + k_1 A^{n-1} + ... + k_{n-1} A + k_n I = 0, \qquad ...(iii)$$

for the terms on the left cancel in pairs. This proves the theorem.

Cor. Another method of finding the inverse.

Multiplying (*iii*) by A^{-1}, we get

$$(-1)^n A^{n-1} + k_1 A^{n-2} + ... + k_{n-1} I + k_n A^{-1} = 0$$

whence
$$A^{-1} = -\frac{1}{k_n}[(-1)^n A^{n-1} + k_1 A^{n-2} + ... + k_{n-1} I].$$

*See footnote on p.17. *William Rowan Hamilton* (1805–1865) an Irish mathematician who is known for his work in dynamics.

This result gives the inverse of A in terms of $\overline{n-1}$ *powers of A and is considered as a practical method for the computation of the inverse of the large matrices.* As a by-product of the computation, the characteristic equation and the determinant of the matrix are also obtained.

Example 2.45. *Verify Cayley-Hamilton theorem for the matrix* $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ *and find its inverse.*

Also express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ *as a linear polynomial in A.* (*Bhopal, 2009*)

Solution. The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^2 - 4\lambda - 5 = 0 \qquad ...(i)$$

By Cayley-Hamilton theorem, A must satisfy its characteristic equation (i), so that

$$A^2 - 4A - 5I = 0 \qquad ...(ii)$$

Now

$$A^2 - 4A - 5I = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

This verifies the theorem.

Multiplying (ii) by A^{-1}, we get $A - 4I - 5A^{-1} = 0$

or

$$A^{-1} = \frac{1}{5}(A - 4I) = \frac{1}{5}\left\{\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$$

Now dividing the polynomial $\lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10I$ by the polynomial $\lambda^2 - 4\lambda - 5$, we obtain

$$\lambda^5 - 4\lambda^4 - 7\lambda^3 - \lambda - 10I = (\lambda^2 - 4\lambda - 5)(\lambda^3 - 2\lambda + 3) + \lambda + 5$$

$$= \lambda + 5 \qquad [\text{By } (i)]$$

Hence $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = A + 5$, which is a linear polynomial in A.

Example 2.46. *Find the characteristic equation of the matrix* $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ *and hence find its inverse.*

Solution. The characteristic equation is $\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 3-\lambda & -3 \\ -2 & -4 & -4-\lambda \end{vmatrix} = 0$, *i.e.*, $\lambda^3 - 20\lambda + 8 = 0$.

By Cayley-Hamilton theorem, $A^3 - 20A + 8I = 0$, whence $A^{-1} = \frac{5}{2}I - \frac{1}{8}A^2$,

$$= \frac{5}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{8}\begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix} \qquad [\text{cf. Ex. 2.21}$$

Example 2.47. *Find the characteristic equation of the matrix,* $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ *and hence compute* A^{-1}.

(*U.T.U., 2010*)

Also find the matrix represented by

$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I.$ (*Anna, 2009 ; Rajasthan, 2005 ; U.P.T.U., 2003*)

Solution. The characteristic equation of the matrix A is

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0 \quad \text{or} \quad [\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0]$$

According to Cayley-Hamilton theorem, we have $A^3 - 5A^2 + 7A - 3I = 0$ ...(*i*)

Multiplying (*i*) by A^{-1}, we get

$$A^2 - 5A + 7I - 3A^{-1} = 0 \quad \text{or} \quad A^{-1} = \frac{1}{3}[A^2 - 5A + 7I] \qquad ...(ii)$$

But
$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4+0+1 & 2+1+1 & 2+0+2 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 2+0+2 & 1+1+2 & 1+0+4 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$\therefore \quad A^2 - 5A + 7I = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - 5\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

Hence from (*ii*), $A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$

Now
$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$$
$$= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) + A^2 + A + I$$
$$= A^2 + A + I \qquad [\because \; A^3 - 5A^2 + 7A - 3I = 0]$$
$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}.$$

PROBLEMS 2.9

1. Find the sum and product of the eigen values of $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$. (*Madras, 2006*)

2. Find the eigen values and eigen vectors of the matrices :

   (*a*) $\begin{bmatrix} 4 & 3 \\ 2 & 9 \end{bmatrix}$ (*W.B.T.U., 2005*) (*b*) $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$ (*Bhopal, 2002 S*)

3. Find the latent roots and the latent vectors of the matrices :

   (*a*) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ (*Bhopal, 2008 ; Nagarjuna, 2008 ; S.V.T.U., 2008 ; J.N.T.U., 2006*)

   (*b*) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ (*J.N.T.U., 2005 ; Kurukshetra, 2005*)

   (*c*) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ (*Mumbai, 2006 ; B.P.T.U., 2006 ; U.P.T.U., 2006*)

   (*d*) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ (*Kurukshetra, 2006*) (*e*) $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$ (*Madras, 2006*)

4. If λ be an eigen value of a non-singular matrix A, show that $|A|/\lambda$ is an eigen value of the matrix *adj A*. (*U.P.T.U., 2001*)

5. Find the eigen values of adj A and of $A^2 - 2A + I$, where $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ (*Mumbai, 2006*)

6. Two eigen values of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are = 1 each. Find the eigen values of A^{-1}.

7. Show that if $\lambda_1, \lambda_2, ..., \lambda_n$ are the latent roots of a matrix A, then A^2 has the latent roots $\lambda_1^2, \lambda_2^2, ... \lambda_n^2$. (*P.T.U., 2005*)

8. For a symmetrical square matrix, show that the eigen vectors corresponding to two unequal eigen values are orthogonal.
9. Using Cayley-Hamilton theorem, find the inverse of

(i) $\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ (*Osmania, 2000 S*)

(iii) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ 2 & -4 & -4 \end{bmatrix}$ (*Bhopal, 2002 S*) (iv) $\begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (*U.P.T.U., 2006*)

10. Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$. Show that the equation is satisfied by A and hence obtain the inverse of the given matrix. (*Bhopal, 2008 ; Anna, 2005 ; Kerala, 2005*)
11. Verify Cayley-Hamilton theorem for the matrix A and find its inverse.

(i) $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ (*Anna, 2009 ; S.V.T.U., 2008 ; Madras, 2006*)

(ii) $\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$ (*Coimbatore, 2001*) (iii) $\begin{bmatrix} 3 & 2 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix}$ (*P.T.U., 2006*)

12. Using Cayley-Hamilton theorem, find A^8, if $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$. (*Anna, 2003*)
13. If $A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, find A^4. (*Madras, 2006*)
14. Using Cayley-Hamilton theorem, find A^{-2}, where $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. (*Bhopal, 2008*)
15. If $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$, evaluate A^{-1}, A^{-2} and A^{-3}.
16. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, show that $A^n = A^{n-2} + A^2 - 1$. Hence find A^{50}. (*Mumbai, 2006*)

2.16 (1) REDUCTION TO DIAGONAL FORM

If a square matrix A of order n has n linearly independent eigen vectors, then a matrix P can be found such that $P^{-1}AP$ is a diagonal matrix.

[This result will be proved for a square matrix of order 3 but the method will be capable of easy extension to matrices of any order.]

Let A be a square matrix of order 3. Let $\lambda_1, \lambda_2, \lambda_3$ be its eigen values and

$$X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \text{ and } X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} \text{ be the corresponding eigen vectors.}$$

Denoting the square matrix $[X_1 X_2 X_3] = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$ by P, we have

$$AP = A[X_1 X_2 X_3] = [AX_1, AX_2, AX_3] = [\lambda_1 X_1, \lambda_2 X_2, \lambda_3 X_3]$$

$$= \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \\ \lambda_1 y_1 & \lambda_2 y_2 & \lambda_3 y_3 \\ \lambda_1 z_1 & \lambda_2 z_2 & \lambda_3 z_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \times \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = PD, \text{ where } D \text{ is the diagonal matrix.}$$

$\therefore$ $P^{-1}AP = P^{-1}PD = D$, which proves the theorem.

Obs. 1. *The matrix P which diagonalises A is called the* **modal matrix** *of A and the resulting diagonal matrix D is known as the* **spectral matrix** *of A.*

2. *The diagonal matrix has the eigen values of A as its diagonal elements.*

3. *The matrix P, which diagonalise A, constitutes the eign vectors of A.*

(2) Similarity of matrices. *A square matrix $\hat{A}$ of order n is called* **similar** *to a square matix A of order n if*

$$\hat{A} = P^{-1}AP \text{ for some non-singular } n \times n \text{ matrix } P.$$

This transformation of a matrix A by a non-singular matrix P to $\hat{A}$ is called a **similarity transformation.**

Obs. *If the matrix $\hat{A}$ is similar to the matrix A, then $\hat{A}$ has the same eigen values as A.*

If **x** *is an eigen vector of A, then $y = P^{-1}$* **x** *is an eigen vector of $\hat{A}$ corresponding to the same eigen value.*

(3) Powers of a matrix. *Diagonalisation of a matrix is quite useful for obtaining powers of a matrix.*

Let A be the square matrix. Then a non-singular matrix P can be found such that

$$D = P^{-1}AP$$

$\therefore$ $D^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}A^2P$ $[\because PP^{-1} = I]$

Similarly, $D^3 = P^{-1}A^3P$ and in general, $D^n = P^{-1}A^nP$ ...(*i*)

To obtain A^n, premultiply (*i*) by P and post-multiply by P^{-1}.

Then $PD^nP^{-1} = PP^{-1}A^nPP^{-1} = A^n$ which gives A^n.

Thus, $$A^n = PD^nP^{-1} \text{ where, } D^n = \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix}$$

Working procedure :

1. Find the eigen values of the square matrix A.
2. Find the corresponding eigen vectors and write the modal matrix P.
3. Find the diagonal matrix D from $D = P^{-1}AP$
4. Obtain A^n from $A^n = PD^nP^{-1}$.

Example 2.48. *Reduce the matrix* $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ *to the diagonal form.*

(V.T.U., 2011 ; U.T.U., 2010 ; Bhopal, 2009 ; U.P.T.U., 2006)

Solution. The characteristic equation of A is

$$\begin{bmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{bmatrix} = 0 \quad \text{or} \quad \lambda^3 - \lambda^2 - 5\lambda + 5 = 0.$$

Solving, we get $\lambda_1 = 1, \lambda_2 = \sqrt{5}, \lambda_3 = -\sqrt{5}$ as the eigen values of A.

When $\lambda = 1$, the corresponding eigen vector is given by

$$-2x + 2y - 2z = 0, x + y + z = 0, -x - y - z = 0$$

Solving the first two equations, we get $\frac{x}{2} = \frac{y}{0} = \frac{z}{-2}$ giving the eigen vector $(1, 0, -1)$

When $\lambda = \sqrt{5}$, the corresponding eigen vector is given by

$$(-1-\sqrt{5})x + 2y - 2z = 0, x + (2-\sqrt{5})y + z = 0, -x - y - \sqrt{5}z = 0.$$

Solving 2nd and 3rd equations, we get

$$\frac{x}{6-2\sqrt{5}}=\frac{y}{-1+\sqrt{5}}=\frac{z}{1-\sqrt{5}} \quad \text{or} \quad \frac{x}{\sqrt{5}-1}=\frac{y}{1}=\frac{z}{-1}$$

giving the eigen vector ($\sqrt{5}-1, 1, -1$).

Similarly the eigen vector corresponding to $\lambda = -\sqrt{5}$, is ($\sqrt{5}+1, -1, 1$).

Writing the three eigen vectors as the three columns, we get the transformation (*modal*) matrix as

$$P=\begin{bmatrix}1 & \sqrt{5}-1 & \sqrt{5}+1\\ 0 & 1 & -1\\ 1 & -1 & 1\end{bmatrix}.$$

Hence the diagonal matrix is

$$D=\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & \sqrt{5} & 0\\ 0 & 0 & -\sqrt{5}\end{bmatrix}.$$

Example 2.49. *Find the matrix P which transforms the matrix* $A=\begin{bmatrix}1 & 1 & 3\\ 1 & 5 & 1\\ 3 & 1 & 1\end{bmatrix}$ *to the diagonal form. Hence calculate* A^4.

Solution. The eigen values of A (found in Ex. 2.43) are $-2, 3, 6$ and the eigen vectors are $(-1, 0, 1)$, $(1, -1, 1)$, $(1, 2, 1)$. Writing these eigen vectors as the three columns, the required transformation matrix (*modal matrix*) is

$$P=\begin{bmatrix}-1 & 1 & 1\\ 0 & -1 & 2\\ 1 & 1 & 1\end{bmatrix}$$

To find P^{-1}, $\quad |P|=\begin{vmatrix}-1 & 1 & 1\\ 0 & -1 & 2\\ 1 & 1 & 1\end{vmatrix}=\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}$ (say)

$$A_1=-3, B_1=2, C_1=1, A_2=0, B_2=-2, C_2=2, A_3=3, B_3=2, C_3=1$$

Also $\quad |P| = a_1A_1 + b_1B_1 + c_1C_1 = 6$

$$\therefore \quad P^{-1}=\frac{1}{|P|}\begin{bmatrix}A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{bmatrix}=\frac{1}{6}\begin{bmatrix}-3 & 0 & 3\\ 2 & -2 & 2\\ 1 & 2 & 1\end{bmatrix}$$

Thus

$$D=P^{-1}AP=\begin{bmatrix}-2 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 6\end{bmatrix}$$

$$\therefore \quad D^4=\begin{bmatrix}(-2)^4 & 0 & 0\\ 0 & 3^4 & 0\\ 0 & 0 & 6^4\end{bmatrix}=\begin{bmatrix}16 & 0 & 0\\ 0 & 81 & 0\\ 0 & 0 & 1296\end{bmatrix}$$

Hence

$$A^4=PD^4P^{-1}=\frac{1}{6}\begin{bmatrix}-1 & 1 & 1\\ 0 & -1 & 2\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}16 & 0 & 0\\ 0 & 81 & 0\\ 0 & 0 & 1296\end{bmatrix}\begin{bmatrix}-3 & 0 & 3\\ 2 & -2 & 2\\ 1 & 2 & 1\end{bmatrix}$$

$$=\begin{bmatrix}-1 & 1 & 1\\ 0 & -1 & 2\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}-8 & 0 & 8\\ 27 & -27 & 27\\ 216 & 512 & 216\end{bmatrix}=\begin{bmatrix}251 & 485 & 235\\ 485 & 1051 & 485\\ 235 & 485 & 251\end{bmatrix}$$

Example 2.50. *Find* e^A *and* 4^A *if* $A = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$. (*Mumbai, 2006*)

Solution. The characteristic equation of A is

$$\begin{vmatrix} 3/2-\lambda & 1/2 \\ 1/2 & 3/2-\lambda \end{vmatrix} = 0, \quad i.e., (3/2-\lambda)^2 - 1/4 = 0.$$

$$\therefore \qquad \lambda^2 - 3\lambda + 2 = 0 \quad \text{whence } \lambda = 1, 2.$$

When $\lambda = 1$, $[A - \lambda I]\, X = 0$, gives

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad [\text{By } 2R_1, 2R_2]$$

or
$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad [\text{By } R_2 - R_1]$$

$\therefore$ $x_1 + x_2 = 0$. If $x_2 = -1, x_1 = 1$, *i.e.*, the eigen vector is $[1, -1]'$.

When $\lambda = 2$, $[A - \lambda I]\, X = 0$, gives $\begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

or
$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} \text{By } 2R_1 \\ 2R_2 \end{bmatrix}$$

or
$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad [\text{By } R_2 - R_1]$$

$$\therefore \qquad -x_1 + x_2 = 0, \quad i.e., \quad x_1 = x_2$$

If $x_2 = 1, x_1 = 1$, *i.e.*, the eigen vector is $[1, 1]'$

Now $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$$\therefore \qquad P^{-1} = \frac{adj\, P}{|P|} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

If
$$f(A) = e^A, f(D) = e^D = \begin{bmatrix} e^1 & 0 \\ 0 & e^2 \end{bmatrix}$$

$$\therefore \qquad e^A = P\, f(D)\, P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e^1 & 0 \\ 0 & e^2 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} e & e^2 \\ -e & e^2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e+e^2 & -e+e^2 \\ -e+e^2 & e+e^2 \end{bmatrix}$$

Replacing e by 4, we get

$$4^A = \frac{1}{2}\begin{bmatrix} 20 & 12 \\ 12 & 20 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}.$$

2.17 REDUCTION OF QUADRATIC FORM TO CANONICAL FORM

A homogeneous expression of the second degree in any number of variables is called a *quadratic form.*

For instance, if $\quad A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $X' = [x \quad y \quad z]$, then

$$X'AX = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy \qquad ...(i)$$

which is a *quadratic form.*

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of the matrix A and

$$X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}, X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$

be its corresponding eigen vectors in the normalized form (*i.e.*, each element is divided by square root of sum of the squares of all the three elements in the eigen vector).

Then by § 2.16(1), $P^{-1} AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ where $P = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$

Hence the *quadratic form (i) is reduced to a* **canonical form (or sum of squares form** or **Principal axes form).**

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2$$

and P is the **matrix of transformation** which is an orthogonal matrix.

Note. Congruent (or orthogonal) transformation. The diagonal matrix D and the matrix A are called *congruent matrices* and *the above method of reduction is called* **congruent (or orthogonal) transformation.**

Remember that the matrix A corresponding to the quadratic form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

is $$\begin{bmatrix} \text{coeff. of } x^2 & \frac{1}{2}\text{coeff. of } yz & \frac{1}{2}\text{coeff. of } zx \\ \frac{1}{2}\text{coeff. of } yz & \text{coeff. of } y^2 & \frac{1}{2}\text{coeff. of } xy \\ \frac{1}{2}\text{coeff. of } zx & \frac{1}{2}\text{coeff. of } xy & \text{coeff. of } z^2 \end{bmatrix}, \quad i.e., \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}.$$

Example 2.51. *Reduce the quadratic form $3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy$ to the canonical form and specify the matrix of transformation.* (*Bhopal, 2009; Kurukshetra, 2006*)

Solution. The matrix of the given quadratic form is $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Its characteristic equation is $| A - \lambda I | = 0$, *i.e.*, $\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$

which gives $\lambda = 2, 3, 6$ as its eigen values. Hence the given quadratic form reduces to the canonical form

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2, \qquad i.e., \qquad 2x^2 + 3y^2 + 6z^2.$$

To find the matrix of transformation

From $[A - \lambda I]\, X = 0$, we obtain the equations

$$(3 - \lambda)\, x - y + z = 0\,;\, - x + (5 - \lambda)\, y - z = 0\,;\, x - y + (3 - \lambda)\, z = 0.$$

Now corresponding to $\lambda = 2$, we get $x - y + z = 0, - x + 3y - z = 0$, and $x - y + z = 0$,

whence $$\frac{x}{1} = \frac{y}{0} = \frac{z}{-1}$$

$\therefore$ The eigen vector is X_1 $(1, 0, - 1)$ and its normalised form is $(1/\sqrt{2}, 0, - 1/\sqrt{2})$.

Similarly, corresponding to $\lambda = 3$, the eigen vector is X_2 $(1, 1, 1)$ and its normalised form is $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$.

Finally, corresponding to $\lambda = 6$, the eigen vector is X_3 $(1, - 2, 1)$ and its normalised form is $(1/\sqrt{6}, - 2/\sqrt{6}, 1/\sqrt{6})$.

Hence the matrix of transformation is $P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$.

2.18 NATURE OF A QUADRATIC FORM

Let $Q = X'AX$ be a quadratic form in n variables $x_1, x_2, \ldots x_n$.

Index. *The number of positive terms in its canonical form is called the index of the quadratic form.*

Signature (S) *of the quadratic form is the difference of positive and negative terms in the canonical form. If the rank of the matrix A is r and the signature of the quadratic form Q is s, then the quadratic form is said to be*

(*i*) *positive definite* if $r = n$ and $s = n$

(*ii*) *negative definite* if $r = n$ and $s = 0$

(*iii*) *positive semidefinite* if $r < n$ and $s = r$

(*iv*) *negative semidefinite* if $r < n$ and $s = 0$

(*v*) *indefinite* in all other cases.

In other words *a real quadratic form X'AX in a variable is said to be*

(*i*) **positive definite** *if all the eigen values of* $A > 0$.

(*ii*) **negative definite** *if all the eigen values of* $A < 0$.

(*iii*) **positive semidefinite** *if all the eigen values of* $A \geq 0$ *and at least one eigen value* $= 0$.

(*iv*) **negative semidefinite** *if all the eigen values of* $A \leq 0$ *and at least one eigen value* $= 0$.

(*v*) **indefinite** *if some of the eigen values of A are positive and others negative.*

Example 2.52. *Reduce the quadratic form* $2x_1x_2 + 2x_1x_3 - 2x_2x_3$ *to a canonical form by an orthogonal reduction and discuss its nature.* (*Madras, 2006*)

Also find the modal matrix.

Solution. (*i*) The matrix of the given quadratic form is $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$

Its characteristic equation is $[A - \lambda I] = 0$, *i.e.*, $\begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & -1 \\ 1 & -1 & -\lambda \end{bmatrix} = 0$

which gives $\lambda^3 - 3\lambda + 2 = 0$

Solving, we get $\lambda = 1, 1, -2$ as the eigen values. Hence the given quadratic form reduces to the canonical form

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = 0, \quad i.e., \quad x^2 + y^2 - 2z^2 = 0$$

(*ii*) Since some of the eigen values of A are positive and others are negative, the given quadratic form is *Indefinite*.

(*iii*) *To find the matrix of transformation*

From $[A - \lambda I]\, X = 0$, we get the equations

$$-\lambda x + y + z = 0, x - \lambda y + z = 0, x - y - \lambda z = 0$$

When $\lambda = -2$, we get $\quad 2x + y + z = 0, x + 2y - z = 0, x - y + 2z = 0.$

Solving first and second equations, we get

$$\frac{x}{-1} = \frac{y}{1} = \frac{z}{1}$$

$\therefore$ The corresponding eigen vector $X_1 = (-1, 1, 1)$ and its normalised form is $(-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$

When $\lambda = 1$, we get $\quad -x + y + z = 0, x - y - z = 0, x - y - z = 0.$

These equations are same. Take $y = 0$ so that $x = z$.

$\therefore$ The corresponding eigen vector $X_2 = (1, 0, 1)$ and its normalised form is $(1/\sqrt{2}, 0, 1/\sqrt{2})$

To find the eigen vector $X_3 = (l, m, n)$ (say)

Since X_3 is orthogonal to X_1, $\quad \therefore \quad -l + m + n = 0$

Since X_3 is orthogonal to X_2, $\quad \therefore \quad l + n = 0$

These equations give $\dfrac{l}{1} = \dfrac{m}{2} = \dfrac{n}{-1}$.

$\therefore$ The eigen vector $X_3 = (1, 2, -1)$ and normalised form is $(1/\sqrt{6}, 2/\sqrt{6}, -1/\sqrt{6})$.

Hence the modal matrix is

$$P = \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}.$$

PROBLEMS 2.10

1. If $A = \begin{bmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $P = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, show that $P^{-1}AP$ is a diagonal matrix.
2. Show that the linear transformation

   $H = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, where $\theta = \frac{1}{2}\tan^{-1}\frac{2h}{a-b}$, changes the matrix

   $C = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$ to the diagonal form $D = HCH'$.
3. Reduce the matrix $A = \begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix}$ to the diagonal form. *(B.P.T.U., 2005)*
4. If $A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$, find A^n and A^4. *(Mumbai, 2006)*
5. If $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$, calculate A^4. *(Coimbatore, 2001)*
6. If $A = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$, then prove that $3\tan A = A\tan 3$. *(Mumbai, 2006)*
7. Find the eigen vectors of the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ and hence reduce $6x^2 + 3y^2 + 3z^2 - 2yz + 4zx - 4xy$ to a 'sum of squares'. Also write the nature of the matrix. *(Calicut, 2005)*
8. Reduce the quadratic form $2xy + 2yz + 2zx$ into canonical form. *(Anna, 2009 ; Kurukshetra, 2006 ; Mumbai, 2003)*
9. (*a*) Find the eigen values, eigen vectors and the modal of matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$.

   (*b*) Reduce the quadratic form $x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$ to a canonical form. *(Anna, 2009)*
10. Reduce the following quadratic forms into a 'sum of squares' by an orthogonal transformation and give the matrix of transformation. Also state the nature of each of these.

    (*i*) $3x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3$.

    (*ii*) $8x^2 + 7y^2 + 3z^2 - 12xy - 8yz + 4zx$ *(Anna, 2002 S)*
11. Find the index and signature of the quadratic form $x_1^2 + 2x_2^2 - 3x_3^2$. *(Madras, 2006)*
12. Find the nature of the quadratic form $x^2 + 5y^2 + z^2 + 2xy + 2yz + 6zx$. *(Bhopal, 2009)*
13. Show that the form $5x_1^2 + 26x_2^2 + 10x_3^2 + 4x_2x_3 + 14x_3x_1 + 6x_1x_2$ is a positive semi-definite and find a non-zero set of values of x_1, x_2, x_3 which make the form zero. *(P.T.U., 2003)*

2.19 COMPLEX MATRICES

So far, we have considered matrices whose elements were real numbers. The elements of a matrix can, however, be complex numbers also.

(1) Conjugate of a matrix. *If the elements of a matrix* $A = [a_{rs}]$ *are complex numbers* $\alpha_{rs} + i\,\beta_{rs}$, α_{rs} *and* β_{rs} *being real, then the matrix*

$\overline{A} = [\overline{a}_{rs}] = [\alpha_{rs} - i\beta_{rs}]$ *is called the* **conjugate matrix** *of* A.

The transpose of a conjugate of a matrix A is denoted by A^* or A^θ, *i.e.*, $(\overline{A})' = A^*$.

(2) Hermitian matrix. *A square matrix A such that* $A' = \overline{A}$ *is said to be a* **Hermitian matrix***. The elements of the leading diagonal of a Hermitian matrix are evidently real, while every other element is the complex conjugate of the element in the transposed position. For instance $A = \begin{bmatrix} 2 & 3+4i \\ 3-4i & -5 \end{bmatrix}$ is a Hermitian matrix, since $A' = \begin{bmatrix} 2 & 3-4i \\ 3+4i & -5 \end{bmatrix} = \overline{A}$

(3) Skew-Hermitian matrix. *A square matrix A such that* $A' = -\overline{A}$ *is said to be a* **skew-Hermitian matrix.** This implies that the leading diagonal elements of a skew-Hermitian matrix are either all zeros or all purely imaginary.

Obs. *A Hermitian matrix is a generalisation of a real symmetric matrix* as every real symmetric matrix is Hermitian. Similarly, *a skew-Hermitian matrix is a generalisation of a real skew-symmetric matrix.*

Properties

I. Any square matrix A can be written as the sum of a Hermitian and skew-Hermitian matrices. *(Mumbai, 2007)*

Take $$B = \frac{1}{2}(A + \overline{A}') \text{ and } C = \frac{1}{2}(A - \overline{A}')$$

Then $$B' = \frac{1}{2}(A + \overline{A}')' = \frac{1}{2}(A' + \overline{A})$$

and $$\overline{B} = \frac{1}{2}\overline{(A + \overline{A}')} = \frac{1}{2}(\overline{A} + A') = B'$$

i.e., B is a Hermitian matrix.

Again, $$C' = \frac{1}{2}(A - \overline{A}')' = \frac{1}{2}(A' - \overline{A})$$

and $$\overline{C} = \frac{1}{2}\overline{(A - \overline{A}')} = \frac{1}{2}(\overline{A} - A') = -C'$$

$\therefore$ $C' = -C$, *i.e.*, C is a skew-Hermitian matrix.

Thus, $$A = \frac{1}{2}(A + \overline{A}') + \frac{1}{2}(A - \overline{A}') = B + C$$

Hence the result.

II. If A is a Hermitian matrix, then (iA) is a skew-Hermitian matrix. *(Mumbai, 2007)*

We have $$(\overline{iA})' = (\overline{i}\,\overline{A})' = (-i\overline{A})' = -i\overline{A}'$$
$$= -iA \qquad [\because \overline{A}' = A]$$

Thus (iA) is a skew-Hermitian matrix.

Similarly *if A is a skew-Hermitian matrix then (iA) is a Hermitian matrix.*

III. The eigen values of a Hermitian matrix are real. (see Fig. 2.1)

Let λ be the eigen value and X the corresponding eigen vector of a Hermitian matrix A, so that
$$AX = \lambda X$$
$$\overline{X}'AX = \overline{X}'\lambda X = \lambda\overline{X}'X \quad \text{or} \quad \lambda = \overline{X}'AX / \overline{X}'X$$

Since $\overline{X}'X = \overline{x}_1x_1 + \overline{x}_2x_2 + \dots + \overline{x}_nx_n = |x_1|^2 + |x_2|^2 + \dots + |x_n|^2$ is real and non-zero. Also $\overline{X}'AX$ is a Hermitian form which is always real.

$\therefore$ λ, the eigen value of a Hermitian matrix is real.

IV. The eigen values of a skew-Hermitian matrix are purely imaginary or zero.

* Named after the French mathematician *Charles Hermite* (1822–1901), known for his contributions to algebra and number theory.

Let λ be the eigen value and X the corresponding eigen vector of a skew-Hermitian matrix B so that $BX = \lambda X$.

$$\therefore \quad \overline{X}'BX = \overline{X}'\lambda X = \lambda\overline{X}'X \quad \text{or} \quad \lambda = \overline{X}'BX / \overline{X}'X$$

Since $\overline{X}'X$ is real and non-zero. Also $\overline{X}BX$ is a skew-Hermitian form which is purely imaginary or zero.

$\therefore$ λ, the eigen value of a skew-Hermitian matrix is purely imaginary or zero.

4. **Unitary matrix.** A square matrix U such that $\overline{U'} = U^{-1}$ is called a **unitary matrix.** For a unitary matrix, U, $U \cdot U^* = U^* \cdot U = I$.

This is a generalisation of the orthogonal matrix in the complex field.

Properties

I. Inverse of a unitary matrix is unitary

If U is a unitary matrix, then

$$\overline{U'} = U^{-1}$$

or
$$U' = \overline{U^{-1}}$$

$$\therefore \quad [(U^{-1})^{-1}]' = \overline{U^{-1}}$$

Writing $U^{-1} = V$, we have

$$[V^{-1}]' = \overline{V} \quad \text{or} \quad V^{-1} = \overline{V}'$$

Thus $V\ (= U^{-1})$ is also unitary.

Cor. *Inverse of an orthogonal matrix is orthogonal.*

II. Transpose of a unitary matrix is unitary

If U is a unitary matrix, $\quad \overline{U}' = U^{-1}$

or
$$(\overline{U}') = U^{-1}$$

or
$$[(\overline{U}')]' = [U^{-1}]' = [U']^{-1}$$

Writing $U' = V$, we have $\overline{V}' = V^{-1}$

Thus V (*i.e.*, U') is also unitary.

Cor. *Transpose of an orthogonal matrix is orthogonal.*

III. Product of two unitary matrices is a unitary matrix.

If U and V are unitary matrices then

$$U' = \overline{U}^{-1},\ V' = \overline{V}^{-1}$$

Now,
$$(\overline{UV})^{-1} = (\overline{U}\overline{V})^{-1} = \overline{V}^{-1}\overline{U}^{-1}$$
$$= V'U'$$
$$= (UV)' \qquad [\because \ U, V \text{ are unitary.}]$$

Thus, UV is a unitary matrix.

Cor. *Product of two orthogonal matrices is an orthogonal matrix.*

IV. The eigen value of a unitary matrix has absolute value 1. *(U.T.U., 2010)*

If U is a unitary matrix then $\quad UX = \lambda X$ ...(1)

Taking conjugate transpose of (1),

$$(\overline{UX})' = (\overline{U}\,\overline{X})' = \overline{X}'\overline{U}' = \overline{X}'\overline{U}^{-1}$$

Also
$$(\overline{UX})' = (\overline{\lambda}\overline{X})' = \overline{\lambda}\overline{X}'$$

i.e.,
$$\overline{X}'U^{-1} = \overline{\lambda}\overline{X}' \qquad ...(2)$$

Post-multiplying (2) by (1), we get

$$(\overline{X}'U^{-1})(UX) = (\overline{\lambda}\overline{X}') = (\lambda X)$$
$$\overline{X}'(U^{-1}U)X = (\overline{\lambda}\lambda)(\overline{X}'X) \qquad [\because \ U^{-1}U = I]$$
$$\overline{X}'X = (\lambda\lambda')\,\overline{X}'X$$

Thus
$$\lambda\lambda' = |\lambda|^2 = 1. \qquad [\because \ \overline{X}X \neq 0]$$

Hence the result.

Cor. *The eigen value of an orthogonal matrix has absolute value 1.*

Example 2.53. *If* $A = \begin{bmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix}$, *show that* AA^* *is a Hermitian matrix, where* A^* *is the conjugate transpose of A..* *(J.N.T.U., 2005 ; U.P.T.U., 2003)*

Solution. We have $A' = \begin{bmatrix} 2+i & -5 \\ 3 & i \\ -1+3i & 4-2i \end{bmatrix}$

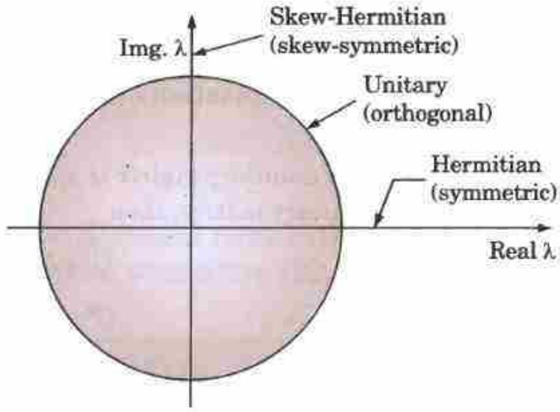


Fig. 2.1. Eigen values of various matrices.

and $$A^* = \begin{bmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4+2i \end{bmatrix}$$

$$\therefore \quad AA^* = \begin{bmatrix} 2-i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix} \begin{bmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4+2i \end{bmatrix}$$

$$= \begin{bmatrix} 4-i^2+9+1-9i^2 & -10-5i-3i-10+10i \\ -10+5i+3i-10-10i & 25-i^2+16-4i^2 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & -20+2i \\ -20-2i & 46 \end{bmatrix}$$, which is a Hermitian matrix.

Example 2.54. *Prove that the matrix* $A = \begin{bmatrix} \frac{1}{2}(1+i) & \frac{1}{2}(-1+i) \\ \frac{1}{2}(1+i) & \frac{1}{2}(1-i) \end{bmatrix}$ *is unitary and find* A^{-1}. *(Mumbai, 2006)*

Solution. Conjugate of A, *i.e.*, $\overline{A} = \begin{bmatrix} \frac{1}{2}(1-i) & \frac{1}{2}(-1-i) \\ \frac{1}{2}(1-i) & \frac{1}{2}(1+i) \end{bmatrix}$

$\therefore$ Transpose of $\overline{A}$, *i.e.*, $$A^\theta = \begin{bmatrix} \frac{1}{2}(1-i) & \frac{1}{2}(1-i) \\ \frac{1}{2}(-1-i) & \frac{1}{2}(1+i) \end{bmatrix}$$

Now $$A^\theta . A = \begin{bmatrix} \frac{1}{2}(1-i) & \frac{1}{2}(1-i) \\ \frac{1}{2}(-1-i) & \frac{1}{2}(1+i) \end{bmatrix} \begin{bmatrix} \frac{1}{2}(1+i) & \frac{1}{2}(-1+i) \\ \frac{1}{2}(1+i) & \frac{1}{2}(1-i) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4}(1+1)+\frac{1}{4}(1+1) & -\frac{1}{4}(1-i)^2+\frac{1}{4}(1-i)^2 \\ -\frac{1}{4}(1+i)^2+\frac{1}{4}(1+i)^2 & \frac{1}{4}(1+1)+\frac{1}{4}(1+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Similarly, $AA^\theta = I$.
Hence A is a unitary matrix.

Also $$A^{-1} = A^\theta = \begin{bmatrix} \frac{1}{2}(1-i) & \frac{1}{2}(1-i) \\ -\frac{1}{2}(1+i) & \frac{1}{2}(1+i) \end{bmatrix}$$

Example 2.55. *Given that* $A = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$, *show that* $(1 - A)(1 + A)^{-1}$ *is a unitary matrix.*

(Mumbai, 2007)

Solution. $I + A = \begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$, $|I + A| = 1 - (-1 - 4) = 6$

$$(I + A)^{-1} = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} \div 6. \text{ Also } I - A = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$\therefore \quad (I - A)(I + A)^{-1} = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} \div 6 = \frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix} \qquad ...(i)$$

Its conjugate-transpose $= \dfrac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$ ...(ii)

$\therefore$ Product of (*i*) and (*ii*) $= \dfrac{1}{36}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix} = \dfrac{1}{36}\begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix} = I.$

Hence the result.

PROBLEMS 2.11

1. Prove that every Hermitian matrix can be written as $A + iB$, where A is real and symmetric and B is real and skew-symmetric : *(P.T.U., 1999)*
2. Show that every square matrix can be uniquely expressed as $P + iQ$, where P and Q are Hermitian matrices. *(Mumbai, 2008 ; Bhopal, 2002 S)*
3. Show that a Hermitian matrix remains Hermitian when transformed by an orthogonal matrix.
4. Show that the matrix $\begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix}$ is a unitary matrix, if $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$. *(U.P.T.U., 2006)*
5. Show that $\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$ is a Hermitian matrix.
6. If $A = \begin{bmatrix} -1 & 2+i & 5-3i \\ 2-i & 7 & 5i \\ 5+3i & -5i & 2 \end{bmatrix}$, show that A is a Hermitian matrix and iA is a skew-Hermitian matrix. *(Sambalpur, 2002)*
7. Show that the following matrix is unitary

   (*i*) $\dfrac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ *(U.P.T.U., 2002)* (*ii*) $\begin{bmatrix} \frac{2+i}{3} & \frac{2i}{3} \\ \frac{2i}{3} & \frac{2-i}{3} \end{bmatrix}$ *(Mumbai, 2008)*
8. Express $A = \begin{bmatrix} 2i & 2+i & 1-i \\ -2+i & -i & 3i \\ -1-i & 3i & 0 \end{bmatrix}$ as $P + iQ$ where P is real and skew-symmetric and Q is real and symmetric. *(Mumbai, 2006)*
9. If $S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix}$, where $a = e^{2i\pi/3}$, prove that $S^{-1} = \frac{1}{3}\bar{S}$. *(Kurukshetra, 2006 ; J.N.T.U., 2001)*

2.20 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 2.12

Choose the correct answer or fill up the blanks in the following problems :

1. To multiply a matrix by scalar k, multiply

   (*a*) any row by k (*b*) every element by k (*c*) any column by k.

2. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then A^n is

   (*a*) $\begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$ (*b*) $\begin{bmatrix} 3^n & (-4)^n \\ 1 & (-1)^n \end{bmatrix}$ (*c*) $\begin{bmatrix} 1+3n & 1-4n \\ 1+n & 1-n \end{bmatrix}$ (*d*) $\begin{bmatrix} 1+2n & -4n \\ 1+n & 1-2n \end{bmatrix}$

3. The inverse of the matrix $\begin{bmatrix} -0.5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is

   (*a*) $\begin{bmatrix} 0.5 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (*b*) $\begin{bmatrix} 0.5 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (*c*) $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (*d*) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -0.25 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

4. If $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$, then the determinant AB has the value

   (*a*) 4 (*b*) 8 (*c*) 16 (*d*) 32

5. The system of equations $x + 2y + z = 9$, $2x + y + 3z = 7$ can be expressed as

   (*a*) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ (*b*) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

   (*c*) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix}$ (*d*) none of the above.

6. If $\begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$, then X equals

   (*a*) $\begin{bmatrix} -3 & -14 \\ 4 & 17 \end{bmatrix}$ (*b*) $\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$ (*c*) $\begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}$ (*d*) $\begin{bmatrix} 3 & -14 \\ 4 & -17 \end{bmatrix}$

7. If $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$, then A (adj A) equals

   (*a*) $\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$ (*b*) $\begin{bmatrix} 0 & 10 \\ 10 & 0 \end{bmatrix}$ (*c*) $\begin{bmatrix} 10 & 1 \\ 1 & 10 \end{bmatrix}$ (*d*) none of the above.

8. If $3x + 2y + z = 0$, $x + 4y + z = 0$, $2x + y + 4z = 0$, be a system of equations, then

   (*a*) it is inconsistent

   (*b*) it has only the trivial solution $x = 0, y = 0, z = 0$.

   (*c*) it can be reduced to a single equation and so a solution does not exist.

   (*d*) determinant of the matrix of coefficients is zero.

9. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then

   (*a*) $C = A\cos\theta - B\sin\theta$ (*b*) $C = A\sin\theta + B\cos\theta$

   (*c*) $C = A\sin\theta - B\cos\theta$ (*d*) $C = A\cos\theta + B\sin\theta$.

10. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & \gamma & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}$, then

(*a*) A is row equivalent to B only when $\alpha = 2, \beta = 3$, and $\gamma = 4$

(*b*) A is row equivalent to B only when $\alpha \neq 0, \beta \neq 0$, and $\gamma = 0$

(*c*) A is not row equivalent to B

(*d*) A is row equivalent to B for all value of α, β, γ.

11. If $A \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then A is

(*a*) $\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$ (*b*) $\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$ (*c*) $\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ (*d*) $\begin{bmatrix} 2 & 1 \\ -1/2 & -1/2 \end{bmatrix}$

12. Matrix has a value. This statement

(*a*) is always true (*b*) depends upon the matrices

(*c*) is false

13. If A is a square matrix such that $AA' = I$, then value of $A'A$ is

(*a*) A^2 (*b*) I (*c*) A^{-1}

14. If every minor of order r of a matrix A is zero, then rank of A is

(*a*) greater than r (*b*) equal to r (*c*) less than or equal to r (*d*) less than r.

15. A square matrix A is called orthogonal if

(*a*) $A = A^2$ (*b*) $A' = A^{-1}$ (*c*) $AA^{-1} = I$

16. The rank of matrix $\begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 1 \\ 5 & 2 & 4 & 3 \end{bmatrix}$ is ....

17. The sum of the eigen values of a matrix is the .... of the elements of the principal diagonal.

18. The sum and product of the eigen values of the matrix $\begin{bmatrix} 2 & -3 \\ 4 & -2 \end{bmatrix}$ are .... and .... respectively. (*Anna*, 2009)

19. Inverse of $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ is $\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & k \\ 2 & 2 & 5 \end{bmatrix}$ then k is ....

20. Using Cayley-Hamilton theorem, the value of $A^4 - 4A^3 - 5A^2 - A + 2I$ when $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$ is .... (*Anna, 2009*)

21. If two eigen values of $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ are 3 and 15, then the third eigen value is ....

22. A quadratic form is positive semi-definite when ....

23. $A_{m \times n}$ and $B_{p \times q}$ are two matrices. When will

(*a*) $A \,.\, B$ exist (*b*) $A + B$ exist ?

24. The product of the eigen values of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ is ....

25. The quadratic form corresponding to the diagonal matrix diag $(\lambda_1, \lambda_2, ..., \lambda_n)$ is

(*a*) $x_1^2 + x_2^2 + ... + x_n^2$ (*b*) $\lambda_1 x_1^2 + \lambda_2 x_2^2 + ... + \lambda_n x_n^2$

(*c*) $\lambda_1^2 x_1^2 + \lambda_2^2 x_2^2 + ... + \lambda_n^2 x_n^2$

26. An example of a 3×3 matrix of rank one is ....

27. The quadratic form corresponding to the symmetric matrix $\begin{bmatrix} 1 & 2 \\ 2 & -4 \end{bmatrix}$ is ....

28. Solving the equations $x + 2y + 3z = 0$, $3x + 4y + 4z = 0$, $7x + 10y + 12z = 0$, $x =$ ...., $y =$ ...., $z =$ ....

29. The eigen values of the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ are ....

30. A matrix A is *idempotent* if ....

31. The rank of the matrix $\begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ is ....

32. If A = $\begin{bmatrix} -1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{bmatrix}$, then the eigen values of A^2 are ....

33. The sum of the eigen values of $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & -1 & -1 \end{bmatrix}$ is

(*i*) – 2 (*ii*) 3 (*iii*) 6 (*iv*) 7 (*S.V.T.U., 2009*)

34. The maximum value of the rank of a 4 × 5 matrix is ....

35. The sum of two eigen values and trace of a 3 × 3 matrix are equal, then the value of $|A|$ is .... (*Anna, 2009*)

36. If the sum of the eigen values of the matrix of the quadratic form is zero, then the nature of the quadratic form is ....

37. The eigen values of matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{bmatrix}$ are ....

38. The eigen values of a triangular matrix are ....

39. If the product of two eigen values of the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is 16, then the third eigen value is ....

40. If λ_i, $i = 1, 2, \ldots\ldots\ldots n$ are the eigen values of a square matrix A, then the eigen values of A^T are ....

41. By applying elementary transformations to a matrix, its rank

(*a*) increases (*b*) decreases (*c*) does not change

42. If λ is an eigen value of A, then it is an eigen value of B, only if B = ...

43. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$, then eigen values of A^{-1} are ....

44. The characteristic equation of $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ is ....

45. If $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, then eigen values of A^{-1} are ....

46. Matrix $\begin{bmatrix} x & 2 \\ 1 & x-1 \end{bmatrix}$ is singular for x = ...

47. Every Hermitian matrix can be written as $A + iB$, where A is real and .... and B is real and ....

48. The sum and product of the eigen values of $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are .... and ....

49. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then A^3 = ...

50. The product of the eigen values of a matrix is equal to ....

51. The eigen values of $A = \begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix}$ are the roots of the equation ....

52. A system of linear non-homogeneous equations is consistent, if and only if the rank of coefficient matrix is equal to rank of ....
53. The matrix of the quadratic form $q = 4x^2 - 2y^2 + z^2 - 2xy + 6zx$ is ....
54. If $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of a matrix A, then A^3 has the eigen values ....
55. If λ is an eigen value of a non-singular matrix A, then the eigen value of A^{-1} is ....
56. The matrix corresponding to the quadratic form $x^2 + 2y^2 - 7z^2 - 4xy + 8xy + 5yz$ is ....
57. The sum of the squares of the eigen values of $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ is ....
58. If the rank of a matrix A is 2, then the rank of A' is ....
59. The index and signature of the quadratic form $x_1^2 + 2x_2^2 - 3x_3^2$ are respectively .... and ....
60. The equations $x + 2y = 1$, $7x + 14y = 12$ are consistent. (True or False)
61. If rank $(A) = 2$, rank $(B) = 3$, then rank $(AB) = 6$. (True or False)
62. Any set of vectors which includes the zero vector is linearly independent. (True or False)
63. If λ is an eigen value of a symmetric matrix,then λ is real. (True or False)
64. Every square matrix does not satisfy its own characteristic equation. (True or False)
65. If λ is an eigen value of an orthogonal matrix, then $1/\lambda$ is also its eigen value. (True or False)
66. If the rank of a matrix A is 3, then the rank of $3A^T$ is 1. (True or False)
67. The vectors $[1, 1, -1, 1]$, $[1, -1, 2, -1]$, $[3, 1, 0, 1]$ are linearly dependent. (True or False)
68. The eigen values of a skew-symmetric matrix are real. (True or False)
69. Inverse of a unitary matrix is a unitary matrix. (True or False)
70. A is a non-zero column matrix and B is a non-zero row matrix, then rank of AB is one. (True or False)
71. The sum of the eigen values of A equals to the trace of $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$. (True or False)

CHAPTER 3

Vector Algebra & Solid Geometry

VECTOR ALGEBRA

3.1 (1) VECTORS

A quantity which is completely specified by its magnitude only is called a *scalar*. Length, time, mass, volume, temperature, work, electric charge and numerical data in Statistics are all examples of scalar quantities.

A quantity which is completely specified by its magnitude and direction is called a **vector.** Weight, displacement, velocity, acceleration and electric current density are all vector quantities for each involves magnitude and direction.

A vector is represented by a directed line segment. Thus $\vec{PQ}$ represents a vector whose magnitude is the length PQ and direction is from P (starting point) to Q (end point). We denote a vector by a single letter in capital bold type (or with an arrow on it) and its magnitude (length) by the corresponding small letter in italics type. Thus if $\mathbf{V}$ is the velocity vector, its magnitude is v, the speed.

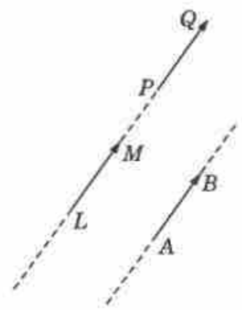


Fig. 3.1

A vector of unit magnitude is called a *unit vector*. The idea of unit vector is often used to represent concisely the direction of any vector. Unit vector corresponding to the vector $\mathbf{A}$ is written as $\hat{\mathbf{A}}$.

A vector of zero magnitude (which can have no direction associated with it) is called a *zero (or null) vector* and is denoted by **0**–a thick zero.

The vector $\vec{QP}$ represents the negative of $\vec{PQ}$, *i.e.*, $-\mathbf{A}$.

Two vectors **A** *and* **B** *having the same magnitude and the same* (or parallel) directions are said to be equal and we write $\mathbf{A} = \mathbf{B}$. Clearly the vectors $\vec{AB}$, $\vec{LM}$ and $\vec{PQ}$ are all equal (Fig. 3.1).

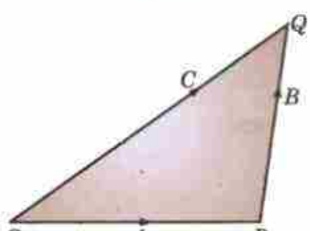


Fig. 3.2

(2) Addition of vectors. Vectors are added according to the *triangle law of addition*, which is a matter of common knowledge. Let **A** and **B** be represented by two vectors $\vec{OP}$ and $\vec{PQ}$ respectively then $\vec{OQ} = \mathbf{C}$ is called the sum or resultant of **A** and **B**. Symbolically, we write,

$$\mathbf{C} = \mathbf{A} + \mathbf{B}$$

(3) Subtraction of vectors. *The subtraction of a vector* **B** *from* **A** *is taken to be the addition of* –**B** *to* **A** *and we write*

$$\mathbf{A} + (-\mathbf{B}) = \mathbf{A} - \mathbf{B}$$

(4) Multiplication of vectors by scalars.

We have just seen that $\quad \mathbf{A} + \mathbf{A} = 2\mathbf{A}$

and $\quad -\mathbf{A} + (-\mathbf{A}) = -2\,\mathbf{A}$

where both 2**A** and – 2**A** denote vectors of magnitude twice that of **A** ; the former having the same direction as **A** and the latter the opposite direction.

In general, *the product m***A** *of a vector* **A** *and a scalar m is a vector whose magnitude is m times that of* **A** *and direction is the same or opposite to* **A** *according as m is positive or negative.*

Thus $\quad \mathbf{A} = a\,\hat{\mathbf{A}}$.

Example 3.1. *If* **A** *and* **B** *are the vectors determined by two adjacent sides of a regular hexagon. What are the vectors represented by the other sides taken in order ?*

Solution. Let *ABCDEF* be the given hexagon, such that

$$\vec{AB} = \mathbf{A} \text{ and } \vec{BC} = \mathbf{B}$$

$\therefore \quad \vec{AC} = \vec{AB} + \vec{BC} = \mathbf{A} + \mathbf{B}$

Also $\quad \vec{AD} = 2\vec{BC} = 2\mathbf{B}$

$\therefore \quad \vec{CD} = \vec{AD} - \vec{AC} = 2\mathbf{B} - (\mathbf{A} + \mathbf{B}) = \mathbf{B} - \mathbf{A}$

Now $\quad \vec{DE} = -\vec{AB} = -\mathbf{A} \qquad [\because \; AB = \text{and} \parallel ED]$

$\quad \vec{EF} = -\vec{BC} = -\mathbf{B} \qquad [\because \; BC = \text{and} \parallel FE]$

and $\quad \vec{FA} = -\vec{CD} = -(\mathbf{B} - \mathbf{A}) = \mathbf{A} - \mathbf{B} \qquad [\because \; CD = \text{and} \parallel AF]$

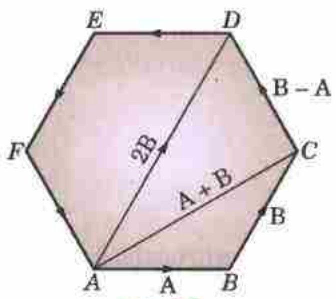


Fig. 3.3

3.2. (1) Space coordinates. Let $X'OX$ and $Y'OY$, $Z'OZ$ be three mutually perpendicular lines which intersect at O. Then O is called the origin.

X′OX is called the **x-axis,** *Y′OY the* **y-axis**, *Z′OZ the* **z-axis** *and taken together these are called the* **coordinate axes.**

The plane YOZ is called the **yz-plane,** *the plane ZOX the* **zx-plane,** *the plane XOY the* **xy-plane** *and taken together these are called the* **coordinate planes.**

Let P be any point in space, Draw PL, PM, PN ⊥s to the yz, zx and xy-planes. Then LP, MP, NP are respectively called the coordinates of P (Fig. 3.4). In practice, if $OA = x$, AN y, $NP = z$, then (x, y, z) are the coordinates of P which are positive along OX, OY, OZ respectively and negative along OX', OY', OZ'.

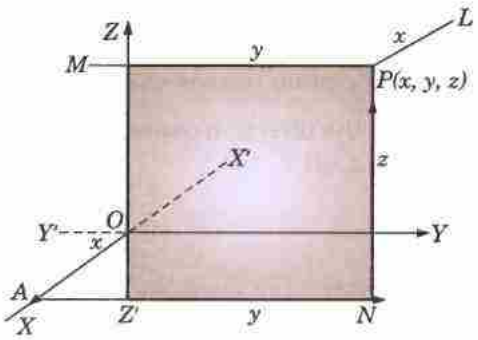


Fig. 3.4

The three coordinate planes divide the space into eight compartments called **octants.** The octant *OXYZ* in which all the coordinates are positive is called the **positive** or **first octant.**

Note. *Three non-coplanar vectors* **A, B, C** *are said to form a* **right-handed** (*or a* **left-handed**) *system according as a right threaded screw rotated through an angle less than* 180° *from* **A** *to* **B** *will advance along (or opposite to)* **C** as shown in Fig. 3.5.

An area of a closed curve described in a given manner is represented by a vector whose magnitude is the given area and direction normal to the plane of the area. Thus the vector **A** representing the area is taken to be positive or negative according as the direction of description of the boundary of the curve and the sense of **A** correspond to a right-handed or a left-handed system.

We have explained the most commonly used system of coordinates namely the *Rectangular Cartesian Coordinates.* The other two systems of coordinates often used to locate a point in space are the *Polar spherical coordinates and Cylindrical coordinates,* which are explained in § 8.21 and 8.20.

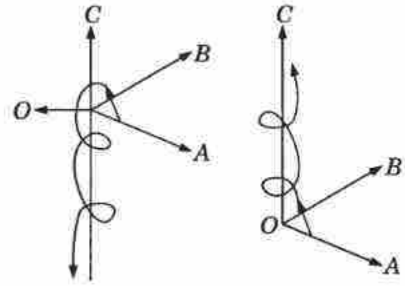


Fig. 3.5

(2) Resolution of vectors. Let **I, J, K** denote unit vectors along OX, OY, OZ respectively. Let $P(x, y, z)$ be a point in space. On OP as diagonal, construct a rectangular parallelopiped with edges OA, OB, OC along the axes so that

$$\vec{OA} = x\mathbf{I},\ \vec{OB} = y\mathbf{J},\ \vec{OC} = z\mathbf{K}$$

Then
$$\mathbf{R} = \vec{OP} = \vec{OC'} + \vec{C'P}$$
$$= \vec{OA} = \vec{AC'} + \vec{OC} = \vec{OA} + \vec{OB} + \vec{OC}$$

Hence $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ is called the *position vector* of P relative to origin O and.

$$r = |\mathbf{R}| = \sqrt{(x^2 + y^2 + z^2)}$$
$$[\because\ r^2 = OP^2 = OC'^2 + C'P^2 = OA^2 + AC'^2 + C'P^2]$$

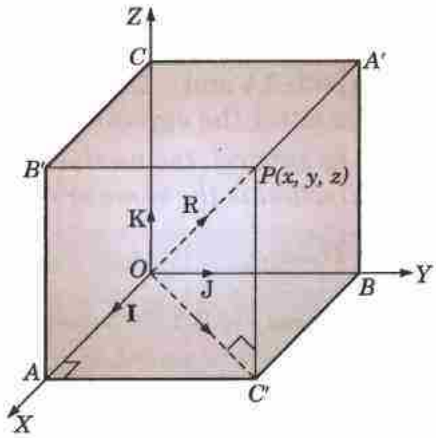


Fig. 3.6

(3) Direction cosines. Let any line L or its parallel OP, make angles α, β, γ with OX, OY, OZ respectively, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ *are called the direction cosines of this line* which are usually denoted by l, m, n.

If l, m, n are direction cosines of a vector **R**, then

(*i*) $\hat{\mathbf{R}} = l\mathbf{I} + m\mathbf{J} + n\mathbf{K}$, (*ii*) $\mathbf{l^2 + m^2 + n^2 = 1}$

Proof. Let D be the foot of the perpendicular from $P(x, y, z)$ on OY. Then

$$y = OD = r\cos\beta = mr. \text{ Similarly, } z = nr \text{ and } x = lr.$$

$\therefore$
$$\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K} = r(l\mathbf{I} + m\mathbf{J} + n\mathbf{K})$$

or
$$\hat{\mathbf{R}} = \frac{\mathbf{R}}{r} = l\mathbf{I} + m\mathbf{J} + n\mathbf{K}$$

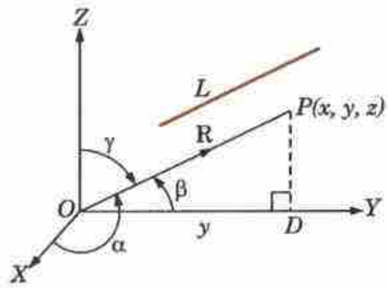


Fig. 3.7

which expresses a unit vector in terms of its direction cosines.

Also
$$1 = |\hat{\mathbf{R}}| = \sqrt{(l^2 + m^2 + n^2)} \text{ thus } \mathbf{l^2 + m^2 + n^2 = 1}$$

i.e.,
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$
(*V.T.U., 2010*)

Obs. Directions ratios. If the direction cosines of a line be proportional to a, b, c, then these are called proportional direction cosines or direction ratios of the line.

If the direction cosines be l, m, n, then

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{\sqrt{(l^2 + m^2 + n^2)}}{\sqrt{(a^2 + b^2 + c^2)}} = \frac{1}{\sqrt{(\Sigma a^2)}}$$

$\therefore$
$$l = \frac{a}{\sqrt{(\Sigma a^2)}},\ m = \frac{b}{\sqrt{(\Sigma a^2)}},\ n = \frac{c}{\sqrt{(\Sigma a^2)}}$$

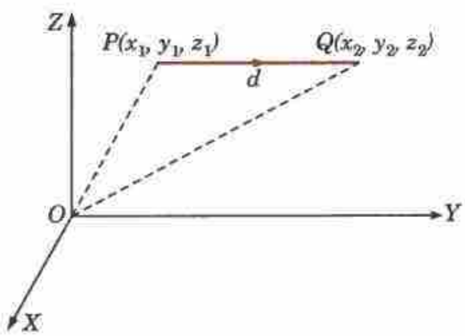


Fig. 3.8

(4) Distance between two points $P(x_1, y_1, z_1)$ *and* $Q(x_2, y_2, z_2)$ is

$$\sqrt{[(\mathbf{x_2 - x_1})^2 + (\mathbf{y_2 - y_1})^2 + (\mathbf{z_2 - z_1})^2]}$$

and **direction ratios** *of* $\vec{PQ}$ *are* $\mathbf{x_2 - x_1, y_2 - y_1, z_2 - z_1}$

We have

$$\vec{OP} = x_1\mathbf{I} + y_1\mathbf{J} + z_1\mathbf{K}$$

and
$$\vec{OQ} = x_2\mathbf{I} + y_2\mathbf{J} + z_2\mathbf{K}$$

$\therefore$
$$\vec{PQ} = \vec{OQ} - \vec{OP}$$

$$= (x_2 - x_1)\mathbf{I} + (y_2 - y_1)\mathbf{J} + (z_2 - z_1)\mathbf{K}$$

Thus, $$d = |\,\vec{PQ}\,| = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$$

and direction cosines of $\vec{PQ}$ are proportional to $x_2 - x_1, y_2 - y_1, z_2 - z_1$.

Example 3.2. *Show that the points A(–4, 9, 6), B(–1, 6, 6) and C(0, 7, 10) form a right angled isosceles triangle. Also find the direction cosines of AB.*

Solution. We have $$AB = \sqrt{[(-1+4)^2 + (6-9)^2 + (6-6)^2]} = 3\sqrt{2}$$

$$BC = \sqrt{[(0+1)^2 + (7-6)^2 + (10-6)^2]} = 3\sqrt{2}$$

and $$CA = \sqrt{[(-4-0)^2 + (9-7)^2 + (6-10)^2]} = 6$$

Since $AB^2 + BC^2 = CA^2$ and $AB = BC$, it follows that ΔABC is a right-angled isosceles triangle. The direction ratios of $\vec{AB}$ are $-1+4, 6-9, 6-6$.

$\therefore$ Its direction cosines are $\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}, 0$.

3.3 SECTION FORMULAE

The point $\mathbf{R}$ (x, y, z) dividing the join of the points $\mathbf{A}(x_1, y_1, z_1)$ *and* $\mathbf{B}(x_2, y_2, z_2)$ *in the ratio* $m_1 : m_2$ *is*

$$\mathbf{R} = \frac{m_1\mathbf{B} + m_2\mathbf{A}}{m_1 + m_2}, \quad i.e., \quad \left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, \frac{m_1z_2 + m_2z_1}{m_1 + m_2}\right) \quad ...(i)$$

Let $P(\mathbf{A})$ and $Q(\mathbf{B})$ be the given points referred to origin O. Let $R(\mathbf{R})$ be the point dividing the line joining P and Q in the ratio $m_1 : m_2$ so that

$$\frac{PR}{RQ} = \frac{m_1}{m_2}, \quad i.e., \quad m_2 \,.\, PR = m_1 \,.\, RQ$$

$\therefore$ We have $$m_2\vec{PR} = m_1\vec{RQ}$$

or $$m_2(\vec{OR} - \vec{OP}) = m_1(\vec{OQ} - \vec{OR})$$

or $$m_2(\mathbf{R} - \mathbf{A}) = m_1(\mathbf{B} - \mathbf{R})$$

whence $$\mathbf{R} = \frac{m_1\mathbf{B} + m_2\mathbf{A}}{m_1 + m_2}$$

Since $$\mathbf{A} = x_1\mathbf{I} + y_1\mathbf{J} + z_1\mathbf{K}, \ \mathbf{B} = x_2\mathbf{I} + y_2\mathbf{J} + z_2\mathbf{K}$$

and $$\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$$

$$\therefore \quad x\mathbf{I} + y\mathbf{J} + z\mathbf{K} = \frac{m_1(x_2\mathbf{I} + y_2\mathbf{J} + z_2\mathbf{K}) + m_1(x_1\mathbf{I} + y_1\mathbf{J} + z_1\mathbf{K})}{m_1 + m_2}$$

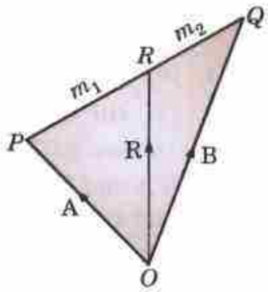


Fig. 3.9

Equating coefficient of **I, J, K,** we get the desired results (i).

Cor. 1. *Mid-point of* $P(\mathbf{A})$ *and* $Q(\mathbf{B})$ *is* $\frac{1}{2}(\mathbf{A} + \mathbf{B})$.

2. Point R dividing the join of $P(\mathbf{A})$ and $Q(\mathbf{B})$ in the ratio $m_1 : m_2$ externally is $\mathbf{R} = \dfrac{m_1\mathbf{B} - m_2\mathbf{A}}{m_1 - m_2}$.

Obs. Rewriting (i) as $m_2\mathbf{A} + m_1\mathbf{B} - (m_1 + m_2)\mathbf{R} = 0$, we note that the sum of the coefficients of **A, B** and **R** is zero. Hence it follows that any *three points with position vectors* **A**, **B** *and* **C** *are collinear if*

$$\lambda\mathbf{A} + \mu\mathbf{B} + \gamma\mathbf{C} = 0, \text{ where } \lambda + \mu + \gamma = 0.$$

Example 3.3. *In a trapezium, prove that the straight line joining the mid-points of the diagonals is parallel to the parallel sides and half their difference.*

Solution. Consider a trapezium $OABC$ with parallel sides OA and BC. Take O as the origin and let the other vertices be $A(\mathbf{A})$, $B(\mathbf{B})$, $C(\mathbf{C})$.

Since CB is parallel to OA, therefore,

$$\mathbf{B} - \mathbf{C} = \vec{CB} = \lambda \vec{OA} = \lambda \mathbf{A}.$$

The mid-points of the diagonals OB and AC are $D(\mathbf{B}/2)$ and $E(\mathbf{A} + \mathbf{C})/2$.

$$\therefore \quad \vec{DE} = \vec{OE} - \vec{OD} = \frac{1}{2}(\mathbf{A} + \mathbf{C}) - \frac{1}{2}\mathbf{B} = \frac{1}{2}[\mathbf{A} - (\mathbf{B} - \mathbf{C})] \qquad ...(i)$$

$$= \frac{1}{2}(1 - \lambda)\mathbf{A} \qquad ...(ii)$$

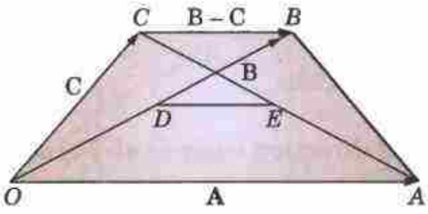


Fig. 3.10

From (ii), it is clear that $\vec{DE}$ is parallel to $\vec{OA}$; from (i), it follows that $DE = \frac{1}{2}(OA - CB)$.

Hence the result.

Example 3.4. *Show that the line joining one vertex of a parallelogram to the mid-point of an opposite side trisects the diagonal and is itself trisected there at.*

Solution. Consider a parallelogram $OABC$. Take O as the origin and let the other vertices be $A(\mathbf{A})$, $B(\mathbf{B})$ and $C(\mathbf{C})$.

The mid-point D of OA is $\mathbf{A}/2$.

Now since OA is equal to and parallel to CB.

$$\therefore \quad \vec{OA} = \vec{CB}, \text{ i.e., } \mathbf{A} = \mathbf{B} - \mathbf{C}$$

which may be written as $\dfrac{2(\mathbf{A}/2) + 1 \cdot \mathbf{C}}{2 + 1} = \dfrac{\mathbf{B}}{3} = \mathbf{P}$ so that P trisects DC and OB.

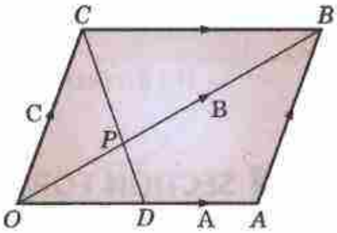


Fig. 3.11

PROBLEMS 3.1

1. Given $\mathbf{R}_1 = 5\mathbf{I} - 2\mathbf{J} + 4\mathbf{K}$ and $\mathbf{R}_2 = \mathbf{I} + 3\mathbf{J} + 7\mathbf{K}$, find the magnitude and direction cosines of the vectors $\mathbf{R}_1 + \mathbf{R}_2$ and $2\mathbf{R}_1 - \mathbf{R}_2$.
2. Show that the points (0, 4, 1) ; (2, 3, –1) ; (4, 5, 0) and (2, 6, 2) are the vertices of a square. *(Osmania, 1999 S)*
3. A straight line is inclined to the axes of x and y at angles of 30° and 60°. Find the inclination of the line to the z-axis. *(Madras, 2003)*
4. If a line makes angles α, β, γ with the axes, prove that
   (i) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$. *(V.T.U., 2000 ; Osmania, 1999)*
   (ii) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1$.
5. If $\mathbf{A}$ and $\mathbf{B}$ are non-collinear vectors and $\mathbf{P} = (2x + 3y - 2)\mathbf{A} + (3x + 2y + 5)\mathbf{B}$ and $\mathbf{Q} = (-x + 4y - 2)\mathbf{A} + (3x - 4y + 7)\mathbf{B}$, find x, y such that $7\mathbf{P} = 3\mathbf{Q}$.
6. Prove that the line joining the mid-points of the two sides of a triangle is parallel to the third side and half of it.
7. Prove that (i) the diagonals of a parallelogram bisect each other ;
   (ii) a quadrilateral whose diagonals bisect each other is a parallelogram.
8. In a skew quadrilateral, prove that :
   (i) the figure formed by joining the mid-points of the adjacent sides is a parallelogram.
   (ii) the joins of the mid-points of opposite sides bisect each other.
9. In a trapezium, prove that the straight line joining the mid-points of the non-parallel sides is parallel to the parallel sides and half their sum.
10. Prove that the vectors $\mathbf{A} = 3\mathbf{I} + \mathbf{J} - 2\mathbf{K}$, $\mathbf{B} = -\mathbf{I} + 3\mathbf{J} + 4\mathbf{K}$, $\mathbf{C} = 4\mathbf{I} - 2\mathbf{J} - 6\mathbf{K}$ can form the sides of a triangle. Also find the length of the median bisecting the vector $\mathbf{C}$. *(J.N.T.U., 1995 S)*
11. Find the ratio in which the line joining (2, 4, 16) and (3, 5, – 4) is divided by the plane $2x - 3y + z + 6 = 0$. *(Mysore, 1995)*
12. Show that the three points $\mathbf{I} - 2\mathbf{J} + 3\mathbf{K}$, $2\mathbf{I} + 3\mathbf{J} - 4\mathbf{K}$, $-7\mathbf{J} + 10\mathbf{K}$ are collinear.
13. If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be the position vectors of the vertices A, B, C of the triangle ABC, show that the three
    (i) medians concur at the point $\frac{1}{3}(\mathbf{A} + \mathbf{B} + \mathbf{C})$, called the *centroid*.
    (ii) internal bisectors of the angles concur at the point $\dfrac{a\mathbf{A} + b\mathbf{B} + c\mathbf{C}}{a + b + c}$, called the *incentre*.

14. Show that the coordinates of the *centroid of the triangle* whose vertices are $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ are

$$\left[\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right].$$

15. Show that the coordinates of the *centroid of the tetrahedron* whose vertices are $(x_r, y_r, z_r) : r = 1, 2, 3, 4$ are

$$\left[\frac{1}{4}(x_1+x_2+x_3+x_4), \frac{1}{4}(y_1+y_2+y_3+y_4); \frac{1}{4}(z_1+z_2+z_3+z_4)\right].$$

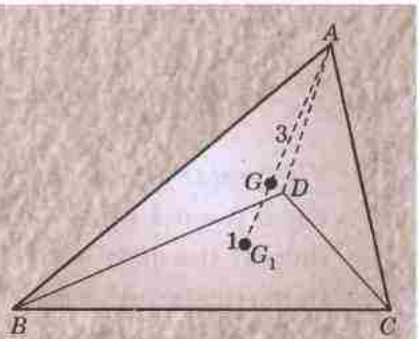


Fig. 3.12

[**Def.** *A* **tetrahedron** *is a solid bounded by four triangular faces.* Thus the tetrahedron $ABCD$ has four faces—the Δs ABC, ACD, ADB, BCD. (Fig. 3.12.)

It has four **vertices** A, B, C, D and three pairs of **opposite edges** AB, CD ; BC, AD ; CA, BD.

The centroid of the tetrahedron divides the join of each vertex to the centroid of the opposite triangular face in the ratio 3 : 1].

16. M and N are the mid-points of the diagonals AC and BD respectively of a quadrilateral $ABCD$. Show that the resultant of the vectors $\vec{AB}, \vec{AD}, \vec{CB}, \vec{CD}$ is $4\,\vec{MN}$. (*Cochin, 1999*)

3.4 PRODUCTS OF TWO VECTORS

Unlike the product of two scalars or that of a vector by a scalar, the product of two vectors is sometimes seen to result in a scalar quantity and sometimes in a vector. As such, we are led to define two types of such products, called the *scalar product* and the *vector product* respectively.

The scalar and vector products of two vectors **A** and **B** are usually written as **A . B** and **A × B** respectively and are read as **A** dot **B** and **A** cross **B**. In view of this notation, the former is sometimes called the *dot product* and the latter the *cross product*.

In vector algebra, the division of a vector by another vector is not defined.

3.5 SCALAR OR DOT PRODUCT

(1) Definition. *The scalar or dot product of two vectors* **A** *and* **B** *is defined as the scalar ab* cos θ, *where* θ *is the angle between* **A** *and* **B**.

Thus $\qquad \mathbf{A.B} = ab\cos\theta$.

(2) Geometrical interpretation. A.B *is the product of the length of one vector and the length of the projection of the other in the direction of the former.*

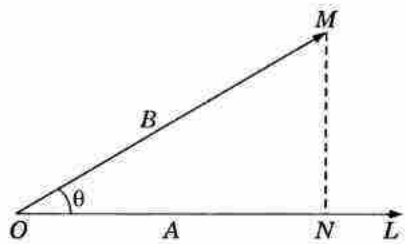


Fig. 3.13

Let $\qquad \vec{OL} = \mathbf{A}, \ \vec{OM} = \mathbf{B} \qquad$ then

$\mathbf{A}\,.\,\mathbf{B} = ab\cos\theta = a(OM\cos\theta) = a(ON) = |\,A\,|$ Proj. of $|\,\mathbf{B}\,|$ in the direction of **A**.

Similarly, $\mathbf{A}\,.\,\mathbf{B} = |\,\mathbf{B}\,|$ Proj. of $|\,\mathbf{A}\,|$ in the direction of **B**.

(3) Properties and other results.

I. Scalar product of two vectors is commutative.

i.e., $\mathbf{A}\,.\,\mathbf{B} = \mathbf{B}\,.\,\mathbf{A}$ for $\mathbf{A}\,.\,\mathbf{B} = ab\cos\theta = ba\cos(-\theta) = \mathbf{B}\,.\,\mathbf{A}$

II. The necessary and sufficient condition for two vectors to be perpendicular is that their scalar product should be zero.

When the vectors **A** and **B** are perpendicular, $\mathbf{A}\,.\,\mathbf{B} = ab\cos 90° = 0$.

Conversely, when $\mathbf{A}\,.\,\mathbf{B} = 0$, $ab\cos\theta = 0$, *i.e.*, $\cos\theta = 0$. $\qquad (\because \ a \neq 0, b \neq 0)$, or $\theta = 90°$.)

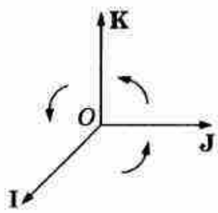


Fig. 3.14

III. $\mathbf{A}\,.\,\mathbf{A} = a^2$ which is written as $\mathbf{A}^2$. Thus *the square of a vector is a scalar which stands for the square of its magnitude.*

IV. For the mutually perpendicular unit vectors, **I, J, K,** we have the relations.

$$\mathbf{I}\,.\,\mathbf{J} = \mathbf{J}\,.\,\mathbf{K} = \mathbf{K}\,.\,\mathbf{I} = 0$$

and
$$\mathbf{I}^2 = \mathbf{J}^2 = \mathbf{K}^2 = 1$$

which are of great utility.

V. Scalar product of two vectors is distributive i.e.,

$$(\mathbf{A} + \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{C} + \mathbf{B} \cdot \mathbf{C}$$

VI. Schwarz inequality :* $|\mathbf{A} \cdot \mathbf{B}| \le |\mathbf{A}|\, |\mathbf{B}|$

$$|\mathbf{A} \cdot \mathbf{B}| = |\mathbf{A}|\, |\mathbf{B}|\, |\cos\theta| \le |\mathbf{A}|\, |\mathbf{B}| \qquad [\because \ |\cos\theta| \le 1]$$

VII. Scalar product of two vectors is equal to the sum of the products of their corresponding components.

For if $\mathbf{A} = a_1\mathbf{I} + a_2\mathbf{J} + a_3\mathbf{K}$, $\mathbf{B} = b_1\mathbf{I} + b_2\mathbf{J} + b_3\mathbf{K}$

then by the distributive law, $\mathbf{A} \cdot \mathbf{B} = a_1b_1 + a_2b_2 + a_3b_3$

In particular, $\mathbf{A}^2 = a_1^2 + a_2^2 + a_3^2$.

VIII. **Angle between two lines** *whose direction cosines are l, m, n and l′, m′, n′ is* $\mathbf{\cos^{-1}(ll' + mm' + nn')}$.

The unit vectors in the direction of the given lines are $\mathbf{U} = l\mathbf{I} + m\mathbf{J} + n\mathbf{K}$ and $\mathbf{U'} = l'\mathbf{I} + m'\mathbf{J} + n'\mathbf{K}$.

If θ be the angle between the lines, then

$$\mathbf{U} \cdot \mathbf{U'} = (l\mathbf{I} + m\mathbf{J} + n\mathbf{K}) \cdot (l'\mathbf{I} + m'\mathbf{J} + n'\mathbf{K})$$

or
$$1 \cdot 1 \cdot \cos\theta = ll' + mm' + nn' \qquad \textit{(V.T.U., 2008)}$$

Hence
$$\cos\theta = ll' + mm' + nn' \qquad ...(i)$$

Cor. 1.
$$\sin^2\theta = 1 - \cos^2\theta = 1 - (ll' + mm' + nn')^2$$
$$= (l^2 + m^2 + n^2)(l'^2 + m'^2 + n'^2) - (ll' + mm' + nn')^2$$
$$= (mn' - nm')^2 + (nl' - ln')^2 + (lm' - ml')^2$$

∴
$$\sin\theta = \pm\sqrt{\Sigma(mn' - nm')^2}. \qquad ...(ii)$$

Cor. 2. *The condition that the lines whose direction cosines are l, m, n and l′, m′, n′ should be* **perpendicular** *is*

$$\mathbf{ll' + mm' + nn' = 0} \qquad ...(iii)$$

and **parallel** *is*
$$\mathbf{l = l',\ m = m',\ n = n'} \qquad ...(iv)$$

These conditions easily follow from (*i*) and (*ii*).

Cor. 3. *The angle θ between two lines whose direction ratios are a, b, c, and a′, b′, c′ is given by*

$$\cos\theta = \frac{aa' + bb' + cc'}{\sqrt{(\Sigma a^2)}\sqrt{(\Sigma a'^2)}}$$

or
$$\sin\theta = \frac{\sqrt{[(bc' - cb')^2 + (ca' - ac')^2 + (ab' - ba')^2}}{\sqrt{(\Sigma a^2)}\sqrt{(\Sigma a'^2)}}$$

These lines are (i) perpendicular if $aa' + bb' + cc' = 0$, *(ii) parallel if* $a/a' = b/b' = c/c'$.

IX. **Projection of the line joining two points** (x_1, y_1, z_1) *and* (x_2, y_2, z_2) *on a line whose direction cosines are l, m, n is*
$$l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1).$$

Let
$$\vec{OP} = x_1\mathbf{I} + y_1\mathbf{J} + z_1\mathbf{K},\ \vec{OQ} = x_2\mathbf{I} + y_2\mathbf{J} + z_2\mathbf{K}$$

∴
$$\vec{PQ} = (x_2 - x_1)\mathbf{I} + (y_2 - y_1)\mathbf{J} + (z_2 - z_1)\mathbf{K}$$

Also unit vector $\mathbf{U}$ along the given lines is $l\mathbf{I} + m\mathbf{J} + n\mathbf{K}$.

∴ Projection of PQ on the given line = $\vec{PQ} \cdot \mathbf{U}$.

$$= l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1).$$

Example 3.5. *Find the sides and angles of the triangle whose vertices are* $\mathbf{I} - 2\mathbf{J} + 2\mathbf{K}$, $2\mathbf{I} + \mathbf{J} - \mathbf{K}$, *and* $3\mathbf{I} - \mathbf{J} + 2\mathbf{K}$.

Solution. Let $\vec{OA} = \mathbf{I} - 2\mathbf{J} + 2\mathbf{K}$, $\vec{OB} = 2\mathbf{I} + \mathbf{J} - \mathbf{K}$, $\vec{OC} = 3\mathbf{I} - \mathbf{J} + 2\mathbf{K}$

Then
$$\vec{BC} = \mathbf{I} - 2\mathbf{J} + 3\mathbf{K}$$
$$\vec{CA} = -2\mathbf{I} - \mathbf{J}$$

* Named after the German mathematician *Hermann Amandus Schwarz* (1843—1921) who is known for his work in conformal mapping, calculus of variations and differential geometry. He succeeded *Weierstrass* in Berlin University.

and $\qquad \vec{AB} = \mathbf{I} + 3\mathbf{J} - 3\mathbf{K}$

$\therefore \qquad BC = \sqrt{14}, CA = \sqrt{5}, AB = \sqrt{19}.$

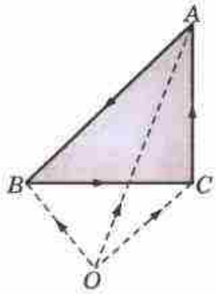


Fig. 3.15

Now d.c.'s of AB and AC being

$$1/\sqrt{19}, 3/\sqrt{19}, -3/\sqrt{19} \text{ and } 2/\sqrt{5}, 1/\sqrt{5}, 0,$$

We have $\qquad \cos A = \frac{1}{\sqrt{19}} \cdot \frac{2}{\sqrt{5}} + \frac{3}{\sqrt{19}} \cdot \frac{1}{\sqrt{5}} + \frac{-3}{\sqrt{19}} \cdot 0 = \sqrt{(5/19)}$

i.e., $\angle A = \cos^{-1} \sqrt{(5/19)}$. Again d.c.'s of BC and BA being

$$1/\sqrt{14}, -2/\sqrt{14}, 3/\sqrt{14} \text{ and } -1/\sqrt{19}, -3/\sqrt{19}, 3/\sqrt{19};$$

we have $\qquad \cos B = \frac{1}{\sqrt{14}} \cdot \frac{-1}{\sqrt{19}} + \frac{-2}{\sqrt{14}} \cdot \frac{-3}{\sqrt{19}} + \frac{3}{\sqrt{14}} \cdot \frac{3}{\sqrt{19}} = \sqrt{(14/19)}$, *i.e.*, $\angle B = \cos^{-1} \sqrt{(14/19)}$

Finally, d.c.'s of CA and CB being $-2/\sqrt{5}, -1/\sqrt{5}, 0$ and $-1/\sqrt{14}, 2/\sqrt{14}, -3/\sqrt{14}$;

we have $\qquad \cos C = \frac{-2}{\sqrt{5}} \cdot \frac{-1}{\sqrt{14}} + \frac{-1}{\sqrt{5}} \cdot \frac{2}{\sqrt{14}} + 0 \cdot \frac{-3}{\sqrt{14}} = 0$, *i.e.*, $\angle C = 90°$

Example 3.6. *Prove that the right -bisectors of the sides of a triangle concur at its circumcentre.*

Solution. Let $A(\mathbf{A}), B(\mathbf{B}), C(\mathbf{C})$ be the vertices of any triangle ABC. The mid-points of the sides BC, CA and AB are

$$D\left(\frac{\mathbf{B}+\mathbf{C}}{2}\right), E\left(\frac{\mathbf{C}+\mathbf{A}}{2}\right), F\left(\frac{\mathbf{A}+\mathbf{B}}{2}\right)$$

Let the perpendicular at D and E to BC and CA respectively intersect at the point $P(\mathbf{R})$. Then $\vec{DP} \cdot \vec{BC} = 0$

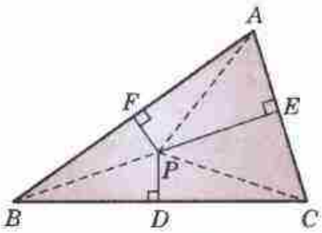


Fig. 3.16

i.e., $\qquad \left(\mathbf{R} - \frac{\mathbf{B}+\mathbf{C}}{2}\right) \cdot (\mathbf{C} - \mathbf{B}) = 0 \qquad ...(i)$

and $\qquad \vec{EP} \cdot \vec{CA} = 0$, *i.e.*, $\left(\mathbf{R} - \frac{\mathbf{C}+\mathbf{A}}{2}\right) \cdot (\mathbf{A} - \mathbf{C}) = 0 \qquad ...(ii)$

Adding (*i*) and (*ii*), we get $\left(\mathbf{R} - \frac{\mathbf{A}+\mathbf{B}}{2}\right) \cdot (\mathbf{A} - \mathbf{B}) = 0$

which shows that FP is perpendicular to AB. Hence the result.

Further $\qquad PA = PB$ if $|\mathbf{A} - \mathbf{R}| = |\mathbf{B} - \mathbf{R}|$

or if, $\qquad (\mathbf{A} - \mathbf{R})^2 = (\mathbf{B} - \mathbf{R})^2$ or if, $\mathbf{A}^2 - 2\mathbf{A} \cdot \mathbf{R} = \mathbf{B}^2 - 2\mathbf{B} \cdot \mathbf{R}$

of if, $\qquad \left(\mathbf{R} - \frac{\mathbf{A}+\mathbf{B}}{2}\right) \cdot (\mathbf{A} - \mathbf{B}) = 0$, which is true.

Example 3.7. *If the distance between two points P and Q is d and the lengths of the projections of PQ on the coordinate planes d_1, d_2, d_3, show that $2d^2 = d_1^2 + d_2^2 + d_3^2$.*

Solution. Let P be (x_1, y_1, z_1) and Q be (x_2, y_2, z_2), then

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2.$$

The feet of the perpendiculars drawn from P and Q on the XY-plane are the projections of P and Q on this plane. If these are L and M, then L is $(x_1, y_1, 0)$ and M is $(x_2, y_2, 0)$.

$\therefore \qquad d_1$ = projection of PQ on XY-plane, *i.e.*, LM

or $\qquad d_1^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

Similarly, $\qquad d_2^2 = (y_1 - y_2)^2 + (z_1 - z_2)^2$ and $d_3^2 = (z_1 - z_2)^2 + (x_1 - x_2)^2$

$\therefore \qquad d_1^2 + d_2^2 + d_3^2 = 2[(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2] = 2d^2.$

Example 3.8. *A line makes angles $\alpha, \beta, \gamma, \delta$ with diagonals of a cube, prove that*

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + \cos^2 \delta = 4/3. \qquad \text{(V.T.U., 2006 ; Osmania, 2000 S)}$$

Solution. Take O, a corner of the cube as origin and OA, OB, OC the three edges through it, as the axes. Let $OA = OB = OC = a$. Then the coordinates of the corners are as shown in Fig. 3.17.

The four diagonals are OP, AA', BB' and CC'.

Clearly, direction cosines of OP are

$$\frac{a-0}{\sqrt{(\Sigma a^2)}}, \frac{a-0}{\sqrt{(\Sigma a^2)}}, \frac{a-0}{\sqrt{(\Sigma a^2)}} \text{ i.e., } \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}.$$

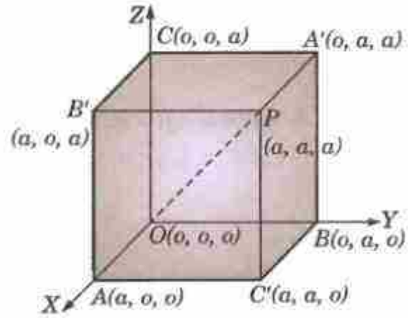


Fig. 3.17

Similarly, direction cosines of AA' are $-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$;

Similarly, direction cosines of BB' are $\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$;

and Similarly direction cosines of CC' are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$.

Let l, m, n be the direction cosines of the given line which makes angles $\alpha, \beta, \gamma, \delta$ with OP, AA', BB', CC' respectively. Then

$$\cos \alpha = \frac{1}{\sqrt{3}}(l + m + n) \,;\, \cos \beta = \frac{1}{\sqrt{3}}(-l + m + n)$$

$$\cos \gamma = \frac{1}{\sqrt{3}}(l - m + n); \cos \delta = \frac{1}{\sqrt{3}}\,(l + m - n)$$

Squaring and adding, we get

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + \cos^2 \delta = \frac{1}{3}\,[(l + m + n)^2 + (-l + m + n)^2 + (l - m + n)^2 + (l + m - n)^2]$$

$$= \frac{1}{3}\,[4(l^2 + m^2 + n^2)] = \frac{4}{3}. \qquad [\because \; l^2 + m^2 + n^2 = 1]$$

Example 3.9. *If the edges of a rectangular parallelopiped are a, b, c, show that the angle between the four diagonals are $\cos^{-1}\left(\dfrac{\pm a^2 \pm b^2 \pm c^2}{a^2 + b^2 + c^2}\right)$.*

Solution. Let $OA = a, OB = b, OC = c$ be the edges of the rectangular parallelopiped. Then the coordinates of the corners are as shown in Fig. 3.18. The four diagonals taken in pairs are (*i*) (OP, AA'), (*ii*) (OP, BB'), (*iii*) (OP, CC'), (*iv*) (AA', BB'), (*v*) (AA', CC') and (*vi*) (BB', CC').

Let the angles between these pairs of diagonals be $\theta_1, \theta_2, \ldots \theta_6$ respectively. Clearly d.r.'s OP are a, b, c ; d.r.'s, of AA' are $-a, b, c$, d.r.'s of BB' are $a, -b, c$ and d.r.'s of CC' are $a, b, -c$.

$\therefore$ For the pair (*i*) *i.e.*, (OP, AA') ;

$$\cos \theta_1 = \frac{-a^2 + b^2 + c^2}{\sqrt{(a^2 + b^2 + c^2}\;\sqrt{(a^2 + b^2 + c^2)}} = \frac{-a^2 + b^2 + c^2}{a^2 + b^2 + c^2}$$

Similarly, $\cos \theta_2 = \dfrac{a^2 - b^2 + c^2}{a^2 + b^2 + c^2}$, $\qquad \cos \theta_3 = \dfrac{a^2 + b^2 - c^2}{a^2 + b^2 + c^2}$;

$$\cos \theta_4 = \frac{-a^2 - b^2 + c^2}{a^2 + b^2 + c^2}; \qquad \cos \theta_5 = \frac{-a^2 + b^2 - c^2}{a^2 + b^2 + c^2};$$

$$\cos \theta_6 = \frac{a^2 - b^2 - c^2}{a^2 + b^2 + c^2}$$

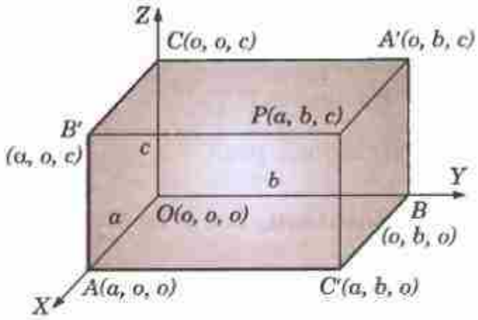


Fig. 3.18

Thus, noting that at least one term in the numerator is negative, we have in general

$$\cos \theta = \frac{\pm a^2 \pm b^2 \pm c^2}{a^2 + b^2 + c^2}.$$

Example 3.10. *Prove that the lines whose direction cosines are given by the relations $al + bm + cn = 0$ and $mn + nl + lm = 0$ are*

(i) Perpendicular if $a^{-1} + b^{-1} + c^{-1} = 0$ (Burdwan, 2003)

(ii) parallel if $\sqrt{a} + \sqrt{b} + \sqrt{c} = 0$.

Solution. Eliminating n from the given relations, we have

$$(m + l)\left(-\frac{al + bm}{c}\right) + lm = 0 \quad \text{or} \quad al^2 + (c - a - b)lm + bm^2 = 0$$

or $$a(l/m)^2 + (c - a - b)(l/m) + b = 0 \qquad ...(1)$$

If l_1, m_1, n_1 ; l_2, m_2, n_2, are the direction cosines of these lines then l_1/m_1, l_2/m_2 are the roots of the quadratic (1).

$$\therefore \quad \frac{l_1}{m_1}\cdot\frac{l_2}{m_2} = \frac{b}{a} \quad \text{or} \quad \frac{l_1 l_2}{1/a} = \frac{m_1 m_2}{1/b} = \frac{n_1 n_2}{1/c} \quad \text{(by symmetry)} = k \text{ (say).}$$

The lines will be perpendicular if $l_1 l_2 + m_1 m_2 + n_1 n_2 = k\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 0$

or if, $$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0.$$

The lines will be parallel if $l_1 = l_2$, $m_1 = m_2$, $n_1 = n_2$.

i.e., if, $\quad l_1/m_1 = l_2/m_2$; *i.e.* if, $\quad (c - a - b)^2 = 4ab$

or if, $\quad c - a - b = \pm 2\sqrt{(ab)}$ or if, $\quad c = a + b \pm 2\sqrt{(ab)} = (\sqrt{a} \pm \sqrt{b})^2$

or if, $\quad \pm\sqrt{c} = \sqrt{a} \pm \sqrt{b}$ or if, $\quad \sqrt{a} + \sqrt{b} + \sqrt{c} = 0$ [Taking necessary signs]

Example 3.11. *Find the angle between the lines whose direction cosines are given by the equation $l + 3m + 5n = 0$ and $5lm - 2mn + 6nl = 0$.*

Solution. Let us eliminate l from the given relations, by substituting $l = -3m - 5n$ in the second relation

$$5m(-3m - 5n) - 2mn + 6n(-3m - 5n) = 0$$

i.e., $$15m^2 + 45mn + 30n^2 = 0 \quad \text{or} \quad m^2 + 3mn + 2n^2 = 0$$

or $$(m + n)(m + 2n) = 0, \quad i.e., \quad m + n = 0 \text{ or } m + 2n = 0$$

Now let us first solve the equations $l + 3m + 5n = 0$ and $m + n = 0$

These give $m = -n$ and $l = -2n$, *i.e.*, $= \dfrac{l}{-2} = \dfrac{m}{-1} = \dfrac{n}{1}$ ...(*i*)

Similarly, solving the equations $l + 3m + 5n = 0$ and $m + 2n = 0$,

We get $$\frac{l}{1} = \frac{m}{-2} = \frac{n}{1} \qquad ...(ii)$$

(*i*) and (*ii*) give the direction ratios of the two lines.

If θ be the angle between these two lines, then

$$\cos\theta = \frac{(-2)\times 1 + (-1)\times(-2) + 1\times 1}{\sqrt{(2^2 + 1^2 + 1^2)}\sqrt{(1^2 + 2^2 + 1^2)}} = \frac{1}{6}, \quad i.e., \quad \theta = \cos^{-1}\left(\frac{1}{6}\right).$$

PROBLEMS 3.2

1. If $\mathbf{A} = \mathbf{I} + 2\mathbf{J} + 3\mathbf{K}$, $\mathbf{B} = -\mathbf{I} + 2\mathbf{J} + \mathbf{K}$ and $\mathbf{C} = 3\mathbf{I} + \mathbf{J}$, find t such that $\mathbf{A} + t\mathbf{B}$ is perpendicular to $\mathbf{C}$.

2. (*i*) Show that $\left(\dfrac{\mathbf{A}}{a^2} - \dfrac{\mathbf{B}}{b^2}\right)^2 = \left(\dfrac{\mathbf{A} - \mathbf{B}}{ab}\right)^2$.

   (*ii*) Interpret geometrically $(\mathbf{C} - \mathbf{A}) \cdot (\mathbf{B} - \mathbf{C}) = 0$.

3. If $|\mathbf{A} + \mathbf{B}| = |\mathbf{A} - \mathbf{B}|$, show that $\mathbf{A}$ and $\mathbf{B}$ are mutually perpendicular.

4. If $\mathbf{A} = \mathbf{I} + 2\mathbf{J} - 3\mathbf{K}$ and $\mathbf{B} = 3\mathbf{I} - \mathbf{J} + 2\mathbf{K}$, show that $\mathbf{A} + \mathbf{B}$ is perpendicular to $\mathbf{A} - \mathbf{B}$. Also calculate the angle between $2\mathbf{A} + \mathbf{B}$ and $\mathbf{A} + 2\mathbf{B}$.
5. Show that the three concurrent lines with direction cosines (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) are coplanar if
$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0.$$
6. Find the projection of the vector $\mathbf{I} - 2\mathbf{J} + \mathbf{K}$ on $4\mathbf{I} - 4\mathbf{J} + 7\mathbf{K}$.
7. The projection of a line on the coordinate axes are 12, 4, 3. Find the length and direction cosines of the line. *(Rajasthan, 2006)*
8. Show (by vector methods) that the mid-point of the hypotenuse of a right-angled triangle is equidistant from its vertices.
9. Prove (by vector methods) that the angle in a semi-circle is a right angle.
10. Show (by vector methods) that the diagonals of a rhombus intersect at right angles.
11. Show that the altitudes of a triangle meet in a point (called the *orthocentre*).
12. *ABCD* is a tetrahedron having the edges *BC* and *AC* at right angles to opposite edges *AD* and *BD* respectively. Show that the third pair of opposite edges *AB* and *CD* are also at right angles.
13. Find the angle between the lines whose direction cosines are given by the equations $l + m + n = 0$, $l^2 + m^2 + n^2 = 0$. *(Rajasthan, 2005)*
14. Show that the lines whose direction cosines are given by the equations $4lm - 3mn - nl = 0$, and $3l + m + 2n = 0$ are perpendicular. *(Anna, 2005)*
15. Show that the lines whose direction cosines are given by the equations $l + m + n = 0$, $al^2 + bm^2 + cn^2 = 0$ are (*i*) perpendicular, if $a + b + c = 0$, (*ii*) parallel, if $a^{-1} + b^{-1} + c^{-1} = 0$.
16. Show that the straight lines whose direction cosines are given by the equations
$$al + bm + cn = 0, fmn + gnl + hlm = 0 \text{ are } (i) \text{ perpendicular if } \frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$$ *(Osmania, 2003)*
(*ii*) parallel if $\sqrt{af} \pm \sqrt{bg} \pm \sqrt{ch} = 0$.
17. Show that the angle between any two diagonals of a cube is $\cos^{-1} 1/3$. *(V.T.U., 2009 ; Assam, 1999)*
18. (l_1, m_1, n_1), (l_2, m_2, n_2) and (l_3, m_3, n_3) are the direction cosines of three mutually perpendicular lines. Prove that the line whose d.c.'s are proportional to $l_1 + l_2 + l_3$, $m_1 + m_2 + m_3$, $n_1 + n_2 + n_3$ makes equal angles with the axes. *(V.T.U. 2003)*
19. *AB*, *BC* are the diagonals of adjacent faces of a rectangular box with its centre at the origin *O*, its edges are parallel to the axes. If the angles *BOC*, *COA* and *AOB* are equal to θ, ϕ, ψ respectively, prove that
$$\cos\theta + \cos\phi + \cos\psi = -1.$$

3.6 VECTOR, OR CROSS PRODUCT

(1) Definition. *The vector, or cross product of two vectors* **A** *and* **B** *is defined as a vector such that*

(*i*) *its magnitude is ab* sin θ, θ *being the angle between* **A** *and* **B**,

(*ii*) *its direction is perpendicular to the plane of* **A** *and* **B**,

and (*iii*) *it forms with* **A** *and* **B** *a right-handed system.*

If **N** be a unit vector normal to the plane of **A** and **B** (**A, B, N** forming a right-handed system), then
$$\mathbf{A} \times \mathbf{B} = ab \sin\theta\, \mathbf{N}.$$

(2) Geometrical interpretation. **A × B** *represents twice the vector area of the triangle having the vectors* **A** *and* **B** *as its adjacent sides.*

If **N** be a unit vector normal to the plane of the triangle *OAB*, then
$$\mathbf{A} \times \mathbf{B} = ab \sin\theta\, \mathbf{N}$$
$$= 2\left(\frac{1}{2} ab \sin\theta\right) \mathbf{N} = 2\Delta\, OAB\, \mathbf{N} = 2\Delta\, \vec{OA}\, B.$$

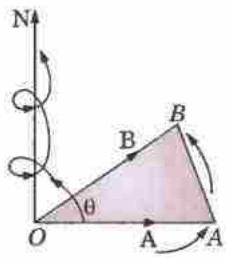


Fig. 3.19

(3) Properties and other results

I. Vector product of two vectors is not commutative,
$$\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}. \text{ In fact, } \mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}.$$

for $\mathbf{A} \times \mathbf{B} = ab \sin \theta\, \mathbf{N}$ or $2\Delta\, \overrightarrow{OAB}$.

and $\mathbf{B} \times \mathbf{A} = ab \sin(-\theta)\, \mathbf{N} = -ab \sin \theta\, \mathbf{N}$ or $2\Delta\, \overrightarrow{OBA}$.

II. The necessary and sufficient condition for two non-zero vectors to be parallel is that their vector product should be zero.

When the vectors **A** and **B** are parallel, the angle θ between them is 0 and 180° so that $\sin\theta = 0$, and as such $\mathbf{A} \times \mathbf{B} = \mathbf{0}$.

Conversely, when $\mathbf{A} \times \mathbf{B} = \mathbf{0}$; $ab \sin \theta = 0$

i.e., $\sin \theta = 0$ $(\because\ a \neq 0, b \neq 0)$

or $\theta = 0$ or 180°. In particular, $\mathbf{A} \times \mathbf{A} = 0$.

III. For the orthonormal vector trial **I, J, K,** we have the relations :

$$\mathbf{I} \times \mathbf{I} = \mathbf{J} \times \mathbf{J} = \mathbf{K} \times \mathbf{K} = \mathbf{0}$$
$$\mathbf{I} \times \mathbf{J} = \mathbf{K}, \quad \mathbf{J} \times \mathbf{I} = -\mathbf{K}$$
$$\mathbf{J} \times \mathbf{K} = \mathbf{I}, \quad \mathbf{K} \times \mathbf{J} = -\mathbf{I}$$
$$\mathbf{K} \times \mathbf{I} = \mathbf{J}, \quad \mathbf{I} \times \mathbf{K} = -\mathbf{J}.$$

IV. Relation between scalar and vector products.

We have $(\mathbf{A} \cdot \mathbf{B})^2 = a^2b^2 \cos\theta = a^2b^2 - a^2b^2 \sin^2\theta = a^2b^2 - (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{B})$

$$(\mathbf{A} \times \mathbf{B})^2 = \mathbf{A}^2\mathbf{B}^2 - (\mathbf{A} \cdot \mathbf{B})^2.$$

V. Vector product of two vectors is distributive

i.e., $(\mathbf{A} + \mathbf{B}) \times \mathbf{C} = \mathbf{A} \times \mathbf{C} + \mathbf{B} \times \mathbf{C}$.

VI. Analytical expression for the vector product.

If $\mathbf{A} = a_1\mathbf{I} + a_2\mathbf{J} + a_3\mathbf{K}, \mathbf{B} = b_1\mathbf{I} + b_2\mathbf{J} + b_3\mathbf{K}$ *then* $\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

For we get

$$\mathbf{A} \times \mathbf{B} = (a_2b_3 - a_3b_2)\mathbf{I} + (a_3b_1 - a_1b_3)\mathbf{J} + (a_1b_2 - a_2b_1)\mathbf{K}$$

whence follows the required result.

Example 3.12. *If* $\mathbf{A} = 4\mathbf{I} + 3\mathbf{J} + \mathbf{K}$, $\mathbf{B} = 2\mathbf{I} - \mathbf{J} + 2\mathbf{K}$, *find a unit vector* **N** *perpendicular to vectors* **A** *and* **B** *such that* **A, B, N** *form a right handed system. Also find the angle between the vectors A and B.*

Solution. Since $\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ 4 & 3 & 1 \\ 2 & -1 & 2 \end{vmatrix} = 7\mathbf{I} - 6\mathbf{J} - 10\mathbf{K}$

and $|\mathbf{A} \times \mathbf{B}| = \sqrt{[(7)^2 + (-6)^2 + (-10)^2]} = \sqrt{185}$

$\therefore$ Unit vector $\mathbf{N} \perp$ to **A** and **B** $= \dfrac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|} = (7\mathbf{I} - 6\mathbf{J} - 10\mathbf{K})/\sqrt{185}$

Also $a = \sqrt{4^2 + 3^2 + 1^2} = \sqrt{26}$ and $b = 3$.

If θ be the angle between **A** and **B**, then $|\mathbf{A} \times \mathbf{B}| = ab \sin\theta$, *i.e.*, $\sin\theta = |\mathbf{A} \times \mathbf{B}|/ab$

Thus $\sin\theta = \sqrt{185}/3\sqrt{26}$ whence $\theta = 62° 40'$.

Example 3.13. (*i*) *Prove that the area of the triangle whose vertices are* **A, B, C** *is*

$$\frac{1}{2}\ |\mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A} + \mathbf{A} \times \mathbf{B}|$$

(*ii*) *Calculate the area of the triangle whose vertices are A(1, 0 – 1), B(2, 1, 5) and C(0, 1, 2).*

Solution. (*i*) Let $A(\mathbf{A})$, $B(\mathbf{B})$, $C(\mathbf{C})$ be the vertices of the triangle ABC (Fig. 3.20) and O, the origin so that

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \mathbf{C} - \mathbf{B}$$

and $$\vec{BA} = \vec{OA} - \vec{OB} = \mathbf{A} - \mathbf{B}$$

$\therefore$ Vector area of ΔABC

$$= \frac{1}{2}[\vec{BC} \times \vec{BA}] = \frac{1}{2}[(\mathbf{C} - \mathbf{B}) \times (\mathbf{A} - \mathbf{B})]$$

$$= \frac{1}{2}[\mathbf{C \times A - C \times B - B \times A + B \times B}]$$

$$= \frac{1}{2}[\mathbf{B \times C + C \times A + A \times B}] \quad [\because \ \mathbf{B \times B = 0}]$$

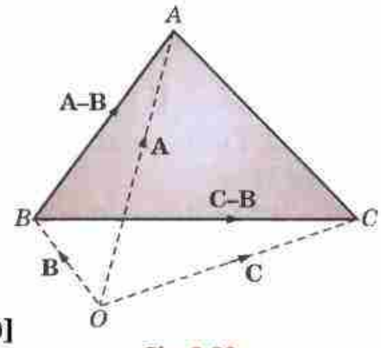


Fig. 3.20

Thus area of $$\Delta ABC = \frac{1}{2}\,|\mathbf{B \times C + C \times A + A \times B}|.$$

(*ii*) Let O be the origin so that

$$\vec{OA} = \mathbf{I} - \mathbf{K},\ \vec{OB} = 2\mathbf{I} + \mathbf{J} + 5\mathbf{K} \text{ and } \vec{OC} = \mathbf{J} + 2\mathbf{K}$$

Then $$\vec{BC} = \vec{OC} - \vec{OB} = -2\mathbf{I} - 3\mathbf{K}$$

and $$\vec{BA} = \vec{OA} - \vec{OB} = -\mathbf{I} - \mathbf{J} - 6\mathbf{K}$$

$$\therefore \text{ Vector area of } \Delta ABC = \frac{1}{2}(\vec{BC} \times \vec{BA}) = \frac{1}{2}\begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ -2 & 0 & -3 \\ -1 & -1 & -6 \end{vmatrix}$$

Thus area of $$\Delta ABC = \frac{1}{2}\,|-3\mathbf{I} - 9\mathbf{J} + 2\mathbf{K}| = \frac{1}{2}\sqrt{94}.$$

Example 3.14. *In a triangle ABC; D, E, F are the mid-points of the sides BC, CA, AB; prove that*

$$\Delta DEF = \Delta FCE = \frac{1}{4}\Delta ABC.$$

Solution. Take B as the origin and let the position vectors of C and A be $\mathbf{C}$ and $\mathbf{A}$ (Fig 3.21); so that the position vectors of D, E, F are

$$\mathbf{C}/2,\ (\mathbf{C} + \mathbf{A})/2,\ \mathbf{A}/2.$$

$$\therefore \quad \Delta \vec{DEF} = \frac{1}{2}(\vec{DE} \times \vec{DF}) = \frac{1}{2}\left(\frac{\mathbf{C}+\mathbf{A}}{2} - \frac{\mathbf{C}}{2}\right)\left(\frac{\mathbf{A}}{2} - \frac{\mathbf{C}}{2}\right)$$

$$= \frac{1}{8}[\mathbf{A} \times (\mathbf{A} - \mathbf{C})] = \frac{1}{8}\ \mathbf{C} \times \mathbf{A} = \frac{1}{4}\Delta \vec{ABC}$$

$$\Delta \vec{FCE} = \frac{1}{2}(\vec{FC} \times \vec{FE}) = \frac{1}{2}[\mathbf{C} - \mathbf{A}/2) \times \mathbf{C}/2]$$

$$= \frac{1}{8}\ \mathbf{C} \times \mathbf{A} = \frac{1}{4}\Delta \vec{ABC}. \text{ Hence the result.}$$

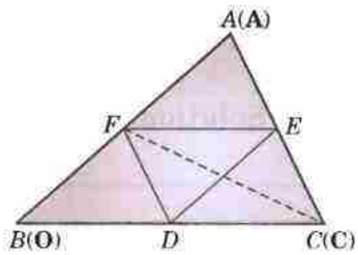


Fig. 3.21

Example 3.15. *Prove that*
(*i*) *sin (A + B) = sin A cos B + cos A sin B.*
(*ii*) *cos (A + B) = cos A cos B – sin A sin B.*

Solution. Let $\mathbf{I}, \mathbf{J}$ denote unit vectors along two perpendicular lines OX, OY so that

$$\mathbf{I}^2 = \mathbf{J}^2 = 1,\ \mathbf{I}.\ \mathbf{J} = 0$$

and $$\mathbf{I} \times \mathbf{I} = \mathbf{J} \times \mathbf{J} = \mathbf{0}$$

Let $$\angle POX = A \text{ and } \angle XOQ = B,$$

so that $$\angle POQ = A + B.$$

If $OP = p$ and $OQ = q$, then the coordinates of P are $(p \cos A, -p \sin A)$ and those of Q are $(q \cos B, q \sin B)$ so that

$$\overrightarrow{OP} = (p \cos A)\mathbf{I} - (p \sin A)\mathbf{J}$$

$$\overrightarrow{OQ} = (q \cos B)\mathbf{I} + (q \sin B)\mathbf{J}$$

Then $\quad |\overrightarrow{OP} \times \overrightarrow{OQ}| = |[(p \cos A)\mathbf{I} - (p \sin A)\mathbf{J}] \times [(q \cos B)\mathbf{I} + (q \sin B)\mathbf{J}|$

$$= pq\ |\cos A \sin B\ (\mathbf{I} \times \mathbf{J}) - \sin A \cos B\ (\mathbf{J} \times \mathbf{I})|$$

$$= pq\ (\cos A \sin B + \sin A \cos B) \text{ for } |\ \mathbf{I} \times \mathbf{J}\ | = 1$$

Also $\quad |\overrightarrow{OP} \times \overrightarrow{OQ}| = pq \sin (A + B)$. Equating the two expressions, we get (*i*).

Similarly, (*ii*) follows from $\overrightarrow{OP} \cdot \overrightarrow{OQ} = pq \cos (A + B)$.

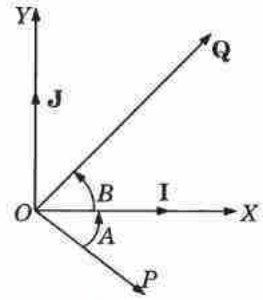


Fig. 3.22

Example 3.16. *In any triangle ABC, prove that*

(*i*) $a/\sin A = b/\sin B = c/\sin C$. (*Sine formula*)

(*ii*) $a = b \cos C + c \cos B$. (*Projection formula*)

(*iii*) $a^2 = b^2 + c^2 - 2bc \cos A$. (*Cosine formula*)

Solution. From ΔABC, we have $\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB} = 0$

or $$\overrightarrow{CA} + \overrightarrow{AB} = -\overrightarrow{BC} \qquad ...(\lambda)$$

(*i*) Multiplying (λ) vectorially by $\overrightarrow{AB}$, we get

$$\overrightarrow{CA} \times \overrightarrow{AB} = -\overrightarrow{BC} \times \overrightarrow{AB}$$

or $$|\overrightarrow{CA} \times \overrightarrow{AB}| = |\overrightarrow{BC} \times \overrightarrow{AB}|$$

$\therefore$ $$bc \sin (\pi - A) = ac \sin (\pi - B)$$

or $$a/\sin A = b/\sin B.$$

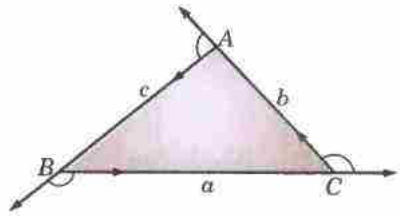


Fig. 3.23

Similarly, multiplying (λ) vectorially by $\overrightarrow{CA}$, we get

$a/\sin A = c/\sin C$, whence follows the result.

(*ii*) Multiplying (λ) scalarly by $\overrightarrow{BC}$, we get $\overrightarrow{CA} \cdot \overrightarrow{BC} + \overrightarrow{AB} \cdot \overrightarrow{BC} = -(\overrightarrow{BC})^2$

$\therefore \quad ba \cos (\pi - C) + ca \cos (\pi - B) = -a^2 \quad$ or $\quad a = b \cos C + c \cos B$.

(*iii*) Squaring (λ), we get

$$(\overrightarrow{CA})^2 + (\overrightarrow{AB})^2 + 2\overrightarrow{CA} \cdot \overrightarrow{AB} = (\overrightarrow{BC})^2$$

i.e., $$b^2 + c^2\ 2bc \cos (\pi - A) = a^2 \text{ or } a^2 = b^2 + c^2 - 2bc \cos A.$$

PROBLEMS 3.3

1. Given $\mathbf{A} = 2\mathbf{I} + 2\mathbf{J} - \mathbf{K}$, $\mathbf{B} = 6\mathbf{I} - 3\mathbf{J} + 2\mathbf{K}$, find $\mathbf{A} \times \mathbf{B}$ and the unit vector perpendicular to both **A** and **B**. Also determine the sine of the angle between **A** and **B**.
2. If **A** and **B** are unit vectors and θ is the angle between them, show that $\sin \dfrac{\theta}{2} = \dfrac{1}{2} |\mathbf{A} - \mathbf{B}|$.
3. Find a unit vector normal to the plane of $\mathbf{A} = 3\mathbf{I} - 2\mathbf{J} + 4\mathbf{K}$ and $\mathbf{B} = \mathbf{I} + \mathbf{j} - 2\mathbf{K}$.
4. For any vector **A**, show that $|\mathbf{A} \times \mathbf{I}|^2 + |\mathbf{A} \times \mathbf{J}|^2 + |\mathbf{A} \times \mathbf{K}|^2 = 2\ |\mathbf{A}|^2$.
5. By vector method, find the area of the triangle whose vertices are $(3, -1, 2)$, $(1, -1, -3)$ and $(4, -3, 1)$.
6. (*a*) Prove that the vector area of the quadrilateral *ABCD* is $\dfrac{1}{2}\ \overrightarrow{AC} \times \overrightarrow{BD}$.

   (*b*) If $3\mathbf{I} + \mathbf{J} - 2\mathbf{K}$ and $\mathbf{I} - 3\mathbf{J} - 4\mathbf{K}$ are the diagonals of a parallelogram. Find its area.

7. Given vectors $\mathbf{A} = \mathbf{I} - 3\mathbf{J} + 2\mathbf{K}$ and $\mathbf{B} = 2\mathbf{I} + \mathbf{J} - \mathbf{K}$. Find the projection of $\mathbf{A} \times \mathbf{B}$ parallel to $5\mathbf{I} - \mathbf{K}$.
8. If $\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{0}$, prove that $\mathbf{A} \times \mathbf{B} = \mathbf{B} \times \mathbf{C} = \mathbf{C} \times \mathbf{A}$, and interpret it geometrically.
9. Show that the perpendicular distance of the point **C** from the line joining **A** and **B** is $|\mathbf{B} \times \mathbf{C} + \mathbf{C} \times \mathbf{A} + \mathbf{A} \times \mathbf{B}| \div |\mathbf{B} - \mathbf{A}|$
10. In AC, diagonal of the parallelogram $ABCD$, a point **P** is taken. Prove that $\Delta BAP = \Delta DAP$.
11. Prove by vector methods, that
(i) $\sin (A - B) = \sin A \cos B - \cos A \sin B$; (ii) $\cos (A - B) = \cos A \cos B + \sin A \sin B$. (*Cochin, 1999*)
12. In any triangle ABC, prove by vector methods, that
(i) $b = c \cos A + a \cos C$; (ii) $c^2 = a^2 + b^2 - 2ab \cos C$.

3.7 PHYSICAL APPLICATIONS

(1) Work done as a scalar product. If constant force **F** acting on a particle displaces it from the position A to position B, then

Work done = (resolved part of F in the direction of AB). AB

$$= F \cos \theta \,.\, AB = \mathbf{F} \,.\, \overrightarrow{AB}$$

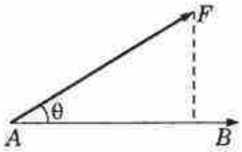


Fig. 3.24

Thus, the work done by a constant force is the scalar (or dot) product of the vectors representing the force and the displacement.

Example 3.17. *Constant forces* $\mathbf{P} = 2\mathbf{I} - 5\mathbf{J} + 6\mathbf{K}$ *and* $\mathbf{Q} = -\mathbf{I} + 2\mathbf{J} - \mathbf{K}$ *act on a particle. Determine the work done when the particle is displaced from A to B the position vectors of A and B being* $4\mathbf{I} - 3\mathbf{J} - 2\mathbf{K}$ *and* $6\mathbf{I} + \mathbf{J} - 3\mathbf{K}$ *respectively.*

Solution. Resultant force $\quad \mathbf{F} = \mathbf{P} + \mathbf{Q} = \mathbf{I} - 3\mathbf{J} + 5\mathbf{K}$

and $\quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (6\mathbf{I} + \mathbf{J} - 3\mathbf{K}) - (4\mathbf{I} - 3\mathbf{J} - 2\mathbf{K}) = 2\mathbf{I} + 4\mathbf{J} - \mathbf{K}$

$\therefore$ *Work done* $\quad = \mathbf{F} \,.\, \overrightarrow{AB} = (\mathbf{I} - 3\mathbf{J} + 5\mathbf{K}) \,.\, (2\mathbf{I} + 4\mathbf{J} - \mathbf{K})$

$\quad = 1 \,.\, 2 - 3 \,.\, 4 + 5 \,.\, (-1) = -15$ units.

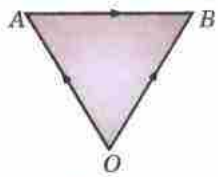


Fig. 3.25

(2) Normal flux. Consider the flow of a liquid through an element of area δs with a velocity **V** inclined at an angle θ to the out ward unit normal **N** to the surface δs (Fig. 3.26).

$\therefore$ Normal flux of the liquid through δs in unit time

$$\mathbf{V} \cos \theta \,.\, \delta s = \mathbf{V} \,.\, \mathbf{N} \delta s.$$

Thus, *the rate of normal flux per unit area* = $\mathbf{V} \,.\, \mathbf{N}$

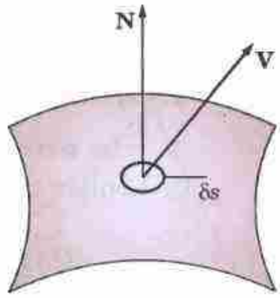


Fig. 3.26

Obs. We can also apply this result to the case of electric or magnetic flux.

(3) Moment of a force about a point. Suppose the moment of the force **F** acting at the point P about the point A is required.

Draw $AM \perp$ the line of action of **F** (Fig. 3.27). If θ be the angle between $\overrightarrow{AP}$ and **F** and **N** be a unit vector $\perp$ to their plane, then $\overrightarrow{AP} \times \mathbf{F} = (AP \,.\, F \sin \theta)\, \mathbf{N} = F(AP \sin \theta)\, \mathbf{N} = (F \,.\, AM)\, \mathbf{N}$

Clearly, (i) the magnitude of $\overrightarrow{AP} \times \mathbf{F} = F \,.\, AM$ which is the numerical measure of the moment of **F** about A.

and (ii) the direction of $\overrightarrow{AP} \times \mathbf{F}$ is the direction of the moment of **F** about A.

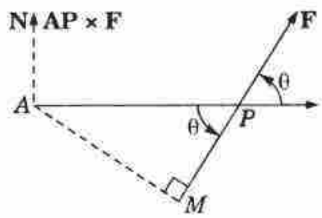


Fig. 3.27

Hence the moment (or torque) of **F** about A is $\overrightarrow{AP} \times \mathbf{F}$.

Example 3.18. *Find the torque about the point* $2\mathbf{I} + \mathbf{J} - \mathbf{K}$ *of a force represented by* $4\mathbf{I} + \mathbf{K}$ *acting through the point* $\mathbf{I} - \mathbf{J} + 2\mathbf{K}$.

Solution. Let O be the origin and P be the point, moment about which of the force $\overrightarrow{AB}$ through A, is required (Fig. 3.28).

$\therefore$ $\quad \vec{OP} = 2\mathbf{I} + \mathbf{J} - \mathbf{K},$

$\vec{OA} = \mathbf{I} - \mathbf{J} + 2\mathbf{K}$, and $\vec{AB} = 4\mathbf{I} + \mathbf{K}$.

Then, $\quad \vec{PA} = \vec{OA} - \vec{OP} = -\mathbf{I} - 2\mathbf{J} + 3\mathbf{K}$

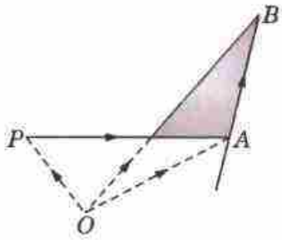


Fig. 3.28

$\therefore$ Moment of the force $\vec{AB}$ about P

$$= \vec{PA} \times \vec{AB} = (-\mathbf{I} - 2\mathbf{J} + 3\mathbf{K}) \times (4\mathbf{I} + \mathbf{K})$$

$$= \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ -1 & -2 & 3 \\ 4 & 0 & 1 \end{vmatrix} = -2\mathbf{I} + 13\mathbf{J} + 8\mathbf{K}$$

$\therefore$ Magnitude of the moment $= \sqrt{(4 + 169 + 64)} = 15.4$

(4) Moment of a force about a line.

Def. *The moment of a force* **F** *about a line* **D** *is the resolved part along* **D** *of the moment of* **F** *about any point on* **D**.

Example 3.19. *Find the moment about a line through the origin having direction of* $2\mathbf{I} + 2\mathbf{J} + \mathbf{K}$*, due to a 30 kg force acting at a point* $(-4, 2, 5)$ *in the direction of* $12\mathbf{I} - 4\mathbf{J} - 3\mathbf{K}$.

Solution. Let **D** be the given line through the origin O and **F** the force through **A**$(-4, 2, 5)$.

Clearly, $\quad \vec{OA} = -4\mathbf{I} + 2\mathbf{J} + 5\mathbf{K}$

and the force $\quad \mathbf{F} = 30\left(\dfrac{12\mathbf{I} - 4\mathbf{J} - 3\mathbf{K}}{13}\right)$

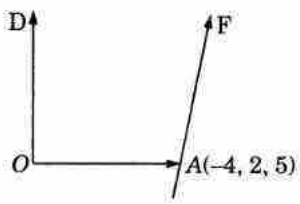


Fig. 3.29

$\therefore$ Moment of **F** about $O = \vec{OA} \times \mathbf{F}$

$$= \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ -4 & 2 & 5 \\ \dfrac{360}{13} & \dfrac{-120}{13} & \dfrac{-90}{13} \end{vmatrix} = \frac{60}{13}(7\mathbf{I} + 24\mathbf{J} - 4\mathbf{K})$$

Thus the moment of **F** *about the line* **D**

= resolved part of the moment of **F** about O along **D**,

i.e., $\quad \dfrac{60}{13}(7\mathbf{I} + 24\mathbf{J} - 4\mathbf{K}) \cdot \hat{\mathbf{D}}$

$$= \frac{60}{13}(7\mathbf{I} + 24\mathbf{J} - 4\mathbf{K}) \cdot \frac{2\mathbf{I} + 2\mathbf{J} + \mathbf{K}}{\sqrt{(4 + 4 + 1)}} = \frac{20}{13}(7 \times 2 + 24 \times 2 - 4 \times 1) = 89.23.$$

(5) Angular velocity of a rigid body

Let a rigid body be rotating about the axis OM with angular velocity ω radians per second (Fig. 3.30). Let P be a point of the body such that $\vec{OP} = \mathbf{R}$ and $\angle MOP = \theta$. Draw $PM \perp OM$.

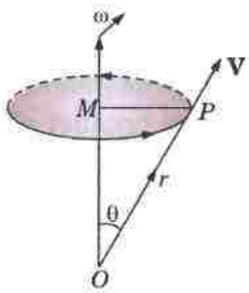


Fig. 3.30

Now if **N** be a unit vector $\perp \vec{\omega}\, \mathbf{R}$ then

$$\vec{\omega} \times \mathbf{R} = \omega r \sin\theta \,.\, \mathbf{N} = \omega MP \,.\, \mathbf{N}$$

= (speed of **P**) **N**

= velocity **V** of **P** in a direction $\perp$ to the plane MOP.

Hence $\quad \mathbf{V} = \vec{\omega} \times \mathbf{R}$.

Example 3.20. *A rigid body is spinning with angular velocity 27 radians per second about an axis parallel to* $2\mathbf{I} + \mathbf{J} - 2\mathbf{K}$ *passing through the point* $\mathbf{I} + 3\mathbf{J} - \mathbf{K}$. *Find the velocity of the point of the body whose position vector is* $4\mathbf{I} + 8\mathbf{J} + \mathbf{K}$.

Solution. Unit vector along the direction of $\vec{\omega} = \dfrac{2\mathbf{I} + \mathbf{J} - 2\mathbf{K}}{\sqrt{(4+1+4)}} = \dfrac{1}{3}(2\mathbf{I} + \mathbf{J} - 2\mathbf{K})$

$\therefore$ Angular velocity $\vec{\omega} = \dfrac{27}{3}(2\mathbf{I} + \mathbf{J} - 2\mathbf{K}) = 9(2\mathbf{I} + \mathbf{J} - 2\mathbf{K})$

Let A be the point $\mathbf{I} + 3\mathbf{J} - \mathbf{K}$ and the point P of the body be $(4\mathbf{I} + 8\mathbf{J} - \mathbf{K})$ so that

$$\vec{AP} = (4\mathbf{I} + 8\mathbf{J} + \mathbf{K}) - (\mathbf{I} + 3\mathbf{J} - \mathbf{K}) = 3\mathbf{I} + 5\mathbf{J} + 2\mathbf{K}$$

$\therefore$ Velocity vector of $P = \mathbf{V} = \vec{\omega} \times \vec{AP} = 9(2\mathbf{I} + \mathbf{J} - 2\mathbf{K}) \times (3\mathbf{I} + 5\mathbf{J} + 2\mathbf{K})$

$$= 9\begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ 2 & 1 & -2 \\ 3 & 5 & 2 \end{vmatrix} = 9(12\mathbf{I} - 10\mathbf{J} + 7\mathbf{K})$$

and its magnitude $9\sqrt{(144+100+49)} = 9\sqrt{293}$.

PROBLEMS 3.4

1. A particle acted on by constant forces $4\mathbf{I} + \mathbf{J} - 3\mathbf{K}$ and $3\mathbf{I} + \mathbf{J} - \mathbf{K}$ is displace from the point $\mathbf{I} + 2\mathbf{J} + 3\mathbf{K}$ to the point $5\mathbf{I} + 4\mathbf{J} + \mathbf{K}$. Find the total work done by the forces.
2. Forces $2\mathbf{I} - 5\mathbf{J} + 6\mathbf{K}$, $-\mathbf{I} + 2\mathbf{J} - \mathbf{K}$ and $2\mathbf{I} + 7\mathbf{J}$ act on a particle P whose position vector is $4\mathbf{I} - 3\mathbf{J} - 2\mathbf{K}$. Determine the work done by the forces in a displacement of the particle to the point Q (6, 1, –3).
   Also find the vector moment of the resultant of three forces acting at P about the point Q.
3. Forces of magnitudes 5, 3 1 units act in the directions $6\mathbf{I} + 2\mathbf{J} + 3\mathbf{K}$, $3\mathbf{I} - 2\mathbf{J} + 6\mathbf{K}$, $2\mathbf{I} - 3\mathbf{J} - 6\mathbf{K}$ respectively on a particle which is displaced from the point (2, 1, –3) to (5, –1, 1). Find the work done by the forces.
4. The point of application of the force (–2, 4, 7) is displaced from the point (3, –5, 1) to the point (5, 9, 7). But the force is suddenly halved when the point of application moves half the distance. Find the work done.
5. A force $\mathbf{F} = 3\mathbf{I} + 2\mathbf{J} - 4\mathbf{K}$ is applied at the point (1, – 1, 2). Find the moment of the force about the point (2, – 1, 3). *(Assam, 1999)*
6. A force with components (5, – 4, 2) acts at a point P which is at a distance 3 units from the origin. If the moment of the force about origin has components (12, 8, – 14), find the co-ordinates of P.
7. Find the moment of the force $\mathbf{F} = 2\mathbf{I} + 2\mathbf{J} - \mathbf{K}$ acting at the point (1, – 2, 1) about z-axis.
8. A force of 10 kg acts in a direction equally inclined to the co-ordinate axes through the point (3, – 2, 5). Find the magnitude of the moment of the force about a line through the origin and whose direction ratios are (2, –3, 6).
9. A rigid body is rotating at 2.5 radians per second about an axis OR, where R is the point $2\mathbf{I} - 2\mathbf{J} + \mathbf{K}$ relative to O. Find the velocity of the particle of the body at the point $4\mathbf{I} + \mathbf{J} + \mathbf{K}$. (All lengths are in cm).

3.8 PRODUCTS OF THREE OR MORE VECTORS

With any three vectors **A**, **B**, **C**, we can form the products (**A** . **B**) **C**, (**A** × **B**) . **C** and (**A** × **B**) × **C**. The first being the product of a scalar **A** . **B** and a vector **C**, represents a vector in the direction of **C**. The second being the scalar product of vectors **A** × **B** and **C**, represents a scalar and is usually called the *scalar product of three vectors*. The third being the vector product of the vectors **A** × **B** and **C**, represents a vector and is usually known as the *vector product of three vectors*.

The reader must, however, note that the products of the form **A** . (**B** . **C**), **A** × (**B** . **C**) and **A**(**B** × **C**) are meaningless.

In practical applications, we seldom come across products of more than three vectors. Such products if they occur can, in general, be reduced by using successively the expansion formula for vector triple products. As an illustration, we shall consider two products (**A** × **B**). (**C** × **D**) and (**A** × **B**) × (**C** × **D**) of any four vectors, the former being a scalar and a latter a vector.

3.9 SCALAR PRODUCT OF THREE VECTORS

(1) Definition. *If* **A, B, C** *be any three vectors then the scalar or dot product of* **A × B** *with* **C** *is called the scalar product of the three vectors* **A, B, C** *and is written as* **(A × B) . C** *or* **[ABC]**.

No ambiguity can arise by omitting the brackets in $(A \times B) . C$ as $A \times (B . C)$ would be meaningless .

(2) Geometrical interpretation. *The Product* **A × B . C** *represents numerically the volume of a parallelopiped having* **A, B, C** *as coterminous edges.*

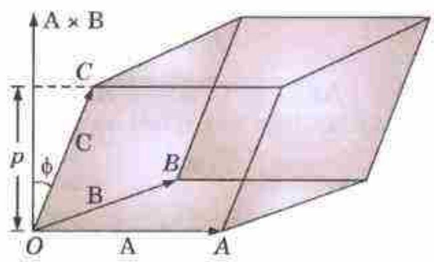


Fig. 3.31

Consider a parallelopiped with $\vec{OA} = \mathbf{A}$, $\vec{OB} = \mathbf{B}$, $\vec{OC} = \mathbf{C}$ as coterminous edges (Fig. 3.31).

Let V be its volume, α the area of each of the two faces parallel to the vectors **A** and **B** and p the perpendicular distance between these faces.

Then $|\mathbf{A} \times \mathbf{B}| = \alpha$ and $|\mathbf{C}| \cos \phi = p$ or $-p$ according as **A, B, C** form a right-handed or left-handed triad.

$$\therefore \qquad \mathbf{A} \times \mathbf{B} . \mathbf{C} = |\mathbf{A} \times \mathbf{B}| . |\mathbf{C}| \cos \phi = \pm \alpha p = \pm V.$$

Thus **[ABC]** *= V or – V according as* **A, B, C** *form a right-handed or left-handed triad.*

(Kerala, 1990; J.N.T.U., 1988)

In particular, for an orthonormal right-handed vector triad **I, J, K,**

$$[\mathbf{IJK}] = \mathbf{I} \times \mathbf{J} . \mathbf{K} = \mathbf{K} . \mathbf{K} = I.$$

(3) Properties and other results.

I. The condition for three vectors to be coplanar is that their scalar triple product should vanish.

If three vectors **A, B, C** anti coplanar, then the volume of the parallelopiped so formed is zero, *i.e.*, $[\mathbf{ABC}] = 0$.

II. If any two vectors of a scalar triple product are equal, the product vanishes, i.e., **[ABC]** = 0 when either **A = B** or **B = C**, or **C = A**, for in this case the parallelopiped has zero volume.

III. **Two important rules** *(for evaluating a scalar triple product). Every scalar triple product*

(i) is independent of the position of the dot or cross.

and (ii) depends upon the cyclic order of the vectors.

It is easy to note that if **A, B, C** is a right-handed triad so are **B, C, A** and **C, A, B.**

Moreover a parallelopiped having **A, B, C** as coterminous edges is the same as that having **B, C, A** or **C, A, B** as coterminous edges.

Thus, if V be the volume of this parallelopiped,

$$\mathbf{A} \times \mathbf{B} . \mathbf{C} = V, \mathbf{B} \times \mathbf{C} . \mathbf{A} = V, \mathbf{C} \times \mathbf{A} . \mathbf{B} = V$$

Also, since **A . B = B . A**, we have

$$\mathbf{C} . \mathbf{A} \times \mathbf{B} = \mathbf{A} \times \mathbf{B} . \mathbf{C} = V$$
$$\mathbf{A} . \mathbf{B} \times \mathbf{C} = \mathbf{B} \times \mathbf{C} . \mathbf{A} = V$$
$$\mathbf{B} . \mathbf{C} \times \mathbf{A} = \mathbf{C} \times \mathbf{A} . \mathbf{B} = V$$

Thus
$$\left.\begin{array}{l} \mathbf{AA} \times \mathbf{B} . \mathbf{C} = \mathbf{A} . \mathbf{B} \times \mathbf{C} \\ \mathbf{B} \times \mathbf{C} . \mathbf{A} = \mathbf{B} . \mathbf{C} \times \mathbf{A} \\ \mathbf{C} \times \mathbf{A} . \mathbf{B} = \mathbf{C} . \mathbf{A} \times \mathbf{B} \end{array}\right\} = V \qquad ...(\alpha)$$

Further a right-handed triad becomes left-handed when the cyclic order of the vectors is changed. Therefore **A, C, B; B, A, C; C, B, A** being left-handed triads, it follows that

$$\mathbf{A} \times \mathbf{C} . \mathbf{B} = -V, \mathbf{B} \times \mathbf{A} . \mathbf{C} = -V, \mathbf{C} \times \mathbf{B} . \mathbf{A} = -V.$$

Thus
$$\left.\begin{array}{l} \mathbf{AA} \times \mathbf{C} . \mathbf{B} = \mathbf{A} . \mathbf{C} \times \mathbf{B} \\ \mathbf{B} \times \mathbf{A} . \mathbf{C} = \mathbf{B} . \mathbf{A} \times \mathbf{C} \\ \mathbf{C} \times \mathbf{B} . \mathbf{A} = \mathbf{C} . \mathbf{B} \times \mathbf{A} \end{array}\right\} = -V \qquad ...(\beta)$$

Obs. In support of the above rules, our notation **[ABC]** indicates the cyclic order of the factors and has nothing to do with position of the dot or the cross.

$\therefore$ The relations (α) and (β) can be compactly written as

$$[\mathbf{ABC}] = [\mathbf{BCA}] = [\mathbf{CAB}] = V \quad \text{and} \quad [\mathbf{ACB}] = [\mathbf{BAC}] = [\mathbf{CBA}] = -V.$$

IV. Scalar triple product is distributive

i.e., $$[\mathbf{A}, \mathbf{B}+\mathbf{C}, \mathbf{D}-\mathbf{E}] = [\mathbf{ABD}] - [\mathbf{ABE}] + [\mathbf{ACD}] - [\mathbf{ACE}]$$

V. If $\mathbf{A} = a_1\mathbf{I} + a_2\mathbf{J} + a_3\mathbf{K}$, $\mathbf{B} = b_1\mathbf{I} + b_2\mathbf{J} + b_3\mathbf{K}$, $\mathbf{C} = c_1\mathbf{I} + c_2\mathbf{J} + c_3\mathbf{K}$

then $$[\mathbf{ABC}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

As $\mathbf{A}\times\mathbf{B} = (a_2b_3 - a_3b_2)\,\mathbf{I} + (a_3b_1 - a_1b_3)\,\mathbf{J} + (a_1b_2 - a_2b_1)\mathbf{K}$

$\therefore$ $[\mathbf{ABC}] = [a_2b_3 - a_3b_2)\,\mathbf{I} + (a_3b_1 - a_1b_3)\,\mathbf{J} + (a_1b_2 - a_2b_1)\,\mathbf{K}]\,.\,(c_1\mathbf{I} + c_2\mathbf{J} + c_3\mathbf{K})$

$= c_1(a_2b_3 - a_3b_2) + c_2(a_3b_1 - a_1b_3) + c_3(a_1b_2 - a_2b_1)$ which is the required result.

Obs. Linear dependence of vectors. Any three vectors **A, B, C** are said to be *linearly dependent* if one of these can be expressed as a linear combination of other two *i.e.*,

$$\mathbf{A} = m\mathbf{B} + n\mathbf{C}$$

where m, n are constants. This means that **A** lies in the plane of **B, C** *i.e.*, $[\mathbf{ABC}] = 0$. Thus *three vectors are linearly dependent if their scalar triple product is zero. Otherwise these vectors are linearly independent.*

Example 3.21. *Show that the points* $-6\mathbf{I} + 3\mathbf{J} + 2\mathbf{K}$, $3\mathbf{I} - 2\mathbf{J} + 4\mathbf{K}$, $5\mathbf{I} + 7\mathbf{J} + 3\mathbf{K}$ *and* $-13\mathbf{I} + 17\mathbf{J} - \mathbf{K}$ *are coplanar.*

Solution. Let $\vec{OA} = -6\mathbf{I} + 3\mathbf{J} + 2\mathbf{K}$, $\vec{OB} = 3\mathbf{I} - 2\mathbf{J} + 4\mathbf{K}$, $\vec{OC} = 5\mathbf{I} + 7\mathbf{J} + 3\mathbf{K}$

and $\vec{OD} = -13\mathbf{I} + 17\mathbf{J} - \mathbf{K}$. Then $\vec{AB} = \vec{OB} - \vec{OA} = 9\mathbf{I} - 5\mathbf{J} + 2\mathbf{K}$

Similarly, $\vec{AC} = 11\mathbf{I} + 4\mathbf{J} + \mathbf{K}$, and $\vec{AD} = -7\mathbf{I} + 14\mathbf{J} - 3\mathbf{K}$.

The given points will be coplanar if $\vec{AB}, \vec{AC}, \vec{AD}$ are coplanar, *i.e.*, if their scalar triple product is zero. Now

$$[\vec{AB}, \vec{AC}, \vec{AD}] = \begin{vmatrix} 9 & -5 & 2 \\ 11 & 4 & 1 \\ -7 & 14 & -3 \end{vmatrix} = 9(-12-14) + 5(-33+7) + 2(154+28) = 0$$

Hence the points A, B, C, D are coplanar.

Example 3.22. *Show that the volume of the tetrahedron ABCD is* $\frac{1}{6}[\vec{AB}, \vec{AC}, \vec{AD}]$.

Hence find the volume of the tetrahedron formed by the points (1, 1, 1), (2, 1, 3), (3, 2, 2) and (3, 3, 4).

Solution. (*i*) Volume of the tetrahedron $ABCD$

$$= \frac{1}{3}(\text{area of } \Delta ABC) \times (\text{height } h \text{ of } D \text{ above the plane } ABC)$$

$$= \frac{1}{6}(2 \text{ area of } \Delta ABC)h$$

$$= \frac{1}{6}(\text{volume of the parallelopiped with } AB, AC, AD \text{ as coterminus edges})$$

$$= \frac{1}{6}[\vec{AB}, \vec{AC}, \vec{AD}].$$

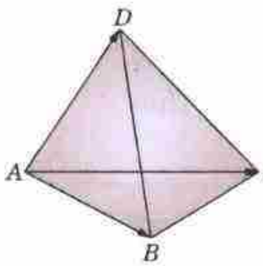


Fig. 3.32

(*ii*) Let $\vec{OA} = \mathbf{I} + \mathbf{J} + \mathbf{K}$, $\vec{OB} = 2\mathbf{I} + \mathbf{J} + 3\mathbf{K}$, $\vec{OC} = 3\mathbf{I} + 2\mathbf{J} + 2\mathbf{K}$ and $\vec{OD} = 3\mathbf{I} + 3\mathbf{J} + 4\mathbf{K}$.

Then $\vec{AB} = \vec{OB} - \vec{OA} = \mathbf{I} + 2\mathbf{K}$

Similarly, $AC = 2\mathbf{I} + \mathbf{J} + \mathbf{K}$ and $\vec{AD} = 2\mathbf{I} + 2\mathbf{J} + 3\mathbf{K}$

$\therefore$ Volume of the tetrahedron $ABCD = \frac{1}{6}[\vec{AB}, \vec{AC}, \vec{AD}] = \frac{1}{6}\begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 2 & 2 & 3 \end{vmatrix} = \frac{5}{6}$.

3.10 VECTOR PRODUCT OF THREE VECTORS

(1) Definition. *If* **A, B, C** *be any three vectors, then the vector or cross product of* **A** × **B** *with* **C** *is called the vector product of three vectors* **A, B, C** *and is written as* (**A** × **B**) × **C**.

Here the brackets are essential as **A** × (**B** × **C**) ≠ (**A** × **B**) × **C**, expressing the fact that vector triple product is not associative.

(2) Expansion formula. *If* **A, B, C** *be any three vectors,* (**A** × **B**) × **C** = (**C** . **A**)**B** – (**C** . **B**) **A**

In words (extreme × adjacent) × outer = (outer . extreme) adjacent – (outer . adjacent) extreme.

The vector (**A** × **B**) × **C** is perpendicular to the vector **A** × **B** and the latter is perpendicular to the plane containing **A** and **B**. Hence (**A** × **B**) × **C** lies in the plane of **A** and **B**. As such we can write

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = l\mathbf{A} + m\mathbf{B} \qquad ...(1)$$

where l and m are some scalars.

Multiply both sides scalarly by **C**, then $\mathbf{C}\,.\,(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = l\mathbf{C}\,.\,A + m\mathbf{C}\,.\,\mathbf{B}$

The scalar triple product on the left-hand side is zero, since two of its vectors are equal.

$$\therefore \quad l(\mathbf{C}\,.\,\mathbf{A}) + m(\mathbf{C}\,.\,\mathbf{B}) = 0$$

or

$$\frac{l}{\mathbf{C}\,.\,\mathbf{B}} = \frac{m}{-\mathbf{C}\,.\,\mathbf{A}} = n, \text{ say.}$$

Substituting the values of l and m in (1), we get

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = n(\mathbf{C}\,.\,\mathbf{B})\mathbf{A} - n(\mathbf{C}\,.\,\mathbf{A})\mathbf{B} \qquad ...(2)$$

Evidently n is some numerical constant. To find it, take the special case **A** = **I**, **B** = **C** = **J**. Then (2) gives

$$(\mathbf{I}\times\mathbf{J})\times\mathbf{J} = n(\mathbf{J}\,.\,\mathbf{J})\mathbf{I} - n(\mathbf{J}\,.\,\mathbf{I})\mathbf{J}$$

i.e.,

$$\mathbf{K}\times\mathbf{J} = n\mathbf{I} \text{ or } -\mathbf{I} = n\mathbf{I}.$$

This gives $n = -1$. Hence (2) reduces to the required result.

Similarly, it can be shown that **A** × (**B** × **C**) = (**A** . **C**)**B** – (**A** . **B**)**C**

Cor. A × (B × C) + B × (C × A) + C × (A × B) = 0.

For L.H.S. = (**A** . **C**)**B** – (**A** . **B**)**C** + (**B** . **A**)**C** – (**B** . **C**)**A** + (**C** . **B**)**A** – (**C** . **A**)**B** which vanishes identically.

Example 3.23. *If A, B, C, D be any four vectors, prove that*

(*i*) $(A\times B)\,.\,(C\times D) = \begin{vmatrix} A\,.\,C & B\,.\,C \\ A\,.\,D & B\,.\,D \end{vmatrix}$ (*ii*) $(A\times B)\times(C\times D) = [ACD]\,B - [BCD]A$

Solution. (*i*) $(\mathbf{A}\times\mathbf{B})\,.\,(\mathbf{C}\times\mathbf{D}) = [\mathbf{A}\times\mathbf{B})\times\mathbf{C}]\,.\,\mathbf{D}$ (interchanging the dot and cross)

$$= [(\mathbf{A}\,.\,\mathbf{C})\mathbf{B} - (\mathbf{B}\,.\,\mathbf{C})\mathbf{A}]\,.\,\mathbf{D}$$

$$= (\mathbf{A}\,.\,\mathbf{C})(\mathbf{B}\,.\,\mathbf{D}) - (\mathbf{A}\,.\,\mathbf{D})(\mathbf{B}\,.\,\mathbf{C}) \text{ whence follows the result.}$$

In particular, we have $(\mathbf{A}\times\mathbf{B})^2 = \mathbf{A}^2\mathbf{B}^2 - (\mathbf{A}\,.\,\mathbf{B})^2$ which has already been proved in § 3.6 (3) – IV.

(*ii*) $(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\times\mathbf{B})\times\mathbf{P}$, where $\mathbf{P} = \mathbf{C}\times\mathbf{D}$

$$= (\mathbf{A}\,.\,\mathbf{P})\mathbf{B} - (\mathbf{B}\,.\,\mathbf{P})\mathbf{A} = (\mathbf{A}\,.\,\mathbf{C}\times\mathbf{D})\mathbf{B} - (\mathbf{B}\,.\,\mathbf{C}\times\mathbf{D})\mathbf{A}$$

$$= [\mathbf{ACD}]\mathbf{B} - [\mathbf{BCD}]\mathbf{A}.$$

Example 3.24. *Show that the components of a vector* **B** *along and perpendicular to a vector* **A**, *in the plane of* **A** *and* **B**, *are*

$$\frac{\mathbf{A}\,.\,\mathbf{B}}{\mathbf{A}^2} \text{ and } \frac{(\mathbf{A}\times\mathbf{B})\times\mathbf{A}}{\mathbf{A}^2}.$$

Solution. Let $\vec{\mathbf{OA}} = \mathbf{A}$, $\vec{\mathbf{OB}} = \mathbf{B}$ and **OM** be the projection of **B** on **A** (Fig. 3.33)

∴ Component of **B** along **A** = *OM* (unit vector along **A**)

$$= (\mathbf{B}\,.\,\hat{\mathbf{A}})\hat{\mathbf{A}} = \left(\frac{\mathbf{B}\,.\,\mathbf{A}}{a}\right)\frac{\mathbf{A}}{a} \qquad [\because \ \mathbf{A} = a\,\hat{\mathbf{A}}]$$

$$= \frac{\mathbf{B}\,.\,\mathbf{A}}{\mathbf{A}^2}\mathbf{A} \qquad [\because \ a^2 = \mathbf{A}^2]$$

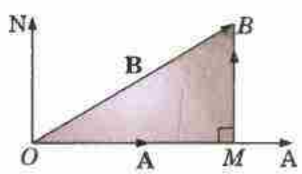


Fig. 3.33

Also component of $\mathbf{B} \perp \mathbf{A} = \vec{\mathbf{MB}}$

$$= \vec{OB} - \vec{OM} = \mathbf{B} - \frac{\mathbf{B}\,.\,\mathbf{A}}{\mathbf{A}^2}\,\mathbf{A} = \frac{(\mathbf{A}\,.\,\mathbf{A})\mathbf{B} - (\mathbf{A}\,.\,\mathbf{B})\mathbf{A}}{\mathbf{A}^2} = \frac{(\mathbf{A}\times\mathbf{B})\times\mathbf{A}}{\mathbf{A}^2}.$$

Example 3.25. *Prove the formula*

$$(\mathbf{B}\times\mathbf{C})\,.\,(\mathbf{A}\times\mathbf{D}) + (\mathbf{C}\times\mathbf{A})\,.\,(\mathbf{B}\times\mathbf{D}) + (\mathbf{A}\times\mathbf{B})\,.\,(\mathbf{C}\times\mathbf{D}) = 0.$$

and hence show that $\sin(\theta+\phi)\sin(\theta-\phi) = \sin^2\theta - \sin^2\phi$.

Solution. We know that

$$(\mathbf{B}\times\mathbf{C})\,.\,(\mathbf{A}\times\mathbf{D}) = (\mathbf{B}\,.\,\mathbf{A})(\mathbf{C}\,.\,\mathbf{D}) - (\mathbf{B}\,.\,\mathbf{D})(\mathbf{C}\,.\,\mathbf{A})$$
$$(\mathbf{C}\times\mathbf{A})\,.\,(\mathbf{B}\times\mathbf{D}) = (\mathbf{C}\,.\,\mathbf{B})(\mathbf{A}\,.\,\mathbf{D}) - (\mathbf{C}\,.\,\mathbf{D})(\mathbf{A}\,.\,\mathbf{B})$$
$$(\mathbf{A}\times\mathbf{B})\,.\,(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\,.\,\mathbf{C})(\mathbf{B}\,.\,\mathbf{D}) - (\mathbf{A}\,.\,\mathbf{D})(\mathbf{B}\,.\,\mathbf{C})$$

Adding, we get

$$(\mathbf{B}\times\mathbf{C})\,.\,(\mathbf{A}\times\mathbf{D}) + (\mathbf{C}\times\mathbf{A})\,.\,(\mathbf{B}\times\mathbf{D}) + (\mathbf{A}\times\mathbf{B})\,.\,(\mathbf{C}\times\mathbf{D}) = 0 \quad ...(i)$$

Let the vectors **A, B, C, D** be acting along coplanar lines OA, OB, OC, OD respectively (Fig. 3.34).

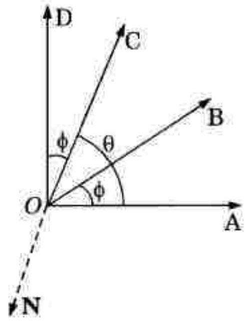


Fig. 3.34

Take $\angle AOC = \theta$ and $\angle AOB = \angle COD = \phi$,

so that $\angle AOD = \theta + \phi$ and $\angle BOC = \theta - \phi$

If **N** be a unit vector normal to the plane of these lines, then

$$(\mathbf{B}\times\mathbf{C})\,.\,(\mathbf{A}\times\mathbf{D}) = [bc\sin(\theta-\phi)\mathbf{N}]\,.\,[ad\sin(\theta+\phi)\mathbf{N}]$$
$$= abcd\sin(\theta+\phi)\sin(\theta-\phi) \quad ...(ii)$$
$$(\mathbf{C}\times\mathbf{A})\,.\,(\mathbf{B}\times\mathbf{D}) = [ca\sin(-\theta)\mathbf{N}]\,.\,[bd\sin\theta\mathbf{N}]$$
$$= -abcd\sin^2\theta \quad ...(iii)$$

and

$$(\mathbf{A}\times\mathbf{B})\,.\,(\mathbf{C}\times\mathbf{D}) = [ab\sin\phi\mathbf{N}]\,.\,[cd\sin\phi\mathbf{N}]$$
$$= abcd\sin^2\phi \quad ...(iv)$$

Substituting the values from (*ii*), (*iii*), (*iv*) in (*i*), we get

$abcd\sin(\theta+\phi)\sin(\theta-\phi) - abcd\sin^2\theta + abcd\sin^2\phi = 0$ whence follows the required result.

Example 3.26. *Prove that*

(*i*) $[\mathbf{B}\times\mathbf{C}, \mathbf{C}\times\mathbf{A}, \mathbf{A}\times\mathbf{B}] = [\mathbf{ABC}]^2$. *(Nagpur, 2009)*

(*ii*) $\mathbf{A}\times\{\mathbf{B}\times(\mathbf{C}\times\mathbf{D})\} = \mathbf{B}\,.\,\mathbf{D}(\mathbf{A}\times\mathbf{C}) - \mathbf{B}\,.\,\mathbf{C}\,(\mathbf{A}\times\mathbf{D})$.

Solution. (*i*) $[\mathbf{B}\times\mathbf{C}, \mathbf{C}\times\mathbf{A}, \mathbf{A}\times\mathbf{B}] = (\mathbf{B}\times\mathbf{C})\,.\,(\mathbf{C}\times\mathbf{A})\times(A\times B)$

$$= (\mathbf{B}\times\mathbf{C})\,.\,\{[(\mathbf{C}\times\mathbf{A})\,.\,\mathbf{B}]\mathbf{A} - [(\mathbf{C}\times\mathbf{A})\,.\,\mathbf{A}]\mathbf{B}\}$$
$$= (\mathbf{B}\times\mathbf{C})\,.\,\{[\mathbf{B}\,.\,(\mathbf{C}\times\mathbf{A})]\mathbf{A}\} \qquad [\because\ [(\mathbf{C}\times\mathbf{A})\,.\,\mathbf{A}] = 0$$
$$= [\mathbf{B}\times\mathbf{C})\,.\,\mathbf{A}]\,[(\mathbf{B}\times\mathbf{C})\,.\,\mathbf{A}] = [\mathbf{BCA}]^2 = [\mathbf{ABC}]^2 \qquad [\because\ [\mathbf{BCA}] = [\mathbf{ABC}]$$

(*ii*)

$$\mathbf{A}\times\{\mathbf{B}\times(\mathbf{C}\times\mathbf{D})\} = \mathbf{A}\times\{(\mathbf{B}\,.\,\mathbf{D})\,\mathbf{C} - (\mathbf{B}\,.\,\mathbf{C})\mathbf{D}\}$$
$$= (\mathbf{A}\times\mathbf{C})(\mathbf{B}\,.\,\mathbf{D}) - (\mathbf{A}\times\mathbf{D})(\mathbf{B}\,.\,\mathbf{C}) = (\mathbf{B}\,.\,\mathbf{D})\,(\mathbf{A}\times\mathbf{C}) - \mathbf{B}\,.\,\mathbf{C}(A\times\mathbf{D}).$$

PROBLEMS 3.5

1. Find the volume of the parallelopied whose edges are represented by the vectors $\mathbf{A} = 2\mathbf{I} - 3\mathbf{J} + 4\mathbf{K}$, $\mathbf{B} = \mathbf{I} + 2\mathbf{J} - \mathbf{K}$, $\mathbf{C} = 3\mathbf{I} - \mathbf{J} + 2\mathbf{K}$.
2. Find a such that the vectors $2\mathbf{I} - \mathbf{J} + \mathbf{K}$, $\mathbf{I} + 2\mathbf{J} - 3\mathbf{K}$ and $3\mathbf{I} + a\mathbf{J} + 5\mathbf{K}$ are coplanar.
3. (*i*) Prove that the vectors $\mathbf{I} - 2\mathbf{J} + 3\mathbf{K}$, $-2\mathbf{I} + 3\mathbf{J} - 4\mathbf{K}$ and $\mathbf{I} - 3\mathbf{J} + 5\mathbf{K}$ are coplanar.
   (*ii*) Do the points (4, – 2, 1), (5, 1, 6), (2, 2 – 5) and (3, 5, 0) lie in a plane.
4. (*a*) Test the linear dependency of the vectors (1, 2, 1), (2, 1, 4), (4, 5, 6) and (1, 8, – 5).
   (*b*) Verify whether the following set of vectors are linearly independent (4, 2, 9), (3, 2, 1), (– 4, 6, 9). *(B.P.T.U., 2005)*
5. Find the volume of the tetrahedron, three of whose coterminus edges are $3\mathbf{I} - \mathbf{J} + 2\mathbf{K}$, $2\mathbf{I} + \mathbf{J} - \mathbf{K}$ and $\mathbf{I} - 2\mathbf{J} + 2\mathbf{K}$.

6. Find the volume of the tetrahedron formed by the points
   (*i*) (1, 3, 6), (3, 7, 12), (8, 8, 9) and (2, 2, 8). *(B.P.T.U., 2005)*
   (*ii*) (2, 1, 1), (1, – 1, 2), (0, 1, – 1) and (1, – 2, 1).
7. If $\mathbf{A} \cdot \mathbf{N} = 0$, $\mathbf{B} \cdot \mathbf{N} = 0$, $\mathbf{C} \cdot \mathbf{N} = 0$, prove that $[\mathbf{ABC}] = 0$. Interpret this result geometrically.
8. (*a*) Prove that $[\mathbf{A}+\mathbf{B}, \mathbf{B}+\mathbf{C}, \mathbf{C}+\mathbf{A}] = 2[\mathbf{ABC}]$.
   (*b*) Show that volume of the tetrahedron having $\mathbf{A}+\mathbf{B}$, $\mathbf{B}+\mathbf{C}$ and $\mathbf{C}+\mathbf{A}$ as concurrent edges is twice the volume of the tetrahedron having $\mathbf{A}, \mathbf{B}, \mathbf{C}$ as concurrent edges.
9. If $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$, show that $(\mathbf{A} \times \mathbf{C}) \times \mathbf{B} = 0$.
10. Show that $\mathbf{I} \times (\mathbf{R} \times \mathbf{I}) + \mathbf{J} \times (\mathbf{R} \times \mathbf{J}) + \mathbf{K} \times (\mathbf{R} \times \mathbf{K}) = 2\mathbf{R}$. *(Assam, 1999)*
11. If $\mathbf{A} = \mathbf{I} - 2\mathbf{J} - 3\mathbf{K}$, $\mathbf{B} = 2\mathbf{I} + \mathbf{J} - \mathbf{K}$, $\mathbf{C} = \mathbf{I} + 3\mathbf{J} - \mathbf{K}$, find
    (*i*) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ (*ii*) $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{B} \times \mathbf{C})$.
12. (*a*) Given $\mathbf{A} = 2\mathbf{I} - \mathbf{J} + 3\mathbf{K}$, $\mathbf{B} = -\mathbf{I} + 3\mathbf{J} + 3\mathbf{K}$, $\mathbf{C} = \mathbf{I} + \mathbf{J} - 2\mathbf{K}$, find the reciprocal triad $(\mathbf{A}', \mathbf{B}', \mathbf{C}')$ and verify that $[\mathbf{ABC}]\,[\mathbf{A'B'C'}] = 1$.
    (*b*) Prove that $\mathbf{A} \times \mathbf{A}' + \mathbf{B} \times \mathbf{B}' + \mathbf{C} \times \mathbf{C}' = 0$
13. Prove that (*i*) $[\mathbf{A} \times \mathbf{B}, \mathbf{C} \times \mathbf{D}, \mathbf{E} \times \mathbf{F}] = [\mathbf{ABD}][\mathbf{CEF}] - [\mathbf{ABC}][\mathbf{DEF}]$
    (*ii*) $\{(\mathbf{A}+\mathbf{B}+\mathbf{C}) \times (\mathbf{B}+\mathbf{C})\} \cdot \mathbf{C} = [\mathbf{ABC}]$.
14. Show that
    (*i*) $(\mathbf{B} \times \mathbf{C}) \times (\mathbf{A} \times \mathbf{D}) + (\mathbf{C} \times \mathbf{A}) \times (\mathbf{B} \times \mathbf{D}) + (\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = -2[\mathbf{ABC}]\mathbf{D}$. *(Mumbai, 2007)*
    (*ii*) $\mathbf{A} \times [(\mathbf{F} \times \mathbf{B}) \times (\mathbf{G} \times \mathbf{C})] + \mathbf{B} \times [(\mathbf{F} \times \mathbf{C}) \times (\mathbf{G} \times \mathbf{A}) + \mathbf{C} \times [(\mathbf{F} \times \mathbf{A}) \times (\mathbf{G} \times \mathbf{B})] = 0$,
15. (*a*) Prove that $[\mathbf{LMN}][\mathbf{ABC}] = \begin{vmatrix} \mathbf{L}\cdot\mathbf{A} & \mathbf{L}\cdot\mathbf{B} & \mathbf{L}\cdot\mathbf{C} \\ \mathbf{M}\cdot\mathbf{A} & \mathbf{M}\cdot\mathbf{B} & \mathbf{M}\cdot\mathbf{C} \\ \mathbf{N}\cdot\mathbf{A} & \mathbf{N}\cdot\mathbf{B} & \mathbf{N}\cdot\mathbf{C} \end{vmatrix}$
    (*b*) The length of the edges OA, OB, OC of the tetrahedron $OABC$ are a, b, c and the angles BOC, COA, AOB are θ, ϕ, ψ, find its volume.

SOLID GEOMETRY

3.11 (1) EQUATION OF A PLANE

Let $P(x, y, z)$ be any point on the plane through $A(x_1, y_1\, z_1)$ which is normal to the vector $\mathbf{N} = a\mathbf{I} + b\mathbf{J} + c\mathbf{K}$.

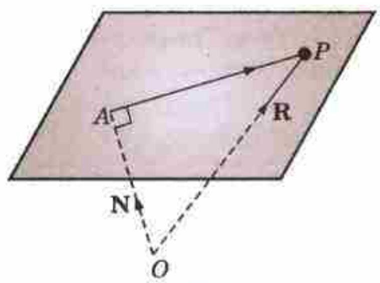


Fig. 3.35

Then $\vec{OP} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ and $\vec{OA} = x_1\mathbf{I} + y_1\mathbf{J} + z_1\mathbf{K}$

Clearly the vectors $\vec{AP} = (x - x_1)\,\mathbf{I} + (y - y_1)\,\mathbf{J} + (z - z_1)\,\mathbf{K}$ and $\mathbf{N}$ are perpendicular to each other.

$$\therefore \quad \vec{AP} \cdot \mathbf{N} = 0 \qquad ...(i)$$

or $$[(x - x_1)\,\mathbf{I} + (y - y_1)\,\mathbf{J} + (z - z_1)\,\mathbf{K}] \cdot (a\mathbf{I} + b\mathbf{J} + c\mathbf{K}) = 0$$

or $$\mathbf{a(x - x_1) + b(y - y_1) + c(z - z_1) = 0} \qquad ...(ii)$$

which is *the equation of any plane through the point* (x_1, y_1, z_1).

Obs. Equation (*ii*) written as $ax + by + cz + d = 0$ *is the general equation of a plane.*
Conversely, *every linear equation in x, y, z represents a plane and the coefficients of x, y, z are the direction ratios of the normal to the plane.*

Cor. IF l, m, n be the direction cosines of the normal to the plane, then

$$\mathbf{lx + my + nz = p} \qquad ...(iii)$$

which is called the **normal form** *of the equation of the plane* where p is the perpendicular distance from the origin.

(2) Angle between two planes. Def. *The angle between two planes is equal to the angle between their normals.*

Let the two planes be

$$ax + by + cz + d = 0 \quad \text{and} \quad a'x + b'y + c'z + d' = 0.$$

Now the direction ratio of their normals are a, b, c and a', b', c'.

Hence the angle θ between the planes is given by $\cos\theta = \dfrac{aa' + bb' + cc'}{\sqrt{(a^2+b^2+c^2)}\sqrt{a'^2+b'^2+c'^2}}$

The planes will be perpendicular (if their normal are parallel), *i.e., if* $aa' + bb' + cc' = 0$

The planes will be parallel (if their normals are parallel), *i.e., if* $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{c}{c'}$.

Cor. *Any plane parallel to the plane* $ax + by + cz + d = 0$

is $\mathbf{ax + by + cz + k = 0}$ (k being any constant)

for the direction-ratios of their normals are the same.

(3) Perpendicular distance of the point (x_1, y_1, z_1) *from the plane*

$$ax + by + cz + d = 0 \quad ...(i)$$

is

$$\frac{\mathbf{ax_1 + by_1 + cz_1 + d}}{\sqrt{(\mathbf{a^2 + b^2 + c^2})}}$$

Let PL be the perpendicular distance of $P(x_1, y_1, z_1)$ from the plane (i) so that the direction cosines of $\vec{LP}$ are

$$\frac{a}{\sqrt{(\Sigma a^2)}}, \frac{b}{\sqrt{(\Sigma a^2)}}, \frac{c}{\sqrt{(\Sigma a^2)}}.$$

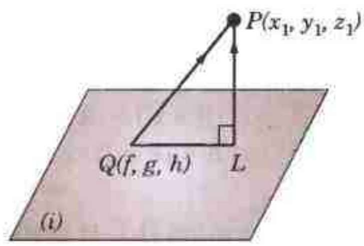


Fig. 3.36

If $Q(f, g, h)$ be a point on (i) then

$$af + bg + ch + d = 0 \quad ...(ii)$$

$$\therefore \quad PL = \text{projection of } \vec{QP} \text{ on } \vec{LP} = \vec{QP} \cdot \vec{LP}$$

$$= \frac{(x_1 - f)a + (y_1 - g)b + (z_1 - h)c}{\sqrt{(a^2 + b^2 + c^2)}} \quad \text{[By } IX \text{ p. 82]}$$

$$= \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{(a^2 + b^2 + c^2)}} \text{ by virtue of } (ii) \quad ...(iii)$$

The sign of the radical in (iii) is taken to be positive or negative according as d is positive or negative.

Obs. The perpendicular to a plane from two points are taken to be of the same sign if the points lie on the same side and of different signs if they lie on the opposite sides of the plane.

∴ *The two points* (x_1, y_1, z_1) *and* (x_2, y_2, z_2) *lie on the same side or on opposite sides of the plane* $ax + by + cz + d = 0$, according as $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ are of the same sign or of opposite signs.

Cor. Planes bisecting the angles between two planes.

Let $$ax + by + cz + d = 0 \quad ...(i)$$

and $$a'x + b'y + c'z + d' = 0 \quad ...(ii)$$

be the given planes.

Let $P(x, y, z)$ be any point on either of the planes bisecting the angles between the planes (i) and (ii).

Then ⊥ distance of P from (i) = ⊥ distance of P from (ii),

$$\therefore \quad \frac{ax + by + cz + d}{\sqrt{(a^2 + b^2 + c^2)}} = \pm \frac{a'x + b'y + c'z + d'}{\sqrt{(a'^2 + b'^2 + c'^2)}}$$

which are the required equations of the bisecting planes.

Example 3.27. *Find the equation of the plane which*

(i) cuts off intercepts a, b, c from the axes.

(ii) passes through the points A(0, 1, 1), B(1, 1, 2) and C(–1, 2, –2).

Solution. (i) **Intercept form** *of the equation of the plane.* Let the required equation of the plane be

$$\alpha x + \beta y + \gamma z + \delta = 0 \quad ...(1)$$

The plane cuts the axes at A, B, C such that $OA = a, OB = b, OC = c$,

i.e., it passes through the points $A(a, 0, 0), B(0, b, 0), C(0, 0, c)$.

$\therefore$ $\alpha a + \delta = 0, \beta b + \delta = 0, \gamma c + \delta = 0$

whence $\alpha = -\delta/a, \beta = -\delta/b, \gamma = -\delta/c$

Substituting these values of α, β, γ in (1), $-\frac{\delta}{a}x - \frac{\delta}{b}y - \frac{\delta}{c}z + \delta = 0$ or $\frac{\mathbf{x}}{\mathbf{a}} + \frac{\mathbf{y}}{\mathbf{b}} + \frac{\mathbf{z}}{\mathbf{c}} = 1.$

(*ii*) **Three points form** *of the equation of the plane.*

Any plane through (0, 1, 1) is $a(x - 0) + b(y - 1) + c(z - 1) = 0$ ...(2)

It will pass through (1, 1, 2) and (–1, 2, – 2), if $a + c = 0$ and $-a + b - 3c = 0$.

By cross-multiplication, $\frac{a}{-1} = \frac{b}{2} = \frac{c}{1}$.

Substituting these values in (2), we get $-1 \cdot x + 2(y - 1) + 1(z - 1) = 0$

or $x - 2y - z + 3 = 0$, which is the required equation of the plane.

Example 3.28. *Find the equation of the plane which passes through the point (3, – 3, 1) and is*

(*i*) *parallel to the plane $2x + 3y + 5z + 6 = 0$.*

(*ii*) *normal to the line joining the points (3, 2, – 1) and (2, – 1, 5).* (*V.T.V., 2006*)

(*iii*) *Perpendicular to the planes $7x + y + 2z = 6$ and $3x + 5y - 6z = 8$.* (*Cochin, 2005 ; V.T.U., 2005*)

Solution. (*i*) Any plane parallel to the given plane is

$2x + 3y + 5z + k = 0$ which goes through (3, – 3, 1), if $k = -2$

Thus the required plane is $2x + 3y + 5z - 2 = 0$

(*ii*) Any plane through (3, – 3, 1) is $a(x - 3) + b(y + 3) + c(z - 1) = 0$

The direction cosines of the line joining the points (3, 2, – 1) and (2, – 1, 5) are proportional to 1, 3, – 6.

This line is normal to the plane (1). $\therefore$ a, b, c are proportional to 1, 3, – 6.

Substituting these values in (1), the required equation is

$$1(x - 3) + 3(y + 3) - 6(z - 1) = 0 \quad \text{or} \quad x + 3y - 6z + 12 = 0.$$

(*iii*) Any plane through (3, – 3, 1) is

$a(x - 3) + b(y + 3) + c(z - 1) = 0$ which will be $\perp$ to the planes

$$7x + y + 2z = 6 \text{ and } 3x + 5y - 6z = 8$$

if $7a + b + 2c = 0$ and $3a + 5b - 6c = 0$.

Solving these by cross-multiplication, we get $\frac{a}{1} = \frac{b}{-3} = \frac{c}{-2}$.

Hence the required equation is $1(x - 3) - 3(y + 3) - 2(z - 1) = 0$ or $x - 3y - 2z - 10 = 0$.

Example 3.29. *The plane $4x + 5y - z = 7$ is rotated through a right angle about its line of intersection with the plane $2x + 3y - 3z = 5$. Find the equation of this plane in its new position.*

Solution. Any plane through the line of intersection of

$$4x + 5y - z = 7 \qquad ...(i)$$

and $$2x + 3y - 3z = 5 \qquad ...(ii)$$

is $$4x + 5y - z - 7 + k(2x + 3y - 3z - 5) = 0$$

i.e., $$(4 + 2k)x + (5 + 3k)y - (1 + 3k)z - (7 + 5k) = 0 \qquad ...(iii)$$

Then new position of (*i*) when rotated through a right angle, is such that (*i*) and (*iii*) are perpendicular. This requires that

$$4(4 + 2k) + 5(5 + 3k) + (1 + 3k) = 0$$

i.e., $26k + 42 = 0$ or $k = -21/13$

Substituting $k = -21/13$ in (*iii*), we get $10x + 2y + 50z + 14 = 0$.

or $5x + y + 25x + 7 = 0$, which is the required plane.

Example 3.30. *Find the distance between the parallel planes $2x - 2y + z + 3 = 0$ and $4x - 4y + 2z + 9 = 0$. Find also the equation of the parallel plane that lies mid-way between the given planes.* (*Madras, 2003*)

Solution. The distance between the given planes is the perpendicular distance of any point on one of the planes from the other.

A point on the first plane is (0, 0, – 3).

$\therefore$ Required distance = $\perp$ distance of (0, 0, – 3) from $4x - 4y + 2z + 9 = 0$

$$= \frac{-6+9}{\sqrt{(16+16+4)}} = \frac{3}{6} = \frac{1}{2}$$

Let the equation of the parallel plane that lies mid-way between the given planes be

$$2x - 2y + z + k = 0 \qquad ...(i)$$

Now distance of any point (0, 0, – 3) on the first plane from (*i*) should be 1/4.

$$\therefore \qquad \pm \frac{-3+k}{\sqrt{(4+4+1)}} = 1/4 \quad i.e., \quad k = 15/4 \text{ or } 9/4.$$

Thus the required plane is $2x - 2y + z + 15/4 = 0$.

Assume that $k = 15/4$ and verify that the distance of a point on this plane $4x - 4y + 2z + 9 = 0$ is also 1/4.

A point on this plane is (0, 0, – 9/4). Its distance from the plane (*i*) = $\dfrac{-9/2 + 15/4}{3} = \dfrac{1}{4}$ (in magnitude)

Thus $k = 9/4$ is not admissible.

$\therefore$ The required plane is $2x - 2y + z + 15/4 = 0$.

Example 3.31. *A variable plane is at a constant distance p from the origin and meets the axes at A, B, C. Find the locus of the centroid of the tetrahedron OABC.*

Solution. As the given plane is at a $\perp$ distance p from the origin, therefore its equation is of the form

$$lx + my + nz = p \qquad ...(i)$$ where l, m, n are the d.c's of the $\perp$ from the origin.

(*i*) may be rewritten as $\dfrac{x}{(p/l)} + \dfrac{y}{(p/m)} + \dfrac{z}{(p/n)} = 1$

so that $OA = p/l$, $OB = p/m$, $OC = p/n$.

$$\therefore \qquad A = (p/l, 0, 0), B = (0, p/m, 0), C = (0, 0, p/n).$$

Thus the coordinates of the centroid G of the tetrahedron $OABC$ are

$$(x_1, y_1, z_1) = (p/4l, p/4m, p/4n)$$ [See p. 81]

$$\therefore \qquad \frac{1}{x_1^2} + \frac{1}{y_1^2} + \frac{1}{z_1^2} = \frac{16}{p^2}(l^2 + m^2 + n^2) = \frac{16}{p^2}$$

Thus the locus of G is $x^{-2} + y^{-2} + z^{-2} = 16p^{-2}$.

Example 3.32. *A variable plane at a constant distance p from the origin meets the axes in A, B, C. Planes are drawn through A, B, C parallel to the coordinate planes, Show that the locus of their point of intersection is given by* $x^{-2} + y^{-2} + z^{-2} = p^{-2}$.

Solution. Let the variable plane be $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$.

Its distance from origin = $\dfrac{1}{\sqrt{(a^{-2} + b^{-2} + c^{-2})}} = p$ (given)

i.e.,
$$a^{-2} + b^{-2} + c^{-2} = p^{-2} \qquad ...(i)$$

Since $OA = a$, $OB = b$ and $OC = c$, therefore equations of the planes through A, B, C parallel to yz, zx and xy-planes are $x = a, y = b, z = c$

Let the point of intersection of these three planes be (x_1, y_1, z_1).

Then
$$x_1 = a, y_1 = b, z_1 = c \qquad ...(ii)$$

Substituting (*ii*) in (*i*), we get $x_1^{-2} + y_1^{-2} + z_1^{-2} = p^{-2}$

Thus the locus of (x_1, y_1, z_1) is $x^{-2} + y^{-2} + z^{-2} = p^{-2}$.

Example 3.33. *A variable plane passes through the fixed point (a, b, c) and meets the coordinate axes in A, B, C. Show that the locus of the point common to the planes through A, B, C parallel to the coordinate planes is*

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1.$$

Solution. Let ABC be any plane through the fixed point $H(a, b, c)$ such that $OA = x_1$, $OB = y_1$, $OC = z_1$. Then its equation is

$$\frac{x}{x_1}+\frac{y}{y_1}+\frac{z}{z_1}=1 \qquad \text{[See Ex. 3.27 (i)]}$$

Since H lies on it.

$$\therefore \qquad \frac{a}{x_1}+\frac{b}{y_1}+\frac{c}{z_1}=1. \qquad ...(1)$$

The planes through A, B, C parallel to the coordinate planes are $x = x_1$, $y = y_1$, $z = z_1$, which meet in $P(x_1, y_1, z_1)$.

Thus changing x_1 to x, y_1, to y and z, to z in the locus of the P is

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1.$$

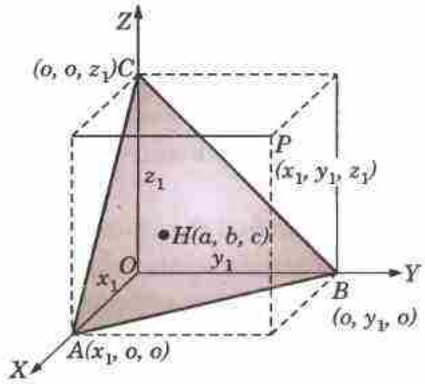


Fig. 3.37

Example 3.34. *Find the equations to the two planes which bisect the angles between the planes $3x - 4y + 5z = 3$, $5x + 3y - 4z = 9$.*

Also point out which of the planes bisects the acute angle. (V.T.U., 2007)

Solution. The equations of the planes bisecting the angles between the given planes are

$$\frac{3x-4y+5z-3}{\sqrt{[3^2+(-4)^2+5^2]}}=\pm\frac{5x+3y-4z-9}{\sqrt{[5^2+3^2+(-4)^2]}}$$

or $\qquad 2x + 7y - 9z - 6 = 0 \qquad ...(i)$

and $\qquad 8x - y + z - 12 = 0 \qquad ...(ii)$

which are the required planes.

Let θ be the angle between (*i*) and either of the given planes, say:

$$5x + 3y - 4z = 9.$$

Then,
$$\cos\theta=\frac{2\times5+7\times3\,(-9)\times(-4)}{\sqrt{[2^2+7^2+(-9)^2]}\sqrt{[5^2+3^2+(-4)^2]}}=\frac{67}{5\sqrt{(268)}}$$

$$\therefore \qquad \tan\theta=\frac{\sqrt{2211}}{67} \text{ which is less than 1.}$$

i.e., $\qquad \theta < 45°.$

Now θ is half the angle between the given planes, so that (*i*) bisects that angle between the planes which is $2\theta < 90°$.

Hence the plane $2x + 7y - 9z = 6$, bisects the acute angle.

PROBLEMS 3.6

1. Find the equation of the plane passing through the point (1, 2, 3) and having the vector $\mathbf{N} = 2\mathbf{I} - \mathbf{J} + 3\mathbf{K}$ normal to it.
2. Find the equation of the plane through the points (3, – 1, 1), (1, 2, – 1) and (1, 1, 1).
3. Find a unit vector normal to the plane through the points (– 1, 2, 3), (1, 1, 1) and (2, – 1, 3).
4. Find the distance of the point (1, 4, 5) from the plane passing through the points (2, – 1, 5), (0, – 4, 1) and (2, – 6, 0). (Rajasthan, 2006)
5. Show that the four points (0, – 1, 0), (2, 1, – 1), (1, 1, 1) and (3, 3, 0) are coplanar. Find the equation of the plane through them. (V.T.U., 2008)

6. Show that the point $(-1/2, 2, 0)$ is the circumcentre of the triangle formed by the points $(1, 1, 0)$, $(1, 2, 1)$, $(-2, 2, -1)$.

[**Hint.** Show that the point $(-1/2, 2, 0)$ lies in the plane of the triangle and is equidistant from its vertices.]

7. Find the equation of the plane through the point $(2, 1, 0)$ and perpendicular to the planes $2x - y - z = 5$ and $x + 2y - 3z = 5$.
8. Find the equations of the plane through $(0, 0, 0)$ parallel to the plane $x + 2y = 3z + 4$. *(Madras, 2006)*
9. Find the equation of the plane which bisects the join of the points (x_1, y_1, z_1) and (x_2, y_2, z_2) at right angles.
10. Find the equation of the plane through the points $(-1, 2, 1)$, $(-3, 2, -3)$ and parallel to y-axis *(V.T.U., 2010)*
11. Find the equation of the plane through the points $(2, 2, 1)$ and $(9, 3, 6)$ and perpendicular to the plane $2x + 6y + 6z = 9$. *(V.T.U., 2004 ; Osmania, 1999)*
12. A plane contains the points $A(-4, 9, -9)$ and $B(5, -9, 6)$ and is perpendicular to the line which joins B and $C(4, -6, k)$. Evaluate k and find the equation of the plane.
13. Find the distance between the parallel planes
$$2x - 3y + 6z + 12 = 0 \text{ and } 6x - 9y + 18z - 6 = 0.$$
Also find the equation of the parallel plane that lies mid-way between the given planes.
14. Find the angle between the plane $x + y + z = 8$ and $2x + y - z = 3$. *(B.P.TU., 2006)*
15. Two planes are given by $x + 2y - 3z - 2 = 0$ and $2x + y + z + 3 = 0$, find
(*i*) direction cosines of their line of intersection,
(*ii*) acute angle between the planes, and
(*iii*) equation of the plane perpendicular to both of them through the point $(2, 2, 1)$.
16. The plane $lx + my = 0$ is rotated about its line of intersection with the plane $z = 0$, through an angle α.
Prove that the equation of the plane is $lx + my + z\sqrt{(l^2 + m^2)}\tan\alpha = 0$. *(Anna, 2005 S)*
17. Find the equations of the two planes through the points $(0, 4, -3)$, $(6, -4, 3)$ other than the plane through the origin which cut off from the axes intercepts whose sum is zero.
18. A plane meets the coordinates axes at A, B, C, such that the centroid of the triangle ABC is the point (a, b, c), show that the equation of the plane is $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 3$. *(Assam, 1999)*
19. A plane passes through a fixed point (a, b, c), show that the locus of the foot of the perpendicular from the origin on the plane is a sphere. *(P.T.U., 2005)*
20. A variable plane is at a constant distance p from the origin and meets the axes at A, B, C. Find the locus of the centroid of the triangle ABC. *(Rajasthan, 2005)*
21. A variable plane makes with the coordinate axes a tetrahedron of constant volume $64\ k^3$. Find the locus of the centroid of the tetrahedron. *(Rajasthan, 2006 ; Osmania, 2003)*
22. Find equations of the planes bisecting the angle between the planes
$$x + 2y + 2z = 9,\ 4x - 3y + 12z + 12 = 0$$
and specify the one which bisects the acute angle.

3.12 EQUATIONS OF A STRAIGHT LINE

(1) General form. Two linear equations in x, y, z

i.e., $$ax + by + cz + d = 0 \quad ...(i)$$
and $$a'x + b'y + c'z + d' = 0 \quad ...(ii)$$
taken together represent a straight line which is the line of intersection of the planes (*i*) and (*ii*). (Fig. 3.38).

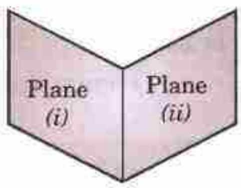


Fig. 3.38

(2) Symmetrical form. *Equations of the line through the point $A(x_1, y_1, z_1)$ and having direction cosines l, m, n are*
$$\frac{\mathbf{x - x_1}}{\mathbf{l}} = \frac{\mathbf{y - y_1}}{\mathbf{m}} = \frac{\mathbf{z - z_1}}{\mathbf{n}}$$
Let $P(x, y, z)$ be any point on the given line through $A(x_1, y_1, z_1)$ and parallel to the unit vector $\mathbf{U} = l\mathbf{I} + m\mathbf{J} + n\mathbf{K}$.

Since $\overrightarrow{AP}$ is parallel to $\mathbf{U}$, we can write $\overrightarrow{AP} = t\mathbf{U}$, where t is a parameter. ...(*i*)

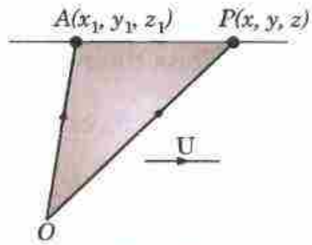


Fig. 3.39

or $$(x - x_1)\,\mathbf{I} + (y - y_1)\,\mathbf{J} + (z - z_1)\,\mathbf{K} = t\,(l\mathbf{I} + m\mathbf{J} + n\mathbf{k})$$

$\therefore$ $$x - x_1 = tl,\ y - y_1 = tm,\ z - z_1 = tn \qquad ...(ii)$$

Every point P on the line is given by (*ii*) for some value of t. Thus these are the parametric equations of the given line. Eliminating t, we get

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \qquad ...(iii)$$

which are the *symmetrical form of the equations of the line.*

Obs. Any point on the line (*iii*) is $(x_1 + lt, y_1 + mt, z_1 + nt)$.

Cor. *The equations of the line joining the points* (x_1, y_1, z_1) *and* (x_2, y_2, z_2) *are*

$$\frac{\mathbf{x} - \mathbf{x_1}}{\mathbf{x_2} - \mathbf{x_1}} = \frac{\mathbf{y} - \mathbf{y_1}}{\mathbf{y_2} - \mathbf{y_1}} = \frac{\mathbf{z} - \mathbf{z_1}}{\mathbf{z_2} - \mathbf{z_1}}$$

for the direction-ratios of the line joining the given points are

$$x_2 - x_1, y_2 - y_1, z_2 - z_1.$$

[**To reduce the general equation of a line of the symmetrical form:**

(*i*) *find a point on the line,* by putting $z = 0$ in the the given equations and solving the resulting equations for x and y.

(*ii*) *find the direction cosines of the line,* from the fact that it is perpendicular to the normals to the given planes.

(*iii*) *write the equations of the line in the symmetrical form.*]

Example 3.35. *Find in symmetrical form, the equations of the line*

$$x + y + z + 1 = 0,\ 4x + y - 2z + 2 = 0. \qquad \text{(Osmania, 1999)}$$

Solution. (*i*) To find a point on the line.

Putting $z = 0$ in the given equations, we have

$$x + y + 1 = 0;\ 4x + y + 2 = 0$$

Solving, $\dfrac{x}{1} = \dfrac{y}{2} = \dfrac{1}{-3}$ $\quad\therefore$ A point on the line is $(-1/3, -2/3, 0)$.

(*ii*) *To find the direction cosines l, m, n of the line.*

Since the line lies on both the given planes.

$\therefore$ It is perpendicular to their normals whose direction cosines are proportional to 1, 1, 1 and 4, 1, – 2.

i.e., $$l + m + n = 0;\ 4l + m - 2n = 0.$$

Solving, $$\frac{l}{-1} = \frac{m}{2} = \frac{n}{-1}$$

$\therefore$ The direction cosines of the given line are proportional to – 1, 2, – 1.

(*iii*) Thus the equations of the line in the symmetrical form are

$$\frac{x + 1/3}{-1} = \frac{y + 2/3}{2} = \frac{z}{-1}.$$

Example 3.36. *Find the distance of the point (1, – 2, 3) from the plane x – y + z = 5 measured parallel to the line*

$$\frac{x}{2} = \frac{y}{3} = \frac{z}{-6} \qquad \text{(Calicut, 1999)}$$

Solution. The line through $P(1, -2, 3)$ having direction ratios $(2, 3, -6)$ is

$$\frac{x - 1}{2} = \frac{y + 2}{3} = \frac{z - 3}{-6} = r.$$

Any point on this line is $(2r + 1, 3r - 2, 3 - 6r)$.

This point will lie on the plane $x - y + z = 5$

if $$2r + 1 - (3r - 2) + 3 - 6r = 5 \quad \text{or} \quad r = 1/7.$$

$\therefore$ The point of intersection is $Q(9/7, -11/7, 15/7)$

Thus the required distance = $PQ = \sqrt{\left(\frac{4}{49} + \frac{9}{49} + \frac{36}{49}\right)} = 1$

$x + 2y + 2z = 9$, $4x - 3y + 12z + 12 = 0$ and specify the one which bisects the acute angle.

Example 3.37. (*a*) *Find the image (reflection) of the point* (p, q, r) *in the plane* $2x + y + z = 6$.

(*b*) *Find the image (reflection) of the line* $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{4}$ *in the same plane.* (*Delhi, 2002*)

[*If two points P, P′ be such that the line PP′ is bisected perpendicularly by a plane then either of the points is the image (or reflection) of the other in the plane.*]

Solution. (*a*) Let $P'(p', q', r')$ be the image of $P(p, q, r)$. Then the mid-point of PP' must lie on the given plane.

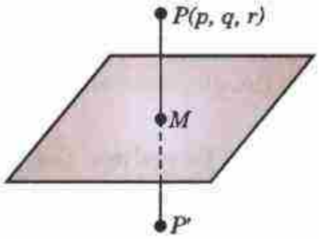


Fig. 3.40(a)

$$\therefore \qquad \frac{p+p'}{2} + \frac{q+q'}{2} + \frac{r+r'}{2} = 6 \qquad ...(i)$$

Also the line PP' must be perpendicular to the plane. The direction ratios of PP' being $p - p'$, $q - q'$, $r - r'$, we therefore, have

$$\frac{p-p'}{2} = \frac{q-q'}{1} = \frac{r-r'}{1} = k \text{ (say)}$$

whence $p' = p - 2k$, $q' = q - k$, $r' = r - k$.

Substituting these in (*i*) and solving, we get

$$k = \frac{1}{3}(2p + q + r - 6).$$

Hence P' is

$$\left[\frac{1}{3}(12 - p - 2q - 2r), \frac{1}{3}(6 - 2p + 2q - r), \frac{1}{3}(6 - 2p - q + 2r)\right] \qquad ...(ii)$$

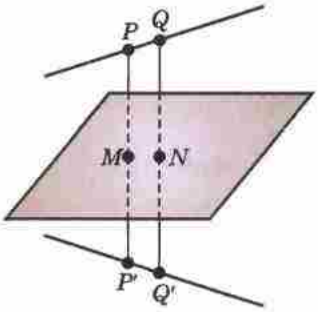


Fig. 3.40(b)

(*b*) Any two points on the given line are evidently $P(1, 2, 3)$ and (on putting $z = 7$) $Q(3, 3, 7)$. Their images are [by using (*ii*)] $P'\left(\frac{1}{3}, \frac{5}{3}, \frac{8}{3}\right)$ and $Q'\left(-\frac{11}{3}, -\frac{1}{3}, \frac{11}{3}\right)$. The line joining P' and Q' is, therefore

$$\frac{x - \frac{1}{3}}{-\frac{11}{3} - \frac{1}{3}} = \frac{y - \frac{5}{3}}{-\frac{1}{3} - \frac{5}{3}} = \frac{z - \frac{8}{3}}{\frac{11}{3} - \frac{8}{3}}, \text{ i.e., } \frac{3x-1}{4} = \frac{3y-5}{2} = \frac{3z-8}{-1}$$

which is the required image of the given line PQ [Fig. 3.40(*b*)].

Example 3.38. *Find the angle between the line*

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

and the plane $ax + by + cz + d = 0$.

Solution. If θ be the angle between the line and the plane, then $90° - \theta$ is the angle between the line and the normal to the plane (Fig. 3.41).

Now the direction ratios of the line are l, m, n and the direction ratios of the normal to the plane are a, b, c.

$$\therefore \qquad \cos(90° - \theta) = \frac{la + mb + nc}{\sqrt{(l^2 + m^2 + n^2)}\sqrt{(a^2 + b^2 + c^2)}}$$

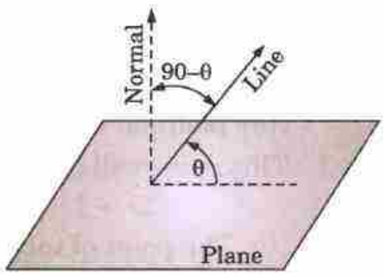


Fig. 3.41

or $$\sin\theta = \frac{la + mb + nc}{\sqrt{(\Sigma l^2)}\sqrt{(\Sigma a^2)}}$$

Hence the required angle $\theta = \sin^{-1}\left(\frac{al + bm + cn}{\sqrt{(\Sigma l^2)}\sqrt{(\Sigma a^2)}}\right)$

Cor. *If the line is parallel to the plane, sin θ = 0*

$\therefore$ **al + bm + cn = 0**

If the line is perpendicular to the plane, it will be parallel to its normal.

$\therefore$ $l/a = m/b = n/c.$

Example 3.39. *Find the equations of the two straight lines through the origin, each of which intersects the straight line* $\frac{1}{2}(x-3) = y - 3 = z$ *and is inclined at an angle of 60° to it.*

Solution. Let AB be the given line so that any point A on it is $(2r + 3, r + 3, r)$. (Fig. 3.42)

$\therefore$ Direction ratios of OA are $2r + 3 - 0, r + 3 - 0, r - 0$.

Angle between AO and AB has to be 60°,

$$\therefore \quad \cos 60° = \frac{2(2r+3) + 1(r+3) + 1.(r)}{\sqrt{2^2 + 1^2 + 1^2)}\sqrt{[2r+3)^2 + (r+3)^2 + r^2]}}$$

or $$\frac{1}{2} = \frac{6r+9}{\sqrt{[6(6r^2 + 18r + 18)]}} \text{ or } r^2 + 3r + 2 = 0 \text{ i.e., } r = -1, -2$$

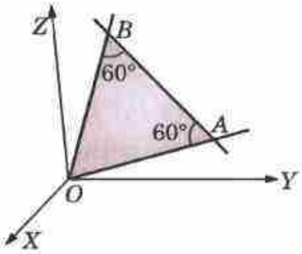


Fig. 3.42

$\therefore$ Coordinates of A and B are $(1, 2, -1)$ and $(-1, 1, -2)$.

Hence the equations of the required lines OA and OB are $\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$ and $\frac{x}{-1} = \frac{y}{1} = \frac{z}{-2}$

PROBLEMS 3.7

1. Prove that the points (3, 2, 4), (4, 5, 2) and (5, 8, 0) are collinear. Find the equations of the line through them.
2. Find the angle between the line of intersection of the planes
   $2x + 2y - z + 15 = 0$, $4y + z + 29 = 0$ and the line $\frac{x+4}{4} = \frac{y-3}{-3} = \frac{z+2}{1}$. (*V.T.U., 2003 S*)
3. Find the angle between the line of intersection of the planes $3x + 2y + z = 5$ and $x + y - 2z = 3$ and the line of intersection of the plane $2x = y + z$ and $7x + 10y = 8z$.
4. Find the equation of the line through the point $(-2, 3, 4)$ and parallel to the planes $2x + 3y + 4z = 5$ and $4x + 3y + 5z = 6$.
5. Show that the line $\frac{x-1}{3} = \frac{y+2}{-2} = \frac{z-1}{2}$ is parallel to the plane $2x + 2y - z = 6$, and find the distance between them.
6. Find the equation of the line through $(1, 2, -1)$ perpendicular to each of the lines
   $\frac{x}{1} = \frac{y}{0} = \frac{z}{-1}$ and $\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$.
7. Find the equation of the lines bisecting the angle between the lines
   $\frac{x-1}{2} = \frac{y+2}{-2} = \frac{z-3}{1}$; $\frac{x-1}{12} = \frac{y+2}{4} = \frac{z-3}{-3}$.
8. Find the foot of the perpendicular from (1, 1, 1) to the line joining the points (1, 4, 6) and (5, 4, 4). (*V.T.U., 2010*)
9. Find the perpendicular distance of the point (1, 1, 1) from the line
   $\frac{x-2}{2} = \frac{y+3}{2} = \frac{z}{-1}$.

10. Find the distance of the point (3, – 4, 5) from the plane $2x + 5y - 6z = 16$, measured parallel to the line $x/2 = y/1 = z/-2$. (V.T.U., 2002)
11. Find the reflection (image) of the point
(*i*) (1, 2, 3) in the plane $x + y + z = 9$. (V. T.U., 2010)
(*ii*) (2, – 1, 3) in the plane $3x - 2y - z - 9 = 0$.
12. Find the angle between the line $\frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6}$ and the plane $3x + y + z = 7$.
13. Find the equation of the plane through the points (1, 0, – 1), (3, 2, 2) and parallel to the line
$$x - 1 = \frac{1}{2}(1 - y) = \frac{1}{3}(z - 2). \quad \text{(V.T.U., 2000)}$$
14. Find the equations of the straight line which passes through the point (2, – 1, – 1), is parallel to the plane $4x + y + z + 2 = 0$ and is perpendicular to the line $2x + y = 0 = x - z + 5$.

3.13 CONDITIONS FOR A LINE TO LIE IN A PLANE

To find the conditions that the line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ ...(1)

may lie in the plane $ax + by + cz + d = 0$ ...(2)

Any point on the line (1) is $(lr + x_1, mr + y_1, nr + z_1)$ which will lie on the plane (2), if
$$a(lr + x_1) + b(mr + y_1) + c(nr + z_1) + d = 0.$$
or if
$$(al + bm + cn)\,r + (ax_1 + by_1 + cz_1 + d) = 0 \quad ...(3)$$
The line (1) will lie in the plane (2), if every point of the line lies in the plane so that (3) is satisfied by all values of r.

∴ The coefficient of $r = 0$ and the constant term = 0.

i.e.,
$$\mathbf{al + bm + cn = 0} \quad ...(4)$$
and
$$\mathbf{ax_1 + by_1 + cz_1 + d = 0} \quad ...(5)$$
These are the required conditions which state that

(*i*) *the line should be parallel to the plane,* (*ii*) *a point of line should lie in the plane.*

Thus the equation of any plane through the line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$

is
$$\mathbf{a(x - x_1) + b(y - y_1) + c(z - z_1) = 0} \textit{ where } \mathbf{al + bm + cn = 0.}$$

Obs. The equation of any plane through the line of intersection of the planes
$$ax + by + cz + d = 0 \quad ...(i)$$
and
$$a'x + b'y + c'z + d' = 0. \quad ...(ii)$$
is
$$\mathbf{ax + by + cz + d + k(a'x + b'y + c'z + d') = 0.}$$
For (*i*) is an equation of the first degree in x, y, z representing a plane and (*ii*) it is satisfied by the coordinates of the points which satisfy both the given planes, *i.e.*, it contains all the points common to these planes.

Example 3.40. *Obtain the equation of a plane passing through the line of intersection of the planes $7x - 4y + 7z + 16 = 0$ and $4x + 3y - 2z + 13 = 0$ and perpendicular to the plane $x - y - 2z + 5 = 0$.* (V.T.U., 2009)

Solution. The equation of any plane through the line of intersection of the two given planes is
$$7x - 4y + 7z + 16 + k(4x + 3y - 2z + 13) = 0$$
or
$$(7 + 4k)x + (-4 + 3k)y + (7 - 2k)z + (16 + 13k) = 0 \quad ...(i)$$
The plane (*i*) will be perpendicular to the plane
$$x - y - 2z + 5 = 0 \text{ if their normals are perpendicular,}$$
i.e., if
$$(7 + 4k)\,.\,1 + (-4 + 3k)\,.\,(-1) + (7 - 2k)\,.\,(-2) = 0 \quad \text{or} \quad \text{if, } k = 3/5.$$
Substituting this value of k in (*i*), we get
$$(7 + 12/5)x + (-4 + 9/5)y + (7 - 6/5)x + (16 + 39/5) = 0$$
or
$$47x - 11y + 29z + 119 = 0 \text{ which is the required equation.}$$

Example 3.41. *Find the equation in the symmetrical form of the projection of the line* $\frac{x-1}{2} = -(y+1) = \frac{z-3}{4}$ *on the plane* $x + 2y + z = 12$.

Solution. Any plane through the given line is

$$A(x-1) + B(y+1) + C(z-3) = 1 \qquad ...(i)$$

where

$$2A - B + 4C = 0 \qquad ...(ii)$$

The plane (i) will be perpendicular to the given plane, if

$$A + 2B + C = 0 \qquad ...(iii)$$

Solving (ii) and (iii), we get $\frac{A}{-9} = \frac{B}{2} = \frac{C}{5}$.

Substituting these values in (i), we get $9x - 2y - 5z + 4 = 0$ ...(iv)

which cuts the given plane $x + 2y + z = 12$ ...(v)

along the required line of projection.

One point on this line is got by putting $z = 0$ in (iv) and (v) and solving, it is $(4/5, 28/5, 0)$.

The direction ratios of the line are found, by solving

$$l + 2m + n = 0 \quad \text{and} \quad 9l - 2m - 5n = 0$$

to be $4, -7, 10$.

Hence the required equations of the line of projection are

$$\frac{x - 4/5}{4} = \frac{y - 28/5}{-7} = \frac{z}{10}$$

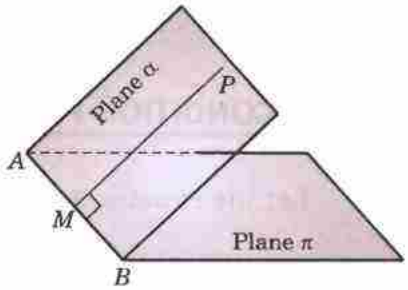


Fig. 3.43

[**The line of greatest slope** in a plane is a line which lies in the plane and is perpendicular to the line of intersection of the plane with the horizontal plane.

In Fig. 3.43, AB is the line of intersection of the given plane α with the horizontal plane π. Then PM drawn perpendicular to AB, is the line of greatest slope on the plane α through the point P.]

Example 3.42. *Assuming the line* $x/4 = y/-3 = z/7$ *as vertical, find the equations of the line of greatest slope in the plane* $2x + y - 5z = 12$ *and passing through the point* $(2, 3, -1)$.

Solution. The equation of the horizontal plane through the origin is $4x - 3y + 7z = 0$ ...(i)

[The direction ratios of the normal are those of the given vertical line.]

If l, m, n be the direction ratios of the line of intersection of the plane (i) and

$$2x + y - 5z = 12 \qquad ...(ii)$$

then solving, $4l - 3m + 7n = 0$ and $2l + m - 5n = 0$, we have $l/4 = m/17 = n/5$ ...(iii)

Let l', m', n' be the direction ratios of the line of greatest slope which lies in the plane (ii).

$$\therefore \quad 2l' + m' - 5n' = 0 \qquad ...(iv)$$

Also the line of greatest slope is perpendicular to the line of intersection of the planes (i) and (ii).

$$\therefore \quad 4l' + 17m' + 5n' = 0 \qquad ...(v)$$

Solving (iv) and (v), $\frac{l'}{3} = \frac{m'}{-1} = \frac{n'}{1}$.

Hence the equations of the line of greatest slope through $(2, 3, -1)$ and having direction ratios $3, -1, 1$ are

$$\frac{x-2}{3} = \frac{y-3}{-1} = \frac{z+1}{1}.$$

PROBLEMS 3.8

1. Find the equation of the plane which contains the line $\frac{x-1}{2} = y + 1 = \frac{z-3}{4}$ and is perpendicular to the plane $x + 2y + z = 12$. *(V.T.U., 2006)*

2. Find the equation of the plane through the line $\frac{x-1}{3}=\frac{y-4}{2}=\frac{z-4}{-2}$ and parallel to the line $\frac{x+1}{3}=\frac{y-1}{-4}=\frac{z+2}{1}$.
3. Find the equation of the plane passing through the line of intersection of the planes $x+y+z=1$ and $2x+3y-z+4=0$ and perpendicular to the plane $2y-3z=4$.
4. Find the equation of the plane which contains the line of intersection of the planes $x+y+z=3$ and $2x-y+3z=4$ and is parallel to the line joining the points (2, 1, 1) and (3, 2, 4). *(Madras, 2006)*
5. Find in symmetric form the equations of the line which lies in the plane $2x-y-3z=4$ and is perpendicular to the line
$$\frac{x+1}{3}=\frac{y-1}{3}=\frac{z+4}{2}$$
at the point where the line pierces the plane.
6. A plane is drawn through the line $x+y=1, z=0$ to make an angle $\sin^{-1}(1/3)$ with the plane $x+y+z=0$. Prove that two such planes can be drawn and find their equations. Prove also that the angle between the planes is $\cos^{-1}(7/9)$.
7. Find the equations of the projection of the line $3x-y+2z-1=x+2y-z-2=0$ on the plane $3x+2y+z=0$ in the symmetrical form.
8. Assuming the plane $4x-3y+7z=0$ to be horizontal, find the equations of the line of greatest slope through the point (2, 1, 1) in the plane $2x+y-5z=0$. *(Roorkee, 2000)*

3.14 CONDITION FOR THE TWO LINES TO INTERSECT (OR TO BE COPLANAR)

Let the equations of the lines be $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ ...(1)

$$\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2} \quad ...(2)$$

The equation of any plane through the line (1) is $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$ ...(3)

where $al_1+bm_1+cn_1=0$ ...(4)

The line (2) will lie in the plane (3), if it is parallel to the plane and its point (x_2, y_2, z_2) lies on this plane.

$\therefore$ $al_2+bm_2+cn_2=0$ ...(5)

and $a(x_2-x_1)+b(y_2-y_1)+c(z_2-z_1)=0$ ...(6)

Eliminating a, b, c from (6), (4) and (5), we get

$$\begin{vmatrix} \mathbf{x_2-x_1} & \mathbf{y_2-y_1} & \mathbf{z_2-z_1} \\ \mathbf{l_1} & \mathbf{m_1} & \mathbf{n_1} \\ \mathbf{l_2} & \mathbf{m_2} & \mathbf{n_2} \end{vmatrix} = 0 \text{ which is the required condition.}$$

Also eliminating a, b, c from (3), (4) and (5), we get $\begin{vmatrix} \mathbf{x-x_1} & \mathbf{y-y_1} & \mathbf{z-z_1} \\ \mathbf{l_1} & \mathbf{m_1} & \mathbf{n_1} \\ \mathbf{l_2} & \mathbf{m_2} & \mathbf{n_2} \end{vmatrix} = \mathbf{0}$

which is the equation of the plane containing the lines (1) and (2).

Example 3.43. *Show that the lines* $\frac{x-5}{4}=\frac{y-7}{4}=\frac{z+3}{-5}$; $\frac{x-8}{7}=\frac{y-4}{1}=\frac{z-5}{3}$ *are coplanar; find their common point and the equation of the plane in which they lie.* *(Madurai, 2002)*

Solution. Any point on the first line is $(5+4r, 7+4r, -3-5r)$ ...(*i*)

which lies on the second line if $\frac{-3+4r}{7}=\frac{3+4r}{7}=\frac{-8-5r}{3}$ ...(*ii*)

$\therefore$ From $\frac{-3+4r}{7}=3+4r$, we have $r=-1$.

This value clearly satisfies the equation $\frac{3+4r}{7}=\frac{-8-5r}{3}$

Hence the lines intersect, (*i.e.*, are coplanar) and from (*i*) their point of intersection is (1, 3, 2).

The equation of the plane in which they lie, is $\begin{vmatrix} x-5 & y-7 & z+3 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix} = 0$

i.e., $17x - 47y - 24z + 172 = 0.$

Example 3.44. *Show that the lines*

$$\frac{x+4}{3} = \frac{y+6}{5} = \frac{z-1}{-2} \text{ and } 3x - 2y + z + 5 = 0 = 2x + 3y + 4z - 4$$

are coplanar. Find their point of intersection and the plane in which they lie.

Solution. Any point on the first line is $P(3r - 4, 5r - 6, -2r + 1)$, which lie in the plane

$$3x - 2y + z + 5 = 0$$

if $$3(3r - 4) - 2(5r - 6) + (-2r + 1) + 5 = 0 \quad \text{or} \quad r = 2,$$

The point P will also lie in the plane $2x + 3y + 4z - 4 = 0$

if $$2(3r - 4) + 3(5r - 6) + 4(-2r + 1) - 4 = 0 \quad \text{or} \quad r = 2.$$

Since the two values of r are equal, the given lines intersect, *i.e.*, are coplanar.

Putting $r = 2$ in the coordinates of P, we get $(2, 4, -3)$ as their point of intersection.

The equation of a plane containing the second line is

$$3x - 2y + z + 5 + k(2x + 3y + 4z - 4) = 0$$

which will contain the first line if its point $(-4, -6, 1)$ lies on it.

$$\therefore \quad -12 + 12 + 1 + 5 + k(-8 - 18 + 4 - 4) = 0$$

i.e., $$k = 3/13$$

Substituting this value of k, (i) becomes $45x - 17y + 25z + 53 = 0$, which is the required plane.

Example 3.45. *Find the equations of the line drawn through the point $(1, 0, -1)$ and intersecting the lines*

$$x = 2y = 2z \quad \text{and} \quad 3x + 4y = 1, 4x + 5z = 2.$$ *(V.T.U., 2007)*

Solution. The required line will comprise of

(a) the plane containing the first line and the point $(1, 0, -1)$.

(b) the plane containing the second line and the point $(1, 0, -1)$.

The equation of any plane containing the first line

i.e., $$\frac{x}{2} = \frac{y}{1} = \frac{z}{1}$$

is $$a(x - 0) + b(y - 0) + c(z - 0) = 0 \quad ...(i)$$

where $$2a + b + c = 0 \quad ...(ii)$$

Also $(1, 0, -1)$ lies on (i) $\therefore$ $$a - c = 0 \quad ...(iii)$$

Solving (ii) and (iii), we have $\dfrac{a}{1} = \dfrac{b}{-3} = \dfrac{c}{1}$.

Substituting these values in (i), we get $x - 3y + z = 0$ ...(iv)

Again, the equation of any plane containing the second line is

$$3x + 4y - 1 + k(4x + 5z - 2) = 0. \text{ Also } (1, 0, -1) \text{ lies on it.} \quad ...(v)$$

$$\therefore \quad 3 + 0 - 1 + k(4 - 5 - 2) = 0, \quad i.e., \quad k = \frac{2}{3}.$$

Substituting $k = 2/3$ in (v), we get $17x + 12y + 10z - 7 = 0$ ...(iv)

Hence (iv) and (vi) constitute the required line.

PROBLEMS 3.9

1. Prove that the lines

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \text{ and } \frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$$

are coplanar and find the equation of the plane containing them.

2. Prove that the lines $\frac{x-4}{1}=\frac{y+3}{-4}=\frac{1+z}{7}$ and $\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}$ intersect and find the coordinates of their point of intersection. *(V.T.U., 2000 S ; Andhra, 1999)*

3. Find the condition that the lines $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ and $ax+by+cz+d=0=a'x+b'y+c'z+d'$ are coplanar.

4. Show that the lines $\frac{x+1}{1}=\frac{y+1}{2}=\frac{z+1}{3}$ and $x+2y+3z-8=0=2x+3y+4z-11$ intersect. Find their point of intersection and the equation of the plane containing them. *(V.T.U., 2009)*

5. Show that the lines $x-3y+2z-4=0=2x+y+4z+1$ and
$3x+2y+5z-1=0=2y-z$, are coplanar. *(Andhra, 2000)*

6. Prove that the lines $x=ay+b=cz+d$ and $x=\alpha y+\beta=\gamma z+\delta$ are coplanar if
$(\gamma-c)(a\beta-b\alpha)-(\alpha-a)(c\delta-d\gamma)=0$ *(Rajasthan, 2006)*

7. Obtain the equations of the straight line lying in the plane.

$$x-2y+4z-51=0$$

and interesecting the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-6}{7}$ at right angles.

8. Find the equation of the straight line perpendicular to both the lines

$$\frac{x-1}{1}=\frac{y-1}{2}=\frac{z+2}{3} \quad\text{and}\quad \frac{x+2}{2}=\frac{y-5}{-1}=\frac{z+3}{2}$$

and passing through their point of intersection.

9. A line with direction cosines proportional to 2, 7, – 5 is drawn to intersect the lines

$$\frac{x-8}{3}=\frac{y-6}{-1}=\frac{z+1}{1} \quad\text{and}\quad \frac{x+3}{-3}=\frac{y-3}{2}=\frac{z-6}{4}.$$

Find the coordinates of the point of intersection and the length intercepted.

3.15 SHORTEST DISTANCE BETWEEN TWO LINES

[Two straight lines which do not lie in one plane are called *skew lines*. Such lines possess a common perpendicular which is the *shortest distance* between them.

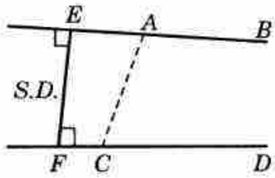


Fig. 3.44

Let the given skew lines AB and CD be

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \quad\text{and}\quad \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$$

so that $A\equiv(x_1, y_1, z_1)$ and $C\equiv(x_2, y_2, z_2)$.

Let l, m, n be the direction cosines of the shortest distance EF.

Since $EF \perp$ to both AB and CD.

$\therefore$ $ll_1+mm_1+nn_1=0$ and $ll_2+mm_2+nn_2=0$.

Solving,
$$\frac{l}{m_1n_2-m_2n_1}=\frac{m}{n_1l_2-n_2l_1}=\frac{n}{l_1m_2-l_2m_1}$$

$$=\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{[\Sigma(m_1n_2-m_2n_1)^2]}}=\frac{1}{\sin\theta} \qquad ...(1)$$

where θ is the angle between the lines AB and CD.

$\therefore$ *Length of S.D.* (EF) = projection of AC on EF

$= l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)$ where l, m, n have the values as given by (1).

To find the equations of the line of shortest distance, we observe that it is coplanar with both AB and CD.

Plane containing the lines AB and EF is, $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0$ ...(2)

Plane containing the lines CD and EF is $\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix} = 0$ ...(3)

Hence (2) and (3) are the equations of the line of shortest distance.

Obs. The condition for the given lines to be coplanar is also obtained by equating the shortest distance (EF) to zero.

Example 3.46. *Find the magnitude and the equations of the shortest distance between the lines*

$$\frac{x}{2} = \frac{y}{-3} = \frac{z}{1} \quad \text{and} \quad \frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}.$$ *(V.T.U., 2009 ; Cochin, 2005)*

Solution. Let l, m, n be the direction cosines of the shortest distance EF.

$\because$ $EF \perp$ to both AB and CD,

$\therefore$ $2l - 3m + n = 0, 3l - 5m + 2n = 0.$

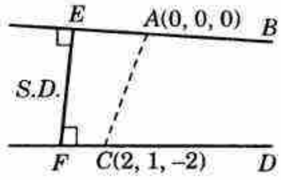


Fig. 3.45

Solving $\frac{l}{1} = \frac{m}{1} = \frac{n}{1} = \frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{(1+1+1)}} = \frac{1}{\sqrt{3}}.$

$\therefore$ *Length of S.D.* (EF) = projection of AC on EF

$$= (2-0)\frac{1}{\sqrt{3}} + (1-0)\frac{1}{\sqrt{3}} + (-2-0)\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

The equations of the line of shortest distance (EF) are

$$\begin{vmatrix} x & y & z \\ 2 & -3 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0 \text{ and } \begin{vmatrix} x-2 & y-1 & z+2 \\ 3 & -5 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

i.e., $4x + y - 5z = 0$ and $7x + y - 8z = 31.$

Example 3.47. *Find the points on the lines*

$$\frac{x-6}{3} = \frac{y-7}{-1} = \frac{z-4}{1} \quad ...(i)$$

$$\frac{x}{-3} = \frac{y+9}{2} = \frac{z-2}{4} \quad ...(ii)$$

which are nearest to each other. Hence find the shortest distance between the lines and its equations.

(V.T.U., 2004 ; Burdwan, 2003 ; Osmania, 2003)

Solution. Any point on the line (*i*) is $E(6 + 3r, 7 - r, 4 + r)$ ...(*iii*)

and any point on the line (*ii*) is $F(-3r', -9 + 2r', 2 + 4r')$ ...(*iv*)

Then the direction cosines of EF are proportional to $6 + 3r + 3r', 16 - r - 2r', 2 + r - 4r'$

Since $EF \perp$ both the lines (*i*) and (*ii*), $\therefore$ $3(6 + 3r + 3r') - (16 - r - 2r') + (2 + r - 4r') = 0$

and $-3(6 + 3r + 3r') + 2(16 - r - 2r') + 4(2 + r - 4r') = 0$

or $11r + 7r' + 4 = 0, 7r + 29r' - 22 = 0$, whence $r = -1, r' = 1.$

Substituting $r = -1$ in (*iii*) and $r' = 1$ in (*iv*), we get $E = (3, 8, 3)$ and $F = (-3, -7, 6)$ which are the points on (*i*) and (*ii*) nearest to each other.

$\therefore$ Length of the shortest distance (EF) = $\sqrt{[(3+3)^2 + (8+7)^2 + (3-6)^2]} = 3\sqrt{30}$

The equations of the shortest distance (EF) is $\frac{x-3}{2} = \frac{y-8}{5} = \frac{z-3}{-1}.$

Obs. This method is sometimes very convenient and is especially useful when the points of intersection of the line of shortest distance with the given lines are required.

Example 3.48. *Two control cables in the form of straight lines AB and CD are laid such that the coordinates of A, B, C and D are respectively (1, 2, 3), (2, 1, 1), (– 1, 1, 2) and (2, – 1, –3). Determine the amount of clearance between the cables.*

Solution. The direction ratios of AB are 1, – 1, – 2 and those of CD are 3, – 2, – 5.

The amount of clearance between AB and CD is nothing but the shortest distance PQ between the cables. If the direction cosines of PQ be l, m, n then

$$l - m - 2n = 0 \text{ and } 3l - 2m - 5n = 0$$

$$\therefore \qquad \frac{l}{1} = \frac{m}{-1} = \frac{n}{1} \qquad [\because \ PQ \perp \text{ to both } AB + CD].$$

Thus the clearance between the cables

$$= \text{shortest distance between } AB \text{ and } CD$$
$$= \text{projection of } AC \text{ (or } BD\text{) on } PQ$$
$$= \frac{1(-1-1) - 1(1-2) + 1(2-3)}{\sqrt{(1+1+1)}} = \frac{2}{\sqrt{3}} \text{ (in magnitude)}$$

Example 3.49. *Find the equation of the plane through the line*

$$\frac{x-1}{3} = \frac{y-4}{2} = \frac{z-4}{-2} \qquad ...(i)$$

and parallel to the line $\frac{x+1}{2} = \frac{y-1}{-4} = \frac{z+2}{1}$ ...(*ii*)

Hence find the shortest distance between them (*Hazaribagh, 2009*)

Solution. The equation of the plane containing the line (*i*) and parallel to (*ii*) is

$$\begin{vmatrix} x-1 & y-4 & z-4 \\ 3 & 2 & -2 \\ 2 & -4 & 1 \end{vmatrix} = 0$$

or
$$6x + 7y + 16z = 98 \qquad ...(iii)$$

Now the shortest distance between the lines (*i*) and (*ii*)

= Length of the perpendicular drawn from the point (– 1, 1, – 2) of (*ii*) on the plane (*iii*)

$$= \frac{-6+7-32-98}{\sqrt{(6^2+7^2+16^2)}} = \frac{120}{\sqrt{(341)}}, \text{ numerically.}$$

Example 3.50. *Show that the shortest distance between z-axis and the line* $ax + by + cz + d = 0 = a'x + b'y + c'z + d'$ *is* $\dfrac{dc' - d'c}{\sqrt{[(ac' - a'c)^2 + (bc' - b'c)^2]}}$.

Solution. The plane containing the given line is

$$(ax + by + cz + d) + k(a'x + b'y + c'z + d') = 0 \qquad ...(i)$$

or
$$(a + ka')x + (b + kb')y + (c + kc')z + (d + kd') = 0$$

This plane is parallel to the z-axis ($d.\ c's$, 0, 0, 1) if $c + kc' = 0$ or $k = -c/c'$. Then (*i*) becomes

$$(ac' - a'c)x + (bc' - b'c)y + (dc' - d'c) = 0 \qquad ...(ii)$$

A point on the z-axis is the origin.

$\therefore$ $\perp$ distance of the origin from the plane (*ii*)

$$= \frac{dc' - d'c}{\sqrt{[(ac' - a'c)^2 + (bc' - b'c)^2]}} \text{ which is the required S.D.}$$

Example 3.51. *A square ABCD of diagonal 2a is folded along the diagonal AC, so that the planes DAC and BAC are at right angles. Find the shortest distance between DC and AB.*

Solution. Let the diagonals AC and BD intersect at O the folded position of the square. Let OB, OC and OD be the axes. Then equations of DC are

$$\frac{x-0}{0-0}=\frac{y-a}{a-0}=\frac{z-0}{0-a} \quad \text{or} \quad \frac{x}{0}=\frac{y-a}{a}=\frac{z}{-a}$$

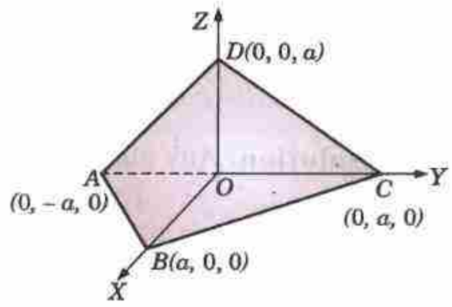


Fig. 3.46

and those of AB are $\dfrac{x-a}{a}=\dfrac{y}{a}=\dfrac{z}{0}$

The equation of the plane through DC and parallel to AB is

$$\begin{vmatrix} x & y-a & z \\ 0 & a & -a \\ a & a & 0 \end{vmatrix}=0 \quad \text{or} \quad x-y-z+a=0 \qquad ...(i)$$

A point on the line AB is $(a, 0, 0)$.

Hence required S.D. = $\perp$ distance of $(a, 0, 0)$ from the plane (i)

$$=\frac{a+a}{\sqrt{1+1+1}}=\frac{2a}{\sqrt{3}}.$$

PROBLEMS 3.10

1. Find the magnitude and the equations of the shortest distance between the lines.
$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \text{ and } \frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}.$$
(V.T.U., 2008 ; Rajasthan, 2005 ; Madras, 2003)

2. Find the magnitude and equations of the shortest distance between the lines
$$\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1} \text{ and } \frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$$ *(Anna, 2005 S ; Osmania, 2000 S)*
Find also the points where it intersects the lines.

3. Find the shortest distance and the equation of the line of shortest distance between the line $\dfrac{x-1}{2}=\dfrac{y-2}{3}=\dfrac{z-3}{4}$ and the y-axis. *(V.T.U., 2010)*

4. Show that the shortest distance between the lines $y - mx = 0 = z - c$ and $y + mx = 0 = z + c$ is c units.

5. If the shortest distance between the lines $\dfrac{y}{b}+\dfrac{z}{c}=1, x=0$ and $\dfrac{x}{a}-\dfrac{z}{c}=1, y=0$ be $2d$, then show that $d^{-2}=a^{-2}+b^{-2}+c^{-2}$.

6. Show that the shortest distance between x-axis and the line $ax+by+cz+d=0=a'x+b'y+c'z+d'$ is
$$\frac{da'-d'a}{\sqrt{[(ba'-b'a)^2+(ca'-c'a)^2]}}.$$

7. Show that the shortest distance between a diagonal of a rectangular parallelopiped whose edges are a, b, c and the edges not meeting it, are
$bc/(b^2+c^2)^{1/2}, ca/(c^2+a^2)^{1/2}, ab/(a^2+b^2)^{1/2}$.

8. Show that the shortest distance between two opposite edges of the tetrahedron formed by the planes $x+y=0$, $y+z=0, z+x=0$ and $x+y+z=a$ is $2a/\sqrt{6}$.

3.16 INTERSECTION OF THREE PLANES

Any three planes (no two of which are parallel) intersect in one of the following ways :

(1) *The planes may meet in a point,* if the line of section of two of them is not parallel to the third.

(2) *The planes may have a common line of section,* if the line of section of two of them lies on the third (Fig. 3.47).

(3) *The planes may form a triangular prism,* if the line of section of two of them is parallel to the third but does not lie on it. (See Fig. 3.48)

Example 3.52. *Prove that the planes*

(i) $12x - 15y + 16z - 28 = 0$, *(ii)* $6x + 6y - 7z - 8 = 0$. *and (iii)* $2x + 35y - 39z + 92 = 0$,

have a common line of intersection. Prove that the point in which the line $\frac{x-1}{3} = \frac{y}{-2} = \frac{z-3}{1}$ *meets the third plane is equidistant from other two planes.*

Solution. Any plane through the line of intersection of the planes (*i*) and (*ii*) is

$$12x - 15y + 16z - 28 + \lambda(6x + 6y - 7z - 8) = 0$$

or $$(12 + 6\lambda)x + (-15 + 6\lambda)y + (16 - 7\lambda)z - (28 + 8\lambda) = 0 \quad ...(iv)$$

Three planes will intersect in a common line if the planes (iii) and (iv) represent the same plane.

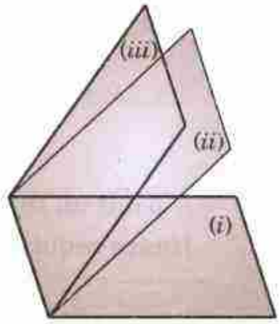


Fig. 3.47

$$\therefore \quad \frac{12 + 6\lambda}{2} = \frac{-15 + 6\lambda}{35} = \frac{16 - 7\lambda}{-39} = \frac{-28 - 8\lambda}{12} \quad ...(v)$$

From $\frac{12 + 6\lambda}{2} = \frac{-15 + 6\lambda}{35}$, we have $\lambda = \frac{-25}{11}$ which satisfies all the equations (*v*).

Hence the given planes intersect in a line.

Any point on the line $\frac{x-1}{3} = \frac{y}{-2} = \frac{z-3}{1} = r$ (say) ...(*vi*)

is $(3r + 1, -2r, r + 3)$ which lies in the plane (*iii*)

if $2(3r + 1) + 35(-2r) - 39(r + 3) + 12 = 0$, *i.e.* if $r = -1$.

$\therefore$ The coordinates of the point P in which (*vi*) meets (*iii*) are $(-2, 2, 2)$.

Distance of P from plane (*i*) $= \frac{12(-2) - 15(2) + 16(2) - 28}{\sqrt{144 + 225 + 256}} = \frac{-50}{\sqrt{(625)}} = 2$ (in magnitude)

Distance of P from plane (*ii*) $= \frac{6(-2) + 6(2) - 7(2) - 8}{\sqrt{36 + 36 + 49}} = 2$ (in magnitude)

Hence the point P is equidistant from the planes (*i*) and (*ii*).

Example 3.53. *Prove that the three planes*

(i) $2x + y + z = 3$, *(ii)* $x - y + 2z = 4$, *(iii)* $x + z = 2$,

form a triangular prism and find the area of the normal section of the prism.

Solution. Let l, m, n be the direction cosines of the line of intersection of the planes (*ii*) and (*iii*) so that $l - m + 2n = 0, l + n = 0$,

whence $$\frac{l}{1} = \frac{m}{-1} = \frac{n}{-1}.$$

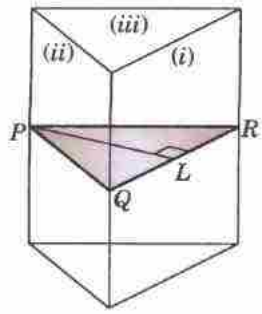


Fig. 3.48

To find a point P on this line, put $x = 0$ in (*ii*) and (*iii*), $-y + 2z = 4$ and $z = 2$. Thus the point P is $(0, 0, 2)$.

Now the line of intersection of (*ii*) and (*iii*) is parallel to the plane (*i*).

$[\because \quad 2 \times 1 + 1 \times (-1) + 1 \times (-1) = 0]$

Also the point P does not lie on the plane (*i*).

Hence the given planes form a triangular prism.

Let ΔPQR be its normal section through P.

The equation of the plane through P perpendicular to the line of intersection of the planes (*i*) and (*iii*) is,

$$1(x - 0) - 1(y - 0) - 1(z - 2) = 0$$

or $$x - y - z + 2 = 0 \quad ...(iv)$$

Solving the equations (*i*), (*ii*) and (*iv*), we get

$$Q \equiv \left(\frac{1}{3}, \frac{1}{3}, 2\right).$$

Solving the equation (*i*), (*iii*) and (*iv*), we get

$$R \equiv \left(\frac{1}{3}, \frac{2}{3}, \frac{5}{3}\right).$$

$$\therefore \qquad QR = \sqrt{\left\{\left(\frac{1}{3}-\frac{1}{3}\right)^2+\left(\frac{1}{3}-\frac{2}{3}\right)^2+\left(2-\frac{5}{3}\right)^2\right\}} = \sqrt{\left(\frac{2}{9}\right)}$$

Also $PL \perp$ from P on the plane (*i*) $= \dfrac{3-2}{\sqrt{(4+1+1)}} = \dfrac{1}{\sqrt{6}}$.

Hence the area of $\Delta PQR = \dfrac{1}{2} QR \times PL = \dfrac{1}{2} \cdot \dfrac{\sqrt{2}}{3} \cdot \dfrac{1}{\sqrt{6}} = \dfrac{1}{6\sqrt{3}}$.

PROBLEMS 3.11

1. Prove that the three planes $2x - 3y - 7z = 0$, $3x - 14y - 13z = 0$, $8x - 31y - 33z = 0$ pass through one line.
2. Prove that the planes $x = cy + bz$, $y = az + cx$, $z = bx + ay$ intersect in a line if $a^2 + b^2 + c^2 + 2abc = 1$ and show that the equations of this line are
$$\frac{x}{\sqrt{1-a^2}} = \frac{y}{\sqrt{1-b^2}} = \frac{z}{\sqrt{1-c^2}}$$
(*Rajasthan, 2005*)
3. Show that the planes $x + 2y - 3 = 0$, $3x - 4y + z - 4 = 0$ and $4x + 3y - 2z - 24 = 0$ from a triangular prism.
4. Prove that the planes $2x + 3y + 4z = 6$, $3x + 4y + 5z = 20$, $x + 2y + 3z = 0$ form a prism : obtain the equation of one of its edges in the symmetrical form.

3.17 SPHERE

(1) Def. *A* ***sphere*** *is the locus of a point which remains at a constant distance from a fixed point.*

The fixed point is called the *centre* and the constant distance the *radius* of the sphere

(2) The equation of the sphere whose centre is (a, b, c) and radius r, is

$$(\mathbf{x} - \mathbf{a})^2 + (\mathbf{y} - \mathbf{b})^2 + (\mathbf{z} - \mathbf{c})^2 = \mathbf{r}^2.$$

For the distance of any point $P(x, y, z)$ on the sphere from the centre $C(a, b, c)$ = the radius r.

In particular the *equation of the sphere whose centre is the origin and radius a, is*

$$\mathbf{x}^2 + \mathbf{y}^2 + \mathbf{z}^2 = \mathbf{a}^2$$

(3) The equation $\mathbf{x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0}$ represents a sphere whose centre is $(-\mathbf{u}, -\mathbf{v}, -\mathbf{w})$ and radius

$$= \sqrt{\mathbf{u}^2 + \mathbf{v}^2 + \mathbf{w}^2 - \mathbf{d}}.$$

For on writing it as $(x^2 + 2ux) + (y^2 + 2vy) + (z^2 + 2wz) = -d$

or as $\qquad (x + u)^2 + (y + v)^2 + (z + w)^2 = u^2 + v^2 + w^2 - d$

and comparing with $\qquad (x - a)^2 + (y - b)^2 + (z - c)^2 = r^2,$

it clearly represents a sphere whose centre is

$$(a, b, c) = (-u, -v, -w) \text{ and radius } r = \sqrt{(u^2 + v^2 + w^2 - d)}$$

Thus *the general equation of a sphere is such that*

(*i*) *it is the second degree in x, y, z,*

(*ii*) *the coefficient of x^2, y^2, z^2 are equal,*

and (*iii*) *there are no terms containing yz, zx or xy*

(4) Section of a sphere by a plane *is a circle and the section of a sphere by a plane through its centre is called a* **great circle.**

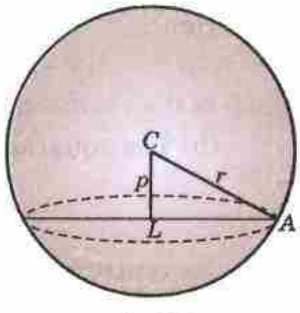


Fig. 3.49

Thus the equations $\qquad x^2 + y^2 + z^2 + 2ux\ 2vy + 2wz + d = 0$ [Sphere]

and $\qquad Ax + By + Cz + D = 0$ [Plane]

taken together represent a circle (Fig. 3.49) having centre L and radius $LA = \sqrt{(r^2 - p^2)}$.

(5) The equation of any sphere through the circle *of intersection of*

the sphere $S = 0$

and the plane $U = 0$

is $S + kU = 0$

For the equation $S + kU = 0$

represents a sphere and the points of intersection of the sphere $S = 0$ and the plane $U = 0$ satisfy it.

Example 3.54. *Find the equation of the sphere through the points (0, 0, 0), (0, 1, – 1), (– 1, 2, 0) and (1, 2, 3). Locate its centre and find the radius.*

Solution. Let the required equation of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad ...(i)$$

It passes through (0, 0, 0), (0, 1, – 1), (– 1, 2, 0) and (1, 2, 3).

$\therefore$ $d = 0,$

$$1 + 1 + 2v - 2w + d = 0 \quad \text{or} \quad v - w + 1 = 0 \qquad ...(ii)$$

$$1 + 4 - 2u + 4v + d = 0 \quad \text{or} \quad -2u + 4v + 5 = 0 \qquad ...(iii)$$

$$1 + 4 + 9 + 2u + 4v + 6w + d = 0 \quad \text{or} \quad u + 2v + 3w + 7 = 0 \qquad ...(iv)$$

Multiplying (*ii*) by (*iii*) and adding to (*iv*), we get

$$u + 5v + 10 = 0 \qquad ...(v)$$

Solving (*iii*) and (*v*), we get $u = -\dfrac{15}{14}, v = -\dfrac{25}{14}$

From (*ii*), $w = v + 1 = \dfrac{-25}{14} + 1 = \dfrac{-11}{14}$

Substituting these values of u, v, w, d in (*i*), we get

$$x^2 + y^2 + z^2 - \frac{15}{7}x - \frac{25}{7}y - \frac{11}{7}z = 0 \qquad ...(vi)$$

which is the required equaton of the sphere.

Its centre is (15/14, 25/14, 11/14) $[(-u, -v, -w)]$

and the radius = $[(-15/14)^2] + (-25/14)^2 + (-11/14)^2 - 0] = \sqrt{971/14}$.

Example 3.55. (*a*) *Find the equation of the sphere which has (x_1, y_1, z_1) and (x_2, y_2, z_2) as the extremities of a diameter.*

(*b*) *Deduce the equation of the sphere described on the line joining the points (2, – 1, 4) and (– 2, 2, – 2) as diameter. Find the area of the circle in which the sphere is intersected by the plane $2x + y - z = 3$.*

(*Anna, 2009 ; Hazaribagh, 2009*)

Solution. (*a*) Let $P(x, y, z)$ be any point on the sphere having $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ as ends of diameter (Fig. 3.50. (*a*)). Then AP and BP are at right angles.

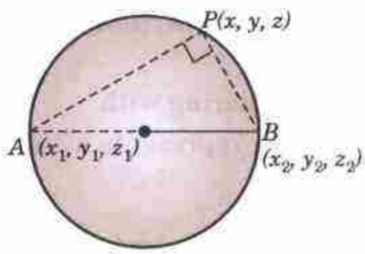


Fig. 3.50 (*a*)

Now direction ratio of AP are $x - x_1, y - y_1, z - z_1$ and those of BP are $x - x_2, y - y_2, z - z_2$.

Hence

$$(\mathbf{x} - \mathbf{x_1})(\mathbf{x} - \mathbf{x_2}) + (\mathbf{y} - \mathbf{y_1})(\mathbf{y} - \mathbf{y_2}) + (\mathbf{z} - \mathbf{z_1})(\mathbf{z} - \mathbf{z_2}) = 0$$

which is the required equation.

(*b*) The equation of the required sphere is

$$(x - 2)(x + 2) + (y + 1)(y - 2) + (z - 4)(z + 2) = 0$$

or

$$x^2 + y^2 + z^2 - y - 2z - 14 = 0 \qquad ...(i)$$

Its centre is $C(0, 1/2, 1)$

and radius $(r) = \sqrt{(0, 1/4 + 1 + 14)} = \sqrt{(61/4)}$.

Let the given plane $2x + y - z - 3 = 0 \qquad ...(ii)$

cut the sphere (1) in the circle PP' having centre L.

$\therefore$ p = perpendicular CL from C on the plane (2)

$$= \frac{1/2-1-3}{\sqrt{4+1+1}} = \frac{7}{2\sqrt{6}} \text{ (in magnitude)}$$

If a be the radius of the circle PP', then

$$a^2 = r^2 - p^2 = \frac{61}{4} - \frac{49}{24} = \frac{317}{24}$$

Hence the area of circle $PP' = \pi a^2 = \frac{317}{24}\pi$.

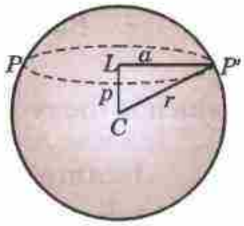


Fig. 3.50 (b)

Example 3.56. *A plane passes through a fixed point (a, b, c) and cuts the axes in A, B, C. Show that the locus of the centre of the sphere OABC is*

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2. \quad \textit{(P.T.U., 2010)}$$

Solution. Let the centre of the sphere $OABC$ be $P(f, g, h)$ so that its radius $OP = \sqrt{(f^2 + g^2 + h^2)}$.

$\therefore$ The equation of the sphere is

$$(x - f)^2 + (y - g)^2 + (z - h)^2 = f^2 + g^2 + h^2$$

or $$x^2 + y^2 + z^2 - 2fx - 2gy - 2hz = 0 \quad ...(i)$$

To find OA, putting $y = 0, z = 0$ in (i), we have

$$x^2 - 2fx = 0, \text{ i.e., } OA = x = 2f. \text{ Similarly, } OB = 2g, OC = 2h.$$

Thus the equation of the plane ABC is $\frac{x}{2f} + \frac{y}{2g} + \frac{z}{2h} = 1$

Since the plane passes through (a, b, c) $\therefore$ $\frac{a}{2f} + \frac{b}{2g} + \frac{c}{2h} = 1.$

Hence the locus of the centre (f, g, h) of the sphere is,

$$\frac{a}{2x} + \frac{b}{2y} + \frac{c}{2z} = 1 \quad \text{or} \quad \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2.$$

Example 3.57. *Find the equation of the sphere having the circle*

$$x^2 + y^2 + z^2 + 10y - 4z - 8 = 0, x + y + z = 3$$

as a great circle. *(Anna, 2009 ; Madras, 2001 S)*

Solution. The equation of any sphere through the given circle is

$$x^2 + y^2 + z^2 + 10y - 4z - 8 + k(x + y + z - 3) = 0$$

i.e., $$x^2 + y^2 + z^2 + kx + (10 + k)y - (4 - k)z - (8 + 3k) = 0 \quad ...(i)$$

In order that (i) may have the given circle as its great circle, its centre $[-k/2, -(10 + k)/2, (4 - k)/2]$ must lie on the plane $x + y + z = 3$

$\therefore$ $$-\frac{k}{2} - \frac{10+k}{2} + \frac{4-k}{2} = 3, \text{ i.e., } k = -4$$

whence (i) becomes, $x^2 + y^2 + z^2 - 4x + 6y - 8z + 4 = 0$ which is the required equation.

Example 3.58. *Find the equation of the smallest sphere which contains the circle $x^2 + y^2 + z^2 + 2x + 6y + 4z - 11 = 0$ and $2x + 2y + z + 1 = 0$.*

Solution. Equation of any sphere containing the given circle is

$$x^2 + y^2 + z^2 + 2x + 6y + 4z - 11 + \lambda(2x + 2y + z + 1) = 0$$

or $$x^2 + y^2 + z^2 + (2 + 2\lambda)x + (6 + 2\lambda)y + (4 + \lambda)z - 11 + \lambda = 0 \quad ...(i)$$

Its radius r is given by

$$r^2 = (1 + \lambda)^2 + (3 + \lambda)^2 + (2 + \frac{1}{2}\lambda)^2 - (\lambda - 11) = \frac{9}{4}\left[\lambda^2 + 4\lambda + \frac{100}{9}\right] = \frac{9}{4}\left[(\lambda + 2)^2 + \frac{64}{9}\right]$$

Now r^2 has the least value when $\lambda = -2$.

$\therefore$ Substituting $\lambda = -2$ in (*i*), we get

$$x^2 + y^2 + z^2 - 2x + 2y + 2z - 13 = 0$$

which is the required smallest sphere.

Example 3.59. *Prove that the circles $x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 = 0$, $5y + 6z + 1 = 0$ and $x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 = 0$, $x + 2y - 7z = 0$ lie on the same sphere and find its equation.*

Solution. Equation of any sphere containing the first circle is

$$x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 + \lambda(5y + 6z + 1) = 0$$

or
$$x^2 + y^2 + z^2 - 2x + (3 + 5\lambda)y + (4 + 6\lambda)z - 5 + \lambda = 0 \quad ...(i)$$

Similarly equation of any sphere containing the second given circle is

$$x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 + \lambda'(x + 2y - 7z) = 0$$

or
$$x^2 + y^2 + z^2 + (-3 + \lambda')x + (-4 + 2\lambda')y + (5 - 7\lambda')z - 6 = 0 \quad ...(ii)$$

(*i*) and (*ii*) will represent the same sphere when

$$-2 = -3 + \lambda' \quad ...(iii); \qquad 3 + 5\lambda = -4 + 2\lambda' \quad ...(iv)$$

$$4 + 6\lambda = 5 - 7\lambda' \quad ...(v); \qquad -5 + \lambda = -6 \quad ...(vi)$$

Now (*iii*) gives $\lambda' = 1$ and (*vi*) gives $\lambda = -1$.

Clearly $\lambda = -1$ and $\lambda' = 1$ also satisfy (*iv*) and (*v*). This shows that the given circles lie on the same sphere.

Substituting $\lambda = -1$ in (*i*) or $\lambda' = 1$ in (*ii*), we get

$$x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0$$

which is the desired sphere.

PROBLEMS 3.12

1. Find the equation of the sphere through the points (2, 0, 1), (1, – 5, – 1), (0, –2, 3) and (4, – 1, 2). Also find its centre and radius.
2. Find the equation of the sphere whose diameter is the line joining the origin to the point (2, – 2, 4). Also find its centre and radius.
3. Obtain the equation of the sphere which passes through the points (1, 0, 0), (0, 1, 0), (0, 0, 1) and
   (*a*) has its centre on the plane $x + y + z = 6$.
   (*b*) has its radius as small as possible.
4. A sphere of constant radius k passes through the origin and meets the axes in A, B, C. Prove that the centroid of the
   (*i*) triangle ABC lies on the sphere $9(x^2 + y^2 + z^2) = 4k^2$. *(Assam, 1999)*
   (*ii*) tetrahedron $OABC$ lies on the sphere $x^2 + y^2 + z^2 = k^2/4$.
5. A plane passes through a fixed points (a, b, c), show that the locus of the foot of the perpendicular from the origin on the plane is the sphere $x^2 + y^2 + z^2 - ax - by - cz = 0$.
6. A sphere of constant radius r passes through the origin O and cuts the axes in A, B, C. Prove that the locus of the foot of the perpendicular from O on the plane ABC is given by
   $$(x^2 + y^2 + z^2)^2 (x^{-2} + y^{-2} + z^{-2}) = 4r^2.$$
7. A plane cuts the coordinate axes at A, B, C. If $OA = a$, $OB = b$, $OC = c$, find the equation of the
   (*i*) circumsphere of the tetrahedron $OABC$, *(Assam, 1999)*
   (*ii*) circum-circle of the triangle ABC. Also obtain the coordinates of its centre.
8. Find the centre and radius of the circle $x^2 + y^2 + z^2 - 2y - 4z = 11$, $x + 2y + 2z = 15$.
   (P.T.U., 2009 S ; Burdwan, 2003 ; Cochin, 2001)
9. Show that the points (2, – 6, 0), (4, – 9, 6), (5, 0, 2), (7, – 3, 8) are concyclic.
10. Find the equation of the sphere for which the circle $x^2 + y^2 + z^2 - 3x + 4y - 2z - 5 = 0$, $5x - 2y + 4z + 7 = 0$ is a great circle.
11. Find the equation of the sphere having its centre on the plane $4x - 5y - z = 3$ and passing through the circle $x^2 + y^2 + z^2 - 2x - 3y + 4z + 8 = 0$, $x - 2y + z = 8$. *(Delhi, 2001)*
12. Prove that the plane $x + 2y - z = 4$ cuts the sphere $x^2 + y^2 + z^2 - x - z - 2 = 0$ in a circle of radius unity. Find also the equation of the sphere which has this circle as one of its great circles. *(Nagpur, 2009)*
13. Find the equation of the sphere passing through the circle $x^2 + y^2 + z^2 + 2x + 3y + 6 = 0$, $x - 2y + 4z = 9$ and the centre of the sphere $x^2 + y^2 + z^2 - 2x + 4y - 6z + 5 = 0$. *(Anna, 2009)*

3.18 EQUATION OF THE TANGENT PLANE

The equation of the tangent plane at any point (x_1, y_1, z_1) *of the sphere*

$$x^2 + y^2 + z^2 = a^2 \text{ is } \mathbf{xx_1 + yy_1 + zz_1 = a^2}.$$

If $P(x, y, z)$ be any point on the tangent plane at $P_1(x_1, y_1, z_1)$ to the given sphere, the direction ratios of P_1P are $x - x_1, y - y_1, z - z_1$. Also the direction ratios of radius OP_1 are $x_1 - 0, y_1 - 0, z_1 - 0$.

Since OP_1 is normal to the tangent plane at P_1, $OP_1 \perp P_1P$.

$\therefore$ $$x_1(x - x_1) + y_1(y - y_1) + z_1(z - z_1) = 0$$

or $$xx_1 + yy_1 + zz_1 = x_1^2 + y_1^2 + z_1^2 = a^2 \qquad [\because\ P_1(x_1, y_1, z_1) \text{ lies on the sphere.}$$

This is the desired equation of the tangent plane.

Similarly, *the tangent plane at* (x_1, y_1, z_1) *to the sphere*

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

is $$\mathbf{xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0}$$

Thus to write the equation of the tangent plane at (x_1, y_1, z_1) *to a sphere, change* x^2 *to* xx_1, y^2 to yy_1, z^2 *to* zz_1, $2x$ *to* $x + x_1$, $2y$ *to* $y + y_1$, $2z$ *to* $z + z_1$.

Obs. *The condition for a plane (or a line) to touch a sphere is that the perpendicular distance of the centre from the plane (or the line) = the radius.*

Example 3.60. *Find the equations of the spheres passing through the circle* $x^2 + y^2 + z^2 - 6x - 2z + 5 = 0$, $y = 0$ *and touching the plane* $3y + 4z + 5 = 0$.

Solution. The equation of any sphere through the given circle is

$$x^2 + y^2 + z^2 - 6x - 2z + 5 + ky = 0$$

or $$x^2 + y^2 + z^2 - 6x + ky - 2z + 5 = 0 \qquad ...(i)$$

$\therefore$ Its centre $= (3, -k/2, 1)$ and radius $= \sqrt{[9 + (k^2/4) + 1 - 5]} = \sqrt{(5 + k^2/4)}$.

The sphere (i) will touch the plane $3y + 4z + 5 = 0$, if $\perp$ distance of the centre $(3, -k/2, 1)$ from the plane = radius.

i.e., $$\frac{3(-k/2) + 4 + 5}{\sqrt{(9 + 16)}} = \sqrt{\left(5 + \frac{k^2}{4}\right)} \quad \text{or if,} \quad 4k^2 + 27k + 44 = 0$$

$\therefore$ $$k = \frac{-27 \pm \sqrt{[(27)^2 - 704]}}{8} = -\frac{11}{4} \text{ or } -4$$

Substituting the value of k in (1), we get

$$x^2 + y^2 + z^2 - 6x - \frac{11}{4}y + 2z + 5 = 0 \text{ and } x^2 + y^2 + z^2 - 6x - 4y - 2z + 5 = 0$$

as the two required spheres.

Example 3.61. *Find the equation of the sphere which touches the plane* $x - 2y - 2z = 7$ *at the point* $L(3, -1, -1)$ *and passes through the point* $M(1, 1, -3)$.

Solution. If C is the centre of the sphere, then CL is perpendicular to the given plane $x - 2y - 2z = 7$.

$\therefore$ The direction ratios of CL being $1, -2, -2$, the equation of CL is

$$\frac{x - 3}{1} = \frac{y + 1}{-2} = \frac{z + 1}{-2} = k \text{ (say)}$$

Any point on CL is $(k + 3, -2k - 1, -2k - 1)$ which will represent C for some value of k.

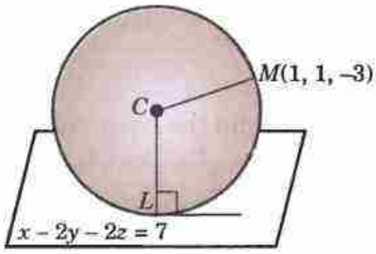


Fig. 3.51

Since M lies on the sphere, therefore its radius $CL = CM$ or $(CL)^2 = (CM)^2$

i.e. $$(k + 3 - 3)^2 + (-2k - 1 + 1)^2 + (-2k - 1 + 1)^2$$
$$= (k + 3 - 1)^2 + (-2k - 1 - 1)^2 + (-2k - 1 + 3)^2$$

or $$4k = -12 \quad \text{or} \quad k = -3.$$

$\therefore$ The centre C is $(0, 5, 5)$ and radius $CL = \sqrt{(9 + 36 + 36)} = 9$.

Hence the required equation of the sphere is

$$(x-0)^2+(y-5)^2+(2-5)^2=(9)^2$$

or $$x^2+y^2+z^2-10y-10z-31=0$$

[**Orthogonal spheres.** Two spheres are said to cut orthogonally if the tangent planes at a point of intersection are at right angles (Fig. 3.52).

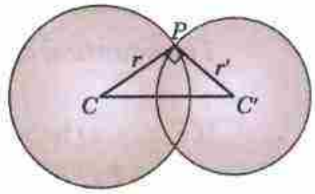


Fig. 3.52

The radii of such spheres through their point of intersection P, being $\perp$ to the tangent planes at P are also at right angles. *Thus two spheres cut orthogonally, if the square of the distance between their centre = sum of the squares of their radii*].

Example 3.62. *Show that the condition for spheres*

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

and $$x^2+y^2+z^2+2u'x+2v'y+2w'z+d'=0$$

to cut orthogonally is $\mathbf{2uu'+2vv'+2ww'=d+d'}$ *(Anna, 2002 S)*

Solution. The centres of the spheres are

$C(-u,-v,-w)$, $C'(-u',-v',-w')$ and their radii are

$$r=\sqrt{(u^2+v^2+w^2-d)},$$

$$r'=\sqrt{(u'^2+v'^2+w'^2-d')}.$$

Now these spheres will cut orthogonally, if $(CC')^2=r^2+r'^2$

i.e., $$(u-u')^2+(v-v')^2+(w-w')^2$$
$$=u^2+v^2+w^2-d+u'^2+v'^2+w'^2-d'$$

or $2uu'+2vv'+2ww'=d+d'$ which is the required condition.

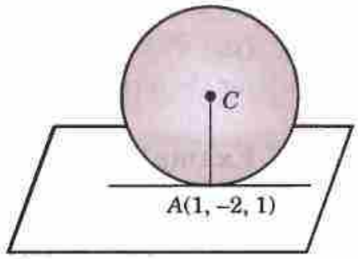


Fig. 3.53

Example 3.63. *Find the equation of the sphere which touches the plane $3x+2y-z+2=0$ at the point $(1,-2,1)$ and cuts the sphere $R^2-2(2I-3J).R+4=0$ orthogonally.* *(Roorkee, 2000)*

Solution. The given plane $3x+2y-z+2=0$ ...(*i*)

will touch the required sphere at $A(1,-2,1)$ if its centre lies on the normal to (*i*) at A (Fig. 3.53). The equations of the normal to (*i*) at A are $\dfrac{x-1}{3}=\dfrac{y+2}{2}=\dfrac{z-1}{-1}$

Any point on this line is $C(3r+1, 2r-2, \pi r+1)$

Also radius (AC) of the required sphere.

$$=\sqrt{[(3r)^2+(2r)^2+(-r)^2]}=r\sqrt{14}.$$

Since the required sphere cuts the given sphere

$x^2+y^2+z^2-4x+6y+4=0$ [Centre $=(2,-3,0)$ and $+$ radius $=3$

orthogonally, therefore (distance between their centres)$^2=\Sigma$ of squares of their radii

i.e., $$(3r+1-2)^2+(2r-2+3)^2+(-r+1)^2=14r^2+9 \text{ or } r=-3/2.$$

Thus centre C is $(-7/2,-5,5/2)$ and radius $=\dfrac{3\sqrt{14}}{2}$.

Hence the required sphere is

$$(x+7/2)^2+(y+5)^2+(z-5/2)^2=(3\sqrt{14/2})^2$$

or $$x^2+y^2+z^2+7x+10y-5z+12=0.$$

PROBLEMS 3.13

1. Find the equations of the tangent planes to the sphere
   (*i*) $x^2+y^2+z^2-4x+2y-6z+11=0$ which are parallel to the plane $x=0$. *(Anna, 2009)*
   (*ii*) $x^2+y^2+z^2=9$ which pass through the line $x+y=6$, $x-2z=3$. *(Madras, 2006)*

2. Find the equations of the spheres which pass through the circle
$$x^2 + y^2 + z^2 = 5x + 2y + 3z = 3, \text{ and touch the plane } 4x + 3y = 15.$$ *(Anna, 2009)*
3. Find the equation of the sphere which is tangential to the plane $x - 2y - 2z = 7$ at $(3, -1, -1)$ and passes through $(1, 1, -3)$.
4. (*i*) Prove that the equation of the sphere which lies in the first octant and touches the coordinates planes is of the form $(x^2 + y^2 + z^2) - 2\lambda(x + y + z) + 2\lambda^2 = 0$.
   (*ii*) Find the equation of the sphere passing through $(1, 4, 9)$ and touching the coordinate planes.
5. Tangent plane at any point of the sphere $x^2 + y^2 + z^2 = r^2$ meets the coordinates axes at A, B, C. Show that the locus of the point of intersection of the planes drawn parallel to the coordinate planes through A, B, C is the surface $x^{-2} + y^{-2} + z^{-2} = r^{-2}$. *(Rajasthan, 2006)*
6. Find the equation of the tangent line to the circle $x^2 + y^2 + z^2 = 3$, $3x - 2y + 4z + 3 = 0$ at the point $(1, 1, -1)$.
7. Show that the sphere $x^2 + y^2 + z^2 - 2x + 6y + 14z + 3 = 0$ divides the line joining the points $(2, -1, -4)$ and $(5, 5, 5)$ internally and externally in the ratio $1 : 2$.
8. Find the shortest and the longest distance from the point $(1, 2, -1)$ to the sphere $x^2 + y^2 + z^2 = 24$.
9. Show that the spheres $x^2 + y^2 + z^2 + 6y + 14z + 8 = 0$ and $x^2 + y^2 + z^2 + 6x + 8y + 4z + 20 = 0$, intersect at right angles. Find their plane of intersection.
10. Show that the spheres $x^2 + y^2 + z^2 = 25$ and $x^2 + y^2 + z^2 - 24x - 40y - 18z + 225 = 0$ touch externally and find their point of contact.

3.19 (1) CONE

Def. *A cone is a surface generated by a straight line which passes through a fixed point and satisfies one more condition e.g.*, it may intersect a given curve (called the *guiding curve*).

The fixed point is called the **vertex** *and the straight line in any position is called a* **generator.**

The degree of the equation of a cone depends upon the nature of its guiding curve. In case the guiding curve is a conic, the equation of the cone shall be of the second degree. Such cones are called *Quadric cones.* In what follows, we shall be conncerned only with quadric cones.

Example 3.64. *Find the equation of the cone whose vertex is (3, 1, 2) and base the circle*
$$2x^2 + 3y^2 = 1, z = 1.$$

Solution. Any line through $(3, 1, 2)$ is
$$\frac{x-3}{l} = \frac{y-1}{m} = \frac{z-2}{n} \qquad ...(i)$$
It meets $z = 1$, where $\dfrac{x-3}{l} = \dfrac{y-1}{m} = \dfrac{-1}{n}$

whence $x = 3 - l/n$, $y = 1 - m/n$.

Substituting these values of x and y in $2x^2 + 3y^2 = 1$,
$$2(3 - l/n)^2 + 3(1 - m/n)^2 = 1 \qquad ...(ii)$$
Eliminating l, m, n from (*i*) and (*ii*), the locus of the line (*i*) is
$$2\left(3 - \frac{x-3}{z-2}\right)^2 + 3\left(1 - \frac{y-1}{z-2}\right)^2 = 1$$
or $\quad 2x^2 + 3y^2 + 20z^2 - 6yz - 12xz + 12x + 6y - 38z + 17 = 0$ which is the required equation.

Example 3.65. *Find the equation of the cone whose vertex is at the origin and guiding curve is* $\dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{1} = 1, x + y + z = 1.$

Solution. Any line through $(0, 0, 0)$ is $x/l = y/m = z/n$ ...(*i*)

Any point on it is $P(lr, mr, nr)$.

If (*i*) intersects the given curve, the coordinates of P should satisfy its equations.

$$\therefore \qquad \frac{l^2r^2}{4} + \frac{m^2r^2}{9} + \frac{n^2r^2}{1} = 1 \text{ and } lr + mr + nr = 1.$$

Eliminating r, $\quad \left(\frac{l^2}{4} + \frac{m^2}{9} + n^2\right) \Big/ (l + m + n)^2 = 1.$

Simplifying, $27l^2 + 32m^2 + 72(lm + mn + nl) = 0$ ...(*ii*)

Eliminating l, m, n from (*i*) and (*ii*), the locus of the line (*i*) is

$27x^2 + 32y^2 + 72(xy + yz + zx) = 0$ which is the required equation.

Obs. *The equation of a cone with vertex at the origin is a homogeneous equation of the second degree in x, y, z (i.e., all terms are of the same degree).* The reason is that every generator will have the equation of the form (*i*) above. So the point (lr, mr, nr) will satisfy the equation of the cone for every value of r. This is possible only if the equation is homogeneous.

Example 3.66. *A variable plane parallel to the plane* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0$ *meets the coordinate axes in A, B, C. Find the equation of the cone whose vertex is the origin and guiding curve the circle ABC.*

Solution. Let the plane $\quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = k$ ...(*i*)

meet the axes at A, B, C, so that $A = (ka, 0, 0)$, $B = (0, kb, 0)$ and $C = (0, 0, kc)$.

$\therefore$ The equation of the sphere through $O\ (0, 0, 0)$ and A, B, C is

$$x^2 + y^2 + z^2 - k(ax + by + cz) = 0 \qquad \text{...(ii)}$$

Since the equation of the cone with vertex at O is a homogeneous equation of the second degree, therefore, it must be satisfied by points lying on the circle ABC, *i.e.*, on (*i*) and (*ii*) both.

$\therefore$ Making (*ii*) homogeneous with the help of (*i*), we have

$$x^2 + y^2 + z^2 - (ax + by + cz)\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) = 0$$

or $\quad yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{c}{a} + \frac{a}{c}\right) + xy\left(\frac{a}{b} + \frac{b}{a}\right) = 0$ which is the required equation.

Example 3.67. *Show that the general equation of the cone of the second degree which passes through the axes is of the form $fyz + gzx + hxy = 0$.*

Solution. Any cone which passes through the axes will have origin V as its vertex. The general equation of a cone of the second degree having vertex at the origin is of the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \qquad \text{...(i)}$$

Since it passes through x-axis

$\therefore$ The direction cosines of x-axis (*i.e.*, 1, 0, 0) must satisfy (*i*). This gives $a = 0$.

As the cone passes through y-axis, $b = 0$.

Similarly, as the cone passes through z-axis, $c = 0$.

Hence (*i*) reduces to $fyz + gzx + hxy = 0$.

(2) **Right circular cone. Def.** *A right circular cone is a surface generated by a straight line which passes through a fixed point (vertex) and makes a constant angle with a fixed line* (Fig. 3.54).

The constant angle ($\angle AVC$) *is called its* **semi-vertical angle** *and the fixed line (VC) is called the* **axis.** The section of a right circular cone by a plane perpendicular to its axis is a circle.

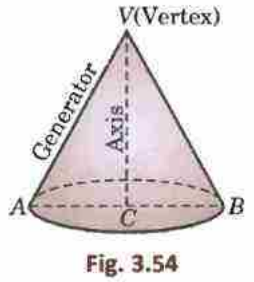


Fig. 3.54

Example 3.68. *Find the equation of the right circular cone whose vertex is the origin, whose axis is the line $x/1 = y/2 = z/3$ and which has semi-vertical angle of 30°.* (Anna, 2009)

Solution. Let $P(x, y, z)$ be any point on the cone with vertex O and axis (OC)

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3}, \text{ so that } \angle POC = 30°. \qquad \text{(Fig. 3.55)}$$

Now the direction ratios of OP are x, y, z and those of OC are 1, 2, 3.

$$\therefore \qquad \cos 30° = \frac{x(1) + y(2) + z(3)}{\sqrt{(x^2 + y^2 + z^2)} \cdot \sqrt{(1+4+9)}}$$

or
$$\frac{\sqrt{3}}{2} = \frac{x + 2y + 3z}{\sqrt{[14(x^2 + y^2 + z^2)]}}$$

Squaring $\quad 3 \times 14(x^2 + y^2 + z^2) = 4(x + 2y + 3z)^2$

or
$$19x^2 + 13y^2 + 3z^2 - 8xy - 24yz - 12zx = 0$$

which is the required equation of the cone.

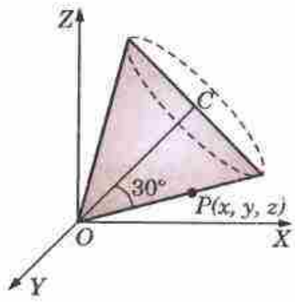


Fig. 3.55

Example 3.69. *Find the equation of the right circular cone generated when the straight line $2y + 3z = 6$, $x = 0$ revolves about z-axis.* *(Hazaribagh, 2009)*

Solution. The vertex is the point of intersection of the line $2y + 3z = 6$, $x = 0$ and the z-axis, *i.e.*, $x = 0$, $y = 0$ (Fig. 3.56).

$\therefore$ Vertex is $A(0, 0, 2)$. A generator of the cone is

$$\frac{x}{0} = \frac{y}{3} = \frac{z-2}{-2}$$

$\therefore$ Direction ratios of the generator are $0, 3, -2$ and the axis (z-axis) are $0, 0, 1$. The semi-vertical angle α is, therefore, given by

$$\cos \alpha = \frac{0.0 + 3.0 + (-2).1}{\sqrt{13}} = \frac{-2}{\sqrt{13}}$$

Let $P(x, y, z)$ be any point on the cone so that the direction ratios of AP are $x, y, z - 2$. Since AP makes an angle α with AZ, we have

$$\cos \alpha = \frac{x.0 + y.0 + (z-2).1}{\sqrt{[x^2 + y^2 + (z-2)^2]}}$$

Thus
$$\frac{(z-2)^2}{x^2 + y^2 + (z-2)^2} = \cos^2 \alpha = \frac{4}{13}$$

or
$$4x^2 + 4y^2 - 9z^2 + 36z - 36 = 0$$

which is the required equation of the cone.

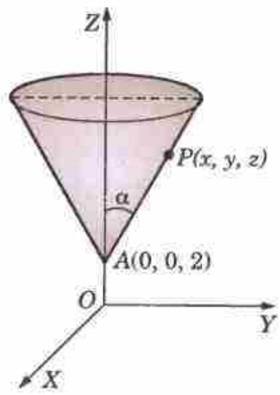


Fig. 3.56

Example 3.70. *Find the equations to the lines in which the plane $2x + y - z = 0$ cuts the cone $4x^2 - y^2 + 3z^2 = 0$.*

Solution. Let $x/l = y/m = z/n$ be one of the two lines in which the given plane $2x + y - z = 0$ ...(*i*)

cuts the given cone $\qquad 4x^2 - y^2 + 3z^2 = 0$ ...(*ii*)

$\because$ This line lies on (*i*), $\therefore \quad 2l + m - n = 0$ ...(*iii*)

and it lies on (*ii*), $\therefore \quad 4l^2 - m^2 + 3n^2 = 0$ ...(*iv*)

To eliminate n from (*iii*) and (*iv*), put $n = 2l + m$ in (*iv*).

$$4l^2 - m^2 + 3(2l + m)^2 = 0 \quad \text{or} \quad (4l + m)(2l + m) = 0$$

$\therefore$ Either $4l + m = 0$ | or $2l + m = 0$

From (*iii*) $2l + m - n = 0$ | and $2l + m - n = 0$

$\therefore \quad \dfrac{l}{-1} = \dfrac{m}{4} = \dfrac{n}{2}$ | $\therefore \quad \dfrac{l}{-1} = \dfrac{m}{2} = \dfrac{n}{0}$

Hence the required lines are

$$\frac{x}{-1} = \frac{y}{4} = \frac{z}{2} \text{ and } \frac{x}{-1} = \frac{y}{2} = \frac{z}{0}.$$

Example 3.71. *Find the equation of the enveloping cone of the sphere $x^2 + y^2 + z^2 = a^2$ with vertex at the point (x_1, y_1, z_1).*

Solution. The equation of any generator through $V(x_1, y_1, z_1)$ having direction ratios l, m, n is

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r \text{ (say)} \qquad ...(i)$$

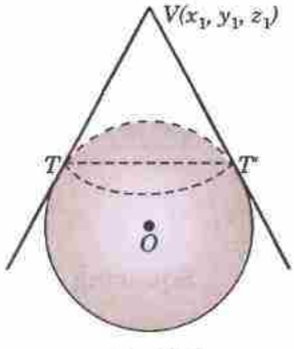


Fig. 3.57

Any point on (*i*) is $P(x_1 + lr, y_1 + mr, z_1 + nr)$.

It lies on the given sphere if

$$(x_1 + lr)^2 + (y_1 + mr)^2 + (z_1 + nr)^2 = a^2$$

or $$(l^2 + m^2 + n^2)\, r^2 + 2\,(lx_1 + my_1 + nz_1)\, r + x_1^2 + y_1^2 + z_1^2 - a^2 = 0 \qquad ...(ii)$$

The line (*i*) will touch the given sphere if (*ii*) has equal roots.

$$\therefore \quad (lx_1 + my_1 + nz_1)^2 = (l^2 + m^2 + n^2)(x_1^2 + y_1^2 + z_1^2 - a^2) \qquad ...(iii)$$

The locus of all such lines is the enveloping cone of the given sphere which is obtained by eliminating l, m, n from (*i*) and (*iii*).

Thus $[(x - x_1)\, x_1 + (y - y_1) y_1 + (z - z_1) z_1]^2 = [(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2]\, (x_1^2 + y_1^2 + z_1^2 - a^2)$

which is the equation of the enveloping cone. (Fig. 3.57)

Obs. It can be reduced to the form $SS_1 = T^2$

where $S = x^2 + y^2 + z^2 - a^2$, $S_1 = x_1^2 + y_1^2 + z_1^2 - a^2$, $T = xx_1 + yy_1 + zz_1 - a^2$.

Thus the enveloping cone of the surface $S = 0$ with vertex (x_1, y_1, z_1) is $SS_1 = T^2$

PROBLEMS 3.14

1. Find the equation of the cone with vertex (α, β, γ) and base $y^2 - 4ax = 0, z = 0$.
2. Find the equation of the cone whose vertex is (3, 4, 5) and base is the conic $3y^2 + 4z^2 = 16, z + 2x = 0$.
3. Find the equation of the cone whose vertex is (1, 2, 3) and whose guiding curve is the circle $x^2 + y^2 + z^2 = 4$, $x + y + z = 1$. (*P.T.U., 2010*)
4. The generators of a cone pass through the point (1, 1, 1) and their direction cosines l, m, n satisfy the relation $l^2 + m^2 = 3n^2$. Obtain the equation of the cone.
5. Find the equation of the right circular cone whose vertex is at the origin and semi-vertical angle is α and having axis of z as its axis. (*V.T.U., 2006 ; Rajasthan, 2005*)
6. Find the equation of the cone whose vertical angle is $\pi/2$, which has its vertex at the origin and its axis along the line $x = -2y = z$. (*V.T.U., 2005*)
   Also show that the plane $z = 0$ cuts the cone in two straight lines inclined at an angle $\cos^{-1} 4/5$.
7. Find the equation of the circular cone which passes through the point (1, 1, 2) and has its vertex at the origin and axis the line $x/2 = -y/4 = z/3$. (*Cochin, 2005 ; Rajasthan, 2005 ; V.T.U., 2004*)
8. Find the equation of the right circular cone generated by revolving the line $x = 0, y - z = 0$ about the axis $x = 0, x = 2$. (*Anna, 2009*)
9. Find the equation of the right circular cone passing through the coordinate axes having vertex at the origin. Obtain the semi-vertical angle and the equation of the axis.
10. Find the semi-vertical angle and the equation of the right circular cone having its vertex at the origin and passing through the circle $y^2 + z^2 = 25, x = 4$. (*Anna, 2009*)
11. Find the equation of the right circular cone which has its vertex at (0, 0, 10) whose intersection with the *XY*-plane is a circle of radius 5. (*Nagpur, 2009*)
12. Find the equations to the lines in which the plane $3x + y + 5z = 0$ cuts the cone $6yz - 2zx + 5xy = 0$.
13. Prove that the plane $ax + by + cz = 0$ meets the cone $yz + zx + xy = 0$ in perpendicular lines if $a^{-1} + b^{-1} + c^{-1} = 0$.
14. Find the equation of the enveloping cone of the sphere $x^2 + y^2 + z^2 + 2x - 4y + 2z - 1 = 0$ with vertex at (1, 1, 1).

3.20 (1) CYLINDER

Def. *A cylinder is a surface generated by a straight line which is parallel to a fixed line and satisfies one more condition e.g.,* it may intersect a given curve (called the *guiding curve*).

The straight line in any position is called the *generator* and the fixed line the *axis* of the cylinder.

Example 3.72. *Find the equation of a cylinder whose generating lines have the direction cosines l, m, n and which pass through the circumference of the fixed circle $x^2 + z^2 = a^2$ in the ZOX plane.*

Solution. Let $P(x_1, y_1, z_1)$ be any point of the cylinder so that the equation of the generator through P is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \quad ...(i)$$

Given guiding circle is $x^2 + z^2 = a^2, y = 0$ ...(ii)

The generator (i) cuts the plane $y = 0$, where

$$\frac{x - x_1}{l} = \frac{-y_1}{m} = \frac{z - z_1}{n}$$

i.e., where $x = x_1 - \dfrac{ly_1}{m}$ and $z = z_1 - \dfrac{ny_1}{m}$

But these values of x and z satisfy $x^2 + z^2 = a^2$

$$\therefore \quad \left(x_1 - \frac{ly_1}{m}\right)^2 + \left(z_1 - \frac{ny_1}{m}\right)^2 = a^2$$

Hence the locus of (x_1, y_1, z_1) is

$(mx - ly)^2 + (mz - ny)^2 = a^2 m^2$, which is the required equation of the cylinder.

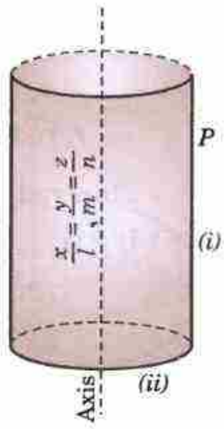


Fig. 3.58

(2) Right circular cylinder. Def. *A right circular cylinder is a surface generated by a straight line which is parallel to a fixed line and is at a constant distance from it.*

The constant distance is called the *radius of the cylinder.*

Example 3.73. *The radius of a normal section of a right circular cylinder is 2 units ; the axis lies along the straight line*

$$\frac{x - 1}{2} = \frac{y + 3}{-1} = \frac{z - 2}{5}, \text{ find its equation.} \qquad (P.T.U., 2005)$$

Solution. A point on the axis of the cylinder is $A(1, -3, 2)$ and its direction ratios are $2, -1, 5$.

$\therefore$ Its actual direction cosines are $\dfrac{2}{\sqrt{30}}, \dfrac{-1}{\sqrt{30}}, \dfrac{5}{\sqrt{30}}$.

Let $P(x, y, z)$ be any point on the cylinder. Draw $PM \perp$ to the axis AM. Then $MP = 2$. Now AM = Projection of AP on AM (axis)

$$= (x - 1)\frac{2}{\sqrt{30}} + (y + 3)\frac{-1}{\sqrt{30}} + (z - 2)\frac{5}{\sqrt{30}}$$

$$= \frac{2x - y + 5z - 15}{\sqrt{30}}$$

Also $\quad AP = \sqrt{(x-1)^2 + (y+3)^2 + (z-2)^2}$

$\therefore$ From the rt. $\angle d\ \Delta AMP$, $(AM)^2 + (MP)^2 = (AP)^2$

or $\quad \dfrac{1}{30}(2x - y + 5z - 15)^2 + 4 = (x - 1)^2 + (y + 3)^2 + (z - 2)^2$

or $\quad 26x^2 + 29y^2 + 5z^2 + 4xy + 10yz - 20zx + 150y + 30z + 75 = 0.$

This is the required equation of the right circular cylinder. (Fig. 3.59)

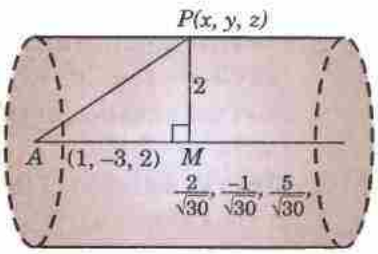


Fig. 3.59

Example 3.74. *Find the equation of the circular cylinder having for its base the circle $x^2 + y^2 + z^2 = 9$, $x - y + z = 3$.* (P.T.U., 2006 ; Cochin, 2005)

Solution. The axis of the cylinder is the line through the centre L of the given circle (or through O (0, 0, 0) the centre of the sphere) (Fig. 3.60) and perpendicular to the plane of the circle.

i.e. $\qquad x - y + z = 3 \qquad ...(i)$

$\therefore$ Axis of the cylinder is $\dfrac{x}{1} = \dfrac{y}{-1} = \dfrac{z}{1}$

Also $OL \perp$ from O (0, 0, 0) on (i)

$$= \frac{3}{\sqrt{(1+1+1)}} = \sqrt{3}.$$

$\therefore$ r, radius of the circle $= \sqrt{(OA^2 - OL^2)} = \sqrt{(9-3)} = \sqrt{6}$

Thus radius of the cylinder $(= r) = \sqrt{6}$

If $P(x, y, z)$ be any point on the cylinder, then

$$OP^2 = OM^2 + MP^2$$

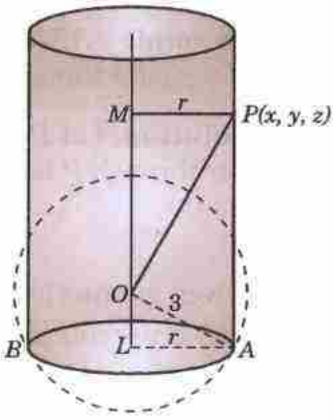


Fig. 3.60

i.e., $$x^2 + y^2 + z^2 = \left[\frac{1}{\sqrt{3}}(x-0) - \frac{1}{\sqrt{3}}(y-0) + \frac{1}{\sqrt{3}}(z-0)\right]^2 + 6$$

i.e., $x^2 + y^2 + z^2 + xy + yz - zx - 9 = 0$ which is the required equation.

Example 3.75. *Find the equation of the enveloping cylinder of the sphere $x^2 + y^2 + z^2 = 9$ having generator parallel to the line $x/3 = y/2 = z/1$.*

Solution. If $P(x_1, y_1, z_1)$ be a point on the enveloping cylinder, then the equation of the generator is

$$\frac{x - x_1}{3} = \frac{y - y_1}{2} = \frac{z - z_1}{1} = r(\text{say}). \qquad ...(i)$$

Any point on (i) is $(x_1 + 3r, y_1 + 2r, z_1 + r)$. It lies on the sphere $x^2 + y^2 + z^2 = 9$. ...(ii)

Then $$(x_1 + 3r)^2 + (y_1 + 2r)^2 + (z_1 + r)^2 = 9$$

or $$14r^2 + 2(3x_1 + 2y_1 + z_1)\,r + x_1^2 + y_1^2 + z_1^2 - 9 = 0 \qquad ...(iii)$$

In order that (i) touches (ii), the equation (iii) must have equal roots for which

$$4(3x_1 + 2y_1 + z_1)^2 = 4 \times 14\,(x_1^2 + y_1^2 + z_1^2 - 9) \qquad [\because\ b^2 = 4ac]$$

or $$5x_1^2 + 10y_1^2 + 13z_1^2 + 12x_1y_1 + 4y_1z_1 + 6z_1x_1 = 126$$

$\therefore$ The locus of (x_1, y_1, z_1) is

$$5x^2 + 10y^2 + 13z^2 + 12xy + 4yz + 6zx = 126$$

which is the required equation of the enveloping cylinder.

PROBLEMS 3.15

1. Find the equation of the right circular cylinder whose axis is the line $x = 2y = -z$ and radius 4. *(Anna, 2009)*
2. Find the equation of the cylinder whose generators are parallel to the line $x = -y/2 = z/3$ and whose guiding curve is the ellipse $x^2 + 2y^2 = 1$, $z = 3$. *(Rajasthan, 2005 ; Roorkee, 2000)*
3. Find the equation of the right circular cylinder of radius 2 whose axis passes through (1, 2, 3) and has direction ratios (2, – 3, 6). *(V.T.U., 2006 ; Anna, 2005 S)*
4. Find the equation of the right circular cylinder describe on the circle through the points (a, 0, 0), (0, a, 0), (0, 0, a) guiding curve.
5. Find the equation of the cylinder whose directing curve is $x^2 + z^2 - 4x - 2z + 4 = 0$, $y = 0$ and whose axis contains the point (0, 3, 0). Find also the area of the section of the cylinder by a plane parallel to xz-plane.
6. Find the equation of the enveloping cylinder of the sphere $x^2 + y^2 + z^2 - 2x - 4y - 6z - 2 = 0$ whose generators are perpendicular to the lines $\dfrac{x}{3} = \dfrac{y}{-1} = \dfrac{z}{0}$ and $\dfrac{x}{1} = \dfrac{y}{2} = \dfrac{z}{0}$.
7. Find the equation to the cylinder whose generators intersect the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, $z = 0$ and are parallel to the line $x/l = y/m = z/n$.

3.21 QUADRIC SURFACES

The surface represented by general equation of the second degree in x, y, z is called a **quadric surface** *or a* **conicoid.**

Thus the general equation of a *quadric surface* is of the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$$

which can be reduced to any of the following standard forms so useful in engineering problems. We now, proceed to study their shapes.

(1) Ellipsoid : $\mathbf{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1}$.

(*i*) It is symmetrical about each of the coordinate planes for only even powers of x, y, z occur in its equation.

(*ii*) It meets the x-axis at $A\ (a, 0, 0), A'\ (-a, 0, 0)$;
the y-axis at $B\ (0, b, 0), B'\ (0, -b, 0)$;
and the z-axis at $C(0, 0, c), C'\ (0, 0, -c)$.

(*iii*) Its sections by the coordinate planes are ellipses. For the section by the yz-plane $(x = 0)$ is the ellipse.

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \text{ etc.}$$

(*iv*) The surface is generated by a variable ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2}, z = k ;$$

(as k varies from $-c$ to c) and is limited in every direction.

Hence its shape is as shown in Fig. 3.61 which is like that of an *egg*.

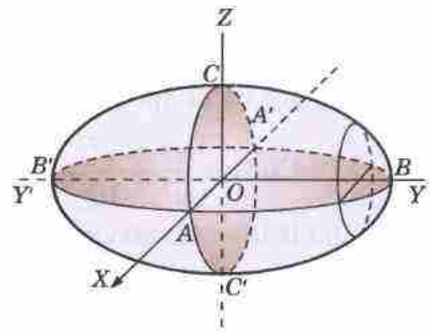


Fig. 3.61

(2) Hyperboloid of one sheet : $\mathbf{\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1}$.

(*i*) It is symmetrical about each of the coordinate planes for only even powers of x, y, z occur in its equation.

(*ii*) It meets the x-axis at $A(a, 0, 0), A'\ (-a, 0, 0)$; the y-axis at $B\ (0, b, 0), B'\ (0, -b, 0)$; and the z-axis in imaginary points.

(*iii*) Its section by the yz-plane $(x = 0)$ is the hyperbola $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, (*i.e.*, $DE, D'E'$)

Its section by the zx-plane $(y = 0)$ is the hyperbola $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$. (*i.e.*, $FG, F'G'$)

Its section by the xy-plane $(z = 0)$ is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(*iv*) The surface is generated by a variable ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{k^2}{c^2}, z = k$ (as k varies from $-\infty$ to ∞) and extends to infinity on both sides of the xy-plane.

Hence its shape is as shown in Fig. 3.62 which is like that of *juggler's dabru*.

(3) Hyperboloid of two sheets : $\mathbf{\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1}$.

(*i*) It is symmetrical about each of the coordinate planes for only even powers of x, y, z occur in its equation.

(*ii*) It meets the z-axis at $C(0, 0, c), C'(0, 0, -c)$ and the x and y-axes in imaginary points.

(*iii*) Its section by the yz-plane $(x = 0)$ is the hyperbola $\frac{z^2}{c^2} - \frac{y^2}{b^2} = 1$. (*i.e.*, $ACB, A'C'B'$)

Its section by the zx-plane $(y = 0)$ is the hyperbola $\frac{z^2}{c^2} - \frac{x^2}{a^2} = 1$. (*i.e.*, $DCE, D'C'E'$)

Its section by the xy-plane ($z = 0$), is the imaginary ellipse $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$.

Its section by the xy-plane ($z = 0$) is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(iv) The surface is generated by a variable ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{k^2}{c^2} - 1, z = k$,

(as k varies from $-\infty$ to $-c$ and c to $+\infty$) and extends to infinity on both sides of the xy-plane.

Hence its shape is as shown in Fig. 3.63.

(4) Cone : $\mathbf{\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.}$

(i) It is symmetrical about each of the coordinate planes.

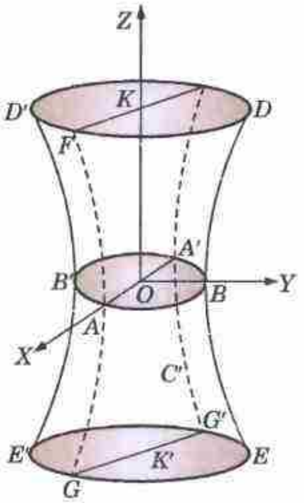


Fig. 3.62

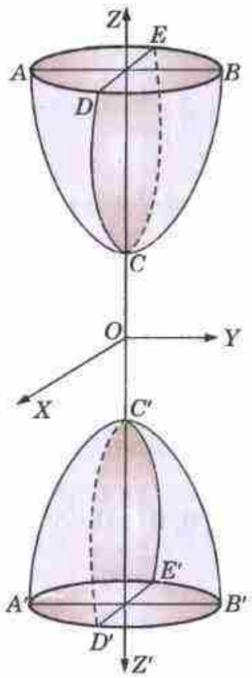


Fig. 3.63

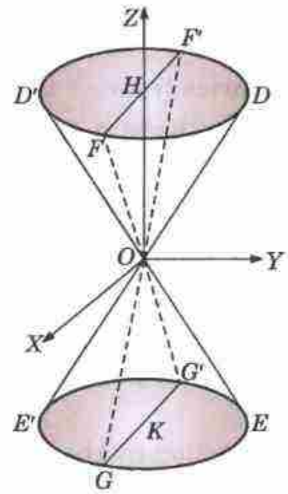


Fig. 3.64

(ii) It meets the axes only at the origin.

(iii) Its section by the yz-plane ($x = 0$) is the pair of straight lines

$$y = \pm \frac{b}{c} z \qquad (i.e., DOE' \text{ and } D'OE).$$

Its section by the zx-plane ($y = 0$) is the pair of straight lines

$$x = \pm \frac{a}{c} z \qquad (i.e., FOG' \text{ and } F'OG).$$

Its section by the zx-plane ($z = 0$) is the point ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$.

(iv) The surface is generated by a variable ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{k^2}{c^2}$, $z = k$ (k varies)

and extends to infinity on both sides of the xy-plane. Hence its shape is as shown in Fig. 3.64.

(5) Elliptic paraboloid : $\mathbf{\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2z}{c}}$

(i) It is symmetrical about yz- and zx-planes for only even powers of x and y occur in its equation

(ii) It meets the axes at the origin only and touches the xy-plane threat.

(iii) Its section by the yz-plane ($x = 0$) is the parabola $y^2 = \frac{2b^2}{c} z$, ($i.e.$, DOD').

Its section by the zx-plane ($y = 0$) is the parabola $x^2 = \frac{2a^2}{c} z$ (*i.e.*, EOE').

Its section by the xy-plane ($z = 0$) is the point ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$

(*iv*) The surface is generated by a variable ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2k}{c}$, $z = k$ (as k varies from 0 to ∞) and it extends to infinity above the xy-plane.

Hence its shape is as shown in Fig. 3.65 and is like that of *tabla*.

(6) Hyperbolic paraboloid : $\frac{\mathbf{x}^2}{\mathbf{a}^2} - \frac{\mathbf{y}^2}{\mathbf{b}^2} = \frac{\mathbf{2z}}{\mathbf{c}}$.

(*i*) It is symmetrical about the yz and zx-planes for only even powers of x and y occur in its equation.

(*ii*) It meets the axes only at the origin and touches the xy-plane threat.

(*iii*) Its section by the yz-plane ($x = 0$) is the parabola $y^2 = -\frac{2b^2}{c} z$. (*i.e.*, DOD')

Its section by the zx-plane ($y = 0$) is the parabola $x^2 = \frac{2a^2}{c} z$ (*i.e.*, EOE').

Its section by the xy-plane ($z = 0$) is the part of lines $y = \pm \frac{b}{a} x$ (not shown in Fig. 3.66.)

(*iv*) The surface is generated by a variable hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2k}{c}$, $z = k$

and it extends to infinity on both sides of xy-plane. Hence its shape is as shown in Fig. 3.66.

(7)· Cylinder. An equation of the form $f(x, y) = 0$ represents a cylinder generated by a straight line which is parallel to the z-axis and its section by the xy-plane is the curve $f(x, y) = 0$ (Fig. 3.67).

In particular (*i*) $y^2 = 4ax$ represents a *parabolic cylinder*,

(*ii*) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ represents an *elliptic cylinder*, (*iii*) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ represents a *hyperbolic cylinder*.

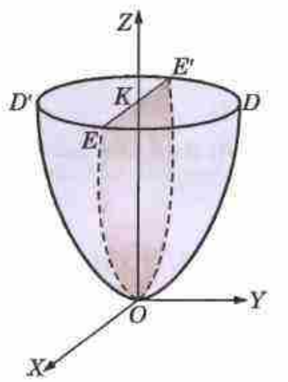


Fig. 3.65

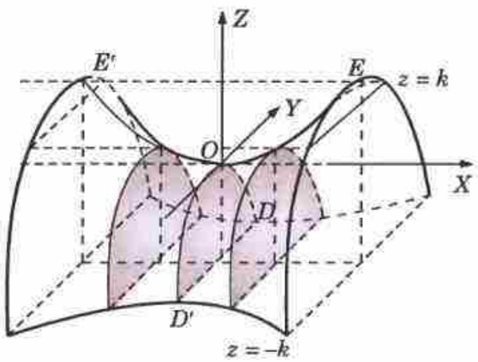


Fig. 3.66

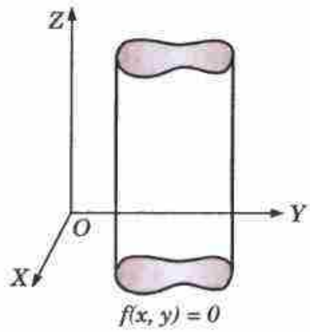


Fig. 3.67

3.22 SURFACES OF REVOLUTION

Let $P(x, y)$ be any point on the curve $y = f(x)$ in the xy-plane. Draw $PM \perp$ to x-axis so that $OM = x$ and $MP = y$. Thus the equation of this curve can be written as

$$MP = f(OM) \qquad ...(1)$$

As this curve revolves about the x-axis, the point P describe a circle with centre M and radius MP. Let $Q(x, y, z)$ be any other position of P. Draw $QN \perp$ to zx-plane and join MN so that $OM = x$, $MN = z$, $NQ = y$

and $\angle MNQ = 90°$. $\therefore$ $MP^2 = MQ^2 = MN^2 + NQ^2 = z^2 + y^2$.

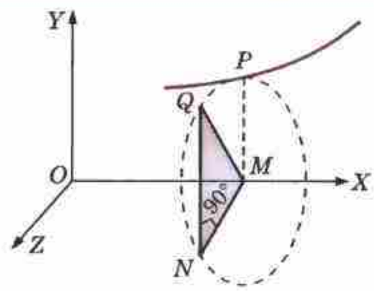


Fig. 3.68

Now substituting the values of MP and MO in (1), we have

$$\sqrt{(y^2 + z^2)} = f(x) \quad \text{or} \quad \mathbf{y^2 + z^2 = [f(x)]^2}$$

which is *the equation of the surface generated by the revolution of the curve $y = f(x)$ about the x-axis* (Fig. 3.68)

Similarly, *the surface generated by the revolution of the curve*

(*i*) $x = f(y)$ *about y-axis is* $\mathbf{z^2 + x^2 = [f(y)]^2}$, (*ii*) $x = f(z)$ about z-axis is $\mathbf{x^2 + y^2 = [f(z)]^2}$

The given revolving curve is called the *generating curve.*

Some standard surfaces of revolution :

Let the generating curve be $y = f(x)$ in the xy-plane and the axis of rotation be the x-axis ; then the surface generated is $y^2 + z^2 = [f(x)]^2$.

(1) Right-circular cylinder. *When $f(x) = a$, the generating curve is a straight line* ($y = a$) *parallel to the x-axis.*

$\therefore$ The surface generated is $\mathbf{y^2 + z^2 = a^2}$

which represents a *right-circular cylinder of radius a and axis as x-axis* (Fig. 3.69).

(2) Right-circular cone. *When* $f(x) = mx$, the generating curve is a straight line ($y = mx$) passing through the origin.

$\therefore$ The surface generated is $y^2 + z^2 = m^2x^2$ or $\mathbf{y^2 + z^2 = x^2 \tan^2 \alpha}$

which represents *a right-circular cone of semi-vertical angle α and axis as the x-axis* (Fig. 3.70).

(3) Sphere. *When* $f(x) = \sqrt{(a^2 - x^2)}$, the generating curve is a circle ($x^2 + y^2 = a^2$).

$\therefore$ The surface generated is

$$y^2 + z^2 = a^2 - x^2 \quad i.e., \quad \mathbf{x^2 + y^2 + z^2 = a^2},$$

which is *a sphere of radius a and centre (0, 0, 0).*

(4) Ellipsoid of revolution. *When* $f(x) = b\sqrt{(1 - x^2/a^2)}$, the generating curve is an ellipse

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right). \qquad \therefore \text{ The surface generated is } y^2 + z^2 = b^2(1 - x^2/a^2)$$

or $$\frac{\mathbf{x^2}}{\mathbf{a^2}} + \frac{\mathbf{y^2}}{\mathbf{b^2}} + \frac{\mathbf{z^2}}{\mathbf{b^2}} = \mathbf{1},$$ which is called an *ellipsoid of revolution.*

If $a^2 > b^2$, the major axis of the generating ellipse is along the x-axis—the axis of revolution and the surface generated, in this case, is called a **prolate spheroid** (Fig. 3.71).

If $a^2 < b^2$, the minor axis of the ellipse lies along the x-axis—the axis of revolution and the surface thus generated is called an **oblate spheroid** (Fig. 3.72).

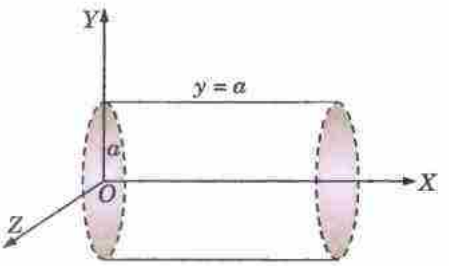


Fig. 3.69

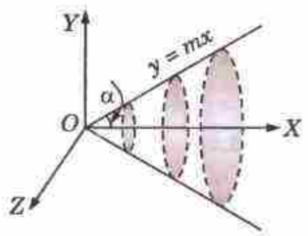
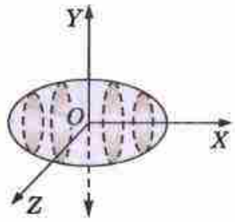
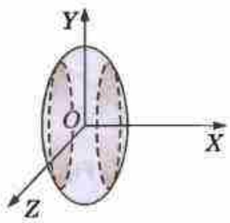


Fig. 3.70



Prolate spheroid

Fig. 3.71



Oblate spheroid

Fig. 3.72

(5) Hyperboloids of revolution

(*i*) *When* $f(x) = b\sqrt{(1 + x^2/a^2)}$, the generating curve is $\dfrac{y^2}{b^2} - \dfrac{z^2}{a^2} = 1$ which represents a hyperbola having conjugate axis along the x-axis.

$\therefore$ The surface generated is $y^2 + z^2 = b^2(1 + x^2/a^2)$

or $$\frac{\mathbf{y^2}}{\mathbf{b^2}} + \frac{\mathbf{z^2}}{\mathbf{b^2}} - \frac{\mathbf{x^2}}{\mathbf{a^2}} = \mathbf{1}$$ which is called a *hyperboloid of revolution of one sheet* (Fig. 3.73).

(*ii*) When $f(x) = b\sqrt{(x^2/a^2 - 1)}$, the generating curve is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ which represents a hyperbola having transverse axis along the x-axis.

$\therefore$ The surface generated is $y^2 + z^2 = b^2(x^2/a^2 - 1)$

or $$\mathbf{\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{b^2} = 1},$$ which is called a *hyperboloid of revolution of two sheets* (Fig. 3.74).

(6) Paraboloid of revolution. When $f(x) = \sqrt{(ax)}$, the generating curve is a parabola ($y^2 = ax$). The surface generated is $\mathbf{y^2 + z^2 = ax}$. which is called a *paraboloid of revolution* (Fig. 3.75).

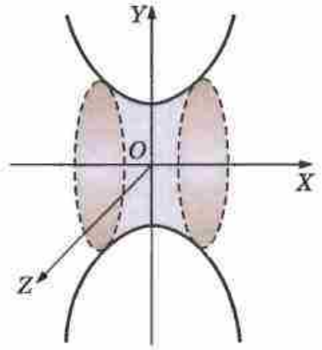


Fig. 3.73

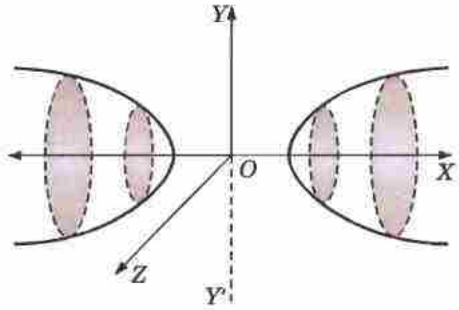


Fig. 3.74

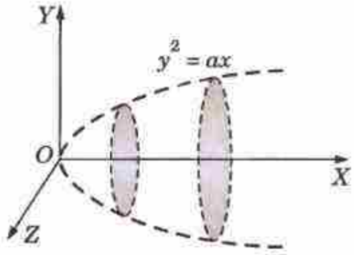


Fig. 3.75

PROBLEMS 3.16

1. What surface is represented by $4x^2 + 9y^2 + 16z^2 = 144$? Trace it roughly. Find the area of the plane curve in which $y = 2$ cuts it.
2. Sketch (roughly) the surface $5(x^2 + z^2) - y^2 = 6$.

   In what curve does the plane $z = 2$ intersect it ? Find the area of the curve of intersection ? What surfaces are represented by the following equations ? Draw diagrams to show their shapes.
3. $x^2 + y^2 = 16$.
4. $x^2/2 - y^2/3 = z$.
5. $z^2 = 4(1 + x^2 + y^2)$.
6. $y^2 = 4z - 8$ (*Andhra, 2000*)
7. $x^2 + y^2 = 5 - 2y$.
8. $x^2 + y^2 = 9z^2$.
9. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = \dfrac{2z}{c}$. (*P.T.U., 2009*)
10. $4x^2 - y^2 - 16z^2 = 36$.

Note. For the equations of the tangent plane and the normal line to *a* surface refer to § 5.8 (2).

3.23 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 3.17

Select the correct answer or fill up the blanks in each of the following questions :

1. The line $x = ay + b, z = cy + d$ and $x = a'y + b', z = c'y + d'$ are perpendicular if
   (*a*) $aa' + cc' = 1$ (*b*) $aa' + cc' = -1$ (*c*) $bb' + dd' = 1$ (*d*) $bb' + dd' = -1$.
2. The coordinates of the point of intersection of the line $\dfrac{x+1}{1} = \dfrac{y+3}{3} = \dfrac{z+2}{-2}$ with the plane $3x + 4y + 5z = 5$ is
   (*a*) (5, 15, − 14) (*b*) (3, 4, 5) (*c*) (1, 3, − 2) (*d*) (3, 12, − 10).
3. The equation of a right circular cylinder, whose axis is the z-axis and radius a is
   (*a*) $x^2 + y^2 + z^2 = a^2$ (*b*) $z^2 + y^2 = a^2$ (*c*) $x^2 + y^2 = a^2$ (*d*) $z^2 + x^2 = a^2$.
4. The equation $\sqrt{fx} + \sqrt{gy} + \sqrt{hz} = 0$ represent a
   (*a*) sphere (*b*) cylinder (*c*) cone (*d*) pair of planes.

5. The plane $ax + by + cz = 0$ cuts the cone $xy + yz + zx = 0$ in perpendicular lines if

(a) $a + b + c = 0$ (b) $1/a + 1/b + 1/c = 0$
(c) $a^2 + b^2 + c^2 = 0$ (d) $abc = 0$.

6. The equation of the cylinder which intersects the curve $x^2 + y^2 + z^2 = 1$, $x + y + z = 1$ and whose generators are parallel to the axis of z, is

(a) $x^2 + y^2 + xy - x - y = 0$ (b) $x^2 + y^2 + xy + x + y = 0$
(c) $x^2 + y^2 - xy - x - y = 0$ (d) $x^2 + y^2 - xy + x + y = 0$.

7. The equation $x^2 + y^2 + z^2 + xy + yz - zx = 9$ represents

(a) a sphere with $x^2 + y^2 + z^2 = 9$, $x - y + z = 3$ as a great circle
(b) a cone with $x^2 + y^2 + z^2 = 9$, $x - y + z = 3$ as a guiding circle
(c) a cylinder with $x^2 + y^2 + z^2 = 9$, $x - y + z = 3$ as a guiding circle
(d) none of the above.

8. The sum of the direction cosines of a straight line is

(a) zero (b) one (c) constant (d) n one of the above.

9. The angle between the planes $x - y + 2z - 9 = 0$ and $2x + y + z = 7$ is

(a) 30° (b) 90° (c) 60° (d) 120° *(V.T.U., 2010 S)*

10. The equation of the right circular cone whose axis is $x = y = z$, vertex is the origin and the semi-vertical angle is 45° is given as

(a) $x^2 + y^2 + z^2 = 0$ (b) $2(x^2 + y^2 + z^2) = 3(x + y + z)^2$
(c) $3(x^2 + y^2 + z^2) = 2(x + y + z)^2$ (d) $x^2 + y^2 + z^2 + xy + yz + zx = 0$.

11. The equation of a straight line parallel to the x-axis is given by

(a) $\dfrac{x-a}{1} = \dfrac{y-b}{1} = \dfrac{z-c}{1}$ (b) $\dfrac{x-a}{0} = \dfrac{y-b}{1} = \dfrac{z-c}{1}$
(c) $\dfrac{x-a}{0} = \dfrac{y-b}{0} = \dfrac{z-c}{1}$ (d) $\dfrac{x-a}{1} = \dfrac{y-b}{0} = \dfrac{z-c}{0}$.

12. The equation of the plane passing through $(4, -2, 1)$ and perpendicular to the line with direction ratios $7, 2, -3$ is

(a) $x + 3y - 4z - 8 = 0$ (b) $2x + 7y - 3z - 24 = 0$
(c) $7x + 2y - 3z - 21 = 0$. (d) $7x + 3y - 2z + 21 = 0$. *(V.T.U., 2009 S)*

13. The equation to the axis of the right circular cylinder whose guiding circle is $x^2 + y^2 + z^2 = 9$, $x - y + z = 3$ is given by

(a) $x = y = z$ (b) $x = -y = z$ (c) $x = y = -z$ (d) $x = -y = -z$.

14. In three dimensions, the equation $x^2 - y^2 = a^2$ represents

(a) a pair of straight lines (b) a hyperbola
(c) a cylinder (d) a cone.

15. Section of a sphere by a plane is

(a) parabola (b) ellipse (c) circle.

16. A line makes angles α, β, γ with the coordinate axes, then $\cos 2\alpha + \cos 2\beta + \cos 2\gamma$ is equal to

(a) 1 (b) 2 (c) -1 (d) -2. *(V.T.U., 2010 S)*

17. Three lines are coplanar if

(a) they are concurrent
(b) a line is perpendicular to each of them
(c) they are concurrent and a line is perpendicular to each of them.

18. The distance between the planes $2x + 2y + z - 6 = 0$ and $4x + 4y + 2z - 7 = 0$ is

(a) 1/3 (b) 5/6 (c) 13/3.

19. The line $\dfrac{x-1}{1} = \dfrac{y-2}{2} = \dfrac{z-3}{3}$ and the plane $x + y - z = 0$ are

(a) parallel (b) perpendicular (c) such that the line lies in the plane.

20. The radius of a great circle of a sphere is

(a) greater than the radius of the sphere (b) less than the radius of the sphere
(c) equal to the radius of the sphere.

21. Which of the following lines are generators of the cone $yz + 4zx + 3xy = 0$?

(a) $x = y = z$ (b) $x = -y = z$ (c) $x = 2y = -z$.

22. The semi-vertical angle of the cone generated by revolving the line $x + y = 0, z = 0$ about the x-axis is
(a) 90° (b) 45° (c) 30°.
23. All cones passing through the coordinate axes are given by the equation
(a) $x^2 + y^2 + z^2 - yz - zx - xy = 0$ (b) $ax^2 + by^2 + cz^2 - yz - zx - xy = 0$
(c) $ayz + bzx + cxy = 0$.
24. The line $\frac{x+1}{3} = \frac{y-2}{6} = \frac{z-3}{9}$ is perpendicular to the plane $ax + by + cz + d = 0$, if
(a) $a = 2b, b = 3c$ (b) $2a = b, b = 3c$ (c) $2a = b, 3b = 2c$ (d) $a = 3b, 2b = c$.
25. The equation $2(x^2 + y^2 + z^2) - 2xy + 2yz + 2zx = 3a^2$ represents a
(a) cone (b) right-circular cylinder
(c) sphere (d) pair of planes.
26. The equation of the plane through the point $(2, -3, 1)$ and parallel to the plane $3x - 4y + 2z = 5$ is
(a) $3x - 4y + 2z - 20 = 0$ (b) $3x + 4y - 2z + 20 = 0$
(c) $3x - 4y - 2z + 20 = 0$ (d) $3x + 4y + 2z - 20 = 0$.
27. The direction cosines of a line which is equally inclined to the coordinate axes are ....
28. The direction cosines of the line $x = 0 = y$ are ....
29. The equation of the axis of the cylinder $x^2 + y^2 = 25$ is ......
30. The image of the point $(3, 2, -1)$ in the YOZ plane is ......
31. The plane $x - 2y - 2z = k$ touches the sphere $x^2 + y^2 + z^2 - 2x + 4y - 6z + 5 = 0$ for $k =$ ...... (P.T.U., 2010)
32. The condition for the three concurrent lines to be coplanar is ......
33. The equation of the cone whose vertex is at the origin and base the circle $x = a, y^2 + z^2 = b^2$ is given by ....
34. The plane through points $(2, 2, 1)$, $(9, 3, 6)$ and perpendicular to the plane $2x + 6y + 6z = 9$ is ......
35. Volume of the sphere $x^2 + y^2 + z^2 + 2x - 4y + 8z - 2 = 0$ is ......
36. Angle between the planes $x - y + z = 1$ and $2x - 3y + z = 7$ is ......
37. The equation of the cone whose vertex is the origin and ginding curve is $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{1} = 1, x + y + z = 1$, is ...... (Anna, 2009)
38. Any two points on the line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ other than $(1, 2, 3)$ are ......
39. The equation of the line joining the points $(1, 2, 3)$ and $(2, 1, -3)$ is ......
40. The equation of the sphere on the line joining $(1, 5, 6)$ and $(-2, 1, 1)$ as diameter is ......
41. The conditions for the line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ to lie on the plane $ax + by + cz + d = 0$ are ......
42. The distance between the planes $4x + 3y + z + 4 = 0$ and $8x + 6y + 2z + 12 = 0$ is ....
43. The centre and radius of the sphere $2x^2 + 2y^2 + 2z^2 - 6x + 8y - 8z - 1 = 0$ are ......
44. The radius of the circle $x^2 + y^2 + z^2 - 2x - 4y - 11 = 0, x + 2y + 2z = 15$ is ......
45. The symmetric form of the line $x + y + z + 1 = 0 = 4x + y - 2z + 2$ is ......
46. The equation $y^2 = 4z - 8$ represents a ....
47. The equation $x^2 + y^2 = \frac{1}{4}z^2 - 1$ represents a ......
48. Angle between the lines whose d.r.s. are 1, 2, 3 and -1, 1, 2 is ......
49. The intercepts of the plane $2x - 3y + z = 12$ on the coordinate axes are ....
50. The radius of the sphere whose centre is $(4, 4, -2)$ and which passes through the origin is ......
51. The points $(0, 4, 1)$, $(2, 3, -1)$, $(4, 5, 0)$ and $(2, 6, 2)$ are the vertices of a square. (True or False)
52. The points $(3, -1, 1)$, $(5, -4, 2)$ and $(11, -13, 5)$ are collinear. (True or False)
53. The plane $5x + 6y + 7z = 110$, $2x + 3y - 4z = 29$ are perpendicular to each other. (True or False)
54. In three dimensional space, $9x^2 + 16y^2 = 144$ represents ....
55. Equation of the right circular cone with vertex at origin and passing through the curve $x^2 + y^2 + z^2 = 9, x + y + z = 1$ is ....
56. A unit vector perpendicular to the vectors $-2\mathbf{I} + 3\mathbf{J} + \mathbf{K}$ and $4\mathbf{I} + 2\mathbf{J}$ is ....

CHAPTER 4

Differential Calculus & Its Applications

4.1 (1) SUCCESSIVE DIFFERENTIATION

The reader is already familiar with the process of differentiating a function $y = f(x)$. For ready reference, a list of derivatives of some standard functions is given in the beginning.

The derivative dy/dx is, in general, another function of x which can be differentiated. The derivative of dy/dx is called the *second derivative* of y and is denoted by d^2y/dx^2. Similarly, the derivative of d^2y/dx^2 is called the *third derivative* of y and is denoted by d^3y/dx^3. In general, the nth derivative of y is denoted by d^ny/dx^n.

Alternative notations for the successive derivatives of $y = f(x)$ are

$$Dy, D^2y, D^3y, ..., D^ny ;$$

or $$y_1, y_2, y_3, ..., y_n ;$$

or $$f'(x), f''(x), f'''(x) ..., f^n(x).$$

The nth derivative of $y = f(x)$ at $x = a$ is denoted by $(d^ny/dx^n)_a$, $(y_n)_a$ or $f^n(a)$.

Example 4.1. *If $y = e^{ax} \sin bx$, prove that $y_2 - 2ay_1 + (a^2 + b^2)y = 0$.* *(Cochin, 2005)*

Solution. We have $$y = e^{ax} \sin bx \qquad ...(i)$$

$\therefore$ $$y_1 = e^{ax} (\cos bx \,.\, b) + \sin bx\, (e^{ax} \,.\, a) = be^{ax} \cos bx + ay \qquad [\text{By } (i)]$$

or $$y_1 - ay = be^{ax} \cos bx \qquad ...(ii)$$

Again differentiating both sides,

$$y_2 - ay_1 = be^{ax} (-\sin bx \,.\, b) + b \cos bx\, (e^{ax} \,.\, a) = -b^2y + a(y_1 - ay)$$

or $$y_2 - 2ay_1 + (a^2 + b^2)y = 0.$$

Example 4.2. *If $x = a (\cos t + t \sin t)$, $y = a (\sin t - t \cos t)$, find d^2y/dx^2.*

Solution. We have $$\frac{dx}{dt} = a(-\sin t + t \cos t + \sin t) = at \cos t$$

and $$\frac{dy}{dt} = a(\cos t + t \sin t - \cos t) = at \sin t$$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{dy}{dt} \Big/ \dfrac{dx}{dt} = \dfrac{at \sin t}{at \cos t} = \tan t$

$\therefore \quad \dfrac{d^2y}{dx^2} = \dfrac{d}{dt}(\tan t) \cdot \dfrac{dt}{dx} = \sec^2 t \cdot \dfrac{1}{at \cos t} = 1/at \cos^3 t.$

Example 4.3. *Given $y^2 = f(x)$, a polynomial of third degree, then evaluate* $\dfrac{d}{dx}\left(y^3 \dfrac{d^2y}{dx^2}\right)$.

Solution. Differentiating $y^2 = f(x)$ w.r.t. x, we get

$$2y \frac{dy}{dx} = f'(x) \qquad ...(i)$$

Differentiating (i) w.r.t. x again, we obtain

$$2\left(\frac{dy}{dx} \cdot \frac{dy}{dx} + y \frac{d^2y}{dx^2}\right) = f''(x) \quad \text{or} \quad 2\left(\frac{dy}{dx}\right)^2 + 2y \frac{d^2y}{dx^2} = f''(x)$$

Again differentiating, we get

$$4 \cdot \frac{dy}{dx} \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} \frac{d^2y}{dx^2} + 2y \frac{d^3y}{dx^3} = f'''(x)$$

or $$3y^2 \frac{dy}{dx} \frac{d^2y}{dx^2} + y^3 \frac{d^3y}{dx^3} = \frac{1}{2} y^2 f'''(x) \qquad \text{[Multiplying by } y^2\text{]}$$

Hence $$\frac{d}{dx}\left(y^3 \frac{d^2y}{dx^2}\right) = \frac{1}{2} f(x) f'''(x). \qquad [\because \; y^2 = f(x)]$$

Example 4.4. *If $ax^2 + 2hxy + by^2 = 1$, prove that* $\dfrac{d^2y}{dx^2} = \dfrac{h^2 - ab}{(hx + by)^3}$.

Solution. Differentiating the given equation w.r.t. x,

$$2ax + 2h\left(x \frac{dy}{dx} + y\right) + 2by \frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{ax + hy}{hx + by} \qquad ...(i)$$

Differentiating both sides of (i) w.r.t. x,

$$\frac{d^2y}{dx^2} = -\frac{(hx + by)(a + h\,dy/dx) - (ax + hy)(h + b\,dy/dx)}{(hx + by)^2}$$

[Substituting the value of dy/dx from (i)]

$$= -\frac{(hx + by)\left(a - h \cdot \dfrac{ax + hy}{hx + by}\right) - (ax + hy)\left(h - b \cdot \dfrac{ax + hy}{hx + by}\right)}{(hx + by)^2}$$

$$= \frac{(h^2 - ab)(ax^2 + 2hxy + by^2)}{(hx + by)^3}$$

$$= (h^2 - ab)/(hx + by)^3 \qquad [\because \; ax^2 + 2hxy + by^2 = 1]$$

PROBLEMS 4.1

1. If $y = (ax + b)/(cx + d)$, show that $2y_1y_3 = 3y_2^{\,2}$.
2. If $y = \sin(\sin x)$, prove that $\dfrac{d^2y}{dx^2} + \tan x \dfrac{dy}{dx} + y \cos^2 x = 0$.
3. If $y = e^{-kt} \cos(lt + c)$, show that $\dfrac{d^2y}{dx^2} + 2k \dfrac{dy}{dx} + n^2y = 0$, where $n^2 = k^2 + l^2$.

4. If $y = \sinh [m \log \{x + \sqrt{(x^2+1)}\}]$, show that $(x^2+1)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = m^2 y$.

5. If $y = \sin^{-1} x$, show that $(1-x^2)\, y_5 - 7xy_4 - 9y_3 = 0$. *(Madras, 2000 S)*

6. If $x = \dfrac{1}{2}\left(t + \dfrac{1}{t}\right), y = \dfrac{1}{2}\left(t - \dfrac{1}{t}\right)$, find $\dfrac{d^2y}{dx^2}$. *(Cochin, 2005)*

7. If $x = 2\cos t - \cos 2t$, $y = 2\sin t - \sin 2t$, find the value of d^2y/dx^2 when $t = \pi/2$.

8. If $x = a(\cos t + \log \tan t/2)$, $y = a \sin t$, find d^2y/dx^2.

9. If $x = \sin t$, $y = \sin pt$, prove that $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + p^2 y = 0$.

10. If $x^3 + y^3 = 3axy$, prove that $\dfrac{d^2y}{dx^2} = -\dfrac{2a^2xy}{(y^2-ax)^3}$.

(2) Standard Results

We have (1) $\mathbf{D^n (ax+b)^m = m(m-1)(m-2) \ldots (m-n+1)\, a^n (ax+b)^{m-n}}$

(2) $\mathbf{D^n \left(\dfrac{1}{ax+b}\right) = \dfrac{(-1)^n (n!)\, a^n}{(ax+b)^{n+1}}}$ (3) $\mathbf{D^n \log (ax+b) = \dfrac{(-1)^{n-1} (n-1)!\, a^n}{(ax+b)^n}}$

(4) $\mathbf{D^n (a^{mx}) = m^n (\log a)^n . a^{mx}}$ (5) $\mathbf{D^n (e^{mx}) = m^n e^{mx}}$

(6) $\mathbf{D^n \sin (ax+b) = a^n \sin (ax+b+n\pi/2)}$ (7) $\mathbf{D^n \cos (ax+b) = a^n \cos (ax+b+n\pi/2)}$

(8) $\mathbf{D^n [e^{ax} \sin (bx+c)] = (a^2+b^2)^{n/2} e^{ax} \sin (bx+c+n \tan^{-1} b/a)}$

(9) $\mathbf{D^n [e^{ax} \cos (bx+c)] = (a^2+b^2)^{n/2} e^{ax} \cos (bx+c+n \tan^{-1} b/a)}$

To prove (1), let $y = (ax+b)^m$

$$y_1 = m \,.\, a(ax+b)^{m-1}$$
$$y_2 = m(m-1)a^2 (ax+b)^{m-2}$$
$$y_3 = m(m-1)(m-2)\, a^3 (ax+b)^{m-3}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Hence $y_n = m(m-1)(m-2) \ldots (m-n+1)\, a^n (ax+b)^{m-n}$

In particular, $\mathbf{D^n (x^n) = n!}$.

(2) follows from (1) by taking $m = -1$. The proof of (3) is left as an exercise for the student.

To prove (4), let $y = a^{mx}$

$$y_1 = m \log a \,.\, a^{mx},\ y_2 = (m \log a)^2 a^{mx}, \text{ etc.}$$

In general $y_n = (m \log a)^n a^{mx}$.

(5) follows from (4) by taking $a = e$.

To prove (6), let $y = \sin (ax+b)$

$\therefore$
$$y_1 = a \cos (ax+b) = a \sin (ax+b+\pi/2)$$
$$y_2 = a^2 \cos (ax+b+\pi/2) = a^2 \sin (ax+b+2\pi/2)$$
$$y_3 = a^3 \cos (ax+b+2\pi/2) = a^3 \sin (ax+b+3\pi/2)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

In general, $y_n = a^n \sin (ax+b+n\pi/2)$.

The proof of (7), is left as an exercise for the reader.

To prove (8), let $y = e^{ax} \sin (bx+c)$

$\therefore$
$$y_1 = e^{ax} \cos (bx+c) \,.\, b + ae^{ax} \sin (bx+c)$$
$$= e^{ax} [a \sin (bx+c) + b \cos (bx+c)]$$

Put $a = r\cos\alpha$, $b = r\sin\alpha$ so that $r = \sqrt{(a^2+b^2)}$, $\alpha = \tan^{-1} b/a$

$\therefore$
$$y_1 = re^{ax} [\sin (bx+c) \cos\alpha + \cos (bx+c) \sin\alpha]$$
$$= re^{ax} \sin (bx+c+\alpha)$$

Similarly,
$$y_2 = r^2 e^{ax} \sin (bx+c+2\alpha)$$
$$y_3 = r^3 e^{ax} \sin (bx+c+3\alpha)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

In general, $y_n = r^n e^{ax} \sin (bx + c + n\alpha)$ *(V.T.U., 2000)*

where $r = \sqrt{(a^2 + b^2)}$ and $\alpha = \tan^{-1} b/a$.

Proceeding as in (8), the student should prove (9) himself.

(3) **Preliminary transformations.** Quite often preliminary simplification reduces the given function to one of the above standard forms and then the nth derivative can be written easily.

To find the nth derivative of the powers of sines or cosines or their products, we first express each of these as a series of sines or cosines of multiple angles and then use the above formulae (6) and (7).

Example 4.5. *If* $y = x \log \dfrac{x-1}{x+1}$, *show that* $y_n = (-1)^{n-2} (n-2)! \left[\dfrac{x-n}{(x-1)^n} - \dfrac{x+n}{(x+1)^n}\right]$. *(U.P.T.U., 2003)*

Solution. Differentiating y w.r.t. x, we have

$$y_1 = \log \frac{x-1}{x+1} + x\left[\frac{1}{x-1} - \frac{1}{x+1}\right]$$

$$= \log (x-1) - \log (x+1) + \frac{1}{x-1} + \frac{1}{x+1} \qquad ...(i)$$

Now differentiating (i) $(n-1)$ times w.r.t. x,

$$y_n = \frac{(-1)^{n-2} (n-2)!}{(x-1)^{n-1}} - \frac{(-1)^{n-2} (n-2)!}{(x+1)^{n-1}} + \frac{(-1)^{n-1} (n-1)!}{(x-1)^n} + \frac{(-1)^{n-1} (n-1)!}{(x+1)^n}$$

$$= (-1)^{n-2} (n-2)! \left\{\frac{x-1}{(x-1)^n} - \frac{x+1}{(x+1)^n} + \frac{-(n-1)}{(x-1)^n} + \frac{-(n-1)}{(x+1)^n}\right\}$$

$$= (-1)^{n-2} (n-2)! \left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right].$$

Example 4.6. *Find the nth derivative of (i)* $\cos x \cos 2x \cos 3x$ *(S.V.T.U., 2009)*
(ii) $e^{2x} \cos^2 x \sin x$.

Solution. (i) $\quad y = \cos x \cos 2x \cos 3x = \frac{1}{2} \cos x (\cos 5x + \cos x)$

$$= \tfrac{1}{4} (2 \cos x \cos 5x + 2 \cos^2 x) = \tfrac{1}{4} [(\cos 6x + \cos 4x) + (1 + \cos 2x)]$$

$$= \tfrac{1}{4} (1 + \cos 2x + \cos 4x + \cos 6x)$$

$$\therefore \quad y_n = \tfrac{1}{4} [2^n \cos (2x + n\pi/2) + 4^n \cos (4x + n\pi/2 + 6^n \cos (6x + n\pi/2)]$$

(ii) $\quad \cos^2 x \sin x = \cos x (\sin x \cos x) = \cos x \,.\, \frac{1}{2} \sin 2x$

$$= \tfrac{1}{4} (2 \sin 2x \cos x) = \tfrac{1}{4} (\sin 3x + \sin x)$$

$$\therefore \quad D^n(e^{2x} \cos^2 x \sin x) = \tfrac{1}{4} [D^n (e^{2x} \sin 3x) + D^n (e^{2x} \sin x)]$$

$$= \tfrac{1}{4} [(2^2 + 3^2)^{n/2} \sin (3x + n \tan^{-1} 3/2) + (2^2 + 1^2)^{n/2} \sin (x + n \tan^{-1} \tfrac{1}{2})]$$

$$= \tfrac{1}{4} [(13)^{n/2} \sin (3x + n \tan^{-1} 3/2) + (5)^{n/2} \sin (x + n \tan^{-1} \tfrac{1}{2})].$$

(4) Use of partial fractions. To find the nth derivative of any rational algebraic fraction, we first split it up into partial fractions. Even when the denominator cannot be resolved into real factors, the method of partial fractions can still be used after breaking the denominator into complex linear factors. Then to put the result back in a real form, we apply De Moivre's theorem (p. 647).

Example 4.7. *Find the nth derivative of* $\dfrac{x}{(x-1)(2x+3)}$.

Solution. $$\frac{x}{(x-1)(2x+3)} = \frac{1}{(x-1)(2.1+3)} + \frac{-3/2}{(-3/2-1)(2x+3)}$$
$$= \frac{1}{5}\cdot\frac{1}{x-1} + \frac{3}{5}\cdot\frac{1}{2x+3}$$

Hence $$D^n\left[\frac{x}{(x-1)(2x+3)}\right] = \frac{1}{5}\cdot\frac{(-1)^n\, n!}{(x-1)^{n+1}} + \frac{3}{5}\cdot\frac{(-1)^n\,(n!)2^n}{(2x+3)^{n+1}}$$
$$= \frac{(-1)^n\, n!}{5}\left\{\frac{1}{(x-1)^{n+1}} + \frac{3\,.\,2^n}{(2x+3)^{n+1}}\right\}.$$

Example 4.8. *Find the nth derivative of* $\frac{1}{x^2+a^2}$.

Solution. We have $$y = \frac{1}{x^2+a^2} = \frac{1}{(x+ia)(x-ia)} = \frac{1}{2ia}\left(\frac{1}{x-ia} - \frac{1}{x+ia}\right)$$

$\therefore$ $$y_n = \frac{1}{2ia}\left\{\frac{(-1)^n\, n!}{(x-ia)^{n+1}} - \frac{(-1)^n\, n!}{(x+ia)^{n+1}}\right\}$$

[Put $x = r\cos\theta$, $a = r\sin\theta$ so that $r = \sqrt{(x^2+a^2)}$, $\theta = \tan^{-1}(a/x)$]

$$= \frac{(-1)^n\, n!}{2ia}\left\{\frac{1}{r^{n+1}(\cos\theta - i\sin\theta)^{n+1}} - \frac{1}{r^{n+1}(\cos\theta + i\sin\theta)^{n+1}}\right\}$$
$$= \frac{(-1)^n\, n!}{2iar^{n+1}}\{(\cos\theta - i\sin\theta)^{-(n+1)} - (\cos\theta + i\sin\theta)^{-(n+1)}\}$$
$$= \frac{(-1)^n\, n!}{2iar^{n+1}}[\cos(n+1)\theta + i\sin(n+1)\theta - [\cos(n+1)\theta - i\sin(n+1)\theta]\}$$

[By De Moivre's theorem]

$$= \frac{(-1)^n\, n!}{2iar^{n+1}}\,.\, 2i\sin(n+1)\theta \qquad \left[\text{Put } \frac{1}{r} = \frac{\sin\theta}{a}\right]$$
$$= \frac{(-1)^n\, n!}{a^{n+2}}\sin(n+1)\theta\,\sin^{n+1}\theta.$$

PROBLEMS 4.2

Find the nth derivative of (1 to 11) :

1. $\log(4x^2-1)$ *(V.T.U., 2010)*
2. $\frac{x+2}{x+1} + \log\frac{x+2}{x+1}$
3. $\sin^3 x\cos^2 x$ *(V.T.U., 2006)*
4. $\cos^9 x$ *(Mumbai, 2008)*
5. $\sinh 2x\sin 4x$ *(V.T.U., 2010 S)*
6. $e^{5x}\cos x\cos 3x$ *(Mumbai, 2007)*
7. $\frac{x+3}{(x-1)(x+2)}$ *(V.T.U., 2009)*
8. $\frac{x^2}{2x^2+7x+6}$ *(V.T.U., 2005)*
9. $\frac{1}{1+x+x^2+x^3}$ *(Mumbai, 2009)*
10. $\frac{x}{x^2+a^2}$ *(Mumbai, 2007)*
11. Find the nth derivative of $\tan^{-1}\frac{2x}{1-x^2}$ in terms of r and θ. *(U.P.T.U., 2002)*

4.2 LEIBNITZ'S THEOREM for the *n*th Derivative of the product of two functions*

If u, v be two function of x possessing derivatives of the nth order, then

$$(\mathbf{uv})_n = \mathbf{u}_n\,\mathbf{v} + {}^n\mathbf{C}_1\,\mathbf{u}_{n-1}\,\mathbf{v}_1 + {}^n\mathbf{C}_2\,\mathbf{u}_{n-2}\,\mathbf{v}_2 + \dots + {}^n\mathbf{C}_r\mathbf{u}_{n-r}\,\mathbf{v}_r + \dots + {}^n\mathbf{C}_n\,\mathbf{uv}_n$$

We shall prove this theorem by mathematical induction.

Step I. By actual differentiation,

$$(uv)_1 = u_1 v + uv_1$$
$$(uv)_2 = (u_2 v + u_1 v_1) + (u_1 v_1 + uv_2)$$
$$= u_2 v + {}^2C_1\,u_1 v_1 + {}^2C_2\,uv_2 \qquad [\because\ 2 = {}^2C_1,\ 1 = {}^2C_2]$$

Thus we see that the theorem is true for $n = 1, 2$.

Step II. Assume the theorem to be true for $n = m$ (say) so that

$$(uv)_m = u_m v + {}^mC_1\,u_{m-1}v_1 + {}^mC_2\,u_{m-2}v_2 + \dots + {}^mC_{r-1}u_{m-r+1}v_{r-1} + {}^mC_r\,u_{m-r}v_r + \dots + {}^mC_m\,uv_m$$

Differentiating both sides,

$$(uv)_{m+1} = (u_{m+1}v + u_m v_1) + {}^mC_1\,(u_m v_1 + u_{m-1}v_2) + {}^mC_2\,(u_{m-1}v_2 + u_{m-2}v_3) + \dots$$
$$+ {}^mC_{r-1}\,u_{m-r+2}v_{r-1} + u_{m-r+1}v_r) + {}^mC_r\,(u_{m-r+1}v_r + u_{m-r}v_{r+1}) + \dots$$
$$+ {}^mC_m\,(u_1 v_m + uv_{m+1})$$
$$= u_{m+1}v + (1 + {}^mC_1)\,u_m v_1 + ({}^mC_1 + {}^mC_2)\,u_{m-1}v_2 + \dots + ({}^mC_{r-1} + {}^mC_r)\,u_{m-r+1}v_r + \dots + {}^mC_m\,uv_{m+1}$$

But $\quad 1 + {}^mC_1 = {}^mC_0 + {}^mC_1 = {}^{m+1}C_1,\ {}^mC_1 + {}^mC_2 = {}^{m+1}C_2 \dots$

$${}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r, \dots \quad \text{and} \quad {}^mC_m = 1 = {}^{m+1}C_{m+1}$$

$$\therefore \quad (uv)_{m+1} = u_{m+1}v + {}^{m+1}C_1\,u_m v_1 + {}^{m+1}C_2\,u_{m-1}v_2 + \dots + {}^{m+1}C_r\,u_{m+1-r}v_r + \dots + {}^{m+1}C_{m+1}uv_{m+1}$$

which is of exactly the same form as the given formula with n replaced by $m + 1$. Hence if the theorem is true for $n = m$, it is also true for $n = m + 1$.

Step III. In step I, the theorem has been seen to be true for $n = 2$, and by step II, it must be true for $n = 2 + 1$ *i.e.*, 3 and so for $n = 3 + 1$ *i.e.*, 4 and so on.

Hence the theorem is true for all positive integral values of n.

Example 4.9. *Find the nth derivative of $e^x (2x + 3)^3$.*

Solution. Take $u = e^x$ and $v = (2x + 3)^3$, so that $u_n = e^x$ for all integral values of n, and $v_1 = 6\,(2x + 3)^2$, $v_2 = 24(2x + 3)$, $v_3 = 48$, v_4, v_5 etc. are all zero.

$\therefore$ By Leibnitz's theorem,

$$(uv)_n = u_n v + {}^nC_1\,u_{n-1}v_1 + {}^nC_2\,u_{n-2}v_2 + {}^nC_3\,u_{n-3}v_3$$

i.e.,
$$[e^x (2x + 3)^3]_n = e^x (2x + 3)^3 + ne^x[6\,(2x + 3)^2]$$
$$+ \frac{n(n-1)}{1\,.\,2} e^x\,[24(2x + 3)] + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3} e^x\,[48]$$
$$= e^x\,\{(2x + 3)^3 + 6n(2x + 3)^2 + 12n\,(n - 1)\,(2x + 3) + 8n(n - 1)\,(n - 2)\}.$$

Example 4.10. *If $y = (\sin^{-1} x)^2$, show that $(1 - x^2)\,y_{n+2} - (2n + 1)\,xy_{n+1} - n^2 y_n = 0$. Hence find $(y_n)_0$.* (U.P.T.U., 2005)

Solution. We have $\quad y = (\sin^{-1} x)^2$

Differentiating, $\quad y_1 = \dfrac{2\sin^{-1} x}{\sqrt{(1 - x^2)}}$ or $(1 - x^2)\,y_1^2 = 4\,(\sin^{-1} x)^2 = 4y \qquad \dots(i)$

Again differentiating, $\quad (1 - x^2)\,2y_1 y_2 + (-2x)\,y_1^2 = 4y_1 \qquad \dots(ii)$

Dividing by $2y_1$, $(1 - x^2)\,y_2 - xy_1 - 2 = 0$

Differentiating it n times by Leibnitz's theorem,

*Named after the German mathematician and philosopher *Gottfried Wilhelm Leibnitz* (1646–1716) who invented the differential and integral calculus independent of Sir Issac Newton.

$$(1-x^2)\,y_{n+2} + n(-2x)\,y_{n+1} + \frac{n(n-1)}{2}(-2)y_n - [xy_{n+1} + n(1)\,y_n] = 0$$

or $$(1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} - n^2\,y_n = 0$$

which is the required result.

Putting $x = 0$, $\qquad (y_{n+2})_0 = n^2(y_n)_0 \qquad$ ...(*iii*)

From (*i*), $\qquad (y_1)_0 = 0$. From (*ii*), $(y_2)_0 = 2$.

Putting $n = 1, 3, 5, 7, \ldots$ in (*iii*), $\quad 0 = y_1 = y_3 = y_5 = y_7 = \ldots$

i.e., if n is odd, $\qquad (y_n)_0 = 0$

Again putting $n = 2, 4, 6, \ldots$ in (*iii*)

$$(y_4)_0 = 2^2\,(y_2)_0 = 2\,.\,2^2$$
$$(y_6)_0 = 4^2\,(y_6)_0 = 2.\,2^2\,.\,4^2$$
$$(y_8)_0 = 6^2\,(y_6)_0 = 2.\,2^2\,.\,4^2\,.\,6^2$$

In general, if n is even, $\qquad (y_n)_0 = 2\,.\,2^2\,.\,4^2\,.\,6^2 \ldots (n-2)^2,\ (n \neq 2).$

Example 4.11. *If* $y = e^{a\sin^{-1}x}$, *prove that* $(1-x^2)\,y_{n+2} - (2n+1)\,x\,y_{n+1} - (n^2+a^2)y_n = 0$. *Hence find the value of* y_n *when* $x = 0$. (V.T.U., 2003)

Solution. We have $\qquad y = e^{a\sin^{-1}x} \qquad$ ...(*i*)

Differentiating, $\qquad y_1 = e^{a\sin^{-1}x}\,\dfrac{a}{\sqrt{(1-x^2)}} = \dfrac{ay}{\sqrt{(1-x^2)}} \qquad$ ...(*ii*)

or $$(1-x^2)\,y_1^{\,2} = a^2\,y^2.$$

Again differentiating, $(1-x^2)\,2y_1y_2 + (-2x)y_1^{\,2} = 2a^2\,yy_1.$

Dividing by $2y_1$, $(1-x^2)y_2 - xy_1 - a^2y = 0 \qquad$ ...(*iii*)

Differentiating it n times by Leibnitz's theorem,

$$(1-x^2)\,y_{n+2} + n\,.\,(-2x)\,y_{n+1} + \frac{n(n-1)}{2}\,.\,(-2)\,y_n - [xy_{n+1} + n\,.\,1\,.\,y_n] - a^2\,y_n = 0$$

or $$(1-x^2)y_{n+2} - (2n+1)\,xy_{n+1} - (n^2+a^2)\,y_n = 0$$

which is the required result.

Putting $x = 0$, $\qquad (y_{n+2})_0 = (n^2+a^2)\,(y_n)_0 \qquad$ ...(*iv*)

From (*i*), (*ii*), (*iii*) : $\qquad (y)_0 = 1,\ (y_1)_0 = a,\ (y_2)_0 = a^2$

Putting $n = 1, 2, 3, 4 \ldots$ in (*iv*),

$$(y_3)_0 = (1^2+a^2)\,(y_1)_0 = a(1^2+a^2)$$
$$(y_4)_0 = (2^2+a^2)\,(y_2)_0 = a^2(2^2+a^2)$$
$$(y_5)_0 = (3^2+a^2)\,(y_3)_0 = a(1^2+a^2)\,(3^2+a^2)$$
$$(y_6)_0 = (4^2+a^2)\,(y_4)_0 = a^2(2^2+a^2)\,(4^2+a^2).$$

Hence in general, $\qquad (y_n)_0 = a(1^2+a^2)\,(3^2+a^2) \ldots [(n-2)^2+a^2], \qquad$ when n is odd.

$\qquad\qquad = a^2(2^2+a^2)\,(4^2+a^2) \ldots\ldots [(n-2)^2+a^2], \qquad$ when n is even.

Example 4.12. *If* $y^{1/m} + y^{-1/m} = 2x$, *prove that*

$$(x^2-1)y_{n+2} + (2n+1)\,xy_{n+1} + (n^2-m^2)\,y_n = 0.$$

(V.T.U., 2008 S ; Mumbai, 2007 ; S.V.T.U., 2007)

Solution. We have $\qquad y^{1/m} + \dfrac{1}{y^{1/m}} = 2x$

or $$(y^{1/m})^2 - 2x(y^{1/m}) + 1 = 0$$

$\therefore$ $$y^{1/m} = \frac{2x \pm \sqrt{(4x^2-4)}}{2} = x \pm \sqrt{(x^2-1)}$$

Hence $$y = [x \pm \sqrt{(x^2-1)}\,]^m$$

Taking logarithm, $\qquad \log y = m\log[x \pm \sqrt{(x^2-1)}\,]$

Differentiating both sides w.r.t. x,

$$\frac{1}{y}y_1 = m \cdot \frac{1}{x \pm \sqrt{(x^2-1)}} \cdot \left\{1 \pm \frac{x}{\sqrt{(x^2-1)}}\right\} = \pm \frac{m}{\sqrt{(x^2-1)}}$$

Squaring, $\quad y_1^2 (x^2 - 1) = m^2 y^2$

Again differentiating, $\quad (x^2 - 1)\, 2y_1 y_2 + y_1^2 (2x) = m^2 \cdot 2y \cdot y_1$

Dividing by $2y_1$, $\quad (x^2 - 1) y_2 + xy_1 - m^2 y = 0$

Differentiating it n times by Leibnitz's theorem,

$$(x^2-1) y_{n+2} + ny_{n+1}(2x) + \frac{n(n-1)}{2} y_n (2) + xy_{n+1} + n \cdot y_n (1) - m^2 y_n = 0$$

or
$$(x^2 - 1) y_{n+2} + (2n+1) xy_{n+1} + (n^2 - m^2) y_n = 0.$$

PROBLEMS 4.3

1. Find the nth derivative of (*i*) $x^2 \log 3x$. (*ii*) $2^x \cos^9 x$. (*Mumbai, 2009*)
2. If $y = a \cos(\log x) + b \sin(\log x)$, show that $x^2 y_2 + xy_1 + y = 0$ and $x^2 y_{n+2} + (2n+1) xy_{n+1} + (n^2+1) y_n = 0$. (*U.P.T.U., 2004 ; Madras, 2000*)
3. If $y = \sin^{-1} x$, prove that $(1 - x^2) y_{n+2} - (2n+1) xy_{n+1} - n^2 y_n = 0$. Also find (y_n). (*S.V.T.U., 2009*)
4. If $\cos^{-1}(y/b) = \log(x/n)^n$, prove that $x^2 y_{n+2} + (2n+1) xy_{n+1} + 2n^2 y_n = 0$. (*U.P.T.U., 2006*)
5. If $y = \tan^{-1} x$, prove that $(1 + x^2) y_{n+1} + 2nxy_n + n(n-1) y_{n-1} = 0$. Find $y_{n(0)}$.
6. If $y = \cos(m \sin^{-1} x)$, prove that $(1 - x^2) y_{n+2} - (2n+1) xy_{n+1} + (m^2 - n^2) y_n = 0$. (*Mumbai, 2008 S*)
7. If $y = \sin(m \sin^{-1} x)$, prove that $(1 - x^2) y_2 - xy_1 + m^2 y = 0$
   and $\quad (1 - x^2) y_{n+2} - 2(n+1) xy_{n+1} + (m^2 - n^2) y_n = 0$. (*V.T.U., 2009 ; Cochin, 2005*)
   Also find $(y_n)_0$. (*U.P.T.U., 2005*)
8. If $y = e^{m \cos^{-1} x}$, prove that (*i*) $(1 - x^2) y_2 - xy_1 = m^2 y$
   (*ii*) $(1 - x^2) y_{n+2} - (2n+1) xy_{n+1} - (n^2 + m^2) y_n = 0$. Also find $(y_n)_0$.
9. If $y = (x^2 - 1)^n$, prove that $(x^2 - 1) y_{n+2} + 2xy_{n+1} - n(n+1) y_n = 0$. (*U.T.U., 2010*)
10. If $\sin^{-1} y = 2 \log(x+1)$, prove that $(x+1)^2 y_{n+2} + (2n+1)(x+1) y_{n+1} + (x^2 + 4) y_n = 0$. (*V.T.U., 2003*)
11. If $y = x^n \log x$, prove that $y_{n+1} = n\,!/x$. (*Mumbai, 2008*)
12. If $V_n = \dfrac{d^n}{dx^n}(x^n \log x)$, show that $V_n = nV_{n-1} + (n-1)\,!$
    Hence, show that $V_n = n\,!\left(\log x + 1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots + \dfrac{1}{n}\right)$. (*V.T.U., 2001*)
13. Show that $\dfrac{d^n}{dx^n}\left(\dfrac{\log x}{x}\right) = \dfrac{(-1)^n n\,!}{x^{n+1}}\left\{\log x - 1 - \dfrac{1}{2} - \dfrac{1}{3} - \ldots - \dfrac{1}{n}\right\}$. (*V.T.U., 2006*)
14. If $y = x \log\left(\dfrac{x-1}{x+1}\right)$, show that $y_n = (-1)^{n-2}(n-2)\,!\left[\dfrac{x-n}{(x-1)^n} - \dfrac{x+n}{(x+1)^n}\right]$. (*U.P.T.U., 2003*)
15. If $x = \sin t$, $y = \cos pt$, show that $(1 - x^2) y_2 - xy_1 + p^2 y = 0$. Hence prove that
    $(1 - x^2) y_{n+2} - (2n+1) xy_{n+1} - (n^2 - p^2) y_n = 0$. (*Raipur, 2005 ; V.T.U., 2005*)
16. If $y = \log\{x + \sqrt{(1+x^2)}\}^2$, prove that $(1 + x^2) y_{n+2} + (2n+1) xy_{n+1} + n^2 y_n = 0$. (*V.T.U., 2007 ; Bhillai, 2005*)
    Hence show that $(y_{2k})_0 = (-1)^{k-1} \cdot 2^k \cdot [(k-1)\,!]^2$, where k is positive integer.
17. If $y = [x + \sqrt{(x^2+1)}\,]^m$, prove that (*i*) $(x^2 + 1) y_2 + xy_1 - m^2 y = 0$, (*ii*) $y_{n+2} + (n^2 - m^2) y_n = 0$ at $x = 0$. (*V.T.U., 2009 S*)
    Hence find $y_n(0)$. (*Madras, 2000*)
18. If $y = \sin \log(x^2 + 2x + 1)$, prove that (*i*) $(x+1)^2 y_2 + (x+1) y_1 + 4y = 0$
    (*ii*) $(x+1)^2 y_{n+2} + (2n+1)(x+1) y_{n+1} + (n^2 + 4) y_n = 0$. (*U.P.T.U., 2006*)

19. If $y = \dfrac{\sinh^{-1} x}{\sqrt{(1+x^2)}}$, show that $(1 + x^2) y_{n+2} + (2n + 3) xy_{n+1} + (n + 1)^2 y_n = 0.$ (*V.T.U., 2010*)

20. If $y = \sinh [m \log (x + \sqrt{x^2 + 1})]$, prove that $(x^2 + 1) y_{n+2} + (2n + 1) xy_{n+1} + (n^2 - m^2) y_n = 0.$ (*V.T.U., 2010 S*)

4.3 FUNDAMENTAL THEOREMS

(1) Rolle's Theorem

If (*i*) *f*(*x*) *is continuous in the closed interval [a, b], (ii) f ′(x) exists for every value of x in the open interval (a, b) and (iii) f (a) = f (b), then there is at least one value c of x in (a, b) such that f ′(c) = 0.*

Consider the portion AB of the curve $y = f(x)$, lying between $x = a$ and $x = b$, such that

(*i*) it goes continuously from A to B,

(*ii*) it has a tangent at every point between A and B, and

(*iii*) ordinate of A = ordinate of B.

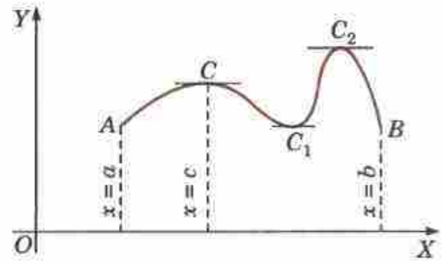


Fig. 4.1

From the Fig. 4.1, it is self-evident that there is at least one point C (may be more) of the curve at which the tangent is parallel, to the x-axis.

i.e., slope of the tangent at C $(x = c) = 0$

But the slope of the tangent at C is the value of the differential coefficient of $f(x)$ w.r.t. x thereat, therefore $f'(c) = 0$.

Hence the theorem is proved.

Example 4.13. *Verify Rolle's theorem for (i) $\sin x / e^x$ in $(0, \pi)$.* (*J.N.T.U., 2003*)
(ii) $(x - a)^m (x - b)^n$ where m, n are positive integers in [a, b]. (*V.T.U., 2010 ; Nagarjuna, 2008*)

Solution. (*i*) Let $f(x) = \sin x/e^x$.

$f(x)$ is derivable in $(0, \pi)$.

Also $f(0) = f(\pi) = 0.$

Hence the conditions of Rolle's theorem are satisfied.

$$\therefore \quad f'(x) = \frac{e^x \cos x - e^x \sin x}{e^{2x}} \quad \text{vanishes where } e^x (\cos x - \sin x) = 0$$

or $\tan x = 1$ *i.e.*, $x = \pi/4$.

The value $x = \pi/4$ lies in $(0, \pi)$, so that Rolle's theorem is verified.

(*ii*) Let $f(x) = (x - a)^m (x - b)^n$.

Since every polynomial is continuous for all values, $f(x)$ is also continuous in $[a, b]$.

$$f'(x) = m(x - a)^{m-1} (x - b)^n + (x - a)^m \,.\, n(x - b)^{n-1}$$
$$= (x - a)^{m-1} (x - b)^{n-1} [(m + n) x - (mb + na)]$$

which exists, *i.e.*, $f(x)$ is derivable in (a, b).

Also $f(a) = 0 = f(b)$.

Thus all the conditions of Rolle's theorem are satisfied and there exists c in (a, b) such that $f'(c) = 0$.

$$\therefore \quad (c - a)^{m-1} (c - b)^{n-1} [(m + n) c - (mb + na)] = 0 \quad \text{or} \quad c = (mb + na)/(m + n).$$

Hence, Rolle's theorem is verified.

(2) Lagrange's Mean-Value Theorem*

First form. *If (i) f(x) is continuous in the closed interval [a, b], and*
(ii) f′(x) exists in the open interval (a, b), then there is at least one value c of x in (a, b), such that

$$\mathbf{\frac{f(b) - f(a)}{b - a} = f'(c).}$$

*Named after the great French mathematician *Joseph Louis Lagrange* (1736–1813) who became professor at Military Academy, Turin when he was just 19 and director of Berlin Academy in 1766. His important contribution are to algebra, number theory, differential equations, mechanics, approximation theory and calculus of variations.

Consider the function $\phi(x) = f(x) - \dfrac{f(b) - f(a)}{b - a}\, x$

Since $f(x)$ is continuous in $[a, b]$; $\therefore$ $\phi(x)$ is also continuous in $[a, b]$.

Since $f'(x)$ exists in (a, b) ;

$\therefore$ $\phi'(x)$ also exists in (a, b) and $= f'(x) - \dfrac{f(b) - f(a)}{b - a}$ ...(i)

Clearly, $\phi(a) = \dfrac{b\,f(a) - a\,f(b)}{b - a} = \phi(b)$.

Thus $\phi(x)$ satisfies all the conditions of Rolle's theorem.

$\therefore$ There is at least one value c of x between a and b such that $\phi'(c) = 0$. Substituting $x = c$ in (1), we get

$$f'(c) = \frac{f(b) - f(a)}{b - a} \qquad ...(2)$$

which proves the theorem.

Second form. If we write $b = a + h$, then since $a < c < b$,

$$c = a + \theta h \text{ where } 0 < \theta < 1.$$

Thus the mean value theorem may be stated as follows :

If (i) $f(x)$ is continuous in the closed interval $[a, a + h]$ and (ii) $f'(x)$ exists in the open interval $(a, a + h)$, then there is at least one number $\theta(0 < \theta < 1)$ such that

$$\mathbf{f(a + h) = f(a) + hf'(a + \theta h)}$$

Geometrical Interpretation. Let A, B be the points on the curve $y = f(x)$ corresponding to $x = a$ and $x = b$ so that $A = [a, f(a)]$ and $B = [b, f(b)]$. (Fig. 4.2)

$\therefore$ Slope of chord $AB = \dfrac{f(b) - f(a)}{b - a}$

By (2), the slope of the chord $AB = f'(c)$, the slope of the tangent of the curve at $C(x = c)$.

Hence the Lagrange's mean value theorem asserts that *if a curve AB has a tangent at each of its points, then there exists at least one point C on this curve, the tangent at which is parallel to the chord AB.*

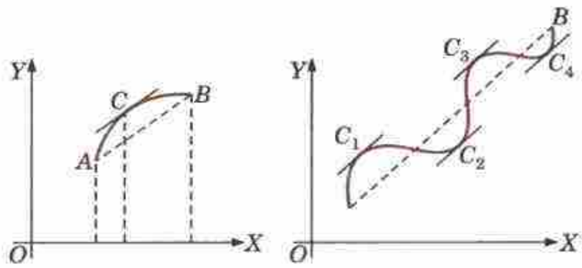


Fig. 4.2

Cor. *If $f'(x) = 0$ in the interval (a, b) then $f(x)$ is constant in $[a, b]$.* For, if x_1, x_2 be any two values of x in (a, b), then by (2),

$$f(x_2) - f(x_1) = (x_2 - x_1) f'(c) = 0 \ (x_1 < c < x_2)$$

Thus, $f(x_1) = f(x_2)$ *i.e.*, $f(x)$ has the same value for every value of x in (a, b).

Example 4.14. *In the Mean value theorem $f(b) - f(a) = (b - a) f'(c)$,* ...(i)
determine c lying between a and b, if $f(x) = x(x - 1)(x - 2)$, $a = 0$ and $b = 1/2$. (Gorakhpur, 1999)

Solution. $f(a) = 0$, $f(b) = \dfrac{1}{2}\left(-\dfrac{1}{2}\right)\left(-\dfrac{3}{2}\right) = \dfrac{3}{8}$

$f'(x) = 3x^2 - 6x + 2$, $f'(c) = 3c^2 - 6c + 2$

Substituting in (i), $\dfrac{3}{8} - 0 = \left(\dfrac{1}{2} - 0\right)(3c^2 - 6c + 2)$

or $12c^2 - 24c + 5 = 0$

whence $c = \dfrac{24 \pm \sqrt{(24)^2 - 12 \times 5 \times 4}}{24} = 1 \pm 0.764 = 1.764 ; 0.236.$

Hence $c = 0.236$, since it only lies between 0 and 1/2.

Example 4.15. *Prove that (if $0 < a < b < 1$), $\dfrac{b - a}{1 + b^2} < \tan^{-1} b - \tan^{-1} a < \dfrac{b - a}{1 + a^2}$.*

Hence show that $\dfrac{\pi}{4} + \dfrac{3}{25} < \tan^{-1}\dfrac{4}{3} < \dfrac{\pi}{4} + \dfrac{1}{6}$. (Mumbai, 2009 ; V.T.U., 2006)

Solution. Let $f(x) = \tan^{-1} x$, so that $f'(x) = \dfrac{1}{1+x^2}$.

By Mean value theorem, $\dfrac{\tan^{-1} b - \tan^{-1} a}{b-a} = \dfrac{1}{1+c^2}, a < c < b$ ...(*i*)

Now $a < c < b$, $\therefore$ $1 + a^2 < 1 + c^2 < 1 + b^2$.

$$\therefore \quad \frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2} \text{ i.e., } \frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2}$$

i.e., $$\frac{1}{1+b^2} < \frac{\tan^{-1} b - \tan^{-1} a}{b-a} < \frac{1}{1+a^2}$$ [By (*i*)]

Hence $$\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$$

Now let $a = 1$, $b = 4/3$.

Then $$\frac{1/3}{1+16/9} < \tan^{-1}\frac{4}{3} - \frac{\pi}{4} < \frac{1/3}{1+1}$$

i.e., $$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}.$$

Example 4.16. *Prove that* $\log(1+x) = x/(1+\theta x)$, *where* $0 < \theta < 1$ *and hence deduce that*

$\dfrac{x}{1+x} < \log(1+x) < x, x > a$ *(Mumbai, 2008)*

Solution. Let $f(x) = \log(1+x)$, then by second form of Lagrange's mean value theorem

$$f(a+h) = f(a) + h f'(a+\theta h), \quad (0 < \theta < 1)$$

we have $$f(x) = f(0) + x f'(\theta x)$$ [Taking $a = 0$, $h = x$]

or $$\log(1+x) = \log(1) + x \,.\, 1/(1+\theta x)$$ [$\because$ $f'(x) = 1/1 + x$)

Hence $$\log(1+x) = x/(1+\theta x)$$ ...(*i*) [$\because$ $\log(1) = 0$

Since $0 < \theta < 1$, $\therefore$ $0 < \theta x < x$ for $x > 0$.

or $$1 < 1 + \theta x < 1 + x \quad \text{or} \quad 1 > \frac{1}{1+\theta x} > \frac{1}{1+x}$$

or $$x > \frac{x}{1+\theta x} > \frac{x}{1+x}$$

or $$\frac{x}{1+x} < \log(1+x) < x, x > 0.$$ [By (*i*)]

(3) Cauchy's Mean-value Theorem*

If (i) $f(x)$ and $g(x)$ *be continuous in [a, b]*

(*ii*) $f'(x)$ and $g'(x)$ exist in (a, b)

and (*iii*) $g'(x) \neq 0$ *for any value of x in (a, b),*

then there is at least one value c of x in (a, b), such that $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(c)}{f'(c)}$

Consider the function $\phi(x) = f(x) - \dfrac{f(b) - f(a)}{g(b) - g(a)} g(x)$

Since $f(x)$ and $g(x)$ are continuous in $[a, b]$

$\therefore$ $\phi(x)$ is also continuous in $[a, b]$.

Again since $f'(x)$ and $g'(x)$ exist in (a, b).

*Named after the great French mathematician *Augustin-Louis Cauchy* (1789–1857) who is considered as the father of modern analysis and creator of complex analysis. He published nearly 800 research papers of basic importance. Cauchy is also well known for his contributions to differential equations, infinite series, optics and elasticity.

$\therefore$ $\phi'(x)$ also exists in (a, b) and $= f'(x) - \dfrac{f(b) - f(a)}{g(b) - g(a)}\, g'(x)$

Clearly, $\phi(a) = \phi(b)$.

Thus, $\phi(x)$ satisfies all the conditions of Rolle's theorem. There is therefore, at least one value c of x between a and b, such that $\phi'(c) = 0$

i.e., $$0 = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)}\, g'(c) \text{ whence follows the result.}$$

(P.T.U., 2007 S ; V.T.U., 2006)

Obs. Cauchy's mean value theorem is a generalisation of Lagrange's mean value theorem, where $g(x) = x$.

Example 4.17. *Verify Cauchy's Mean-value theorem for the functions e^x and e^{-x} in the interval (a, b).*

Solution. $f(x) = e^x$ and $g(x) = e^{-x}$ are both continuous in $[a, b]$ and both functions are differentiable in (a, b).

$\therefore$ $$f'(x) = e^x,\ g'(x) = -e^{-x}$$

By Cauchy's mean value theorem.

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

$\therefore$ $$\frac{e^b - e^a}{e^{-b} - e^{-a}} = \frac{e^c}{-e^{-c}} \quad i.e., c = \frac{1}{2}(a + b)$$

Thus c lies in (a, b) which verifies the Cauchy's Mean value theorem.

(4) Taylor's Theorem* (Generalised mean value theorem)

If (i) $f(x)$ and its first $(n - 1)$ derivatives be continuous in $[a, a + h]$, and (ii) $f^n(x)$ exists for every value of x in $(a, a + h)$, then there is at least one number θ $(0 < \theta < 1)$, such that

$$\mathbf{f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^n}{n!} f^n(a + \theta h)} \qquad ...(1)$$

which is called Taylor's theorem with Lagrange's form remainder, the remainder R_n being $\dfrac{h^n}{n!} f^n(a + \theta h)$.

Proof. Consider the function

$$\phi(x) = f(x) + (a + h - x) f'(x) + \frac{(a + h - x)^2}{2!} f''(x) + ... + \frac{(a + h - x)^n}{n!} K$$

where K is defined by

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^n}{n!} K \qquad ...(2)$$

(*i*) Since $f(x), f'(x), ..., f^{n-1}(x)$ are continuous in $[a, a + h]$, therefore $\phi(x)$ is also continuous in $[a, a + h]$,

(*ii*) $\phi'(x)$ exists and $= \dfrac{(a + h - x)^{n-1}}{(n - 1)!}\, [f^n(x) - K]$

(*iii*) Also $\phi(a) = \phi(a + h)$. [By (2)]

Hence $\phi(x)$ satisfies all the conditions of Rolle's theorem, and therefore, there exists at least one number $\theta (0 < \theta < 1)$, such that $\phi'(a + \theta h) = 0$ *i.e.*, $K = f^n(a + \theta h)$ $(0 < \theta < 1)$

Substituting this value of K in (2), we get (1).

Cor. 1. Taking $n = 1$ in (1), *Taylor's theorem reduces to Lagrange's Mean-value theorem.*

Cor. 2. Putting $a = 0$ and $h = x$ in (1), we get

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + ... + \frac{x^n}{n!} f^n(\theta x). \qquad ...(3)$$

which is known as *Maclaurin's theorem with Lagrange's form of remainder.*

*Named after an English mathematician, *Brooke Taylor* (1685–1731).

Example 4.18. *Find the Maclaurin's theorem with Lagrange's form of remainder for $f(x) = \cos x$.* (*J.N.T.U., 2003*)

Solution. $f^n(x) = \dfrac{d^n}{dx^n}(\cos x) = \cos\left(\dfrac{n\pi}{2} + x\right)$ so that $f^n_{(0)} = \cos(n\pi/2)$

Thus $f(0) = 1$,

$$f^{2n}(0) = \cos(2n\pi/2) = (-1)^n$$
$$f^{2n+1}(0) = \cos[(2n+1)\pi/2] = 0$$

Substituting these values in the Maclaurin's theorem with Lagrange's form of remainder *i.e.*,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots + \frac{x^{2n}}{(2n)!}f^{2n}(0) + \frac{x^{2n+1}}{(2n+1)!}f^{2n+1}(\theta x)$$

We get $$\cos x = 1 + 0 + \frac{x^2}{2!}(-1) + 0 + \dots + \frac{x^{2n}}{(2n)!}(-1)^n + \frac{x^{2n+1}}{(2n+1)!}(-1)^n(-1)\cos(\theta x)$$

i.e., $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + (-1)^n\frac{x^{2n}}{(2n)!} + \frac{(-1)^{n+1}x^{2n+1}}{(2n+1)!}\cos(\theta x)$$

Example 4.19. *If $f(x) = \log(1 + x)$, $x > 0$, using Maclaurin's theorem, show that for $0 < \theta < 1$,*

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3}.$$

Deduce that $\log(1 + x) < x - \dfrac{x^2}{2} + \dfrac{x^3}{3}$ for $x > 0$. (*J.N.T.U., 2005*)

Solution. By Maclaurin's theorem with remainder R_3, we have

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(\theta x) \quad \dots(i)$$

Here $f(x) = \log(1 + x)$, $\quad f(0) = 0$

$\therefore$ $f'(x) = \dfrac{1}{1+x}$, $\quad f'(0) = 1$

$f''(x) = \dfrac{-1}{(1+x)^2}$, $\quad f''(0) = -1$

and $f'''(x) = \dfrac{2}{(1+x)^3}$, $\quad f'''(\theta x) = \dfrac{2}{(1+\theta x)^3}$

Substituting in (*i*), we get $\log(1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3(1+\theta x)^3}$ ...(*ii*)

Since $x > 0$ and $\theta > 0$, $\theta x > 0$

or $(1 + \theta x)^3 > 1$ *i.e.*, $\dfrac{1}{(1+\theta x)^3} < 1$

$$\therefore \quad x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3} < x - \frac{x^2}{2} + \frac{x^3}{3}$$

Hence $\log(1 + x) < x - \dfrac{x^2}{2} + \dfrac{x^3}{3}$ [By (*ii*)]

PROBLEMS 4.4

1. Verify Rolle's theorem for (*i*) $f(x) = (x + 2)^3 (x - 3)^4$ in $(-2, 3)$.
(*ii*) $y = e^x(\sin x - \cos x)$ in $(\pi/4, 5\pi/4)$. (*iii*) $f(x) = x(x + 3)e^{-1/2x}$ in $(-3, 0)$.
(*iv*) $f(x) = \log\left\{\dfrac{x^2 + ab}{x(a+b)}\right\}$ in (a, b). (*V.T.U., 2005*)

2. Using Rolle's theorem for $f(x) = x^{2m-1}(a-x)^{2n}$, find the value of x between o and a where $f'(x) = 0$.
3. Verify Lagrange's Mean value theorem for the following functions and find the appropriate value of c in each case :
   (*i*) $f(x) = (x-1)(x-2)(x-3)$ in $(0, 4)$ (*V.T.U., 2009*)
   (*ii*) $f(x) = \sin x$ in $[0, \pi]$ (*Nagpur, 2008*)
   (*iii*) $f(x) = \log_e x$ in $[1, e]$. (*Burdwan, 2003*)
   (*iv*) $f(x) = e^x$ in $[0, 1]$. (*V.T.U., 2007*)
4. By applying Mean value theorem to
   $f(x) = \log 2 \,.\, \sin \frac{\pi x}{2} + \log x$, prove that $\frac{\pi}{2} \log 2 \,.\, \cos \frac{\pi x}{2} + \frac{1}{x} = 0$ for some x between 1 and 2.
5. In the Mean value theorem : $f(x+h) = f(x) + h f(x + \theta h)$,
   show that $\theta = 1/2$ for $f(x) = ax^2 + bx + c$ in $(0, 1)$.
6. If $f(h) = f(0) + h f'(0) + \frac{h^2}{2!} f''(\theta h)$, $0 < \theta < 1$, find θ when $h = 1$ and $f(x) = (1-x)^{5/2}$.
7. If x is positive, show that $x > \log(1+x) > x - \frac{1}{2}x^2$. (*V.T.U., 2000*)
8. If $f(x) = \sin^{-1} x$, $0 < a < b < 1$, use Mean value theorem to prove that
   $$\frac{b-a}{\sqrt{(1-a^2)}} < \sin^{-1} b - \sin^{-1} a < \frac{b-a}{\sqrt{(1-b^2)}}$$
9. Prove that $\frac{b-a}{b} < \log\left(\frac{b}{a}\right) < \frac{b-a}{b}$ for $0 < a < b$.
   Hence show that $\frac{1}{4} < \log \frac{4}{3} < \frac{1}{3}$. (*Mumbai, 2008*)
10. Verify the result of Cauchy's mean value theorem for the functions
    (*i*) $\sin x$ and $\cos x$ in the interval $[a, b]$. (*J.N.T.U., 2006 S*)
    (*ii*) $\log_e x$ and $1/x$ in the interval $[1, e]$.
11. If $f(x)$ and $g(x)$ are respectively e^x and e^{-x}, prove that 'c' of Cauchy's mean value theorem is the arithmetic mean between a and b. (*Mumbai, 2008*)
12. Verify Maclaurin's theorem $f(x) = (1-x)^{5/2}$ with Lagrange's form of remainder upto 3 terms where $x = 1$.
13. Using Taylor's theorem, prove that
    $$x - \frac{x^3}{6} < \sin x < x - \frac{x^3}{6} + \frac{x^5}{120}, \quad \text{for } x > 0.$$

4.4 EXPANSIONS OF FUNCTIONS

(1) Maclaurin's series. *If $f(x)$ can be expanded as an infinite series, then*

$$\mathbf{f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots \infty} \qquad \ldots(1)$$

If $f(x)$ possess derivatives of all orders and the remainder R_n in (3) on page 145 tends to zero as $n \to \infty$, then the Maclaurin's theorem becomes the Maclaurin's series (1).

Example 4.20. *Using Maclaurin's series, expand tan x upto the term containing x^5.* (*V.T.U., 2006*)

Solution. Let $f(x) = \tan x$ $\qquad f(0) = 0$

$\therefore$ $f'(x) = \sec^2 x = 1 + \tan^2 x$ $\qquad f'(0) = 1$

$$f''(x) = 2\tan x \sec^2 x = 2\tan x(1 + \tan^2 x)$$
$$= 2\tan x + 2\tan^3 x \qquad f''(0) = 0$$
$$f'''(0) = 2\sec^2 x + 6\tan^2 x \sec^2 x$$
$$= 2(1 + \tan^2 x) + 6\tan^2 x(1 + \tan^2 x)$$
$$= 2 + 8\tan^2 x + 6\tan^4 x \qquad f'''(0) = 2$$
$$f^{iv}(0) = 16\tan x \sec^2 x + 24\tan^3 x \sec^2 x$$

$$= 16 \tan x\,(1 + \tan^2 x) + 24 \tan^3 x\,(1 + \tan^2 x)$$

$$= 16 \tan x + 40 \tan^3 x + 24 \tan^5 x \qquad f^{iv}(0) = 0$$

$$f^{v}(0) = 16 \sec^2 x + 120 \tan^2 x \sec^2 x + 120 \tan^4 x \sec^2 x. \qquad f^{v}(0) = 16$$

and so on.

Substituting the values of $f(0)$, $f'(0)$, etc. in the Maclaurin's series, we get

$$\tan x = 0 + x\,.\,1 + \frac{x^2}{2!} + \frac{x^3}{3!}\,.\,2 + \frac{x^4}{4!}\,.\,0 + \frac{x^5}{5!}\,.\,16 + \ldots = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$$

(2) Expansion by use of known series. When the expansion of a function is required only upto first few terms, it is often convenient to employ the following well-known series :

1. $\sin\theta = \theta - \dfrac{\theta^3}{3!} + \dfrac{\theta^5}{5!} - \dfrac{\theta^7}{7!} + \ldots$

2. $\sinh\theta = \theta + \dfrac{\theta^3}{3!} + \dfrac{\theta^5}{5!} + \dfrac{\theta^7}{7!} + \ldots$

3. $\cos\theta = 1 - \dfrac{\theta^2}{2!} + \dfrac{\theta^4}{4!} - \dfrac{\theta^6}{6!} + \ldots$

4. $\cosh\theta = 1 + \dfrac{\theta^2}{2!} + \dfrac{\theta^4}{4!} + \dfrac{\theta^6}{6!} + \ldots$

5. $\tan\theta = \theta + \dfrac{\theta^3}{3} + \dfrac{2}{15}\theta^5 + \ldots$

6. $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \ldots$

7. $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \ldots$

8. $\log(1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots$

9. $\log(1 - x) = -\left(x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{4} + \ldots\right)$

10. $(1 + x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + \ldots$

Example 4.21. *Expand $e^{\sin x}$ by Maclaurin's series or otherwise upto the term containing x^4.*

(Bhopal, 2009; V.T.U., 2011)

Solution. We have $e^{\sin x} = 1 + \sin x + \dfrac{(\sin x)^2}{2!} + \dfrac{(\sin x)^3}{3!} + \dfrac{(\sin x)^4}{4!} + \ldots$

$$= 1 + \left(x - \frac{x^3}{3!} + \ldots\right) + \frac{1}{2!}\left(x - \frac{x^3}{3!} + \ldots\right)^2 + \frac{1}{3!}\left(x - \frac{x^3}{3!} + \ldots\right)^3 + \frac{1}{4!}(x - \ldots)^4 + \ldots$$

$$= 1 + \left(x - \frac{x^3}{6} + \ldots\right) + \frac{1}{2}\left(x^2 - \frac{x^4}{3} + \ldots\right) + \frac{1}{6}\left(x^3 - \ldots\right) + \frac{1}{24}(x^4 + \ldots) + \ldots$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \ldots$$

Otherwise, let $f(x) = e^{\sin x}$ $\qquad f(0) = 1$

$\therefore$ $f'(x) = e^{\sin x}\cos x = f(x)\,.\,\cos x$ $\qquad f'(0) = 1$

$f''(x) = f'(x)\cos x - f(x)\sin x,$ $\qquad f''(0) = 1$

$f'''(x) = f''(x)\cos x - 2f'(x)\sin x - f(x)\cos x,$ $\qquad f'''(0) = 0$

$f^{iv}(x) = f'''(x)\cos x - 3f''(x)\sin x - 3f'(x)\cos x + f(x)\sin x,$ $\qquad f^{iv}(0) = -3$

and so on.

Substituting the values of $f(0)$, $f'(0)$ etc., in the Maclaurin's series, we obtain

$$e^{\sin x} = 1 + x\,.\,1 + \frac{x^2}{2!}\,.\,1 + \frac{x^3}{3!}\,.\,0 + \frac{x^4}{4!}\,.\,(-3) + \ldots$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \ldots$$

Example 4.22. *Expand log $(1 + \sin^2 x)$ in powers of x as far as the term in x^6.* (*Hissar, 2005 S*)

Solution. We have $\sin^2 x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)^2 = \left[x - \left(\frac{x^3}{6} - \frac{x^5}{120} + \dots\right)\right]^2$

$$= x^2 - 2x\left(\frac{x^3}{6} - \frac{x^5}{120} + \dots\right) + \left(\frac{x^3}{6} - \frac{x^5}{120} + \dots\right)^2$$

$$= x^2 - \frac{x^4}{3} + \frac{x^6}{60} + \frac{x^6}{36} + \dots = x^2 - \frac{x^4}{3} + \frac{2x^6}{45} + \dots = t, \text{ say.}$$

Now $\log(1 + \sin^2 x) = \log(1 + t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \dots$

Substituting the value of t, we get

$$\log(1 + \sin^2 x) = x^2 - \frac{x^4}{3} + \frac{2x^6}{45} + \dots - \frac{1}{2}\left(x^2 - \frac{x^4}{3} + \dots\right)^2 - \frac{1}{3}(x^2 - \dots)^3 - \dots$$

$$= x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \frac{1}{2}\left(x^4 - \frac{2x^6}{3} + \dots\right) + \frac{1}{3}(x^6 + \dots) + \dots$$

$$= x^2 - \frac{5}{6}x^4 + \frac{32}{45}x^6 + \dots$$

Obs. As it is very cumbersome to find the successive derivatives of $\log(1 + \sin^2 x)$, therefore the above method is preferable to Maclaurin's series method.

Example 4.23. *Expand $e^{a \sin^{-1} x}$ in ascending powers of x.*

Solution. Let $y = e^{a \sin^{-1} x}$. In Ex. 4.9, we have shown that

$$(y)_0 = 1, (y_1)_0 = a, (y_2)_0 = a^2, (y_3)_0 = a(1 + a^2), (y_4)_0 = a^2(2^2 + a^2)$$

and so on.

Substituting these values in the Maclaurin's series

$$y = (y)_0 + \frac{(y_1)_0}{1!}x + \frac{(y_2)_0}{2!}x^2 + \frac{(y_3)_0}{3!}x^3 + \frac{(y_4)_0}{4!}x^4 + \dots$$

we get $$e^{a \sin^{-1} x} = 1 + ax + \frac{a^2}{2!}x^2 + \frac{a(1^2 + a^2)}{3!}x^3 + \frac{a^2(2^2 + a^2)}{4!}x^4 + \dots$$

(3) Taylor's series. *If $f(x + h)$ can be expanded as an infinite series, then*

$$f(x + h) = f(x) + h f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots \infty \quad \dots(1)$$

If $f(x)$ possesses derivatives of all orders and the remainder R_n in (1) on page 147, tends to zero as $n \to \infty$, then the Taylor's theorem becomes the *Taylor's series* (1).

Cor. Replacing x by a and h by $(x - a)$ in (1), we get

$$f(x) = f(a) + (x - a)f'(a) + \frac{(x - a)^2}{2!}f''(a) + \frac{(x - a)^3}{3!}f'''(a) + \dots \infty$$

Taking $a = 0$, we get *Maclaurin's series.*

Example 4.24. *Expand $\log_e x$ in powers of $(x - 1)$ and hence evaluate $\log_e 1{\cdot}1$ correct to 4 decimal places.* (*Bhopal, 2007 ; Kurukshetra 2006*)

Solution. Let $f(x) = \log_e x \qquad f(1) = 0$

$\therefore \qquad f'(x) = \frac{1}{x}, \qquad f'(1) = 1$

$f''(x) = -\frac{1}{x^2}, \qquad f''(1) = -1$

$$f'''(x) = \frac{2}{x^3}, \qquad f'''(1) = 2$$

$$f^{iv}(x) = -\frac{6}{x^4}, \qquad f^{iv}(0) = -6$$

etc. etc.

Substituting these values in the Taylor's series

$$f(x) = f(1) + (x-1) f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) + \ldots,$$

we get $$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots$$

Now putting $x = 1.1$, so that $x - 1 = 0.1$, we have

$$\log(1.1) = 1.1 - \tfrac{1}{2}(0.1)^2 + \tfrac{1}{3}(0.1)^3 - \tfrac{1}{4}(0.1)^4 + \ldots$$

$$= 0.1 - 0.005 + 0.0003 - 0.00002 + \ldots = 0.0953.$$

Example 4.25. *Use Taylor's series, to prove that*

$$\tan^{-1}(x+h) = \tan^{-1} x + (h \sin z) \cdot \frac{\sin z}{1} - (h \sin z)^2 \cdot \frac{\sin 2z}{2} + (h \sin z)^3 \cdot \frac{\sin 3z}{3} - \ldots$$

where $z = \cot^{-1} x$. *(Bhillai, 2005)*

Solution. We have $\cot z = x$ ...(*i*)

$\therefore$ $-\operatorname{cosec}^2 z \cdot dz/dx = 1$ or $dz/dx = -\sin^2 z$ ...(*ii*)

Now let $f(x+h) = \tan^{-1}(x+h)$, so that $f(x) = \tan^{-1} x$

$$\therefore \quad f'(x) = \frac{1}{1+x^2} = \frac{1}{1+\cot^2 z} = \sin^2 z \qquad \text{[By } (i)]$$

$$f''(x) = 2 \sin z \cos z \frac{dz}{dx} = \sin 2z \cdot (-\sin^2 z) \qquad \text{[By } (ii)]$$

$$f'''(x) = -[2 \cos 2z \cdot \sin^2 z + \sin 2z \cdot 2 \sin z \cos z] \frac{dz}{dx}$$

$$= -2 \sin z [\sin z \cos 2z + \sin 2z \cos z](-\sin^2 z) = 2 \sin^3 z \sin 3z$$

and so on.

Substituting these values in the Taylor's series

$$f(x+h) = f(x) + hf(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots,$$

we get the required result.

PROBLEMS 4.5

Using Maclaurin's series, expand the following functions :

1. $\log(1+x)$. Hence deduce that $\log \sqrt{\frac{1+x}{1-x}} = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots$
2. $\sin x$ *(P.T.U., 2005)*
3. $\sqrt{(1 + \sin 2x)}$ *(V.T.U., 2010)*
4. $\sin^{-1} x$ *(Mumbai, 2007)*
5. $\tan^{-1} x$
6. $\log \sec x$ *(Mumbai, 2009 S ; V.T.U., 2009)*

Prove that :

7. $\sec x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \ldots$
8. $x \operatorname{cosec} x = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \ldots$ *(Mumbai, 2007)*
9. $\sin^{-1} \frac{2x}{1+x^2} = 2\left\{x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\right\}$
10. $\tan^{-1} \frac{\sqrt{(1+x^2)} - 1}{x} = \frac{1}{2}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\right)$

11. $\sin^{-1}(3x-4x^3)=3\left(x+\frac{x^3}{3}+\frac{3x^5}{40}+...\right)$

12. $e^x\cos x=1+x-\frac{2x^3}{3!}-\frac{2^2x^4}{4!}$ ... *(Raipur, 2005)*

13. $e^{x\sin x}=1+x^2+\frac{x^4}{3}+\frac{x^6}{120}+...$ *(Kurukshetra, 2009)*

14. $e^{\cos^{-1}x}=e^{\pi/2}\left(1-x+\frac{x^2}{3}-\frac{x^3}{3}+...\right)$ *(Mumbai, 2008)*

15. $\log\frac{\sin x}{x}=-\left(\frac{x^2}{6}+\frac{x^4}{180}+\frac{x^6}{2835}+...\right)$

16. $\log(1+\sin x)=x-\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^4}{12}+...$ *(S.V.T.U. 2009 ; J.N.T.U., 2006 S)*

17. $\sqrt{(1+\sin x)}=1+\frac{x}{2}-\frac{x^2}{2}-\frac{x^3}{48}+\frac{x^4}{384}+...$ *(V.T.U., 2006)*

18. $\log(1+e^x)=\log 2+\frac{x}{2}+\frac{x^2}{8}-\frac{x^4}{192}+...$ *(Bhopal, 2008)*

19. $\frac{e^x}{e^x+1}=\frac{1}{2}+\frac{x}{4}-\frac{x^3}{48}+...$ *(Bhopal, 2008 S)*

20. $\frac{x}{2}\left(\frac{e^x+1}{e^x-1}\right)=1+\frac{1}{6}\cdot\frac{x^2}{2!}-\frac{1}{30}\cdot\frac{x^4}{4!}+...$ *(Mumbai, 2007)*

21. $\sin x\cosh x=x+\frac{x^3}{3}-\frac{x^5}{30}+...$

By forming a differential equation, show that

22. $(\sin^{-1}x)^2=2\frac{x^2}{2!}+2\cdot 2^2\frac{x^4}{4!}+2\cdot 2^2\cdot 4^2\cdot\frac{x^6}{6!}+...$

23. $\log[1+\sqrt{(1+x^2)}]=x-\frac{1}{2}\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5}-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7}+...$

24. If $y=\sin(m\sin^{-1}x)$, show that $(1-x^2)y_2-xy_1+m^2y=0$
Hence expand $\sin m\theta$ in powers of $\sin\theta$. *(S.V.T.U., 2008)*

25. Using Taylor's theorem, express the polynomial $2x^3+7x^2+x-6$ in powers of $(x-1)$ *(Burdwan, 2003)*

26. Expand (*i*) e^x *(Cochin., 2005)* (*ii*) $\tan^{-1}x$, in powers of $(x-1)$ upto four terms.

27. Expand $\sin x$ in powers of $(x-\pi/2)$. Hence find the value of sin 91° correct to 4 decimal places. *(Rohtak, 2003)*

28. Prove that $\log\sin x=\log\sin a+(x-a)\cot a-\frac{1}{2}(x-a)^2\operatorname{cosec}^2 a+...$

29. Find the Taylor's series expansion for $\log\cos x$ about the point $\pi/3$.

30. Compute to four decimal places, the value of cos 32°, by the use of Taylor's series. *(Kurukshetra, 2006)*

31. Calculate approximately (*i*) $\log_{10}404$, given $\log 4=0.6021$. *(Rohtak, 2005 S)*
(*ii*) $(1.04)^{3.01}$ *(Mumbai, 2007)*

4.5 INDETERMINATE FORMS

In general $\underset{x\to a}{\text{Lt}}[f(x)/\phi(x)]=\underset{x\to a}{\text{Lt}}f(x)/\underset{x\to a}{\text{Lt}}\phi(x)$. But when $\underset{x\to a}{\text{Lt}}f(x)$ and $\underset{x\to a}{\text{Lt}}\phi(x)$ are both zero, then the quotient reduces to the indeterminate form 0/0. This does not imply that $\underset{x\to a}{\text{Lt}}[f(x)/\phi(x)]$ is meaningless or it does not exist. In fact, in many cases, it has a finite value. We shall now, study the methods of evaluating the limits in such and similar other cases :

(1) Form 0/0. *If $f(a)=\phi(a)=0$, then*

$$\underset{x\to a}{\text{Lt}}\frac{f(x)}{\phi(x)}=\underset{x\to a}{\text{Lt}}\frac{f'(x)}{\phi'(x)}$$

By Taylor's series,

$$\underset{x\to a}{\text{Lt}}\frac{f(x)}{\phi(x)}=\underset{x\to a}{\text{Lt}}\frac{f(a)+(x-a)f'(a)+\frac{1}{2!}(x-a)^2f''(a)+...}{\phi(a)+(x-a)\phi'(a)+\frac{1}{2!}(x-a)^2\phi''(a)+...}$$

$$= \underset{x \to a}{\text{Lt}} \frac{f'(a) + \frac{1}{2}(x-a) f'(a) + \dots}{\phi'(a) + \frac{1}{2}(x-a)\phi'(a) + \dots}$$

$$= \frac{f'(a)}{\phi'(a)} = \underset{x \to a}{\text{Lt}} \frac{f'(x)}{\phi'(x)} \qquad \dots(1)$$

This is known as *L'Hospital's rule.*

In general, if

$$f(a) = f'(a) = f''(a) = \dots = f^{n-1}(a) = 0, \text{ but } f^n(a) \neq 0,$$

and
$$\phi(a) = \phi'(a) = \phi''(a) = \dots = \phi^{n-1}(a) = 0, \text{ but } \phi^n(a) \neq 0,$$

then from (1),

$$\underset{x \to a}{\textbf{Lt}} \frac{\mathbf{f(x)}}{\boldsymbol{\phi}\mathbf{(x)}} = \frac{\mathbf{f^n(a)}}{\boldsymbol{\phi}^\mathbf{n}\mathbf{(a)}} = \underset{x \to a}{\textbf{Lt}} \frac{\mathbf{f^n(x)}}{\boldsymbol{\phi}^\mathbf{n}\mathbf{(x)}}$$

[**Rule to evaluate Lt $[f(x)/\phi(x)]$ in 0/0 form :**

Differentiating the numerator and denominator separately as many times as would be necessary to arrive a determinate form].

Example 4.26. *Evaluate (i)* $\underset{x \to 0}{Lt} \dfrac{xe^x - \log(1+x)}{x^2}$. *(V.T.U., 2004 ; Osmania, 2000 S)*

(ii) $\underset{x \to 1}{Lt} \dfrac{x^x - x}{x - 1 - \log x}$

Solution. *(i)*

$$\underset{x \to 0}{\text{Lt}} \frac{xe^x - \log(1+x)}{x^2} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{(xe^x + e^x \cdot 1) - 1/(1+x)}{2x} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{xe^x + e^x + e^x + 1/(1+x)^2}{2} = \frac{0+1+1+1}{2} = 1\frac{1}{2}.$$

(ii)

$$\underset{x \to 1}{\text{Lt}} \frac{x^x - x}{x - 1 - \log x} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \underset{x \to 1}{\text{Lt}} \frac{d(x^x)/dx - 1}{1 - 0 - 1/x}$$

$$= \underset{x \to 1}{\text{Lt}} \frac{x^x(1 + \log x) - 1}{1 - 1/x}$$

$$\left(\text{form } \frac{0}{0}\right)$$

Let $y = x^x$ so that

$\log y = x \log x$

$\therefore \quad \dfrac{1}{y}\dfrac{dy}{dx} = x \cdot \dfrac{1}{x} + 1 \,.\, \log x$

or $\quad \dfrac{d}{dx}(x^x) = x^x(1 + \log x) \qquad \dots(i)$

$$= \underset{x \to 1}{\text{Lt}} \frac{d(x^x)/dx \,.\, (1 + \log x) + x^x(1/x) - 0}{1/x^2}$$

$$= \underset{x \to 1}{\text{Lt}} \frac{x^x(1 + \log x)^2 + x^x(1/x)}{x^{-2}} \qquad [\text{By } (i)]$$

$$= \frac{1(1+0)^2 + 1 \cdot 1}{1} = 2.$$

Example 4.27. *Find the values of a and b such that* $\underset{x \to 0}{Lt} \dfrac{x(a + b\cos x) - c\sin x}{x^5} = 1.$ *(Mumbai., 2007)*

Solution. $$\underset{x\to 0}{\text{Lt}} \frac{x(a+b\cos x)-c\sin x}{x^5} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \underset{x\to 0}{\text{Lt}} \frac{a+b\cos x - bx\sin x - c\cos x}{5x^4} \qquad ...(i)$$

As the denominator is 0 for $x = 0$, (i) will tend to a finite limit if and only if the numerator also becomes 0 for $x = 0$. This requires $a + b - c = 0$ ...(ii)

With this condition, (i) assumes the form 0/0.

$$\therefore \qquad (i) = \underset{x\to 0}{\text{Lt}} \frac{-b\sin x - b(\sin x + x\cos x) + c\sin x}{20x^3}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{(c-2b)\sin x - bx\cos x}{20x^3} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \underset{x\to 0}{\text{Lt}} \frac{(c-2b)\cos x - b(\cos x - x\sin x)}{60x^2} \qquad ...(iii)$$

$$= \frac{c-2b-b}{0} = \frac{c-3b}{0} = 1 \qquad \text{(Given)}$$

$\therefore \qquad c - 3b = 0 \quad i.e., \quad c = 3b.$

Now (iii) $$= \underset{x\to 0}{\text{Lt}} \frac{b\cos x - b\cos x + bx\sin x}{60x^2}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{b\sin x}{60x} = \frac{b}{60} \underset{x\to 0}{\text{Lt}} \left(\frac{\sin x}{x}\right) = \frac{b}{60} = 1.$$

$i.e.$, $b = 60$, and $\therefore \quad c = 180$.

Form (ii), $a = 120$.

(2) Form ∞/∞. It can be shown that L'Hospital's rule can also be applied to this case by differentiating the numerator and denominator separately as many times as would be necessary.

Example 4.28. *Evaluate* $\underset{x\to 0}{Lt} \dfrac{\log x}{\cot x}$.

Solution. $$\underset{x\to 0}{\text{Lt}} \frac{\log x}{\cot x} = \underset{x\to 0}{\text{Lt}} \frac{1/x}{-\text{cosec}^2 x} = -\underset{x\to 0}{\text{Lt}} \frac{\sin^2 x}{x} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= -\underset{x\to 0}{\text{Lt}} \frac{2\sin x\cos x}{1} = 0$$

Obs. Use of known series and standard limits. In many cases, it would be found more convenient to use expansions of known functions and standard limits for evaluating the indeterminate forms. For this purpose, remember the series of § 4.4 (2) and the following limits :

$$\underset{x\to 0}{\text{Lt}} \frac{\sin x}{x} = 1, \qquad \underset{x\to 0}{\text{Lt}} (1+x)^{1/x} = e$$

Example 4.29. *Evaluate* $\underset{x\to 0}{Lt} \dfrac{e^x \sin x - x - x^2}{x^2 + x\log(1-x)}$.

Solution. Using the expansions of e^x, $\sin x$ and $\log(1-x)$, we get

$$\underset{x\to 0}{\text{Lt}} \frac{e^x\sin x - x - x^2}{x^2 + x\log(1-x)}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + ...\right)\left(x - \frac{1}{3!}x^3 + ...\right) - x - x^2}{x^2 + x\left(-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - ...\right)}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\left(x + x^2 + \frac{1}{3}x^3 - 0 \cdot x^4 + \ldots\right) - x - x^2}{x^2 - \left(x^2 + \frac{1}{2}x^3 + \frac{1}{3}x^4 + \ldots\right)} = \underset{x \to 0}{\text{Lt}} \frac{\frac{1}{3}x^3 - 0 \cdot x^4 + \ldots}{-\frac{1}{2}x^3 - \frac{1}{3}x^4 - \ldots} = \underset{x \to 0}{\text{Lt}} \frac{\frac{1}{3} + \ldots}{-\frac{1}{2} - \frac{1}{3}x - \ldots} = -\frac{2}{3}.$$

Example 4.30. *Evaluate* $\underset{x \to 0}{Lt} \dfrac{(1 + x)^{1/x} - e}{x}$.

Solution. Let $\qquad y = (1 + x)^{1/x}$

$\therefore \qquad \log y = \dfrac{1}{x} \log (1 + x) = \dfrac{1}{x}\left(x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \ldots\right) = 1 - \dfrac{x}{2} + \dfrac{x^2}{3} - \ldots$

or $\qquad y = e^{1 - \frac{1}{2}x + \frac{1}{3}x^2 - \ldots} = e \cdot e^{-\frac{1}{2}x + \frac{1}{3}x^2 - \ldots}$

$$= e\left[1 + \left(-\frac{1}{2}x + \frac{1}{3}x^2 - \ldots\right) + \frac{1}{2!}\left(-\frac{1}{2}x + \frac{1}{3}x^2 - \ldots\right)^2 + \ldots\right] = e\left(1 - \frac{x}{2} + \frac{11}{24}x^2 + \ldots\right)$$

$$\therefore \qquad \underset{x \to 0}{\text{Lt}} \frac{(1 + x)^{1/x} - e}{x} = \underset{x \to 0}{\text{Lt}} \frac{e\left(1 - \frac{x}{2} + \frac{11}{24}x^2 + \ldots\right) - e}{x}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{e\left(-\frac{1}{2}x + \frac{11}{24}x^2 + \ldots\right)}{x} = \underset{x \to 0}{\text{Lt}} \left(\frac{-e}{2} + \frac{11}{24}ex + \ldots\right) = -\frac{e}{2}.$$

PROBLEMS 4.6

Evaluate the following limits :

1. $\underset{x \to 0}{\text{Lt}} \dfrac{a^x - b^x}{x}$ *(V.T.U., 2008)*
2. $\underset{x \to 0}{\text{Lt}} \dfrac{x \cos x - \sin x}{x^2 \sin x}$ *(J.N.T.U., 2006 S)*
3. $\underset{\theta \to 0}{\text{Lt}} \dfrac{\theta - \sin \theta}{\sin \theta (1 - \cos \theta)}$
4. $\underset{x \to \pi/2}{\text{Lt}} \dfrac{a^{\sin x} - a}{\log_e \sin x}$
5. $\underset{x \to 0}{\text{Lt}} \dfrac{2 \sin x - \sin 2x}{x^3}$
6. $\underset{x \to 0}{\text{Lt}} \dfrac{\sin x \sin^{-1} x - x^2}{x^6}$
7. $\underset{x \to 0}{\text{Lt}} \dfrac{x \cos x - \log (1 + x)}{x^2}$
8. $\underset{x \to 0}{\text{Lt}} \dfrac{\log \sec x - \frac{1}{2} x^2}{x^4}$
9. $\underset{x \to 0}{\text{Lt}} \dfrac{x e^x - \log (1 + x)}{\cosh x - \cos x}$
10. $\underset{x \to 0}{\text{Lt}} \dfrac{\cos x - \log (1 + x) - 1 + x}{\sin^2 x}$
11. $\underset{x \to 0}{\text{Lt}} \dfrac{e^x + 2 \sin x - e^{-x} - 4x}{x^5}$
12. $\underset{x \to 0}{\text{Lt}} \dfrac{\log (x - a)}{\log (e^x - e^a)}$
13. $\underset{x \to 0}{\text{Lt}} \dfrac{\tan x - x}{x^2 \tan x}$
14. $\underset{x \to 0}{\text{Lt}} \dfrac{e^x - e^{-x} - 2x}{x - \sin x}$
15. $\underset{x \to 0}{\text{Lt}} \dfrac{e^x - e^{\sin x}}{x - \sin x}$
16. $\underset{x \to 0}{\text{Lt}} \dfrac{\sin \{(\log (1 + x)\}}{\log (1 + \sin x)}$
17. $\underset{x \to 0}{\text{Lt}} \dfrac{e^x + \sin x - 1}{\log (1 + x)}$
18. $\underset{x \to 0}{\text{Lt}} \dfrac{(1 + x)^{1/x} - e + \frac{1}{2} ex}{x^2}$
19. If $\underset{x \to 0}{\text{Lt}} \dfrac{\sin 2x + a \sin x}{x^3}$ is finite, find the value of a and the limit. *(Nagpur, 2009)*
20. Find a, b if $\underset{x \to 0}{\text{Lt}} \dfrac{a \sinh x + b \sin x}{x^3} = \dfrac{5}{3}$. *(Mumbai, 2009)*
21. Find a, b, c so that $\underset{x \to 0}{\text{Lt}} \dfrac{ae^x - b \cos x + ce^{-x}}{x \sin x} = 2$. *(Mumbai, 2008)*

(3) Forms reducible to 0/0 form. Each of the following indeterminate forms can be easily reduced to the form 0/0 (or ∞/∞) by suitable transformation and then the limits can be found as usual.

I. Form $0 \times \infty$. If $\underset{x \to 0}{Lt} f(x) = 0$ and $\underset{x \to \infty}{Lt} \phi(x) = \infty$, then

$\underset{x \to a}{\text{Lt}} [f(x) . \phi(x)]$ assumes the form $0 \times \infty$.

To evaluate this limit, we write

$$f(x) \cdot \phi(x) = f(x)/[1/\phi(x)] \text{ to take the form } 0/0.$$
$$= \phi(x)/[1/f(x)] \text{ to take the form } \infty/\infty.$$

Example 4.31. *Evaluate* $\underset{x \to 0}{Lt} (\tan x \log x)$ *(V.T.U., 2009)*

Solution. $$\underset{x \to 0}{\text{Lt}} (\tan x \log x) = \underset{x \to 0}{\text{Lt}} \left(\frac{\log x}{\cot x}\right) \qquad \left(\text{form } \frac{\infty}{\infty}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \left(\frac{1/x}{-\operatorname{cosec}^2 x}\right) = -\underset{x \to 0}{\text{Lt}} \left(\frac{\sin^2 x}{x}\right) \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2 \sin x \cos x}{1} = 0.$$

II. Form $\infty - \infty$. If $\underset{x \to a}{Lt} f(x) = \infty = \underset{x \to a}{Lt} \phi(x)$, then $\underset{x \to a}{\text{Lt}} [f(x) - \phi(x)]$ assumes the form $\infty - \infty$.

It can be reduced to the from 0/0 by writing

$$f(x) - \phi(x) = \left[\frac{1}{\phi(x)} - \frac{1}{f(x)}\right] \Big/ \frac{1}{f(x)\,\phi(x)}$$

Example 4.32. *Evaluate* $\underset{x \to 0}{Lt} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$.

Solution. $$\underset{x \to 0}{\text{Lt}} \left(\frac{1}{\sin x} - \frac{1}{x}\right) = \underset{x \to 0}{\text{Lt}} \frac{x - \sin x}{x \sin x} = \underset{x \to 0}{\text{Lt}} \frac{1 - \cos x}{x \cos x + \sin x} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\sin x}{x(-\sin x) + \cos x + \cos x} = \frac{0}{0 + 1 + 1} = 0.$$

III. Forms 0^0, 1^∞, ∞^0. If $y = \underset{x \to a}{\text{Lt}} [f(x)]^{\phi(x)}$ assumes one of these forms, then $\log y = \underset{x \to a}{\text{Lt}} \phi(x) \log f(x)$ takes the form $0 \times \infty$, which can be evaluated by the method given in I above. If $\log y = l$, then $y = e^l$.

Example 4.33. *Evaluate* (*i*) $\underset{x \to \pi/2}{Lt} (\sin x)^{\tan x}$ (*ii*) $\underset{x \to 0}{Lt} \left(\frac{a^x + b^x + c^x}{3}\right)^{1/x}$ *(V.T.U., 2011)*

(*iii*) $\underset{x \to 0}{Lt} \left(\frac{\tan x}{3}\right)^{1/x^2}$

Solution. (*i*) Let $$y = \underset{x \to \pi/2}{\text{Lt}} (\sin x)^{\tan x}.$$

$$\therefore \quad \log y = \underset{x \to \pi/2}{\text{Lt}} \tan x \log \sin x = \underset{x \to \pi/2}{\text{Lt}} \frac{\log \sin x}{\cot x} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \underset{x \to \pi/2}{\text{Lt}} \frac{(1/\sin x) \cos x}{-\operatorname{cosec}^2 x} = -\underset{x \to \pi/2}{\text{Lt}} (\sin x \cos x) = 0$$

Hence $$y = e^0 = 1.$$

(*ii*) Let $$y = \underset{x \to 0}{\text{Lt}} \left(\frac{a^x + b^x + c^x}{3} \right)^{1/x}$$

so that $$\log y = \underset{x \to 0}{\text{Lt}} \frac{\log (a^x + b^x + c^x) - \log 3}{x} \qquad \text{form } \left(\frac{0}{0}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{(a^x + b^x + c^x)^{-1} (a^x \log a + b^x \log b + c^x \log c)}{1}$$

$$= (1 + 1 + 1)^{-1} (\log a + \log b + \log c) = \frac{1}{3} \log (abc) = \log (abc)^{1/3}.$$

$\therefore$ $$y = (abc)^{1/3}$$

(*iii*) $$\underset{x \to 0}{\text{Lt}} \left(\frac{\tan x}{x} \right)^{1/x^2} = \underset{x \to 0}{\text{Lt}} \left(\frac{x + \frac{x^3}{3} + \frac{2}{15} x^5 + \ldots}{x} \right)^{1/x^2}$$

$$= \underset{x \to 0}{\text{Lt}} \left(1 + \frac{x^2}{3} + \frac{2}{15} x^4 + \ldots \right)^{1/x^2}$$

$$= \underset{x \to 0}{\text{Lt}} (1 + tx^2)^{1/x^2} \qquad \text{where } t = \frac{1}{3} + \frac{2}{15} x^2 + \ldots$$

$$= \underset{x \to 0}{\text{Lt}} [(1 + tx^2)^{1/tx^2}]^t = \underset{x \to 0}{\text{Lt}}\, e^t = e^{1/3}. \qquad \left[\because \underset{z \to 0}{\text{Lt}} (1 + z)^{1/z} = e\right]$$

PROBLEMS 4.7

Evaluate the following limits :

1. $\underset{x \to 0}{\text{Lt}} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x} \right)$

2. $\underset{x \to 0}{\text{Lt}} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right)$ (*Burdwan, 2003*)

3. $\underset{x \to 1}{\text{Lt}} (2x \tan x - \pi \sec x)$ (*V.T.U., 2008*)

4. $\underset{x \to 0}{\text{Lt}} \left(\frac{\cot x - 1/x}{x} \right)$

5. $\underset{x \to 0}{\text{Lt}} \left(\frac{1}{x^2} - \cot^2 x \right)$

6. $\underset{x \to 1}{\text{Lt}} (x)^{1/(1-x)}$

7. $\underset{x \to 0}{\text{Lt}} (a^x + x)^{1/x}$ (*V.T.U., 2007*)

8. $\underset{x \to \pi/2}{\text{Lt}} (\sec x)^{\cot x}$

9. $\underset{x \to 0}{\text{Lt}} (1 + \sin x)^{\cot x}$

10. $\underset{x \to 0}{\text{Lt}} (\cos x)^{1/x^2}$

11. $\underset{x \to \pi/2}{\text{Lt}} (\tan x)^{\tan 2x}$ (*V.T.U., 2004*)

12. $\underset{x \to 0}{\text{Lt}} (\cot x)^{1/\log x}$

13. $\underset{x \to \pi/2}{\text{Lt}} (\cos x)^{\frac{\pi}{2} - x}$

14. $\underset{x \to 0}{\text{Lt}} \left(\frac{\sin x}{x} \right)^{1/x}$

15. $\underset{x \to 0}{\text{Lt}} \left(\frac{\sin x}{x} \right)^{1/x^2}$ (*V.T.U., 2001*)

16. $\underset{x \to 1}{\text{Lt}} (1 - x^2)^{1/\log (1-x)}$

17. $\underset{x \to a}{\text{Lt}} \left(2 - \frac{x}{a} \right)^{\tan (\pi x/2a)}$ (*V.T.U., 2010 ; Nagpur, 2009*)

18. $\underset{x \to 0}{\text{Lt}} \left\{ \frac{2 (\cosh x - 1)^{1/x^2}}{x^2} \right\}$

19. $\underset{x \to 2}{\text{Lt}} \left\{ \frac{1}{x - 2} - \frac{1}{\log (x - 1)} \right\}$ (*Osmania, 2000 S*)

20. $\underset{x \to 0}{\text{Lt}} \left(\frac{1^x + 2^x + 3^x}{3} \right)^{1/x}$. (*V.T.U., 2008*)

4.6 TANGENTS AND NORMALS – CARTESIAN CURVES

(1) Equation of the tangent *at the point* (x, y) *of the curve* $y = f(x)$ *is*

$$\mathbf{Y - y = \frac{dy}{dx}(X - x).}$$

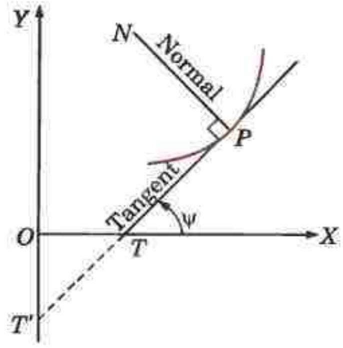


Fig. 4.3

The equation of any line through $P(x, y)$ is

$$Y - y = m(X - x)$$

where X, Y are the current coordinates of any point on the line (Fig. 4.3).

If this line is the tangent PT, then

$$m = \tan\psi = dy/dx$$

Hence the equation of the tangent at (x, y) is

$$Y - y = \frac{dy}{dx}(X - x) \qquad ...(2)$$

Cor. Intercepts. Putting $Y = 0$ in (2)

$$-y = \frac{dy}{dx}(X - x) \quad \text{or} \quad X = x - y/\frac{dy}{dx}$$

$\therefore$ *Intercept which the tangent cuts off from x-axis* $(= OT) = \mathbf{x - y\, dx/dy}$

Similarly putting $X = 0$ *in (2), we see that the intercept which the tangent cuts off from the y-axis*

$$(= OT') = \mathbf{y - x\frac{dy}{dx}}$$

(2) Equation of the normal *at the point* (x, y) *of the curve* $y = f(x)$ is

$$\mathbf{Y - y = -\frac{dx}{dy}(X - x)}$$

A normal *to the curve* $y = f(x)$ *at* $P(x, y)$ *is a line through P perpendicular to the tangent there at.*

$\therefore$ Its equation is $Y - y = m'(X - x)$

where $\qquad m' \cdot dy/dx = -1 \quad \text{or} \quad m' = -1/\frac{dy}{dx} = -dx/dy$

Hence the equation of the normal at (x, y) is $Y - y = -\dfrac{dx}{dy}(X - x)$.

Example 4.34. *Find the equation of the tangent at any point* (x, y) *to the curve* $x^{2/3} + y^{2/3} = a^{2/3}$. *Show that the portion of the tangent intercepted between the axes is of constant length.*

Solution. Equation of the curve is $x^{2/3} + y^{2/3} = a^{2/3}$. $\qquad ...(i)$

Differentiating (i) w.r.t. x,

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$\therefore$ Slope of the tangent at $(x, y) = \dfrac{dy}{dx} = -\left(\dfrac{y}{x}\right)^{1/3}$

$\therefore$ Equation of the tangent at (x, y) is

$$Y - y = -(y/x)^{1/3}(X - x) \qquad ...(ii)$$

Put $Y = 0$ in (ii). Then $\quad X = x + x^{1/3} \,.\, y^{2/3}$

i.e., Intercept on x-axis $\quad = (x^{2/3} + y^{2/3})x^{1/3} = a^{2/3} \,.\, x^{1/3}$ $\qquad$ [By (i)]

Put $X = 0$ in (ii). Then $\quad Y = y + y^{1/3} \,.\, x^{2/3}$

i.e., Intercept on y-axis $\quad = (x^{2/3} + y^{2/3})\,y^{1/3} = a^{2/3} \,.\, y^{1/3}$ $\qquad$ [By (i)]

Thus the portion of the tangent intercepted between the axes

$$= \sqrt{[(\text{Intercept on } x\text{-axis})^2 + (\text{Intercept on } y\text{-axis})^2]}$$

$$= \sqrt{[(a^{2/3} \cdot x^{1/3})^2 + (a^{2/3} \cdot y^{1/3})^2]}$$

$$= \sqrt{[a^{4/3}(x^{2/3} + y^{2/3})]} = a^{2/3}\sqrt{(a)^{2/3}} \quad \text{[By } (i)\text{]}$$

$= a$, which is a constant length.

Example 4.35. *Show that the conditions for the line $x \cos \alpha + y \sin \alpha = p$ to touch the curve $(x/a)^m + (y/b)^m = 1$ is $(a \cos \alpha)^{m/(m-1)} + (b \sin \alpha)^{m/(m-1)} = p^{m/(m-1)}$.*

Solution. Equation of the curve is $\dfrac{x^m}{a^m} + \dfrac{y^m}{b^m} = 1$ ...(i)

Differentiating (i) w.r.t. x, $\dfrac{mx^{m-1}}{a^m} + \dfrac{my^{m-1}}{b^m}\dfrac{dy}{dx} = 0$

$\therefore$ Slope of the tangent at $(x, y) = \dfrac{dy}{dx} = -\left(\dfrac{b}{a}\right)^m \left(\dfrac{x}{y}\right)^{m-1}$

$\therefore$ Equation of the tangent at (x, y) is

$$Y - y = -\left(\frac{b}{a}\right)^m \left(\frac{x}{y}\right)^{m-1} (X - x)$$

or
$$\frac{x^{m-1}X}{a^m} + \frac{y^{m-1}Y}{b^m} = \frac{x^m}{a^m} + \frac{y^m}{b^m} = 1 \quad \text{...(ii) [By } (i)\text{]}$$

If the given line touches (i) at (x, y) then (ii) must be same as $X \cos \alpha + Y \sin \alpha = p$ ...(iii)

Comparing coefficients in (ii) and (iii),

$$\frac{x^{m-1}}{a^m} \bigg/ \cos \alpha = \frac{y^{m-1}}{b^m} \bigg/ \sin \alpha = \frac{1}{p}$$

or
$$\left(\frac{x}{a}\right)^{m-1} = \frac{a \cos \alpha}{p}, \left(\frac{y}{b}\right)^{m-1} = \frac{b \sin \alpha}{p}$$

or
$$\left(\frac{a \cos \alpha}{p}\right)^{\frac{m}{m-1}} + \left(\frac{b \sin \alpha}{p}\right)^{\frac{m}{m-1}} = \left(\frac{x}{a}\right)^m + \left(\frac{y}{b}\right)^m = 1 \quad \text{[By } (i)\text{]}$$

whence follows the required condition.

Example 4.36. *Find the equation of the normal at any point θ to the curve $x = a (\cos \theta + \theta \sin \theta)$, $y = a (\sin \theta - \theta \cos \theta)$. Verify that these normals touch a circle with its centre at the origin and whose radius is constant.*

Solution. We have
$$\frac{dx}{d\theta} = a(-\sin \theta + \sin \theta + \theta \cos \theta) = a\theta \cos \theta$$
$$\frac{dy}{d\theta} = a(\cos \theta - \cos \theta + \theta \sin \theta) = a\theta \sin \theta$$

$\therefore$
$$\frac{dy}{dx} = \frac{dy}{d\theta} \bigg/ \frac{dx}{d\theta} = \frac{\sin \theta}{\cos \theta}$$

$\therefore$ Slope of the normal at $\theta = -\dfrac{\cos \theta}{\sin \theta}$

Hence the equation of the normal at θ

$$y - a(\sin \theta - \theta \cos \theta) = -\frac{\cos \theta}{\sin \theta}[x - a(\cos \theta + \theta \sin \theta)]$$

i.e., $y \sin \theta - a \sin^2 \theta + a\, \theta \sin \theta \cos \theta = -x \cos \theta + a \cos^2 \theta + a\, \theta \sin \theta \cos \theta$

i.e., $x \cos \theta + y \sin \theta = a(\cos^2 \theta + \sin^2 \theta) = a.$

Now the perpendicular distance of this normal from $(0, 0) = a$, which is a constant. Hence it touches a circle of radius a having its centre at $(0, 0)$.

(3) Angle of intersection of two curves *is the angle between the tangents to the curves at their point of intersection.*

To find this angle θ, proceed as follows :

(*i*) Find P, the point of intersection of the curves by solving their equations simultaneously.

(*ii*) Find the values of dy/dx at P for the two curves (say : m_1, m_2).

(*iii*) Find $\angle\theta$, using the $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2}$.

When $m_1 m_2 = -1$, $\theta = 90°$ *i.e.*, the curves cut orthogonally.

Example 4.37. *Find the angle of intersection of the curves* $x^2 = 4y$ ...(*i*)

and $y^2 = 4x$. ...(*ii*)

Solution. We have $x^4 = 16y^2 = 16.4\,x = 64x$

or $x(x^3 - 64) = 0$ whence $x = 0$ and 4.

Substituting these values in (*i*), $y = 0$ and 4.

∴ The curves intersect at (0, 0) and (4, 4).

For the curve (*i*), $dy/dx = x/2$. For the curve (*ii*), $dy/dx = 2/y$

At (0, 0), slope of tangent to (*i*) $(= m_1) = 0/2 = 0$ and slope of tangent to (*ii*) $(= m_2) = 2/0 = \infty$.

Evidently the curves intersect at right angles.

At (4, 4), slope of tangent to (*i*) $(= m_1) = 4/2 = 2$ and slope of tangent to (*ii*) $(= m_2) = 2/4 = \frac{1}{2}$

∴ Angle of intersection of the curves

$$= \tan^{-1}\frac{m_1 - m_2}{1 + m_1 m_2} = \tan^{-1}\frac{2 - \frac{1}{2}}{1 + 2\cdot\frac{1}{2}} = \tan^{-1}\frac{3}{4}.$$

Example 4.38. *Show that the condition that the curves* $ax^2 + by^2 = 1$ *and* $a'x^2 + b'y^2 = 1$ *should intersect orthogonally is that*

$$\frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}.$$

Solution. Given curves are $ax^2 + by^2 = 1$ ...(*i*) and $a'x^2 + b'y^2 = 1$ ...(*ii*)

Let $P(h, k)$ be a point of intersection of (*i*) and (*ii*) so that

$$ah^2 + bk^2 = 1 \quad\text{and}\quad a'h^2 + b'k^2 = 1$$

$$\therefore \qquad \frac{h^2}{-b + b'} = \frac{k^2}{-a' + a} = \frac{1}{ab' - a'b}$$

or $$h^2 = (b' - b)/(ab' - a'b),\ k^2 = (a - a')/(ab' - a'b) \qquad ...(iii)$$

Differentiating (*i*) w.r.t. x,

$$2ax + 2by\, dy/dx = 0 \quad\text{or}\quad dy/dx = -ax/by.$$

Similarly for (*ii*), $dy/dx = -a'x/b'y$

∴ m_1 = slope of tangent to (*i*) at $P = -ah/bk$; m_2 = slope of tangent to (*ii*) at $P = -a'h/b'k$

For orthogonal intersection, we should have $m_1 m_2 = -1$.

i.e., $$\frac{-ah}{bk} \times \frac{-a'h}{b'k} = 1 \text{ i.e., } aa'h^2 + bb'k^2 = 0$$

Substituting the values of h^2 and k^2 from (*iii*),

$$\frac{aa'(b' - b)}{ab' - a'b} + \frac{bb'(a - a')}{ab' - a'b} = 0 \quad\text{or}\quad \frac{b' - b}{bb'} + \frac{a - a'}{aa'} = 0$$

i.e., $$\frac{1}{b} - \frac{1}{b'} = \frac{1}{a} - \frac{1}{a'}$$ which leads to the required condition.

(4) Lengths of tangent, normal, subtangent and subnormal. Let the tangent and the normal at any point $P(x, y)$ of the curve meet the x-axis at T and N respectively. (Fig. 4.4). Draw the ordinate PM. Then PT and PN are called *the lengths of the tangent and the normal respectively. Also TM and MN are called the subtangent and subnormal respectively.*

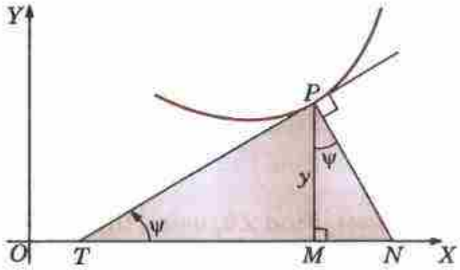


Fig. 4.4

Let $\angle MTP = \psi$ so that $\tan \psi = dy/dx$.

Clearly, $\angle MPN = \psi$.

(1) **Tangent** $= TP = MP \operatorname{cosec} \psi = y\sqrt{(1+\cot^2 \psi)} = \mathbf{y\sqrt{[1+(dx/dy)^2]}}$

(2) **Normal** $= NP = MP \sec \psi = y\sqrt{(1+\tan^2 \psi)} = \mathbf{y\sqrt{[1+(dy/dx)^2]}}$

(3) **Subtangent** $= TM = y \cot \psi = \mathbf{y\, dx/dy}$

(4) **Subnormal** $= MN = y \tan \psi = \mathbf{y\, dy/dx}$.

Example 4.39. *For the curve $x = a(\cos t + \log \tan t/2)$, $y = a \sin t$, prove that the portion of the tangent between the curve and x-axis is constant.*

Also find its subtangent.

Solution. Differentiating with respect to t,

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan t/2}\cdot \sec^2 \frac{t}{2}\cdot\frac{1}{2}\right) = a\left(-\sin t + \frac{\cos t/2}{2\sin t/2}\cdot\frac{1}{\cos^2 t/2}\right)$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right) = \frac{a(1-\sin^2 t)}{\sin t} = a\cos^2 t/\sin t\ ;\ \frac{dy}{dt} = a\cos t$$

$$\therefore \qquad \frac{dy}{dx} = \frac{dy}{dt}\bigg/\frac{dx}{dt} = a\cos t\cdot\frac{\sin t}{a\cos^2 t} = \tan t.$$

Thus *length of the tangent between the curve and x-axis*

$$= y\sqrt{[1+(dx/dy)^2]} = a\sin t\cdot\sqrt{(1+\cot^2 t)} = a\sin t\,.\,\operatorname{cosec} t = a \text{ which is a constant.}$$

Also *subtangent* $= y\dfrac{dx}{dy} = a\sin t\cdot\cot t = a\cos t.$

PROBLEMS 4.8

1. Find the equation of the tangent and the normal to the curve $y(x-2)(x-3) - x + 7 = 0$ at the point where it cuts the x-axis.
2. The straight line $x/a + y/b = 2$ touches the curve $(x/a)^n + (y/b)^n = 2$ for all values of n. Find the point of contact. (*Bhopal, 2008*)
3. Prove that $\dfrac{x}{a} + \dfrac{y}{b} = 1$ touches the curve $y = be^{-x/a}$ at the point where the curve crosses the axis of y. (*Bhopal, 2009*)
4. If $p = x\cos\alpha + y\sin\alpha$, touches the curve $(x/a)^{n/(n-1)} + (y/b)^{n/(n-1)} = 1$, prove that
$$p^n = (a\cos\alpha)^n + (b\sin\alpha)^n.$$
5. Prove that the condition for the line $x\cos\alpha + y\sin\alpha = p$ to touch the curve $x^m y^n = a^{m+n}$, is
$$p^{m+n}\cdot m^m\cdot n^n = (m+n)^{m+n}\, a^{m+n}\cos^m\alpha\,\sin^n\alpha.$$
6. Show that the sum of the intercepts on the axes of any tangent to the curve $\sqrt{x} + \sqrt{y} = a$ is a constant.
7. If x, y be the parts of the axes of x and y intercepted by the tangent at any point (x, y) on the curve $(x/a)^{2/3} + (x/b)^{2/3} = 1$, then show that $(x_1/a)^2 + (y_1/b)^2 = 1$. (*Bhopal, 2008*)
8. If the tangent at (x_1, y_1) to the curve $x^3 + y^3 = a^3$ meets the curve again in (x_2, y_2), show that
$$\frac{x_2}{x_1} + \frac{y_2}{y_1} = -1.$$

9. If the normal to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ makes an angle ϕ with the axis of x, show that its equation is $y \cos \phi - x \sin \phi = a \cos 2\phi$.
10. Find the angle of intersection of the curves $x^2 - y^2 = a^2$ are $x^2 + y^2 = a^2\sqrt{2}$.
11. Show that the parabolas $y^2 = 4ax$ and $2x^2 = ay$ intersect at an angle $\tan^{-1}(3/5)$.
12. Prove that the curves $\dfrac{x^2}{a} + \dfrac{y^2}{b} = 1$ and $\dfrac{x^2}{a'} + \dfrac{y^2}{b'} = 1$ will cut orthogonally if $a - b = a' - b'$.
13. Show that in the exponential curve $y = be^{x/a}$, the subtangent is of constant length and that the subnormal varies as the square of the ordinate. *(Madras, 2000 S)*
14. Find the lengths of the tangent, normal, subtangent and subnormal for the cycloid:
$$x = a(t + \sin t),\ y = a(1 - \cos t),$$
15. For the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$, show that the portion of the tangent intercepted between the point of contact and the x-axis is $y \operatorname{cosec}\theta$. Also find the length of the subnormal.

4.7 POLAR CURVES

(1) Angle between radius vector and tangent. *If ϕ be the angle between the radius vector and the tangent at any point of the curve $r = f(\theta)$,* $\tan \theta = r\dfrac{d\theta}{dr}$.

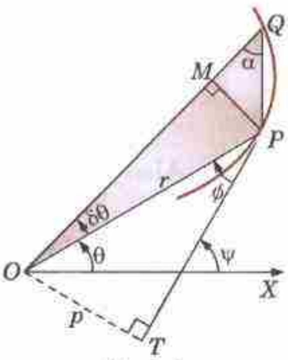


Fig. 4.5

Let $P(r, \theta)$ and $Q(r + \delta r, \theta + \delta\theta)$ be two neighbouring points on the curve (Fig. 4.5). Join PQ and draw $PM \perp OQ$. Then from the rt. angled ΔOMP, $MP = r \sin \delta\theta$, $OM = r\cos\delta\theta$.

$$\therefore \qquad MQ = OQ - OM = r + \delta r - r\cos\delta\theta$$
$$= \delta r + r(1 - \cos\delta\theta) = \delta r + 2r\sin^2\delta\theta/2.$$

If $\angle MQP = \alpha$, then
$$\tan\alpha = \frac{MP}{MQ} = \frac{r\sin\delta\theta}{\delta r + 2r\sin^2\delta\theta/2}$$

In the limit as $Q \to P$ (*i.e.*, $\delta\theta \to 0$), the chord PQ turns about P and becomes the tangent at P and $\alpha \to \phi$.

$$\therefore \qquad \tan\phi = \underset{Q\to P}{\text{Lt}}(\tan\alpha) = \underset{\delta\theta\to 0}{\text{Lt}} \frac{r\sin\delta\theta}{\delta r + 2r\sin^2\delta\theta/2}$$
$$= \underset{\delta\theta\to 0}{\text{Lt}} \frac{r(\sin\delta\theta/\delta\theta)}{(\delta r/\delta\theta) + r\sin\delta\theta/2\cdot(\sin\delta\theta/2 \div \delta\theta/2)}$$
$$= \frac{r\cdot 1}{(dr/d\theta) + r\cdot 0\cdot 1} = r\frac{d\theta}{dr}$$

Cor. Angle of intersection of two curves. *If ϕ_1, ϕ_2 be the angles between the common radius vector and the tangents to the two curves at their point of intersection, then the angle of intersection of these curves is $\phi_1 \sim \phi_2$.*

(2) Length of the perpendicular from pole on the tangent. *If p be the perpendicular from the pole on the tangent, then*

(*i*) $\mathbf{p = r\sin\phi}$ (*ii*) $\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$

From the rt. $\angle$ed ΔOTP, $p = r\sin\phi$

$$\therefore \qquad \frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi = \frac{1}{r^2}(1 + \cot^2\phi)$$
$$= \frac{1}{r^2}\left[1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right] = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \qquad \text{[By (1)]}$$

(3) Polar subtangent and subnormal. Let the tangent and the normal at any point $P(r, \theta)$ of a curve meet the line through the pole perpendicular to the radius vector OP in T and N respectively (Fig. 4.6). Then OT is called the *polar subtangent* and ON the *polar subnormal*.

Let $\angle OTP = \phi$ so that $\tan \phi = r d\theta/dr$

Clearly, $\angle PNO = \phi$.

$\therefore$ (*i*) **Polar subtangent**

$$= OT = r \tan \phi = r \cdot r d\theta/dr = \mathbf{r^2 \frac{d\theta}{dr}}$$

(*ii*) **Polar subnormal**

$$= ON = r \cot \phi = r . \frac{1}{r}\frac{dr}{d\theta} = \mathbf{\frac{dr}{d\theta}}$$

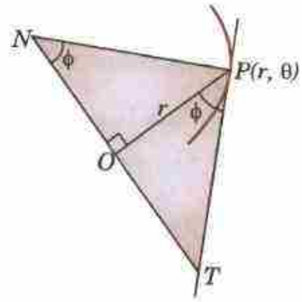


Fig. 4.6

Example 4.40. *For the cardioid $r = a(1 - \cos \theta)$, prove that*

(*i*) $\phi = \theta/2$ (*ii*) $p = 2a \sin^3 \theta/2$

(*iii*) *polar subtangent* $= 2a \sin^2 \dfrac{\theta}{2} \tan \dfrac{\theta}{2}$.

Solution. We have $\dfrac{dr}{d\theta} = a \sin \theta$

$$\therefore \quad \tan \phi = r\frac{d\theta}{dr} = a(1 - \cos \theta) \cdot \frac{1}{a \sin \theta}$$

$$= 2 \sin^2 \theta/2 \div 2 \sin \theta/2 \cos \theta/2 = \tan \theta/2. \text{ Thus } \phi = \theta/2 \qquad ...(i)$$

Also $p = r \sin \phi = a(1 - \cos \theta) \cdot \sin \theta/2 = a \cdot 2 \sin^2 \theta/2 \cdot \sin \theta/2$

$$= 2a \sin^3 \theta/2 \qquad ...(ii)$$

Polar subtangent $= r^2 d\theta/dr = [a(1 - \cos \theta)]^2 \div a \sin \theta$

$$= 4a \sin^4 \theta/2 \div 2 \sin \theta/2 \cos \theta/2 = 2a \sin^2 \theta/2 \tan \theta/2. \qquad ...(iii)$$

Example 4.41. *Find the angle of intersection of the curves $r = \sin \theta + \cos \theta$, $r = 2 \sin \theta$.*

Solution. To find the point of intersection of the curves $r = \sin \theta + \cos \theta$

and $r = 2 \sin \theta$, ...(*ii*), we eliminate r.

Then $2 \sin \theta = \sin \theta + \cos \theta$ or $\tan \theta = 1$ *i.e.*, $\theta = \pi/4$.

For (*i*), $\dfrac{dr}{d\theta} = \cos \theta - \sin \theta$

$$\therefore \quad \tan \phi = r\frac{d\theta}{dr} = \frac{\sin \theta + \cos \theta}{\cos \theta - \sin \theta} \text{ which} \to \infty \text{ at } \theta = \pi/4. \text{ Thus } \phi = \pi/2.$$

For (*ii*), $dr/d\theta = 2 \cos \theta \quad \therefore \quad \tan \phi' = r\dfrac{d\theta}{dr} = \dfrac{2 \sin \theta}{2 \cos \theta} = 1$ at $\theta = \pi/4$. Thus $\phi' = \pi/4$

Hence the angle of intersection of (*i*) and (*ii*) $= \phi - \phi' = \pi/4$.

PROBLEMS 4.9

1. For a curve in Cartesian form, show that $\tan \phi = \dfrac{xy' - y}{x + yy'}$.
2. Show that in the equiangular spiral $r = ae^{\theta \cot \alpha}$, the tangent is inclined at a constant angle to the radius vector.
3. Show that the tangent to the cardioid $r = a(1 + \cos \theta)$ at the points $\theta = \pi/3$ and $\theta = 2\pi/3$ are respectively parallel and perpendicular to the initial line. (*V.T.U., 2006*)
4. Prove that, in the parabola $2a/r = 1 - \cos \theta$,

   (*i*) $\phi = \pi - \theta/2$ (*ii*) $\pi = a \operatorname{cosec} \theta/2$, and (*iii*) polar subtangent $= 2a \operatorname{cosec} \theta$.
5. Show that the angle between the tangent at any point P and the line joining P to the origin is the same at all points of the curve

$$\log(x^2 + y^2) = k \tan^{-1}(y/x).$$

6. Show that in the curve $r = a\theta$, the polar subnormal is constant and in the curve $r\,\theta = a$ the polar subtangent is constant.
7. Find the angle of intersection of the curves
   (i) $r = 2 \sin \theta$, and $r = 2 \cos \theta$ *(Bhopal, 1991)*
   (ii) $r = a/(1 + \cos \theta)$ and $r = b/(1 - \cos \theta)$. *(V.T.U., 2008 S)*
8. Prove that the curves $r = a(1 + \cos \theta)$ and $r = b(1 - \cos \theta)$ intersect at right angles. *(V.T.U., 2011 S)*
9. Show that the curves $r^n = a^n \cos n\theta$ and $r^n = b^n \sin n\theta$ cut each other orthogonally.
10. Show that the angle of intersection of the curves $r = a \log \theta$ and $r = a/\log \theta$ is $\tan^{-1} [2e/(1 - e^2)]$. *(V.T.U., 2005)*

4.8 PEDAL EQUATION

If r be the radius vector of any point on the curve and p, the length of the perpendicular from the pole on the tangent at that point, them *the relation between p and r is called pedal equation of the curve.*

Given the cartesian or polar equation of a curve, we can derive its pedal equation. The method is explained through the following examples.

Example 4.42. *Find the pedal equation of the ellipse* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$ ...(i)

Solution. Equation of the tangent at (x, y) is $\dfrac{Xx}{a^2} + \dfrac{Yy}{b^2} = 1$ ...(ii)

p, length of $\perp$ from (0, 0) on (ii) $= \dfrac{-1}{\sqrt{[(x/a^2)^2 + (y/b^2)^2]}}$

or
$$\frac{1}{p^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4} \qquad ...(iii)$$

Also
$$r^2 = x^2 + y^2 \qquad ...(iv)$$

Substituting the value of y^2 from (iv) in (i),

$$\frac{x^2}{a^2} = \frac{r^2 - b^2}{a^2 - b^2}.$$

Then from (i),
$$\frac{y^2}{b^2} = \frac{a^2 - r^2}{a^2 - b^2}$$

Now substituting these values of x^2/a^2 and y^2/b^2 in (iii),

$$\frac{1}{p^2} = \frac{1}{a^2}\left(\frac{r^2 - b^2}{a^2 - b^2}\right) + \frac{1}{b^2}\left(\frac{a^2 - r^2}{a^2 - b^2}\right)$$

or
$$\frac{a^2 b^2}{p^2} = \frac{r^2 b^2 - b^4 + a^4 - a^2 r^2}{a^2 - b^2} = a^2 + b^2 - r^2$$

Her[illegible] [illegible]e required pedal equation.

Example 4.43. [illegible] *of the curves*
(i) $2a / r = 1$ [illegible] $r^n = a^n \cos n\,\theta$ *(V.T.U., 2010)*

Solution.
Takin[illegible]

log $2a$ – lo[illegible]

Differentiating both sides with respect to θ, we get

$$-\frac{1}{r}\frac{dr}{d\theta}=\frac{1}{1-\cos\theta}\cdot\sin\theta=\cot\frac{\theta}{2}$$

$\therefore$ $$\tan\phi=r\frac{d\theta}{dr}=-\tan\theta/2=\tan(\pi-\theta/2)\ i.e.,\ \phi=\pi-\theta/2$$

Also $$p=r\sin\phi=r\sin(\pi-\theta/2)\ i.e.,\ p=r\sin\theta/2$$

or $$p^2=r^2\sin^2\theta/2=r^2\left(\frac{1-\cos\theta}{2}\right)=r^2.\,a/r \qquad \text{[By } (i)\text{]}$$

Hence $p^2 = ar$, which is the required pedal equation.

(*ii*) From the given equation, $nr^{n-1}\dfrac{dr}{d\theta}=-na^n\sin n\theta$

so that $$\tan\phi=r\,d\theta/dr=r\frac{nr^{n-1}}{-na^n\sin n\theta}=-\cot n\theta=\tan\left(\frac{\pi}{2}+n\theta\right)$$

i.e., $$\phi=\pi/2+n\theta$$

$\therefore$ $$p=r\sin\phi=r\sin\left(\frac{\pi}{2}+n\theta\right)=r\cos n\theta=r\,.\,(r^n/a^n)=r^{n+1}/a^n.$$

Hence $p\,a^n = r^{n+1}$, which is the required pedal equation.

4.9 DERIVATIVE OF ARC

(1) *For the curve y = f (x), we have*

$$\frac{\mathbf{ds}}{\mathbf{dx}}=\sqrt{\left[1+\left(\frac{\mathbf{dy}}{\mathbf{dx}}\right)^2\right]}$$

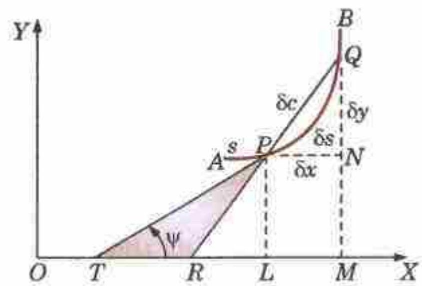


Fig. 4.7

Let $P(x, y)$, $Q(x + \delta x, y + \delta y)$ be two neighbouring points on the curve AB (Fig. 4.7). Let arc $AP = s$, arc $PQ = \delta s$ and chord $PQ = \delta c$.

Draw PL, QM ⊥s on the x-axis and $PN \perp QM$.

$\therefore$ From the rt. ∠ed ΔPNQ,

$$PQ^2 = PN^2 + NQ^2$$

i.e., $$\delta c^2 = \delta x^2 + \delta y^2$$

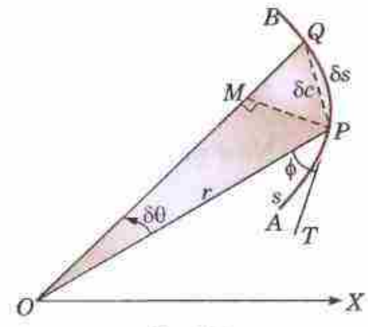


Fig. 4.8

or $$\left(\frac{\delta c}{\delta x}\right)^2=1+\left(\frac{\delta y}{\delta x}\right)^2$$

$\therefore$ $$\left(\frac{\delta s}{\delta x}\right)^2=\left(\frac{\delta s}{\delta c}\cdot\frac{\delta c}{\delta x}\right)^2$$

$$=\left(\frac{\delta s}{\delta c}\right)^2=\left[1+\left(\frac{\delta y}{\delta x}\right)^2\right]$$

Taking limits as $Q \to P$ (*i.e.*, $\delta c \to 0$),

$$\left(\frac{ds}{dx}\right)^2=1\cdot\left[1+\left(\frac{dy}{dx}\right)^2\right] \qquad \left.\frac{\delta s}{\delta c}=1\right]$$

If s increases with x as in Fig. 4.7, dy/dx is positive.

Thus $$\frac{ds}{dx}=\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]},$$ taking positive sign before the radical. ...(1)

Cor. 1. *If the equation of the curve is x = f(y), then*

$$\frac{ds}{dy}=\frac{ds}{dx}\cdot\frac{dx}{dy}=\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\cdot\frac{dx}{dy}$$

$\therefore$ $$\frac{\mathbf{ds}}{\mathbf{dy}} = \sqrt{\left[1 + \left(\frac{\mathbf{dx}}{\mathbf{dy}}\right)^2\right]} \qquad ...(2)$$

Cor. 2. *If the equation of the curve is in parametric form $x = f(t)$, $y = \phi(t)$, then*

$$\frac{ds}{dt} = \frac{ds}{dx} \cdot \frac{dx}{dt} = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]} \cdot \frac{dx}{dt}$$

$$= \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx} \cdot \frac{dx}{dt}\right)^2\right]}$$

$\therefore$ $$\frac{\mathbf{ds}}{\mathbf{dt}} = \sqrt{\left[\left(\frac{\mathbf{dx}}{\mathbf{dt}}\right)^2 + \left(\frac{\mathbf{dy}}{\mathbf{dt}}\right)^2\right]} \qquad ...(3)$$

Cor. 3. We have $$\frac{ds}{dx} = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]} = \sqrt{(1 + \tan^2 \psi)} = \sec \psi$$

$\therefore$ $$\cos \psi = \frac{\mathbf{dx}}{\mathbf{ds}}. \qquad ...(4)$$

Also $$\sin \psi = \tan \psi \cos \psi = \frac{dy}{dx} \cdot \frac{dx}{ds}$$

$\therefore$ $$\sin \psi = \frac{\mathbf{dy}}{\mathbf{ds}} \qquad ...(5)$$

(2) *For the curve $r = f(\theta)$, we have* $\dfrac{\mathbf{ds}}{\mathbf{d\theta}} = \sqrt{\left[\mathbf{r}^2 + \left(\dfrac{\mathbf{dr}}{\mathbf{d\theta}}\right)^2\right]}$.

Let $P(r, \theta)$, $Q(r + \delta r, \theta + \delta\theta)$ be two neighbouring points on the curve AB (Fig. 4.8). Let arc $AP = s$, arc $PQ = \delta s$ and chord $PQ = \delta c$.

Draw $PM \perp OQ$, then

$$PM = r \sin \delta\theta \text{ and } MQ = OQ - OM = r + \delta r - r \cos \delta\theta = \delta r + 2r \sin^2 \delta\theta/2$$

From the rt. $\angle$ed ΔPMQ,

$$PQ^2 = PM^2 + MQ^2$$

or $$\delta c^2 = (r \sin \delta\theta)^2 + (\delta r + 2r \sin^2 \delta\theta/2)^2$$

$\therefore$ $$\left(\frac{\delta s}{\delta\theta}\right)^2 = \left(\frac{\delta s}{\delta c} \cdot \frac{\delta c}{\delta\theta}\right)^2 = \left(\frac{\delta s}{\delta c}\right)^2 \left[\left(\frac{r \sin \delta\theta}{\delta\theta}\right)^2 + \left(\frac{\delta r}{\delta\theta} + \frac{2r \sin^2 \delta\theta/2}{\delta\theta}\right)^2\right]$$

$$= \left(\frac{\delta s}{\delta c}\right)^2 \left[r^2 \left(\frac{\sin \delta\theta}{\delta\theta}\right)^2 + \left(\frac{\delta r}{\delta\theta} + r \sin \frac{\delta\theta}{2} \cdot \frac{\sin \delta\theta/2}{\delta\theta/2}\right)^2\right]$$

Taking limits as $Q \to P$

$$\left(\frac{ds}{d\theta}\right)^2 = 1^2 . \left[r^2 \cdot 1^2 + \left(\frac{dr}{d\theta} + r \cdot 0 \cdot 1\right)^2\right] = r^2 + \left(\frac{dr}{d\theta}\right)^2$$

As s increases with the increase of θ, $ds/d\theta$ is positive. Thus

$$\frac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]} \qquad ...(1)$$

Cor. 1. If the equation of the curve is $\theta = f(r)$, then

$$\frac{ds}{dr} = \frac{ds}{d\theta} \cdot \frac{d\theta}{dr} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]} \cdot \frac{d\theta}{dr}$$

$$\therefore \qquad \mathbf{\frac{ds}{dr} = \sqrt{\left[1 + r^2\left(\frac{d\theta}{dr}\right)^2\right]}} \qquad ...(2)$$

Cor. 2. We have $\frac{ds}{dr} = \sqrt{\left[1 + \left(r\frac{d\theta}{dr}\right)^2\right]} = \sqrt{[1 + \tan^2\phi]}\ \frac{ds}{dr} = \sqrt{\left[1 + \left(r\frac{d\theta}{dr}\right)^2\right]} = \sqrt{[1 + \tan^2\phi]} = \sec\phi$

$$\therefore \qquad \mathbf{\cos\phi = \frac{dr}{ds}} \qquad ...(3)$$

Also $\sin\phi = \tan\phi \,.\, \cos\phi = r\frac{d\theta}{dr}\cdot\frac{dr}{ds}$

$$\therefore \qquad \mathbf{\sin\phi = r\frac{d\theta}{ds}} \qquad ...(4)$$

PROBLEMS 4.10

Prove that the pedal equation of :

1. the parabola $y^2 = 4a(x + a)$ is $p^2 = ar$.
2. the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $a^2b^2/p^2 = r^2 - a^2 + b^2$.
3. the astroid $x = a\cos^3 t, y = a\sin^3 t$ is $r^2 = a^2 - 3p^2$.

Find the pedal equations of the following curves :

4. $r = a\,(1 + \cos\theta)$ (*V.T.U., 2009*)
5. $r^2 = a^2\sin^2\theta$
6. $r^m\cos m\theta = a^m$. (*V.T.U., 2004*)
7. $r^m = a^m\,(\cos m\theta + \sin m\theta)$ (*V.T.U., 2010*)
8. $r = ae^{m\theta}$. (*V.T.U., 2007*)
9. Calculate ds/dx for the following curves :
   (*i*) $ay^2 = x^3$. (*ii*) $y = c\cosh x/c$.
10. Find $ds/d\theta$ for the curve $x = a(\cos\theta + \theta\sin\theta), y = a\,(\sin\theta - \theta\cos\theta)$ (*V.T.U., 2007*)
11. Find $ds/d\theta$ for the following curves :
    (*i*) $r = a\,(1 - \cos\theta)$ (*V.T.U., 2004*) (*ii*) $r^2 = a^2\cos^2 2\theta$
    (*iii*) $r = \frac{1}{2}\sec^2\theta$ (*V.T.U., 2007*)
12. For the curves $\theta = \cos^{-1}(r/k) - \sqrt{(k^2 - r^2)}/r$, prove that $r\frac{ds}{dr}$ = constant. (*V.T.U., 2005*)
13. With the usual meanings for r, s, θ and ϕ for the polar curve $r = f(\theta)$, show that $\frac{d\phi}{d\theta} + r\,\text{cosec}^2\,\theta\,\frac{d^2r}{ds^2} = 0$. (*V.T.U., 2000*)

4.10 CURVATURE

Let P be any point on a given curve and Q a neighbouring point. Let arc $AP = s$ and arc $PQ = \delta s$. Let the tangents at P and Q make angle ψ and $\psi + \delta\psi$ with the x-axis, so that the angle between the tangents at P and $Q = \delta\psi$ (Fig. 4.9).

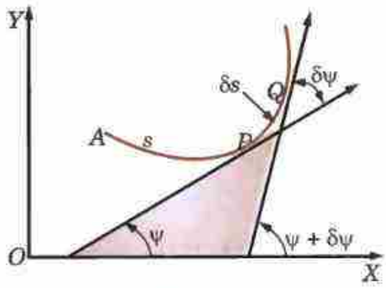


Fig. 4.9

In moving from P to Q through a distance δs, the tangent has turned through the angle $\delta\psi$. This is called the *total bending or total curvature* of the arc PQ.

$\therefore$ The *average curvature* of arc $PQ = \frac{\delta\psi}{\delta s}$

The limiting value of average curvature when Q approaches P (i.e., $\delta s \to 0$) is defined as the curvature of the curve at P.

Thus **curvature** K (at P) $= \mathbf{\frac{d\psi}{ds}}$

Obs. Since $\delta\psi$ is measured in radians, the *unit of curvature is radians per unit length e.g.*, radians per centimetre.

(2) Radius of curvature. *The reciprocal of the curvature of a curve at any point P is called the* **radius of curvature** *at P and is denoted by* ρ, so the $\rho = \mathbf{ds/d\psi}$.

(3) Centre of curvature. *A point C on the normal at any point P of a curve distant* ρ *from it, is called the* **centre of curvature** *at P.*

(4) Circle of curvature. *A circle with centre C (centre of curvature at P) and radius* ρ *is called the* **circle of curvature** *at P.*

4.11 (1) RADIUS OF CURVATURE FOR CARTESIAN CURVE y = f (x), is given by

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

We know that $\tan\psi = dy/dx = y_1$ or $\psi = \tan^{-1}(y_1)$

Differentiating both sides w.r.t. x,

$$\frac{d\psi}{dx} = \frac{1}{1+y_1^2}\cdot\frac{d(y_1)}{dx} = \frac{y_2}{1+y_1^2}$$

$$\therefore \quad \rho = \frac{ds}{d\psi} = \frac{ds}{dx}\cdot\frac{dx}{d\psi} = \sqrt{(1+y_1^2)}\cdot\frac{1+y_1^2}{y_2} = \frac{(1+y_1^2)^{3/2}}{y_2} \qquad \text{...(1)}$$

(2) Radius of curvature for parametric equations

$$x = f(t), \qquad y = \phi(t).$$

Denoting differentiations with respect to t by dashes,

$$y_1 = \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = y'/x'.$$

$$y_2 = \frac{d}{dx}(y_1) = \frac{d}{dt}\left(\frac{y'}{x'}\right)\cdot\frac{dt}{dx} = \frac{x'y'' - y'x''x}{(x')^2}\cdot\frac{1}{x'}$$

Substituting the values of y_1 and y_2 in (1)

$$\rho = \left[1+\left(\frac{y'}{x'}\right)^2\right]^{3/2}\Bigg/\left[\frac{x'y''-y'x''}{(x')^3}\right] = \frac{(x'^2+y'^2)^{3/2}}{x'y''-y'x''} \qquad \textit{(Rajasthan, 2005)}$$

(3) Radius of curvature at the origin. Newton's formulae*

(*i*) *If x-axis is tangent to a curve at the origin, then*

$$\rho \text{ at } (0, 0) = \mathop{\mathbf{Lt}}_{x\to 0}\left(\frac{\mathbf{x^2}}{\mathbf{2y}}\right)$$

Since x-axis is a tangent at (0, 0), $(dy/dx)_0$ or $(y_1)_0 = 0$

Also $$\mathop{\text{Lt}}_{x\to 0}\left(\frac{x^2}{2y}\right) = \mathop{\text{Lt}}_{x\to 0}\left(\frac{2x}{2dy/dx}\right) = \mathop{\text{Lt}}_{x\to 0}\frac{1}{d^2y/dx^2} = \frac{1}{(y_2)_0} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$\therefore \quad \rho \text{ at } (0, 0) = \frac{\{1+(y_1^2)_0\}^{3/2}}{(y_2)_0} = \frac{1}{(y_2)_0} = \mathop{\text{Lt}}_{x\to 0}\frac{x^2}{2y} \qquad \text{[From (1)]}$$

(*ii*) Similarly, *if y-axis is tangent to a curve at the origin, then*

$$\rho \text{ at } (0, 0) = \mathop{\text{Lt}}_{x\to 0}\left(\frac{\mathbf{y^2}}{\mathbf{2x}}\right)$$

* Named after the great English mathematician and physicist *Sir Issac Newton* (1642–1727) whose contributions are of utmost importance. He discovered many physical laws, invented Calculus alongwith *Leibnitz* (see footnote p. 139) and created analytical methods of investigating physical problems. He became professor at Cambridge in 1699, but his '*Mathematical Principles of Natural Philosophy*' containing development of classical mechanics had been completed in 1687.

(*iii*) In case the curve passes through the origin but neither x-axis nor y-axis is tangent at the origin, we write the equation of the curve as

$$y = f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots \qquad \text{[By Maclaurin's series]}$$

$$= px + qx^2/2 + \ldots \qquad [\because \ f(0) = 0]$$

where $p = f'(0)$ and $q = f''(0)$

Substituting this in the equation $y = f(x)$, we find the values of p and q by equating coefficients of like powers of x. Then $\rho\ (0, 0) = (1 + p^2)^{3/2}/q$.

Obs. *Tangents at the origin to a curve are found by equating to zero the lowest degree terms in its equation.*

Example 4.44. *Find the radius of curvature at the point (i) $(3a/2, 3a/2)$ of the Folium $x^3 + y^3 = 3axy$.* (Anna, 2009 ; Kurukshetra, 2009 S ; V.T.U., 2008)

(ii) $(a, 0)$ on the curve $xy^2 = a^3 - x^3$. (Anna, 2009 ; Kerala, 2005)

Solution. (*i*) Differentiating with respect to x, we get

$$3x^2 + 3y^2 \frac{dy}{dx} = 3a\left(y + x\frac{dy}{dx}\right)$$

or $$(y^2 - ax)\frac{dy}{dx} = ay - x^2 \quad \ldots(i) \qquad \therefore \quad \frac{dy}{dx} \text{ at } (3a/2, 3a/2) = -1$$

Differentiating (*i*),

$$\left(2y\frac{dy}{dx} - a\right)\frac{dy}{dx} + (y^2 - ax)\frac{d^2y}{dx^2} = a\frac{dy}{dx} - 2x \quad \therefore \quad \frac{d^2y}{dx^2} \text{ at } (3a/2, 3a/2) = -32/3a$$

Hence ρ at $(3a/2, 3a/2)$ $$= \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = \frac{[1 + (-1)^2]^{3/2}}{-32/3a} = \frac{3a}{8\sqrt{2}} \quad \text{(in magnitude).}$$

(*ii*) We have $$y^2 = a^3 x^{-1} - x^2$$

$\therefore$ $$2y\, dy/dx = -a^3x^{-2} - 2x \quad \text{or} \quad dy/dx = -a^3/(2x^2y) - x/y$$

At $(a, 0)$, $dy/dx \to \infty$, so we find dx/dy from $xy^2 = a^3 - x^3$

$\therefore$ $$x - 2y + y^2\frac{dx}{dy} = -3x^2\frac{dx}{dy}$$

or $$\frac{dx}{dy} = \frac{-2xy}{3x^2 + y^2} \quad \text{or} \quad \frac{dx}{dy} \text{ at } (a, 0) = 0.$$

$\therefore$ $$\frac{d^2x}{dy^2} = \frac{(3x^2 + y^2)\left(-2y\frac{dx}{dy} - 2x\right) - (-2xy)\left(6x\frac{dx}{dy} + 2y\right)}{(3x^2 + y^2)^2}$$

or $$\frac{d^2x}{dy^2} \text{ at } (a, 0) = \frac{(3a^2 + 0)(0 - 2a) - 0}{(3a^2 + 0)^2} = \frac{-2}{3a}$$

Hence $$\rho \text{ at } (a, 0) = \frac{\left[1 + \left(\frac{dx}{dy}\right)_{(a,0)}\right]^{3/2}}{\left(\frac{d^2x}{dy^2}\right)_{(a,0)}} = \frac{(1 + 0)^{3/2}}{(-2/3a)} = -\frac{3a}{2}.$$

Example 4.45. *Show that the radius of curvature at any point of the cycloid $x = a\ (\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ is $4a \cos\theta/2$.* (V.T.U., 2011 ; P.T.U., 2006)

Solution. We have $$\frac{dx}{d\theta} = a\ (1 + \cos\theta), \ \frac{dy}{d\theta} = a \sin\theta.$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{a \sin\theta}{a(1+\cos\theta)} = \frac{2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2} = \tan\theta/2$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\cdot\frac{d\theta}{dx} = \frac{1}{2}\sec^2\frac{\theta}{2}\cdot\frac{1}{a(1+\cos\theta)}$$

$$= \frac{1}{2}\sec^2\frac{\theta}{2}\cdot\frac{1}{2a\cos^2\theta/2} = \frac{1}{4a}\sec^4\frac{\theta}{2}.$$

$$\therefore \quad \rho = \frac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2} = \frac{4a(1+\tan^2\theta/2)^{3/2}}{\sec^4\theta/2}$$

$$= 4a\cdot(\sec^2\theta/2)^{3/2}\cdot\cos^4\theta/2 = 4a\cos\theta/2.$$

Example 4.46. *Prove that the radius of curvature at any point of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$, is three times the length of the perpendicular from the origin to the tangent at that point.*

(J.N.T.U., 2005 ; Bhopal, 2002 S)

Solution. The parametric equation of the curve is

$$x = a\cos^3 t, y = a\sin^3 t.$$

$$\therefore \quad x' (= dx/dt) = -3a\cos^2 t\sin t, y' = 3a\sin^2 t\cos t.$$

$$x'' = -3a(\cos^3 t - 2\cos t\sin^2 t) = 3a\cos t(2\sin^2 t - \cos^2 t)$$

$$y'' = 3a(2\sin t\cos^2 t - \sin^3 t) = 3a\sin t(2\cos^2 t - \sin^2 t)$$

$$x'^2 + y'^2 = 9a^2(\cos^4 t\sin^2 t + \sin^4 t\cos^2 t) = 9a^2\sin^2 t\cos^2 t$$

$$x'y'' - y'x'' = -9a^2\cos^2 t\sin^2 t(2\cos^2 t - \sin^2 t)$$

$$-9a^2\cos^2 t\sin^2 t(2\sin^2 t - \cos^2 t) = -9a^2\sin^2 t\cos^2 t$$

$$\therefore \quad \rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} = \frac{27a^3\sin^3 t\cos^3 t}{-9a^2\sin^2 t\cos^2 t} = -3a\sin t\cos t.$$

Since $\quad dy/dx = y'/x' = -\tan t,$

$\therefore$ Equation of the tangent at $(a\cos^3 t, a\sin^3 t)$ is $y - a\sin^3 t = -\tan t(x - a\cos^3 t)$

i.e., $\quad x\tan t + y - a\sin t = 0 \quad$ ...(*i*)

p, length of $\perp$ from $(0, 0)$ on $(i) = \dfrac{0 + 0 - a\sin t}{\sqrt{(\tan^2 t + 1)}} = -a\sin t\cos t$. Thus $\rho = 3p$.

EExample 4.47. *If ρ_1 and ρ_2 be the radii of curvature at the ends of a focal chord of the parabola $y^2 = 4ax$, then show that $\rho_1^{-2/3} + \rho_2^{-2/3} = (2a)^{-2/3}$.* *(Rohtak, 2006 S ; Kurukshetra, 2005)*

Solution. Given parabola is $y^2 = 4ax$ or $x = at^2, y = 2at$. If dashes denote differentiation w.r.t. t, then

$$x' = 2at, y' = 2a ; x'' = 2a, y'' = 0.$$

$$\therefore \quad \rho \text{ at } (at^2, 2at) = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - x''y'} = \frac{(4a^2t^2 + 4a^2)^{3/2}}{0 - 4a^2} = 2a(1+t^2)^{3/2} \quad \text{(Numerically)}$$

If $P(t_1)$ and $Q(t_2)$ be the extremities of the focal chord of the parabola, then

$$t_1 t_2 = -1 \quad i.e., \quad t_2 = -1/t_1 \quad ...(i)$$

$$\therefore \quad \rho_1 \text{ at } P(t_1) = 2a(1+t_1^2)^{3/2} ; \rho_2 \text{ at } Q(t_2) = 2a(1+t_2^2)^{3/2}$$

Thus $\quad \rho_1^{-2/3} + \rho_2^{-2/3} = (2a)^{-2/3} = [(1+t_1^2)^{-1} + (1+t_2^2)^{-1}]$

$$= (2a)^{-2/3}\left[\frac{1}{1+t_1^2} + \frac{t_1^2}{1+t_1^2}\right] \quad \text{[By (i)]}$$

$$= (2a)^{-2/3}$$

Example 4.48. *Show that the radius of curvature of P on an ellipse $x^2/a^2 + y^2/b^2 = 1$ is CD^3/ab where CD is the semi-diameter conjugate to CP.* *(J.N.T.U., 2002)*

Solution. Two diameters of an ellipse are said to be conjugate if each bisects chords parallel to the other. If CP and CD are two semi-conjugate diameters and P is $(a \cos \theta, b \sin \theta)$ then D is $a \cos\left(\theta + \frac{\pi}{2}\right), b \sin\left(\theta + \frac{\pi}{2}\right)$ *i.e.*, $(-a \sin \theta, b \cos \theta)$.

Also $C(0, 0)$ is the centre of the ellipse.

$$\therefore \qquad CD = \sqrt{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)}$$

At P, we have $x = a \cos \theta, y = b \sin \theta$.

$$\therefore \qquad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b \cos \theta}{-a \sin \theta} = \frac{-b}{a} \cot \theta \,;\, \frac{d^2y}{dx^2} = \frac{b}{a} \operatorname{cosec}^2 \theta \,.\, \frac{d\theta}{dx} = \frac{-b}{a^2} \operatorname{cosec}^3 \theta.$$

$$\therefore \qquad \rho = \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = \frac{\left(1 + \frac{b^2}{a^2} \cot^2 \theta\right)^{3/2}}{-\frac{b}{a^2} \operatorname{cosec}^3 \theta}$$

$$= \frac{a^2}{b \operatorname{cosec}^3 \theta} \frac{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}}{a^3 \sin^3 \theta} \qquad \text{(Numerically)}$$

$$= \frac{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{3/2}}{ab} = \frac{CD^3}{ab}.$$

Example 4.49. *Find ρ at the origin for the curves*
(i) $y^4 + x^3 + a(x^2 + y^2) - a^2 y = 0$ *(ii)* $y - x = x^2 + 2xy + y^2$

Solution. (*i*) Equating to zero the lowest degree terms, we get $y = 0$.

$\therefore$ x-axis is the tangent at the origin. Dividing throughout by y, we have

$$y^3 + x \,.\, \frac{x^2}{y} + a\left(\frac{x^2}{y} + y\right) - a^2 = 0$$

Let $x \to 0$, so that $\qquad \underset{x \to \infty}{\text{Lt}} (x^2/2y) = \rho.$

$$\therefore \qquad 0 + 0.2\rho + a(2\rho + 0) - a^2 = 0 \quad \text{or} \quad \rho = a/2.$$

(*ii*) Equating to zero the lowest degree terms, we get $y = x$, as the tangent at the origin, which is neither of the coordinates axes.

$\therefore$ Putting $y = px + qx^2/2 + \ldots$ in the given equation, we get

$$px + qx^2/2 + \ldots - x = x^2 + 2x(px + qx^2/2 + \ldots) + (px + qx^2/2 + \ldots)^2$$

Equating coefficients of x and x^2,

$$p - 1 = 0, q/2 = 1 + 2p + p^2 \quad i.e., \quad p = 1 \text{ and } q = 2 + 4 \cdot 1 + 2 \cdot 1^2 = 8.$$

$$\therefore \qquad \rho(0, 0) = (1 + p^2)^{3/2}/q = (1 + 1)^{3/2}/8 = 1/2\sqrt{2}.$$

(4) Radius of curvature for polar curve $r = f(\theta)$ is given by

$$\rho = \frac{(\mathbf{r}^2 + \mathbf{r}_1^2)^{3/2}}{\mathbf{r}^2 + 2\mathbf{r}_1^2 - \mathbf{r}\mathbf{r}_2}$$

With the usual notations, we have from Fig. 4.10.

$$\psi = \theta + \phi$$

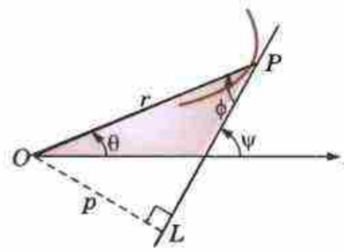


Fig. 4.10

Differentiating w.r.t. s,

$$\frac{1}{\rho} = \frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{d\theta} \cdot \frac{d\theta}{ds}$$

$$= \frac{d\theta}{ds}\left(1 + \frac{d\phi}{d\theta}\right) \qquad \ldots(1)$$

Also we know that

$$\tan \phi = r\frac{d\theta}{dr} = \frac{r}{r_1} \quad \text{or} \quad \phi = \tan^{-1}\left(\frac{r}{r_1}\right) \qquad \text{where } r_1 = \frac{dr}{d\theta}$$

Differentiating w.r.t. θ,

$$\frac{d\phi}{d\theta} = \frac{1}{1+(r/r_1)^2}\cdot\frac{r_1\cdot r_1 - rr_2}{r_1^2} = \frac{r_1^2 - rr_2}{r^2 + r_1^2} \qquad ...(2)$$

Also, $$\frac{ds}{d\theta} = \sqrt{(r^2 + r_1^2)} \qquad ...(3)$$

Substituting the value from (2) and (3) in (1),

$$\frac{1}{\rho} = \frac{1}{\sqrt{r^2 + r_1^2}}\cdot\left(1 + \frac{r_1^2 - rr_2}{r^2 + r_1^2}\right)$$

Hence $$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

(5) Radius of curvature for pedal curve $p = f(r)$ *is given by*

$$\rho = \mathbf{r}\frac{\mathbf{dr}}{\mathbf{dp}}$$

With the usual notation (Fig. 4.10), we have $\psi = \theta + \phi$

Differentiating w.r.t. s,

$$\frac{1}{\rho} = \frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds} \qquad ...(1)$$

Also we know that $p = r \sin \phi$

$$\therefore \quad \frac{dp}{dr} = \sin\phi + r\cos\phi\,\frac{d\phi}{ds}$$

$$= r\frac{d\theta}{ds} + r\frac{dr}{ds}\cdot\frac{d\phi}{dr} \qquad \text{[By (3) and (4) of § 4.9 (2)]}$$

$$= r\left(\frac{d\theta}{ds} + \frac{d\phi}{ds}\right) = \frac{r}{\rho} \qquad \text{[By (1)]}$$

Hence $$\rho = r\frac{dr}{dp}.$$

Example 4.50. *Show that the radius of curvature at any point of the cardioid $r = a\,(1 - \cos\theta)$ varies as $\sqrt{r}$.* *(V.T.U., 2003)*

Solution. Differentiating w.r.t. θ, we get

$$r_1 = a\sin\theta,\ r_2 = a\cos\theta$$

$$\therefore \quad (r^2 + r_1{}^2)^{3/2} = [a^2(1-\cos\theta)^2 + a^2\sin^2\theta]^{3/2} = a^3[2\,(1-\cos\theta)]^{3/2}$$

$$r^2 - rr_2 + 2r_1{}^2 = a^2(1-\cos\theta)^2 - a^2\,(1-\cos\theta)\cos\theta + 2a^2\sin^2\theta = 3a^2\,(1-\cos\theta)$$

Thus $$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 - rr_2 + 2r_1^2} = \frac{a^3\,2\sqrt{2}(1-\cos\theta)^{3/2}}{3a^2(1-\cos\theta)}$$

$$= \frac{2\sqrt{2}}{3}a\,(1-\cos\theta)^{1/2} = \frac{2\sqrt{2}a}{3}\left(\frac{r}{a}\right)^{1/2} \propto \sqrt{r}.$$

Otherwise. The pedal equation of this cardioid is $2ap^2 = r^3$ ...(*i*)

Differentiating w.r.t. p, we get

that $$4ap = 3r^2\frac{dr}{dp} \quad \text{whence } \rho = r\frac{dr}{dp} = \frac{4ap}{3r} = \frac{4ar^{3/2}}{3r\cdot\sqrt{(2a)}} \propto \sqrt{r}.$$

$$\left[\because \quad p = r^{3/2}/\sqrt{(2a)} \text{ from } (i)\right]$$

PROBLEMS 4.11

1. Find the radius of curvature at any point
   (*i*) $(at^2, 2at)$ of the parabola $y^2 = 4ax$.
   (*ii*) (o, c) of the catenary $y = c \cosh x/c$.
   (*iii*) $(a, 0)$ of the curve $y = x^3 (x - a)$. (*V.T.U., 2010*)
2. Show that for (*i*) the rectangular hyperbola $xy = c^2$, $\rho = \dfrac{(x^2 + y^2)^{3/2}}{2c^2}$. (*Rohtak, 2005 ; Madras, 2000*)
   (*ii*) the curve $y = ae^{x/a}$, $\rho = a \sec^2 \theta \operatorname{cosec} \theta$ where $\theta = \tan^{-1} (y/a)$. (*Rajasthan, 2006*)
3. Show that the radius of curvature at
   (*i*) $(a, 0)$ on the curve $y^2 = a^2 (a - x)/x$ is $a/2$. (*V.T.U., 2000 S*)
   (*ii*) $(a/4, a/4)$ on the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is $a/\sqrt{2}$. (*J.N.T.U., 2006 S*)
   (*iii*) $x = \pi/2$ of the curve $y = 4 \sin x - \sin 2x$ is $5\sqrt{5}/4$. (*V.T.U., 2009 S*)
4. For the curve $y = \dfrac{ax}{a + x}$, show that $\left(\dfrac{2\rho}{a}\right)^{2/3} = \left(\dfrac{x}{y}\right)^2 + \left(\dfrac{y}{x}\right)^2$. (*V.T.U., 2008*)
5. Find the radius of curvature at any point on the
   (*i*) *ellipse* : $x = a \cos \theta$, $y = b \sin \theta$. (*V.T.U., 2003*)
   (*ii*) *cycloid* : $x = a (\theta - \sin \theta)$, $y = a (1 - \cos \theta)$.
   (*iii*) *curve* : $x = a (\cos t + t \sin t)$, $y = a (\sin t - t \cos t)$.
6. Show that the radius of curvature (*i*) at the point $(a \cos^3 \theta, a \sin^3 \theta)$ on the curve
   $$x^{2/3} + y^{2/3} = a^{2/3} \text{ is } 3a \sin \theta \cos \theta.$$ (*Anna, 2009*)
   (*ii*) at the point t on the curve $x = e^t \cos t$, $y = e^t \sin t$ is $\sqrt{2}e^t$. (*Calicut, 2005*)
7. If ρ be the radius of curvature at any point P on the parabola, $y^2 = 4ax$ and S be its focus, then show that ρ^2 varies as $(SP)^3$. (*Kurukshetra, 2006*)
8. Prove that for the ellipse in pedal form $\dfrac{1}{p^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2} - \dfrac{1}{a^2 b^2}$, the radius of curvature at the point (p, r) is $\rho = a^2 b^2/p^3$. (*V.T.U., 2010 S*)
9. Show that the radius of curvature at an end of the major axis of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is equal to the semi-latus rectum. (*Osmania, 2000 S*)
10. Show that the radius of curvature at each point of the curve $x = a (\cos t + \log \tan t/2)$, $y = a \sin t$, is inversely proportional to the length of the normal intercepted between the point on the curve and the x-axis. (*J.N.T.U., 2003*)
11. Find the radius of curvature at the origin for
    (*i*) $x^3 + y^3 - 2x^2 + 6y = 0$ (*Burdwan, 2003*)
    (*ii*) $2x^4 + 3y^4 + 4x^2y + xy - y^2 + 2x = 0$
    (*iii*) $y^2 = x^2 (a + x)/(a - x)$.
12. Find the radius of the curvature at the point (r, θ) on each of the curves :
    (*i*) $r = a (1 - \cos \theta)$ (*Kurukshetra, 2005*)
    (*ii*) $r^n = a^n \cos n \theta$. (*P.T.U., 2010; J.N.T.U., 2006*)
13. For the cardioid $r = a(1 + \cos \theta)$, show that ρ^2/r is constant. (*P.T.U., 2005*)
14. Find the radius of curvature for the parabola $2a/r = 1 + \cos \theta$. (*Kurukshetra, 2006*)
15. If ρ_1, ρ_2 be the radii of curvature at the extremities of any chord of the cardioid $r = a (1 + \cos \theta)$ which passes through the pole, show that $\rho_1^2 + \rho_2^2 = 16 a^2/9$.
16. For any curve $r = f(\theta)$, prove that $\dfrac{r}{\rho} = \sin \phi \left(1 + \dfrac{d\phi}{d\theta}\right)$.

4.12 (1) CENTRE OF CURVATURE at any point P (x, y) on the curve y = f (x) is given by

$$\bar{\mathbf{x}} = \mathbf{x} - \frac{\mathbf{y_1(1 + y_1^2)}}{\mathbf{y_2}}, \quad \bar{\mathbf{y}} = \mathbf{y} + \frac{\mathbf{1 + y_1^2}}{\mathbf{y_2}}.$$

Let $C(x, y)$ be the centre of curvature and ρ the radius of curvature of the curve at $P(x, y)$ (Fig. 4.11). Draw PL and CM $\perp$s to OX and $PN \perp CM$. Let the tangent at P make an $\angle\psi$ with the x-axis. Then $\angle NCP = 90° - \angle NPC = \angle NPT = \psi$

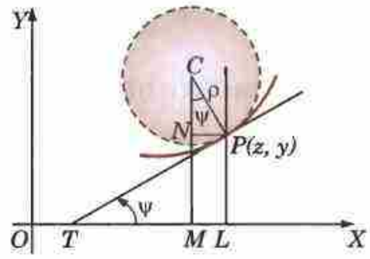


Fig. 4.11

$$\therefore \quad \bar{x} = OM = OL - ML = OL - NP$$

$$= x - \rho \sin\psi = x - \frac{(1+y_1^2)^{3/2}}{y_2} \cdot \frac{y_1}{\sqrt{(1+y_1^2)}}$$

$$\left[\because \quad \tan\psi = y_1, \quad \therefore \quad \sin\psi = \frac{y_1}{\sqrt{(1+y_1^2)}}\right]$$

$$= x - \frac{y_1(1+y_1^2)}{y_2}$$

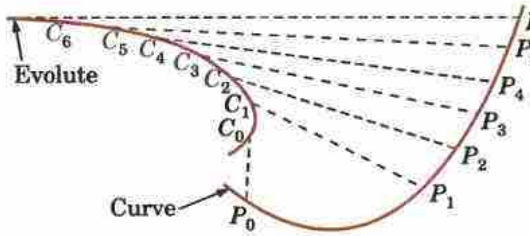


Fig. 4.12

and $$\bar{y} = MC = MN + NC = LP + \rho\cos\psi$$

$$\left[\because \quad \sec\psi = \sqrt{(1+\tan^2\psi)} = \sqrt{(1+y_1^2)}\right]$$

$$= y + \frac{(1+y_1^2)^{3/2}}{y_2} \cdot \frac{1}{\sqrt{(1+y_1^2)}} = y + \frac{1+y_1^2}{y_2}$$

Cor. Equation of the circle of curvature at P is $(x - \bar{x})^2 + (y - \bar{y})^2 = \rho^2$.

(2) Evolute. *The locus of the centre of curvature for a curve is called its* **evolute** *and the curve is called an* **involute** *of its evolute.* (*Fig. 4.12*)

Example 4.51. *Find the coordinates of the centre of curvature at any point of the parabola $y^2 = 4ax$. Hence show that its evolute is*

$$27ay^2 = 4(x - 2a)^3. \qquad \text{(V.T.U., 2000)}$$

Solution. We have $\quad 2yy_1 = 4a \quad$ *i.e.,* $\quad y_1 = 2a/y$

and $$y_2 = -\frac{2a}{y^2} \cdot y_1 = -\frac{4a^2}{y^3}$$

If $(\bar{x}, \bar{y})$ be the centre of curvature, then

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = x - \frac{2a/y(1+4a^2/y^2)}{-4a^2/y^3}$$

$$= x + \frac{y^2+4a^2}{2a} = x + \frac{4ax+4a^2}{2a} = 3x + 2a \qquad [\because \; y^2 = 4ax] \qquad ...(i)$$

and $$\bar{y} = y + \frac{1+y_1^2}{y_2} = y + \frac{1+4a^2/y^2}{-4a^2/y^2}$$

$$= y - \frac{y(y^2+4a^2)}{4a^2} = \frac{-y^3}{4a^2} = -\frac{2x^{3/2}}{\sqrt{a}} \qquad ...(ii)$$

To find the evolute, we have to eliminate x from (i) and (ii)

$$\therefore \quad (\bar{y})^2 = \frac{4x^3}{a} = \frac{4}{a}\left(\frac{\bar{x}-2a}{3}\right)^3 \quad \text{or} \quad 27a(\bar{y})^2 = 4(\bar{x} - 2a)^3.$$

Thus the locus of $(\bar{x}, \bar{y})$ *i.e.,* evolute, is $27ay^2 = 4(x - 2a)^3$.

Example 4.52. *Show that the evolute of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ is another equal cycloid.* (*Madras, 2006*)

Solution. We have $y_1 = \dfrac{dy}{d\theta} \div \dfrac{dx}{d\theta} = \dfrac{a\sin\theta}{a(1-\cos\theta)} = \cot\dfrac{\theta}{2}$.

$$y^2 = \frac{d}{dx}(y_1) = \frac{d}{d\theta}\left(\cot\frac{\theta}{2}\right) \cdot \frac{d\theta}{dx}$$

$$= -\operatorname{cosec}^2\frac{\theta}{2} \cdot \frac{1}{2} \cdot \frac{1}{a(1-\cos\theta)} = -\frac{1}{4a\sin^4\theta/2}$$

If $(\bar{x}, \bar{y})$ be the centre of curvature, then

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = a(\theta - \sin\theta) + \cot\frac{\theta}{2}\left(-4a\sin^4\frac{\theta}{2}\right)\left(1+\cot^2\frac{\theta}{2}\right)$$

$$= a(\theta - \sin\theta) + \frac{\cos\theta/2}{\sin\theta/2} \cdot 4a\sin^4\frac{\theta}{2} \cdot \operatorname{cosec}^2\frac{\theta}{2}$$

$$= a(\theta - \sin\theta) + 4a\sin\theta/2\cos\theta/2 = a(\theta - \sin\theta) + 2a\sin\theta = a(\theta + \sin\theta)$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2} = a(1-\cos\theta) + \left(1+\cot^2\frac{\theta}{2}\right)\left(-4a\sin^4\frac{\theta}{2}\right)$$

$$= a(1-\cos\theta) - 4a\sin^4\theta/2 \cdot \operatorname{cosec}^2\theta/2$$

$$= a(1-\cos\theta) - 4a\sin^2\theta/2$$

$$= a(1-\cos\theta) - 2a(1-\cos\theta) = -a(1-\cos\theta)$$

Hence the locus of $(\bar{x}, \bar{y})$ *i.e.*, the evolute, is given by

$x = a(\theta + \sin\theta)$, $y = -a(1-\cos\theta)$ which is another equal cycloid.

(3) Chord or curvature *at a given point of a curve*

(*i*) parallel to x-axis $= 2\rho\sin\psi$

(*ii*) parallel to y-axis $= 2\rho\cos\psi$

Consider the circle of curvature at a given point P on a curve. Let C be the centre and ρ the radius of curvature at P so that $PQ = 2\rho$. (Fig. 4.13)

Let PL, PM be the chords of curvature parallel to the axes of x and y respectively. Let the tangent PT make an $\angle\psi$ with the x-axis so that $\angle LQP = \angle QPM = \psi$.

Then from the rt. $\angle$ed ΔPLQ,

$$PL = 2\rho\sin\psi\sigma$$

and $$PM = 2\rho\cos\psi.$$

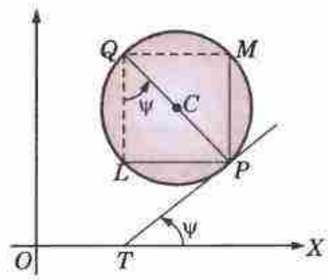


Fig. 4.13

4.13 (1) ENVELOPE

The equation $x\cos\alpha + y\sin\alpha = 1$ ...(1)

represents a straight line for a given value of α. If different values are given to α, we get different straight lines. All these straight lines thus obtained are said to constitute a family of straight lines.

In general, the *curves corresponding to the equation $f(x, y, \alpha) = 0$ for different values of α, constitute a* **family of curves** *and α is called the* **parameter** *of the family.*

The envelope of a family of curves is the curve which touches each member of the family. For example, we know that all the straight lines of the family (1) touch the circle

$$x^2 + y^2 = 1 \qquad ...(2)$$

i.e., the envelope of the family of lines (1) is the circle (2)—Fig. 4.14, which may also be seen as the locus of the ultimate points of intersection of the consecutive members of the family of lines (1). This leads to the following :

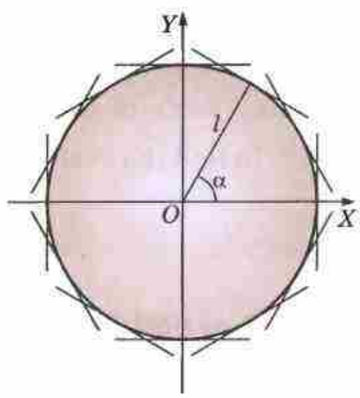


Fig. 4.14

Def. *If $f(x, y, \alpha) = 0$ and $f(x, y, \alpha + \delta\alpha) = 0$ be two consecutive members of a family of curves, then the locus of their ultimate points of intersection is called the* **envelope** *of that family.*

(2) Rule to find the envelope of the family of curves f(x, y, α) = 0 :

Eliminate α *from* $f(x, y, \alpha) = 0$ *and* $\dfrac{\partial f(x, y, \alpha)}{\partial \alpha} = 0.$

Example 4.53. *Find the envelope of the family of lines* $y = mx + \sqrt{(1 + m^2)}$, *m being the parameter.*

Solution. We have $(y - mx)^2 = 1 + m^2$ ...(*i*)

Differentiating (*i*) partially with respect to m,

$$2(y - mx)(-x) = 2m \quad \text{or} \quad m = xy/(x^2 - 1) \qquad ...(ii)$$

Now eliminating m from (*i*) and (*ii*)

Substituting the value of m in (*i*), we get

$$\left(y - \frac{x^2 y}{x^2 - 1}\right)^2 = 1 + \left(\frac{xy}{x^2 - 1}\right)^2 \quad \text{or} \quad y^2 = (x^2 - 1)^2 + x^2y^2$$

or $\qquad x^2 + y^2 = 1 \qquad$ which is the required equation of the envelope.

Obs. Sometimes the equation to the family of curves contains two parameters which are connected by a relation. In such cases, we eliminate one of the parameters by means of the given relation, then proceed to find the envelope.

Example 4.54. *Find the envelope of a system of concentric and coaxial ellipses of constant area.*

Solution. Taking the common axes of the system of ellipses as the coordinate axes, the equation to an ellipse of the family is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ where } a \text{ and } b \text{ are the parameters.} \qquad ...(i)$$

The area of the ellipse = πab which is given to be constant, say = πc^2.

$$\therefore \qquad ab = c^2 \quad \text{or} \quad b = c^2/a. \qquad ...(ii)$$

Substituting in (*i*), $\dfrac{x^2}{a^2} + \dfrac{y^2}{(c^2/a^2)} = 1$ or $x^2 a^{-2} + (y^2/c^4)\, a^2 = 0$ ...(*iii*)

which is the given family of ellipses with a as the only parameter.

Differentiating partially (*iii*) with respect to a,

$$-2x^2 a^{-3} + 2(y^2/c^4)\, a = 0 \quad \text{or} \quad a^2 = c^2\, x/y \qquad ...(iv)$$

Eliminate a from (*iii*) and (*iv*).

Substituting the value of a^2 in (*iii*), we get

$$x^2(y/c^2x) + (y^2/c^4)(c^2x/y) = 1 \text{ or } 2xy = c^2$$

which is the required equation of the envelope. *P*

(3) Evolute *of a curve is the envelope of the normals to that curve* (*Fig. 4.12*)

Example 4.55. *Find the evolute of the parabola* $y^2 = 4ax$. (*Madras, 2003*)

Solution. Any normal to the parabola is $y = mx - 2am - am^3$ ...(*i*)

Differentiating it with respect to m partially,

$$0 = x - 2a - 3am^2 \text{ or } m = [(x - 2a)/3a]^{1/2}$$

Substituting this value of m in (*i*),

$$y = \left(\frac{x - 2a}{3a}\right)^{1/2} \left[x - 2a - a \cdot \frac{x - 2a}{3a}\right]$$

Squaring both sides, we have

$$27ay^2 = 4(x - 2a)^3$$

which is the evolute of the parabola. (*cf.* Example 4.51).

PROBLEMS 4.12

1. Find the coordinates of the centre of curvature at $(at^2, 2at)$ on the parabola $y^2 = 4ax$. *(V.T.U., 2000 S)*
2. If the centre of curvature of the ellipse $x^2/a^2 + y^2/b^2 = 1$ at one end of the minor axis lies at the other end, then show that the eccentricity of the ellipse is $1/\sqrt{2}$. *(Anna, 2005 S ; Madras, 2003)*
3. Show that the equation of the evolute of the
   (i) *parabola* $x^2 = 4ay$ is $4(y - 2a)^3 = 27ax^2$. *(Anna, 2009)*
   (ii) *ellipse* $x = a \cos\theta, y = b \sin\theta$ (*i.e.*, $x^2/a^2 + y^2/b^2 = 1$) is
   $$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}.$$
   (iii) *rectangular hyperbola* $xy = c^2$, (*i.e.*, $x = ct, y = c/t$) is
   $$(x + y)^{2/3} - (x - y)^{2/3} = (4c)^{2/3}.$$ *(Anna, 2003)*
4. Find the evolute of (i) cycloid $x = a(t + \sin t), y = a(1 - \cos t)$
   (ii) the curve $x = a(\cos\theta + \theta \sin\theta), y = a(\sin\theta - \theta\cos\theta)$. *(Anna, 2009 S)*
5. Find the evolute of the curve $x = a \cos^3\theta, y = a \sin^3\theta$ *i.e.*, $x^{2/3} + y^{2/3} = a^{2/3}$. *(Osmania, 2002)*
6. Show that the evolute of the curve $x = a(\cos t + \log\tan t/2), y = a \sin t$ is $y = a \cosh x/a$. *(Anna, 2005 S)*
7. Find the circle of curvature at the point (i) $(a/4, a/4)$ of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$.
   (ii) $(3/2, 3/2)$ of the curve $x^3 + y^3 = 3xy$ *(Anna, 2009 ; Madras, 2006 ; Calicut, 2005)*
8. Show that the circle of curvature at the origin for the curve $x + y = ax^2 + by^2 + ex^3$ is
   $$(a + b)(x^2 + y^2) = 2(x + y).$$ *(Nagpur, 2009)*
9. If C_x, C_y be the chords of curvature parallel to the axes at any point on the curve $y = ae^{x/a}$, prove that
   $$\frac{1}{C_x^2} + \frac{1}{C_y^2} = \frac{1}{2aC_x}.$$
10. In the curve $y = a \cosh x/a$, prove that the chord of curvature parallel to y-axis is the double the ordinate.

Find the envelope of the following family of lines :

11. $y = mx + a/m$, m being the parameter. *(Madras, 2006)*
12. $\dfrac{x}{a}\cos\alpha + \dfrac{y}{b}\sin\alpha = 1$, α being the parameter.
13. $y = mx - 2am - am^3$.
14. $y = mx + \sqrt{(a^2m^2 + b^2)}$, m being the parameter. *(Anna, 2009)*
15. Find the envelope of the family of parabolas $y = x\tan\alpha - \dfrac{gx^2}{2u^2\cos\alpha}$, α being the parameter.
16. Find the envelope of the straight line $x/a + y/b = 1$, where the parameters a and b are connected by the relation :
   (i) $a + b = c$. (ii) $ab = c^2$
   (iii) $a^2 + b^2 = c^2$.
17. Find the envelope of the family of ellipses $x^2/a^2 + y^2/b^2 = 1$ for which $a + b = c$. *(Madras, 2006)*

Prove that the evolute of the

18. *ellipse* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$. *(J.N.T.U., 2006 ; Anna, 2005)*
19. *hyperbola* $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is $(ax)^{2/3} - (by)^{2/3} = (a^2 + b^2)^{2/3}$. *(Anna, 2009)*
20. *parabola* $x^2 = 4by$ is $27bx^2 = 4(y - 2b)^3$.

4.14 (1) INCREASING AND DECREASING FUNCTIONS

In the function $y = f(x)$, *if y increases as x increases* (as at A), *it is called an* **increasing function of** x. On the contrary, *if y decreases as x increases* (as at C), *it is called a* **decreasing function** *of* x.

Let the tangent at any point on the graph of the function make an $\angle\psi$ with the x-axis (Fig. 4.15) so that

$$dy/dx = \tan\psi$$

At any point such as A, where the function is increasing $\angle\psi$ is acute *i.e.*, dy/dx is positive. At a point such as C, where the function is decreasing $\angle\psi$ is obtuse *i.e.*, dy/dx is negative.

Hence the *derivative of an increasing function is + ve*, and *the derivative of a decreasing function is – ve.*

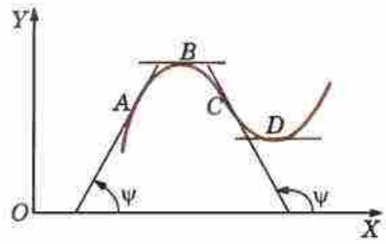


Fig. 4.15

Obs. *If the derivative is zero* (as at B or D), *then y is neither increasing nor decreasing.* In such cases, we say *that the function is* **stationary.**

(2) Concavity, Convexity and Point of Inflexion

(*i*) *If a portion of the curve on both sides of a point, however small it may be, lies above the tangent* (*as at D*), *then the curve is said to be* **concave upwards** at D where d^2y/dx^2 is positive.

(*ii*) *If a portion of the curve on both sides of a point lies below the tangent* (as at B), *then the curve is said to be* **Convex upwards** at B where d^2y/dx^2 is negative.

(*iii*) *If the two portions of the curve lie on different sides of the tangent thereat* (*i.e.*, the curve crosses the tangent (as at C), *then the point C* is said to be a **point of inflexion** of the curve.

At a point of inflexion $\dfrac{d^2y}{dx^2} = 0$ and $\dfrac{d^3y}{dx^3} \neq 0.$

4.15 (1) MAXIMA AND MINIMA

Consider the graph of the continuous function $y = f(x)$ in the interval (x_1, x_2) (Fig. 4.16). Clearly the point P_1 is the highest in its own immediate neighbourhood. So also is P_3. At each of these points P_1, P_3 the function is said to have a *maximum* value.

On the other hand, the point P_2 is the lowest in its own immediate neighbourhood. So also is P_4. At each of these points P_2, P_4 the function is said to have a *minimum* value.

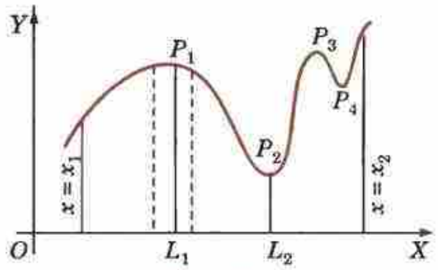


Fig. 4.16

Thus, we have

Def. *A function f(x) is said to have a* **maximum** *value at x = a, if there exists a small number h, however small, such that f(a) > both f(a – h) and f(a + h).*

A function f(x) is said to have a **minimum** *value at x = a, if there exists a small number h, however small, such that f(a) < both f(a – h) and f(a + h).*

Obs. 1. *The maximum and minimum values of a function taken together are called its* **extreme** *values and the points at which the function attains the extreme values are called the* **turning points** *of the function.*

Obs. 2. *A maximum or minimum value of a function is not necessarily the greatest or least value of the function in any finite interval.* The maximum value is simply the greatest value in the immediate neighbourhood of the maxima point or the minimum value is the least value in the immediate neighbourhood of the minima point. In fact, *there may be several maximum and minimum values of a function in an interval and a minimum value may be even greater than a maximum value.*

Obs. 3. It is seen from the Fig. 4.16 that *maxima and minima values occur alternately.*

(2) Conditions for maxima and minima. At each point of extreme value, it is seen from Fig. 4.16 that the tangent to the curve is parallel to the x-axis, *i.e.*, its slope (= dy/dx) is zero. Thus if *the function is maximum or minimum at $x = a$, then $(dy/dx)_a = 0$.*

Around a maximum point say, P_1 $(x = a)$, the curve is increasing in a small interval $(a - h, a)$ before L_1 and decreasing in $(a, a + h)$ after L_1 where h is positive and small.

i.e., in $(a - h, a)$, $dy/dx \geq 0$; at $x = a$, $dy/dx = 0$ and in $(a, a + h)$, $dy/dx \leq 0$.

Thus dy/dx (which is a function of x) changes sign from positive to negative in passing through P_1, *i.e.*, it is a decreasing function in the interval $(a - h, a + h)$ and therefore, its derivative d^2y/dx^2 is negative at $P_1(x = a)$.

Similarly, around a minimum point say P_2, dy/dx changes sign from negative to positive in passing through P_2, *i.e.*, it is an increasing function in the small interval around L_2 and therefore its derivative d^2y/dx^2 is positive at P_2.

Hence (*i*) *$f(x)$ is maximum at $x = a$ if $f'(a) = 0$ and $f''(a)$ is – ve [i.e., $f'(a)$ changes sign from + ve to – ve]*

(*ii*) *$f(x)$ is minimum at $x = a$, if $f'(a) = 0$ and $f''(a)$ is + ve [i.e., $f'(a)$ changes sign from – ve to + ve]*

Obs. A maximum or a minimum value is a stationary value but a stationary value may neither be a maximum nor a minimum value.

(3) Procedure for finding maxima and minima

(*i*) *Put the given function = $f(x)$*

(*ii*) *Find $f'(x)$ and equate it to zero. Solve this equation and let its roots be a, b, c, ...*

(*iii*) *Find $f''(x)$ and substitute in it by turns $x = a, b, c, ...$*

If $f''(a)$ is – ve, $f(x)$ is maximum at $x = a$.

If $f''(a)$ is + ve, $f''(x)$ is minima at $x = a$.

(*iv*) Sometimes $f''(x)$ may be difficult to find out or $f''(x)$ may be zero at $x = a$. In such cases, see if *$f'(x)$ changes sign from + ve to – ve as x passes through a, then $f(x)$ is maximum at $x = a$.*

If $f'(x)$ changes sign from – ve to + ve as x passes through a, $f(x)$ is minimum at $x = a$.

If $f'(x)$ does not change sign while passing through $x = a$, $f(x)$ is neither maximum nor minimum at $x = a$.

Example 4.56. *Find the maximum and minimum values of $3x^4 - 2x^3 - 6x^2 + 6x + 1$ in the interval (0, 2).*

Solution. Let $f(x) = 3x^4 - 2x^3 - 6x^2 + 6x + 1$

Then $f'(x) = 12x^3 - 6x^2 - 12x + 6 = 6(x^2 - 1)(2x - 1)$

$\therefore$ $f'(x) = 0$ when $x = \pm 1, \frac{1}{2}$.

So in the interval (0, 2) $f(x)$ can have maximum or minimum at $x = \frac{1}{2}$ or 1.

Now $f''(x) = 36x^2 - 12x - 12 = 12(3x^2 - x - 1)$ so that $f''\left(\frac{1}{2}\right) = -9$ and $f''(1) = 12$.

$\therefore$ $f(x)$ has a maximum at $x = \frac{1}{2}$ and a minimum at $x = 1$.

Thus the maximum value $= f\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right)^4 - 2\left(\frac{1}{2}\right)^3 - 6\left(\frac{1}{2}\right)^2 + 6\left(\frac{1}{2}\right) + 1 = 2\frac{7}{16}$

and the minimum value $= f(1) = 3(1)^4 - 2(1)^3 - 6(1)^2 + 6(1) + 1 = 2$.

Example 4.57. *Show that $\sin x (1 + \cos x)$ is a maximum when $x = \pi/3$.* (*Bhopal, 2009 ; Rajasthan, 2005*)

Solution. Let $f(x) = \sin x (1 + \cos x)$

Then $f'(x) = \cos x (1 + \cos x) + \sin x (-\sin x)$

$= \cos x (1 + \cos x) - (1 - \cos^2 x) = (1 + \cos x)(2 \cos x - 1)$

$\therefore$ $f'(x) = 0$ when $\cos x = \frac{1}{2}$ or -1 *i.e.*, when $x = \pi/3$ or π.

Now $f''(x) = -\sin x (2 \cos x - 1) + (1 + \cos x)(-2 \sin x) = -\sin x (4 \cos x + 1)$

so that $f''(\pi/3) = -3\sqrt{2}/2$ and $f''(\pi) = 0$.

Thus $f(x)$ has a maximum at $x = \pi/3$.

Since $f''(\pi)$ is 0, let us see whether $f'(x)$ changes sign or not.

When x is slightly $< \pi$, $f'(x)$ is – ve, then when x is slightly $> \pi$, $f'(x)$ is again – ve *i.e.*, $f'(x)$ does not change sign as x passes through π. So $f(x)$ is neither maximum nor minimum at $x = \pi$.

(4) Practical Problems

In many problems, the function (whose maximum or minimum value is required) is not directly given. It has to be formed from the given data. If the function contains two variables, one of them has to be eliminated with the help of the other conditions of the problem. A number of problems deal with triangles, rectangles, circles, spheres, cones, cylinders etc. The student is therefore, advised to remember the formulae for areas, volumes, surfaces etc. of such figures.

Example 4.58. *A window has the form of a rectangle surmounted by a semi-circle. If the perimeter is 40 ft., find its dimensions so that the greatest amount of light may be admitted.* (*Madras, 2000 S*)

Solution. The greatest amount of light may be admitted means that the area of the window may be maximum.

Let x ft. be the radius of the semi-circle so that one side of the rectangle is $2x$ ft. (Fig. 4.17). Let the other side of the rectangle y ft. Then the perimeter of the whole figure

$$= \pi x + 2x + 2y = 40 \text{ (given) and the area } A = \tfrac{1}{2}\pi x^2 + 2xy. \qquad ...(i)$$

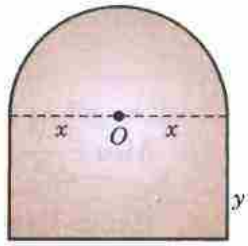


Fig. 4.17

Here A is a function of two variables x and y. To express A in terms of one variable x (say), we substitute the value of y from (i) in it.

$$\therefore \qquad A = \tfrac{1}{2}\pi x^2 + x[40 - (\pi + 2)x] = 40\,x - \left(\frac{\pi}{2} + 2\right)x^2$$

Then $$\frac{dA}{dx} = 40 - (\pi + 4)x$$

For A to be maximum or minimum, we must have $dA/dx = 0$ *i.e.*, $40 - (\pi + 4)x = 0$

or $$x = 40/(\pi + 4)$$

$\therefore$ From (i), $$y = \tfrac{1}{2}[40 - (\pi + 2)x] = \tfrac{1}{2}[40 - (\pi + 2)\,40/(\pi + 4)] = 40/(\pi + 4) \textit{ i.e., } x = y$$

Also $$\frac{d^2A}{dx^2} = -(\pi + 4), \text{ which is negative.}$$

Thus the area of the window is maximum when the radius of the semi-circle is equal to the height of the rectangle.

Example 4.59. *A rectangular sheet of metal of length 6 metres and width 2 metres is given. Four equal squares are removed from the corners. The sides of this sheet are now turned up to form an open rectangular box. Find approximately, the height of the box, such that the volume of the box is maximum.*

Solution. Let the side of each of the squares cut off be x m so that the height of the box is x m and the sides of the base are $6 - 2x$, $2 - 2x$ m (Fig. 4.18).

$\therefore$ Volume V of the box

$$= x(6 - 2x)(2 - 2x) = 4(x^3 - 4x^2 + 3x)$$

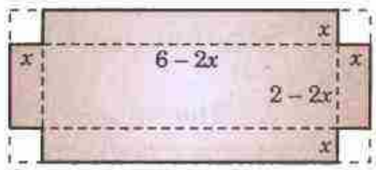


Fig. 4.18

Then $$\frac{dV}{dx} = 4(3x^2 - 8x + 3)$$

For V to be maximum or minimum, we must have

$$dV/dx = 0 \textit{ i.e., } 3x^2 - 8x + 3 = 0$$

$$\therefore \qquad x = \frac{8 \pm \sqrt{[64 - 4 \times 3 \times 3]}}{6} = 2.2 \text{ or } 0.45 \text{ m.}$$

The value $x = 2.2$ m is inadmissible, as no box is possible for this value.

Also $\dfrac{d^2V}{dx^2} = 4(6x - 8)$, which is – ve for $x = 0.45$ m.

Hence the volume of the box is maximum when its height is 45 cm.

Example 4.60. *Show that the right circular cylinder of given surface (including the ends) and maximum volume is such that its height is equal to the diameter of the base.*

Solution. Let r be the radius of the base and h, the height of the cylinder.

Then given surface $\quad S = 2\pi rh + 2\pi r^2 \quad ...(i) \quad$ and the volume $V = \pi r^2 h \quad ...(ii)$

Hence V is a function of two variables r and h. To express V in terms of one variable only (say r), we substitute the value of h from (i) in (ii).

Then $$V = \pi r^2\left(\frac{S - 2\pi r^2}{2\pi r}\right) = \frac{1}{2}Sr - \pi r^3 \quad \therefore \quad \frac{dV}{dr} = \frac{1}{2}S - 3\pi r^2.$$

For V to be maximum or minimum, we must have $dV/dr = 0$,

i.e., $$\tfrac{1}{2}S - 3\pi r^2 = 0 \quad \text{or} \quad r = \sqrt{(S/6\pi)}.$$

Also $\dfrac{d^2V}{dr^2} = -6\pi r$, which is negative for $r = \sqrt{(S/6\pi)}$.

Hence V is maximum for $r = \sqrt{(S/6\pi)}$.

i.e., for $6\pi r^2 = S = 2\pi rh + 2\pi r^2$ *i.e.*, for $h = 2r$, which proves the required result. [By (*i*)]

Example 4.61. *Show that the diameter of the right circular cylinder of greatest curved surface which can be inscribed in a given cone is equal to the radius of the cone.*

Solution. Let r be the radius OA of the base and α the semi-vertical angle of the given cone (Fig. 4.19). Inscribe a cylinder in it with base-radius $OL = x$.

Then the height of the cylinder LP

$$= LA \cot\alpha = (r - x)\cot\alpha$$

$\therefore$ The curved surface S of the cylinder

$$= 2\pi x.\,LP = 2\pi x(r - x)\cot\alpha$$
$$= 2\pi\cot\alpha\,(rx - x^2)$$

$\therefore$ $$\frac{dS}{dx} = 2\pi\cot\alpha\,(r - 2x) = 0 \text{ for } x = r/2.$$

and $$\frac{d^2S}{dx^2} = -4\pi\cot\alpha$$

Hence S is maximum when $x = r/2$.

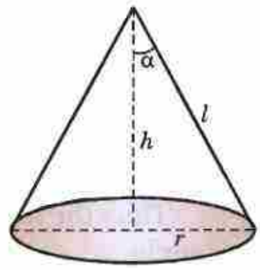


Fig. 4.19

Example 4.62. *Find the altitude and the semi-vertical angle of a cone of least volume which can be circumscribed to a sphere of radius a.*

Solution. Let h be the height and α the semi-vertical angle of the cone so that its radius $BD = h\tan\alpha$ (Fig. 4.20).

$\therefore$ The volume V of the cone is given by

$$V = \frac{1}{3}\pi(h\tan\alpha)^2 h = \frac{1}{3}\pi h^3\tan^2\alpha.$$

Now we must express $\tan\alpha$ in terms of h.

In the rt. $\angle d\ \Delta AEO$,

$$EA = \sqrt{(OA^2 - a^2)} = \sqrt{[(h-a)^2 - a^2]} = \sqrt{(h^2 - 2ha)}$$

$\therefore$ $$\tan\alpha = \frac{EO}{EA} = \frac{a}{\sqrt{(h^2 - 2ha)}}$$

Thus $$V = \frac{1}{3}\pi h^3 \cdot \frac{a^2}{h^2 - 2ha} = \frac{1}{3}\pi a^3 \cdot \frac{h^2}{h - 2a}$$

$\therefore$ $$\frac{dV}{dh} = \frac{1}{3}\pi a^2 \cdot \frac{(h-2a)2h - h^2 \cdot 1}{(h-2a)^2} = \frac{1}{3}\pi a^2 \cdot \frac{h(h-4a)}{(h-2a)^2}$$

Thus $\dfrac{dV}{dh} = 0$ for $h = 4a$, the other value ($h = 0$) being not possible.

Also dV/dh is – ve when h is slightly $< 4a$, and it is + ve when h is slightly $> 4a$.

Hence V is minimum (*i.e.* least) when $h = 4a$

and $$\alpha = \sin^{-1}\left(\frac{a}{OA}\right) = \sin^{-1}\left(\frac{a}{3a}\right) = \sin^{-1}\frac{1}{3}.$$

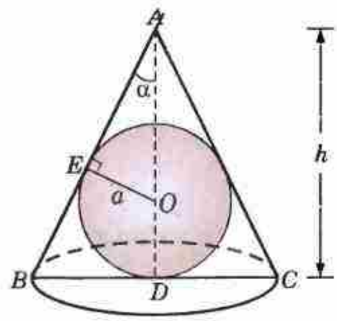


Fig. 4.20

Example 4.63. *Find the volume of the largest possible right-circular cylinder that can be inscribed in a sphere of radius a.*

Solution. Let O be the centre of the sphere of radius a. Construct a cylinder as shown in Fig. 4.21. Let $OA = r$.

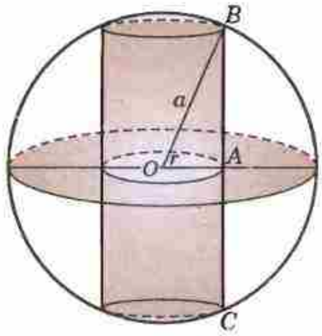


Fig. 4.21

Then $$AB = \sqrt{(OB^2 - OA^2)} = \sqrt{(a^2 - r^2)}$$

$\therefore$ Height h of the cylinder $= 2 \cdot AB = 2\sqrt{(a^2 - r^2)}$.

Thus volume V of the cylinder

$$= \pi r^2 h = 2\pi r^2 \sqrt{(a^2 - r^2)}$$

$$\therefore \quad \frac{dV}{dr} = 2\pi \left\{2r\sqrt{(a^2 - r^2)} + r^2 \cdot \frac{1}{2}(a^2 - r^2)^{-1/2}(-2r)\right\}$$

$$= \frac{2\pi r(2a^2 - 3r^2)}{\sqrt{(a^2 - r^2)}}$$

The $dV/dr = 0$ when $r^2 = 2a^2/3$, the other value ($r = 0$) being not admissible.

Now $$\frac{d^2V}{dr^2} = 2\pi \frac{\sqrt{(a^2 - r^2)}(2a^2 - 9r^2) - r(2a^2 - 3r^2) \times \frac{1}{2}(a^2 - r^2)^{-1/2} . (-2r)}{(a^2 - r^2)}$$

$$= 2\pi \frac{(a^2 - r^2)(2a^2 - 9r^2) + r^2(2a^2 - 3r^2)}{(a^2 - r^2)^{3/2}}$$ which is – ve for $r^2 = 2a^2/3$.

Hence V is maximum for $r^2 = 2a^2/3$ and maximum volume

$$= 2\pi r^2 \sqrt{(a^2 - r^2)} = 4\pi a^3/3\sqrt{3}.$$

Example 4.64. *Assuming that the petrol burnt (per hour) in driving a motor boat varies as the cube of its velocity, show that the most economical speed when going against a current of c miles per hour is $\frac{3}{2}$c miles per hour.*

Solution. Let v m.p.h. be the velocity of the boat so that its velocity relative to water (when going against the current) is $(v - c)$ m.p.h.

$\therefore$ Time required to cover a distance of s miles $= \dfrac{s}{v - c}$ hours.

Since the petrol burnt per hour $= kv^3$, k being a constant.

$\therefore$ The total petrol burnt, y, is given by

$$y = k\frac{v^3 s}{v - c} = ks\frac{v^3}{v - c} \quad \therefore \quad \frac{dy}{dv} = ks \cdot \frac{(v - c)3v^2 - v^3 \cdot 1}{(v - c)^2}$$

$$= ks \cdot \frac{v^2(2v - 3c)}{(v - c)^2}$$

Thus $dy/dv = 0$ for $v = 3c/2$, the other value ($v = 0$) is inadmissible.

Also dy/dv is – ve, when v is slightly < $3c/2$ and it is + ve, when v is slightly > $3c/2$.

Hence y is minimum for $v = 3c/2$.

PROBLEMS 4.13

1. (*i*) Test the curve $y = x^4$ for points of inflexion ? (*Burdwan, 2003*)

   (*ii*) Show that the points of inflexion of the curve $y^2 = (x - a)^2 (x - b)$ lie on the straight line

   $3x + a = 4b$. (*Rajasthan, 2005*)

2. The function $f(x)$ defined by $f(x) = a/x + bx, f(2) = 1$, has an extremum at $x = 2$. Determine a and b. Is this point (2, 1), a point of maximum or minimum on the graph of $f(x)$?
3. Show that $\sin^p\theta \cos^q\theta$ attains a maximum when $\theta = \tan^{-1}(p/q)$. *(Rajasthan, 2006)*
4. If a beam of weight w per unit length is built-in horizontally at one end A and rests on a support O at the other end, the deflection y at a distance x from O is given by
$$EIy = \frac{w}{48}(2x^4 - 3lx^3 + l^3x),$$
where l is the distance between the ends. Find x for y to be maximum.
5. The horse-power developed by an aircraft travelling horizontally with velocity v feet per second is given by
$$H = \frac{aw^2}{v} + bv,$$
where a, b and w are constants. Find for what value of v the horse-power is maximum.
6. The velocity of waves of wave-length λ on deep water is proportional to $\sqrt{(\lambda/a + a/\lambda)}$, where a is a certain constant, prove that the velocity is minimum when $\lambda = a$.
7. In a submarine telegraph cable, the speed of signalling varies as $x^2 \log_e(1/x)$, where x is the ratio of the radius of the core to that of the covering. Show that the greatest speed is attained when this ratio is $1/\sqrt{e}$.
8. The efficiency e of a screw-jack is given by $e = \tan\theta/\tan(\theta + \alpha)$, where α is a constant. Find θ if this efficiency is to be maximum. Also find the maximum efficiency.
9. Show that of all rectangles of given area, the square has the least parameter.
10. Find the rectangle of greatest perimeter that can be inscribed in a circle of radius a.
11. A gutter of rectangular section (open at the top) is to be made by bending into shape of a rectangular strip of metal. Show that the capacity of the gutter will be greatest if its width is twice its depth.
12. Show that the triangle of maximum area that can be inscribed in a given circle is an equilateral triangle.
13. An open box is to be made from a rectangular piece of sheet metal 12 cms × 18 cms, by cutting out equal squares from each corner and folding up the sides. Find the dimensions of the box of largest volume that can be made in this manner.
14. An open tank is to be constructed with a square base and vertical sides to hold a given quantity of water. Find the ratio of its depth to the width so that the cost of lining the tank with lead is least.
15. A corridor of width b runs perpendicular to a passageway of width a. Find the longest beam which can be moved in a horizontal plane along the passageway into the corridor ?
16. One corner of a rectangular sheet of paper of width a is folded so as to reach the opposite edge of the sheet. Find the minimum length of the crease.
17. Show that the height of closed cylinder of given volume and least surface is equal to its diameter.
18. Prove that a conical vessel of a given storage capacity requires the least material when its height is $\sqrt{2}$ times the radius of the base. *(Warangal, 1996)*
19. Show that the semi-vertical angle of a cone of maximum volume and given slant height is $\tan^{-1}\sqrt{2}$.
20. The shape of a hole bored by a drill is cone surmounting a cylinder. If the cylinder be of height h and radius r and the semi-vertical angle of the cone be α where $\tan\alpha = h/r$, show that for a total fixed depth H of the hole, the volume removed is maximum if $h = \frac{H}{6}(1 + \sqrt{7})$. *(Raipur, 2005)*
21. A cylinder is inscribed in a cone of height h. If the volume of the cylinder is maximum, show that its height is $h/3$.
22. Show that the volume of the biggest right circular cone that can be inscribed in a sphere of given radius is 8/27 times that of the sphere.
23. A given quantity of metal is to be cast into a half-cylinder with a rectangular base and semi-circular ends. Show that in order that the total surface area may be a minimum, the ratio of the length of the cylinder to the diameter of its semi-circular ends is $\pi/(\pi + 2)$.
24. A person being in a boat a miles from the nearest point of the beach, wishes to reach as quickly as possible a point b miles from that point along the shore. The ratio of his rate of walking to his rate of rowing is $\sec\alpha$. Prove that he should land at a distance $b - a\cot\alpha$ from the place to be reached.
25. The cost per hour of propelling a steamer is proportional to the cube of her speed through water. Find the relative speed at which the steamer should be run against a current of 5 km per hour to make a given trip at the least cost.

4.16 ASYMPTOTES

(1) Def. *An asymptote of a curve is a straight line at a finite distance from the origin, to which a tangent to the curve tends as the point of contact recedes to infinity.*

In other words, an asymptote is a straight line which cuts a curve on two points, at an infinite distance from the origin and yet is not itself wholly at infinity.

(2) Asymptotes parallel to axes. Let the equation of the curve arranged according to powers of x be

$$a_0x^n + (a_1y + b_1)x^{n-1} + (a_2y^2 + b_2y + c_2)x^{n-2} + \dots 0 \qquad \dots(1)$$

If $a_0 = 0$ and y be so chosen that $a_1y + b_1 = 0$, then the coefficients of two highest powers of x in (1) vanish and therefore, two of its roots are infinite. Hence $a_1y + b_1 = 0$ is an asymptote of (1) which is parallel to x-axis.

Again if a_0, a_1, b_1 are all zero and if y be so chosen that $a_2y^2 + b_2y + c_2 = 0$, then three roots of (1) become infinite. Therefore, the two lines represented by $a_2y^2 + b_2y + c_2 = 0$ are the asymptotes of (1) which are parallel to x-axis, and so on.

Similarly, for asymptotes parallel to y-axis.

Thus we have the following **rules :**

I. *To find the asymptotes parallel to x-axis, equate to zero the coefficient of the highest power of x in the equation,* provided this is not merely a constant.

II. *To find the asymptotes parallel to y-axis, equate to zero the coefficient of the highest power of y in the equation,* provided this is not merely a constant.

Example 4.65. *Find the asymptotes of the curve*

$$x^2y^2 - x^2y - xy^2 + x + y + 1 = 0.$$

Solution. The highest power of x is x^2 and its coefficient is $y^2 - y$.

$\therefore$ The asymptotes parallel to the x-axis are given by

$$y(y-1) = 0 \text{ i.e., by } y = 0 \text{ and } y = 1.$$

The highest power of y is y^2 and its coefficient is $x^2 - x$.

$\therefore$ The asymptotes parallel to the y-axis are given by

$$x(x-1) = 0 \text{ i.e., by } x = 0 \text{ and } x = 1.$$

Hence the asymptotes are $x = 0, x = 1, y = 0$ and $y = 1$.

(3) Inclined asymptotes. Let the equation of the curve be of the form

$$x^n\phi_n(y/x) + x^{n-1}\phi_{n-1}(y/x) + x^{n-2}\phi_{n-2}(y/x) + \dots = 0 \qquad \dots(1)$$

where $\phi_r(y/x)$ is an expression of degree r is y/x.

To find where this curve is cut by the line $y = mx + c$, $\qquad \dots(2)$

put $y/x = m + c/x$ in (1). The resulting equation is

$$x^n\phi_n(m + c/x) + x^{n-1}\phi_{n-1}(m + c/x) + x^{n-2}\phi_{n-2}(m + c/x) + \dots = 0$$

which gives the abscissae of the points of intersection.

Expanding each of the ϕ-functions by Taylor's series,

$$x^n\left\{\phi_n(m) + \frac{c}{x}\phi_n'(m) + \frac{c^2}{2!\,x^2}\phi_n''(m) + \dots\right\} + x^{n-1}\left\{\phi_{n-1}(m) + \frac{c}{x}\phi'_{n-1}(m) + \dots\right\}$$
$$+ x^{n-2}\{\phi_{n-2}(m) + \dots\} = 0$$

or

$$x^n\phi_n(m) + x^{n-1}\{c\phi'_n(m) + \phi_{n-1}(m)\}$$
$$+ x^{n-2}\left\{\frac{c^2}{2!}\phi_n''(m) + c\phi'_{n-1}(m) + \phi_{n-2}(m)\right\} + \dots = 0 \qquad \dots(3)$$

If the line (2) is an asymptote to the curve, it cuts the curve in two points at infinity *i.e.*, the equation (3) has two infinite roots for which the coefficients of two highest terms should be zero.

i.e., $\qquad \phi_n(m) = 0 \quad \dots(4) \quad$ and $\quad c\,\phi'_n(m) + \phi_{n-1}(m) = 0 \qquad \dots(5)$

If the roots of (4) be $m_1, m_2, \dots m_n$, then the corresponding values of c (*i.e.* $c_1, c_2, \dots\dots c_n$) are given by (5). Hence the asymptotes are

$$y = m_1x + c_1, y = m_2x + c_2, \dots\dots, y = m_nx + c_n.$$

Obs. If (4) gives two equal values of m, then the corresponding values of c cannot be found from (5). Then c is determined by equating to zero the coefficient of x^{n-2} *i.e.*, from

$$\frac{c^2}{2!}\phi''_n(m) + c\,\phi'_{n-1}(m) + \phi_{n-2}(m) = 0 \qquad ...(6)$$

In this case, *there will be two parallel asymptotes.*

Working rule :

1. Put $x = 1$, $y = m$ in the highest degree terms, thus getting $\phi_n(m)$. Equate it to zero and solve for m. Let its roots be $m_1, m_2, ...$
2. Form $\phi_{n-1}(m)$ by putting $x = 1$ and $y = m$ in the $(n-1)$th degree terms.
3. Find the values of c (*i.e.* $c_1, c_2, ...$) by substituting $m = m_1, m_2, ...$ in turn in the formula
$$c = -\phi_{n-1}(m)/\phi'_n(m)$$
[Sometimes it take (0/0) form, then find c from (6).]
4. Substitute the values of m and c in $y = mx + c$ in turn.

Example 4.66. *Find the asymptotes of the curve*
(i) $y^3 - 2xy^2 - x^2y + 2x^3 + 3y^2 - 7xy + 2y^2 + 2y + 2x + 1 = 0.$
(ii) $x^3 + 3x^2y - 4y^3 - x + y + 3 = 0.$
(iii) $(x + y)^2(x + y + 2) = x + 9y - 2.$ *(Rohtak, 2005)*

Solution. (*i*) Putting $x = 1$ and $y = m$ in the third degree terms,

$$\phi_3(m) = m^3 - 2m^2 - m + 2, \quad \therefore \quad \phi_3(m) = 0 \text{ gives } m^3 - 2m^2 - m + 2 = 0$$

or $$(m^2 - 1)(m - 2) = 0 \text{ whence } m = 1, -1, 2.$$

Also putting $x = 1$ and $y = m$ in the 2nd degree terms, $\phi_2(m) = 3m^2 - 7m + 2$

$$\therefore \quad c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{3m^2 - 7m + 2}{3m^2 - 4m - 1}$$

$$= -1 \text{ when } m = 1, = -2 \text{ when } m = -1, = 0 \text{ when } m = 2.$$

Hence the asymptotes are $y = x - 1$, $y = -x - 2$ and $y = 2x$.

(*ii*) Putting $x = 1$ and $y = m$ in the third degree terms,

$$\phi_3(m) = 1 + 3m - 4m^3$$

$$\therefore \quad \phi_3(m) = 0 \text{ gives } 4m^3 - 3m - 1 = 0, \quad \text{or} \quad (m-1)(2m+1)^2 = 0$$

whence $$m = 1, -1/2, -1/2.$$

Similarly, $$\phi_2(m) = 0$$

$$\therefore \quad c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{0}{3 - 12m^2}$$

$$= 0 \text{ when } m = 1, = \frac{0}{0} \text{ form when } m = -\frac{1}{2}.$$

Thus $\left(\text{when } m = -\frac{1}{2}\right)$ c is to be obtained from

$$\frac{c^2}{2!}\phi''_3(m) + c\,\phi'_2(m) + \phi_1(m) = 0$$

or $$\frac{c^2}{2}(-24m) + c\,.\,0 + (-1 + m) = 0$$

Putting $m = -1/2$, $6c^2 - 3/2 = 0$ whence $c = \pm 1/2$.

Hence the asymptotes are $y = x$, $y = -\frac{1}{2}x + \frac{1}{2}$, $y = -\frac{1}{2}x - \frac{1}{2}$.

(*iii*) Putting $x = 1$ and $y = m$ in the third degree terms, $\phi_3(m) = (1 + m)^3$.

$$\therefore \quad \phi_3(m) = 0 \text{ gives } (m + 1)^3 = 0 \text{ whence } m = -1, -1, -1.$$

Similarly, $$\phi_2(m) = 2(1 + m)^2, \ \phi_1(m) = -1 - 9m, \ \phi_0(m) = 2.$$

For these three equal values of $m = -1$, values of c are obtained from

$$\frac{c^3}{3!}\phi_3'''(m) + \frac{c^2}{2!}\phi_2''(m) + c\,\phi_1'(m) + \phi_0(m) = 0$$

or $$\frac{c^3}{6}(6) + \frac{c^2}{2}(4) + c(-9) + 2 = 0 \quad \text{or} \quad c^3 + 2c^2 - 9c + 2 = 0.$$

Solving for c, we have $c = 2, -2 \pm \sqrt{5}$.

Hence the three asymptotes are

$$y = -x + 2, y = -x - 2 + \sqrt{5}, y = -x - 2 - \sqrt{5}.$$

4. Asymptotes of polar curves. It can be shown that an *asymptote of the curve* $1/r = f(\theta)$ *is*

$$r \sin(\theta - \alpha) = 1/f'(\alpha),$$

where α *is a root of the equation* $f(\theta) = 0$
and $f'(\alpha)$ *is the derivative of* $f(\theta)$ *w.r.t.* θ *at* $\theta = \alpha$.

Example 4.67. *Find the asymptote of the spiral* $r = a/\theta$.

Equation of the curve can be written as $1/r = \theta/a = f(\theta)$, say.

$$f(\theta) = 0, \text{ if } \theta = 0\ (= \alpha). \text{ Also } f'(\theta) = 1/a \quad \therefore \quad f'(\alpha) = 1/a.$$

$\therefore$ The asymptote is $r \sin(\theta - 0) = 1/f'(0)$ or $r \sin\theta = a$.

PROBLEMS 4.14

Find the asymptotes of

1. $x^3 + y^3 = 3axy$ (*Agra, 2002*)
2. $(x^2 - a^2)(y^2 - b^2) = a^2 b^2$ (*Osmania, 2002*)
3. $(a/x)^2 + (b/y)^2 = 1$ (*Burdwan, 2003*)
4. $x^2y + xy^2 + xy + y^2 + 3x = 0$.
5. $4x^3 + 2x^2 - 3xy^2 - y^3 - 1 - xy - y^2 = 0$. (*U.P.T.U., 2001*)
6. $x^2(x - y)^2 - a^2(x^2 + y^2) = 0$ (*Kurukshetra, 2006*)
7. $(x + y)^2(x + 2y + 2) = (x + 9y - 2)$ (*Rajasthan, 2006*)
8. Show that the asymptotes of the curve $x^2 y^2 = a^2(x^2 + y^2)$ form a square of side $2a$.
9. Find the asymptotes of the curve $x^2y - xy^2 + xy + y^2 + x - y = 0$ and show that they cut the curve again in three points which lie on the line $x + y = 0$. (*Kurukshetra, 2006*)

Find the asymptotes of the following curves :

10. $r = a \tan\theta$. (*Rohtak, 2006 S*)
11. $r = a(\sec\theta + \tan\theta)$
12. $r \sin\theta = 2\cos 2\theta$. (*Kurukshetra, 2009 S*)
13. $r \sin n\theta = a$.

4.17 (1) CURVE TRACING

In many practical applications, a knowledge about the shapes of given equations is desirable. On drawing a sketch of the given equation, we can easily study the behaviour of the curve as regards its symmetry asymptotes, the number of branches passing through a point etc.

A point through which two branches of a curve pass is called a **double point.** At such a point P, the curve has two tangents, one for each branch.

If the tangents are real and distinct, the double point is called a **node** [Fig. 4.22 (a)].

If the tangents are real and coincident, the double point is called a **cusp** [Fig. 4.22 (b)].

If the tangents are imaginary, the double point is called a *conjugate point* (or an *isolated point*). Such a point cannot be shown in the figure.

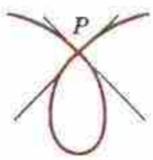


Fig. 4.22 (*a*)

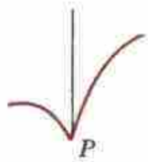


Fig. 4.22 (*b*)

(2) Procedure for tracing cartesian curves.

1. Symmetry. *See if the curve is symmetrical about any line.*

(*i*) *A curve is symmetrical about the x-axis, if only even powers of y occur in its equation.* (*e.g.*, $y^2 = 4ax$ is symmetrical about x-axis).

(*ii*) *A curve is symmetrical about the y-axis, if only even powers of x occur in its equation.* (*e.g.*, $x^2 = 4ay$ is symmetrical about y-axis).

(*iii*) *A curve is symmetrical about the line* $y = x$, *if on interchanging* x and y *its equation remains unchanged,* (*e.g.*, $x^3 + y^3 = 3axy$ *is symmetrical about the line* $y = x$).

2. Origin. (*i*) *See if the curve passes through the origin.*

(A curve passes through the origin if there is no constant term in its equation).

(*ii*) *If it does, find the equation of the tangents thereat,* by equating to zero the lowest degree terms.

(*iii*) *If the origin is a double point, find whether the origin is a node, cusp or conjugate point.*

3. Asymptotes. (*i*) *See if the curve has any asymptote parallel to the axes* (p. 183).

(*ii*) *Then find the inclined asymptotes,* if need be. (p. 183).

4. Points. (*i*) *Find the points where the curve crosses the axes and the asymptotes.*

(*ii*) *Find the points where the tangent is parallel or perpendicular to the x-axis,* (*i.e.* the points where $dy/dx = 0$ or ∞).

(*iii*) *Find the region (or regions) in which no portion of the curve exists.*

Example 4.68. *Trace the curve* $y^2 (2a - x) = x^3$.
(*P.T.U., 2010, V.T.U., 2008 ; Rajasthan, 2006 ; U.P T.U., 2005*)

Solution. (*i*) *Symmetry*: The curve is symmetrical about the x-axis.

[$\because$ only even powers of y occur in the equation.

(*ii*) *Origin :* The curve passes through the origin

[$\because$ there is no constant term in its equation.

The tangents at the origin are $y = 0, y = 0$ [Equating to zero the lowest degree terms.]

$\therefore$ Origin is a *cusp*

(*iii*) *Asymptotes :* The curve has an asymptote $x = 2a$.

[$\because$ co-eff. of y^3 is absent, co-eff. of y^2 is an asymptote.

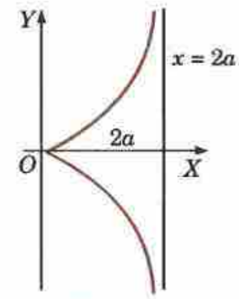


Fig. 4.23

(*iv*) *Points* : (*a*) curve meets the axes at (0, 0) only. (*b*) $y^2 = x^3/(2a - x)$

When x is –ve, y^2 is –ve (*i.e.* y is imaginary) so that no portion of the cuve lies to the left of the y-axis. Also when $x > 2a$, y^2 is again –ve, so that no portion of the curve lies to the right of the line $3x = 2a$.

Hence, the shape of the curve is as shown in Fig. 4.23. This curve is known as *Cissoid.*

Example 4.69. *Trace the curve* $y^2 (a - x) = x^2(a + x)$. (*V.T.U., 2010 ; B.P.T.U., 2005*)

Solution. (*i*) *Symmetry* : The curve is symmetrical about the x-axis.

(*ii*) *Origin* : The curve passes through the origin and the tangents at the origin are $y^2 = x^2$,

i.e. $y = x$ and $y = -x$. $\therefore$ Origin is a *node.*

(*iii*) *Asymptotes* : The curve has an asymptote $x = a$

(*iv*) *Points* : (*a*) When $x = 0, y = 0$; when $y = 0, x = 0$ or $-a$.

$\therefore$ The curve crosses the axes at (0, 0) and $(-a, 0)$.

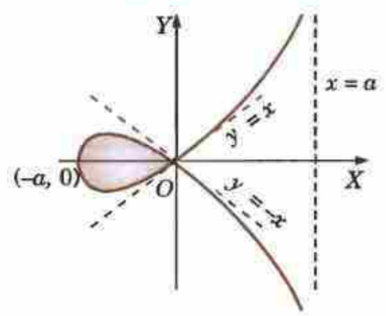


Fig. 4.24

We have $y = \pm x \sqrt{\left(\dfrac{a + x}{a - x}\right)}$

When $x > a$ or $< -a$, y is imaginary.

$\therefore$ No portion of the curve lies to the right of the line $x = a$ or to the left of the line $x = -a$.

Hence the shape of the curve is as shown in Fig. 4.24. This curve is known as *Strophoid.*

Example 4.70. *Trace the curve* $y = x^2 / (1 - x^2)$.

Solution. (*i*) *Symmetry* : The curve is symmetrical about y-axis.

(*ii*) *Origin* : It passes through the origin and the tangent at the origin is $y = 0$ (*i.e.*, x-axis).

(*iii*) *Asymptotes* : The asymptotes are given by $1 - x^2 = 0$ or $x = \pm 1$ and $y = -1$.

(*iv*) *Points* : (*a*) The curve crosses the axes at the origin only. (*b*) When $x \to 1$ from left, $y \to \infty$

When $x \to 1$ from right $y \to -\infty$

When $x > 1$, y is – ve

Hence the curve is as shown in Fig. 4.25.

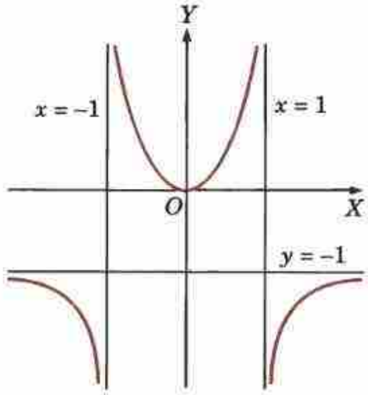


Fig. 4.25

Example 4.71. *Trace the curve* $a^2y^2 = x^2(a^2 - x^2)$. (*P.T.U., 2009 ; V.T.U., 2008 S*)

Solution. (*i*) *Symmetry.* The curve is symmetrical about x-axis, y-axis and origin.

(*ii*) *Origin.* The curve passes through the origin and the tangents at the origin are $a^2y^2 = a^2x^2$ *i.e.*, $y = \pm x$.

(*iii*) *Asymptotes.* The curve has no asymptote.

(*iv*) *Points.* (*a*) The curve cuts x-axis ($y = 0$) at $x = 0, \pm a$. and cuts y-axis ($x = 0$) at $y = 0$ *i.e.*, (0, 0) only.

(*b*) $\dfrac{dy}{dx} = \dfrac{x(a^2 - 2x^2)}{a^2 y} \to \infty$ at $(a, 0)$

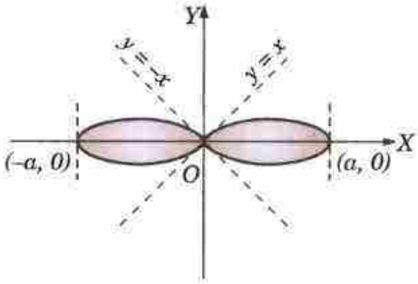


Fig. 4.26

i.e., tangent to the curve at $(a, 0)$ is parallel to y-axis. Similarly the tangent at $(-a, 0)$ is parallel to y-axis.

(*c*) We have $y = \dfrac{x}{a}\sqrt{a^2 - x^2}$ which is real for $x^2 < a^2$ *i.e.*, $-a < x < a$.

$\therefore$ The curve lies between $x = a$ and $x = -a$

Hence the shape of the curve is as shown is Fig. 4.26.

Example 4.72. *Trace the curve* $y = x^3 - 12x - 16$. (*P.T.U., 2008*)

Solution. (*i*) *Symmetry*. The curve has no symmetry.

(*ii*) *Origin.* It doesn't pass through the origin.

(*iii*) *Asymptotes* : The curve has no asymptote.

(*iv*) *Points.* (*a*) The curve cuts x-axis ($y = 0$) at (– 2, 0), (4, 0) and cuts y-axis ($x = 0$) at (0, – 16).

(*b*) $\dfrac{dy}{dx} = 3x^2 - 12$

At (– 2, 0), $\dfrac{dy}{dx} = 0$ *i.e.*, tangent is parallel to x-axis at (– 2, 0).

At (4, 0), $\dfrac{dy}{dx} = 36$ *i.e.*, $\tan\theta = 36$ *i.e.*, tangent makes an acute angle $\tan^{-1} 36$ with x-axis at (4, 0).

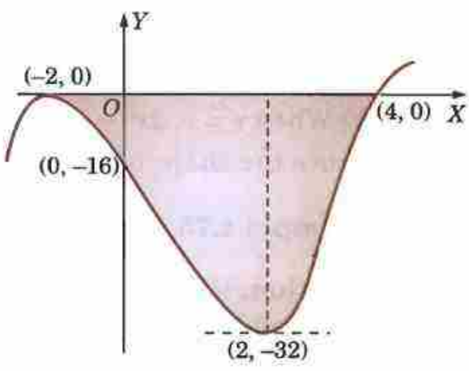


Fig. 4.27

Also $\dfrac{dy}{dx} = 0$ at $3x^2 - 12 = 0$ or $x = \pm 2$ *i.e.*, tangent is also parallel to x-axis at (2, – 32).

(*c*) $y \to \infty$ as $x \to \infty$ and $y \to -\infty$ as $x \to -\infty$; y is + *ve* for $x > 4$ and y is – *ve* for $x < 4$.

Hence the shape of the curve is as shown in Fig. 4.27.

Example 4.73. *Trace the curve* $9ay^2 = (x - 2a)(x - 5a)^2$ (*J.N.T.U., 2008*)

Solution. (*i*) *Symmetry.* The curve is symmetrical about the x-axis.

(*ii*) *Origin.* The curve does'nt pass through the origin.

(*iii*) *Asymptotes*. It has no asymptotes.

(*iv*) *Points*. (*a*) The curve cuts the x-axis ($y = 0$) at $x = 2a$, and $x = 5a$. *i.e.*, at A $(2a, 0)$ and $B(5a, 0)$.

It cuts the y-axis ($x = 0$) at $y^2 = -50\, a^2/9$, *i.e.*, y is imaginary.

So the curve doesn't cut the y-axis.

(*b*) $y = \dfrac{(x-5a)\sqrt{(x-2a)}}{3\sqrt{a}}$ *i.e.*, y is imaginary for $x < 2a$. So the curve exists only for $x \geq 2a$.

(*c*) $\dfrac{dy}{dx} = \pm \dfrac{x-3a}{2\sqrt{a}\,\sqrt{(x-2a)}}$

At A $(2a, 0)$, $\dfrac{dy}{dx} \to \infty$ *i.e.*, tangent is parallel to y-axis.

At B $(5a, 0)$, $\dfrac{dy}{dx} = \pm \dfrac{1}{\sqrt{3}}$ *i.e.*, there are two distinct tangents.

So there is a node at B $(5a, 0)$.

Hence the shape of the curve is as shown in Fig. 4.28.

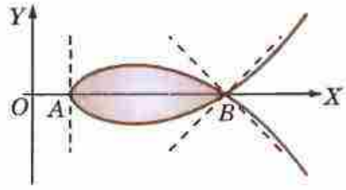


Fig. 4.28

Example 4.74. *Trace the curve* $x^3 + y^3 = 3axy$ (*Kurukshetra, 2005 ; U.P.T.U., 2003*)

or $r = \dfrac{3a \sin\theta \cos\theta}{\sin^3\theta + \cos^3\theta}$.

Solution. (*i*) *Symmetry* : The curve is symmetrical about the line $y = x$.

(*ii*) *Origin* : It passes through the origin and tangents at the origin are $xy = 0$, *i.e.*, $x = 0, y = 0$.

$\therefore$ Origin is a *node*.

(*iii*) *Asymptotes* : (*a*) It has no asymptote parallel to the axes.

(*b*) Putting $y = m$ and $x = 1$ in the third degree terms,

$$\phi_3(m) = 1 + m^3,\ \phi_3(m) = 0 \text{ gives } m = -1.$$

$$\therefore \quad c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\left(\frac{-3am}{3m^2}\right) = \frac{a}{m}.$$
$$= -a, \text{ when } m = -1.$$

Hence $y = -x - a$ $\left(i.e., \ \dfrac{x}{-a} + \dfrac{y}{-a} = 1\right)$ is an asymoptote.

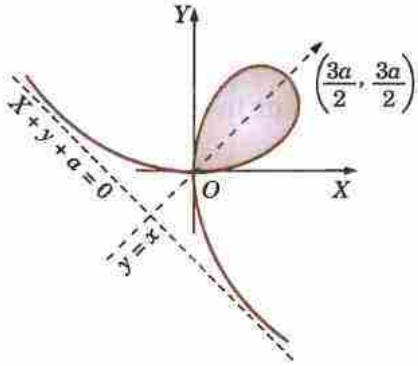


Fig. 4.29

(*iv*) *Points* : (*a*) It meets the axes at the origin only.

(*b*) When $y = x$, $2x^3 = 3ax^2$, *i.e.* $x = 0$ or $3a/2$. *i.e.*, the curve crosses the line $y = x$ at $(3a/2, 3a/2)$.

Hence the shape of the curve is as shown in Fig. 4.29. This curve is known as *Folium of Descartes*.

Example 4.75. *Trace the curve* $x^3 + y^3 = 3ax^2$.

Solution. (*i*) *Symmetry* : The curve has no symmetry.

(*ii*) *Origin* : The curve passes through the origin and the tangents at the origin are $x = 0$ and $x = 0$.

$\therefore$ The origin is a *cusp*.

(*iii*) *Asymptotes* : (*a*) The curve has no asymptote parallel to the axes.

(*b*) Putting $x = 1, y = m$ in the third degree terms, we get

$$\phi_3(m) = m^3 + 1\ ; \quad \therefore \quad \phi_3(m) = 0, \text{ gives } m = -1.$$

$$\therefore \quad c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{-3a}{3m^2} = a \text{ for } m = 1.$$

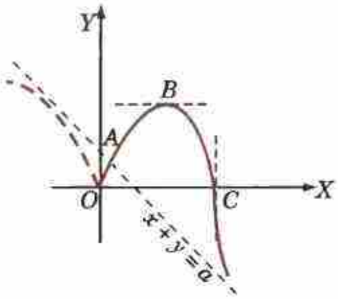


Fig. 4.30

Thus $x + y = a$ is the only asymptote.

The curve lies above the asymptote when x is positive and large and it lies below the asymptote when x is negative.

(*iv*) *Points.* (*a*) The curve crosses the axes at O (0, 0) and $C(3a, 0)$. It crosses the asymptote at $A(a/3, 2a/3)$.

(*b*) Since $y^2\, dy/dx = x(2a - x)$. $\therefore$ $dy/dx = 0$ for $x = 2a$.

(*c*) Now $y = [x^2 (3a - x)]^{1/3}$.

When $0 < x < 3a$, y is positive. As x increases from 0, y also increases till $x = 2a$ where the tangent is parallel to the x-axis. As x increases from $2a$ to $3a$, y constantly decreases to zero.

When $x > 3a$, y is negative.

When $x < 0$, y is positive and constantly increases as x varies from 0 to $-\infty$.

Combining all these facts we see that the shape of the curve is as shown in Fig. 4.30.

Example 4.76. *Trace the curve* $y^2(x - a) = x^2(x + a)$.

Solution. (*i*) *Symmetry* : The curve is symmetrical about the x-axis.

(*ii*) *Origin* : The curve passes through the origin and the tangents at the origin are $y^2 = -x^2$ *i.e.*, $y = \pm ix$, which are imaginary lines. $\therefore$ The origin is an *isolated point.*

(*iii*) *Asymptotes* : (*a*) $x = a$ is the only asymptote parallel to the y-axis.

(*b*) Putting $x = 1$ and $y = m$ in the third degree terms, we get

$$\phi_3(m) = m^2 - 1.$$

$$\therefore \quad \phi_3(m) = 0 \text{ gives } m = \pm 1$$

$$\therefore \quad c = \frac{\phi_2(m)}{\phi_3'(m)} = -\frac{-a(m^2+1)}{2m} = \pm a \text{ for } m = \pm 1.$$

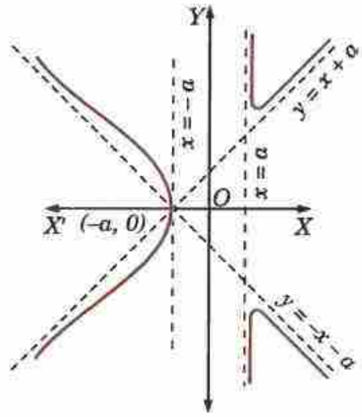


Fig. 4.31

Thus the other two asymptotes are $y = x + a$; $y = -x - a$.

(*iv*) *Points* : (*a*) The curve crosses the axes at $(-a, 0)$ and $(0, 0)$.

It crosses the asymptotes $y = x + a$ and $y = -x - a$ at $(-a, 0)$.

(*b*) $y = \pm x\sqrt{\left(\dfrac{x+a}{x-a}\right)}$

When $x < a$ and $x > -a$, y is imaginary.

$\therefore$ no portion of the curve lies between the lines $x = a$ and $x = -a$. Thus the vertical asymptote must be approached from the right.

(*c*) $\dfrac{dy}{dx} = \pm \dfrac{x^2 - ax + a^2}{(x-a)^{3/2}(x+a)^{1/2}}$

$\therefore$ $dy/dx = 0$, when $x = \frac{1}{2}(1 + \sqrt{5})a = 1.6a$ approx.

[rejecting the value $\frac{1}{2}(1 - \sqrt{5})a$ which lies between $-a$ and a]

and $dy/dx \to \infty$, when $x = \pm a$.

Thus the tangent is parallel to the x-axis at $x = 1.6a$ and perpendicular to the x-axis at $x = \pm a$.

Hence the shape of the curve is as shown in Fig. 4.31.

4.17 (3) PROCEDURE FOR TRACING CURVES IN PARAMETRIC FORM : $x = f(t)$ and $y = \phi(t)$

1. Symmetry. *See if the curve has any symmetry.*

(*i*) *A curve is symmetrical about the x-axis, if on replacing t by – t, f(t) remains unchanged and φ(t) changes to – φ(t).*

(*ii*) *A curve is symmetrical about the y-axis if on replacing t by – t, f(t) changes to – f(t) and φ(t) remains unchanged.*

(*iii*) *A curve is symmetrical in the opposite quadrants, if on replacing t by – t, both f(t) and φ(t) remains unchanged.*

2. **Limits.** *Find the greatest and least values of x and y so as to determine the strips, parallel to the axes, within or outside which the curve lies.*
3. **Points.** (*a*) *Determine the points where the curve crosses the axes.*

The points of intersection of the curve with the x-axis given by the roots of $\phi(t) = 0$, while those with the y-axis are given by the roots of $f(t) = 0$.

(*b*) *Giving t a series of value, plot the corresponding values of x and y, noting whether x and y increase or decrease for the intermediates values of t.* For this purpose, we consider the sign of dx/dt and dy/dt for the different values of t.

(*c*) *Determine the points where the tangent is parallel or perpendicular to the x-axis,* (*i.e.*, where $dy/dx = 0$ or $\to \infty$).

(*d*) *When x and y are periodic functions of t with a common period, we need to study the curve only for one period,* because the other values of t will repeat the same curve over and over again.

Obs. Sometimes it is convenient to eliminate t between the given equations and use the resulting cartesian equation to trace the curve.

Example 4.77. *Trace the curve* $x = a\cos^3 t$, $y = a\sin^3 t$ *or* $x^{2/3} + y^{2/3} = a^{2/3}$.
(*P.T.U., 2009 S ; U.P.T.U., 2005 ; V.T.U., 2003*)

Solution. (*i*) *Symmetry.* The curve is symmetrical about the x-axis.

[$\because$ On changing t to $-t$, x remains unchanged but y changed to $-y$]

(*ii*) *Limits.* $\because$ $|x| \le a$ and $|y| \le a$.

$\therefore$ The curve lies entirely within the square bounded by the lines $x = \pm a$, $y = \pm a$.

(*iii*) *Points* : We have
$$\frac{dx}{dt} = -3a\cos^2 t \sin t,$$
$$\frac{dy}{dt} = 3a\sin^2 t\cos t,\ \frac{dy}{dx} = -\tan t.$$
$$\therefore \quad dy/dx = 0 \text{ when } t = 0 \text{ or } \pi$$
and $dy/dx \to \infty$, when $t = \pi/2$.

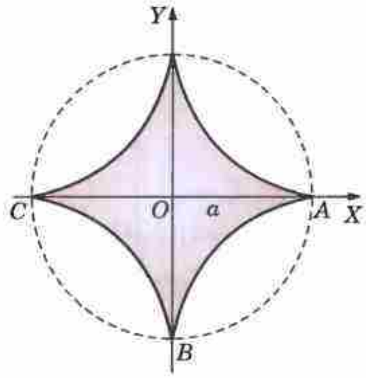


Fig. 4.32

The following table gives the corresponding values of t, x, y and dy/dx.

As t increases	*x*	*y*	*dy/dx varies*	*Portion traced*
from 0 to $\pi/2$	+ve and decreases from a to 0	+ve and increases from 0 to a	from 0 to ∞	A to B
from $\pi/2$ to π	+ ve and increases numerically from 0 to $-a$	+ve and decreases from a to 0	from ∞ to 0	B to C

As t increases from π to 2π, we get the reflection of the curve ABC in the x-axis. The values of $t > 2\pi$ give no new points.

Hence the shape of the curve is as shown in Fig. 4.32 and is known as **Astroid.**

Example 4.78. *Trace the curve* $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$. (*J.N.T.U., 2009 S*)

Solution. (*i*) *Symmetry.* The curve is symmetrical about the y-axis.

[$\because$ On changing θ to $-\theta$, x changes to $-x$ and y remains unchanged]

Thus we may consider the curve only for positive value of x, *i.e.*, for $\theta > 0$.

(*ii*) *Limits.* The greatest value of y is $2a$ and the least value is zero.

Hence the curve lies entirely between the lines $y = 2a$ and $y = 0$.

(*iii*) *Points.* We have
$$\frac{dx}{d\theta} = a(1 + \cos\theta),\ \frac{dy}{d\theta} = a\sin\theta \text{ and } \frac{dy}{dx} = -\tan\theta/2.$$

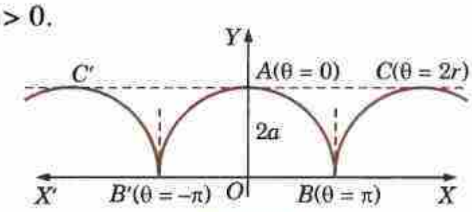


Fig. 4.33

$\therefore \quad dy/dx = 0$ when $\theta = 0$ or 2π and $dy/dx \to \infty$ when $\theta = \pi$.

The following table gives the corresponding values of θ, x, y and $\dfrac{dy}{dx}$:

As θ increases	*x*	*y*	*dy/dx varies*	*Portion traced*
from 0 to π	increases from 0 to $a\pi$	decreases from $2a$ to 0	from 0 to ∞	A to B
from π to 2π	increases from $a\pi$ to $2a\pi$	increases from 0 to $2a$	from ∞ to 0	B to C

As θ decreases from 0 to -2π, we get the reflection of the curve ABC in the y-axis.

The curve consists of congruent arches extending to infinity in both the directions of the x-axis in the intervals ... $(-3\pi, -\pi)$ $(-\pi, \pi)$ $(\pi, 3\pi)$, ...

Hence the shape of the curve is as shown in Fig. 4.33 and is known as **Cycloid.**

Obs. 1. Cycloid is the curve described by a point on the circumference of a circle which rolls without sliding on a fixed straight line. This fixed line (x-axis) is called the *base* and the farthest point (A) from it the *vertex* of the cycloid.

The complete cycloid consists of the arch $B'AB$ and its endless repetitions on both sides.

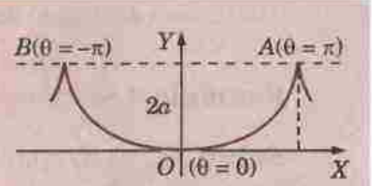


Fig. 4.34

2. Inverted cycloid: $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$.

The complete inverted cycloid consists of the arch BOA and an endless repetitions of the same on both sides. Here AB is the base and O the *vertex* of this cycloid. (Fig. 4.34).

4.17 (4) PROCEDURE FOR TRACING POLAR CURVES

1. Symmetry. *See if the curve is symmetrical about any line.*

(*i*) *A curve is symmetrical about the initial line OX, if only* $\cos\theta$ *(or* $\sec\theta$*) occur in its equation.* (*i.e.*, it remains unchanged when θ is changed to $-\theta$) *e.g.*, $r = a(1 + \cos\theta)$ is symmetrical about the initial line.

(*ii*) *A curve is symmetrical about the line through the pole* $\perp$ *to the initial line (i.e., OY), if only* $\sin\theta$ *(or* $\operatorname{cosec}\theta$*) occur in its equation.* (*i.e.*, it remains unchanged when θ is changed to $\pi - \theta$) *e.g.*, $r = a\sin 3\theta$ is symmetrical about OY.

(*iii*) *A curve is symmetrical about the pole, if only even powers of r occur in the equation* (*i.e.*, it remains unchanged when r is changed to $-r$) *e.g.*, $r^2 = a^2\cos 2\theta$ is symmetrical about the pole.

2. Limits. See if r and θ are confined between certain limits.

(*i*) *Determine the numerically greatest value of r, so as to notice whether the curve lies within a circle or not e.g.*, $r = a\sin 3\theta$ lies wholly within the circle $r = a$.

(*ii*) *Determine the region in which no portion of the curve lies by finding those values of* θ *for which r is imaginary e.g.*, $r^2 = a^2\cos 2\theta$ does not lie between the lines $\theta = \pi/4$ and $\theta = 3\pi/4$.

3. Asymptotes. *If the curve possesses an infinite branch, find the asymptotes* *(p. 183)*.

4. Points. (*i*) *Giving successive values to* θ, *find the corresponding values of r.*

(*ii*) *Determine the points where the tangent coincides with the radius vector or is perpendicular to it* (*i.e.*, the points where $\tan\phi = r\,d\theta/dr = 0$ or ∞).

Example 4.79. *Trace the curve* $r = a\sin 3\theta$. *(U.P.T.U., 2002)*

Solution. (*i*) *Symmetry.* The curve is symmetrical about the line through the pole $\perp$ to the initial line.

(*ii*) *Limits.* The curve wholly lies within the curve $r = a$. $(\because \; r$ is never $> a)$

(*iii*) *Asymptotes.* It has no asymptotes.

(*iv*) *Points.* (*a*) $\tan\phi = r\dfrac{d\theta}{dr} = \dfrac{a\sin 3\theta}{3a\cos 3\theta} = \dfrac{1}{3}\tan 3\theta$

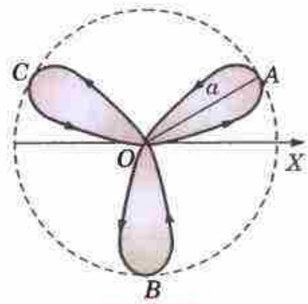


Fig. 4.35

$\therefore$ $\phi = 0$, when $\theta = 0, \pi/3, ...$

$\phi = \pi/2$, when $\theta = \pi/6, \pi/2, ...$

Hence the curve of the curve

(*b*) The following table gives the variations of r, θ and ϕ :

As θ varies from	*r varies from*	*ϕ varies from*	*Portion traced from*
0 to $\pi/6$	0 to a	0 to $\pi/2$	O to A
$\pi/6$ to $\pi/3$	a to 0	$\pi/2$ to 0	A to O
$\pi/3$ to $\pi/2$	0 to $-a$	0 to $\pi/2$	O to B

As θ increases from $\pi/2$ to π, portions of the curve from B to O, O to C and C to O are traced by symmetry about the line $\theta = \pi/2$.

Hence the curve consists of three loops as shown in Fig. 4.35 and is known as *three-leaved rose.*

Obs. *The curves of the form $r = a \sin n\theta$ or $r = a \cos n\theta$ are called* ***Roses*** *having*

(*i*) n leaves (loops) when n is odd,

(*ii*) $2n$ leaves (loops) when n is even.

Example 4.80. *Trace the curve $r = a \sin 2\theta$.(Four Leaved Rose)* (V.T.U., 2009)

Solution. (*i*) *Symmetry.* The curve is symmetrical about the line through the pole, $\perp$ to the initial line.

(*ii*) *Limits* : The curve lies wholly within the circle $r = a$ ($\because$ r is never $> a$)

(*iii*) *Points* : (*a*)

As θ increases from	*r varies from*	*Loop*	
0 to $\frac{\pi}{4}$	0 to a		
		no : 1,	
$\frac{\pi}{4}$ to $\frac{\pi}{2}$	a to 0		
$\frac{\pi}{2}$ to $\frac{3\pi}{4}$	0 to $-a$		
		no : 2,	
$\frac{3\pi}{4}$ to $\frac{\pi}{2}$	$-a$ to 0		etc. etc.

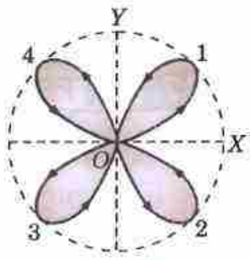


Fig. 4.36

(*b*) $$\tan \phi = r\frac{d\theta}{dr} = \frac{1}{2} \tan 2\theta \;;$$

$\therefore$ $$\phi = 0, \text{ when } \theta = 0, \frac{\pi}{2}, \pi, 3\frac{\pi}{2}, 2\pi \ldots$$

$$\phi = \frac{\pi}{2}, \text{ when } \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \ldots$$

Hence, the shape of the curve is as shown in Fig. 4.36.

Example 4.81. *Trace the curve $r^2 = a^2 \cos 2\theta$.* (V.T.U., 2007 ; Kurukshetra, 2006 ; B.P.T.U., 2005)

Solution. (*i*) *Symmetry.* The curve is symmetrical about the pole.

(*ii*) *Limits* : (*a*) The curve lies wholly within the circle $r = a$.

(*b*) No portion of the curve lies between the lines $\theta = \pi/4$ and $\theta = 3\pi/4$.

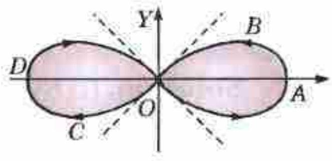


Fig. 4.37

(*iii*) *Points* : (*a*) $\tan \phi = r \dfrac{d\theta}{dr} = -\cot 2\theta = \tan\left(\dfrac{\pi}{2} + 2\theta\right)$

i.e., $$\phi = \frac{\pi}{2} + 2\theta \quad \therefore \quad \phi = 0, \text{ when } \theta = -\pi/4 \;;\; \phi = \pi/2 \text{ when } \theta = 0.$$

Thus, the tangent at O is $\theta = -\pi/4$ and the tangent at A is $\perp$ to the intial line.

(*b*) The variations of r and θ are given below :

As θ varies from	*r varies from*	*Portion traced*
0 to $\pi/4$	a to 0	*ABO*
$3\pi/4$ to π	0 to a	*OCD*

As θ increase from π to 2π, we get the reflection of the arc *ABOCD* in the initial line. Hence the shape of the curve is as shown in Fig. 4.37. This curve is known as *Lemniscate of Bernoulli.*

Example 4.82. *Trace the curve $r = a + b \cos \theta$ (Limacon)*

Solution. (*i*) *Symmetry.* It is symmetrical about the initial line.

(*ii*) *Limits* : The curve wholly lies within the circle $r = a + b$

($\because$ r is never $> a + b$)

(*iii*) *Points* : (α) when $a > b$.

As θ increases from 0 to $\pi/2$; r decreases from $a + b$ to a

As θ increases from $\pi/2$ to π ; r decreases from a to $a - b$

The shape of the curve is as shown in Fig. 4.38 (*i*).

(β) when $a < b$.

As θ increases from 0 to $\pi/2$; r decreases from $a + b$ to a

As θ increases from $\pi/2$ to α ; r decreases from a to 0

As θ increases from α to π ; r decreases from 0 to $a - b$

when $\alpha = \cos^{-1}\left(-\dfrac{a}{b}\right)$

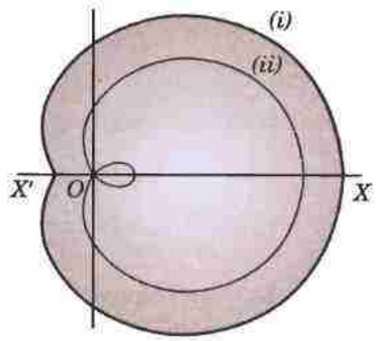


Fig. 4.38

In this case, the curve consists of two parts, one of which forms a loop within the other and the shape is as shown in Fig. 4.38 (*ii*).

Example 4.83. *Trace the curve $r\theta = a$.* *(Spiral)*

Solution. (*i*) *Symmetry.* There is no symmetry.

(*ii*) *Limits* : There are no limits to the values of r.

The curve does not pass through the pole for r does not become zero for any real value of θ.

(*iii*) Asymptotes : $\dfrac{1}{r} = \dfrac{\theta}{a} = f(\theta)$

$f(\theta) = 0$ for $\theta = 0$; $f'(\theta) = 1/a, f'(0) = 1/a$.

$\therefore$ Asymptote is $r \sin(\theta - 0) = 1/f'(0)$

i.e., $y = r \sin \theta = a$ is an asymptote.

(*iv*) *Points* : As θ increases from 0 to ∞, r to positive and decreases from ∞ to 0.

Hence the space of the curve is as shown in Fig. 4.39.

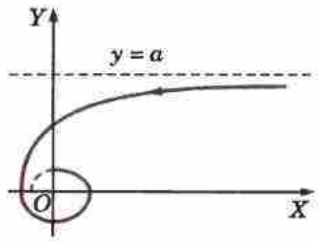


Fig. 4.39

Example 4.84. *Trace the curve $x^5 + y^5 = 5ax^2y^2$.*

Solution. (*i*) *Symmetry.* The curve is symmetrical about the line $y = x$.

$\because$ On interchanging x and y, it remains unchanged.]

(*ii*) *Origin* : It passes through the origin and the tangents at the origin are given by

$$x^2 y^2 = 0, \text{ i.e., } x = 0, x = 0 ; y = 0, y = 0.$$

Hence the curve has both *node* and the *cusp* at the origin.

(*iii*) *Asymptotes* : (*a*) It has no asymptotes parallel to the axes.

(*b*) Putting $x = 1, y = m$ in the fifth degree terms, we get

$$\phi_5(m) = 1 + m^5. \quad \therefore \quad \phi_5(m) = 0 \text{ gives } m = -1.$$

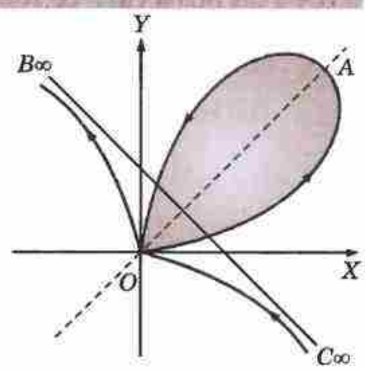


Fig. 4.40

$$\therefore \qquad c = -\frac{\phi_4(m)}{\phi_5'(m)} = -\frac{-5am^2}{5m^4} = a \text{ for } m = -1.$$

Hence $y = -x + a$ or $x + y = a$ is an asymptote.

(*iv*) *Points* : Since it is not convenient to express y as a function of x or *vice versa*, hence we change the equation into polar coordinates by putting, $x = r \cos\theta$ and $y = r \sin\theta$. The equation of the curve becomes :

$$r = \frac{5a \sin^2\theta \cos^2\theta}{\cos^5\theta + \sin^5\theta} = \frac{5a}{4} \cdot \frac{\sin^5 2\theta}{\cos^5\theta + \sin^5\theta}$$

As θ increases from	*r*	*Portion traced from*
0 to $\pi/4$	is +ve and increases from 0 to $\frac{5\sqrt{2}}{2}a$	0 to A
$\pi/4$ to $\pi/2$	is +ve and decreases from $\frac{5\sqrt{2}}{2}a$ to 0	A to 0
$\pi/2$ to $3\pi/4$	is +ve and increases from 0 to ∞	0 to B_∞
$3\pi/4$ to π	is –ve and decreases from ∞ to 0	C_∞ to 0

As θ increases from π to 2π, the curve will retraced.

Hence the shape of the curve is as shown in Fig. 4.40.

PROBLEMS 4.15

Trace the following curves :

1. $y^2(a + x) = x^2(a - x)$. (*S.V.T.U., 2008 ; U.P.T.U., 2006 ; Rajasthan, 2005*)
2. $y^2(a^2 + x^2) = x^2(a^2 - x^2)$ (*V.T.U., 2010*)
3. $y = (x^2 + 1)/(x^2 - 1)$
4. $ay^2 = x^2(\alpha - x)$ (*Kuruksehtra, 2009 S ; V.T.U., 2004*)
5. $x^2y^2 = a^2(y^2 - x^2)$
6. $x = a\cos^3\theta, y = b\sin^3\theta$
7. $x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$ $(0 < \theta < 2\pi)$
8. $x = (a\cos t + \log\tan t/2), y = a\sin t$.
9. $r = a\cos 2\theta$
10. $r = a\cos 3\theta$
11. $r = a(1 - \cos\theta)$
12. $r = 2 + 3\cos\theta$
13. $r^2\cos 2\theta = a^2$. (*S.V.T.U., 2009*)

[**Hint.** Changing to Cartesian form $x^2 - y^2 = a^2$. This is a rectangular hyperbola with asymptotes $x + y = 0$ and $x - y = 0$]

4.18 OBJECTIVE TYPES OF QUESTIONS

PROBLEMS 4.16

Select the correct answer or fill up the blanks in each of the following questions :

1. The radius of curvature of the catenary $y = c\cosh x/c$ at the point where it crosses the y-axis is ......
2. The envelope of the family of straight lines $y = mx + am^2$, (m being the parameter) is ......
3. The curvature of the circle $x^2 + y^2 = 25$ at the point (3, 4) is ......
4. The value of $\underset{x \to \pi/2}{\text{Lt}} \dfrac{\log\sin x}{(\pi/2 - x)^2}$ is

   (*a*) zero (*b*) 1/2 (*c*) – 1/2 (*d*) – 2. (*V.T.U., 2010*)
5. Taylor's expansion of the function $f(x) = \dfrac{1}{1 + x^2}$ is

(a) $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ for $-1 < x < 1$ (b) $\sum_{n=0}^{\infty} x^{2n}$ for $-1 < x < 1$

(c) $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ for any real x (d) $\sum_{n=0}^{\infty} (-1)^n x^n$ for $-1 < x \le 1$.

6. A triangle of maximum area inscribed in a circle of radius r
 (a) is a right angled triangle with hypotenuse measuring $2r$
 (b) is an equilateral triangle
 (c) is an isosceles triangle of height r
 (d) does *not* exist.
7. The extreme value of $(x)^{1/x}$ is
 (a) e (b) $(1/e)^e$ (c) $(e)^{1/e}$ (d) 1.
8. The percentage error in computing the area of an ellipse when an error of 1 per cent is made in measuring the major and minor axes is
 (a) 0.2% (b) 2% (c) 0.02%.
9. The length of subtangent of the rectangular hyperbola $x^2 - y^2 = a^2$ at the point $(a, \sqrt{2}a)$ is
 (a) $\sqrt{2}\,a$ (b) $2a$ (c) $\frac{1}{2a}$ (d) $\frac{a^{3/2}}{\sqrt{2}}$.
10. The length of subnormal to the curve $y = x^2$ at (2, 8) is
 (a) 2/3 (b) 32 (c) 96 (d) 64.
11. If the normal to the curve $y^2 = 5x - 1$ at the point $(1, -2)$ is of the form $ax - 5y + b = 0$, then a and b are
 (a) 4, 14 (b) 4, – 14 (c) – 4, 14 (d) – 4, – 14.
12. The radius of curvature of the curve $y = e^x$ at the point where it crosses the y-axis is
 (a) 2 (b) $\sqrt{2}$ (c) $2\sqrt{2}$ (d) $\frac{1}{2}\sqrt{2}$.
13. The equation of the asymptotes of $x^3 + y^3 = 3axy$, is
 (a) $x + y - a = 0$ (b) $x - y + a = 0$ (c) $x + y + a = 0$ (d) $x - y - a = 0$.
14. If ϕ be the angle between the tangent and radius vector at any point on the curve $r = f(\theta)$, then $\sin\phi$ equals to
 (a) $\frac{dr}{ds}$ (b) $r\frac{d\theta}{ds}$ (c) $r\frac{d\theta}{dr}$.
15. Envelope of the family of lines $x = my + 1/m$ is ...
16. The chord of curvature parallel to y-axis for the curve $y = a \log \sec x/a$ is ....
17. $\sinh x = \ldots x + \ldots x^3 + \ldots x^5 + \ldots$
18. The nth derivative of $(\cos x \cos 2x \cos 3x)$ = ...
19. If $x^3 + y^3 - 3axy = 0$, then d^2y/dx^2 at $(3a/2, 3a/2)$ = ...
20. When the tangent at a point on a curve is parallel to x-axis, then the curvature at that point is same as the second derivative at that point. *(True or False)*
21. If $x = at^2, y = 2at$, t being the parameter, then $xy\, d^2y/dx^2$ = ...
22. The radius of curvature for the parabola $x = a, y = 2at$ at any point t = ...
23. If (a, b) are the coordinates of the centre of curvature whose curvature is k, then the equation of the circle of curvature is ....
24. Evolute is defined as the .... of the normals for a given curve.
25. Envelope of the family of lines $\frac{x}{t} + yt = 2c$ (where t is the parameter) is ....
26. The angle between the radius vector and tangent for the curve $r = ae^{\theta \cot \alpha}$ is ....
27. The subnormal of the parabola $y^2 = 4ax$ is ....
28. The fourth derivative of $(e^{-x} x^3)$ is ....

29. If $y^2 = P(x)$, a polynomial of degree 3, then $\frac{2d}{dx}\left(y^3 \frac{d^2y}{dx^2}\right)$ equals ....

(a) $P'''(x) + P'(x)$ (b) $P''(x) + P'''(x)$ (c) $P(x)\, P'''(x)$.

30. The envelope of the family of straight line $\frac{x}{a} \cos\theta + \frac{y}{b} \sin\theta = 1$.

31. Curvature of a straight line is

(A) ∞ (B) zero (C) Both (A) and (B) (D) None of these.

32. The value of 'c' of the Cauchy's Mean value theorem for $f(x) = e^x$ and $g(x) = e^{-x}$ in [2, 3] is ....

33. If the equation of a curve remains unchanged when x and y are interchanged, then the curve is symmetrical about ....

34. For the curve $y^2(1 + x) = x^2(1 - x)$, the origin is a .... (node/cusp/conjugate point).

35. The number of loops of $r = a \sin 2\theta$ are .... and these of $r = a \cos 3\theta$ are ....

36. Tangents at the origin for the curve $y^2(x^2 + y^2) + a^2(x^2 - y^2) = 0$ are ....

37. The asymptote to the curve $y^2(4 - x) = x^3$ is ....

38. The points of inflexion of the curve $y^2 = (x - a)^2 (x - b)$ lie on the line $3x + a =$ ....

39. The curve $r = a/(1 + \cos\theta)$ intersects orthogonally with the curve

(A) $r = b/(1 - \cos\theta)$ (B) $r = b/(1 + \sin\theta)$ (C) $r = b/(1 + \sin^2\theta)$ (D) $r = b/(1 + \cos^2\theta)$. (V.T.U., 2010)

40. The region where the curve $r = a \sin\theta$ does not lie is ....

41. If $f(x)$ is continuous in the closed interval $[a, b]$, differentiable in (a, b) and $f(a) = f(b)$, then there exists at least one value c of x in (a, b) such that $f'(c)$ is equal to

(A) 1 (B) -1 (C) 2 (D) 0. (V.T.U., 2009)

42. If two curves intersect orthogonally in cartesian form, then the angle between the same two curves in polar form is

(A) $\pi/4$ (B) Zero (C) 1 radian (D) None of these.

43. If the angle between the radius vector and the tangent is constant, then the curve is,

(A) $r = a\cos\theta$ (B) $r^2 = a^2\cos^2\theta$ (C) $r = ae^{b\theta}$. (V.T.U., 2009)

CHAPTER 5

Partial Differentiation and Its Applications

5.1 (1) FUNCTIONS OF TWO OR MORE VARIABLES

We often come across quantities which depend on two or more variables. For example, the area of a rectangle of length x and breadth y is given by $A = xy$. For a given pair of values of x and y, A has a definite value. Similarly, the volume of a parallelopiped $(= xyh)$ depends on the three variables x(= length), y(= breadth) and h(=height).

Def. *A symbol z which has a definite value for every pair of values of x and y is called a function of two independent variables x and y and we write* $z = f(x, y)$ *or* $\phi(x, y)$.

We may interpret (x, y) as the coordinates of a point in the XY-plane and z as the height of the surface $z = f(x, y)$. We have come across several examples of such surfaces in Chapter 4.

The set R of points (x, y) such that any two points P_1 and P_2 of R can be so joined that any arc P_1P_2 wholly lies in R, is called as *region* in the XY-plane. A region is said to be a *closed region* if it includes all the points of its boundary, otherwise it is called an *open region.*

A set of points lying within a circle having centre at (a, b) and radius $\delta > 0$, is said to be *neighbourhood* of (a, b) in the circular region $R : (x - a)^2 + (y - b)^2 < \delta^2$.

When z is a function of three or more variables $x, y, t, \ldots$, we represent the relation by writing $z = f(x, y, t, \ldots)$. For such functions, no geometrical representation is possible. However, the concepts of a region and neighbourhood can easily be extended to functions of three or more variables.

(2) Limits. *The function f(x, y) is said to tend to the limit l as* $x \to a$ *and* $y \to b$ *if and only if the limit l is independent of the path followed by the point (x, y) as* $x \to a$ *and* $y \to b$ *and we write*

$$\underset{\substack{x \to a \\ y \to b}}{Lt} f(x, y) = l$$

In terms of a circular neighbourhood, we have the following *definition of the limit :*

The function f (x, y) defined in a region R, is said to tend to the limit l as $x \to a$ *and* $y \to b$ *if and only if corresponding to a positive number* ε*, there exists another positive number* δ *such that* $| f(x, y) - l | < \varepsilon$ *for* $0 < (x - a)^2 + (y - b)^2 < \delta^2$ *for every point (x, y) in R.*

(3) Continuity. *A function f(x, y) is said to be continuous at the point (a, b) if*

$$\underset{\substack{x \to a \\ y \to b}}{Lt} f(x, y) \text{ exists and } = f(a, b)$$

If a function is continuous at all points of a region, then it is said to be *continuous in that region.* A function which is not continuous at a point is said to be *discontinuous* at that point.

Obs. *Usually, the limit is the same irrespective of the path along which the point (x, y) approaches (a, b) and*

$$\underset{x \to a}{Lt}\left\{\underset{y \to b}{Lt} f(x, y)\right\} = \underset{y \to b}{Lt}\left\{\underset{x \to a}{Lt} f(x, y)\right\}$$

But it is not always so, as the following examples show :

$$\text{Lt}\left(\frac{x-y}{x+y}\right) \text{ as } (x, y) \to (0, 0) \text{ along the line } y = mx$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x - mx}{x + mx} = \frac{1-m}{1+m} \text{ which is different for lines with different slopes.}$$

Also $\underset{x \to 0}{\text{Lt}}\left[\underset{y \to 0}{\text{Lt}}\left(\frac{x-y}{x+y}\right)\right] = \underset{x \to 0}{\text{Lt}}\left(\frac{x}{x}\right) = 1$, whereas $\underset{y \to 0}{\text{Lt}}\left[\underset{x \to 0}{\text{Lt}}\left(\frac{x-y}{x+y}\right)\right] = \underset{y \to 0}{\text{Lt}}\left(\frac{-y}{y}\right) = 1.$

$\therefore$ As (x, y) is made to approach $(0, 0)$ along different paths, $f(x, y)$ approaches different limits. Hence the two repeated limits are not equal and $f(x, y)$ is discontinuous at the origin.

Also the function is not defined at $(0, 0)$ since $f(x, y) = 0/0$ for $x = 0, y = 0$.

(4) *As in the case of functions of one variable, the following results hold :*

I. *If* $\underset{\substack{x \to a \\ y \to b}}{Lt} f(x, y) = l$ and $\underset{\substack{x \to a \\ y \to b}}{Lt} g(x, y) = m,$

then (*i*) *If* $\underset{\substack{x \to a \\ y \to b}}{Lt} [f(x, y) \pm g(x, y)] = l \pm m$ (*ii*) $\underset{\substack{x \to a \\ y \to b}}{Lt} [f(x, y) . g(x, y)] = l . m$

(*iii*) $\underset{\substack{x \to a \\ y \to b}}{Lt} [f(x, y)/ g(x, y)] = l/m$ $(m \neq 0)$

II. *If* $f(x, y), g(x, y)$ *are continuous at (a, b) then so also are the functions*

$f(x, y) \pm g(x, y), f(x, y) . g(x, y)$ and $f(x, y)/g(x, y)$

provided $g(x, y) \neq 0$ in the last case.

PROBLEMS 5.1

Evaluate the following limits :

1. $\underset{\substack{x \to 1 \\ y \to 2}}{\text{Lt}} \frac{2x^2 y}{x^2 + y^2 + 1}$ 2. $\underset{\substack{x \to 0 \\ y \to 0}}{\text{Lt}} \frac{xy}{x^2 + y^2}$ 3. $\underset{\substack{x \to \infty \\ y \to 2}}{\text{Lt}} \frac{xy + 1}{x^2 + 2y^2}$ 4. $\underset{\substack{x \to 1 \\ y \to 1}}{\text{Lt}} \frac{x(y-1)}{y(x-1)}$

5. If $f(x, y) = \dfrac{x-y}{2x+y}$, show that $\underset{x \to 0}{\text{Lt}}\left[\underset{y \to 0}{\text{Lt}} f(x, y)\right] \neq \underset{y \to 0}{\text{Lt}}\left[\underset{x \to 0}{\text{Lt}} f(x, y)\right]$

   Also show that the function is discountinuous at the origin.

6. Show that the function $f(x, y) = x^2 + 2y,\quad (x, y) \neq (1, 2)$

   $3(x, y) = (1, 2) \quad = 0$

   is discontinuous at (1, 2).

7. Investigate the continuity of the function

   $$f(x, y) = xy/(x^2 + y^2), \quad (x, y) \neq (0, 0)$$
   $$= 0 \quad (x, y) = (0, 0)$$

   at the origin.

Note. In whatever follows, all the functions considered are continuous and their partial derivatives (as defined below) exist.

5.2 PARTIAL DERIVATIVES

Let $z = f(x, y)$ be a function of two variables x and y.

If we keep y as constant and vary x alone, then z is a function of x only. The derivative of z with respect to x, treating y as constant, is called the *partial derivative of z with respect to x* and is denoted by one of the symbols

$$\frac{\partial z}{\partial x}, \frac{\partial f}{\partial x}, f_x(x, y), D_x f. \qquad \text{Thus} \quad \frac{\partial z}{\partial x} = \underset{\delta x \to 0}{\text{Lt}} \frac{f(x + \delta x, y) - f(x, y)}{\delta x}$$

Similarly, the derivative of z with respect to y, keeping x as constant, is called the *partial derivative of z with respect to y* and is denoted by one of the symbols.

$$\frac{\partial z}{\partial y}, \frac{\partial f}{\partial y}, f_y(x, y), D_y f. \qquad \text{Thus} \quad \frac{\partial z}{\partial y} = \underset{\partial y \to 0}{\text{Lt}} \frac{f(x, y + \delta y) - f(x, y)}{\delta y}$$

Similarly, if z is a function of three or more variables $x_1, x_2, x_3, \ldots$ *the partial derivative of z with respect to x_1, is obtained by differentiating z with respect to x_1, keeping all other variables constant* and is written as $\partial z/\partial x_1$.

In general f_x and f_y are also functions of x and y and so these can be differentiated further partially with respect to x and y.

Thus $$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \text{ or } \frac{\partial^2 z}{\partial x^2} \text{ or } f_{xx}, \ \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} \text{ or } \frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{yx}{}^*$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} \text{ or } \frac{\partial^2 f}{\partial y \partial x} \text{ or } f_{xy}, \text{ and } \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} \text{ or } \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy}.$$

It can easily be verified that, in all ordinary cases,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}.$$

Sometimes we use the following notation

$$\frac{\partial z}{\partial x} = p, \ \frac{\partial z}{\partial y} = q, \ \frac{\partial^2 z}{\partial x^2} = r, \ \frac{\partial^2 z}{\partial x \partial y} = s, \ \frac{\partial^2 z}{\partial y^2} = t.$$

Example 5.1. *Find the first and second partial derivatives of $z = x^3 + y^3 - 3axy$.*

Solution. We have $z = x^3 + y^3 - 3axy$.

$$\therefore \quad \frac{\partial z}{\partial x} = 3x^2 + 0 - 3ay(1) = 3x^2 - 3ay, \text{ and } \frac{\partial z}{\partial y} = 0 + 3y^2 - 3ax(1) = 3y^2 - 3ax$$

Also $$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(3x^2 - 3ay) = 6x, \ \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}(3x^2 - 3ay) = -3a$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(3y^2 - 3ax) = 6y, \text{ and } \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}(3y^2 - 3ax) = -3a.$$

We observe that $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y}$.

Example 5.2. *If $u = x^2 \tan^{-1} \dfrac{y}{x} - y^2 \tan^{-1} \dfrac{x}{y}$,*

show that $$\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}.$$ *(Mumbai, 2008 S)*

and $$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$ *(Madras, 2000)*

Solution. We have $$\frac{\partial u}{\partial y} = x^2 \cdot \frac{1}{1 + (y/x)^2} \cdot \frac{1}{x} - \left\{2y \cdot \tan^{-1} \frac{x}{y} + y^2 \cdot \frac{1}{1 + (x/y)^2} \cdot \left(-\frac{x}{y^2}\right)\right\}$$

$$= \frac{x^3}{x^2 + y^2} - 2y \tan^{-1} \frac{x}{y} + \frac{xy^2}{x^2 + y^2} = x - 2y \tan^{-1} \frac{x}{y}.$$

*It is important to note that in the subscript notation the subscripts are written in the same order in which we differentiate whereas in the '∂' notation the order is opposite.

$\therefore$ $$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left\{x - 2y \tan^{-1}\frac{x}{y}\right\} = 1 - 2y \cdot \frac{1}{1+(x/y)^2} \cdot \frac{1}{y} = 1 - \frac{2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}.$$

Similarly, $$\frac{\partial u}{\partial x} = 2x \tan^{-1} y/x - y$$

and $$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left\{2x \tan^{-1}\frac{y}{x} - y\right\} = \frac{x^2-y^2}{x^2+y^2}.$$ Hence the result.

Example 5.3. *If $z = f(x+ct) + \phi(x-ct)$, prove that*

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$$ *(J.N.T.U., 2006 ; V.T.U., 2003 S)*

Solution. We have $\dfrac{\partial z}{\partial x} = f'(x+ct) \cdot \dfrac{\partial}{\partial x}(x+ct) + \phi'(x-ct)\dfrac{\partial}{\partial x}(x-ct) = f'(x+ct) + \phi'(x-ct)$

and $$\frac{\partial^2 z}{\partial x^2} = f''(x+ct) + \phi''(x-ct) \qquad ...(i)$$

Again $$\frac{\partial z}{\partial t} = f'(x+ct)\frac{\partial}{\partial t}(x+ct) + \phi'(x-ct)\frac{\partial}{\partial t}(x-ct) = cf'(x-ct) - c\phi'(x-ct)$$

and $$\frac{\partial^2 z}{\partial t^2} = c^2 f''(x+ct) + c^2\phi''(x-ct) = c^2[f''(x+ct) + \phi''(x-ct)] \qquad ...(ii)$$

From (*i*) and (*ii*), it follows that $\dfrac{\partial^2 z}{\partial t^2} = c^2\dfrac{\partial^2 z}{\partial x^2}$.

Obs. This is an important partial differential equation, known as *wave equation* (§ 18.4).

Example 5.4. *If $\theta = t^n e^{-r^2/4t}$, what value of n will make $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial\theta}{\partial r}\right) = \dfrac{\partial\theta}{\partial t}$?*

(Nagpur, 2009 ; Kurukshetra, 2006 ; U.P.T.U., 2006)

Solution. We have $\dfrac{\partial\theta}{\partial r} = t^n \cdot e^{-r^2/4t} \cdot \left(\dfrac{-2r}{4t}\right) = -\dfrac{r}{2}t^{n-1}e^{-r^2/4t}$

$\therefore$ $$r^2\frac{\partial\theta}{\partial r} = -\frac{r^3}{2}t^{n-1} \cdot e^{-r^2/4t}$$

and $$\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = -\frac{3r^2}{2}t^{n-1}e^{-r^2/4t} - \frac{r^3}{2}t^{n-1} \cdot e^{-r^2/4t}\left(-\frac{2r}{4t}\right)$$

$\therefore$ $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \left(-\frac{3}{2}t^{n-1} + \frac{r^2}{4}t^{n-2}\right)e^{-r^2/4t}$$

Also $$\frac{\partial\theta}{\partial t} = nt^{n-1} \cdot e^{-r^2/4t} + t^n \cdot e^{-r^2/4t} \cdot \frac{r^2}{4t^2} = \left(nt^{n-1} + \frac{1}{4}r^2 t^{n-2}\right)e^{-r^2/4t}$$

Since $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial\theta}{\partial r}\right) = \dfrac{\partial\theta}{\partial t}$,

$\therefore$ $$\left(-\frac{3}{2}t^{n-1} + \frac{1}{4}r^2 t^{n-2}\right)e^{-r^2/4t} = \left(nt^{n-1} + \frac{1}{4}r^2 t^{n-2}\right)e^{-r^2/4t} \quad \text{or} \quad \left(n + \frac{3}{2}\right)t^{n-1}e^{-r^2/4t} = 0.$$

Hence $n = -3/2$.

Example 5.5. *If $v = (x^2 + y^2 + z^2)^{-1/2}$, prove that*

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0.$$ *(Laplace equation)** *(V.T.U., 2006 ; Osmania, 2003 S)*

*See footnote p. 18.

Solution. We have $\frac{\partial v}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} . 2x = -x(x^2+y^2+z^2)^{-3/2}$

and
$$\frac{\partial^2 v}{\partial x^2} = -1[1 . (x^2+y^2+z^2)^{-3/2} + x(-3/2)(x^2+y^2+z^2)^{-5/2} . 2x]$$
$$= -(x^2+y^2+z^2)^{-5/2}[x^2+y^2+z^2-3x^2] = (x^2+y^2+z^2)^{-5/2}(2x^2-y^2-z^2)$$

Similarly, $\frac{\partial^2 v}{\partial y^2} = (x^2+y^2+z^2)^{-5/2}(-x^2+2y^2-z^2)$ and $\frac{\partial^2 v}{\partial z^2} = (x^2+y^2+z^2)^{-5/2}(-x^2-y^2+2z^2)$

Hence $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = (x^2+y^2+z^2)^{-5/2} . (0) = 0.$

Obs. *A function v satisfying the Laplace equation is said to be a* **harmonic function.**

Example 5.6. *If* $u = \log(x^3+y^3+z^3-3xyz)$, *show that* $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$.

(P.T.U., 2010 ; Anna, 2009 ; Bhopal, 2008 ; U.P.T.U., 2006)

Solution. We have $\frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$, $\frac{\partial u}{\partial y} = \frac{3y^2-3zx}{x^3+y^3+z^3-3xyz}$, $\frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$

$$\therefore \quad \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}$$
$$= \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} = \frac{3}{x+y+z}$$ *(V.T.U., 2009)*

Now
$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$
$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right) = \left(\frac{3}{x+y+z}\right)$$
$$= -\frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} = -\frac{9}{(x+y+z)^2}.$$

Example 5.7. *If* $\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u} = 1$, *prove that*
$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)$$ *(U.P.T.U., 2003)*

Solution. We have $x^2(a^2+u)^{-1} + y^2(b^2+u)^{-1} + z^2(c^2+u)^{-1} = 1$ ...(*i*)

Differentiating (*i*) partially w.r.t. x, we get
$$2x(a^2+u)^{-1} - x^2(a^2+u)^{-2}\frac{\partial u}{\partial x} - y^2(b^2+u)^{-2}\frac{\partial u}{\partial y} - z^2(c^2+u)^{-2}\frac{\partial u}{\partial z} = 0$$

or
$$\frac{2x}{a^2+u} = \left\{\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{y^2}{(c^2+u)^2}\right\}\frac{\partial u}{\partial x}$$

or
$$\frac{\partial u}{\partial x} = \frac{2x}{(a^2+u)v} \text{ where } v = \frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{y^2}{(c^2+u)^2}$$

Similarly differentiating (*i*) partially w.r.t. y, we get
$$\frac{2y}{b^2+u} = \left\{\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right\}\frac{\partial u}{\partial y} \text{ or } \frac{\partial u}{\partial y} = \frac{2y}{(b^2+u)v}$$

Similarly, differentiating (i) partially w.r.t. z, we get

$$\frac{2z}{(b^2+u)} = \left\{\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right\}\frac{\partial u}{\partial z} \text{ or } \frac{\partial u}{\partial z} = \frac{2z}{(c^2+u)v}$$

$$\therefore \quad \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = \frac{4}{v^2}\left\{\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{y^2}{(c^2+u)^2}\right\} = \frac{4}{v} \qquad ...(ii)$$

Also
$$2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right) = 2\left\{\frac{2x^2}{(a^2+u)v} + \frac{2y^2}{(b^2+u)v} + \frac{2z^2}{(c^2+u)v}\right\}$$

$$= \frac{4}{v}\left\{\frac{x^2}{(a^2+u)} + \frac{y^2}{(b^2+u)} + \frac{z^2}{(c^2+u)}\right\} = \frac{4}{v} \qquad \text{[By } (i)\text{] } ...(iii)$$

Hence the equality of (ii) and (iii) proves the result.

Example 5.8. *If $u = x^y$, show that $\dfrac{\partial^3 u}{\partial x^2 \partial y} = \dfrac{\partial^3 u}{\partial x \partial y \partial x}$.* *(Anna, 2009)*

Solution. We have $\dfrac{\partial u}{\partial y} = x^y \log_e x$ and $\dfrac{\partial^2 u}{\partial x \partial y} = yx^{y-1} . \log x + x^y . \dfrac{1}{x} = x^{y-1}(y \log x + 1)$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2 \partial y} = \frac{\partial}{\partial x}[x^{y-1}(y \log x + 1)] \qquad ...(i)$$

Again $\quad \dfrac{\partial u}{\partial x} = yx^{y-1}$ and $\dfrac{\partial^2 u}{\partial y \partial x} = 1 . x^{y-1} + y\left(\dfrac{1}{x}x^y \log x\right) = x^{y-1}(1 + y \log x)$

$$\therefore \quad \frac{\partial^3 u}{\partial x \partial y \partial x} = \frac{\partial}{\partial x}[x^{y-1}(y \log x + 1)] \qquad ...(ii)$$

From (i) and (ii) follows the required result.

PROBLEMS 5.2

1. Evaluate $\partial z/\partial x$ and $\partial z/\partial y$, if
   (i) $z = x^2 y - x \sin xy$; $\qquad (ii)$ $z = \log(x^2 + y^2)$;
   (iii) $z = \tan^{-1}\{(x^2 + y^2)/(x + y)\}$; $\qquad (iv)$ $x + y + z = \log z$.
2. If $z(x + y) = x^2 + y^2$, show that $\left(\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)^2 = 4\left(1 - \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)$. *(V.T.U., 2003)*
3. If $z = e^{ax+by} f(ax - by)$, prove that $b\dfrac{\partial z}{\partial x} + a\dfrac{\partial z}{\partial y} = 2abz$. *(V.T.U., 2010)*
4. Given $u = e^{r \cos\theta}\cos(r \sin\theta)$, $v = e^{r\cos\theta}\sin(r\sin\theta)$; prove that $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$ and $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$.
5. If $z = \tan(y + ax) - (y - ax)^{3/2}$, show that $\dfrac{\partial^2 z}{\partial x^2} = a^2\dfrac{\partial^2 z}{\partial y^2}$. *(Mumbai, 2009)*
6. Verify that $f_{xy} = f_{yx}$, when f is equal to (i) $\sin^{-1}(y/x)$; (ii) $\log x \tan^{-1}(x^2 + y^2)$.
7. If $f(x, y) = (1 - 2xy + y^2)^{-1/2}$, show that $\dfrac{\partial}{\partial x}\left[(1 - x^2)\dfrac{\partial f}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[y^2\dfrac{\partial f}{\partial y}\right] = 0$. *(Rohtak, 2006 S)*
8. Prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ if (i) $u = \tan^{-1}\left[\dfrac{2xy}{x^2 - y^2}\right]$. (ii) $u = \log(x^2 + y^2) + \tan^{-1}(y/x)$. *(Anna, 2009)*
9. If $v = \dfrac{1}{\sqrt{t}}e^{-x^2/4a^2 t}$, prove that $\dfrac{\partial v}{\partial t} = a^2\dfrac{\partial^2 v}{\partial x^2}$.

10. The equation $\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}$ refers to the conduction of heat along a bar without radiation, show that if $u = Ae^{-gx} \sin (nt - gx)$, where A, g, n are positive constants then $g = \sqrt{(n/2\mu)}$.

11. Find the value of n so that the equation $V = r^n (3 \cos^2 \theta - 1)$ satisfies the relation

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial V}{\partial \theta}\right) = 0.$$

12. If $z = \log (e^x + e^y)$, show that $rt - s^2 = 0$ where $r = \partial^2 z/\partial x^2$, $s = \partial^2 z/\partial x \partial y$, $t = \partial^2 z/\partial y^2$.

13. If $u = \frac{y}{z} + \frac{z}{x}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

14. Let $r^2 = x^2 + y^2 + z^2$ and $V = r^m$, prove that $V_{xx} + V_{yy} + V_{zz} = m(m+1)r^{m-2}$. *(Raipur, 2005)*

15. If $v = \log (x^2 + y^2 + z^2)$, prove that $(x^2 + y^2 + z^2)\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = 2$.

16. If $v = x^y . y^x$, prove that $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = v(x + y + \log v)$. *(Anna, 2005)*

17. If $x^x y^y z^z = c$, show that at $x = y = z$, $\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$. *(Bhopal, 2008)*

18. If $u = e^{xyz}$, find the value of $\frac{\partial^3 u}{\partial x \partial y \partial z}$. *(Rajasthan, 2005 ; Osmania, 2003 S)*

5.3 WHICH VARIABLE IS TO BE TREATED AS CONSTANT

(1) Consider the equation $x = r \cos \theta$, $y = r \sin \theta$ ...(1)

To find $\partial r/\partial x$, we need a relation between r and x. Such a relation will contain one more variable θ or y, for we can eliminate only one variable out of four from the relations (1). Thus the two possible relations are

$$r = x \sec \theta \quad ...(2) \qquad \text{and} \qquad r^2 = x^2 + y^2 \quad ...(3)$$

Now we can find $\partial r/\partial x$ either from (2) by treating θ as constant or from (3) by regarding y as constant. And there is no reason to suppose that the two values of $\partial r/\partial x$ so found, are equal. *To avoid confusion as to which variable is regarded constant, we introduce the following :*

Notation : $(\partial r/\partial x)_\theta$ *means the partial derivative of r with respect to x keeping* θ *constant in* a relation expressing r as a function of x and θ.

Thus from (2), $(\partial r/\partial x)_\theta = \sec \theta$.

When no indication is given regarding the variable to be kept constant, then according to convention $(\partial/\partial x)$ always means $(\partial/\partial x)_y$ and $\partial/\partial y$ means $(\partial/\partial y)_x$. Similarly, $\partial/\partial r$ means $(\partial/\partial r)_\theta$ and $\partial/\partial \theta$ means $(\partial/\partial \theta)_r$.

(2) In thermodynamics, we come across ten variables such as p (pressure), v (volume), T (temperature), W (work), ϕ (entropy) etc. Any one of these can be expressed as a function of other two variables *e.g.*, $T = f(p, v)$, $T = g(p, \phi)$

As we shall see, these respectively give rise to the following results :

$$dT = \frac{\partial T}{\partial p} dp + \frac{\partial T}{\partial v} dv \qquad ...(i)$$

$$dT = \frac{\partial T}{\partial p} dp + \frac{\partial T}{\partial \phi} d\phi \qquad ...(ii)$$

Now, $\partial T/\partial p$ appearing in (*i*), has been obtained from T as function of p and v, treating v as constant, we write it as $(\partial T/\partial p)_v$.

Similarly, $\partial T/\partial p$ occurring in (*ii*), is written as $(\partial T/\partial p)_\phi$.

Example 5.9. *If $u = f(r)$ and $x = r \cos \theta$, $y = r \sin \theta$, prove that*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r).$$

(S.V.T.U., 2008 ; Rajasthan, 2006 ; U.P.T.U., 2005)

Solution. We have $\dfrac{\partial u}{\partial x} = f'(r) \cdot \dfrac{\partial r}{\partial x}$ and $\dfrac{\partial^2 u}{\partial x^2} = f''(r) \cdot \left(\dfrac{\partial r}{\partial x}\right)^2 + f'(r) \cdot \dfrac{\partial^2 r}{\partial x^2}$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = f''(r) \cdot \left(\frac{\partial r}{\partial y}\right)^2 + f'(r) \cdot \frac{\partial^2 r}{\partial y^2}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) \cdot \left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] + f'(r)\left[\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2}\right]$$

Now to find $\dfrac{\partial r}{\partial x}, \dfrac{\partial r}{\partial y}$ etc., we write $r = (x^2 + y^2)^{1/2}$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{1}{2}(x^2 + y^2)^{-1/2} \cdot 2x = \frac{x}{r} \quad \text{and} \quad \frac{\partial^2 r}{\partial x^2} = \frac{r \cdot 1 - x \cdot \partial r/\partial x}{r^2} = \frac{r - x^2/r}{r^2} = \frac{y^2}{r^3}.$$

Similarly, $$\frac{\partial r}{\partial y} = \frac{y}{x} \quad \text{and} \quad \frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3}.$$

Substituting the values of $\partial r/\partial x$ etc. in (*i*), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r)\left[\frac{x^2}{r^2} + \frac{y^2}{r^2}\right] + f'(r)\left[\frac{y^2}{r^3} + \frac{x^2}{r^3}\right] = f''(r) + \frac{1}{r}f'(r).$$

Example 5.10. *If* $x = e^{r\cos\theta}\cos(r\sin\theta)$ *and* $y = e^{r\cos\theta}\sin(r\sin\theta)$, *prove that* $\dfrac{\partial x}{\partial \theta} = -r\dfrac{\partial y}{\partial r}, \dfrac{\partial y}{\partial \theta} = r\dfrac{\partial x}{\partial r}$.

Hence show that $\dfrac{\partial^2 x}{\partial \theta^2} + r\dfrac{\partial x}{\partial r} + r^2\dfrac{\partial^2 x}{\partial r^2} = 0.$

Solution. We have $x = e^{r\cos\theta}\cos(r\sin\theta)$

$$\therefore \quad \frac{\partial x}{\partial \theta} = e^{r\cos\theta}(-r\sin\theta) \cdot \cos(r\sin\theta) + e^{r\cos\theta}[-\sin(r\sin\theta)] \cdot r\cos\theta$$
$$= -re^{r\cos\theta}[\sin\theta\cos(r\sin\theta) + \cos\theta\sin(r\sin\theta)]$$
$$= -re^{r\cos\theta}\sin(\theta + r\sin\theta) \quad ...(i)$$

and $$\frac{\partial x}{\partial r} = e^{r\cos\theta} \cdot \cos\theta \cdot \cos(r\sin\theta) - e^{r\cos\theta}\sin\theta\,(r\sin\theta)\sin\theta$$
$$= e^{r\cos\theta}\cos(\theta + r\sin\theta) \quad ...(ii)$$

Similarly, $y = e^{r\cos\theta}\sin(r\sin\theta)$ gives

$$\frac{\partial y}{\partial \theta} = re^{r\cos\theta}\cos(\theta + r\sin\theta) \quad ...(iii)$$

and $$\frac{\partial y}{\partial r} = e^{r\cos\theta}\sin(\theta + r\sin\theta) \quad ...(iv)$$

From (*i*) and (*iv*), $\dfrac{\partial x}{\partial \theta} = -r\dfrac{\partial y}{\partial r}$ ...(*v*)

From (*ii*) and (*iii*), $\dfrac{\partial y}{\partial \theta} = r\dfrac{\partial x}{\partial r}$ ...(*vi*)

From (*v*), $$\frac{\partial^2 x}{\partial \theta^2} = -r\frac{\partial^2 y}{\partial \theta \partial r} = -r\frac{\partial^2 y}{\partial r \partial \theta}$$

From (*vi*), $$\frac{\partial x}{\partial r} = \frac{1}{r}\frac{\partial y}{\partial \theta} \quad \text{which gives} \quad \frac{\partial^2 x}{\partial r^2} = -\frac{1}{r^2}\frac{\partial y}{\partial \theta} + \frac{1}{r}\frac{\partial^2 y}{\partial r \partial \theta}$$

$$\therefore \quad \frac{\partial^2 x}{\partial \theta^2} + r\frac{\partial x}{\partial r} + r^2\frac{\partial^2 x}{\partial r^2} = -r\frac{\partial^2 y}{\partial r \partial \theta} + \frac{\partial y}{\partial \theta} - \frac{\partial y}{\partial \theta} + r\frac{\partial^2 y}{\partial r \partial \theta} = 0.$$

PROBLEMS 5.3

1. If $x = r\cos\theta$, $y = r\sin\theta$, show that (i) $\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}$ (ii) $\frac{1}{r}\frac{\partial x}{\partial \theta} = r\frac{\partial \theta}{\partial x}$, (iii) $\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = 1$. (*Burdwan, 2003*)

2. If $x^2 = au + bv$, $y^2 = au - bv$, prove that $\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2} = \left(\frac{\partial v}{\partial y}\right)_x \cdot \left(\frac{\partial y}{\partial v}\right)_u$.

3. If $u = lx + my$, $v = mx - ly$, show that $\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right)_v = \frac{l^2}{l^2 + m^2}$, $\left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u = \frac{l^2 + m^2}{l^2}$.

4. If $x = r\cos\theta$, $y = r\sin\theta$, prove that

(i) $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$ (ii) $\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$ $(x \neq 0, y \neq 0)$.

5. If $z = x\log(x + r) - r$ where $r^2 = x^2 + y^2$, prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{x + y}$, $\frac{\partial^3 z}{\partial x^3} = -\frac{x}{r^3}$. (*Mumbai, 2008*)

6. If $u = f(r)$ where $r = \sqrt{(x^2 + y^2 + z^2)}$, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r}f'(r)$.

5.4 (1) HOMOGENEOUS FUNCTIONS

An expression of the form $a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \ldots + a_ny^n$ in which every term is of the nth degree, is called a homogeneous function of degree n. This can be rewritten as

$$x^n[a_0 + a_1(y/x) + a_2(y/x)^2 + \ldots + a_n(y/x)^n].$$

Thus *any function $f(x, y)$ which can be expressed in the form $x^n\phi(y/x)$, is called a* **homogeneous function** of degree n in x and y.

For instance, $x^3\cos(y/x)$ is a homogeneous function of degree 3, in x and y.

In general, a function $f(x, y, z, t, \ldots)$ is said to be a homogeneous function of degree n in $x, y, z, t, \ldots$, if it can be expressed in the form $x^n\phi(y/x, z/x, t/x, \ldots)$.

(2) Euler's theorem on homogeneous functions*. *If u be a homogeneous function of degree n in x and y, then*

$$\mathbf{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.}$$

Since u is a homogeneous function of degree n in x and y, therefore,

$$u = x^n f(y/x)$$

$$\therefore \quad \frac{\partial u}{\partial x} = nx^{n-1}f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right)\cdot y\left(-\frac{1}{x^2}\right) = nx^{n-1}f\left(\frac{y}{x}\right) - yx^{n-2}f'\left(\frac{y}{x}\right)$$

and $$\frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right)\cdot\frac{1}{x} = x^{n-1}f'\left(\frac{y}{x}\right). \text{ Hence } x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n f\left(\frac{y}{x}\right) = nu.$$

In general, if u be a homogeneous function of degree n in $x, y, z, t, \ldots$, then,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + t\frac{\partial u}{\partial t} \ldots = nu.$$

Example 5.11. *Show that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u\log u$ *where* $\log u = (x^3 + y^3)/(3x + 4y)$.

Solution. Since $$z = \log u = \frac{x^3 + y^3}{3x + 4y} = x^2\cdot\frac{1 + (y/x)^3}{3 + 4(y/x)},$$

* After an enormously creative Swiss mathematician *Leonhard Euler* (1707–1783). He studied under John *Bernoulli* and became a professor of mathematics in St. Petersburg, Russia. Even after becoming totally blind in 1771, he contributed to almost all branches of mathematics.

$\therefore$ z is a homogeneous function of degree 2 in x and y.

By Euler's theorem, we get

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z \quad ...(i)$$

But $$\frac{\partial z}{\partial x} = \frac{1}{u}\frac{\partial u}{\partial x} \text{ and } \frac{\partial z}{\partial y} = \frac{1}{u}\frac{\partial u}{\partial y}$$

Hence (i) becomes

$$x \cdot \frac{1}{u}\frac{\partial u}{\partial x} + y \cdot \frac{1}{u}\frac{\partial u}{\partial y} = 2\log u \quad \text{or} \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u\log u.$$

Example 5.12. *If* $u = \sin^{-1}\dfrac{x + 2y + 3z}{x^8 + y^8 + z^8}$, *find the value of* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z}$. *(U.P.T.U., 2004)*

Solution. Here u is not a homogeneous function. We therefore, write

$$\omega = \sin u = \frac{x + 2y + 3z}{x^8 + y^8 + z^8} = x^{-7} \cdot \frac{1 + 2(y/x) + 3(z/x)}{1 + (y/x)^8 + (z/x)^8}$$

Thus ω is a homogeneous function of degree -7 in x, y, z. Hence by Euler's theorem

$$x\frac{\partial \omega}{\partial x} + y\frac{\partial \omega}{\partial y} + z\frac{\partial \omega}{\partial z} = (-7)\,\omega \quad ...(i)$$

But $$\frac{\partial \omega}{\partial x} = \cos u\,\frac{\partial u}{\partial x},\ \frac{\partial \omega}{\partial y} = \cos u\,\frac{\partial u}{\partial y},\ \frac{\partial \omega}{\partial z} = \cos u\,\frac{\partial u}{\partial z}$$

$\therefore$ (i) becomes $$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} + z\cos u\frac{\partial u}{\partial z} = -7\sin u \quad \text{or} \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -7\tan u.$$

Example 5.13. *If* $u = \dfrac{x^3y^3z^3}{x^3 + y^3 + z^3} + \log\left(\dfrac{xy + yz + zx}{x^2 + y^2 + z^2}\right)$, *find the value of* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z}$. *(Mumbai, 2009)*

Solution. Let $$v = \frac{x^3y^3z^3}{x^3 + y^3 + z^3} \text{ and } w = \log\left(\frac{xy + yz + zx}{x^2 + y^2 + z^2}\right) \quad ...(i)$$

so that $$u = v + w$$

Since $v = x^6\,\dfrac{(y/x)^3(z/x)^3}{1 + (y/x)^3 + (z/x)^3}$, therefore v is a homogeneous function of degree 6 in x, y, z.

Hence by Euler's theorem $$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = 6v \quad ...(ii)$$

Since $w = \log\left\{\dfrac{\dfrac{y}{x} + \dfrac{y}{x}\cdot\dfrac{z}{x} + \dfrac{z}{x}}{1 + \left(\dfrac{y}{x}\right)^2 + \left(\dfrac{z}{x}\right)^2}\right\}$ therefore w is a homogeneous function of degree zero in x, y, z.

Hence by Euler's theorem $$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = 0 \quad ...(iii)$$

Addint (ii) and (iii), we obtain

$$x\left(\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right) + y\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\right) + z\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial z}\right) = 6v$$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 6\,\frac{x^3y^3z^3}{x^3 + y^3 + z^3} \quad \text{[By } (i)\text{]}$$

Example 5.14. *If z is a homogeneous function of degree n in x and y, show that*

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial z^2}=n(n-1)\,z.$$ *(Anna, 2009 ; V.T.U., 2007 ; U.P.T.U., 2006)*

Solution. By Euler's theorem, $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz$ ...(*i*)

Differentiating (*i*) partially w.r.t. x, we get $x\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial x}+y\frac{\partial^2 z}{\partial x\partial y}=n\frac{\partial z}{\partial x}$

i.e., $$x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial x\partial y}=(n-1)\frac{\partial z}{\partial x}$$ ...(*ii*)

Again differentiating (*i*) partially w.r.t. y, we get $x\frac{\partial^2 z}{\partial y\partial x}+\frac{\partial z}{\partial y}+y\frac{\partial^2 z}{\partial y^2}=n\frac{\partial z}{\partial y}$

i.e., $$x\frac{\partial^2 z}{\partial x\partial y}+y\frac{\partial^2 z}{\partial y^2}=(n-1)\frac{\partial z}{\partial y}$$ ...(*iii*)

Multiplying (*ii*) by x and (*iii*) by y and adding, we get

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=(n-1)\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)=n(n-1)\,z.$$ [By (*i*)]

Example 5.15. *If* $u=\sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, *prove that* $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\tan u.$

(Rajasthan, 2006 ; Calicut, 2005)

and $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-\frac{\sin u\cos 2u}{4\cos^3 u}.$$ *(P.T.U., 2006)*

Solution. Here u is not a homogeneous function but $z=\sin u=\frac{x+y}{\sqrt{x}+\sqrt{y}}$ is a homogeneous function of degree 1/2 in x and y.

∴ By Euler's theorem, $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{1}{2}z$

or $$x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=\frac{1}{2}\sin u$$

Thus $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\tan u$$ ...(*i*)

Differentiating (*i*) w.r.t. x partially, we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=\frac{1}{2}\sec^2 u\frac{\partial u}{\partial x}\quad\text{or}\quad x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=\left(\frac{1}{2}\sec^2 u-1\right)\frac{\partial u}{\partial x}$$ ...(*ii*)

Again differentiating (*i*) w.r.t. y partially, we get

$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=\frac{1}{2}\sec^2 u\frac{\partial u}{\partial y}\quad\text{or}\quad x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}=\left(\frac{1}{2}\sec^2 u-1\right)\frac{\partial u}{\partial x}$$ ...(*iii*)

Multiplying (*ii*) by x and (*iii*) by y and adding, we obtain

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\left(\frac{1}{2}\sec^2 u-1\right)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

or $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\left(\frac{1}{2}\sec^2 u-1\right)\left(\frac{1}{2}\tan u\right)$$ [By (*i*)]

$$=\frac{1}{4}\frac{\sin u}{\cos^3 u}-\frac{1}{2}\frac{\sin u}{\cos u}=-\frac{\sin u\,(2\cos^2 u-1)}{4\cos^3 u}$$

Hence $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-\frac{\sin u\cos 2u}{4\cos^3 u}.$$

PROBLEMS 5.4

1. Verify Euler's theorem, when (i) $f(x, y) = ax^2 + 2hxy + by^2$
(ii) $f(x, y) = x^2 (x^2 - y^2)^3/(x^2 + y^2)^3$.
(iii) $f(x, y) = 3x^2 yz + 5xy^2z + 4z^4$ (J.N.T.U., 1999)

2. If $u = \sin^{-1} \dfrac{x}{y} + \tan^{-1} \dfrac{y}{x}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$. (Hazaribagh, 2009 ; Osmania, 2003 S)

3. If $u = \sin^{-1} \dfrac{x^2 + y^2}{x + y}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$ (Bhopal, 2009 ; V.T.U., 2003)

4. If $\sin u = \dfrac{x^2y^2}{x + y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3 \tan u$. (Kottayam, 2005 ; V.T.U., 2003 S)

5. If $u = \cos^{-1} \dfrac{x + y}{\sqrt{x} + \sqrt{y}}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = -\dfrac{1}{2} \cot u$. (V.T.U., 2004)

6. Show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u \log u$, where $u = e^{x^2 + y^2}$ (P.T.U., 2010)

7. If $z = f(y/x) + \sqrt{(x^2 + y^2)}$, show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = \sqrt{x^2 + y^2}$. (Mumbai, 2008)

8. If $u = \dfrac{x}{y + z} + \dfrac{y}{z + x} + \dfrac{z}{x + y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$. (V.T.U., 2000 S)

9. If $\sin u = \dfrac{x + 2y + 3z}{\sqrt{(x^8 + y^8 + z^8)}}$, show that $xu_x + yu_y + zu_z + 3 \tan u = 0$. (S.V.T.U., 2009 ; U.T.U., 2009)

10. If $z = x\phi\left(\dfrac{y}{x}\right) + \psi\left(\dfrac{y}{x}\right)$, prove that $x^2\dfrac{\partial^2 z}{\partial x^2} + 2xy\dfrac{\partial^2 z}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 0$. (S.V.T.U., 2009 ; U.P.T.U., 2006)

11. If $u = \tan^{-1} \dfrac{x^3 + y^3}{x + y}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$. (P.T.U., 2009 S)
and $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 2 \cos 3u \sin u$. (Mumbai, 2009 ; Bhopal, 2008 ; S.V.T.U., 2007)

12. Given $z = x^n f_1(y/x) + y^{-n} f_2(x/y)$, prove that $x^2\dfrac{\partial^2 z}{\partial x^2} + 2xy\dfrac{\partial^2 z}{\partial x \partial y} + y^2\dfrac{\partial^2 z}{\partial y^2} + x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = n^2 z$.
(Kurukshetra, 2009 S ; Rohtak, 2003)

13. If $u = x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(x/y)$, evaluate $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2}$. (U.T.U., 2009 ; Hissar, 2005 S)

14. If $u = \tan^{-1}(y^2/x)$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = -\sin^2 u \,.\, \sin 2u$. (Bhillai, 2005 ; P.T.U., 2005)

15. If $u = \operatorname{cosec}^{-1}\left(\dfrac{x^{1/2} + y^{1/2}}{x^{1/3} + y^{1/3}}\right)^{1/2}$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = \dfrac{\tan u}{12}\left(\dfrac{13}{12} + \dfrac{\tan^2 u}{12}\right)$.
(Mumbai, 2008 ; Rohtak, 2006 S)

5.5 (1) TOTAL DERIVATIVE

If $u = f(x, y)$, where $x = \phi(t)$ and $y = \psi(t)$, then we can express u as a function of t alone by substituting the values of x and y in $f(x, y)$. Thus we can find the ordinary derivative du/dt which is called the *total derivative* of u to distinguish it from the partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$.

Now to find du/dt without actually substituting the values of x and y in $f(x, y)$, we establish the following **Chain rule :**

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} \qquad ...(i)$$

Proof. We have $u = f(x, y)$

Giving increment δt to t, let the corresponding increments of x, y and u be δx, δy and δu respectively.

Then $\quad u + \delta u = f(x + \delta x, y + \delta y)$

Subtracting, $\quad \delta u = f(x + \delta x, y + \delta y) - f(x, y)$

$$= \{f(x + \delta x, y + \delta y) - f(x, y + \delta y)\} + \{f(x, y + \delta y) - f(x, y)\}$$

$$\therefore \quad \frac{\delta u}{\delta t} = \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x} \cdot \frac{\delta x}{\delta t} + \frac{f(x, y + \delta y) - f(x, y)}{\delta y} \cdot \frac{\delta y}{\delta t}$$

Taking limits as $\delta t \to 0$, δx and δy also $\to 0$, we have

$$\frac{du}{dt} = \underset{\delta y \to 0}{\text{Lt}} \left[\underset{\delta y \to 0}{\text{Lt}} \left\{ \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x} \right\} \right] \frac{dx}{dt} + \underset{\delta y \to 0}{\text{Lt}} \left\{ \frac{f(x, y + \delta y) - f(x, y)}{\delta y} \right\} \frac{dy}{dt}$$

$$= \underset{\delta y \to 0}{\text{Lt}} \left\{ \frac{\partial f(x, y + \delta y)}{\delta y} \right\} \cdot \frac{dx}{dt} + \frac{\partial f(x, y)}{\partial y} \cdot \frac{dy}{dt}$$

[Supposing $\partial f(x, y)/\partial x$ to be a continuous function of y]

$$= \frac{\partial f(x, y)}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f(x, y)}{\partial y} \cdot \frac{dy}{dt} \text{ which is the desired formula.}$$

Cor. Taking $t = x$, (i) becomes, $\dfrac{du}{dx} = \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} \cdot \dfrac{dy}{dx}$ ...(ii)

Obs. If $u = f(x, y, z)$, where x, y, z are all functions of a variable t, then **Chain rule** is

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt} \qquad ...(iii)$$

(2) Differentiation of implicit functions. If $f(x, y) = c$ be an implicit relation between x and y which defines as a differentiable function of x, then (ii) becomes

$$0 = \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

This gives the *important formula* $\mathbf{\dfrac{dy}{dx} = -\dfrac{\partial f}{\partial x} \Big/ \dfrac{\partial f}{\partial y}}$ $\qquad \left[\dfrac{\partial f}{\partial y} \neq 0\right]$

for the first differential coefficient of an implicit function.

Example 5.16. *Given $u = \sin(x/y)$, $x = e^t$ and $y = t^2$, find du/dt as a function of t. Verify your result by direct substitution.*

Solution. We have $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial u}{\partial y} \cdot \dfrac{dy}{dt} = \left(\cos\dfrac{x}{y}\right)\dfrac{1}{y} \cdot e^t + \left(\cos\dfrac{x}{y}\right)\left(-\dfrac{x}{y^2}\right)2t$

$$= \cos(e^t/t^2) \cdot e^t/t^2 - 2\cos(e^t/t^2) \cdot e^t/t^3 = \{(t-2)/t^3\}e^t \cos(e^t/t^2)$$

Also $u = \sin(x/y) = \sin(e^t/t^2)$

$$\therefore \quad \frac{du}{dt} = \cos\left(\frac{e^t}{t^2}\right) \cdot \frac{t^2 e^t - e^t \cdot 2t}{t^4} = \frac{t-2}{t^3} e^t \cos\left(\frac{e^t}{t^2}\right) \text{ as before.}$$

Example 5.17. *If x increases at the rate of 2 cm/sec at the instant when x = 3 cm. and y = 1 cm., at what rate must y be changing in order that the function $2xy - 3x^2y$ shall be neither increasing nor decreasing ?*

Solution. Let $\quad u = 2xy - 3x^2y$, so that

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} = (2y - 6xy)\frac{dx}{dt} + (2x - 3x^2)\frac{dy}{dt} \qquad ...(i)$$

when $x = 3$ and $y = 1$, $dx/dt = 2$, and u is neither increasing nor decreasing, *i.e.*, $du/dt = 0$.

$\therefore$ (i) becomes $\qquad 0 = (2 - 6 \times 3)\, 2 + (2 \times 3 - 3 \times 9)\dfrac{dy}{dt}$

or $\qquad \dfrac{dy}{dt} = -\dfrac{32}{21}$ cm/sec. Thus y is decreasing at the rate of 32/21 cm/sec.

Example 5.18. *If $u = x \log xy$ where $x^3 + y^3 + 3xy = 1$, find du/dx.* (V.T.U., 2009)

Solution. From $f(x, y) = x^3 + y^3 + 3xy - 1$, we have

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y} = -\frac{3x^2+3y}{3y^2+3x} = -\frac{x^2+y}{y^2+x} \quad ...(i)$$

Also $$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} = (1 \cdot \log xy + x \cdot 1/x) + (x/y)\cdot dy/dx.$$

Hence $$du/dx = 1 + \log xy - x(x^2+y)/y(y^2+x) \quad \text{[By } (i)]$$

Example 5.19. *If $u = u\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0.$* (U.P.T.U., 2005)

Solution. Let $$v = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y} \quad \text{and} \quad w = \frac{z-x}{xz} = \frac{1}{x} - \frac{1}{z} \quad ...(i)$$

so that $$u = u(v, w)$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial x} = \frac{\partial u}{\partial v}\left(-\frac{1}{x^2}\right) + \frac{\partial u}{\partial w}\left(-\frac{1}{x^2}\right) \quad \text{[Using } (i)]$$

or $$x^2\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w} \quad ...(ii)$$

Also $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v}\cdot\frac{\partial u}{\partial y} + \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial y} = \frac{\partial u}{\partial v}\left(\frac{1}{y^2}\right) + \frac{\partial u}{\partial w}(0) \quad \text{[Using } (i)]$$

or $$y^2\frac{\partial u}{\partial y} = \frac{\partial u}{\partial v} \quad ...(iii)$$

Similarly $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial v}\ \frac{\partial v}{\partial z} + \frac{\partial u}{\partial w}\ \frac{\partial w}{\partial z} = \frac{\partial u}{\partial v}(0) + \frac{\partial u}{\partial w}\left(\frac{1}{z^2}\right) \quad \text{[Using } (i)]$$

or $$z^2\frac{\partial u}{\partial z} = \frac{\partial u}{\partial w} \quad ...(iv)$$

Adding (*ii*), (*iii*) and (*iv*), we have

$$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0.$$

Example 5.20. *Formula for the second differential coefficient of an implicit function.*
If $f(x, y) = 0$, show that

$$\frac{d^2y}{dx^2} = -\frac{q^2r - 2pqs + p^2t}{q^3}$$ (Kurukshetra, 2006)

Solution. We have $$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{p}{q} \quad ...(i)$$

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{dy}{dx}\right) = -\frac{d}{dx}\left(\frac{p}{q}\right) = -\frac{q(dp/dx) - p(dq/dx)}{q^2} \quad ...(ii)$$

Using the notations : $r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial p}{\partial x}$, $s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial q}{\partial x}$, $t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial q}{\partial y}$,

we have $$\frac{dp}{dx} = \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y}\cdot\frac{dy}{dx} = r + s(-p/q) = -\frac{qr - ps}{q}$$

and $$\frac{dq}{dx} = \frac{\partial q}{\partial x} + \frac{\partial q}{\partial y}\cdot\frac{dy}{dx} = s + t(-p/q) = \frac{qs - pt}{q}$$

Substituting the values of dp/dx and dq/dx in (*ii*), we get

$$\frac{d^2y}{dx^2} = -\frac{1}{q^2}\left[q\left(\frac{qr-ps}{q}\right) - p\left(\frac{qs-pt}{q}\right)\right] = -\frac{q^2r - 2pqs + p^2t}{q^3}.$$

PROBLEMS 5.5

1. If $z = u^2 + v^2$ and $u = at^2$, $v = 2at$, find dz/dt. (*P.T.U., 2005*)
2. If $u = \tan^{-1}(y/x)$ where $x = e^t - e^{-t}$, and $y = e^t + e^{-t}$, find du/dt. (*V.T.U., 2003*)
3. Find the value of $\frac{du}{dt}$ given $u = y^2 - 4ax$, $x = at^2$, $y = 2at$. (*Anna, 2009*)
4. At a given instant the sides of a rectangle are 4 ft. and 3 ft. respectively and they are increasing at the rate of 1.5 ft./sec. and 0.5 ft./sec. respectively, find the rate at which the area is increasing at that instant.
5. If $z = 2xy^2 - 3x^2y$ and if x increases at the rate of 2 cm. per second and it passes through the value $x = 3$ cm., show that if y is passing through the value $y = 1$ cm., y must be decreasing at the rate of $2\frac{2}{15}$ cm. per second, in order that z shall remain constant.
6. If $u = x^2 + y^2 + z^2$ and $x = e^{2t}$, $y = e^{2t}\cos 3t$, $z = e^{2t}\sin 3t$. Find $\frac{du}{dt}$ as a total derivative and verify the result by direct substitution.
7. If $\phi(cx - az, cy - bz) = 0$, show that $\frac{a\partial z}{\partial x} + \frac{b\partial z}{\partial y} = c$.
8. If $f(x, y) = 0$, $\phi(y, z) = 0$, show that $\frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial z}\cdot\frac{dz}{dx} = \frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y}$.
9. If the curves $f(x, y) = 0$ and $\phi(y, z) = 0$ touch, show that at the point of contact, $\frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y} = \frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial \gamma}$.
10. If $f(x, y) = 0$, show that $\left(\frac{\partial f}{\partial y}\right)^3\frac{d^2y}{dx^2} = 2\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial^2 f}{\partial x\partial y}\right) - \left(\frac{\partial f}{dy}\right)^2\frac{\partial^2 f}{\partial x^2} - \left(\frac{\partial f}{\partial x}\right)^2\left(\frac{\partial^2 f}{\partial y^2}\right)$.

5.6 CHANGE OF VARIABLES

If $\quad u = f(x, y)$ ...(1)

where $\quad x = \phi(s, t)$ and $y = \Psi(s, t)$ ...(2)

it is often necessary to change expressions involving $u, x, y, \partial u/\partial x, \partial u/\partial y$ etc. to expressions involving $u, s, t, \partial u/\partial s, \partial u/\partial t$ etc.

The necessary formulae for the change of variables are easily obtained. If t is regarded as a constant, then x, y, u will be functions of s alone. Therefore, by (*i*) of page 208, we have

$$\frac{\partial \mathbf{u}}{\partial \mathbf{s}} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}}\cdot\frac{\partial \mathbf{x}}{\partial \mathbf{s}} + \frac{\partial \mathbf{u}}{\partial \mathbf{y}}\cdot\frac{\partial \mathbf{y}}{\partial \mathbf{s}} \quad ...(3)$$

where the ordinary derivatives have been replaced by the partial derivatives because x, y are functions of two variables s and t.

$\because$ Similarly, regarding s as constant, we obtain $\frac{\partial \mathbf{u}}{\partial \mathbf{t}} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}}\cdot\frac{\partial \mathbf{x}}{\partial \mathbf{t}} + \frac{\partial \mathbf{u}}{\partial \mathbf{y}}\cdot\frac{\partial \mathbf{y}}{\partial \mathbf{t}}$ ...(4)

On solving (3) and (4) as simultaneous equations in $\partial u/\partial x$ and $\partial u/\partial y$, we get their values in terms of $\partial u/\partial s$, $\partial u/\partial t, u, s, t$.

If instead of the equations (2), s and t are given in terms of x and y, say: $s = \xi(x, y)$ and $t = \eta(x, y)$, ...(5)

then it is easier to use the formulae $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$ ...(6)

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial y} \qquad ...(7)$$

The higher derivatives of u can be found by repeated application of formulae (3) and (4) or of (6) and (7).

Example 5.21. *If $u = F(x - y, y - z, z - x)$, prove that*

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$ *(V.T.U., 2010 ; U.T.U. 2009 ; U.P.T.U., 2003)*

Solution. Put $x - y = r, y - z = s$ and $z - x = t$, so that $u = f(r, s, t)$.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r} . (1) + \frac{\partial x}{\partial s} . (0) + \frac{\partial u}{\partial t} . (-1) = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} \qquad ...(i)$$

Similarly,
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial y} = - \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} \qquad ...(ii)$$

and
$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial z} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial z} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial z} = - \frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} \qquad ...(iii)$$

Adding (*i*), (*ii*) and (*iii*), we get the required result.

Example 5.22. *If $z = f(x, y)$ and $x = e^u \cos v$, $y = e^u \sin v$, prove that* $x \dfrac{\partial z}{\partial v} + y \dfrac{\partial z}{\partial u} = e^{2u} \dfrac{\partial z}{\partial y}$

and
$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial u}{\partial v}\right)^2\right]$$ *(Mumbai, 2009)*

Solution. We have $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x} \cdot \dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y} \cdot \dfrac{\partial y}{\partial u}$

$$= \frac{\partial z}{\partial x}(e^u \cos v) + \frac{\partial z}{\partial y}(e^u \sin v) \qquad ...(i)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x}(- e^u \sin v) + \frac{\partial x}{\partial y}(e^u \cos v) \qquad ...(ii)$$

$$\therefore \quad x \frac{\partial x}{\partial v} + y \frac{\partial z}{\partial u} = (e^u \cos v)\left[- e^u \sin v \frac{\partial z}{\partial x} + e^u \cos v \frac{\partial z}{\partial y}\right] + (e^u \sin v)\left[e^u \cos v \frac{\partial z}{\partial x} + e^u \sin v \frac{\partial z}{\partial y}\right]$$

$$= (e^{2u} \cos^2 v + e^{2u} \sin^2 v) \frac{\partial z}{\partial y} = e^{2u} \frac{\partial z}{\partial y}$$

Now squaring (*i*) and (*ii*) and adding, we get

$$\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = e^{2u}\left(\cos v \frac{\partial z}{\partial x} + \sin v \frac{\partial z}{\partial y}\right)^2 + e^{2u}\left(- \sin v \frac{\partial z}{\partial x} + \cos v \frac{\partial z}{\partial y}\right)^2$$

or
$$e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right] = \cos^2 v\left(\frac{\partial z}{\partial x}\right)^2 + \sin^2 v\left(\frac{\partial z}{\partial y}\right)^2 + 2 \sin v \cos v \frac{\delta z}{\delta z} \frac{\delta z}{\delta y}$$

$$+ \sin^2 v \left(\frac{\partial z}{\partial x}\right)^2 + \cos^2 v \left(\frac{\partial z}{\partial y}\right)^2 - 2 \sin v \cos v \frac{\partial z}{\partial x} \frac{\partial z}{\partial y}$$

$$= (\cos^2 v + \sin^2 v) \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

Hence $$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right].$$

Example 5.23. *If $x + y = 2e^\theta \cos\phi$ and $x - y = 2ie^\theta \sin\phi$, show that*

$$\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial \phi^2} = 4xy\frac{\partial^2 u}{\partial x \partial y}$$ *(Nagpur, 2009 ; U.P.T.U., 2002)*

Solution. We have $x = e^\theta(\cos\phi + i\sin\phi) = e^\theta . e^{i\phi}$ [See p. 205]

and $y = e^\theta(\cos\phi - i\sin\phi) = e^\theta . e^{-i\phi}$

Here u is a composite function of θ and ϕ.

$$\therefore \quad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial u}{\partial x}.(e^\theta . e^{i\phi}) + \frac{\partial u}{\partial y}(e^\theta . e^{-i\phi}) = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

or $$\frac{\partial}{\partial \theta} = x\frac{\partial}{\partial u} + y\frac{\partial}{\partial y} \quad ...(i)$$

Also $$\frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \phi} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \phi} = \frac{\partial u}{\partial x}.(e^\theta . ie^{i\phi}) + \frac{\partial u}{\partial y}(e^\theta . -ie^{-i\phi}) = ix\frac{\partial u}{\partial x} - iy\frac{\partial u}{\partial y}$$

$$\frac{\partial}{\partial \phi} = ix\frac{\partial}{\partial x} - iy\frac{\partial}{\partial y} \quad ...(ii)$$

Using the operator (i), we have

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right) = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$= x\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) + x\frac{\partial}{\partial x}\left(y\frac{\partial u}{\partial y}\right) + y\frac{\partial}{\partial y}\left(x\frac{\partial u}{\partial x}\right) + y\frac{\partial}{\partial y}\left(y\frac{\partial u}{\partial y}\right)$$

$$= x\left(x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\right) + xy\frac{\partial^2 u}{\partial x\partial y} + yx\frac{\partial^2 u}{\partial y \partial x} + y\left(y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}\right)$$

$$= x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \quad ...(iii)$$

Similarly using (ii), $$\frac{\partial^2 u}{\partial \phi^2} = \frac{\partial}{\partial \phi}\left(\frac{\partial u}{\partial \phi}\right) = \left(ix\frac{\partial}{\partial x} - iy\frac{\partial}{\partial y}\right)\left(ix\frac{\partial u}{\partial x} - iy\frac{\partial u}{\partial y}\right)$$

$$= -x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} - y^2\frac{\partial^2 u}{\partial y^2} - x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} \quad ...(iv)$$

Adding (iii) and (iv), we get $$\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial \phi^2} = 4xy\frac{\partial^2 u}{\partial x \partial y}$$

Example 5.24. *Transform the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ into polar coordinates.* *(P.T.U., 2010)*

Solution. We have $x = r\cos\theta$, $y = r\sin\theta$ and $r = \sqrt{(x^2 + y^2)}$, $\theta = \tan^{-1}(y/x)$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{(x^2+y^2)}} = \cos\theta \text{ and } \frac{\partial \theta}{\partial x} = -\frac{y}{x^2+y^2} = -\frac{\sin\theta}{r}$$

Thus, $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}$$

i.e., $\dfrac{\partial}{\partial x} = \cos\theta \dfrac{\partial}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial}{\partial \theta}$ Similarly, $\dfrac{\partial}{\partial y} = \sin\theta \dfrac{\partial}{\partial r} + \dfrac{\cos\theta}{r}\dfrac{\partial}{\partial \theta}$.

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$= \cos^2\theta\frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta} \quad ...(i)$$

and
$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$= \sin^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial r} \quad ...(ii)$$

Adding (*i*) and (*ii*), we get $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial\theta^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r}$.

Hence the transformed equation is $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial\theta^2} = 0$.

PROBLEMS 5.6

1. If $z = f(x, y)$ and $x = e^u + e^{-v}$, $y = e^{-u} - e^v$, prove that $\dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v} = x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y}$. *(V.T.U., 2006)*

2. If $u = f(r, s)$, $r = x + at$, $s = y + bt$ and x, y, t are independent variables, show that $\dfrac{\partial u}{\partial t} = a\dfrac{\partial u}{\partial x} + b\dfrac{\partial u}{\partial y}$.

3. If $\phi(z/x^3, y/x) = 0$, prove that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 3z$. *(Mumbai, 2007)*

4. If $u = f(x, y)$ and $x = r\cos\theta$, $y = r\sin\theta$, prove that $\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2 = \left(\dfrac{\partial u}{\partial r}\right)^2 + \dfrac{1}{r^2}\left(\dfrac{\partial u}{\partial\theta}\right)^2$.
(V.T.U., 2010 ; Madras 2006 ; Rohtak, 2005)

5. If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$, prove that $\dfrac{1}{2}\dfrac{\partial u}{\partial x^3} + \dfrac{1}{3}\dfrac{\partial u}{\partial y} + \dfrac{1}{4}\dfrac{\partial u}{\partial z} = 0$. *(U.P.T.U., 2006 ; Raipur, 2005)*

6. If $u = f(e^{y-z}, e^{z-x}, e^{x-y})$, prove that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$ *(Mumbai, 2008 S)*

7. If $u = f(r, s, t)$ and $r = x/y$, $s = y/z$, $t = z/x$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$. *(Anna, 2009 ; Kurukshetra, 2006)*

8. If $x = u + v + w$, $y = vw + wu + uv$, $z = uvw$ and F is a function of x, y, z, show that
$$u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial w} = x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial F}{\partial z}.$$

9. Given that $u(x, y, z) = f(x^2 + y^2 + z^2)$ where $x = r\cos\theta\cos\phi$, $y = r\cos\theta\sin\phi$ and $z = r\sin\theta$, find $\dfrac{\partial u}{\partial\theta}$ and $\dfrac{\partial u}{\partial\phi}$.

10. If the three thermodynamic variables P, V, T are connected by a relation $f(P, V, T) = 0$, show that
$$\left(\frac{\partial P}{\partial T}\right)_V\left(\frac{\partial T}{\partial V}\right)_P\left(\frac{\partial V}{\partial P}\right)_T = -1.$$

11. If by the substitution $u = x^2 - y^2$, $v = 2xy$, $f(x, y) = \theta(u, v)$, show that
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2 + y^2)\left(\frac{\partial^2\theta}{\partial u^2} + \frac{\partial^2\theta}{\partial v^2}\right).$$ *(Anna, 2003)*

12. Transform $\dfrac{\partial^2 z}{\partial x^2} + 2xy^2\dfrac{\partial z}{\partial x} + 2(y - y^3)\dfrac{\partial z}{\partial y} + x^2y^2z = 0$ by the substitution $x = uv$, $y = 1/v$. Hence show that z is the same function of u and v as of x and y.

5.7 (1) JACOBIANS

If u and v are functions of two independent variables x and y, then the determinant

$$\begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} \text{ is called the } Jacobian^* \text{ of } u, v \text{ with respect to } x, y$$

and is written as $\dfrac{\partial(u, v)}{\partial(x, y)}$ or $J\left(\dfrac{u, v}{x, y}\right)$.

Similarly the Jacobian of u, v, w with respect to x, y, z is

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix}$$

Likewise, we can define Jacobians of four or more variables. An important application of Jacobians is in connection with the change of variables in multiple integrals (§ 7.7).

(2) Properties of Jacobians. We give below two of the important properties of Jacobians. For simplicity, the properties are stated in terms of two variables only, but these are evidently true in general.

I. If $J = \partial(u, v)/\partial(x, y)$ and $J' = \partial(x, y)/\partial(u, v)$ then $JJ' = 1$.

Let $u = f(x, y)$ and $v = g(x, y)$.

Suppose, on solving for x and y, we get $x = \phi(u, v)$ and $y = \psi(u, v)$.

Then
$$\left.\begin{aligned} \frac{\partial u}{\partial u} &= 1 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u}, \\ \frac{\partial u}{\partial v} &= 0 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v}, \\ \frac{\partial v}{\partial u} &= 0 = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u}, \\ \text{and} \quad \frac{\partial v}{\partial v} &= 1 = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v}, \end{aligned}\right\} \quad ...(i)$$

$$\therefore \quad JJ' = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} \times \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial y/\partial x & \partial v/\partial y \end{vmatrix} \times \begin{vmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{vmatrix}$$

(Interchanging rows and columns of the 2nd determinant).

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1. \qquad \text{[By virtue of } (i)]$$

II. **Chain rule for Jacobians.** *If u, v are functions of r, s and r, s are functions of x, y, then*

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(r, s)}\cdot\frac{\partial(r, s)}{\partial(x, y)}.$$

$$\frac{\partial(u, v)}{\partial(r, s)}\cdot\frac{\partial(r, s)}{\partial(x, y)} = \begin{vmatrix} \partial u/\partial r & \partial u/\partial s \\ \partial v/\partial r & \partial v/\partial s \end{vmatrix} \times \begin{vmatrix} \partial r/\partial x & \partial s/\partial x \\ \partial r/\partial y & \partial s/\partial y \end{vmatrix}$$

[Interchanging rows and columns of the 2nd det.]

$$= \begin{vmatrix} \dfrac{\partial u}{\partial r}\cdot\dfrac{\partial r}{\partial x} + \dfrac{\partial u}{\partial s}\cdot\dfrac{\partial s}{\partial x} & \dfrac{\partial u}{\partial r}\cdot\dfrac{\partial r}{\partial y} + \dfrac{\partial u}{\partial s}\cdot\dfrac{\partial s}{\partial y} \\ \dfrac{\partial v}{\partial r}\cdot\dfrac{\partial r}{\partial x} + \dfrac{\partial v}{\partial s}\cdot\dfrac{\partial s}{\partial x} & \dfrac{\partial v}{\partial r}\cdot\dfrac{\partial r}{\partial y} + \dfrac{\partial v}{\partial s}\cdot\dfrac{\partial s}{\partial y} \end{vmatrix} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial(u, v)}{\partial(x, y)}.$$

* Called after the German mathematician *Carl Gustav Jacob Jacobi* (1804–1851), who made significant contributions to mechanics, partial differential equations, astronomy, elliptic functions and the calculus of variations.

Example 5.25. *(i) In polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, show that*

$$\frac{\partial(x, y)}{\partial(r, \theta)} = r.$$ *(U.P.T.U., 2006 ; V.T.U., 2004 ; Andhra, 2000)*

(ii) In cylindrical coordinates (Fig. 8.28), $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$, show that

$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho.$$

(iii) In spherical polar coordidnates (Fig. 8.29), $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, show that

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2\sin\theta.$$ *(Anna, 2009 ; Hazaribagh, 2009 ; Rohtak, 2003)*

Solution. (*i*) We have $\quad \dfrac{\partial x}{\partial r} = \cos\theta,\ \dfrac{\partial x}{\partial \theta} = -r\sin\theta \quad$ and $\quad \dfrac{\partial y}{\partial r} = \sin\theta,\ \dfrac{\partial y}{\partial \theta} = -r\cos\theta$

$$\therefore \qquad \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

(*ii*) We have $\quad \dfrac{\partial x}{\partial \rho} = \cos\phi,\ \dfrac{\partial x}{\partial \phi} = -\rho\sin\phi,\ \dfrac{\partial x}{\partial z} = 0,$

$$\frac{\partial y}{\partial \rho} = \sin\phi,\ \frac{\partial y}{\partial \phi} = \rho\cos\phi,\ \frac{\partial y}{\partial z} = 0 \text{ and } \frac{\partial z}{\partial \rho} = 0,\ \frac{\partial z}{\partial \phi} = 0,\ \frac{\partial z}{\partial z} = 1$$

$$\therefore \qquad \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \begin{vmatrix} \cos\phi & -\rho\sin\theta & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho.$$

(*iii*) We have $\quad \dfrac{\partial x}{\partial r} = \sin\theta\cos\theta,\ \dfrac{\partial x}{\partial \theta} = r\cos\theta\cos\phi,\ \dfrac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi,$

$$\frac{\partial y}{\partial r} = \sin\theta\sin\phi,\ \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi,\ \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi,$$

and $$\frac{\partial z}{\partial r} = \cos\theta,\ \frac{\partial z}{\partial \theta} = -r\sin\theta,\ \frac{\partial z}{\partial \phi} = 0.$$

$$\therefore \qquad \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta.$$

Example 5.26. *If $y_1 = \dfrac{x_2x_3}{x_1}$, $y_2 = \dfrac{x_3x_1}{x_2}$, $y_3 = \dfrac{x_1x_2}{x_3}$, show that the Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3 is 4.* *(U.P.T.U., 2006)*

Solution. We have $\quad \dfrac{\partial y_1}{\partial x_1} = -\dfrac{x_2x_3}{x_1^2},\ \dfrac{\partial y_1}{\partial x_2} = \dfrac{x_3}{x_1},\ \dfrac{\partial y_1}{\partial x_3} = \dfrac{x_2}{x_1}$

$$\frac{\partial y_2}{\partial x_1} = \frac{x_3}{x_2},\ \frac{\partial y_2}{\partial x_2} = -\frac{x_3x_1}{x_2^2},\ \frac{\partial y_2}{\partial x_3} = \frac{x_1}{x_2}$$

and $$\frac{\partial y_3}{\partial x_1} = \frac{x_2}{x_3},\ \frac{\partial y_3}{\partial x_2} = \frac{x_1}{x_3},\ \frac{\partial y_3}{\partial x_3} = -\frac{x_1x_2}{x_3^2}$$

$$\therefore \qquad \frac{\partial(y_1y_2y_3)}{\partial(x_1x_2x_3)} = \begin{vmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} & \dfrac{\partial y_1}{\partial x_3} \\ \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} & \dfrac{\partial y_2}{\partial x_3} \\ \dfrac{\partial y_3}{\partial x_1} & \dfrac{\partial y_3}{\partial x_2} & \dfrac{\partial y_3}{\partial x_3} \end{vmatrix} \begin{vmatrix} \dfrac{-x_2x_3}{x_1^2} & \dfrac{x_3}{x_1} & \dfrac{x_2}{x_1} \\ \dfrac{x_3}{x_2} & \dfrac{-x_3x_1}{x_2^2} & \dfrac{x_1}{x_1} \\ \dfrac{x_2}{x_3} & \dfrac{x_1}{x_3} & \dfrac{-x_1x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2x_2^2x_3^2}\begin{vmatrix} -x_2x_3 & x_3x_1 & x_1x_2 \\ x_2x_3 & -x_3x_1 & x_1x_2 \\ x_2x_3 & x_3x_1 & -x_1x_2 \end{vmatrix} = -\frac{x_1^2x_2^2x_3^2}{x_1^2x_2^2x_3^2}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= -1(1-1) - 1(-1-1) + 1(1+1) = 0 + 2 + 2 = 4.$$

Example 5.27. *In $u = x + 3y^2 - z^3$, $v = 4x^2yz$, $w = 2z^2 - xy$, evaluate $\partial(u, v, w)/\partial(x, y, z)$ at $(1, -1, 0)$.* (V.T.U., 2006)

Solution. $$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 6y & -3z^2 \\ 8xyz & 4x^2z & 4x^2y \\ -y & -x & 4z \end{vmatrix}$$

$\therefore$ At the point $(1, -1, 0)$ $$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} 1 & -6 & 0 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix} = 4(-1 + 6) = 20.$$

Example 5.28. *If $u = x^2 - y^2$, $v = 2xy$ and $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial(u, v)}{\partial(r, \theta)}$.* (V.T.U., 2009 ; Madras, 2006)

Solution. We have $$\frac{\partial(u, v)}{\partial(r, \theta)} = \frac{\partial(u, v)}{\partial(x, y)} \times \frac{\partial(x, y)}{\partial(r, \theta)}$$

Since $u = x^2 - y^2$, $u = 2xy$

$$\therefore \quad \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2 + y^2) \qquad ...(ii)$$

Since $x = r\cos\theta$, $y = r\sin\theta$,

$$\therefore \quad \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \qquad ...(iii)$$

Hence, $$\frac{\partial(u, v)}{\partial(r, \theta)} = 4(x^2 + y^2).\, r = 4(r^2\cos^2\theta + r^2\sin^2\theta).\, r = 4r^3$$ [Using (*ii*) & (*iii*)]

(3) Jacobian of Implicit functions. *If u_1, u_2, u_3 instead of being given explicitly in terms x_1, x_2, x_3, be connected with them equations such as*

$$f_1(u_1, u_2, u_3, x_1, x_2, x_3) = 0,\ f_2(u_1, u_2, u_3, x_1, x_2, x_3) = 0,\ f_3(u_1, u_2, u_3, x_1, x_2, x_3) = 0, \text{ then}$$

$$\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = (-1)^3\frac{\partial(f_1, f_2, f_3)}{\partial(x_1, x_2, x_3)} \div \frac{\partial(f_1, f_2, f_3)}{\partial(u_1, u_2, u_3)}$$

Obs. This result can be easily generalised. It bears analogy to the result $\frac{dy}{dx} = -\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}$, where x, y are connected by the relation $f(x, y) = 0$.

Example 5.29. *If $u = x, y, z$, $v = x^2 + y^2 + z^2$, $w = x + y + z$, find $\partial(x, y, z)/\partial(u, v, w)$.* (U.P.T.U., 2003)

Solution. Let $$f_1 = u - x\,y\,z,\ f_2 = v - x^2 - y^2 - z^2,\ f_3 = w - x - y - z.$$

We have $$\frac{\partial(x, y, z)}{\partial(u, v, w)} = (-1)^3 \frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} \div \frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} \quad ...(i)$$

Now, $$\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} = \begin{vmatrix} \partial f_1/\partial x & \partial f_1/\partial y & \partial f_1/\partial z \\ \partial f_2/\partial x & \partial f_2/\partial y & \partial f_2/\partial z \\ \partial f_3/\partial x & \partial f_3/\partial y & \partial f_3/\partial z \end{vmatrix} = \begin{vmatrix} -yz & -xz & -xy \\ -2x & -2y & -2z \\ -1 & -1 & -1 \end{vmatrix}$$
$$= -2(x-y)(y-z)(z-x) \quad ...(ii)$$

and $$\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \quad ...(iii)$$

Substituting values from (*ii*) and (*iii*) in (*i*), we get
$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = (-1) \times 1/[-2(x-y)(y-z)(z-x)] = 1/2(x-y)(y-z)(z-x).$$

(4) Functional relationship. *If u_1, u_2, u_3 be functions of x_1, x_2, x_3 then the necessary and sufficient condition for the existence of a functional relationship of the form $f(u_1, u_2, u_3) = 0$, is*
$$J\left(\frac{u_1, u_2, u_3}{x_1, x_2, x_3}\right) = 0.$$

Example 5.30. *If $u = x\sqrt{(1-y^2)} + y\sqrt{(1-x^2)}$, $v = \sin^{-1} x + \sin^{-1} y$, show that u, v are functionally related and find the relationship.* *(Kurukshetra, 2006)*

Solution. We have $$\frac{\partial u}{\partial x} = \sqrt{(1-y^2)} - \frac{xy}{\sqrt{(1-x^2)}}, \frac{\partial u}{\partial y} = \frac{-xy}{\sqrt{(1-y^2)}} + \sqrt{(1-x^2)}$$

and $$\frac{\partial v}{\partial x} = \frac{1}{\sqrt{(1-x^2)}}, \frac{\partial v}{\partial y} = \frac{1}{\sqrt{(1-y^2)}}$$

$\therefore$ $$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \sqrt{(1-y^2)} - \dfrac{xy}{\sqrt{(1-x^2)}}, & \sqrt{(1-x^2)} - \dfrac{xy}{\sqrt{(1-y^2)}} \\ \dfrac{1}{\sqrt{(1-x^2)}}, & \dfrac{1}{\sqrt{(1-y^2)}} \end{vmatrix}$$
$$= 1 - \frac{xy}{\sqrt{[(1-x^2)(1-y^2)]}} - 1 + \frac{xy}{\sqrt{[(1-x^2)(1-y^2)]}} = 0$$

Hence u and v are functionally related *i.e.*, they are not independent.

We have $$v = \sin^{-1} x + \sin^{-1} y = \sin^{-1} [x\sqrt{(1-y^2)} + y\sqrt{(1-x^2)}]$$

i.e., $$u = \sin v$$

which is the required relationship between u and v.

PROBLEMS 5.7

1. If $x = r\cos\theta$, $y = r\sin\theta$, evaluate $\partial(r, \theta)/\partial(x, y)$, $\partial(x, y)/\partial(r, \theta)$ and prove that $\{\partial(r, \theta)\,\partial(x, y)\}.\{\partial(x, y)/\partial(r, \theta)\} = 1$. *(V.T.U., 2010)*
2. If $x = u(1-v)$, $y = uv$, prove that $JJ' = 1$. *(V.T.U., 2000 S)*
3. If $x = a\cosh\xi\cos\eta$, $y = a\sinh\xi\sin\eta$, show that $\partial(x, y)/\partial(\xi, \eta) = \frac{1}{2}a^2(\cosh 2\xi - \cos 2\eta)$ *(S.V.T.U., 2007)*
4. If $x = e^u \sec v$, $y = e^u \tan v$, find $J = \partial(u, v)/\partial(x, y)$, $J' = \partial(x, y)/\partial(u, v)$. Hence show $JJ' = 1$. *(V.T.U., 2007 S)*
5. If $u = x^2 - 2y^2$, $v = 2x^2 - y^2$ where $x = r\cos\theta$, $y = r\sin\theta$, show that $\dfrac{\partial(u, v)}{\partial(r, \theta)} = 6r^3 \sin 2\theta$.
6. If $u = x^2 + y^2 + z^2$, $v = xy + yz + zx$, $w = x + y + z$, find $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$. *(U.T.U., 2009 ; V.T.U., 2008)*

7. If $F = xu + v - y$, $G = u^2 + vy + w$, $H = zu - v + vw$, compute $\partial(F, G, H)/\partial(u, v, w)$.

8. If $u = x + y + z$, $uv = y + z$, $uvw = z$, show that $\partial(x, y, z)/\partial(u, v, w) = u^2v$.
(*Kurukshetra, 2009; P.T.U., 2009 S; V.T.U., 2003*)

9. If $u^3 + v^3 = x + y$ and $u^2 + v^2 = x^3 + y^3$, show that $\dfrac{\partial(u, v)}{\partial(x, y)} = \dfrac{1}{2}\dfrac{y^2 - x^2}{uv(u - v)}$ (*U.P.T.U., 2006 MCA*)

10. If $u = \dfrac{x + y}{1 - xy}$ and $v = \tan^{-1} x + \tan^{-1} y$, find $\dfrac{\partial(u, v)}{\partial(x, y)}$. Are u and v functionally related. If so, find this relationship.
(*Nagpur, 2008*)

11. If $u = 3x + 2y - z$, $v = x - 2y + z$ and $w = x(x + 2y - z)$, show that they are functionally related, and find the relation.
(*Nagpur, 2009*)

5.8 (1) GEOMETRICAL INTERPRETATION

If $P(x, y, z)$ be the coordinates of a point referred to axes OX, OY, OZ then the equation $z = f(x, y)$ represents a surface. (Fig. 5.1)

Let a plane $y = b$ parallel to the XZ-plane pass through P cutting the surface along the curve APB given by

$$z = f(x, b).$$

As y remains equal to b and x varies then P moves along the curve APB and $\partial z/\partial x$ is the ordinary derivative of $f(x, b)$ w.r.t. x.

Hence *$\partial z/\partial x$ at P is the tangent of the angle which the tangent at P to the section of the surface $z = f(x, y)$ by a plane through P parallel to the plane XOZ, makes with a line parallel to the x-axis.*

Similarly, *$\partial z/\partial y$ at P is the tangent of the angle which the tangent at P to the curve of intersection of the surface $z = f(x, y)$ and the plane $x = a$, makes with a line parallel to the y-axis.*

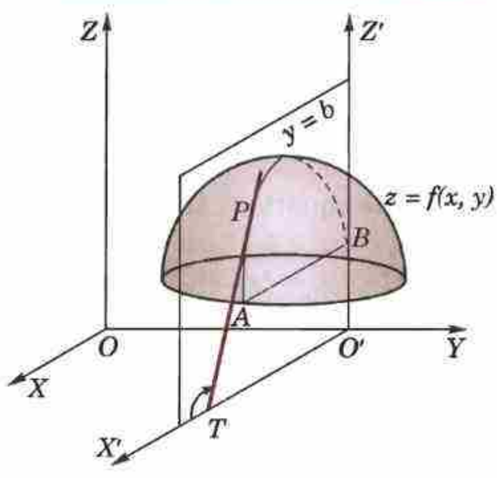


Fig. 5.1

(2) Tangent plane and Normal to a surface. Let $P(x, y, z)$ and $Q(x + \delta x, y + \delta y, z + \delta z)$ be two neighbouring points on the surface $F(x, y, z) = 0$. (Fig. 5.2) ...(*i*)

Let the arc PQ be δs and the chord PQ be δc, so that (as for plane curves)

$$\underset{Q \to P}{\text{Lt}} (\delta s/\delta c) = 1.$$

The direction cosines of PQ are $\dfrac{\delta x}{\delta c}, \dfrac{\delta y}{\delta c}, \dfrac{\delta z}{\delta c}$ *i.e.*, $\dfrac{\delta x}{\delta s} \cdot \dfrac{\delta s}{\delta c}, \dfrac{\delta y}{\delta s} \cdot \dfrac{\delta s}{\delta c}, \dfrac{\delta z}{\delta s} \cdot \dfrac{\delta s}{\delta c}$

When $\delta s \to 0$, $Q \to P$ and PQ tends to tangent line PT. Then noting that the coordinates of any point on arc PQ are functions of s only, the direction cosines of PT are

$$\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds} \qquad ...(ii)$$

Differentiating (*i*) with respect to s, we obtain $\dfrac{\partial F}{\partial x} \cdot \dfrac{dx}{ds} + \dfrac{\partial F}{\partial y} \cdot \dfrac{dy}{ds} + \dfrac{\partial F}{\partial z} \cdot \dfrac{dz}{ds} = 0.$

This shows that the tangent line whose direction cosines are given by (*ii*), is perpendicular to the line having direction ratios

$$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \qquad ...(iii)$$

Since we can take different curves joining Q to P, we get a number of tangent lines at P and the line having direction ratios (*iii*) will be perpendicular to *all* these tangent lines at P. Thus all the tangent lines at P lie in a plane through P perpendicular to line (*iii*).

Hence *the equation of the tangent plane to (i) at the point P is*

$$\frac{\partial \mathbf{F}}{\partial \mathbf{x}} (\mathbf{X} - \mathbf{x}) + \frac{\partial \mathbf{F}}{\partial \mathbf{y}} (\mathbf{Y} - \mathbf{y}) + \frac{\partial \mathbf{F}}{\partial \mathbf{z}} (\mathbf{Z} - \mathbf{z}) = \mathbf{0}$$

where (X, Y, Z) are the current coordinates of any point on this tangent plane.

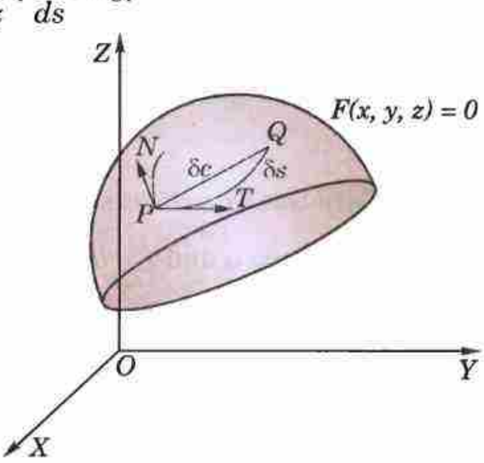


Fig. 5.2

Also *the equation of the normal to the surface at P (i.e., the line through P,* perpendicular to the tangent plane at *P*) is

$$\frac{\mathbf{X}-\mathbf{x}}{\partial \mathbf{F}/\partial \mathbf{x}}=\frac{\mathbf{Y}-\mathbf{y}}{\partial \mathbf{F}/\partial \mathbf{y}}=\frac{\mathbf{Z}-\mathbf{z}}{\partial \mathbf{F}/\partial \mathbf{z}}.$$

Example 5.31. *Find the equations of the tangent plane and the normal to the surface $z^2 = 4(1 + x^2 + y^2)$ at (2, 2, 6).*

Solution. We have $F(x, y, z) = 4x^2 + 4y^2 - z^2 + 4.$

$\therefore$ $\partial F/\partial x = 8x,\ \partial F/\partial y = 8y,\ \partial F/\partial z = -2z$, and at the point (2, 2, 6)

$\partial F/\partial x = 16,\ \partial F/\partial y = 16,\ \partial F/\partial z = -12$

Hence the equation of the tangent plane at (2, 2, 6) is $16(X-2) + 16(Y-2) - 12(Z-6) = 0$

i.e., $4X + 4Y - 3Z + 2 = 0$ ...(*i*)

Also the equation of the normal at (2, 2, 6) [being perpendicular to (*i*)] is

$$\frac{X-2}{4}=\frac{Y-2}{4}=\frac{Z-6}{-3}.$$

PROBLEMS 5.8

Find the equations of the tangent plane and normal to each of the following surfaces at the given points :

1. $2x^2 + y^2 = 3 - 2z$ at $(2, 1, -3)$ (*Assam, 1998*)
2. $x^3 + y^3 + 3xyz = 3$ at $(1, 2, -1)$ (*Osmania, 2003 S*)
3. $xyz = a^2$ at (x_1, y_1, z_1).
4. $2xz^2 - 3xy - 4x = 7$ at $(1, -1, 2)$.
5. Show the plane $3x + 12y - 6z - 17 = 0$ touches the conicoid $3x^2 - 6y^2 + 9z^2 + 17 = 0$. Find also the point of contact.
6. Show that the plane $ax + by + cz + d = 0$ touches the surface $px^2 + qy^2 + 2z = 0$, if $\dfrac{a^2}{p}+\dfrac{b^2}{q}+2cd=0$.
7. Find the equation of the normal to the surface $x^2 + y^2 + z^2 = a^2$. (*P.T.U., 2009 S*)

5.9 TAYLOR'S THEOREM FOR FUNCTIONS OF TWO VARIABLES

Considering $f(x + h, y + k)$ as a function of a single variable x, we have by Taylor's theorem*

$$f(x+h, y+k) = f(x, y+k) + h\,\frac{\partial f(x, y+k)}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 f(x, y+k)}{\partial x^2} + \dots \qquad \dots(i)$$

Now expanding $f(x, y + k)$ as a function of y only,

$$f(x, y+k) = f(x, y) + k\,\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} + \dots$$

$\therefore$ (*i*) takes the form $f(x+h, y+k) = f(x, y) + k\,\dfrac{\partial f(x, y)}{\partial y} + \dfrac{k^2}{2!}\dfrac{\partial^2 f(x, y)}{\partial y^2} + \dots$

$$+ h\,\frac{\partial}{\partial x}\left\{f(x, y) + k\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} + \dots\right\} + \frac{h^2}{2!}\frac{\partial^2}{\partial x^2}\left\{f(x, y) + k\frac{\partial (x, y)}{\partial y} + \dots\right\}$$

Hence, $$f(x+h, y+k) = f(x, y) + h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \dots \qquad \dots(1)$$

In symbols we write it as $f(x+h, y+k) = f(x, y) + \left(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}\right) f + \dfrac{1}{2!}\left(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}\right)^2 f + \dots$

Taking $x = a$ and $y = b$, (1) becomes

$$f(a+h, b+k) = f(a, b) + [hf_x(a, b) + kf_y(a, b)] + \frac{1}{2!}[h^2 f_{xx}(a, b) + 2hkf_{xy}(a, b) + k^2 f_{yy}(a, b)] + \dots$$

---

*See footnote on page 145.

Putting $a + h = x$ and $b + k = y$ so that $h = x - a$, $k = y - b$, we get

$$\mathbf{f(x, y) = f(a, b) + [(x - a)\, f_x\,(a, b) + (y - b)\, f_y\,(a, b)]}$$
$$\mathbf{+ \frac{1}{2!}\,[(x - a)^2\, f_{xx}\,(a, b) + 2(x - a)(y - b)\, f_{xy}\,(a, b) + (y - b)^2\, f_{yy}(a, b)] + \ldots} \quad \ldots(2)$$

This is Taylor's expansion of f(x, y) in powers of (x − a) and (y − b). It is used to expand f(x, y) in the neighbourhood of (a, b).

Cor. Putting $a = 0, b = 0$, in (2), we get

$$\mathbf{f(x, y) = f(0, 0) + [xf_x(0, 0) + yf_y\,(0, 0)] + \frac{1}{2!}\,[x^2\, f_{xx}\,(0, 0) + 2xy\, f_{xy}\,(0, 0) + y^2\, f_{yy}\,(0, 0)] + \ldots} \quad \ldots(3)$$

This is Maclaurin's expansion of $f(x, y)$.

Example 5.32. *Expand $e^x \log(1 + y)$ in powers of x and y upto terms of third degree.*
(V.T.U., 2010 ; P.T.U., 2009 ; J.N.T.U., 2006)

Solution. Here

$$\begin{array}{lcl} f(x, y) = e^x \log(1 + y) & \therefore & f(0, 0) = 0 \\ f_x(x, y) = e^x \log(1 + y) & & f_x(0, 0) = 0 \\ f_y(x, y) = e^x\, \dfrac{1}{1 + y} & & f_y(0, 0) = 1 \\ f_{xx}(x, y) = e^x \log(1 + y) & & f_{xx}(0, 0) = 0 \\ f_{xy}(x, y) = e^x\, \dfrac{1}{1 + y} & & f_{xy}(0, 0) = 1 \\ f_{yy}(x, y) = -\,e^x (1 + y)^{-2} & & f_{yy}(0, 0) = -1 \\ f_{xxx}(x, y) = e^x \log(1 + y) & & f_{xxx}(0, 0) = 0 \\ f_{xxy}(x, y) = e^x\, \dfrac{1}{1 + y} & & f_{xxy}(0, 0) = 1 \\ f_{xyy}(x, y) = -\,e^x (1 + y)^{-2} & & f_{xyy}(0, 0) = -1 \\ f_{yyy}(x, y) = 2e^x (1 + y)^{-3} & & f_{yyy}(0, 0) = 2 \end{array}$$

Now Maclaurin's expansion of $f(x, y)$ gives

$$f(x, y) = f(0, 0) + xf_x(0, 0) + yf_y(0, 0) + \frac{1}{2!}\,\{x^2 f_{xx}(0, 0) + 2xy\, f_{xy}(0, 0) + y^2 f_{yy}(0, 0)\}$$
$$+ \frac{1}{3!}\,\{x^3 f_{xxx}(0, 0) + 3x^2 y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)\} + \ldots$$

$$\therefore \qquad e^x \log(1 + y) = 0 + x(0) + y(1) + \frac{1}{2!}\,\{x^2\,(0) + 2xy(1) + y^2\,(-1)\}$$
$$+ \frac{1}{3!}\,\{x^3(0) + 3x^2y(1) + 3xy^2(-1) + y^3(2)\} + \ldots$$
$$= y + xy - \frac{1}{2}\,y^2 + \frac{1}{2}\,(x^2y - xy^2) + \frac{1}{3}\,y^3 + \ldots$$

Example 5.33. *Expand $x^2y + 3y - 2$ in powers of (x − 1) and (y + 2) using Taylor's theorem.*
(P.T.U., 2010 ; V.T.U., 2008 ; U.P.T.U., 2006 ; Anna, 2005)

Solution. Taylor's expansion of $f(x, y)$ in powers of $(x - a)$ and $(x - b)$ is given by

$$f(x, y) = f(a, b) + [(x - a)\, f_x(a, b) + (y - b)\, f_y(a, b)] + \frac{1}{2!}\,[(x - a)^2 f_{xx}(a, b)$$
$$+ 2\,(x - a)(y - b)\, f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b)] + \frac{1}{3!}\,[(x - a)^3 f_{xxx}(a, b)$$
$$+ 3\,(x - a)^2\,(y - b)\, f_{xxy}(a, b) + 3\,(x - a)\,(y - b)^2 f_{xyy}\,(a, b)$$
$$+ (y - b)^3 f_{yyy}\,(a, b)] + \ldots \quad \ldots(i)$$

Hence $a = 1, b = -2$ and $f(x, y) = x^2y + 3y - 2$

$\therefore \quad f(1,-2) = -10, f_x = 2xy, f_x(1,-2) = -4; f_y = x^2+3, f_y(1,-2) = 4; f_{xx} = 2y,$
$f_{xx}(1,-2) = -4; f_{xy} = 2x, f_{xy}(1,-2) = 2; f_{yy} = 0, f_{yy}(1,-2) = 0; f_{xxx} = 0, f_{xxx} = (1,-2) = 0;$
$f_{xxy}(1,-2) = 2, f_{xyy}(1,-2) = 0, f_{yyy}(1,-2) = 0$

All partial derivatives of higher order vanish.

Substituting these in (*i*), we get

$$x^2y + 3y - 2 = -10 + [(x-1)(-4) + (y+2)4] + \frac{1}{2}[(x-1)^2(-4) + 2(x-1)(y+2)(2)$$
$$+ (y+2)^2(0)] + \frac{1}{6}[(x-1)^3(0) + 3(x-1)^2(y+2)(2) + 3(x-1)(y+2)^2(0) + (y+2)^3(0)]$$
$$= -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2).$$

Example 5.34. *Expand $f(x, y) = \tan^{-1}(y/x)$ in powers of $(x-1)$ and $(y-1)$ upto third-degree terms. Hence compute $f(1.1, 0.9)$ approximately.* (V.T.U., 2010 ; J.N.T.U., 2006 ; U.P.T.U., 2006)

Solution. Here $a = 1, b = 1$ and $f(1, 1) = \tan^{-1}(1) = \pi/4$.

$$f_x = \frac{-y}{x^2+y^2} \qquad f_x(1,1) = -\frac{1}{2}; \qquad f_y = \frac{x}{x^2+y^2}, \qquad f_y(1,1) = \frac{1}{2}$$

$$f_{xx} = \frac{2xy}{(x^2+y^2)^2}, \qquad f_{xx}(1,1) = \frac{1}{2}; \qquad f_{xy} = \frac{y^2-x^2}{(x^2+y^2)^2}, \qquad f_{xy}(1,1) = 0$$

$$f_{yy} = \frac{-2xy}{(x^2+y^2)^2}, \qquad f_{yy}(1,1) = -\frac{1}{2};$$

$$f_{xxx} = \frac{2y^3-6x^2y}{(x^2+y^2)^3}, \qquad f_{xxx}(1,1) = -\frac{1}{2}; \qquad f_{xxy} = \frac{2x^3-6xy^2}{(x^3+y^2)^3}, \qquad f_{xxy}(1,1) = -\frac{1}{2}$$

$$f_{xyy} = \frac{6x^2y-2y^3}{(x^2+y^2)^3}, \qquad f_{xyy}(1,1) = \frac{1}{2}; \qquad f_{yyy} = \frac{6xy^2-2x^3}{(x^2+y^2)^3}, \qquad f_{yyy}(1,1) = \frac{1}{2}$$

Taylor's expansion of $f(x, y)$ in powers of $(x-1)$ and $(y-1)$ is given by

$$f(x,y) = f(1,1) + \frac{1}{1!}\{(x-1)f_x(1,1) + (y-1)f_y(1,1)\} + \frac{1}{2!}\{(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1)$$
$$f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1) + \frac{1}{3!}\{(x-1)^3 f_{xxx}(1,1) + 3(x-1)^2(y-1)f_{xxy}(1,1)$$
$$+ 3(x-1)(y-1)^2 f_{xyy}(1,1) + (y-1)^3 f_{yyy}(1,1)\} + \ldots$$

$$\therefore \quad \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} + \left\{(x-1)\left(-\frac{1}{2}\right) + (y-1)\frac{1}{2}\right\} + \frac{1}{2!}\left\{(x-1)^2\frac{1}{2} + 2(x-1)(y-1)(0) + (y-1)^2\left(-\frac{1}{2}\right)\right\}$$
$$+ \frac{1}{3!}\left\{(x-1)^3\left(-\frac{1}{2}\right) + 3(x-1)^2(y-1)\left(-\frac{1}{2}\right) + 3(x-1)(y-1)^2\frac{1}{2} + (y-1)^3\frac{1}{2}\right\} + \ldots$$
$$= \frac{\pi}{4} - \frac{1}{2}\{(x-1)-(y-1)\} + \frac{1}{4}\{(x-1)^2-(y-1)^2\} - \frac{1}{12}\{(x-1)^3 + 3(x-1)^2(y-1)$$
$$- 3(x-1)(y-1)^2 - (y-1)^3\} + \ldots$$

Putting $x = 1.1$ and $y = 0.9$, we get

$$f(1.1, 0.9) = \frac{\pi}{4} - \frac{1}{2}(0.2) + \frac{1}{4}(0) - \frac{1}{12}\{(0.1)^3 - 3(0.1)^3 - 3(0.1)^3 - (-0.1)^3\}$$
$$= 0.7854 - 0.1000 + 0.0003 = 0.6857.$$

5.10 (1) ERRORS AND APPROXIMATIONS

Let $f(x, y)$ be a continuous function of x and y. If δx and δy be the increments of x and y, then the new value of $f(x, y)$ will be $f(x + \delta x, y + \delta y)$. Hence

$$\delta f = f(x + \delta x, y + \delta y) - f(x, y).$$

Expanding $f(x + \delta x, y + \delta y)$ by Taylor's theorem and supposing δx, δy to be so small that their products, squares and higher powers can be neglected, we get

$$\delta \mathbf{f} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \delta \mathbf{x} + \frac{\partial \mathbf{f}}{\partial \mathbf{y}} \delta \mathbf{y}, \text{ approximately.}$$

Similarly if f be a function of several variables $x, y, z, t, \ldots$, then

$$\delta f = \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial z} \delta z + \frac{\partial f}{\partial t} \delta t + \ldots \text{ approximately.}$$

These formulae are very useful in correcting the effect of small errors in measured quantities.

(2) Total Differential

If u is a function of two variables x and y, the *total differential* of u is defined as

$$\mathbf{du} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \mathbf{dx} + \frac{\partial \mathbf{u}}{\partial \mathbf{y}} \mathbf{dy} \quad \ldots(1)$$

The differentials dx and dy are respectively the increments δx and δy in x and y. If x and y are not independent variables but functions of another variable t even then the formula (1) holds and we write $dx = \frac{dx}{dt} dt$ and $dy = \frac{dy}{dt} dt$. Similar definition can be given for a function of three or more variables.

Example 5.35. *The diameter and altitude of a can in the shape of a right circular cylinder are measured as 4 cm and 6 cm respectively. The possible error in each measurement is 0.1 cm. Find approximately the maximum possible error in the values computed for the volume and the lateral surface.*

Solution. Let x be the diameter and y the height of the can. Then its volume $V = \frac{\pi}{4} x^2 y$

$$\therefore \quad \delta V = \frac{\partial V}{\partial x} \delta x + \frac{\partial V}{\partial y} \delta y = \frac{\pi}{4} (2xy\delta x + x^2 \delta y)$$

When $x = 4$ cm., $y = 6$ cm. and $\delta x = \delta y = 0.1$ cm.

$$\therefore \quad \delta V = \frac{\pi}{4}(2 \times 4 \times 6 \times 0.1 + 4^2 \times 0.1) = 1.6\pi \text{ cm}^3$$

Also its lateral surface $S = \pi xy$

$$\therefore \quad \delta S = \pi(y\, \delta x + x\, \delta y)$$

When $x = 4$ cm., $y = 6$ cm. and $\delta x = \delta y = 0.1$ cm., we have $\delta S = \pi(6 \times 0.1 + 4 \times 0.1) = \pi \text{ cm}^2$.

Example 5.36. *The period of a simple pendulum is $T = 2\pi \sqrt{(l/g)}$, find the maximum error in T due to the possible error upto 1% in l and 2.5% in g.* (U.P.T.U., 2004)

Solution. We have $T = 2\pi \sqrt{(l/g)}$

or
$$\log T = \log 2\pi + \frac{1}{2} \log l - \frac{1}{2} \log g$$

$$\therefore \quad \frac{1}{T} \delta T = 0 + \frac{1}{2} \frac{1}{l} \delta l - \frac{1}{2} \frac{1}{g} \delta g$$

$$\frac{\delta T}{T} 100 = \frac{1}{2}\left(\frac{\delta l}{l} 100 - \frac{\delta g}{g} 100\right) = \frac{1}{2}(1 \pm 2.5) = 1.75 \text{ or } -0.75$$

Thus the maximum error in $T = 1.75\%$

Example 5.37. *A balloon is in the form of right circular cylinder of radius 1.5 m and length 4 m and is surmounted by hemispherical ends. If the radius is increased by 0.01 m and length by 0.05 m, find the percentage change in the volume of balloon.* (U.P.T.U., 2005)

Solution. Let the volume of the balloon (Fig. 5.3) be V, so that

$$V = \pi r^2 h + \frac{2}{3}\pi r^3 + \frac{2}{3}\pi r^3 = \pi r^2 h + \frac{4}{3}\pi r^3$$

$\therefore$
$$\delta V = 2\pi r \delta r h + \pi r^2 \delta h + \frac{4}{3}\pi 3r^2 \delta r$$

or
$$\frac{\delta V}{V} = \frac{\pi r[2h\delta r + r\delta h + 4r\delta r]}{\pi r^2 h + \frac{4}{3}\pi r^3}$$

$$= \frac{2(h+2r)\,\delta r + r\delta h}{rh + \frac{4}{3}r^2} = \frac{2(4+3)(.01) + 1.5(.05)}{1.5 \times 4 + \frac{4}{3}(1.5)^2}$$

$$= \frac{0.14 + 0.075}{6+3} = \frac{0.215}{9}$$

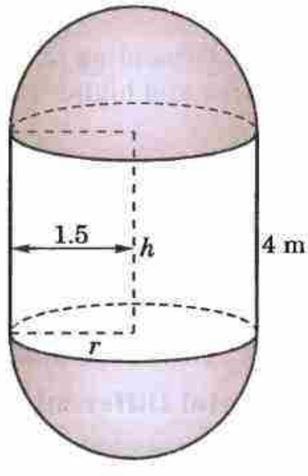


Fig. 5.3

Hence, the percentage change in $V = 100\,\dfrac{\delta V}{V} = \dfrac{21.5}{9} = 2.39\%$

Example 5.38. *In estimating the cost of a pile of bricks measured as 2 m × 15 m × 1.2 m, the tape is stretched 1% beyond the standard length. If the count is 450 bricks to 1 cu. m. and bricks cost ₹ 530 per 1000, find the approximate error in the cost.* (V.T.U., 2001)

Solution. Let x, y and z m be the length, breadth and height of the pile so that its volume $V = xyz$

or
$$\log V = \log x + \log y + \log z \;\therefore\; \frac{\delta V}{V} = \frac{\delta x}{x} + \frac{\delta y}{y} + \frac{\delta z}{z}$$

Since
$$V = 2 \times 15 \times 1.2 = 36 \text{ m}^3, \text{ and } \frac{\delta x}{x} = \frac{\delta y}{y} = \frac{\delta z}{z} = \frac{1}{100}$$

$\therefore$
$$\delta V = 36\left(\frac{3}{100}\right) = 1.08 \text{ m}^3.$$

Number of bricks in $\delta V = 1.08 \times 450 = 486$

Thus error in the cost $= 486 \times \dfrac{530}{1000} =$ ₹ 257.58 which is a loss to the brick seller.

Example 5.39. *The height h and semi-vertical angle α of a cone are measured and from them A, the total area of the surface of the cone including the base is calculated. If h and α are in error by small quantities δh and $\delta\alpha$ respectively, find the corresponding error in the area. Show further that if $\alpha = \pi/6$, an error of + 1% in h will be approximately compensated by an error of – 0.33 degrees in α.*

Solution. If r be the base radius and l the slant height of the cone, (Fig. 5.4), then total area

$$A = \text{area of base} + \text{area of curved surface}$$
$$= \pi r^2 + \pi r l = \pi r(r + l)$$
$$= \pi h \tan\alpha\,(h\tan\alpha + h\sec\alpha)$$
$$= \pi h^2(\tan^2\alpha + \tan\alpha\sec\alpha)$$

$\therefore$
$$\delta A = \frac{\delta A}{\delta h}\delta h + \frac{\delta A}{\delta\alpha}\delta\alpha$$
$$= 2\pi h(\tan^2\alpha + \tan\alpha\sec\alpha)\delta h$$
$$+ \pi h^2(2\tan\alpha\sec^2\alpha + \sec^3\alpha + \tan\alpha\sec\alpha\tan\alpha)\delta\alpha$$

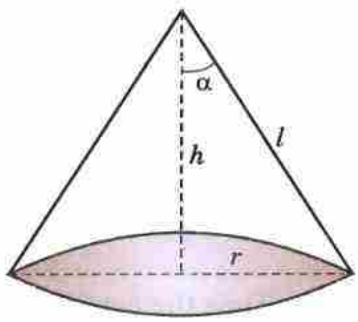


Fig. 5.4

which gives the error in the area A.

Putting $\delta h = h/100$ and $\alpha = \pi/6$, we get

$$\delta A = 2\pi h\left[\left(\frac{1}{\sqrt{3}}\right)^2 + \frac{1}{\sqrt{3}}\cdot\frac{2}{\sqrt{3}}\right]\frac{h}{100} + \pi h^2\left[2.\frac{1}{\sqrt{3}}\cdot\frac{4}{3} + \frac{8}{3\sqrt{3}} + \frac{1}{\sqrt{3}}\cdot\frac{2}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}\right]\delta\alpha$$

$$= \frac{2\pi h^2}{100} + 2\sqrt{3}\pi h^2 \ \delta\alpha$$

The error in h will be compensated by the error in α, when

$$\delta A = 0 \text{ i.e., } \frac{2\pi h^2}{100} + 2\sqrt{3}\pi h^2 \delta\alpha = 0$$

or
$$\delta\alpha = -\frac{1}{100\sqrt{3}} \text{ radians} = -\frac{.01}{1.732} \times 57.3° = -0.33°.$$

Example 5.40. *Show that the approximate change in the angle A of a triangle ABC due to small changes δa, δb, δc in the sides a, b, c respectively, is given by*

$$\delta A = \frac{a}{2\Delta} (\delta a - \delta b \cos C - \delta c \cos B)$$

where Δ is the area of the triangle. Verify that $\delta A + \delta B + \delta C = 0$.

Solution. We know that $a^2 = b^2 + c^2 - 2bc \cos A$

so that
$$2a\delta a = 2b\delta b + 2c\delta c - 2(c\delta b \cos A - b\delta c \cos A + bc \sin A \ \delta A)$$

$\therefore$
$$bc \sin A \ \delta A = a\delta a - (b - c \cos A) \ \delta b - (c - b \cos A) \ \delta c$$

or
$$2 \ \Delta\delta A = a\delta a - (c \cos A + a \cos C - c \cos A) \ \delta b - (a \cos B + b \cos A - b \cos A) \ \delta c$$

[$\because$ $b = c \cos A + a \cos C$ etc. ...(*i*)]

$$= a\delta a - a \cos C \ \delta b - a \cos B \ \delta c$$

or
$$\delta A = \frac{a}{2\Delta} (\delta a - \delta b \cos C - \delta c \cos B)$$

By symmetry, we have

$$\delta B = \frac{b}{2\Delta} (\delta b - \delta c \cos A - \delta a \cos C)$$

$$\delta C = \frac{c}{2\Delta} (\delta c - \delta a \cos B - \delta b \cos A)$$

$\therefore$
$$\delta A + \delta B + \delta C = \frac{1}{2\Delta} (a - b \cos C - c \cos B) \ \delta a + (b - c \cos A - a \cos C) \ \delta b + (c - a \cos B - b \cos A)]$$

$$= \frac{1}{2\Delta} [(a - a) \ \delta a + (b - b) \ \delta b + (c - c) \ \delta c] = 0 \qquad \text{[By (i)]}$$

Example 5.41. *If the sides of a plane triangle ABC vary in such a way that its circumradius remains constant, prove that* $\dfrac{da}{\cos A} + \dfrac{db}{\cos B} + \dfrac{dc}{\cos C} = 0.$

Solution. The circumradius R of ΔABC is given by

$$R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C}$$

$\therefore$
$$a = 2R \sin A \qquad [\because \ R \text{ is constant}$$

Taking differentials, $da = 2R \cos A \ dA$ or $\dfrac{da}{\cos A} = 2RdA$

Similarly, $\dfrac{db}{\cos B} = 2RdB$, $\dfrac{dc}{\cos C} = 2RdC$

$\therefore$
$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 2R \ (dA + dB + dC)$$

Now $A + B + C = \pi$, gives $dA + dB + dC = 0$ ...(*i*)

Thus
$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0 \qquad \text{[By (i)]}$$

PROBLEMS 5.9

1. Expand the following functions as far as terms of third degree :
   (i) $\sin x \cos y$ (*V.T.U., 2009*) (ii) $e^x \sin y$ at $(-1, \pi/4)$ (*Anna, 2009*)
   (iii) $xy^2 + \cos xy$ about $(1, \pi/2)$. (*Hissar, 2005 S ; V.T.U., 2003*)
2. Expand $f(x, y) = x^y$ in powers of $(x - 1)$ and $(y - 1)$. (*U.T.U., 2009*)
3. If $f(x, y) = \tan^{-1} xy$, compute $f(0.9, -1.2)$ approximately.
4. If the kinetic energy $k = wv^2/2g$, find approximately the change in the kinetic energy as w changes from 49 to 49.5 and v changes from 1600 to 1590. (*V.T.U., 2006*)
5. Find the possible percentage error in computing the resistance r from the formula $1/r = 1/r_1 + 1/r_2$, if r_1, r_2 are both in error by 2%.
6. The voltage V across a resistor is measured with an error h, and the resistance R is measured with an error k. Show that the error in calculating the power $W(V, R) = V^2/R$ generated in the resistor, is $VR^{-2}(2Rh - Vk)$. (*V.T.U., 2009*)
7. Find the percentage error in the area of an ellipse if one per cent error is made in measuring the major and minor axes. (*V.T.U., 2011*)
8. The time of oscillation of a simple pendulum is given by the equation $T = 2\pi\sqrt{(l/g)}$. In an experiment carried out to find the value of g, errors of 1.5% and 0.5% are possible in the values of l and T respectively. Show that the error in the calculated value of g is 0.5%. (*Cochin, 2005*)
9. If $pv^2 = k$ and the relative errors in p and v are respectively 0.05 and 0.025, show that the error in k is 10%. (*Mysore, 1999*)
10. If the H.P. required to propel a steamer varies as the cube of the velocity and square of the length. Prove that a 3% increase in velocity and 4% increase in length will require an increase of about 17% in H.P.
11. The range R of a projectile which starts with a velocity v at an elevation α is given by $R = (v^2 \sin 2\alpha)/g$. Find the percentage error in R due to an error of 1% in v and an error of $\frac{1}{2}$% in α. (*Kurukshetra, 2009*)
12. In estimating the cost of a pile of bricks measured as 6 m × 50 m × 4 m, the tape is stretched 1% beyond the standard length. If the count is 12 bricks in 1 m^3 and bricks cost ₹ 100 per 1000, find the approximate error in the cost. (*U.T.U., 2010 ; U.P.T.U., 2005*)
13. The deflection at the centre of a rod of length l and diameter d supported at its ends, loaded at the centre with a weight w varies at wl^3d^{-4}. What is the increase in the deflection corresponding to p% increase in w, q% decrease in l and r% increase in d ?
14. The work that must be done to propel a ship of displacement D for a distance s in time t is proportional to $(s^2D^{2/3}/t^2)$. Find approximately the increase of work necessary when the displacement is increased by 1%, the time is diminished by 1% and the distance diminished by 2%.
15. The indicated horse power I of an engine is calculated from the formula $I = PLAN/33{,}000$, where $A = \pi d^2/4$. Assuming that error of r per cent may have been made in measuring P, L, N and d, find the greatest possible error in I.
16. The torsional rigidity of a length of wire is obtained from the formula $N = 8\pi\, Il/t^2r^4$. If l is decreased by 2%, r is increased by 2%, t is increased by 1.5%, show that the value of N is diminished by 13% approximately. (*V.T.U., 2003*)
17. If $x^2 + y^2 + z^2 - 2xyz = 1$, show that $\dfrac{dx}{\sqrt{(1-x^2)}} + \dfrac{dy}{\sqrt{(1-y^2)}} + \dfrac{dz}{\sqrt{(1-z^2)}} = 0$.
   [**Hint.** $2(x - yz)\,dx + 2(y - zx)\,dy + 2(z - xy)\,dz = 0$. Also $(x - yz)^2 = (1 - y^2)(1 - z^2)$, ...]

5.11 (1) MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

Def. *A function $f(x, y)$ is said to have a* **maximum** *or* **minimum** *at $x = a, y = b$, according as*

$$f(a + h, b + k) < or > f(a, b),$$

for all positive or negative small values of h and k.

In other words, if $\Delta = f(a + h, b + k) - f(a, b)$, *is of the same sign for all small values of h, k, and if this sign is negative, then $f(a, b)$ is a maximum. If this sign is positive, $f(a, b)$ is a minimum.*

Considering $z = f(x, y)$ as a surface, maximum value of z occurs at the top of an elevation (*e.g.*, a dome) from which the surface descends in every direction and a minimum value occurs at the bottom of a depression (*e.g.*, a bowl) from which the surface ascends in every direction. Sometimes the maximum or minimum value may form a *ridge* such that the surface descends or ascends in all directions except that of the ridge. Besides these, we have such a point of the surface, where the tangent plane is horizontal and the surface looks like leather seat on the horse's back [Fig. 5.5 (*c*)] which falls for displacement in certain directions and rises for displacements in other directions. Such a point is called a **saddle point.**

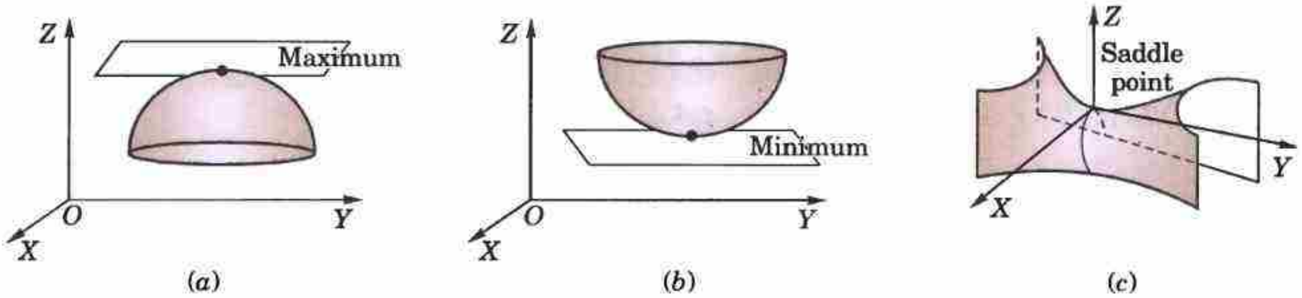


Fig. 5.5

Note. *A maximum or minimum value of a function is called its* **extreme value.**

(2) Conditions for f(x, y) to be maximum or minimum

Using Taylor's theorem page 235, we have $\Delta = f(a + h, b + k) - f(a, b)$

$$= \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)_{a,b} + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \ldots \qquad \ldots(i)$$

For small values of h amd k, the second and higher order terms are still smaller and hence may be neglected. Thus

$$\text{sign of } \Delta = \text{sign of } [hf_x(a, b) + kf_y(a, b)].$$

Taking $h = 0$ we see that the right hand side changes sign when k changes sign. Hence $f(x, y)$ cannot have a maximum or a minimum at (a, b) unless $f_y(a, b) = 0$.

Similarly taking $k = 0$, we find that $f(x, y)$ cannot have a maximum or minimum at (a, b) unless $f_x(a, b) = 0$.

Hence *the necessary conditions for $f(x, y)$ to have a maximum or minimum at (a, b) are that*

$$\mathbf{f_x(a, b) = 0, f_y(a, b) = 0.}$$

If these conditions are satisfied, then for small value of h and k, (i) gives

$$\text{sign of } \Delta = \text{sign of } \left[\frac{1}{2!}(h^2r + 2hks + k^2t)\right] \text{ where } r = f_{xx}(a, b),\ s = f_{xy}(a, b) \text{ and } t = f_{yy}(a, b).$$

Now $$h^2r + 2hks + k^2t = \frac{1}{r}\left[(h^2r^2 + 2hkrs + k^2rt)\right] = \frac{1}{r}\left[(hr + ks)^2 + k^2(rt - s^2)\right]$$

Thus sign of Δ = sign of $\frac{1}{2r}\left\{(hr + ks)^2 + k^2(rt - s^2)\right\}$ $\qquad \ldots(ii)$

In (ii), $(hr + ks)^2$ is always positive and $k^2(rt - s^2)$ will be positive if $rt - s^2 > 0$. In this case, Δ will have the same sign as that of r for all values of h and k.

Hence if $rt - s^2 > 0$, then $f(x, y)$ *has a maximum or a minimum at* (a, b) *according as* $r <$ or > 0.

If $rt - s^2 < 0$, then Δ will change with h and k and hence there is no maximum or minimum at (a, b) *i.e.*, it is a *saddle point*.

If $rt - s^2 = 0$, further investigation is required to find whether there is a maximum or minimum at (a, b) or not.

Note. Stationary value. *$f(a, b)$ is said to be a stationary value of $f(x, y)$, if $f_x(a, b) = 0$* and *$f_y(a, b) = 0$ i.e.* the function is stationary at (a, b).

Thus every extreme value is a stationary value but the converse may not be true.

(3) Working rule to find the maximum and minimum values of f(x, y)

1. Find $\partial f/\partial x$ and $\partial f/\partial y$ *and equate each to zero. Solve these as simultaneous equations in x and y.* Let (a, b), (c, d), ... *be the pairs of values.*
2. *Calculate the value of* $r = \partial^2 f/\partial x^2$, $s = \partial^2 f/\partial x\partial y$, $t = \partial^2 f/\partial y^2$ *for each pair of values.*

3. *(i) If $rt - s^2 > 0$ and $r < 0$ at (a, b), $f(a, b)$ is a max. value.*
(ii) If $rt - s^2 > 0$ and $r > 0$ at (a, b), $f(a, b)$ is a min. value.
(iii) If $rt - s^2 < 0$ at (a, b), $f(a, b)$ is not an extreme value, i.e., (a, b) is a saddle point.
(iv) If $rt - s^2 = 0$ at (a, b), the case is doubtful and needs further investigation.

Similarly examine the other pairs of values one by one.

Example 5.42. *Examine the following function for extreme values :*

$$f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2. \qquad \text{(J.N.T.U., 2003)}$$

Solution. We have $f_x = 4x^3 - 4x + 4y$; $f_y = 4y^3 + 4x - 4y$

and $r = f_{xx} = 12x^2 - 4,\ s = f_{xy} = 4,\ t = f_{yy} = 12y^2 - 4$ ...(i)

Now $f_x = 0, f_y = 0$ give $x^3 - x + y = 0$, ...(i) $y^3 + x - y = 0$ ...(ii)

Adding these, we get $4(x^3 + y^3) = 0$ or $y = -x$.

Putting $y = -x$ in (i), we obtain $x^3 - 2x = 0$, *i.e.* $x = \sqrt{2}, -\sqrt{2}, 0$.

$\therefore$ Corresponding values of y are $-\sqrt{2}, \sqrt{2}, 0$.

At $(\sqrt{2}, -\sqrt{2})$, $rt - s^2 = 20 \times 20 - 4^2 = +$ ve and r is also + ve. Hence $f(\sqrt{2}, -\sqrt{2})$ is a minimum value.

At $(-\sqrt{2}, \sqrt{2})$ also both $rt - s^2$ and r are + ve.

Hence $f(-\sqrt{2}, \sqrt{2})$, is also a minimum value.

At (0, 0), $rt - s^2 = 0$ and, therefore, further investigation is needed.

Now $f(0, 0) = 0$ and for points along the x-axis, where $y = 0$, $f(x, y) = x^4 - 2x^2 = x^2(x^2 - 2)$, which is negative for points in the neighbourhood of the origin.

Again for points along the line $y = x$, $f(x, y) = 2x^4$ which is positive.

Thus in the neighbourhood of (0, 0) there are points where $f(x, y) < f(0, 0)$ and there are points where $f(x, y) > f(0, 0)$.

Hence $f(0, 0)$ is not an extreme value *i.e.*, it is a *saddle point*.

Example 5.43. *Discuss the maxima and minima of $f(x, y) = x^3y^2(1 - x - y)$.*

(Anna, 2009 ; J.N.T.U., 2006 ; Bhopal, 2002)

Solution. We have $f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$; $f_y = 2x^3y - 2x^4y - 3x^3y^2$

and $r = f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3$; $s = f_{xy} = 6x^2y - 8x^3y - 9x^2y^2$; $t = f_{yy} = 2x^3 - 2x^4 - 6x^3y$.

When $f_x = 0, f_y = 0$, we have $x^2y^2(3 - 4x - 3y) = 0$, $x^3y(2 - 2x - 3y) = 0$

Solving these, the stationary points are (1/2, 1/3), (0, 0).

Now $rt - s^2 = x^4y^2[12(1 - 2x - y)(1 - x - 3y) - (6 - 8x - 9y)^2]$

At (1/2, 1/3), $$rt - s^2 = \frac{1}{16} \cdot \frac{1}{9}\left[12\left(1 - 1 - \frac{1}{3}\right)\left(1 - \frac{1}{2} - 1\right) - (6 - 4 - 3)^2\right] = \frac{1}{14} > 0$$

Also $$r = 6\left(\frac{1}{2} \cdot \frac{1}{9} - \frac{2}{4} \cdot \frac{1}{9} - \frac{1}{2} \cdot \frac{1}{27}\right) = -\frac{1}{9} < 0$$

Hence $f(x, y)$ has a maximum at (1/2, 1/3) and maximum value $= \frac{1}{8} \cdot \frac{1}{9}\left(1 - \frac{1}{2} - \frac{1}{3}\right) = \frac{1}{432}$.

At (0, 0), $rt - s^2 = 0$ and therefore further investigation is needed.

For points along the line $y = x$, $f(x, y) = x^5(1 - 2x)$ which is positive for $x = 0.1$ and negative for $x = -0.1$ *i.e.*, in the neighbourhood of (0, 0) there are points where $f(x, y) > f(0, 0)$ and there are points where $f(x, y) < f(0, 0)$. Hence $f(0, 0)$ is not an extreme value.

Example 5.44. *In a plane triangle, find the maximum value of cos A cos B cos C.*

(V.T.U., 2010 ; Nagpur, 2009 ; Anna, 2005 S)

Solution. We have $A + B + C = \pi$ so that $C = \pi - (A + B)$.

$$\cos A \cos B \cos C = \cos A \cos B \cos [\pi - (A + B)]$$
$$= -\cos A \cos B \cos (A + B) = f(A, B), \text{ say.}$$

We get $\dfrac{\partial f}{\partial A} = \cos B\,[\sin A \cos (A + B) + \cos A \sin (A + B)]$

$= \cos B \sin (2A + B)$

and $\dfrac{\partial f}{\partial B} = \cos A \sin (A + 2B)$

$\dfrac{\partial f}{\partial A} = 0, \dfrac{\partial f}{\partial B} = 0$ only when $A = B = \pi/3$.

Also $r = \dfrac{\partial^2 f}{\partial A^2} = 2 \cos B \cos (2A + B), t = \dfrac{\partial^2 f}{\partial B^2} = 2 \cos A \cos (A + 2B)$

$s = \dfrac{\partial^2 f}{\partial A \partial B} = -\sin B \sin (2A + B) + \cos B \cos (2A + B) = \cos (2A + 2B)$

When $A = B = \pi/3$, $r = -1$, $s = -1/2$, $t = -1$ so that $rt - s^2 = 3/4$.

These show that $f(A, B)$ is maximum for $A = B = \pi/3$.

Then $C = \pi - (A + B) = \pi/3$.

Hence $\cos A \cos B \cos C$ is maximum when each of the angles is $\pi/3$ *i.e.*, triangle is equilateral and its maximum value = 1/8.

5.12 LAGRANGE'S METHOD OF UNDERTERMINED MULTIPLIERS

Sometimes it is required to find the stationary values of a function of several variables which are not all independent but are connected by some given relations. Ordinarily, we try to convert the given function to the one, having least number of independent variables with the help of given relations. Then solve it by the above method. When such a procedure becomes impracticable, Lagrange's method* proves very convenient. Now we explain this method.

Let $u = f(x, y, z)$ ...(1)

be a function of three variables x, y, z which are connected by the relation.

$\phi(x, y, z) = 0$ ...(2)

For u to have stationary values, it is necessary that

$\partial u/\partial x = 0, \partial u/\partial y = 0, \partial u/\partial z = 0.$

$\therefore \quad \dfrac{\partial u}{\partial x} dx + \dfrac{\partial u}{\partial y} dy + \dfrac{\partial u}{\partial z} dz = du = 0$ ...(3)

Also differentiating (2), we get $\dfrac{\partial \phi}{\partial x} dx + \dfrac{\partial \phi}{\partial y} dy + \dfrac{\partial \phi}{\partial z} dz = d\phi = 0$ ...(4)

Multiply (4) by a parameter λ and add to (3). Then

$$\left(\frac{\partial u}{\partial x} + \lambda \frac{\partial \phi}{\partial x}\right) dx + \left(\frac{\partial u}{\partial y} + \lambda \frac{\partial \phi}{\partial y}\right) dy + \left(\frac{\partial u}{\partial z} + \lambda \frac{\partial \phi}{\partial z}\right) dz = 0$$

This equation will be satisfied if $\dfrac{\partial u}{\partial x} + \lambda \dfrac{\partial \phi}{\partial x} = 0, \dfrac{\partial u}{\partial y} + \lambda \dfrac{\partial \phi}{\partial y} = 0, \dfrac{\partial u}{\partial z} + \lambda \dfrac{\partial \phi}{\partial z} = 0.$

These three equations together with (2) will determine the values of x, y, z and λ for which u is stationary.

Working rule : **1.** *Write* $F = f(x, y, z) + \lambda\phi(x, y, z)$

2. *Obtain the equations* $\dfrac{\partial F}{\partial x} = 0, \dfrac{\partial F}{\partial y} = 0, \dfrac{\partial F}{\partial z} = 0.$

3. *Solve the above equations together with* $\phi(x, y, z) = 0$.

The values of x, y, z *so obtained will give the stationary value of* $f(x, y, z)$.

Obs. Although the Lagrange's method is often very useful in application yet *the drawback is that we cannot determine the nature of the stationary point*. This can sometimes, be determined from physical considerations of the problem.

*See footnote page 142.

Example 5.45. *A rectangular box open at the top is to have volume of 32 cubic ft. Find the dimensions of the box requiring least material for its construction.* (*Kurukshetra, 2006 ; P.T.U., 2006 ; U.P.T.U., 2005*)

Solution. Let x, y and z ft. be the edges of the box and S be its surface.

Then $$S = xy + 2yz + 2zx \quad ...(i)$$

and $$xyz = 32 \quad ...(ii)$$

Eliminating z from (i) with the help of (ii), we get $S = xy + 2(y + x)\dfrac{32}{xy} = xy + 64\left(\dfrac{1}{x} + \dfrac{1}{y}\right)$

$\therefore$ $$\partial S/\partial x = y - 64/x^2 = 0 \quad \text{and} \quad \partial S/\partial y = x - 64/y^2 = 0.$$

Solving these, we get $x = y = 4$.

Now $r = \partial^2 S/\partial x^2 = 128/x^3$, $s = \partial^2 S/\partial x \partial y = 1$, $t = \partial^2 S/\partial y^2 = 128/y^3$.

At $x = y = 4$, $rt - s^2 = 2 \times 2 - 1 = +$ ve and r is also $+$ ve.

Hence S is minimum for $x = y = 4$. Then from (ii), $z = 2$.

Otherwise (*by Lagrange's method*) :

Write $$F = xy + 2yz + 2zx + \lambda(xyz - 32)$$

Then $$\frac{\partial F}{\partial x} = y + 2z + \lambda yz = 0 \quad ...(iii)$$

$$\frac{\partial F}{\partial y} = x + 2z + \lambda zx = 0 \quad ...(iv)$$

$$\frac{\partial F}{\partial z} = 2y + 2x + \lambda xy = 0 \quad ...(v)$$

Multiplying (iii) by x and (iv) by y and subtracting, we get $2zx - 2zy = 0$ or $x = y$.

[The value $z = 0$ is neglected, as it will not satisfy (ii)]

Again multiplying (iv) by y and (v) by z and subtracting, we get $y = 2z$.

Hence the dimensions of the box are $x = y = 2z = 4$ ...(vi)

Now let us see what happens as z increases from a small value to a large one. When z is small, the box is flat with a large base showing that S is large. As z increases, the base of the box decreases rapidly and S also decreases. After a certain stage, S again starts increasing as z increases. Thus S must be a minimum at some intermediate stage which is given by (vi). Hence S is minimum when $x = y = 4$ ft and $z = 2$ ft.

Example 5.46. *Given $x + y + z = a$, find the maximum valve of $x^m y^n z^p$.* (*Anna, 2009*)

Solution. Let $f(x, y, z) = x^m y^n z^p$ and $\phi(x, y, z) = x + y + z - a$.

Then $$F(x, y, z) = f(x, y, z) + \lambda\, \phi(x, y, z)$$
$$= x^m y^n z^p + \lambda(x + y + z - a).$$

For stationary values of F, $\dfrac{\partial F}{\partial x} = 0, \dfrac{\partial F}{\partial y} = 0, \dfrac{\partial F}{\partial z} = 0$

$\therefore$ $$mx^{m-1} y^n z^p + \lambda = 0,\ nx^m y^{n-1} z^p + \lambda = 0,\ px^m y^n z^{p-1} + \lambda = 0$$

or $$-\lambda = mx^{n-1} y^n z^p = nx^m y^{n-1} z^p = px^m y^n z^{p-1}$$

i.e. $$\frac{m}{x} = \frac{n}{y} = \frac{p}{z} = \frac{m + n + p}{x + y + z} = \frac{m + n + p}{a} \qquad [\because\ x + y + z = a$$

$\therefore$ The maximum value of f occurs when

$$x = am/(m + n + p),\ y = an/(m + n + p),\ z = ap/(m + n + p)$$

Hence the maximum value of $f(x, y, z) = \dfrac{a^{m+n+p} \cdot m^m \cdot n^n \cdot p^p}{(m + n + p)^{m+n+p}}$.

Example 5.47. *Find the maximum and minimum distances of the point (3, 4, 12) from the sphere $x^2 + y^2 + z^2 = 4$.* (*U.T.U., 2010*)

Solution. Let $P(x, y, z)$ be any point on the sphere and $A(3, 4, 12)$ the given point so that

$$AP^2 = (x - 3)^2 + (y - 4)^2 + (z - 12)^2 = f(x, y, z), \text{ say} \quad ...(i)$$

We have to find the maximum and minimum values of $f(x, y, z)$ subject to the condition

$$\phi(x, y, z) = x^2 + y^2 + z^2 - 4 = 0 \quad ...(ii)$$

Let $F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$

$$= (x-3)^2 + (y-4)^2 + (z-12)^2 + \lambda(x^2 + y^2 + z^2 - 4)$$

Then $\dfrac{\partial F}{\partial x} = 2(x-3) + 2\lambda x,\ \dfrac{\partial F}{\partial y} = 2(y-4) + 2\lambda y,\ \dfrac{\partial F}{\partial z} = 2(z-12) + 2\lambda z$

$\therefore$ $\dfrac{\partial F}{\partial x} = 0, \dfrac{\partial F}{\partial y} = 0$ and $\dfrac{\partial F}{\partial z} = 0$ give

$$x - 3 + \lambda x = 0,\ y - 4 + \lambda y = 0,\ z - 12 + \lambda z = 0 \quad ...(iii)$$

which give $\lambda = -\dfrac{x-3}{x} = -\dfrac{y-4}{y} = -\dfrac{z-12}{z}$

$$= \pm \frac{\sqrt{[(x-3)^2 + (y-4)^2 + (z-12)^2]}}{\sqrt{(x^2 + y^2 + z^2)}} = \pm \frac{\sqrt{f}}{1}$$

Substituting for λ in (iii), we get

$$x = \frac{3}{1+\lambda} = \frac{3}{1 \pm \sqrt{f}},\ y = \frac{4}{1 \pm \sqrt{f}},\ z = \frac{12}{1 \pm \sqrt{f}}$$

$\therefore$ $x^2 + y^2 + z^2 = \dfrac{9 + 16 + 144}{(1 \pm \sqrt{f})^2} = \dfrac{169}{(1 \pm \sqrt{f})^2}$

Using (ii), $1 = \dfrac{169}{(1 \pm \sqrt{f})^2}$ or $1 \pm \sqrt{f} = \pm 13,\ \sqrt{f} = 12, 14.$

[We have left out the negative values of $\sqrt{f}$, because $\sqrt{f} = AP$ is + ve by (i)]

Hence maximum $AP = 14$ and minimum $AP = 12$.

Example 5.48. *Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.* *(Kurukshetra, 2006 ; U.P.T.U., 2004)*

Solution. Let $2x$, $2y$, $2z$ be the length, breadth and height of the rectangular solid so that its volume

$$V = 8xyz \quad ...(i)$$

Let R be the radius of the sphere so that $x^2 + y^2 + z^2 = R^2$ ...(ii)

Then $F(x, y, z) = 8xyz + \lambda(x^2 + y^2 + z^2 - R^2)$

and $\partial F/\partial x = 0, \partial F/\partial y = 0$ and $\partial F/\partial z = 0$ give

$$8yz + 2x\lambda = 0,\ 8zx + 2y\lambda = 0,\ 8xy + 2z\lambda = 0$$

or $2x^2\lambda = -8xyz = 2y^2\lambda = 2z^2\lambda$

Thus for a maximum volume $x = y = z$.

i.e., the rectangular solid is a cube.

Example 5.49. *A tent on a square base of side x, has its sides vertical of height y and the top is a regular pyramid of height h. Find x and y in terms of h, if the canvas required for its construction is to be minimum for the tent to have a given capacity.*

Solution. Let V be the volume enclosed by the tent and S be its surface area (Fig. 5.6).

Then V = cuboid $(ABCD, A'B'C'D')$ + pyramid $(K, A'B'C'D')$

$$= x^2y + \frac{1}{3}x^2h = x^2(y + h/3)$$

$$S = 4(ABGF) + 4\Delta\, KGH = 4xy + 4\frac{1}{2}(x \cdot KM)$$

$$= 4xy + x\sqrt{(x^2 + 4h^2)} \qquad [\because KM = \sqrt{(KL^2 + LM^2)} = \sqrt{[h^2 + (x/2)^2]}]$$

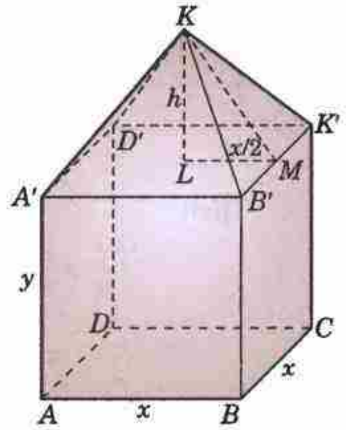


Fig. 5.6

For constant V, we have

$$\delta V = 2x(y + h/3)\,\delta x + x^2(\delta y) + \frac{x^2}{3}\,\delta h = 0$$

For minimum S, we have

$$\delta S = [4y + \sqrt{(x^2 + 4h^2)} + x \cdot \frac{1}{2}(x^2 + 4h^2)^{-1/2} \cdot 2x]\,\delta x$$

$$+ 4x\delta y + x \cdot \frac{1}{2}(x^2 + 4h^2)^{-1/2} \cdot 8h\delta h = 0$$

By Lagrange's method,

$$[4y + \sqrt{(x^2 + 4h^2)} + x^2(x^2 + 4h^2)^{-1/2}] + \lambda \cdot 2x(y + h/3) = 0 \qquad ...(i)$$

$$4x + \lambda \cdot x^2 = 0 \qquad ...(ii)$$

$$4hx(x^2 + 4h^2)^{-1/2} + \lambda \cdot x^2/3 = 0 \qquad ...(iii)$$

(*ii*) gives $\lambda = -4/x$. Then (*iii*) becomes

$$4hx(x^2 + 4h^2)^{-1/2} - 4x/3 = 0 \quad \text{or} \quad x = \sqrt{5}\,h$$

Now putting $x = \sqrt{5}\,h$, $\lambda = -4/x$ in (*i*), we get

$$4y + 3h + \frac{5}{3}h - \frac{4}{x} \cdot 2x(y + h/3) = 0 \quad \text{or} \quad 4y + \frac{14}{3}h - 8y - \frac{8h}{3} = 0, \quad i.e., \quad y = h/2.$$

Example 5.50. *If $u = a^3x^2 + b^3y^2 + c^3z^2$ where $x^{-1} + y^{-1} + z^{-1} = 1$, show that the stationary value of u is given by $x = \Sigma a/a$, $y = \Sigma a/b$, $z = \Sigma a/c$.* (Kerala, 2005)

Solution. Let $u = f(x, y, z) = a^3x^2 + b^3y^2 + c^3z^2$

and $\phi(x, y, z) = x^{-1} + y^{-1} + z^{-1} - 1$

Let
$$F(x, y, z) = f(x, y, z) + \lambda\,\phi(x, y, z)$$
$$= a^3x^2 + b^3y^2 + c^3z^2 + \lambda(x^{-1} + y^{-1} + z^{-1} - 1)$$

Then $\dfrac{\partial F}{\partial x} = 0, \dfrac{\partial F}{\partial y} = 0$ and $\dfrac{\partial F}{\partial z} = 0$ give

$$2a^3x^2 - \lambda/x^2 = 0, \quad 2b^3y - \lambda/y^2 = 0, \quad 2c^3z - \lambda/z^2 = 0$$

or
$$2a^3x^3 = \lambda,\ 2b^3y^3 = \lambda,\ 2c^3z^3 = \lambda$$

which give $ax = by = cz = k$ (say) *i.e.*, $x = k/a$, $y = k/b$, $z = k/c$.

Substituting these in $x^{-1} + y^{-1} + z^{-1} = 1$, we get $k = a + b + c$

Hence the stationary value of u is given by

$$x = \Sigma a/a, \quad y = \Sigma a/b \quad \text{and} \quad z = \Sigma a/c.$$

Example 5.51. *Find the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

(U.T.U., 2010 ; Anna, 2009 ; Madras, 2006)

Solution. Let the edges of the parallelopiped be $2x$, $2y$ and $2z$ which are parallel to the axes. Then its volume $V = 8xyz$.

Now we have to find the maximum value of V subject to the condition that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \qquad ...(i)$$

Write
$$F = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

Then
$$\frac{\partial F}{\partial x} = 8yz + \lambda\left(\frac{2x}{a^2}\right) = 0 \qquad ...(ii)$$

$$\frac{\partial F}{\partial y} = 8zx + \lambda\left(\frac{2y}{b^2}\right) = 0 \quad ...(iii) \qquad \frac{\partial F}{\partial z} = 8xy + \lambda\left(\frac{2z}{c^2}\right) = 0 \quad ...(iv)$$

Equating the values of λ from (*ii*) and (*iii*), we get $x^2/a^2 = y^2/b^2$

Similarly from (*iii*) and (*iv*), we obtain $y^2/b^2 = z^2/c^2 \quad \therefore \quad x^2/a^2 = y^2/b^2 = z^2/c^2$

Substituting these in (*i*), we get $x^2/a^2 = \frac{1}{3}$ *i.e.* $\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = \frac{1}{3}$

These give $x = a/\sqrt{3}$, $y = b/\sqrt{3}$, $z = c/\sqrt{3}$ ...(*v*)

When $x = 0$, the parallelopiped is just a rectangular sheet and as such its volume $V = 0$.

As x increases, V also increases continuously.

Thus V must be greatest at the stage given by (*v*).

Hence the greatest volume $= \frac{8abc}{3\sqrt{3}}$.

PROBLEMS 5.10

1. Find the maximum and minimum values of
   (*i*) $x^3 + y^3 - 3axy$ (*U.P.T.U., 2005*) (*ii*) $xy + a^3/x + a^3/y$.
   (*iii*) $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$ (*Mumbai, 2007*) (*iv*) $2(x^2 - y^2) - x^4 + y^4$ (*Osmania, 2003*)
   (*v*) $\sin x \sin y \sin (x + y)$.
2. If $xyz = 8$, find the values of x, y for which $u = 5xyz/(x + 2y + 4z)$ is a maximum. (*S.V.T.U., 2007 ; Kurukshetra, 2005*)
3. Find the minimum value of $x^2 + y^2 + z^2$, given that
   (*i*) $xyz = a^3$ (*P.T.U., 2009 ; Osmania, 2003*) (*ii*) $ax + by + cz = p$. (*V.T.U., 2010 ; U.P.T.U., 2006*)
   (*iii*) $xy + yz + zx = 3a^2$ (*Anna, 2009*)
4. Find the dimensions of the rectangular box, open at the top, of maximum capacity whose surface is 432 sq. cm. (*Madras, 2000 S*)
5. The sum of three numbers is constant. Prove that their product is maximum when they are equal.
6. Find the points on the surface $z^2 = xy + 1$ nearest to the origin. (*Burdwan, 2003 ; Andhra, 2000*)
7. Show that, if the perimeter of a triangle is constant, the triangle has maximum area when it is equilateral.
8. Find the maximum and minimum distances from the origin to the curve $5x^2 + 6xy + 5y^2 - 8 = 0$.
9. The temperature T at any point (x, y, z) in space is $T = 400\,xyz^2$. Find the highest temperature on the surface of the unit sphere $x^2 + y^2 + z^2 = 1$. (*V.T.U., 2009 ; Hissar, 2005 S*)
10. Divide 24 into three parts such that the continued product of the first, square of the second and the cube of the third may be maximum. (*Bhillai, 2005*)
11. Find the stationary values of $u = x^2 + y^2 + z^2$ subject to $ax^2 + by^2 + cz^2 = 1$ and $lx + my + nz = 0$. (*S.V.T.U., 2008*)

5.13 DIFFERENTIATION UNDER THE INTEGRAL SIGN

If a function $f(x, \alpha)$ of two variables x and α (called a parameter), be integrated with respect to x between the limits a and b, then $\int_a^b f(x, \alpha)\, dx$ is a function of α : $F(\alpha)$, say. To find the derivative of $F(\alpha)$, when it exists, it is not always possible to first evaluate this integral and then to find the derivative. Such problems are solved by the following rules :

(1) Leibnitz's rule*

If $f(x, \alpha)$ and $\frac{\partial f(x, \alpha)}{\partial \alpha}$ *be continuous functions of* x *and* α, *then*

$$\frac{d}{d\alpha}\left[\int_a^b f(x, \alpha)\, dx\right] = \int_a^b \frac{\partial f(x, \alpha)}{\partial \alpha}\, dx \text{ where, } a, b \text{ are constants independent of } \alpha.$$

*See foot note on p. 139.

Let $\int_a^b f(x, \alpha)dx = F(\alpha)$,

then $F(\alpha + \delta\alpha) - F(\alpha) = \int_a^b f(x, \alpha + \delta\alpha)dx - \int_a^b f(x, \alpha)dx = \int_a^b [f(x, \alpha + \delta\alpha) - f(x, \alpha)]dx$

$$= \delta\alpha \int_a^b \frac{\partial f(x, \alpha + \theta\delta\alpha)}{\partial \alpha} dx, \ (0 < \theta < 1) \qquad \left\{\begin{array}{l} \because \ f(x, \alpha + h) - f(x, \alpha) = hf'(x, \alpha + \theta h) \\ \text{where } 0 < \theta < 1, \text{ by Mean Value Theorem} \end{array}\right.$$

Proceeding to limits as $\delta\alpha \to 0$, $\underset{\delta\alpha \to 0}{\text{Lt}} \dfrac{F(\alpha + \delta\alpha) - F(\alpha)}{\delta\alpha} = \int_a^b \dfrac{\partial f(x, \alpha + \theta \cdot 0)}{\partial \alpha} dx$

or $\dfrac{dF}{d\alpha} = \int_a^b \dfrac{\partial f(x, \alpha)}{\partial \alpha} dx$ which is the desired result.

Obs. 1. Leibnitz's rule enables us to derive from the value of a simple definite integral, the value of another definite integral which it may otherwise be difficult, or even impossible, to evaluate.

Obs. 2. The rule for differentiation under the integral sign of an infinite integral is the same as for a definite integral.

Example 5.52. *Evaluate* $\int_0^1 \dfrac{x^\alpha - 1}{\log x} dx$, $\alpha \geq 0$. *(V.T.U., 2010)*

Solution. Let $F(\alpha) = \int_0^1 \dfrac{x^\alpha - 1}{\log x} dx$ ...(*i*)

then $F(\alpha) = \int_0^1 \dfrac{\partial}{\partial \alpha}\left(\dfrac{x^\alpha - 1}{\log x}\right) dx = \int_0^1 \dfrac{x^\alpha \log x}{\log x} dx$

$$= \int_0^1 x^\alpha \, dx = \left| \frac{x^{\alpha+1}}{\alpha + 1} \right|_0^1 = \frac{1}{1 + \alpha} \qquad \left[\because \ \frac{d}{dt}(n^t) = n^t \log n\right]$$

Now integrating both sides w.r.t. α, $F(\alpha) = \log(1 + \alpha) + c$ ...(*ii*)

From (*i*), when $\alpha = 0$, $F(0) = 0$

$\therefore$ From (*ii*), $F(0) = \log(1) + c$, *i.e.*, $c = 0$. Hence (*ii*) gives, $F(\alpha) = \log(1 + \alpha)$.

Example 5.53. *Given* $\int_0^\pi \dfrac{dx}{a + b\cos x} = \dfrac{\pi}{\sqrt{(a^2 - b^2)}}$ $(a > b)$,

evaluate $\int_0^\pi \dfrac{dx}{(a + b\cos x)^2}$ *and* $\int_0^\pi \dfrac{\cos x}{(a + b\cos x)^2} dx$ *(Madras, 2006)*

Solution. We have $\int_0^\pi \dfrac{dx}{a + b\cos x} = \dfrac{\pi}{\sqrt{(a^2 - b^2)}}$ ...(*i*)

Differentiating both sides of (*i*) w.r.t. a,

$$\int_0^\pi \frac{\partial}{\partial a}\left(\frac{1}{a + b\cos x}\right) dx = \frac{\partial}{\partial a}\left\{\frac{\pi}{\sqrt{(a^2 - b^2)}}\right\}$$

i.e.
$$\int_0^\pi \frac{-dx}{(a + b\cos x)^2} = \pi \cdot \left(-\frac{1}{2}\right)(a^2 - b^2)^{-3/2} \cdot 2a$$

$\therefore$
$$\int_0^\pi \frac{dx}{(a + b\cos x)^2} = \frac{\pi a}{(a^2 - b^2)^{3/2}}$$

Now differentiating both sides of (*i*) w.r.t. b,

$$\int_0^\pi -(a + b\cos x)^{-2} \cdot \cos x \, dx = \pi\left(-\frac{1}{2}\right)(a^2 - b^2)^{-3/2} \cdot (-2b)$$

$$\therefore \quad \int_0^{\pi} \frac{\cos x}{(a + b \cos x)^2} dx = \frac{\pi b}{(a^2 - b^2)^{3/2}}.$$

(2) Leibnitz's rule *for variable limits of integration*

If $f(x, \alpha)$, $\dfrac{\partial f(x, \alpha)}{\partial \alpha}$ *be continuous functions of x and* α, *then*

$$\frac{\mathbf{d}}{\mathbf{d}\alpha}\left\{\int_{\phi(\alpha)}^{\psi(\alpha)} \mathbf{f}(\mathbf{x}, \alpha)\,\mathbf{dx}\right\} = \int_{\phi(\alpha)}^{\psi(\alpha)} \frac{\partial \mathbf{f}(\mathbf{x}, \alpha)}{\partial \alpha}\mathbf{dx} + \frac{\mathbf{d}\psi}{\mathbf{d}\alpha}\mathbf{f}[\psi(\alpha), \alpha] - \frac{\mathbf{d}\phi}{\mathbf{d}\alpha}\mathbf{f}[\phi(\alpha), \alpha]$$

provided $\phi(\alpha)$ *and* $\psi(\alpha)$ *possesses continuous first order derivatives w.r.t.* α.

Its proof is beyond the scope of this book.

Example 5.54. *Evaluate* $\int_0^{\alpha} \frac{\log(1 + \alpha x)}{1 + x^2} dx$ *and hence show that*

$$\int_0^1 \frac{\log(1 + x)}{1 + x^2} dx = \frac{\pi}{8} \log_e 2$$

(Hissar, 2005 S)

Solution. Let $\quad F(\alpha) = \int_0^{\alpha} \frac{\log(1 + \alpha x)}{1 + x^2} dx \quad$ ...(*i*)

Then by the above rule, $F'(\alpha) = \int_0^{\alpha} \frac{\partial}{\partial \alpha}\left(\frac{\log(1 + \alpha x)}{1 + x^2}\right) dx + \frac{d(\alpha)}{d\alpha} \cdot \frac{\log(1 + \alpha^2)}{1 + \alpha^2} - 0$

$$= \int_0^{\alpha} \frac{x}{(1 + \alpha x)(1 + x^2)} dx + \frac{\log(1 + \alpha^2)}{1 + \alpha^2} \qquad \text{...(ii)}$$

Breaking the integrand into partial fractions,

$$\int_0^{\alpha} \frac{x\, dx}{(1 + \alpha x)(1 + x^2)} = -\frac{\alpha}{1 + \alpha^2}\int_0^{\alpha} \frac{dx}{1 + \alpha x} + \frac{1}{2(1 + \alpha^2)}\int_0^{\alpha} \frac{2x}{1 + x^2} dx + \frac{\alpha}{1 + \alpha^2}\int_0^{\alpha} \frac{dx}{1 + x^2}$$

$$= -\frac{1}{1 + \alpha^2}\left|\log(1 + \alpha x)\right|_0^{\alpha} + \frac{1}{2(1 + \alpha^2)} \times \left|\log(1 + x^2)\right|_0^{\alpha} + \frac{\alpha}{1 + \alpha^2}\left|\tan^{-1} x\right|_0^{\alpha}$$

$$= -\frac{\log(1 + \alpha^2)}{1 + \alpha^2} + \frac{\log(1 + \alpha^2)}{2(1 + \alpha^2)} + \frac{\alpha \tan^{-1} \alpha}{1 + \alpha^2}$$

Substituting this value in (*ii*), $F'(\alpha) = \dfrac{\log(1 + \alpha^2)}{2(1 + \alpha^2)} + \dfrac{\alpha \tan^{-1} \alpha}{1 + \alpha^2}$

Now integrating both sides w.r.t. α,

$$F(\alpha) = \frac{1}{2}\int \log(1 + \alpha^2) \cdot \frac{1}{1 + \alpha^2} d\alpha + \int \frac{\alpha \tan^{-1} \alpha}{1 + \alpha^2} d\alpha \qquad \text{[Integrating by parts]}$$

$$= \frac{1}{2}\left[\log(1 + \alpha^2) \cdot \tan^{-1} \alpha - \int \frac{2\alpha}{1 + \alpha^2} \cdot \tan^{-1} \alpha\, d\alpha\right] + \int \frac{\alpha \tan^{-1} \alpha}{1 + \alpha^2} d\alpha + c$$

$$= \frac{1}{2} \log(1 + \alpha^2) \cdot \tan^{-1} \alpha + c \qquad \text{...(iii)}$$

But from (*i*), when $\alpha = 0$, $F(0) = 0$.

$\therefore$ From (*iii*), $F(0) = 0 + c$, *i.e.*, $c = 0$. Hence (*iii*) gives, $F(\alpha) = \dfrac{1}{2} \log(1 + \alpha^2) \tan^{-1} \alpha$

Putting $\alpha = 1$, we get $\int_0^1 \frac{\log(1 + x)}{1 + x^2} dx = F(1) = \frac{\pi}{8} \log_e 2.$

PROBLEMS 5.11

1. Differentiating $\int_0^x \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$ under the integral sign, find the value of $\int_0^x \frac{dx}{(x^2+a^2)^2}$.

2. By successive differentiation of $\int_0^1 x^m\, dx = \frac{1}{m+1}$ w.r.t. m, evaluate $\int_0^1 x^m (\log x)^n\, dx$.

3. Evaluate $\int_0^\pi \log(1 + a\cos x)\, dx$, using the method of differentiation under the sign of integration.

4. Given that $\int_0^\pi \frac{dx}{a-\cos x} = \frac{\pi}{\sqrt{(a^2-1)}}$, evaluate $\int_0^\pi \frac{dx}{(a-\cos x)^2}$. (V.T.U., 2009)

Prove that :

5. $\int_0^\infty e^{-x}\cdot\frac{\sin ax}{x}dx = \tan^{-1} a$. $\left[\textbf{Hint.} \text{ Use } \int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$

6. $\int_0^\infty e^{-ax}\frac{\sin x}{x}dx = \tan^{-1}\frac{1}{a}$. Hence show that $\int_0^\infty \frac{\sin x}{x} = \frac{\pi}{2}$. (Rohtak, 2003)

7. $\int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)}dx = \frac{\pi}{2}\log(1+a)$ where $a \geq 0$. (V.T.U., 2010 ; S.V.T.U., 2009 ; Rohtak, 2006 S ; Anna, 2005 S)

8. $\int_0^\infty \frac{e^{-x}}{x}(1-e^{-ax})\, dx = \log(1+a), (a > -1)$.

9. $\int_0^{\pi/2} \log(\alpha^2\cos^2\theta + \beta^2\sin^2\theta)\, d\theta = \pi\log\frac{\alpha+\beta}{2}$ (S.V.T.U., 2008)

10. $\int_0^{\pi/2} \frac{\log(1+y\sin^2 x)}{\sin^2 x}dx = \pi[\sqrt{(1+y)}-1]$ (S.V.T.U., 2008)

11. $\int_0^\pi \frac{\log(1+\alpha\cos x)}{\cos x}dx = \pi\sin^{-1}\alpha$. (V.T.U., 2007)

12. $\int_0^\infty e^{-x^2}\cos\alpha x\, dx = \frac{\sqrt{\pi}}{2}e^{-\alpha^2/4}$ (Mumbai, 2009 S)

13. $\frac{d}{da}\int_0^{a^2}\tan^{-1}\frac{x}{a}dx = 2a\tan^{-1}a - \frac{1}{2}\log(a^2+1)$. Verify your result by direct integration.

14. $\int_{\pi/2-\alpha}^{\pi/2} \sin\theta\cos^{-1}(\cos\alpha\cos\theta)\, d\theta = \frac{\pi}{2}(1-\cos\alpha)$. (Burdwan, 2003)

15. If $y = \int_0^x f(t)\sin[k(x-t)]dt$, prove that y satisfies the differential equation $\frac{d^2y}{dx^2} + k^2y = k f(x)$.

5.14 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 5.12

Select the correct answer or fill up the blanks in each of the following problems :

1. If $u = e^x(x\cos y - y\sin y)$, then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \ldots$

2. If $x = uv$, $y = u/v$, then $\frac{\partial(x, y)}{\partial(u, v)}$ is

   (a) $-2u/v$ (b) $-2v/u$ (c) 0 (d) 1. (V.T.U., 2010)

3. If $J_1 = \dfrac{\partial(u, v)}{\partial(x, y)}$ and $J_2 = \dfrac{\partial(x, y)}{\partial(u, v)}$, then $J_1 J_2 = ...$

4. If $u = f(y/x)$, then

(a) $x\dfrac{\partial u}{\partial x} - y\dfrac{\partial u}{\partial y} = 0$ (b) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$ (c) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u$ (d) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 1.$

5. If $u = x^y$, then $\partial u/\partial x$ is

(a) 0 (b) yx^{y-1} (c) $x^y \log x$.

6. If $x = r\cos\theta, y = r\sin\theta$, then

(a) $\dfrac{\partial x}{\partial r} = 1/\dfrac{\partial r}{\partial x}$ (b) $\dfrac{\partial x}{\partial r} = \dfrac{\partial r}{\partial x}$ (c) $\dfrac{\partial x}{\partial r} = 0.$

7. If $u = x^y$, then $\partial u/\partial y$ is

(a) yx^{y-1} (b) 0 (c) $x^y \log x$.

8. If $u = x^3 + y^3$, then $\dfrac{\partial^2 u}{\partial x \partial y}$ is equal to

(a) -3 (b) 3 (c) 0 (d) $3x + 3y$ (V.T.U., 2010 S)

9. If $u = x^2 + 2xy + y^2 + x + y$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$ is equal to

(a) $2u$ (b) u (c) 0 (d) none of these.

10. If $u = \log \dfrac{x^2}{y}$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$ is equal to

(a) $2u$ (b) $3u$ (c) u (d) 1. (V.T.U., 2010 S)

11. If $x = r\cos\theta, y = r\sin\theta$, then $\dfrac{\partial(x, y)}{\partial(r, \theta)}$ is equal to

(a) 1 (b) r (c) $1/r$ (d) 0. (V.T.U., 2010 S)

12. If $A = f_{xx}(a, b), B = f_{xy}(a, b), C = f_{yy}(a, b)$, then $f(x, y)$ will have a maximum at (a, b) if

(a) $f_x = 0, f_y = 0, \quad AC < B^2$ and $A < 0$ (b) $f_x = 0, f_y = 0, \quad AC = B^2$ and $A > 0$

(c) $f_x = 0, f_y = 0, \quad AC > B^2$ and $A > 0$ (d) $f_x = 0, f_y = 0, \quad AC > B^2$ and $A < 0$.

13. If $z = \sin^{-1} \dfrac{\sqrt{x^2 + y^2}}{x + y}$, then $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y}$ is

(a) 0 (b) 1/2 (c) 1 (d) 2. (Bhopal, 2008)

14. If $u = \sin^{-1}(x/y) + \tan^{-1}(y/x)$, then $x\,\partial u/\partial x + y\,\partial u/\partial y$ equals

(a) $\sin^{-1}(x/y) + \tan^{-1}(y/x)$ (b) $2[\sin^{-1}(x/y) + \tan^{-1}(y/x)]$

(c) $3[\sin^{-1}(x/y) + \tan^{-1}(y/x)]$ (d) zero.

15. If an error of 1% is made in measuring its length and breadth, the percentage error in the area of a rectangle is

(a) 0.2% (b) 0.02% (c) 2% (d) 1%. (V.T.U., 2010)

16. $\dfrac{\partial(u, v)}{\partial(x, y)} \times \dfrac{\partial(x, y)}{\partial(u, v)}$ equals

(a) -1 (b) 1 (c) zero (d) none of these.

17. $\dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$ is a homogeneous function of degree ......

18. If $z = \log(x^3 + y^3 - x^2y - xy^2)$, then $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y}$ is equal to ......

19. If $r = \partial^2 f/\partial x^2$, $s = \partial^2 f/\partial x \partial y$ and $t = \partial^2 f/\partial y^2$, then the condition for the saddle point is ......

20. If $f(x, y) = \dfrac{1}{x^3} + \dfrac{1}{y^3} + \dfrac{1}{x^3 + y^3}$, then $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y}$ is

(a) 0 (b) $3f$ (c) 9 (d) $-3f$. (V.T.U., 2009 S)

21. If $u = x^4 + y^4 + 3x^2y^2$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = ...$

22. If u and v are functions of r, s where r, s are functions of x, y, then $\frac{\partial(u, v)}{\partial(r, s)} \cdot \frac{\partial(r, s)}{\partial(x, y)} = ...$

23. The necessary conditions for a function $f(x, y)$ to have an extreme at (a, b) are ......

24. If $u = (x-y)^4 + (y-z)^4 + (z-x)^4$, then $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$ is

(a) 1 (b) u (c) $4u$ (d) 0. *(V.T.U., 2010)*

25. If u is a composite function of t, defined by the relations $u = f(x, y)$; $x = \phi(t)$, $y = \psi(t)$, then total derivative $\frac{du}{dt} = ...$

26. If $u = \cos^{-1}(x/y) + \tan^{-1}(y/x)$, then $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy}$ is

(a) u (b) $2u$ (c) 0 (d) 1. *(V.T.U., 2010)*

27. If $f(x, y, z) = 0$, then $\frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial x} = ...$

28. If $u = f(x + ay) + g(x - ay)$, then $\frac{\partial^2 u}{\partial y^2}$ equals

(a) $\frac{\partial^2 u}{\partial x^2}$ (b) $a\frac{\partial^2 u}{\partial x^2}$ (c) $a^2\frac{\partial^2 u}{\partial x^2}$ (d) $\frac{\partial^2 u}{\partial x \partial y}$. *(V.T.U., 2010)*

29. If sum of three numbers is constant, then their product is a maximum when the numbers are ......

30. $y = \cosh(\lambda x)\cosh(-\lambda at)$ is a solution of $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$. *(True or False)*

CHAPTER 6

Integral Calculus and Its Applications

1. Reduction formulae. 2. Reduction formulae for $\int \sin^n x\,dx$, $\int \cos^n x\,dx$ and evaluation of $\int_0^{\pi/2} \sin^n x\,dx$, $\int_0^{\pi/2} \cos^n x\,dx$. 3. Reduction formula for $\int \sin^m x \cos^n x\,dx$ and evaluation of $\int_0^{\pi/2} \sin^m x \cos^n x\,dx$. 4. Reduction formulae for $\int \tan^n x\,dx$ $\int \cot^n x\,dx$. 5. Reduction formulae for $\int \sec^n x\,dx$, $\int \operatorname{cosec}^n x\,dx$. 6. Reduction formulae for $\int x^n e^{ax}\,dx$, $\int x^m (\log x)^n\,dx$. 7. Reduction formulae for $\int x^n \sin mx\,dx$, $\int x^n \cos nx\,dx$ and $\int \cos^m x \sin nx\,dx$. 8. Definite integrals. 9. Integral as the limit of a sum. 10. Areas of curves. 11. Lengths of curves. 12. Volumes of revolution. 13. Surface areas of revolution. 14. Objective Type of Questions.

6.1 REDUCTION FORMULAE

The reader is already familiar with some standard methods of integrating functions of a single variable. However, there are some integrals which cannot be evaluated by the afore-said methods. In such cases, the method of reduction formulae proves useful. A reduction formula connects an integral with another of the same type but of lower order. The successive application of the reduction formula enables us to evaluate the given integral. Now we shall derive some standard reduction formulae.

6.2 (1) REDUCTION FORMULAE for

(a) $\int \sin^n x\,dx$ **(b)** $\int \cos^n x\,dx.$

(a) $\int \sin^n x\,dx = \int \sin^{n-1} x \cdot \sin x\,dx$ [Integrated by parts]

$$= \sin^{n-1} x \cdot (-\cos x) - \int (n-1) \sin^{n-2} x \cos x (-\cos x)\,dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x)\,dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\,dx - (n-1) \int \sin^n x\,dx$$

Transposing

$$n \int \sin^n x\,dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\,dx$$

or

$$\int \sin^n x\,dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x\,dx \qquad ...(i)$$

(b) Similarly, $$\int \cos^n x\,dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

Thus we have the required reduction formulae.

Obs. To integrate $\int \sin^n x\,dx$ or $\int \cos^n x\,dx$,

(a) when the index of sin x is odd put cos x = t

when the index of cos x is odd, put sin x = t

(b) when the index is an even positive integer, express the integrand as a series of cosines of multiple angles and integrate term by term if n is small, *otherwise use the method of reduction formulae.*

(2) To show that $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$

$$= \frac{(n-1)(n-3)(n-5)\ldots}{n(n-2)(n-4)\ldots} \times \left(\frac{\pi}{2}, \text{ only if } n \text{ is even}\right)$$

From (i), we have

$$I_n = \int_0^{\pi/2} \sin^n x\,dx = -\left|\frac{\sin^{n-1} x \cos x}{n}\right|_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

i.e. $$I_n = \frac{n-1}{n} I_{n-2}$$

Case I. *When n is odd*

Similarly $$I_{n-2} = \frac{n-3}{n-2} I_{n-4},\quad I_{n-4} = \frac{n-5}{n-4} I_{n-6}$$

....

$$I_5 = \frac{4}{5} I_3,\ I_3 = \frac{2}{3} I_1 = \frac{2}{3}\int_0^{\pi/2} \sin x\,dx = \frac{2}{3}\left| -\cos x \right|_0^{\pi/2} = \frac{2}{3}.$$

Form these, we get $$I_n = \frac{(n-1)(n-3)(n-5)\ldots 2}{n(n-2)(n-4)\ldots 3} \quad \ldots(ii)$$

Case II. *When n is even*

We have $$I_{n-2} = \frac{n-3}{n-2} I_{n-4},\ I_{n-4} = \frac{n-5}{n-4} I_{n-6}$$

....

$$I_4 = \frac{3}{4} I_2,\ I_2 = \frac{1}{2} I_0 = \int_0^{\pi/2} \sin^0 x\,dx = \frac{1}{2}\int_0^{\pi/2} dx = \frac{1}{2}\cdot\frac{\pi}{2}.$$

Form these, we obtain $$I_n = \frac{(n-1)(n-3)(n-5)\ldots 3\cdot 1}{n(n-2)(n-4)\ldots 4\cdot 2}\cdot\frac{\pi}{2} \quad \ldots(iii)$$

Combining (ii) and (iii), we get the required result for $\int_0^{\pi/2} \sin^n x\,dx$.

Proceeding exactly as above, we get the result for $\int_0^{\pi/2} \cos^n x\,dx$.

Example 6.1. *Integrate (i) $\int \sin^4 x\,dx$ (ii) $\int_0^{\pi/2} \cos^6 x\,dx$.*

Solution. (i) We have the reduction formula

$$\int \sin^n x\,dx = \frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

Putting $n = 4, 2$ successively,

$$\int \sin^4 x\,dx = -\frac{\sin^3 x \cos x}{4} + \frac{3}{4}\int \sin^2 x\,dx \quad \ldots(\alpha)$$

$$\int \sin^2 x\,dx = -\frac{\sin x \cos x}{2} + \frac{1}{2}\int (\sin x)^0\,dx$$

But $\int (\sin x)^0\,dx = \int dx = x. \quad \therefore \quad \int \sin^2 x\,dx = -\frac{\sin x \cos x}{2} + \frac{x}{2}$

Substituting this in (α), we get

$$\int \sin^4 x\,dx = -\frac{\sin^3 x \cos x}{4} + \frac{3}{4}\left(-\frac{\sin x \cos x}{2} + \frac{x}{2}\right)$$

(*ii*) We know that $\int_0^{\pi/2} \cos^n x\,dx = \frac{(n-1)(n-3)(n-5)\ldots}{n(n-2)(n-4)\ldots}\left(\frac{\pi}{2} \text{ if } n \text{ is even}\right)$

Putting $n = 6$, we get

$$\int_0^{\pi/2} \cos^6 x\,dx = \frac{5.3.1}{6.4.2}\frac{\pi}{2} = \frac{5\pi}{16}.$$

Example 6.2. *Evaluate*

(*i*) $\int_0^a \frac{x^7\,dx}{\sqrt{(a^2-x^2)}}$ *(V.T.U., 2006)* (*ii*) $\int_0^\pi \frac{\sqrt{(1-\cos x)}}{1+\cos x}\sin^2 x\,dx$ (*iii*) $\int_0^\infty \frac{dx}{(a^2+x^2)^n}$.

Solution. (*i*) $\int_0^a \frac{x^7}{\sqrt{(a^2-x^2)}}dx$ | Put $x = a\sin\theta$, so that $dx = a\cos\theta\,d\theta$. Also when $x = 0, \theta = 0$, when $x = a, \theta = \pi/2$

$$= \int_0^{\pi/2} \frac{a^7\sin^7\theta}{a\cos\theta}.\,a\cos\theta\,d\theta = a^7\int_0^{\pi/2}\sin^7\theta\,d\theta = a^7.\frac{6.4.2}{7.5.3.1} = \frac{16}{35}a^7$$

(*ii*) Putting $x = 2\theta$, we get

$$\int_0^\pi \frac{\sqrt{(1-\cos x)}}{1+\cos x}\sin^2 x\,dx = \int_0^{\pi/2}\frac{\sqrt{(1-\cos 2\theta)}}{1+\cos 2\theta}\sin^2 2\theta\cdot 2d\theta$$

$$= 2\int_0^{\pi/2}\frac{\sqrt{2}\sin\theta}{2\cos^2\theta}\cdot(2\sin\theta\cos\theta)^2\,d\theta = 4\sqrt{2}\int_0^{\pi/2}\sin^3\theta\,d\theta = 4\sqrt{2}\cdot\frac{2}{3} = \frac{8\sqrt{2}}{3}.$$

(*iii*) $\int_0^\infty \frac{dx}{(a^2+x^2)^n}$ | Put $x = a\tan\theta$, so that $dx = a\sec^2\theta\,d\theta$. Also when $x = 0, \theta = 0$, when $x = \infty, \theta = \pi/2$

$$= \int_0^{\pi/2}\frac{a\sec^2\theta\,d\theta}{a^{2n}\sec^{2n}\theta} = \frac{1}{a^{2n-1}}\int_0^{\pi/2}\cos^{2n-2}\theta\,d\theta = \frac{1}{a^{2n-1}}\cdot\frac{(2n-3)(2n-5)\ldots3.1}{(2n-2)(2n-4)\ldots4.2}\cdot\frac{\pi}{2}.$$

Example 6.3. *Evaluate* $\int_0^a \frac{x^n}{\sqrt{(a^2-x^2)}}dx$. *Hence find the value of* $\int_0^1 x^n\sin^{-1}x\,dx$.

Solution. Putting $x = a\sin\theta$, we get

$$\int_0^a \frac{x^n}{\sqrt{(a^2-x^2)}}dx = \int_0^{\pi/2}\frac{(a\sin\theta)^n}{a\cos\theta}(a\cos\theta)\,d\theta = a^n\int_0^{\pi/2}\sin^n\theta\,d\theta$$

$$\left.\begin{aligned} &= \frac{(n-1)(n-3)\ldots.2}{n(n-2)\ldots.3}a^n, \text{ if } n \text{ is odd}\\ &= \frac{(n-1)(n-3)\ldots1}{n(n-2)\ldots2}\cdot\frac{\pi}{2}a^n, \text{ if } n \text{ is even}\end{aligned}\right\} \quad \ldots(i)$$

Now integrating by parts, we have

$$\int_0^1 x^n\sin^{-1}x\,dx = \left|(\sin^{-1}x)\cdot\frac{x^{n+1}}{n+1}\right|_0^1 - \int_0^1\frac{x^{n+1}}{n+1}\frac{1}{\sqrt{(1-x^2)}}dx$$

$$= \frac{1}{(n+1)}\left[\frac{\pi}{2} - \int_0^1 \frac{x^{n+1}}{(1-x^2)}\right] \qquad \text{[Using (i) p. 241]}$$

$$= \frac{1}{n+1}\left\{\frac{\pi}{2} - \frac{n(n-2)(n-4)\ldots 1}{(n+1)(n-1)(n-3)\ldots 2}\,\frac{\pi}{2}\right\} \qquad \text{when } n \text{ is odd}$$

$$= \frac{1}{n+1}\left\{\frac{\pi}{2} - \frac{n(n-2)(n-4)\ldots 2}{(n+1)(n-1)(n-3)\ldots 3}\right\} \qquad \text{when } n \text{ is even}$$

Evaluate 6.4. *Evaluate* $I_n = \int_0^a (a^2 - x^2)^n\, dx$ *where n is a positive integer. Hence show that*

$$I_n = \frac{2n}{2n+1} a^2 I_{n-1}.$$

Solution. Putting $n = a \sin \theta$, we get

$$I_n = \int_0^a (a^2 - x^2)^n\, dx = \int_0^{\pi/2} (a^2 - a^2 \sin^2\theta)^n\, a\cos\theta\, d\theta = a^{2n+1}\int_0^{\pi/2} \cos^{2n+1}\theta\, d\theta$$

$$= a^{2n+1} \cdot \frac{(2n)(2n-2)(2n-4)\ldots 4.2}{(2n+1)(2n-1)(2n-3)\ldots 5.3} \qquad [\because \ (2n+1) \text{ is always odd}]$$

Now replacing n by $n-1$, we get

$$I_{n-1} = a^{2n-1}\frac{(2n-2)(2n-4)\ldots 4.2}{(2n-1)(2n-3)\ldots 5.3} \quad \therefore \quad \frac{I_n}{I_{n-1}} = a^2 \cdot \frac{2n}{2n+1} \quad \text{or} \quad I_n = \frac{2n}{2n+1}a^2 I_{n-1}.$$

which is the second desired result.

6.3 (1) REDUCTION FORMULAE for $\int \sin^m x \cos^n x\, dx$

$$\int \sin^m x \cos^n x\, dx = \int \sin^{m-1} x \cdot \cos^n x \cdot \sin x\, dx \qquad \text{[Integrate by parts]}$$

$$= \sin^{m-1} x \cdot \left(\frac{-\cos^{n+1} x}{n+1}\right) - \int (m-1)\sin^{m-2} x \cos x \cdot \left(-\frac{\cos^{n+1} x}{n+1}\right) dx.$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{m+1}\int \sin^{m-2} x\,(1 - \sin^2 x)\cos^n x\, dx$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2} x \cos^n x\, dx - \frac{m-1}{n+1}\int \sin^m x \cos^n x\, dx$$

Transposing the last term to the left and dividing by $1 + (m-1)/(n+1)$, *i.e.*, $(m+n)/(n+1)$, we obtain the reduction formula

$$\int \sin^m x \cos^n x\, dx = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} x \cos^n x\, dx \qquad \ldots(1)$$

Obs. To integrate $\int \sin^m x \cos^n x\, dx$,

(*a*) *when m is odd,* put $\cos x = t$

when n is odd, put $\sin x = t$

(*b*) *when m and n both are even integers,* express the integrand as a series of cosines of multiple angles and integrate term by term if m and n are small, *otherwise* use the method of reduction formulae.

(2) To show that

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{(m-1)(m-3)\ldots \times (n-1)(n-3)\ldots}{(m+n)(m+n-2)(m+n-4)\ldots} \times \left(\frac{\pi}{2}, \textit{ only if both m and n are even}\right)$$

From (*i*), we have

$$I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x\,dx = \left| -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} \right|_0^{\pi/2} + \frac{m-1}{m+n}\int_0^{\pi/2} \sin^{m-2} x \cos^n x\,dx$$

i.e.,
$$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}.$$

Case I. *When m is odd*

Similarly, $$I_{m-2,n} = \frac{m-3}{m+n-2} I_{m-4,n}, I_{m-4,n} = \frac{m-5}{m+n-4} I_{m-6,n}$$

....

$$I_{5,n} = \frac{4}{n+5} I_{3,n}$$

Finally $$I_{3,n} = \frac{2}{n+3} I_{1,n} = \frac{2}{n+3}\int_0^{\pi/2} \sin x \cos^n x\,dx$$

$$= \frac{2}{n+3}\left| -\frac{\cos^{n+1} x}{n+1} \right|_0^{\pi/2} = \frac{2}{(n+3)(n+1)} \qquad ...(ii)$$

From these, we obtain

$$I_{m,n} = \frac{(m-1)(m-3)(m-5)\ldots 4.2}{(m+n)(m+n-2)(m+n-4)\ldots(n+3)(n+1)}$$

Case II. *When m is even*

We have, $$I_{m-2,n} = \frac{m-3}{m+n-2} I_{m-4,n}, \quad I_{m-4,n} = \frac{m-5}{m+n-4} I_{m-6,n}$$

....

$$I_{4,n} = \frac{3}{n+4} I_{2,n}, I_{2,n} = \frac{1}{n+2} I_{0,n} = \frac{1}{n+2}\int_0^{\pi/2} \cos^n x\,dx$$

From these, we have $$I_{m,n} = \frac{(m-1)(m-3)(m-5)\ldots 1}{(m+n)(m+n-2)(m+n-4)\ldots(n+2)}\int_0^{\pi/2} \cos^n x\,dx$$

$$= \frac{(m-1)(m-3)\ldots 1}{(m+n)(m+n-2)\ldots(n+2)} \cdot \frac{(n-1)(n-3)\ldots}{n(n-2)\ldots} \times (\pi/2 \text{ only if } n \text{ is even}) \qquad ...(iii)$$

Combining (*ii*) and (*iii*), we get the desired result.

Example 6.5. *Integrate* (*i*) $\int \sin^4 x \cos^2 x\,dx$ (*Raipur, 2005*)

(*ii*) $\int_0^\infty \frac{t^6}{(1+t^2)^7}\,dt$ (*iii*) $\int_0^\infty \frac{x^2}{(1+x^2)^{7/2}}\,dx$ (*V.T.U., 2010 S*)

Solution. (*i*) Taking $n = 2$, in (*i*) of page 241, we have the reduction formula :

$$\int \sin^m x \cos^2 x\,dx = \frac{\sin^{m-1} x \cos^3 x}{m+2} + \frac{m-1}{m+2}\int \sin^{m-2} x \cos^2 x\,dx$$

Putting $m = 4, 2$ successively,

$$\int \sin^4 x \cos^2 x\,dx = -\frac{\sin^3 x \cos^3 x}{6} + \frac{3}{6}\int \sin^2 x \cos^2 x\,dx \qquad ...(1)$$

$$\int \sin^2 x \cos^2 x\,dx = -\frac{\sin x \cos^3 x}{4} + \frac{1}{4}\int \cos^2 x\,dx$$

But $$\int \cos^2 x\,dx = \frac{1}{2}\int (1 + \cos 2x)\,dx = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right)$$

$$\therefore \qquad \int \sin^2 x \cos^2 x\, dx = -\frac{\sin x \cos^3 x}{4} + \frac{1}{16}(2x + \sin 2x)$$

Substituting this in (1), we get

$$\int \sin^4 x \cos^2 x\, dx = -\frac{\sin^3 x \cos^3 x}{6} + \frac{1}{2}\left\{-\frac{\sin x \cos^3 x}{4} + \frac{1}{16}(2x + \sin 2x)\right\}$$

(*ii*) Putting $t = \tan\theta$, so that

$$\int_0^\infty \frac{t^6}{(1+t^2)^7}\, dt = \int_0^{\pi/2} \frac{\tan^6\theta}{\sec^{14}\theta}\sec^2\theta\, d\theta = \int_0^{\pi/2} \sin^6\theta \cos^6\theta\, d\theta = \frac{5\cdot3\cdot1\times5\cdot3\cdot1}{12\cdot10\cdot8\cdot6\cdot4\cdot2}\times\frac{\pi}{2} = \frac{5\pi}{2048}.$$

(*iii*) Putting $x = \tan\theta$, so that

$$\int_0^\infty \frac{x^2}{(1+x^2)^{7/2}}\, dx = \int_0^{\pi/2} \frac{\tan^2\theta}{\sec^7\theta}\sec^2\theta\, d\theta = \int_0^{\pi/2} \sin^2\theta\cos^3\theta\, d\theta = \frac{1.2}{53.1} = \frac{2}{15}.$$

Example 6.6. *Evaluate :* (*i*) $\int_0^{\pi/6} \cos^4 3\theta \sin^3 6\theta\, d\theta$ *(V.T.U., 2003 S)*

(*ii*) $\int_0^1 x^4(1-x^2)^{3/2}\, dx$ (*iii*) $\int_0^{2a} x^2\sqrt{(2ax - x^2)}\, dx.$ *(V.T.U., 2010)*

Solution. (*i*) $\int_0^{\pi/6} \cos^4 3\theta \sin^3 6\theta\, d\theta = \int_0^{\pi/6} \cos^4 3\theta\,(2\sin 3\theta\cos 3\theta)^3\, d\theta$

$$= 8\int_0^{\pi/6} \sin^3 3\theta \cos^7 3\theta\, d\theta$$

$$= \frac{8}{3}\int_0^{\pi/2} \sin^3 x \cos^7 x\, dx$$

$$= \frac{8}{3}\cdot\frac{2\times6\cdot4\cdot2}{10\cdot8\cdot6\cdot4\cdot2} = \frac{1}{15}.$$

| Put $3\theta = x$ so that $3d\theta = dx$. Also when $\theta = 0, x = 0$; when $\theta = \pi/6, x = \pi/2$.

(*ii*) $\int_0^1 x^4(1-x^2)^{3/2}\, dx$

| Put $x = \sin t$ so that $dx = \cos t\, dt$. When $x = 0, t = 0$; when $x = 1, t = \pi/2$

$$= \int_0^{\pi/2} \sin^4 t\,(\cos^2 t)^{3/2}\cdot\cos t\, dt = \int_0^{\pi/2} \sin^4 t\cos^4 t\, dt$$

$$= \frac{3\cdot1\times3\cdot1}{8\cdot6\cdot4\cdot2}\cdot\frac{\pi}{2} = \frac{3\pi}{256}.$$

(*iii*) $\int_0^{2a} x^2\sqrt{(2ax - x^2)}\, dx$

$$= \int_0^{\pi/2} x^{5/2}\sqrt{(2a-x)}\, dx$$

| Put $x = 2a\sin^2\theta$ $\therefore\ dx = 4a\sin\theta\cos\theta\, d\theta$

$$= \int_0^{\pi/2} (2a\sin^2\theta)^{5/2}\sqrt{(2a)}\cos\theta\cdot 4a\sin\theta\cos\theta\, d\theta$$

$$= 2^5 a^4\int_0^{\pi/2} \sin^6\theta\cos^2\theta\, d\theta = 32\,a^4\cdot\frac{5\cdot3\cdot1\times1}{8\cdot6\cdot4\cdot2}\cdot\frac{\pi}{2} = \frac{5\pi a^4}{8}.$$

PROBLEMS 6.1

Evaluate :

1. (*i*) $\int_0^{\frac{\pi}{2}} \cos^9 x\, dx$ (*ii*) $\int_0^{\pi/6} \sin^5 3\theta\, d\theta$
2. (*i*) $\int_0^1 \frac{x^9}{\sqrt{(1-x^2)}}\, dx$ (*ii*) $\int_0^1 x^5\sin^{-1} x\, dx$
3. (*i*) $\int_0^\infty \frac{dx}{(1+x^2)^n}\ (n > 1)$ *(V.T.U., 2008 S)* (*ii*) $\int_0^{\pi/4} \sin^2 x\cos^4 x\, dx.$ *(J.N.T.U., 2003)*

4. If $I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x \, dx \; (m > 0, n > 0)$, show that $I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}$.

Hence evaluate $\int_0^{\pi/2} \sin^4 x \cos^8 x \, dx$

Evaluate :

5. (i) $\int_0^{\pi/2} \sin^4 x \cos^6 x \, dx$ *(Cochin, 2005)* (ii) $\int_0^{\pi/2} \sin^{15} x \cos^3 x \, dx$

6. (i) $\int_0^1 x^6 \sqrt{(1-x^2)} \, dx$ (ii) $\int_0^{\pi/2} \cos^4 3\theta \sin^3 6\theta \, d\theta$

7. (i) $\int_0^{2a} x^{7/2} (2a-x)^{-1/2} \, dx$ (ii) $\int_0^{2a} \frac{x^3 \, dx}{\sqrt{(2ax - x^2)}}$ *(Madras, 2000 S)*

8. (i) $\int_0^2 x^{5/2} \sqrt{(2-x)} \, dx$ (ii) $\int_0^4 x^3 \sqrt{(4x - x^2)} \, dx$ *(V.T.U., 2004)*

9. If $I_n = \int x^n \sqrt{(a-x)} \, dx$, prove that $(2n+3) I_n = 2an \, I_{n-1} - 2x^n (a-x)^{3/2}$ *(Marathwada, 2008)*

10. If n is a positive integer, show that $\int_0^{2a} x^n \sqrt{(2ax - x^2)} \, dx = \frac{2n+1}{(n+2)\, n!} \cdot \frac{a^{n+2}}{2n} \pi$ *(V.T.U., 2007)*

6.4 REDUCTION FORMULAE for (a) $\int \tan^n x \, dx$ (b) $\int \cot^n x \, dx$

(a) Let
$$I_n = \int \tan^n x \, dx = \int \tan^{n-2} x \cdot \tan^2 x \, dx = \int \tan^{n-2} x \cdot (\sec^2 x - 1) \, dx$$
$$= \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx$$

Thus, $I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$ which is the required reduction formula.

(b) Let
$$I_n = \int \cot^n x \, dx = \int \cot^{n-2} x \cot^2 x \, dx = \int \cot^{n-2} x \, (\text{cosec}^2 x - 1) \, dx$$
$$= \int \cot^{n-2} x \, \text{cosec}^2 x \, dx - \int \cot^{n-2} x \, dx$$

Thus
$$I_n = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$$
which is the required reduction formula.

Example 6.7. *Evaluate (i)* $\tan^5 x \, dx$ *(ii)* $\int \cot^6 x \, dx$.

Solution. (i) Putting $n = 5, 3$ successively in the reduction formula for $\int \tan^n x \, dx$, we get
$$I_5 = \frac{1}{4} \tan^4 x - I_3 ; \quad I_3 = \frac{1}{2} \tan^2 x - I_1$$

Thus
$$I_5 = \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x + I_1$$

i.e.,
$$\int \tan^5 x \, dx = \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x + \int \tan x \, dx = \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x - \log \cos x.$$

(ii) Putting $n = 6, 4, 2$ successively in the reduction formula for $\int \cot^n x \, dx$, we get
$$I_6 = -\frac{1}{5} \cot^5 x - I_4 ; I_4 = -\frac{1}{3} \cot^3 x - I_2 ; I_2 = -\cot x - I_0$$

Thus
$$I_6 = \frac{-1}{5} \cot^5 x + \frac{1}{3} \cot^3 x - \cot x - \int dx$$

i.e., $$\int \cot^6 x\,dx = -\frac{1}{5}\cot^5 x + \frac{1}{3}\cot^3 x - \cot x - x.$$

Example 6.8. *If* $I_n = \int_0^{\pi/4} \tan^n \theta\,d\theta$, *prove that* $n\,(I_{n-1} + I_{n+1}) = 1.$ *(V.T.U., 2003)*

Solution. The reduction formula for $\int_0^{\pi/4} \tan^n \theta\,d\theta$ is

$$I_n = \frac{1}{n-1}\left|\tan^n x\right|_0^{\pi/4} - I_{n-2} = \frac{1}{n-1} - I_{n-2} \text{ or } I_n + I_{n-2} = \frac{1}{n-1}$$

Changing n to $n+1$, we obtain

$$I_{n+1} + I_{n-1} = \frac{1}{(n+1)} \quad \text{or} \quad (n+1)(I_{n+1} + I_{n-1}) = 1.$$

6.5 REDUCTION FORMULAE for (a) $\int \sec^n x\,dx$ (b) $\int \text{cosec}^n x\,dx$

(*a*) Let $$I_n = \int \sec^n x\,dx = \int \sec^{n-2} x \,.\, \sec^2 x\,dx$$

Integrating by parts, we have

$$I_n = \sec^{n-2} x.\tan x - \int \{(n-2)\sec^{n-3} x \cdot \sec x \tan x\} \tan x\,dx$$
$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \cdot \tan^2 x\,dx$$
$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \cdot (\sec^2 x - 1)\,dx$$
$$= \sec^{n-2} x \tan x - (n-2)\,I_n + (n-2)\,I_{n-2}$$

Transposing, we have

$$(n-1)I_n = \sec^{n-2} x \tan x + (n-2)I_{n-2}$$

Thus $$I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$ which is the desired reduction formula.

(*b*) Let $$I_n = \int \text{cosec}^n x\,dx = \int \text{cosec}^{n-2} x \cdot \text{cosec}^2 x\,dx$$

Integrating by parts, we have

$$I_n = \text{cosec}^{n-2} x.\,(-\cot x) - \int [(n-2)\,\text{cosec}^{n-3} x \cdot (-\text{cosec}\, x \cot x) \cdot (-\cot x)\,dx$$
$$= -\cot x\,\text{cosec}^{n-2} x - (n-2)\int \text{cosec}^{n-2} x\,(\text{cosec}^2 x - 1)\,dx$$
$$= -\cot x\,\text{cosec}^{n-2} x - (n-2)I_n + (n-2)I_{n-2}$$

or $$[1 + (n-2)]I_n = -\cot x\,\text{cosec}^{n-2} x + (n-2)I_{n-2}$$

Thus $$I_n = -\frac{\cot x\,\text{cosec}^{n-2} x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

which is the required reduction formula.

Example 6.9. *Evaluate* (*i*) $\int_0^{\pi/4} \sec^4 x\,dx$ (*ii*) $\int_{\pi/3}^{\pi/2} \text{cosec}^3 \theta\,d\theta.$ *(V.T.U., 2008)*

Solution. (*i*) Putting $n = 4$ in the reduction formula for $\int \sec^n x\,dx$, we get $I_4 = \frac{\sec^2 x \tan x}{3} + \frac{2}{3} I_2$

$$\therefore \quad \int_0^{\pi/4} \sec^4 x\,dx = \left|\frac{\sec^2 x \tan x}{3}\right|_0^{\pi/4} + \frac{2}{3}\int_0^{\pi/4} \sec^2 x\,dx$$
$$= \frac{2}{3} + \frac{2}{3}\left|\tan x\right|_0^{\pi/4} = \frac{2}{3}(1+1) = 4/3.$$

(*ii*) Putting $n = 3$ in the reduction formula for $\int \operatorname{cosec}^n x\,dx$, we get

$$I_3 = -\frac{1}{2}\cot x \operatorname{cosec} x + \frac{1}{2} I_1$$

$$\therefore \quad \int_{\pi/3}^{\pi/2} \operatorname{cosec}^3 x\,dx = -\frac{1}{2}\left|\cot x \operatorname{cosec} x\right|_{\pi/3}^{\pi/2} + \frac{1}{2}\int_{\pi/3}^{\pi/2} \operatorname{cosec} x\,dx$$

$$= -\frac{1}{2}\left(0 - \frac{2}{3}\right) + \frac{1}{2}\left|\log(\operatorname{cosec} x - \cot x)\right|_{\pi/3}^{\pi/2}$$

$$= \frac{1}{3} + \frac{1}{2}\left[\log 1 - \log\left(\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right)\right] = \frac{1}{3} + \frac{1}{4}\log 3.$$

PROBLEMS 6.2

1. Evaluate (*i*) $\int \tan^6 x\,dx$ (*V.T.U., 2007*) (*ii*) $\int \cot^5 x\,dx$.
2. Show that $\int_0^{\pi/4} \tan^7 x\,dx = \frac{1}{12}(5 - 6\log 2)$
3. If $I_n = \int_0^{\pi/4} \tan^n x\,dx$, prove that $(n-1)(I_n + I_{n-2}) = 1$. (*V.T.U., 2009*)
   Hence evaluate I_5. (*Madras, 2000*)
4. If $I_n = \int_{\pi/4}^{\pi/2} \cot^n \theta\,d\theta\ (n > 2)$, prove that $I_n = \frac{1}{n-1} - I_{n-1}$. Hence evaluate I_4. (*Marathwada, 2008*)
5. Obtain the reduction formula for $\int_0^{\pi/4} \sec^n \theta\,d\theta$. (*V.T.U., 2010 S*)
6. Evaluate (*i*) $\int \sec^6 \theta\,d\theta$ (*ii*) $\int_{\pi/6}^{\pi/2} \operatorname{cosec}^5 d\theta$.
7. Evaluate $\int_0^a (a^2 + x^2)^{5/2}\,dx$.
8. If $I_n = \int \frac{t^n}{1+t^2}\,dt$, show that $I_{n+2} = \frac{t^{n+1}}{n+1} - I_n$. Hence evaluate I_6.

6.6 REDUCTION FORMULAE for

(*a*) $\int x^n e^{ax}\,dx$ **(*b*) $\int x^m (\log x)^n\,dx$.**

(*a*) Let $I_n = \int x^n e^{ax}\,dx$

Integrating by parts, we have

$$I_n = x^n \cdot \frac{e^{ax}}{a} - \int n x^{n-1} \cdot \frac{e^{ax}}{a}\,dx$$

or $$I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$$ which is the required reduction formula. (*Madras, 2006*)

(*b*) Let $I_{m,n} = \int x^m (\log x)^n\,dx = \int (\log x)^n \cdot x^m\,dx$

Integrating by parts, we have

$$I_{m,n} = (\log x)^n \cdot \frac{x^{m+1}}{m+1} - \int n(\log x)^{n-1} \cdot \frac{1}{x} \cdot \frac{x^{m+1}}{m+1}\,dx$$

$$= \frac{x^{m+1}}{m+1}(\log x)^n - \frac{n}{m+1}\int x^m (\log x)^{n-1}\,dx \quad \text{or} \quad I_{m,n} = \frac{x^{m+1}}{m+1}(\log x)^n - \frac{n}{m+1} I_{m,n-1}$$

which is the desired reduction formula.

6.7 REDUCTION FORMULAE for

(*a*) $\int \mathbf{x^n \sin mx \, dx}$ (*b*) $\int \mathbf{x^n \cos mx \, dx}$ (*c*) $\int \mathbf{\cos^m x \sin nx \, dx}$

(*a*) Let $I_n = \int x^n \sin mx \, dx$

Integrating by parts, we get

$$I_n = x^n \left(\frac{-\cos mx}{m}\right) - \int n x^{n-1} \left(\frac{-\cos mx}{m}\right) dx$$

$$= -\frac{x^n \cos mx}{m} + \frac{n}{m} \int x^{n-1} \cos mx \, dx \qquad \text{[Again integrate by parts]}$$

$$= -\frac{x^n \cos mx}{m} + \frac{n}{m} \left\{x^{n-1} . \frac{\sin mx}{m} - \left\{\int (n-1)x^{n-2} . \frac{\sin mx}{m} dx\right\}\right\}$$

or
$$I_n = -\frac{x^n \cos mx}{m} + \frac{n}{m^2} x^{n-1} \sin mx - \frac{n(n-1)}{m^2} I_{n-2}$$

which is the desired reduction formula. *(Madras, 2003)*

(*b*) Let $I_n = \int x^n \cos mx \, dx$

Integrating twice by parts as above, we get

$$I_n = \frac{x^n \sin mx}{m} + \frac{n}{m^2} x^{n-1} \cos mx - \frac{n(n-1)}{m^2} I_{n-2}$$

(*c*) Let $I_{m,n} = \int \cos^m x \sin nx \, dx$

Integrating by parts,

$$I_{m,n} = -\cos^m x . \frac{\cos nx}{n} - \int m \cos^{m-1} x (-\sin x) . \left(\frac{-\cos nx}{n}\right) dx$$

$$= -\frac{1}{n} \cos^m x \cos nx - \frac{m}{n} \int \cos^{m-1} x . \cos nx \sin x \, dx$$

$$\left[\begin{array}{l} \because \ \sin (n-1)x = \sin nx \cos x - \cos nx \sin x \\ \text{or } \cos nx \sin x = \sin nx \cos x - \sin (n-1)x \end{array}\right.$$

$$= -\frac{1}{n} \cos^m x \cos nx - \frac{m}{n} \int \cos^{m-1} x \{\sin nx \cos x - \sin (n-1)x\} dx$$

$$= -\frac{1}{n} \cos^m x \cos nx - \frac{m}{n} (I_{m,n} - I_{m-1,n-1})$$

Transposing, we get

$$\left(1 + \frac{m}{n}\right) I_{m,n} = -\frac{1}{n} \cos^m x \cos nx + \frac{m}{n} I_{m-1,n-1}$$

or
$$I_{m,n} = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$$

which is the desired reduction formula.

Example 6.10. *Show that* $\int_0^{\pi/2} \cos^m x \cos nx \, dx = \frac{m}{m+n} \int_0^{\pi/2} \cos^{m-1} x \cos (n-1) x \, dx$

Hence deduce that $\int_0^{\pi/2} \cos^n x \cos nx \, dx = \frac{\pi}{2^{n+1}}$. *(S.V.T.U., 2008)*

Solution. Let $I_{m,n} = \int_0^{\pi/2} \cos^m x \cos nx \, dx$

Integrating by parts

$$I_{m,n} = \left| \cos^m x . \frac{\sin nx}{n} \right|_0^{\pi/2} - \int_0^{\pi/2} m \cos^{m-1} x (-\sin x) \times \frac{\sin nx}{n} dx$$

$$= \frac{m}{n}\int_0^{\pi/2} \cos^{m-1} x \sin nx \sin x \, dx \qquad \left[\begin{array}{l}\because \cos (n-1)x = \cos nx \cos x + \sin nx \sin x \\ \text{or } \sin nx \sin x = \cos (n-1)x - \cos nx \cos x\end{array}\right.$$

$$= \frac{m}{n}\int_0^{\pi/2} \cos^{m-1} x \, [\cos (n-1)x - \cos nx \cos x] \, dx = \frac{m}{n}(I_{m-1,\, n-1} - I_{m,\, n})$$

Transposing and dividing by $(1 + m/n)$, we get

$$I_{m,\,n} = \frac{m}{m+n} I_{m-1,\,n-1}$$

which is the required result.

Putting $\quad m = n, I_n \left(= \int_0^{\pi/2} \cos^n x \cos nx \, dx\right) = \frac{1}{2} I_{n-1}$

Changing n to $n - 1$,

$$I_{n-1} = \frac{1}{2} I_{n-2}$$

$$\therefore \quad I_n = \frac{1}{2}\left(\frac{1}{2} I_{n-2}\right) = \frac{1}{2^2} I_{n-2} = \frac{1}{2^3} I_{n-3} \ldots\ldots = \frac{1}{2^n} I_{n-n} = \frac{1}{2^n} \cdot \int_0^{\pi/2} (\cos x)^0 \, dx$$

Hence $\quad I_n = \frac{1}{2^n} \cdot \frac{\pi}{2} = \frac{\pi}{2^{n+1}}.$

Example 6.11. *Find a reduction formula for* $\int e^{ax} \sin x \, dx$*. Hence evaluate* $\int e^x \sin^3 x \, dx$*.*

Solution. Let $I_n = \int e^{ax} \sin^n x \, dx = \int \underset{I}{\sin^n x} \cdot \underset{II}{e^{ax}} \, dx$

Integrating by parts,

$$I_n = \sin^n x \cdot \frac{e^{ax}}{a} - \int (n \sin^{n-1} x \cos x) \cdot \frac{e^{ax}}{a} \, dx$$

$$= \frac{e^{ax} \sin^n x}{a} - \frac{n}{a}\int (\sin^{n-1} x \cos x) \cdot e^{ax} \, dx \qquad \text{[Again integrating by parts]}$$

$$= \frac{e^{ax} \sin^n x}{a} - \frac{n}{a}\left[\sin^{n-1} x \cos x \cdot \frac{e^{ax}}{a} - \int \{(n-1)\sin^{n-2} x \times \cos x \cdot \cos x + \sin^{n-1} x (-\sin x)\} \frac{e^{ax}}{a} \, dx\right]$$

$$= \frac{e^{ax} \sin^{n-1} x}{a^2}(a \sin x - n \cos x) + \frac{n}{a^2}\int [(n-1)\sin^{n-2} x \times (1 - \sin^2 x) - \sin^n x] \, e^{ax} \, dx$$

$$= \frac{e^{ax} \sin^{n-1} x}{a}(a \sin x - n \cos x) + \frac{n(n-1)}{a^2} I_{n-2} - \frac{n^2}{a^2} I_n$$

Transposing and dividing by $(1 + n^2/a^2)$, we get

$$I_n = \frac{e^{ax} \sin^{n-1} x \, (a \sin x - n \cos x)}{a^2 + n^2} + \frac{n(n-1)}{a^2 + n^2} I_{n-2}$$

which is the required reduction formula.

Putting $a = 1$ and $n = 3$, we get

$$I_3 = \frac{e^x \sin^2 x \, (\sin x - 3 \cos x)}{1^2 + 9} + \frac{3 \cdot 2}{1^2 + 9} I_1$$

But $\quad I_1 = \int e^x \sin x \, dx = \frac{e^{ax}}{\sqrt{2}} \sin (x - \tan^{-1} 1).$

$$\therefore \quad I_3 = \frac{e^x \sin^2 x \, (\sin x - 3 \cos x)}{10} + \frac{3}{5} \cdot \frac{e^{ax}}{\sqrt{2}} \sin (x - \pi/4).$$

PROBLEMS 6.3

1. If $I_n = \int x^n e^x \, dx$, show that $I_n + n I_{n-1} = x^n e^x$. Hence find I_4. *(Madras, 2000)*

2. If $u_n = \int_0^a x^n e^{-x} \, dx$, prove that $u_n - (n + a) u_{n-1} + a(n-1) u_{n-2} = 0$. *(Madras, 2003)*

3. Obtain a reduction formula for $\int x^m (\log x)^n \, dx$. Hence evaluate $\int_0^1 x^5 (\log x)^3 \, dx$. *(S.V.T.U., 2009 ; Bhillai, 2005)*

4. If n is a positive integer, show that $\int_0^1 x^m (\log x)^n \, dx = \frac{(-1)^n n!}{(m+1)^{n+1}}, m > -1$.

5. If $I_n = \int_0^{\pi/2} x \sin^n x \, dx$ $(n > 1)$, prove that $n^2 I_n = n(n-1) I_{n-2} + 1$. Hence evaluate I_5.

6. If $I_n = \int_0^{\pi/2} x \cos^n x \, dx$ $(n > 1)$, prove that $I_n = \frac{n-1}{n} I_{n-2} - \frac{1}{n^2}$. Hence evaluate I_4.

7. If $u_n = \int_0^{\pi/2} x^n \sin x \, dx$, $(n > 1)$, prove that $u_n + n(n-1) u_{n-2} = n(\pi/2)^{n-1}$. Hence evaluate u_2. *(Madras, 2000 S)*

8. If $I_n = \int x^n \sin ax \, dx$, show that $a^2 I_n = -ax^n \cos ax + nx^{n-1} \sin ax - n(n-1) I_{n-2}$. *(Marathwada, 2008)*

9. Prove that $\int_0^{\pi/2} \cos^{n-2} x \sin nx \, dx = \frac{1}{n-1}, n > 1$.

10. If $I_{m,n} = \int_0^{\pi/2} \cos^m x \cos nx \, dx$, prove that $I_{m,n} = \frac{m(m-1)}{m^2 - n^2} I_{m-2,n}$

11. Find a reduction formula for $\int e^{ax} \cos^n x \, dx$. Hence evaluate $\int_0^{\pi/2} e^{2x} \cos^3 x \, dx$.

12. Obtain a reduction formula for $I_m = \int_0^\infty e^{-x} \sin^m x \, dx$ where $m \geq 2$ in the form

    $(1 + m^2) I_m = m(m-1) I_{m-2}$. Hence evaluate I_4. *(Gorakhpur, 1999)*

6.8 DEFINITE INTEGRALS

Property I. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

(*i.e., the value of a definite integral depends on the limits and not on the variable of integration*).

Let $\int f(x)\,dx = \phi(x)$; $\therefore \int_a^b f(x)\,dx = \phi(b) - \phi(a)$.

Then $\int f(t)\,dt = \phi(t)$; $\therefore \int_a^b f(t)\,dt = \phi(b) - \phi(a)$.

Hence the result.

Property II. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(*i.e., the interchange of limits changes the sign of the integral*).

Let $\int f(x)\,dx = \phi(x)$; $\therefore \int_a^b f(x)\,dx = \phi(b) - \phi(a)$

and $-\int_b^a f(x)\,dx = -\left| \phi(x) \right|_b^a = -[\phi(a) - \phi(b)] = \phi(b) - \phi(a)$.

Hence the result.

Property III. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Let $\int f(x)\,dx = \phi(x)$, so that $\int_a^b f(x)\,dx = \phi(b) - \phi(a)$ ...(1)

Also $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = |\,\phi(x)\,|_a^c + |\,\phi(x)\,|_c^b$

$$= [\phi(c) - \phi(a)] + [\phi(b) - \phi(c)] = \phi(b) - \phi(a) \quad ...(2)$$

Hence the result follows from (1) and (2).

Property IV. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Put $x = a - t$, so that $dx = -dt$. Also when $x = 0, t = a$; when $x = a, t = 0$.

$\therefore$ $$\int_0^a f(x)\,dx = -\int_a^0 f(a-t)\,dt = \int_0^a f(a-t)\,dt = \int_0^a f(a-x)\,dx \quad \text{[Prop. II]}$$

Example 6.12. *Evaluate* $\int_0^{\pi/2} \frac{\sqrt{(\sin x)}}{\sqrt{(\sin x)} + \sqrt{(\cos x)}}\,dx$.

Solution. Let $$I = \int_0^{\pi/2} \frac{\sqrt{(\sin x)}}{\sqrt{(\sin x)} + \sqrt{(\cos x)}}\,dx$$

Then $$I = \int_0^{\pi/2} \frac{\sqrt{\left[\sin\left(\frac{1}{2}\pi - x\right)\right]}}{\sqrt{\left[\sin\left(\frac{1}{2}\pi - x\right)\right]} + \sqrt{\left[\cos\left(\frac{1}{2}\pi - x\right)\right]}}\,dx \quad \text{[Prop. IV]}$$

$$= \int_0^{\pi/2} \frac{\sqrt{(\cos x)}}{\sqrt{(\cos x)} + \sqrt{(\sin x)}}\,dx$$

Adding $$2I = \int_0^{\pi/2} \frac{\sqrt{(\sin x)} + \sqrt{(\cos x)}}{\sqrt{(\sin x)} + \sqrt{(\cos x)}}\,dx = \int_0^{\pi/2} dx = |\,x\,|_0^{\pi/2} = \frac{\pi}{2}.$$

Hence $I = \pi/4$.

Example 6.13. *Evaluate* $\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$. *(Cochin, 2005)*

Solution. Let $$I = \int_0^1 \frac{\log(1+x)}{1+x^2}\,dx \quad \left[\begin{array}{l}\text{Put } x = \tan\theta \text{ so that } dx = \sec^2\theta\,d\theta \\ \text{When } x = 0, \theta = 0 \text{ ; when } x = 1, \theta = \pi/4\end{array}\right.$$

$$= \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{1+\tan^2\theta}\,.\sec^2\theta\,d\theta = \int_0^{\pi/4} \log(1+\tan\theta)\,d\theta$$

$$= \int_0^{\pi/4} \log\left[1+\tan\left(\frac{\pi}{4}-\theta\right)\right]d\theta = \int_0^{\pi/4} \log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)d\theta \quad \text{[Prop. IV]}$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right)d\theta = \log 2\int_0^{\pi/4} d\theta - I$$

Transposing, $2I = \log 2 \cdot |\,\theta\,|_0^{\pi/4} = \frac{\pi}{4}\log 2$. Hence $I = \frac{\pi}{8}\log 2$.

Example 6.14. *Evaluate* $\int_0^{\pi} \frac{x\sin^3 x}{1+\cos^2 x}\,dx$. *(Madras, 2006)*

Solution. Let $$I = \int_0^{\pi} \frac{x\sin^3 x}{1+\cos^2 x}\,dx$$

Then $$I = \int_0^\pi \frac{(\pi - x)\sin^3(\pi - x)}{1+\cos^2(\pi - x)}\,dx$$ [Prop. IV]

$$= \int_0^\pi \frac{(\pi - x)\sin^3 x}{1+\cos^2 x}\,dx = \pi\int_0^\pi \frac{\sin^3 x}{1+\cos^2 x}\,dx - \mathrm{I}$$

Transposing, $$2I = \pi\int_0^\pi \frac{\sin^3 x}{1+\cos^2 x}\,dx$$

$$= -\pi\int_1^{-1}(1-t^2)\frac{dt}{1+t^2}$$

Put $\cos x = t$ so that $-\sin x\,dx = dt$
When $x = 0, t = 1$; When $x = \pi, t = -1$;

$$= \pi\int_1^{-1}\frac{-2+(1+t^2)}{1+t^2}\,dt = -2\pi\int_1^{-1}\frac{dt}{1+t^2} + \pi\int_1^{-1} dt$$

$$= -2\pi\left|\tan^{-1} t\right|_1^{-1} + \pi\left|t\right|_1^{-1} = -2\pi\left(-\frac{\pi}{4}-\frac{\pi}{4}\right) - 2\pi.$$ Hence, $I = \pi^2/2 - \pi$.

Property V. $\int_{-a}^{a} \mathbf{f(x)\,dx} = 2\int_0^a \mathbf{f(x)\,dx}$**, if f(x) is an even function,**

$= \mathbf{0}$ **if f(x) is an odd function.** (*Bhopal, 2008*)

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx \qquad ...(1) \quad \text{[Prop. I]}$$

In $\int_{-a}^{0} f(x)\,dx$, put $x = -t$, so that $dx = -dt$

$$\therefore \quad \int_{-a}^{0} f(x)\,dx = -\int_a^0 f(-t)\,dt = \int_0^a f(-t)\,dt = \int_0^a f(-x)\,dx \qquad \text{[Prop. II]}$$

Substituting in (1), we get

$$\int_{-a}^{a} f(x)\,dx = \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx \qquad ...(2)$$

(*i*) *If* $f(x)$ *is an even function,* $f(-x) = f(x)$.

$\therefore$ from (2), $\int_{-a}^{a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$

(*ii*) *If* $f(x)$ *is an odd function,* $f(-x) = -f(x)$.

$\therefore$ from (2), $\int_{-a}^{a} f(x)\,dx = -\int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 0.$

Property VI. $\int_0^{2a} \mathbf{f(x)\,dx} = 2\int_0^a \mathbf{f(x)\,dx}$**, if f (2a – x) = f(x)**

$= \mathbf{0,}$ **if f(2a – x) = – f(x)**

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx \qquad ...(1) \quad \text{[Prop. III]}$$

In $\int_0^{2a} f(x)\,dx$, put $x = 2a - t$, so that $dx = -dt$

Also when $x = a, t = a$; when $x = 2a, t = 0$.

$$\therefore \quad \int_0^{2a} f(x)\,dx = -\int_a^0 f(2a-t)\,dt = \int_0^a f(2a-t)\,dt = \int_0^a f(2a-x)\,dx \qquad \text{[Prop. II]}$$

Subsituting in (1), we get

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx \qquad ...(2)$$

(*i*) *If* $f(2a - x) = f(x)$, then from (2)

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$$

(*ii*) If $f(2a - x) = -f(x)$, then from (2)

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx - \int_0^a f(x)\,dx = 0.$$

Cor. 1. *If n is even,* $\int_0^{\pi} \sin^m x \cos^n x\,dx = 2\int_0^{\pi/2} \sin^m x \cos^n x\,dx$ *and if n is odd,* $\int_0^{\pi} \sin^m x \cos^n x\,dx = 0.$

Cor. 2. *If m is odd,* $\int_0^{2\pi} \sin^m x \cos^n x\,dx = 0$

and *if m is even,* $\int_0^{2\pi} \sin^m x \cos^n x\,dx = 2\int_0^{\pi} \sin^m x \cos^n x\,dx$

$$= 4\int_0^{\pi/2} \sin^m x \cos^n x\,dx, \text{ if } n \text{ is even} = 0, \text{ if } n \text{ is odd.}$$

Example 6.15. *Evaluate* $\int_0^{\pi} \theta \sin^2\theta \cos^4\theta\,d\theta.$ *(V.T.U., 2009 S)*

Solution. Let $I = \int_0^{\pi} \theta \sin^2\theta \cos^4\theta\,d\theta$

Then $I = \int_0^{\pi} (\pi - \theta)\sin^2(\pi-\theta)\cos^4(\pi-\theta)\,d\theta = \pi\int_0^{\pi} \sin^2\theta\cos^4\theta\,d\theta - I$ [Prop. IV]

or $2I = \pi\int_0^{\pi} \sin^2\theta\cos^4\theta\,d\theta = 2\pi\int_0^{\pi/2} \sin^2\theta\cos^4\theta\,d\theta$ [Prop. VI Cor. 2]

$$= 2\pi \cdot \frac{1.3.1}{6.4.2} = \frac{\pi}{2} = \frac{\pi^2}{16}$$

Hence $I = \dfrac{\pi^2}{32}$

Example 6.16. *Evaluate* $\int_0^{\pi/2} \log\sin x\,dx.$ *(Anna, 2005 S)*

Solution. Let $I = \int_0^{\pi/2} \log\sin x\,dx$ ...(*i*)

then $I = \int_0^{\pi/2} \log\sin(\pi/2 - x)\,dx = \int_0^{\pi/2} \log\cos x\,dx$ ...(*ii*)

Adding (*i*) and (*ii*)

$$2I = \int_0^{\pi/2} (\log\sin x + \log\cos x)\,dx$$

$$= \int_0^{\pi/2} \log(\sin x + \cos x)\,dx = \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right)dx$$

$$= \int_0^{\pi/2} \log\sin 2x\,dx - \int_0^{\pi/2} \log 2\,dx = \int_0^{\pi/2} \log\sin 2x\,dx - \log 2\int_0^{\pi/2} dx$$

$$= \int_0^{\pi/2} \log\sin 2x\,dx - \log 2\,|\,x\,|_0^{\pi/2} = I' - \frac{\pi}{2}\log 2 \quad ...(iii)$$

where $I' = \int_0^{\pi/2} \log\sin 2x\,dx$ $\left[\begin{array}{l}\text{Put, } 2x = t, \text{ so that } 2dx = dt \\ \text{When } x = 0, t = 0\text{ ; when } x = \pi/2, t = \pi\end{array}\right.$

$= \dfrac{1}{2}\int_0^{\pi} \log\sin t\,dt = \dfrac{1}{2}\int_0^{\pi} \log\sin x\,dx$ [$\because$ $\log\sin(\pi - x) = \log\sin x$, Prop. IV]

$$= \frac{1}{2} \cdot 2\int_0^{\pi/2} \log\sin x\,dx = I.$$

Thus from (*iii*), $2I = I - (\pi/2)\log 2$, *i.e.*, $I = -(\pi/2)\log 2$.

Obs. The following are its immediate deductions :

$$\int_0^{\pi/2} \log \sin x \, dx = \int_0^{\pi/2} \log \cos x \, dx = -\frac{\pi}{2} \log 2$$

and $$\int_0^{\pi} \log \sin x \, dx = -\pi \log 2.$$

Example 6.17. *Evaluate* $\int_0^1 \frac{\sin^{-1} x}{x} dx$.

Solution. Put $\sin^{-1} x = \theta$ or $x = \sin\theta$ so that $dx = \cos\theta \, d\theta$

Also when $x = 0, \theta = 0$; when $x = 1, \theta = \pi/2$.

$$\therefore \quad \int_0^1 \frac{\sin^{-1} x}{x} dx = \int_0^{\pi/2} \theta \cdot \frac{\cos\theta}{\sin\theta} d\theta \qquad \text{[Integrate by parts]}$$

$$= [\theta \cdot \log \sin\theta]_0^{\pi/2} - \int_0^{\pi/2} 1 \cdot \log \sin\theta \, d\theta$$

$$= -\int_0^{\pi/2} \log \sin\theta \, d\theta = -\left(-\frac{\pi}{2}\log 2\right) = \frac{\pi}{2} \log 2 \qquad \left[\because \operatorname*{Lt}_{x\to 0} (x \log x) = 0\right]$$

PROBLEMS 6.4

Prove that :

1. (*i*) $\int_0^{\pi/2} \log \tan x \, dx = 0$ (*ii*) $\int_0^{\pi/2} \sin 2x \log \tan x \, dx = 0$

2. (*i*) $\int_0^{\infty} \frac{x^7(1-x^{12})}{(1+x)^{28}} dx = 0$ (*ii*) $\int_0^{\pi/4} \log(1+\tan\theta) \, d\theta = \frac{\pi}{8} \log_e 2$ (*Madras, 2000*)

3. (*i*) $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx = \frac{\pi}{4}$ (*ii*) $\int_0^{a} \frac{dx}{x + \sqrt{(a^2 + x^2)}} = \frac{\pi}{4}$

4. (*i*) $\int_0^{\pi/2} \frac{dx}{1 + \sqrt{\cot x}} = \frac{\pi}{4}$. (*ii*) $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} dx = \frac{\pi}{4}$.

5. (*i*) $\int_0^{\pi/2} \frac{x \tan x}{\sec x + \cos x} dx = \frac{\pi^2}{4}$. (*ii*) $\int_0^{\pi} \frac{x}{1+\sin x} dx = \pi$ (*Anna, 2002 S*)

6. (*i*) $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} = \frac{1}{2}\pi(\pi - 2)$. (*ii*) $\int_0^{\pi/2} \frac{x \, dx}{\sin x + \cos x} = \frac{\pi}{2\sqrt{2}} \log(\sqrt{2} + 1)$.

Evaluate :

7. (*i*) $\int_0^{\pi} \sin^4 x \, dx$ (*ii*) $\int_0^{2\pi} \cos^6 x \, dx$

   (*iii*) $\int_0^{\pi} \sin^6 x \cos^4 x \, dx$ (*V.T.U., 2001*) (*iv*) $\int_0^{2\pi} \sin^4 x \cos^6 x \, dx$

8. (*i*) $\int_0^{\pi} x \sin^7 x \, dx$ (*V.T.U., 2009*) (*ii*) $\int_0^{\pi} x \cos^4 x \sin^5 x \, dx$ (*Marathwada, 2008*)

Prove that :

9. (*i*) $\int_0^{\pi} \frac{x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi^2}{2ab}$ (*ii*) $\int_0^{\pi/2} \frac{x \, dx}{2\sin^2 x + \cos^2 x} = \frac{\pi^2}{2\sqrt{2}}$

10. (*i*) $\int_0^{\pi} \frac{x \, dx}{a^2 - \cos^2 x} = \frac{\pi^2}{2a\sqrt{(a^2-1)}}$ $(a > 1)$ (*ii*) $\int_0^{\pi} \frac{x \, dx}{1 + \sin^2 x} = \frac{\pi^2}{2\sqrt{2}}$.

11. $\int_0^{\pi} \log(1+\cos\theta)\, d\theta = -\pi \log_e 2$ (*Madras, 2003*)

12. (*i*) $\int_0^{\infty} \frac{\log(1+x^2)}{1+x^2}\, dx = \pi \log_e 2$ (*ii*) $\int_0^{\infty} \frac{\log(x+1/x)}{1+x^2}\, dx = \pi \log_e 2.$

6.9 (1) INTEGRAL AS THE LIMIT OF A SUM

We have so far considered integration as inverse of differentiation. We shall now define the definite integral as the limit of a sum :

Def. *If $f(x)$ is continuous and single valued in the interval $[a, b]$, then the definite integral of $f(x)$ between the limits a and b is defined by the equation*

$$\int_a^b \mathbf{f(x)\,dx} = \underset{\mathbf{h \to 0}}{\mathbf{Lt}}\ \mathbf{h\,[f(a) + f(a+h) + f(a+2h) + \ldots + f(a+\overline{n-1}\,h)\,]},$$

where $nh = b - a$. ...(1)

(2) Evaluation of limits of series

The summation definition of a definite integral enables us to express the limits of sums of certain types of series as definite integrals which can be easily evaluated. We rewrite (1) as follows :

$$\int_a^b f(x)\,dx = \underset{n \to \infty}{\text{Lt}}\ h \sum_{r=0}^{n-1} f(a+rh), \text{ where } nh = b - a.$$

Putting $a = 0$ and $b = 1$, so that $h = 1/n$, we get

$$\underset{n \to \infty}{\mathbf{Lt}}\ \frac{\mathbf{1}}{\mathbf{n}} \sum_{r=0}^{n-1} \mathbf{f}\left(\frac{\mathbf{r}}{\mathbf{n}}\right) = \int_0^1 \mathbf{f(x)\ dx}$$

Thus to express a given series as definite integral:

(*i*) Write the general term (T_r or T_{r+1} whichever involves r)

i.e., $f(r/n) \cdot 1/n$

(*ii*) Replace r/n by x and $1/n$ *by* dx,

(*iii*) Integrate the resulting expression, taking

the lower limit = $\underset{n \to \infty}{\text{Lt}}\ (r/n)$ where r is as in the first term,

and the upper limit = $\underset{n \to \infty}{\text{Lt}}\ (r/n)$ where r is as in the last term.

Example 6.18. *Find the limit, when $n \to \infty$, of the series*

$$\frac{n}{n^2} + \frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \ldots + \frac{n}{n^2+(n-1)^2}$$

Solution. Here the general term $(= T_{r+1}) = \dfrac{n}{n^2+r^2} = \dfrac{n}{1+(r/n)^2} \cdot \dfrac{1}{n}$

$$= \frac{1}{1+x^2}\, dx \qquad \text{[Putting } r/n = x \text{ and } 1/n = dx]$$

Now for the first term $r = 0$ and for the last term $r = n - 1$

$\therefore$ the lower limit of integration $= \underset{n \to \infty}{\text{Lt}} \left(\frac{0}{n}\right) = 0$

and the upper limit of integration $= \underset{n \to \infty}{\text{Lt}} \left(\frac{n-1}{n}\right) = \underset{n \to \infty}{\text{Lt}} \left(1 - \frac{1}{n}\right) = 1.$

Hence, the required limit $= \int_0^1 \frac{dx}{1+x^2} = \left| \tan^{-1} x \right|_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \pi/4.$

[**To find limit of a product by integration :**

Let $P = \underset{n \to \infty}{\text{Lt}}$ (given product)

Take logs of both sides, so that

$$\log P = \underset{n \to \infty}{\text{Lt}} \text{ (a series) } = k \text{ (say). Then } P = e^k.]$$

Example 6.19. *Evaluate* $\underset{n \to \infty}{Lt} \left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)...\left(1+\frac{n}{n}\right)\right\}^{1/n}$. *(Bhopal, 2008)*

Solution. Let $$P = \underset{n \to \infty}{\text{Lt}} \left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)...\left(1+\frac{n}{n}\right)\right\}^{1/n}.$$

Taking logs of both sides,

$$\log P = \underset{n \to \infty}{\text{Lt}} \frac{1}{n}\left\{\log\left(1+\frac{1}{n}\right) + \log\left(1+\frac{2}{n}\right) + ... + \log\left(1+\frac{n}{n}\right)\right\}$$

Its general term $= \log\left(1+\frac{r}{n}\right).\frac{1}{n} = \log(1+x) . dx$ [Putting $r/n = x$ and $1/n = dx$]

Also for first term $r = 1$ and for the last term $r = n$.

$\therefore$ The lower limit of integration $= \underset{n \to \infty}{\text{Lt}} (1/n) = 0$ and the upper limit $= \underset{n \to \infty}{\text{Lt}} (n/n) = 1$

Hence $$\log P = \int_0^1 \log(1+x)\,dx = \int_0^1 \log(1+x) \,.\, 1\,dx$$ [Integrate by parts]

$$= \left| \log(1+x) . x \right|_0^1 - \int_0^1 \frac{1}{1+x} . x\,dx$$

$$= \log 2 - \int_0^1 \frac{1+x-1}{1+x}dx = \log 2 - \int_0^1 dx + \int_0^1 \frac{dx}{1+x}$$

$$= \log 2 - \left| x \right|_0^1 + \left| \log(1+x) \right|_0^1 = \log 2 - 1 + \log 2$$

$$= \log 2^2 - \log_e e = \log(4/e). \text{ Hence, } P = 4/e.$$

PROBLEMS 6.5

Find the limit, as $n \to \infty$, of the series :

1. $\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + ... + \frac{1}{2n}$. *(Bhopal, 2009)*
2. $\frac{1}{n^3+1} + \frac{4}{n^3+8} + \frac{9}{n^3+27} + ... + \frac{r^2}{n^3+r^3} + ... + \frac{1}{2n}$.
3. $\underset{n \to \infty}{\text{Lt}} \frac{\sqrt{n}}{\sqrt{n^3}} + \frac{\sqrt{n}}{\sqrt{(n+3)^3}} + \frac{\sqrt{n}}{\sqrt{(n+6)^3}} + ... + \frac{\sqrt{n}}{\sqrt{[n+3(n-1)]^3}}$.

Evaluate :

4. $\underset{n \to \infty}{\text{Lt}} \sum_{r=1}^{n-1} \frac{1}{\sqrt{(n^2-r^2)}}$ *(Bhopal, 2008)*
5. $\underset{n \to \infty}{\text{Lt}} \frac{[(n+1)(n+2)...(n+n)]^{1/n}}{n}$.
6. $\underset{n \to \infty}{\text{Lt}} \left\{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)...\left(1+\frac{n^2}{n^2}\right)\right\}^{1/n}$ *(Bhopal, 2008)*

6.10 AREAS OF CARTESIAN CURVES

(1) *Area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$ is* $\int_a^b$ **y dx.**

Let AB be the curve $y = f(x)$ between the ordinates LA $(x = a)$ and MB $(x = b)$. (Fig. 6.1)

Let $P(x, y)$, $P'(x + \delta x, y + \delta y)$ be two neighbouring points on the curve and NP, $N'P'$ be their respective ordinates.

Let the area $ALNP$ be A, which depends on the position of P whose abscissa is x. Then the area $PNN'P' = \delta A$.

Complete the rectangles PN' and $P'N'$.

Then the area $PNN'P'$ lies between the areas of the rectangles PN' and $P'N$.

i.e., δA lies between $y\delta x$ and $(y + \delta y)\,\delta x$

$\therefore$ $\dfrac{\delta A}{\delta x}$ lies between y and $y + \delta y$.

Now taking limits as $P' \to P$ *i.e.*, $\delta x \to 0$ (and $\therefore$ $\delta y \to 0$),

$dA/dx = y$

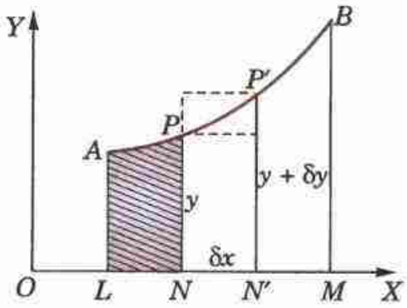


Fig. 6.1

Integrating both sides between the limits $x = a$ and $x = b$, we have

$$|\,A\,|_a^b = \int_a^b y\,dx$$

or (value of A for $x = b$) – (value of A for $x = a$) = $\int_a^b y\,dx$

Thus area $ALMB = \int_a^b y\,dx$.

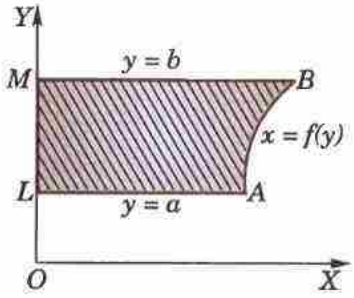


Fig. 6.2

(2) Interchanging x and y in the above formula, we see that *the area bounded by the curve $x = f(y)$, the y-axis and the abscissae $y = a$, $y = b$ is* $\int_a^b$ **x dy.** (Fig. 6.2)

Obs. 1. The area bounded by a curve, the x-axis and two ordinates is called the **area under the curve.** The process of finding the area of plane curves is often called **quadrature.**

Obs. 2. Sign of an area. *An area whose boundary is described in the anti-clockwise direction is considered positive and an area whose boundary is described in the clockwise direction is taken as negative.*

In Fig. 6.3, the area $ALMB \left(= \int_a^b y\,dx\right)$ which is described in the anti-clockwise direction and lies above the x-axis, will give a positive result.

In Fig. 6.4, the area $ALMB \left(= \int_a^b y\,dx\right)$ which is described in the clockwise direction and lies below the x-axis, will give a negative result.

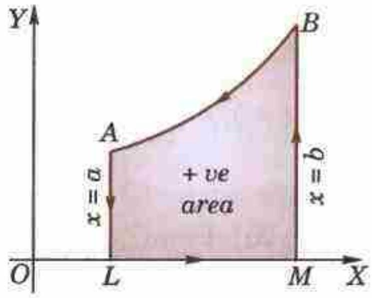


Fig. 6.3

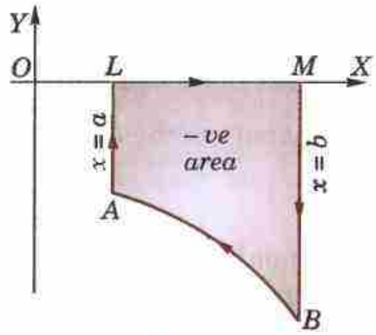


Fig. 6.4

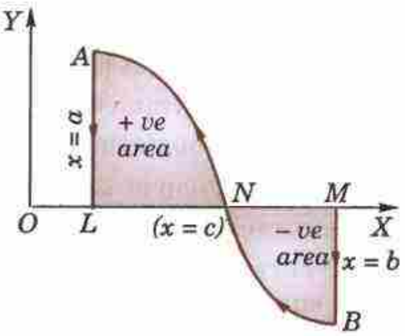


Fig. 6.5

In Fig. 6.5, the area $ALMB \left(= \int_a^b y\,dx\right)$ will not consist of the sum of the area $ALN \left(= \int_a^c y\,dx\right)$ and the area $NMB \left(= \int_c^b y\,dx\right)$, but their difference.

Thus to find the total area in such cases the numerical value of the area of each portion must be evaluated separately and their results added afterwards.

Example 6.20. *Find the area of the loop of the curve $ay^2 = x^2(a - x)$.* (*S.V.T.U., 2009 ; Osmania, 2000*)

Solution. Let us trace the curve roughly to get the limits of integration.

(i) The curve is symmetrical about x-axis.

(*ii*) It passes through the origin. The tangents at the origin are $ay^2 = ax^2$ or $y = \pm x$. $\therefore$ Origin is a node.

(*iii*) The curve has no asymptotes.

(*iv*) The curve meets the x-axis at (0, 0) and (a, 0). It meets the y-axis at (0, 0) only.

From the equation of the curve, we have $y = \dfrac{x}{\sqrt{a}} \sqrt{(a-x)}$

For $x > a$, y is imaginary. Thus no portion of the curve lies to the right of the line $x = a$. Also $x \to -\infty, y \to \infty$.

Thus the curve is as shown in Fig. 6.6.

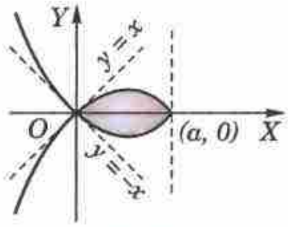


Fig. 6.6

$\therefore$ Area of the loop = 2 (area of upper half of the loop)

$$= 2\int_0^a y\,dx = 2\int_0^a x\sqrt{\left(\frac{a-x}{a}\right)}\,dx = \frac{2}{\sqrt{a}}\int_0^a [a-(a-x)]\sqrt{(a-x)}\,dx$$

$$= \frac{2}{\sqrt{a}}\int_0^a [a(a-x)^{1/2} - (a-x)^{3/2}]\,dx = 2\sqrt{a}\left|\frac{(a-x)^{3/2}}{-3/2}\right|_0^a - \frac{2}{\sqrt{a}}\left|\frac{(a-x)^{5/2}}{-5/2}\right|_0^a$$

$$= -\frac{4}{3}\sqrt{a}\,(0 - a^{3/2}) + \frac{4}{5\sqrt{a}}(0 - a^{5/2}) = \frac{4}{3}a^2 - \frac{4}{5}a^2 = \frac{8}{15}a^2.$$

Example 6.21. *Find the area included between the curve $y^2(2a - x) = x^3$ and its asymptote.* (V.T.U., 2003)

Solution. The curve is as shown in Fig. 4.23.

Area between the curve and the asymptote

$$= 2\int_0^{2a} y\,dx = 2\int_0^{2a}\sqrt{\left(\frac{x^3}{2a-x}\right)}\,dx$$

| Put $x = 2a\sin^2\theta$
| so that $dx = 4a\sin\theta\cos\theta\,d\theta$

$$= 2\int_0^{\pi/2}\sqrt{\left\{\frac{(2a\sin^2\theta)^3}{2a\cos^2\theta}\right\}}\cdot 4a\sin\theta\cos\theta\,d\theta$$

$$= 16a^2\int_0^{\pi/2}\sin^4\theta\,d\theta = 16a^2\cdot\frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2} = 3\pi a^2.$$

Example 6.22. *Find the area enclosed by the curve $a^2x^2 = y^3(2a - y)$.*

Solution. Let us first find the limits of integration.

(*i*) The curve is symmetrical about y-axis.

(*ii*) It passes through the origin and the tangents at the origin are $x^2 = 0$ or $x = 0, x = 0$.

$\therefore$ There is a cusp at the origin.

(*iii*) The curve has no asymptote.

(*iv*) The curve meets the x-axis at the origin only and meets the y-axis at (0, 2a). From the equation of the curve, we have

$$x = \frac{y}{a}\sqrt{\{y(2a-y)\}}$$

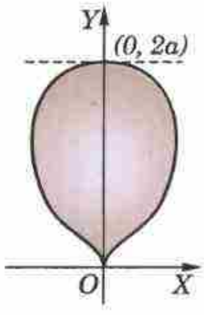


Fig. 6.7

For $y < 0$ or $y > 2a$, x is imaginary. Thus the curve entirely lies between $y = 0$ (x-axis) and $y = 2a$, which is shown in Fig. 6.7.

$$\therefore \text{ Area of the curve } = 2\int_0^{2a} x\,dy = \frac{2}{a}\int_0^{2a} y\sqrt{[y(2a-y)]}\,dy$$

| Put $y = 2a\sin^2\theta$
| $\therefore$ $dy = 4a\sin\theta\cos\theta\,d\theta$

$$= \frac{2}{a}\int_0^{\pi/2} 2a\sin^2\theta\sqrt{[2a\sin^2\theta\,(2a - 2a\sin^2\theta)]}\times 4a\sin\theta\cos\theta\,d\theta$$

$$= 32a^2\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta = 32a^2\,\frac{3\cdot 1\times 1}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2} = \pi a^2.$$

Example 6.23. *Find the area enclosed between one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$; and its base.* (V.T.U., 2000)

Solution. To describe its first arch, θ varies from 0 to 2π *i.e.*, x varies from 0 to $2a\pi$ (Fig. 6.8).

$\therefore$ Required area $= \int_{x=0}^{2\pi a} y\,dx$

where $y = a(1 - \cos\theta)$, $dx = a(1 - \cos\theta)\,d\theta$.

$$= \int_{\theta=0}^{\pi/2} a(1 - \cos\theta) \cdot a(1 - \cos\theta)\,d\theta$$

$$= 2a^2 \int_0^{\pi} (1 - \cos\theta)^2\,d\theta = 8a^2 \int_0^{\pi} \sin^4 \frac{\theta}{2}\,d\theta$$

$$= 16a^2 \int_0^{\pi/2} \sin^4\phi\,d\phi, \text{ putting } \theta/2 = \phi \text{ so that } d\theta = 2d\phi.$$

$$= 16a^2 \cdot \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} = 3\pi a^2.$$

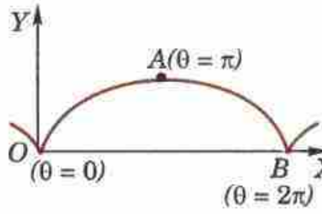


Fig. 6.8

Example 6.24. *Find the area of the tangent cut off from the parabola $x^2 = 8y$ by the line $x - 2y + 8 = 0$.*

Solution. Given parabola is $x^2 = 8y$ ...(*i*)

and the straight line is $x - 2y + 8 = 0$ ...(*ii*)

Substituting the value of y from (*ii*) in (*i*), we get

$$x^2 = 4(x + 8) \text{ or } x^2 - 4x - 32 = 0$$

or $(x - 8)(x + 4) = 0 \quad \therefore \quad x = 8, -4.$

Thus (*i*) and (*ii*) intersect at P and Q where $x = 8$ and $x = -4$. (Fig. 6.9)

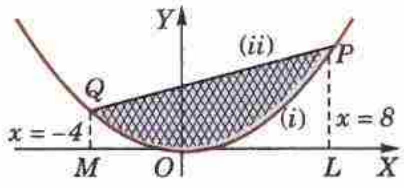


Fig. 6.9

$\therefore$ Required area POQ (*i.e.*, dotted area) = area bounded by straight line (*ii*) and x-axis from $x = -4$ to $x = 8$ – area bounded by parabola (*i*) and x-axis from $x = -4$ to $x = 8$.

$$= \int_{-4}^{8} y\,dx, \text{ from } (ii) - \int_{-4}^{8} y\,dx, \text{ from } (i)$$

$$= \int_{-4}^{8} \frac{x+8}{2}\,dx - \int_{-4}^{8} \frac{x^2}{8}\,dx = \frac{1}{2}\left|\frac{x^2}{2} + 8x\right|_{-4}^{8} - \frac{1}{8}\left|\frac{x^3}{3}\right|_{-4}^{8}$$

$$= \frac{1}{2}\{(32 + 64) - (-24)\} - \frac{1}{24}(512 + 64) = 36.$$

Example 6.25. *Find the area common to the parabola $y^2 = ax$ and the circle $x^2 + y^2 = 4ax$.*

Solution. Given parabola is $y^2 = ax$ ...(*i*)

and the circle is $x^2 + y^2 = 4ax$. ...(*ii*)

Both these curves are symmetrical about x-axis. Solving (*i*) and (*ii*) for x, we have

$$x^2 + ax = 4ax \text{ or } x(x - 3a) = 0$$

or $x = 0, 3a$.

Thus the two curves intersect at the points where $x = 0$ and $x = 3a$. (Fig. 6.10).

Also (*ii*) meets the x-axis at $A(4a, 0)$.

Area common to (*i*) and (*ii*) *i.e.*, the shaded area

$$= 2[\text{Area } ORP + \text{Area } PRA] \qquad \text{(By symmetry)}$$

$$= 2\left[\int_0^{3a} y\,dx, \text{ from } (i) + \int_{3a}^{4a} y\,dx, \text{ from } (ii)\right]$$

$$= 2\left[\int_0^{3a} \sqrt{(ax)}\,dx + \int_{3a}^{4a} \sqrt{(4ax - x^2)}\,dx\right]$$

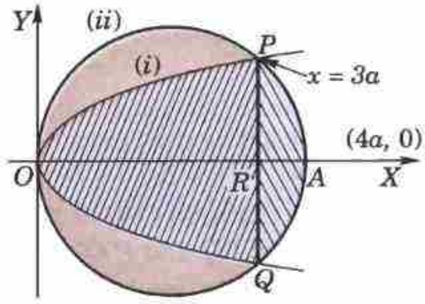


Fig. 6.10

$$= 2\sqrt{a}\left|\frac{x^{3/2}}{3/2}\right|_0^{3a} + 2\int_{3a}^{4a}\sqrt{[4a^2-(x-2a)^2]}\,dx$$

$$= \frac{4\sqrt{a}}{3}(3a)^{3/2} + 2\left[\frac{1}{2}(x-2a)\sqrt{\{4a^2-(x-2a)^2\}} + \frac{4a^2}{2}\sin^{-1}\frac{x-2a}{2a}\right]_{3a}^{4a}$$

$$= 4\sqrt{3}\,a^2 + 2[\{0-\frac{1}{2}a\sqrt{3}a\} + 2a^2\{\pi/2-\pi/6\}]$$

$$= 4\sqrt{3}\,a^2 - \sqrt{3}a^2 + \frac{4}{3}\pi a^2 = \left(3\sqrt{3}+\frac{4}{3}\pi\right)a^2.$$

PROBLEMS 6.6

1. (*i*) Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. (*Kerala, 2005 ; V.T.U., 2003 S*)

   (*ii*) Find the area bounded by the parabola $y^2 = 4ax$ and its latus-rectum.
2. Find the area bounded by the curve $y = x(x-3)(x-5)$ and the x-axis.
3. Find the area included between the curve $ay^2 = x^3$, the x-axis and the ordinates $x = a$.
4. Find the area of the loop of the curve :

   (*i*) $3ay^2 = x(x-a)^2$ (*Rajasthan, 2005*) (*ii*) $x(x^2+y^2) = a(x^2-y^2)$ (*P.T.U., 2010*)
5. Find the whole area of the curve :

   (*i*) $a^2x^2 = y^3(2a-y)$ (*Nagpur, 2009*) (*ii*) $8a^2y^2 = x^2(a^2-x^2)$ (*V.T.U., 2006*)
6. Find the area included between the curve and its asymptotes in each case :

   (*i*) $xy^2 = a^2(a-x)$. (*V.T.U., 2003*) (*ii*) $x^2y^2 = a^2(y^2-x^2)$. (*V.T.U., 2007*)
7. Show that the area of the loop of the curve $y^2(a+x) = x^2(3a-x)$ is equal to the area between the curve and its asymptote.
8. Find the whole area of the *astroid* $x^{2/3}+y^{2/3}=a^{2/3}$ or $x = a\cos^3\theta$, $y = a\sin^3\theta$. (*V.T.U., 2005*)
9. Find the area bounded by the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ and the coordinate axes.
10. Find the area included between the *cycloid* $x = a(\theta+\sin\theta)$, $y = a(1-\cos\theta)$ *and its base.* Also find the area between the curve and the x-axis. (*Gorakhpur, 1999*)
11. Find the area common to the circles $x^2+y^2=4$ and $x^2+y^2=4x$.
12. Prove that the area common to the parabolas $x^2 = 4ay$ and $y^2 = 4ax$ is $16a^2/3$. (*S.V.T.U., 2008 ; Kurukshetra, 2005*)
13. Find the area included between the circle $x^2+y^2=2ax$ and the parabola $y^2 = ax$.
14. Find the area bounded by the parabola $y^2 = 4ax$ and the line $x + y = 3a$.
15. Find the area of the segment cut off from the parabola $y = 4x - x^2$ by the straight line $y = x$. (*V.T.U., 2010 ; S.V.T.U., 2008*)

(2) Areas of polar curves. *Area bounded by the curve $r = f(\theta)$ and the radii vectors*

$$\theta=\alpha,\ \theta=\beta \text{ is } \frac{1}{2}\int_\alpha^\beta \mathbf{r^2\,d\theta}.$$

Let AB be the curve $r = f(\theta)$ between the radii vectors $OA\,(\theta=\alpha)$ and $OB\,(\theta=\beta)$. Let $P(r,\theta)$, $P'(r+\delta r, \theta+\delta\theta)$ be any two neighbouring points on the curve. (Fig. 6.11)

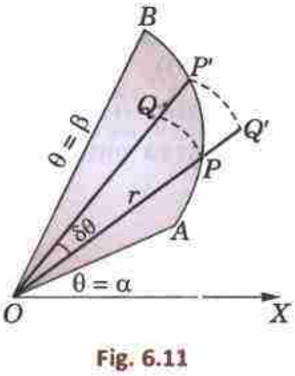


Fig. 6.11

Let the area $OAP = A$ which is a function of θ. Then the area $OPP' = \delta A$. Mark circular arcs PQ and $P'Q'$ with centre O and radii OP and OP'.

Evidently area OPP' lies between the sectors OPQ and $OP'Q'$ *i.e.*, δA lies between $\frac{1}{2}r^2\,\delta\theta$ and $\frac{1}{2}(r+\delta r)^2\,\delta\theta$.

$\therefore$ $\frac{\delta A}{\delta\theta}$ lies between $\frac{1}{2}r^2$ and $\frac{1}{2}(r+\delta r)^2$.

Now taking limits as $\delta\theta \to 0$ ($\because$ $\delta r \to 0$), $\dfrac{dA}{d\theta} = \dfrac{1}{2} r^2$

Integrating both sides from $\theta = \alpha$ to $\theta = \beta$, we get $\left| A \right|_{\alpha}^{\beta} = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$

or (value of A for $\theta = \beta$) – (value of A for $\theta = \alpha$) $= \dfrac{1}{2} \displaystyle\int_{\alpha}^{\beta} r^2 \, d\theta$

Hence the required area $OAB = \dfrac{1}{2} \displaystyle\int_{\alpha}^{\beta} r^2 \, d\theta.$

Example 6.26. *Find the area of the cardioid $r = a(1 - \cos\theta)$.* *(V.T.U., 2004)*

Solution. The curve is as shown in Fig. 6.12. Its upper half is traced from $\theta = 0$ to $\theta = \pi$.

$\therefore$ Area of the curve
$$= 2 \cdot \frac{1}{2} \int_0^{\pi} r^2 \, d\theta = a^2 \int_0^{\pi} (1 - \cos\theta)^2 \, d\theta$$
$$= a^2 \int_0^{\pi} (2\sin^2 \theta/2)^2 \, d\theta = 4a^2 \int_0^{\pi} \sin^4 \theta/2 \cdot d\theta$$
$$= 8a^2 \int_0^{\pi/2} \sin^4 \phi \, d\phi, \text{ putting } \theta/2 = \phi \text{ and } d\theta = 2d\phi.$$
$$= 8a^2 \cdot \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} = \frac{3\pi a^2}{2}.$$

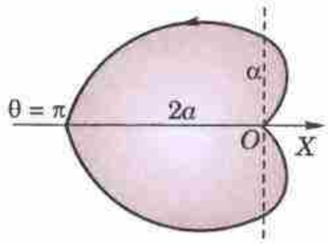


Fig. 6.12

Example 6.27. *Find the area of a loop of the curve $r = a \sin 3\theta$.*

Solution. The curve is as shown in Fig. 4.35. It consists of three loops.
Putting $r = 0$, $\sin 3\theta = 0$ $\therefore$ $3\theta = 0$ or π *i.e.*, $\theta = 0$ or $\pi/3$ which are the limits for the first loop.

$\therefore$ Area of a loop
$$= \frac{1}{2} \int_0^{\pi/3} r^2 \, d\theta = \frac{1}{2} a^2 \int_0^{\pi/3} \sin^2 3\theta \, d\theta = \frac{a^2}{4} \int_0^{\pi/3} (1 - \cos 6\theta) \, d\theta$$
$$= \frac{a^2}{4} \left| \theta - \frac{\sin 6\theta}{6} \right|_0^{\pi/3} = \frac{a^2}{4} \left(\frac{\pi}{3} - 0 \right) = \frac{\pi a^2}{12}.$$

Obs. The limits of integration for a loop of $r = a \sin n\theta$ or $r = a \cos n\theta$ are the two consecutive values of θ when $r = 0$.

Example 6.28. *Prove that the area of a loop of the curve $x^3 + y^3 = 3axy$ is $3a^2/2$.*

Solution. Changing to polar form (by putting $x = r\cos\theta$, $y = r\sin\theta$), $r = \dfrac{3a \sin\theta \cos\theta}{\cos^3\theta + \sin^3\theta}$

Putting $r = 0$, $\sin\theta\cos\theta = 0$.

$\therefore$ $\theta = 0, \pi/2$, which are the limits of integration for its loop.

$\therefore$ Area of the loop
$$= \frac{1}{2} \int_0^{\pi/2} r^2 \, d\theta = \frac{1}{2} \int_0^{\pi/2} \frac{9a^2 \sin^2\theta \cos^2\theta}{(\cos^3\theta + \sin^3\theta)^2} \, d\theta$$
$$= \frac{9a^2}{2} \int_0^{\pi/2} \frac{\tan^2\theta \sec^2\theta}{(1 + \tan^3\theta)^2} \, d\theta \qquad \text{[Dividing num. and denom. by } \cos^6\theta]$$
$$= \frac{3a^2}{2} \int_1^{\infty} \frac{dt}{t^2}, \qquad \text{putting } 1 + \tan^3\theta = t \text{ and } 3\tan^2\theta \sec^2\theta \, d\theta = dt.$$
$$= \frac{3a^2}{2} \left| \frac{t^{-1}}{-1} \right|_1^{\infty} = \frac{3a^2}{2}(-0 + 1) = \frac{3a^2}{2}.$$

Example 6.29. *Find the area common to the circles*
$$r = a\sqrt{2} \text{ and } r = 2a\cos\theta$$

Solution. The equations of the circles are $r = a\sqrt{2}$ ...(*i*) and $r = 2a \cos\theta$ ...(*ii*)

(*i*) represents a circle with centre at (0, 0) and radius $a\sqrt{2}$. (*ii*) represents a circle symmetrical about OX, with centre at $(a, 0)$ and radius a.

The circles are shown in Fig. 6.13. At their point of intersection P, eliminating r from (*i*) and (*ii*),

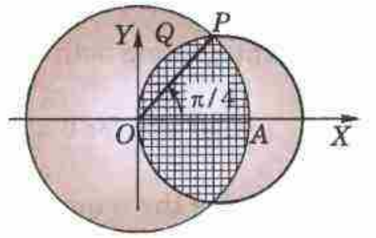


Fig. 6.13

$$a\sqrt{2} = 2a\cos\theta \text{ i.e., } \cos\theta = 1\sqrt{2}$$

or $$\theta = \pi/4$$

$\therefore$ Required area = 2 × area $OAPQ$ (By symmetry)

$$= 2(\text{area } OAP + \text{area } OPQ)$$

$$= 2\left[\frac{1}{2}\int_0^{\pi/4} r^2\, d\theta, \text{ for } (i) + \frac{1}{2}\int_{\pi/4}^{\pi/2} r^2\, d\theta, \text{ for } (ii)\right]$$

$$= \int_0^{\pi/4} (a\sqrt{2})^2\, d\theta + \int_{\pi/4}^{\pi/2} (2a\cos\theta)^2\, d\theta = 2a^2 \left|\, \theta \right|_0^{\pi/4} + 4a^2 \int_{\pi/4}^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= 2a^2(\pi/4 - 0) + 2a^2 \left|\, \theta + \frac{\sin 2\theta}{2} \right|_{\pi/4}^{\pi/2} = \frac{\pi a^2}{2} + 2a^2\left(\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2}\right) = a^2(\pi - 1).$$

Example 6.30. *Find the area common to the cardioids $r = a(1 + \cos\theta)$ and $r = a(1 - \cos\theta)$.*
(Kurukshetra, 2006 ; V.T.U., 2006)

Solution. The cardioid $r = a(1 + \cos\theta)$ is $ABCOB'A$ and the cardioid $r = a(1 - \cos\theta)$ is $OC'BA'B'O$. Both the cardioids are symmetrical about the initial line OX and intersect at B and B' (Fig. 6.14)

$\therefore$ Required area (shaded) = 2 area $OC'BCO$

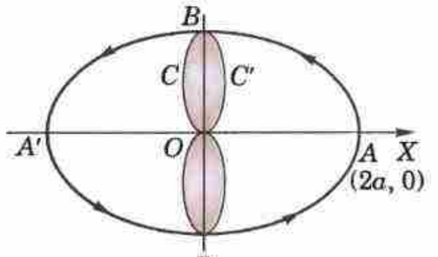


Fig. 6.14

$$= 2\,[\text{area } OC'BO + \text{area } OBCO]$$

$$= 2\left[\left\{\int_0^{\pi/2} \frac{1}{2} r^2\, d\theta\right\}_{r = a(1-\cos\theta)} + \left\{\int_{\pi/2}^{\pi} \frac{1}{2} r^2\, d\theta\right\}_{r = a(1+\cos\theta)}\right]$$

$$= a^2 \int_0^{\pi/2} (1 - \cos\theta)^2\, d\theta + a^2 \int_{\pi/2}^{\pi} (1 + \cos\theta)^2\, d\theta$$

$$= a^2 \left\{\int_0^{\pi/2} (1 - 2\cos\theta + \cos^2\theta)\, d\theta + \int_{\pi/2}^{\pi} [1 + 2\cos\theta + \cos^2\theta]\, d\theta\right\}$$

$$= a^2 \left\{\int_0^{\pi} (1 + \cos^2\theta)\, d\theta - 2\int_0^{\pi/2} \cos\theta\, d\theta + 2\int_{\pi/2}^{\pi} \cos\theta\, d\theta\right\}$$

$$= a^2 \left\{\int_0^{\pi} \left(1 + \frac{1 + \cos 2\theta}{2}\right) d\theta - 2\left|\, \sin\theta \right|_0^{\pi/2} + 2\left|\, \sin\theta \right|_{\pi/2}^{\pi}\right\}$$

$$= a^2 \left\{\left|\, \frac{3}{2}\theta + \frac{\sin 2\theta}{4} \right|_0^{\pi} - 2(1 - 0) + 2(0 - 1)\right\} = \left(\frac{3\pi}{2} - 4\right) a^2.$$

PROBLEMS 6.7

1. Find the whole area of
   (*i*) the *cardioid* $r = a(1 + \cos\theta)$ (*V.T.U., 2008*) (*ii*) the *lemniscate* $r^2 = a^2 \cos 2\theta$; (*V.T.U., 2006*)
2. Find the area of one loop of the curve
   (*i*) $r = a \sin 2\theta$. (*ii*) $r = a \cos 3\theta$.
3. Show that the area included between the folium $x^3 + y^3 = 3axy$ and its asymptote is equal to the area of loop.
4. Prove that the area of the loop of the curve $x^3 + y^3 = 3axy$ is three times the area of the loop of the curve $r^2 = a^2 \cos 2\theta$.
5. Find the area inside the circle $r = a \sin\theta$ and lying outside the *cardioid* $r = a(1 - \cos\theta)$. (*Anna, 2009*)
6. Find the area outside the *circle* $r = 2a \cos\theta$ and inside the *cardioid* $r = a(1 + \cos\theta)$. (*Kurukshetra, 2006*)

6.11 LENGTHS OF CURVES

(1) *The length of the arc of the curve $y = f(x)$ between the points where $x = a$ and $x = b$ is*

$$\int_a^b \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\, dx$$

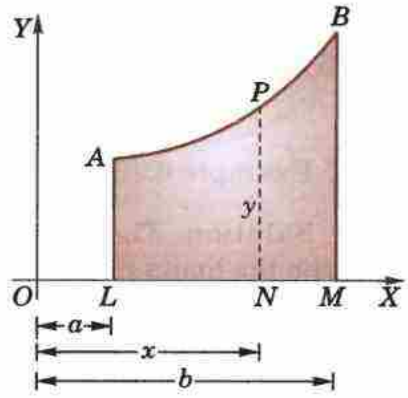


Fig. 6.15

Let AB be the curve $y = f(x)$ between the points A and B where $x = a$ and $x = b$ (Fig. 6.15)

Let $P(x, y)$ be any point on the curve and arc $AP = x$ so that it is a function of x.

Then $$\frac{ds}{dx} = \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}$$ [(1) of p. 164]

$\therefore$ $$\int_a^b \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\, dx = \int_a^b \frac{ds}{dx}\cdot dx = \Big| \, s \, \Big|_{x=a}^{x=b}$$

$$= (\text{value of } s \text{ for } x = b) - (\text{value of } s \text{ for } x = a) = \text{arc } AB - 0$$

Hence, the arc $AB = \displaystyle\int_a^b \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\, dx.$

(2) *The length of the arc of the curve $x = f(y)$ between the points where $y = a$ and $y = b$, is*

$$\int_a^b \sqrt{\left[1+\left(\frac{dx}{dy}\right)^2\right]}\, dy$$ [Use (2) of p. 165]

(3) *The length of the arc of the curve $x = f(t)$, $y = \phi(t)$ between the points where $t = a$ and $t = b$, is*

$$\int_a^b \sqrt{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]}\, dt.$$ [Use (3) p. 165]

(4) *The length of the arc of the curve $r = f(\theta)$ between the points where $\theta = \alpha$ and $\theta = \beta$, is*

$$\int_\alpha^\beta \sqrt{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]}\, d\theta.$$ [Use (1) of p. 165]

(5) *The length of the arc of the curve $\theta = f(r)$ between the points where $r = a$ and $r = b$, is*

$$\int_a^b \sqrt{\left[1+\left(r\frac{d\theta}{dr}\right)^2\right]}\, dr$$ [Use (2) of p. 166]

Example 6.31. *Find the length of the arc of the parabola $x^2 = 4ay$ measured from the vertex to one extremity of the latus-rectum.* *(Delhi, 2002)*

Solution. Let A be the vertex and L an extremity of the latus-rectum so that at A, $x = 0$ and at L, $x = 2a$. (Fig. 6.16).

Now $$y = x^2/4a \text{ so that } \frac{dy}{dx} = \frac{1}{4a}\cdot 2x = \frac{x}{2a}$$

$\therefore$ $$\text{arc } AL = \int_0^{2a} \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\, dx$$

$$= \int_0^{2a} \sqrt{\left[1+\left(\frac{x}{2a}\right)^2\right]}\, dx = \frac{1}{2a}\int_0^{2a} \sqrt{[(2a)^2 + x^2]}\, dx$$

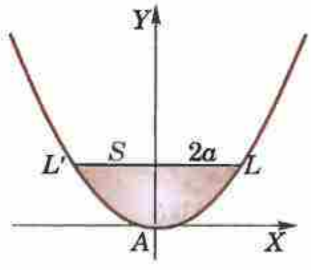


Fig. 6.16

$$= \frac{1}{2a}\left[\frac{x\sqrt{[(2a)^2 + x^2]}}{2} + \frac{(2a)^2}{2}\sinh^{-1}\frac{x}{2a}\right]_0^{2a} = \frac{1}{2a}\left[\frac{2a\sqrt{(8a)^2}}{2} + 2a^2\sinh^{-1}1\right]$$

$$= a[\sqrt{2} + \sinh^{-1}1] = a[\sqrt{2} + \log(1+\sqrt{2})] \qquad [\because \sinh^{-1}x = \log[x + \sqrt{(1+x^2)}]]$$

Example 6.32. *Find the perimeter of the loop of the curve $3ay^2 = x(x-a)^2$.*

Solution. The curve is symmetrical about the x-axis and the loop lies between the limits $x = 0$ and $x = a$. (Fig. 6.17).

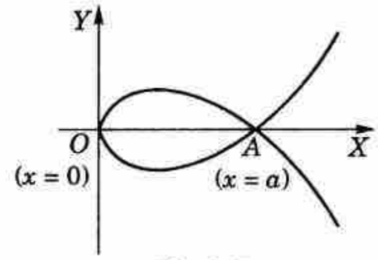


Fig. 6.17

We have $\quad y = \frac{\sqrt{x}(x-a)}{\sqrt{(3a)}}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{(3a)}}\left[\frac{3}{2}x^{1/2} - \frac{a}{2}\cdot x^{-1/2}\right] = \frac{1}{2\sqrt{(3a)}}\frac{3x-a}{\sqrt{x}}$$

$$\therefore \quad \text{Perimeter of the loop} = 2\int_0^a \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}dx \qquad \text{(By symmetry)}$$

$$= 2\int_0^a \sqrt{\left[1 + \frac{(3x-a)^2}{12ax}\right]}dx = 2\int_0^a \frac{\sqrt{(9x^2 + 6ax + a^2)}}{\sqrt{(12ax)}}dx$$

$$= \frac{1}{\sqrt{(3a)}}\int_0^a \frac{3x+a}{\sqrt{x}}dx = \frac{1}{\sqrt{(3a)}}\int_0^a (3x^{1/2} + ax^{-1/2})dx$$

$$= \frac{1}{\sqrt{(3a)}}\left|\frac{3x^{3/2}}{3/2} + a\frac{x^{1/2}}{1/2}\right|_0^a = \frac{1}{\sqrt{(3a)}}(4a^{3/2}) = \frac{4a}{\sqrt{3}}.$$

Example 6.33. *Find the length of one arch of the cycloid*
$x = a(t - \sin t)$, $y = a(1 - \cos t)$. *(P.T.U., 2009 ; V.T.U., 2004)*

Solution. As a point moves from one end O to the other end of its first arch, the parameter t increases from 0 to 2π. [see Fig. 6.8]

Also $\quad \frac{dx}{dt} = a(1-\cos t),\ \frac{dy}{dt} = a\sin t.$

$$\therefore \quad \text{Length of an arch} = \int_0^{2\pi}\sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]}dx$$

$$= \int_0^{2\pi}\sqrt{\{[a(1-\cos t)]^2 + (a\sin t)^2\}}\,dt = a\int_0^{2\pi}\sqrt{[2(1-\cos t)]}\,dt$$

$$= 2a\int_0^{2\pi}\sin t/2\,dt = 2a\left|-\frac{\cos t/2}{1/2}\right|_0^{2\pi} = 4a[(-\cos\pi) - (-\cos 0)] = 8a.$$

Example 6.34. *Find the entire length of the cardioid $r = a(1 + \cos\theta)$.*
(P.T.U., 2010 ; Bhopal, 2008 ; Kurukshetra, 2005)
Also show that the upper half is bisected by $\theta = \pi/3$. *(Bhillai, 2005)*

Solution. The cardioid is symmetrical about the initial line and for its upper half, θ increases from 0 to π (Fig. 6.18)

Also $\quad \frac{dr}{d\theta} = -a\sin\theta.$

$$\therefore \quad \text{Length of the curve} = 2\int_0^{\pi}\sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}d\theta$$

$$= 2\int_0^{\pi} \sqrt{\{[a(1+\cos\theta)]^2 + (-a\sin\theta)^2\}}\, d\theta = 2a\int_0^{\pi} \sqrt{[2(1+\cos\theta)]}\, d\theta$$

$$= 4a\int_0^{\pi} \cos\theta/2\, d\theta = 4a\left|\frac{\sin\theta/2}{1/2}\right|_0^{\pi} = 8a(\sin\pi/2 - \sin 0) = 8a.$$

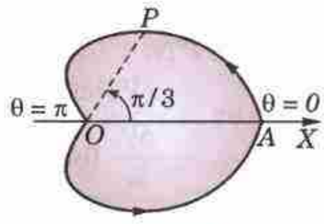


Fig. 6.18

$\therefore$ Length of upper half of the curve is $4a$. Also length of the arc AP from 0 to $\pi/3$.

$$= a\int_0^{\pi/3} \sqrt{[2(1+\cos\theta)]}\, d\theta = 2a\int_0^{\pi/3} \cos\theta/2 \cdot d\theta$$

$$= 4a\left|\sin\theta/2\right|_0^{\pi/3} = 2a = \text{half the length of upper half of the cardioid.}$$

PROBLEMS 6.8

1. Find the length of the arc of the *semi-cubical parabola* $ay^2 = x^3$ from the vertex to the ordinate $x = 5a$.
2. Find the length of the curve (*i*) $y = \log\sec x$ from $x = 0$ to $x = \pi/3$. (*V.T.U., 2010 S ; P.T.U., 2007*)
   (*ii*) $y = \log[(e^x - 1)/(e^x + 1)]$ from $x = 1$ to $x = 2$.
3. Find the length of the arc of the parabola $y^2 = 4ax$ (*i*) from the vertex to one end of the latus-rectum.
   (*ii*) cut off by the line $3y = 8x$. (*V.T.U., 2008 S ; Mumbai, 2006*)
4. Find the perimeter of the loop of the following curves :
   (*i*) $ay^2 = x^2(a - x)$ (*ii*) $9y^2 = (x-2)(x-5)^2$.
5. Find the length of the curve $y^2 = (2x-1)^2$ cut off by the line $x = 4$. (*V.T.U., 2000 S*)
6. Show that the whole length of the curve $x^2(a^2 - x^2) = 8a^2y^2$ is $\pi a\sqrt{2}$.
7. (*a*) Find the length of an arch of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$.
   (*b*) By finding the length of the curve show that the curve $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, is divided in the ratio 1 : 3 at $\theta = 2\pi/3$. (*S.V.T.U., 2009*)
8. Find the whole length of the curve $x = a\cos^3 t$, $y = a\sin^3 t$ *i.e.*, $x^{2/3} + y^{2/3} = a^{2/3}$.
   (*V.T.U., 2010 ; Marathwada, 2008 ; Rajasthan, 2006*)
   Also show that the line $\theta = \pi/3$ divides the length of this *astroid* in the first quadrant in the ratio 1 : 3.
   (*Mumbai, 2001*)
9. Find the length of the loop of the curve $x = t^2$, $y = t - t^3/3$. (*Mumbai, 2001*)
10. For the curve $r = ae^{\theta}\cot\alpha$, prove that s/r = constant, s being measured from the origin.
11. Find the length of the curve $\theta = \frac{1}{2}\left(r + \frac{1}{r}\right)$ from $r = 1$ to $r = 3$. (*Marathwada, 2008*)
12. Find the perimeter of the *cardioid* $r = a(1 - \cos\theta)$. Also show that the upper half of the curve is bisected by the line $\theta = 2\pi/3$.
13. Find the whole length of the *lemniscate* $r^2 = a^2\cos 2\theta$.
14. Find the length of the parabola $r(1 + \cos\theta) = 2a$ as cut off by the latus-rectum. (*J.N.T.U., 2003*)

6.12 (1) VOLUMES OF REVOLUTION

(*a*) **Revolution about x-axis.** *The volume of the solid generated by the revolution about the x-axis, of the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$ is*

$$\int_a^b \pi \mathbf{y^2}\, \mathbf{dx}.$$

Let AB be the curve $y = f(x)$ between the ordinates $LA(x = a)$ and $MB(x = b)$.

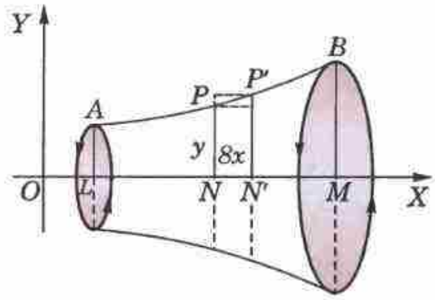


Fig. 6.19

Let $P(x, y)$, $P'(x + \delta x, y + \delta y)$ be two neighbouring points on the curve and NP, $N'P'$ be their respective ordinates (Fig. 6.19).

Let the volume of the solid generated by the revolution about x-axis of the area $ALNP$ be V, which is clearly a function of x. Then the volume of the solid generated by the revolution of the area $PNN'P'$ is δV. Complete the rectangles PN' and $P'N$.

The δV lies between the volumes of the right circular cylinders generated by the revolution of rectangles PN' and $P'N$,

i.e., δV lies between $\pi y^2\, \delta x$ and $\pi(y + \delta y)^2\, \delta x$.

$\therefore$ $\dfrac{\delta V}{\delta x}$ lies between πy^2 and $\pi(y + \delta y)^2$.

Now taking limits as $P' \to P$, *i.e.,* $\delta x \to 0$ (and $\therefore$ $\delta y \to 0$), $\dfrac{dV}{dx} = \pi y^2$

$$\therefore \quad \int_a^b \frac{dV}{dx}\, dx = \int_a^b \pi y^2\, dx \quad \text{or} \quad \left| V \right|_{x=a}^{b} = \int_a^b \pi y^2\, dx$$

or (value of V for $x = b$) – (value of V for $x = a$)

i.e., volume of the solid obtained by the revolution of the area $ALMB = \int_a^b \pi y^2\, dx$.

Example 6.35. *Find the volume of a sphere of radius a.* (S.V.T.U., 2007)

Solution. Let the sphere be generated by the revolution of the semi-circle ABC, of radius a about its diameter CA (Fig. 6.20)

Taking CA as the x-axis and its mid-point O as the origin, the equation of the circle ABC is $x^2 + y^2 = a^2$.

$\therefore$ Volume of the sphere = 2 (volume of the solid generated by the revolution about x-axis of the quadrant OAB)

$$= 2 \int_0^a \pi y^2\, dx = 2\pi \int_0^b (a^2 - x^2)\, dx$$

$$= 2\pi \left| a^2 x - \frac{x^3}{3} \right|_0^a = 2\pi \left[a^3 - \frac{a^3}{3} - (0 - 0) \right] = \frac{4}{3} \pi a^3.$$

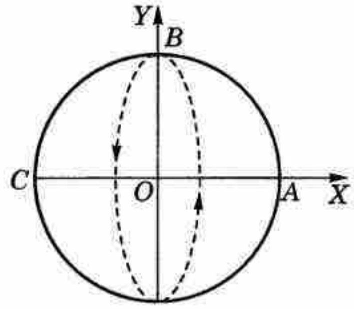


Fig. 6.20

Example 6.36. *Find the volume formed by the revolution of loop of the curve $y^2(a + x) = x^2(3a - x)$, about the x-axis.* (Marathwada, 2008)

Solution. The curve is symmetrical about the x-axis, and for the upper half of its loop x varies from 0 to $3a$ (Fig. 6.21)

$$\therefore \quad \text{Volume of the loop} = \int_0^{3a} \pi y^2\, dx = \pi \int_0^{3a} \frac{x^2 (3a - x)}{a + x}\, dx$$

$$= \pi \int_0^{3a} \frac{-x^3 + 3ax^2}{x + a}\, dx$$

[Divide the numerator by the denominator]

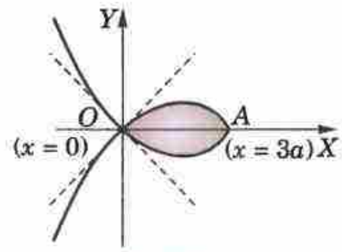


Fig. 6.21

$$= \pi \int_0^{3a} \left[-x^2 + 4ax - 4a^2 + \frac{4a^3}{x + a} \right] dx = \pi \left| -\frac{x^3}{3} + 4a \cdot \frac{x^2}{2} - 4a^2 x + 4a^3 \log (x + a) \right|_0^{3a}$$

$$= \pi \left[\frac{-27a^3}{3} + 2a \cdot 9a^2 - 4a^2 \cdot 3a + 4a^3 \log 4a - (4a^3 \log a) \right]$$

$$= \pi a^3 (-3 + 4 \log 4) = \pi a^3 (8 \log 2 - 3).$$

Example 6.37. *Prove that the volume of the reel formed by the revolution of the cycloid $x = a\,(\theta + \sin \theta)$, $y = a(1 - \cos \theta)$ about the tangent at the vertex is $\pi^2 a^3$.* (V.T.U., 2003)

Solution. The arch AOB of the cycloid is symmetrical about the y-axis and the tangent at the vertex is the x-axis. For half the cycloid OA, θ varies from 0 to π. (Fig. 4.31).

Hence the required volume

$$= 2 \int_{\theta=0}^{\theta=\pi} \pi y^2\, dx = 2\pi \int_0^{\pi} a^2 (1 - \cos \theta)^2 \,.\, a\,(1 + \cos \theta)\, d\theta$$

$$= 2\pi a^3 \int_0^\pi (2\sin^2\theta/2)^2 \cdot (2\cos^2\theta/2)\, d\theta$$

$$= 16\pi a^3 \int_0^\pi \sin^4\theta/2 \cdot \cos^2\theta/2 \cdot d\theta \qquad [\text{Put } \theta/2 = \phi,\ d\theta = 2d\phi]$$

$$= 32\pi a^3 \int_0^{\pi/2} \sin^4\phi\cos^2\phi\, d\phi = 32\pi a^3 \cdot \frac{3.1\times 1}{6.4.2}\cdot\frac{\pi}{2} = \pi^2 a^3.$$

Example 6.38. *Find the volume of the solid formed by revolving about x-axis, the area enclosed by the parabola $y^2 = 4ax$, its evolute $27ay^2 = 4(x-2a)^3$ and the x-axis.*

Solution. The curve $27ay^2 = 4(x-2a)^3$ ...(*i*)
is symmetrical about x-axis and is a semi-cubical parabola with vertex at A $(2a, 0)$. The parabola $y^2 = 4ax$ and (*i*) intersect at B and C where $27a(4ax) = 4(x-2a)^3$ or $x^3 - 6ax^2 - 15a^2x - 8a^3 = 0$ which gives $x = -a, -a, 8a$. Since x is not negative, therefore we have $x = 8a$ (Fig. 6.22).

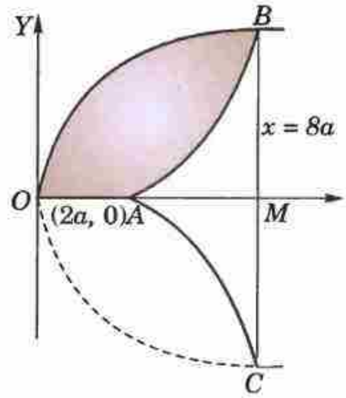


Fig. 6.22

$\therefore$ Required volume = Volume obtained by revolving the shaded area OAB about x-axis = Vol. obtained by revolving area $OMBO$ – Vol. obtained by revolving area $ADBA$

$$= \int_0^{8a} \pi y^2 (= 4ax)\, dx - \int_{2a}^{8a} \pi y^2\, [\text{for } (i)]\, dx$$

$$= 4a\pi \left| \frac{x^2}{2} \right|_0^{8a} - \frac{4\pi}{27a}\int_{2a}^{8a} (x-2a)^3\, dx$$

$$= 128\pi a^3 - \frac{4\pi}{27a}\left| \frac{(x-2a)^4}{4} \right|_{2a}^{8a}$$

$$= 128\pi a^3 - \frac{\pi}{27a}(6a)^4 = 80\pi a^3.$$

(*b*) **Revolution about the y-axis.** Interchanging x and y in the above formula, we see that the *volume of the solid generated by the revolution about y-axis, of the area, bounded by the curve $x = f(y)$, the y-axis and the abscissae $y = a$, $y = b$ is*

$$\int_a^b \pi\, \mathbf{x}^2\, \mathbf{dy}.$$

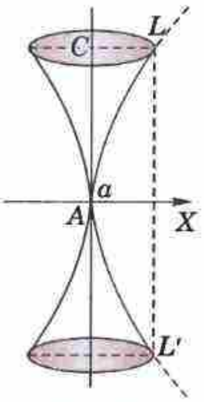


Fig. 6.23

Example 6.39. *Find the volume of the reel-shaped solid formed by the revolution about the y-axis, of the part of the parabola $y^2 = 4ax$ cut off by the latus-rectum.* (*Rohtak, 2003*)

Solution. Given parabola is $x = y^2/4a$.

Let A be the vertex and L one extremity of the latus-rectum. For the arc AL, y varies from 0 to $2a$ (Fig. 6.23).

$\therefore$ required volume = 2 (volume generated by the revolution about the y-axis of the area ALC)

$$= 2\int_0^{2a} \pi x^2\, dy = 2\pi\int_0^{2a} \frac{y^4}{16a^2}\cdot dy = \frac{\pi}{8a^2}\left| \frac{y^5}{5} \right|_0^{2a} = \frac{\pi}{40a^2}(32a^5 - 0) = \frac{4\pi a^3}{5}.$$

(*c*) **Revolution about any axis.** *The volume of the solid generated by the revolution about any axis LM of the area bounded by the curve AB, the axis LM and the perpendiculars AL, BM on the axis, is*

$$\int_{OL}^{OM} \pi(\mathbf{PN})^2\, \mathbf{d(ON)}$$

where O is a fixed point in LM and PN is perpendicular from any point P of the curve AB on LM.

With O as origin, take OLM as the x-axis and OY, perpendicular to it as the y-axis (Fig. 6.24).

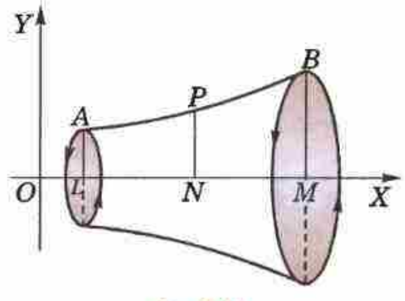


Fig. 6.24

Let the coordinates of P be (x, y) so that $x = ON$, $y = NP$

If $OL = a$, $OM = b$, then required volume $= \int_a^b \pi y^2 \, dx = \int_{OL}^{OM} \pi (PN)^2 \, d(ON)$.

Example 6.40. *Find the volume of the solid obtained by revolving the cissoid $y^2 (2a - x) = x^3$ about its asymptote.* *(V.T.U., 2000)*

Solution. Given curve is $y = \dfrac{x^3}{2a - x}$ ...(*i*)

It is symmetrical about x-axis and the asymptote is $x = 2a$. (See Fig. 4.23). If $P(x, y)$ be any point on it and PN is perpendicular on the asymptote AN then $PN = 2a - x$ and

$$AN = y = \frac{x^{3/2}}{\sqrt{(2a - x)}} \qquad \text{[From } (i)]$$

$$\therefore \quad d(AN) = dy = \frac{\sqrt{(2a - x)}\,(3/2)\sqrt{x} - x^{3/2} \cdot \frac{1}{2}(2a - x)^{-1/2}(-1)}{2a - x}\, dx$$

$$= \frac{3\sqrt{x}(2a - x) + x^{3/2}}{2\,(2a - x)^{3/2}}\, dx = \frac{3ax^{1/2} - x^{3/2}}{(2a - x)^{3/2}}\, dx$$

$$\therefore \quad \text{Required volume} = 2\int_{x=0}^{x=2a} \pi (PN)^2 \, d(AN) = 2\pi \int_0^{2a} (2a - x)^2 \cdot \frac{3ax^{1/2} - x^{3/2}}{(2a - x)^{3/2}} \cdot dx$$

$$= 2\pi \int_0^{2a} \sqrt{(2a - x)}\,(3a - x)\sqrt{x}\, dx \qquad \left[\begin{array}{l}\text{Put } x = 2a \sin^2 \theta \\ \text{then } dx = 4a \sin \theta \cos \theta \, d\theta\end{array}\right.$$

$$= 2\pi \int_0^{\pi/2} \sqrt{(2a)} \cos \theta \, (3a - 2a \sin^2 \theta) \times \sqrt{(2a)} \sin \theta \, . \, 4a \sin \theta \cos \theta \, d\theta$$

$$= 16\pi a^3 \left[3 \int_0^{\pi/2} \sin^2 \theta \cos^2 \theta \, d\theta - 2 \int_0^{\pi/2} \sin^4 \theta \cos^2 \theta \, d\theta\right]$$

$$= 16\pi a^3 \left[3 \cdot \frac{1 \times 1}{4 \cdot 2} \cdot \frac{\pi}{2} - 2 \cdot \frac{3 \cdot 1 \times 1}{6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2}\right] = 2\pi^2 a^3.$$

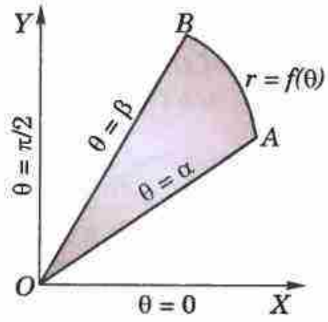


Fig. 6.25

(2) Volumes of revolution (polar curves). *The volume of the solid generated by the revolution of the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$, $\theta = \beta$ (Fig. 6.25)*

(*a*) *about the initial line OX $(\theta = 0)$* $= \displaystyle\int_\alpha^\beta \frac{\mathbf{2\pi}}{\mathbf{3}} \mathbf{r^3} \, \mathbf{\sin \theta \, d\theta}$

(*b*) *about the line OY $(\theta = \pi/2)$* $= \displaystyle\int_\alpha^\beta \frac{\mathbf{2\pi}}{\mathbf{3}} \mathbf{r^3} \, \mathbf{\cos \theta \, d\theta}$.

Example 6.41. *Find the volume of the solid generated by the revolution of the cardioid $r = a(1 + \cos \theta)$ about the initial line.* *(V.T.U., 2010 ; Kurukshetra, 2009 S)*

Solution. The cardioid is symmetrical about the initial line and for its upper half θ varies from 0 to π. [Fig. 6.18].

$$\therefore \quad \text{required volume} = \int_0^\pi \frac{2}{3} \pi r^3 \sin \theta \, d\theta = \frac{2\pi}{3} \int_0^\pi a^3 (1 + \cos \theta)^3 \sin \theta \, d\theta$$

$$= -\frac{2\pi a^3}{3} \int_0^\pi (1 + \cos \theta)^3 \cdot (-\sin \theta) \, d\theta = -\frac{2\pi a^3}{3} \left| \frac{(1 + \cos \theta)^4}{4} \right|_0^\pi = -\frac{\pi a^3}{6} [0 - 16] = \frac{8}{3} \pi a^3.$$

Example 6.42. *Find the volume of the solid generated by revolving the lemniscate $r^2 = a^2 \cos 2\theta$ about the line $\theta = \pi/2$.* *(V.T.U., 2006)*

Solution. The curve is symmetrical about the pole. For the upper half of the R.H.S. loop, θ varies from 0 to $\pi/4$. (Fig. 4.34).

$\therefore$ required volume = 2(volume generated by the half loop in the first quadrant)

$$= 2\int_0^{\pi/4} \frac{2}{3}\pi r^3 \cos\theta\, d\theta = \frac{4\pi}{3}\cdot\int_0^{\pi/4} a^3(\cos 2\theta)^{3/2}\cos\theta\, d\theta \qquad [\because\ r = a(\cos 2\theta)^{1/2}]$$

$$= \frac{4\pi a^3}{3}\int_0^{\pi/4}(1-2\sin^2\theta)^{3/2}\cos\theta\, d\theta \qquad \left[\begin{array}{l}\text{Put } \sqrt{2}\sin\theta = \sin\phi \\ \therefore\ \sqrt{2}\cos\theta\, d\theta = \cos\phi\, d\phi\end{array}\right.$$

$$= \frac{4\pi a^3}{3}\int_0^{\pi/2}(1-\sin^2\phi)^{3/2}\cdot\frac{1}{\sqrt{2}}\cos\phi\, d\phi = \frac{4\pi a^3}{3\sqrt{2}}\int_0^{\pi/2}\cos^4\phi\, d\phi = \frac{4\pi}{3\sqrt{2}}a^3\,\frac{3.1}{4.2}\cdot\frac{\pi}{2} = \frac{\pi a^3}{4\sqrt{2}}.$$

PROBLEMS 6.9

1. Find the volume generated by the revolution of the area bounded by x-axis, the *catenary* $y = c \cosh x/c$ and the ordinates $x = \pm c$, about the axis of x.
2. Find the volume of a spherical segment of height h cut off from a sphere of radius a.
3. Find the volume generated by revolving the portion of the parabola $y^2 = 4ax$ cut off by its latus-rectum about the axis of the parabola. *(V.T.U., 2009)*
4. Find the volume generated by revolving the area bounded by the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$, $x = 0$, $y = 0$ about the x-axis.
5. Find the volume of the solid generated by revolving the ellipse $x^2/a^2 + y^2/b^2 = 1$.
   (*i*) about the major axis. *(Bhopal, 2002 S)* (*ii*) about the minor axis. *(Bhillai, 2005)*
6. Obtain the volume of the frustrum of a right circular cone whose lower base has radius R, upper base is of radius r and altitude is h.
7. Find the volume generated by the revolution of the curve $27ay^2 = 4(x - 2a)^3$ about the x-axis.
8. Find the volume of the solid formed by the revolution, about the x-axis, of the loop of the curve :
   (*i*) $y^2(a - x) = x^2(a + x)$ (*ii*) $2ay^2 = x(x - a)^2$ (*iii*) $y^2 = x(2x - 1)^2$.
9. Find the volume obtained by revolving one arch of the cycloid
   (*i*) $x = a(t - \sin t)$, $y = a(1 - \cos t)$, about its base. *(Kurukshetra, 2006 ; V.T.U., 2005)*
   (*ii*) $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$, about the x-axis.
10. Find the volume of the spindle-shaped solid generated by the revolution of the *astroid* $x^{2/3} + y^{2/3} = a^{2/3}$ about the x-axis. *(P.T.U., 2010 ; S.V.T.U., 2008)*
11. Find the volume of the solid formed by the revolution, about the y-axis, of the area enclosed by the curve $xy^2 = 4a^2(2a - x)$ and its asymptote. *(V.T.U., 2006)*
12. Prove that the volume of the solid formed by the revolution of the curve $(a^2 + x^2) = a^3$, about its asymptote is $\frac{1}{2}\pi^2 a^3$.
13. Find the volume generated by the revolution about the initial line of
    (*i*) $r = 2a\cos\theta$ (*ii*) $r = a(1 - \cos\theta)$. *(P.T.U., 2006)*
14. Determine the volume of the solid obtained by revolving the *lemicon* $r = a + b\cos\theta$ $(a > b)$ about the initial line. *(Gorakhpur, 1999)*
15. Find the volume of the solid formed by revolving a loop of the lemniscate $r^2 = a^2\cos 2\theta$ about the initial line. *(J.N.T.U., 2003 ; Delhi, 2002)*

6.13 SURFACE AREAS OF REVOLUTION

(*a*) **Revolution about x-axis.** *The surface area of the solid generated by the revolution about x-axis, of the arc of the curve $y = f(x)$ from $x = a$ to $x = b$, is*

$$\int_{x=a}^{x=b} \mathbf{2\pi y\, ds.}$$

Let AB be the curve $y = f(x)$ between the ordinates LA $(x = a)$ and MB $(x = b)$. Let $P(x, y)$, $P'(x + \delta x y + \delta y)$ be two neighbouring points on the curve and NP, $N'P'$ be their respective ordinates (Fig. 6.19).

Let the arc $AP = s$ so that arc $PP' = \delta s$. Let the surface-area generated by the revolution about x-axis of the arc AP be S and that generated by the revolution of the arc PP' be δS.

Since δs is small, the surface area δS may be regarded as lying between the curved surfaces of the right cylinders of radii PN and $P'N'$ and of same thickness δs.

Thus δS lies between $2\pi y\ \delta s$ and $2\pi (y + \delta y)\ \delta s$

$\therefore \quad \dfrac{\delta S}{\delta s}$ lies between $2\pi y$ and $2\pi (y + \delta y)$

Taking limits as $P' \to P$, *i.e.*, $\delta s \to 0$ and $\delta y \to 0$, $dS/dx = 2\pi y$

$$\therefore \quad \int_{x=a}^{x=b} \frac{dS}{ds}\, ds = \int_{x=a}^{x=b} 2\pi y\, ds \quad \text{or} \quad \left| S \right|_{x=a}^{x=b} = \int_{x=a}^{x=b} 2\pi y\, ds$$

or (value of S for $x = b$) – (value of S for $x = a$) = $\displaystyle\int_{x=a}^{x=b} 2\pi y\ dx$

or surface area generated by the revolution of the arc $AB - 0 = \displaystyle\int_{x=a}^{x=b} 2\pi y\ ds.$

Hence, the required surface area = $\displaystyle\int_{x=a}^{x=b} 2\pi y\ ds.$

Obs. Practical forms of the formula S = $\displaystyle\int 2\pi y\ ds$.

(*i*) *Cartesian form* [*for the curve* $y = f(x)$]

$$S = \int 2\pi y \frac{ds}{dx}\, dx, \text{ where } \frac{ds}{dx} = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}$$

(*ii*) *Parametric form* [*for the curve* $x = f(t)$, $y = \phi(t)$]

$$S = \int 2\pi y \frac{ds}{dt}\, dt, \text{ where } \frac{ds}{dt} = \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]}$$

(*iii*) *Polar form* [*for the curve* $r = f(\theta)$]

$$S = \int 2\pi y \frac{ds}{d\theta}\, d\theta, \text{ where } y = r \sin\theta,\ \frac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}.$$

Example 6.43. *Find the surface of the solid formed by revolving the cardioid $r = a\ (1 + \cos\theta)$ about the initial line.* (*V.T.U., 2009 ; Rajasthan, 2006 ; J.N.T.U., 2003*)

Solution. The cardioid is symmetrical about the initial line and for its upper half, θ varies from 0 to π (Fig. 6.18).

Also
$$\frac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]} = \sqrt{[a^2(1 + \cos\theta)^2 + a^2 \sin^2\theta]}$$

$$= a\sqrt{[2(1 + \cos\theta)]} = a\sqrt{[2.2\cos^2\theta/2]} = 2a\cos\theta/2$$

$$\therefore \quad \text{required surface} = \int_0^\pi 2\pi y \frac{ds}{d\theta}\, d\theta = 2\pi \int_0^\pi r\sin\theta \cdot 2a\cos\theta/2\ d\theta$$

$$= 4\pi a \int_0^\pi a(1 + \cos\theta)\ \sin\theta \cdot \cos\theta/2\ d\theta = 4\pi a^2 \int_0^\pi 2\cos^2\frac{\theta}{2} . 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} . \cos\frac{\theta}{2}\, d\theta$$

$$= 16\pi a^2 \int_0^\pi \cos^4\frac{\theta}{2}\sin\frac{\theta}{2}\, d\theta = 16\pi a^2 (-2) \int_0^\pi \cos^4\frac{\theta}{2}\left(-\sin\frac{\theta}{2}\cdot\frac{1}{2}\right) d\theta$$

$$= -32\pi a^2 \left| \frac{\cos^5\theta/2}{5} \right|_0^\pi = \frac{-32\pi a^2}{5}(0 - 1) = \frac{32\pi a^2}{5}.$$

(*b*) **Revolution about y-axis.** Interchanging x and y in the above formula, we see that *the surface of the solid generated by the revolution about y-axis, of the arc of the curve $x = f(y)$ from $y = a$ to $y = b$ is*

$$\int_{y=a}^{y=b} 2\pi x\ ds.$$

Example 6.44. *Find the surface area of the solid generated by the revolution of the astroid $x = a\cos^3 t$, $y = a\sin^3 t$, about the y-axis.*

Solution. The astroid is symmetrical about the x-axis, and for its portion in the first quadrant t varies from 0 to $\pi/2$. (Fig. 4.29).

Also $$\frac{dx}{dt} = -3a\cos^2 t \sin t,\ \frac{dy}{dt} = 3a\sin^2 t\cos t.$$

$$\frac{ds}{dt} = \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]} = \sqrt{[9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t]}$$

$$= 3a\sin t\cos t\sqrt{(\cos^2 t + \sin^2 t)} = 3a\sin t\cos t$$

$\therefore$ $$\text{required surface} = 2\int_0^{\pi/2} 2\pi x\frac{ds}{dt}\cdot dt = 4\pi\int_0^{\pi/2} a\cos^3 t\cdot 3a\sin t\cos t\, dt$$

$$= 12\pi a^2\int_0^{\pi/2}\sin t\cos^4 t\, dt = 12\pi a^2\,\frac{3.1}{5.3.1} = \frac{12\pi a^2}{5}.$$

PROBLEMS 6.10

1. Find the area of the surface generated by revolving the arc of the catenary $y = c\cosh x/c$ from $x = 0$ to $x = c$ about the x-axis.
2. Find the area of the surface formed by the revolution of $y^2 = 4ax$ about its axis, by the arc from the vertex to one end of the latus-rectum.
3. Find the surface of the solid generated by the revolution of the *ellipse* $x^2/a^2 + y^2/b^2 = 1$ about the x-axis. *(Raipur, 2005 ; Bhopal, 2002 S)*
4. Find the volume and surface of the *right circular cone* formed by the revolution of a right-angled triangle about a side which contains the right angle.
5. Obtain the surface area of a *sphere* of radius a.
6. Show that the surface area of the solid generated by the revolution of the curve $x = a\cos^3 t$, $y = a\sin^3 t$ about the x-axis, is $12\pi^2/5$.
7. The arc of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ in the first quadrant revolves about x-axis. Show that the area of the surface generated is $6\pi a^2/5$.
8. Find the surface area of the solid generated by revolving the *cycloid* $x = a(t - \sin t)$, $y = a(1 - \cos t)$ about the base. *(Marathwada, 2008 ; Cochin, 2005 ; Kurukshetra, 2005)*
9. Find the surface area of the solid got by revolving the arch of the *cycloid*
   $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ about the base. *(V.T.U., 2010 S)*
10. Prove that the surface and volume of the solid generated by the revolution about the x-axis, of the loop of the curve
    $$x = t^2,\ y = t - t^3/3,\ [\text{or}\ \ 9y^2 = x(x-3)^2],$$
    are respectively 3π and $3\pi/4$.
11. Prove that the surface of the solid generated by the revolution of the *tractrix* $x = a\cos t + \frac{a}{2}\log\tan^2 t/2$, $y = a\sin t$, about x-axis is $4\pi a^2$.
12. Find the surface area of the solid of revolution of the curve $r = 2a\cos\theta$ about the initial line. *(V.T.U., 2009)*
13. Find the surface of the solid generated by the revolution of the *cardioid* $r = a(1 - \cos\theta)$ about the initial line.
14. Find the surface of the solid generated by the revolution of the *lemniscate* $r^2 = a^2\cos 2\theta$ about the initial line. *(V.T.U., 2005)*
15. The part of *parabola* $y^2 = 4ax$ cut off by the latus-rectum revolves about the tangent at the vertex. Find the curved surface of the reel thus formed.

6.14 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 6.11

Choose the correct answer or fill up the blanks in the following problems :

1. If $f(x) = f(2a - x)$, then $\int_0^{2a} f(x)\, dx$ is equal to

(a) $\int_a^0 f(2a-x)\,dx$ (b) $2\int_0^a f(x)\,dx$ (c) $-2\int_0^a f(x)\,dx$ (d) 0.

2. $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$ is equal to

(a) 0 (b) 1 (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$.

3. The value of definite integral $\int_{-a}^{a} |x|\,dx$ is equal to

(a) a (b) a^2 (c) 0 (d) $2a$.

4. $\lim_{n\to\infty}\left[\frac{n}{n^2}+\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\dots+\frac{n}{n^2+(n-1)^2}\right]$ is equal to

(a) $-\frac{\pi}{4}$ (b) 0 (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$.

5. $\int_0^{\pi/2} \frac{\cos 2x}{\cos x+\sin x}\,dx$ equals

(a) -1 (b) 0 (c) 1 (d) 2.

6. $\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{3n}\right)$ equals

(a) $\log_e 2$ (b) $2\log_e 2$ (c) $\log_e 3$ (d) $2\log_e 3$.

7. $\int_0^{\pi} \sin^5\left(\frac{x}{2}\right)$ is equal to

(a) $\frac{16}{15}$ (b) $\frac{15}{16}\pi$ (c) $\frac{16}{15}\pi^2$ (d) $\frac{15}{16}$.

8. $\int_0^{\pi/2} \sin^{99} x \cos x\,dx$ is equal to

(a) $\frac{1}{99}$ (b) $\frac{\pi}{100}$ (c) $\frac{99}{100}$ (d) None of these. *(V.T.U., 2009)*

9. The value of $\int_{-\pi/2}^{\pi/2} \cos^7 x\,dx$ is

(a) $\frac{32\pi}{35}$ (b) $\frac{32}{35}$ (c) zero.

10. The length of the arc of the equiangular spiral $r = ae^{\theta\cot\alpha}$ between the points for which the radii vectors are r_1 and r_2 is

(a) $(r_2-r_1)\operatorname{cosec}\alpha$ (b) $(r_2-r_1)\cos\alpha$ (c) $(r_2-r_1)\sin\alpha$ (d) $(r_2-r_1)\sec\alpha$.

11. The area of the region in the first quadrant bounded by the y-axis and the curves $y=\sin x$ and $y=\cos x$ is

(a) $\sqrt{2}$ (b) $\sqrt{2}+1$ (c) $\sqrt{2}-1$ (d) $2\sqrt{2}-1$.

12. The value of $\int_0^1 x^{3/2}(1-x)^{3/2}\,dx$ is

(a) $\pi/32$ (b) $-\pi/32$ (c) $3\pi/128$ (d) $-3\pi/128$. *(V.T.U., 2010)*

13. The entire length of the cardioid $r = 5(1+\cos\theta)$ is

(a) 40 (b) 30 (c) 20 (d) 5. *(V.T.U., 2009)*

14. The area of the cardioid $r = a(1-\cos\theta)$ is ....

15. If S_1 and S_2 are surface areas of the solids generated by revolving the ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ about the y-axis, then

(a) $S_1 > S_2$ (b) $S_1 < S_2$ (c) $S_1 = S_2$ (d) can't predict.

16. The area of the loop of the curve $r = a\sin 3\theta$ is ....

17. If $I_n = \int_0^{\pi/4} \tan^n\theta\,d\theta$, then $n(I_{n-1}+I_{n+1}) =$ ....

18. $\int_0^2 x^3\sqrt{(2x-x^2)}\,dx =$ ....

19. $\int_0^{\pi/2} \sin 2\theta \log \tan \theta \, d\theta$ is equal to

(a) 1 (b) – 1 (c) 0 (d) $\pi/2$.

20. The area of the loop of the folium of Descartes $x^3 + y^3 - 3xy = 0$ is

(a) π (b) $\pi/2$ (c) 1.5 (d) 3.

21. The volume of the frustrum of a right circular cone whose lower base has radius r_1 and upper base has radius r_2 and altitude is h = ...

22. The length of the arc of the curve $y = \log \sec x$ from $x = 0$ to $x = \pi/4$ is

(a) $\log_e 2$ (b) $\log_e 3$ (c) $\log_e (1 + \sqrt{2})$ (d) $\log_e (1 + \sqrt{3})$. *(Bhopal, 2008)*

23. If v_1 = volume of the solid generated by revolving the area included between x-axis and $x^2 + y^2 = a^2$ about x-axis ; v_2 = volume of the solid generated by revolving the entire area of the circle $x^2 + y^2 = a^2$ about x-axis, then

(a) $v_1 = v_2$ (b) $v_2 = 2v_1$ (c) $v_2 = 4v_1$ (d) $v_2 = 16v_1$.

24. If $f(r, \theta) = f(-r, \theta)$, then the curve is symmetrical about the ....

(a) initial line (b) pole (c) origin (d) tangential line. *(V.T.U., 2010)*

25. The volume generated by the revolution of the curve $y = a^3 (a^2 + x^2)^{-1}$ about its asymptote is

(a) $\pi^2 a^3/2$ (b) $\pi a^3/2$ (c) $\pi a^2/2$ (d) $\pi a/2$. *(V.T.U., 2010)*

CHAPTER 7

Multiple Integrals and Beta, Gamma Functions

7.1 DOUBLE INTEGRALS

The definite integral $\int_a^b f(x)\,dx$ is defined as the limit of the sum

$$f(x_1)\,\delta x_1 + f(x_2)\,\delta x_2 + \dots + f(x_n)\,\delta x_n,$$

where $n \to \infty$ and each of the lengths $\delta x_1, \delta x_2, \dots$ tends to zero. A double integral is its counterpart in two dimensions.

Consider a function $f(x, y)$ of the independent variables x, y defined at each point in the finite region R of the xy-plane. Divide R into n elementary areas $\delta A_1, \delta A_2, \dots, \delta A_n$. Let (x_r, y_r) be any point within the rth elementary area δA_r. Consider the sum

$$f(x_1, y_1)\,\delta A_1 + f(x_2, y_2)\,\delta A_2 + \dots + f(x_n, y_n)\,\delta A_n, \text{ i.e., } \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r$$

The limit of this sum, if it exists, as the number of sub-divisions increases indefinitely and area of each sub-division decreases to zero, is defined as the *double integral of f(x, y) over the region R* and is written as $\iint_R f(x, y)\,dA$.

Thus
$$\iint_R f(x, y)\,dA = \underset{\substack{n \to \infty \\ \delta A \to 0}}{\text{Lt}} \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r \qquad \dots(1)$$

The utility of double integrals would be limited if it were required to take limit of sums to evaluate them. However, there is another method of evaluating double integrals by successive single integrations.

For purpose of evaluation, (1) is expressed as the repeated integral $\int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x, y)\,dxdy.$

Its value is found as follows :

(*i*) *When y_1, y_2 are functions of x and x_1, x_2 are constants*, $f(x, y)$ is first integrated w.r.t. y keeping x fixed between limits y_1, y_2 and then resulting expression is integrated w.r.t. x within the limits x_1, x_2 *i.e.*,

$$\boxed{I_1 = \int_{x_1}^{x_2} \boxed{\int_{y_1}^{y_2} f(x, y)\,dy}\,dx}$$

where integration is carried from the inner to the outer rectangle.

Figure 7.1 illustrates this process. Here AB and CD are the two curves whose equations are $y_1 = f_1(x)$ and $y_2 = f_2(x)$. PQ is a vertical strip of width dx.

Then the inner rectangle integral means that the integration is along one edge of the strip PQ from P to Q (x remaining constant), while the outer rectangle integral corresponds to the sliding of the edge from AC to BD.

Thus the whole region of integration is the area $ABDC$.

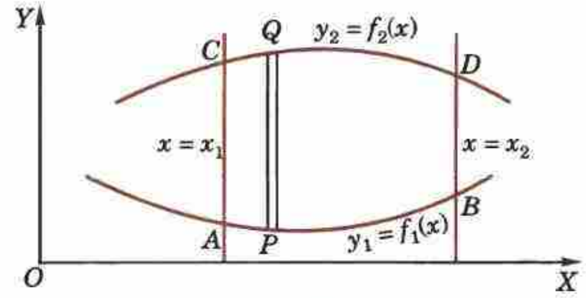


Fig. 7.1

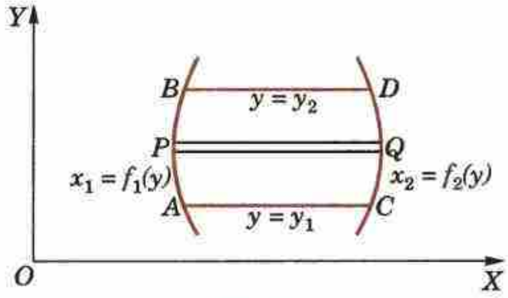


Fig. 7.2

(*ii*) *When x_1, x_2 are functions of y and y_1, y_2 are constants*, $f(x, y)$ is first integrated w.r.t. x keeping y fixed, within the limits x_1, x_2 and the resulting expression is integrated w.r.t. y between the limits y_1, y_2, *i.e.*,

$$I_2 = \boxed{\int_{y_1}^{y_2} \boxed{\int_{x_1}^{x_2} f(x, y)\, dx}\, dy}$$ which is geometrically illustrated by Fig. 7.2.

Here AB and CD are the curves $x_1 = f_1(y)$ and $x_2 = f_2(y)$. PQ is a horizontal strip of width dy.

Then inner rectangle indicates that the integration is along one edge of this strip from P to Q while the outer rectangle corresponds to the sliding of this edge from AC to BD.

Thus the whole region of integration is the area $ABDC$.

(*iii*) *When both pairs of limits are constants*, the region of integration is the rectangle $ABDC$ (Fig. 7.3).

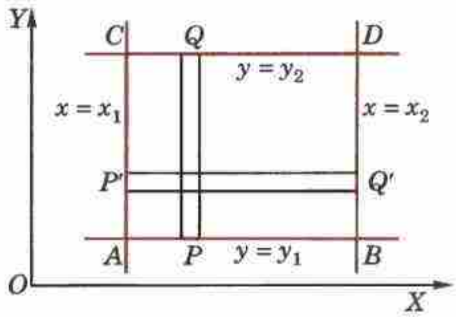


Fig. 7.3

In I_1, we integrate along the vertical strip PQ and then slide it from AC to BD.

In I_2, we integrate along the horizontal strip $P'Q'$ and then slide it from AB to CD.

Here obviously $I_1 = I_2$.

Thus *for constant limits, it hardly matters whether we first integrate w.r.t. x and then w.r.t. y or vice versa.*

Example 7.1. *Evaluate* $\int_0^5 \int_0^{x^2} x(x^2 + y^2)\, dxdy$.

Solution.
$$I = \int_0^5 dx \int_0^{x^2} (x^3 + xy^3)\, dy = \int_0^5 \left[x^3 y + x\,.\,\frac{y^3}{3}\right]_0^{x^2} dx = \int_0^5 \left[x^3\,.\,x^2 + x\,.\,\frac{y^6}{3}\right] dx$$
$$= \int_0^5 \left(x^5 + \frac{x^7}{3}\right) dx = \left|\frac{x^6}{6} + \frac{x^8}{24}\right|_0^5 = 5^6\left[\frac{1}{6} + \frac{5^2}{24}\right] = 18880.2 \text{ nearly.}$$

Example 7.2. *Evaluate* $\iint_A xy\, dx\, dy$, *where A is the domain bounded by x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$.*

Solution. The line $x = 2a$ and the parabola $x^2 = 4ay$ intersect at $L(2a, a)$. Figure 7.4 shows the domain A which is the area OML.

Integrating first over a vertical strip PQ, *i.e.*, w.r.t. y from $P(y = 0)$ to $Q(y = x^2/4a)$ on the parabola and then w.r.t. x from $x = 0$ to $x = 2a$, we have

$$\iint_A xy\, dx\, dy = \int_0^{2a} dx \int_4^{x^2/4a} xy\, dy = \int_0^{2a} x\left[\frac{y^2}{2}\right]_0^{x^2/4a} dx$$

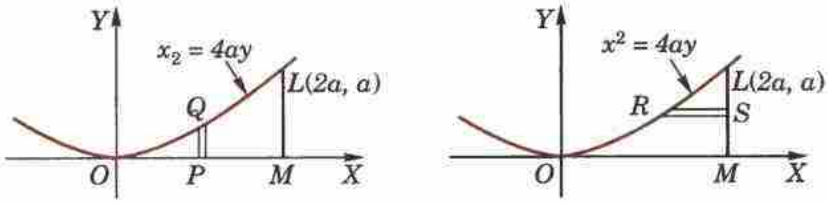


Fig. 7.4

$$= \frac{1}{32a^2}\int_0^{2a} x^5\,dx = \frac{1}{32a^2}\left|\frac{x^6}{6}\right|_0^{2a} = \frac{a^4}{3}.$$

Otherwise integrating first over a horizontal strip RS, *i.e.*, w.r.t. x from, R $(x = 2\sqrt{ay})$ on the parabola to $S(x = 2a)$ and then w.r.t. y from $y = 0$ to $y = a$, we get

$$\iint_A xy\,dx\,dy = \int_0^a dx \int_{2\sqrt{(ay)}}^{2a} xy\,dx = \int_0^a y\left[\frac{x^2}{2}\right]_{2\sqrt{(ay)}}^{2a} dy$$

$$= 2a\int_0^a (ay - y^2)\,dy = 2a\left[\frac{ay^2}{2} - \frac{y^3}{3}\right]_0^a = \frac{a^4}{3}.$$

Example 7.3. *Evaluate* $\iint_R x^2\,dx\,dy$ *where R is the region in the first quadrant bounded by the lines* $x = y$, $y = 0$, $x = 8$ *and the curve* $xy = 16$.

Solution. The line AL $(x = 8)$ intersects the hyperbola $xy = 16$ at A $(8, 2)$ while the line $y = x$ intersects this hyperbola at B $(4, 4)$. Figure 7.5 shows the region R of integration which is the area $OLAB$. To evaluate the given integral, we divide this area into two parts OMB and $MLAB$.

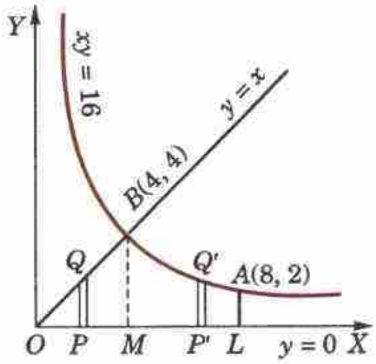


Fig. 7.5

$$\therefore \quad \iint_R x^2\,dxdy = \int_{x \text{ at } 0}^{x \text{ at } M}\int_{y \text{ at } P}^{y \text{ at } Q} x^2\,dxdy + \int_{x \text{ at } M}^{x \text{ at } L}\int_{y \text{ at } P'}^{y \text{ at } Q'} x^2\,dxdy$$

$$= \int_0^4\int_0^x x^2\,dxdy + \int_4^8\int_0^{16/x} x^2\,dxdy$$

$$= \int_0^4 x^2\,dx\,\Big|\,y\,\Big|_0^x + \int_4^8 x^2\,dx\,\Big|\,y\,\Big|_0^{16/x}$$

$$= \int_0^4 x^3\,dx + \int_4^8 16x\,dx = \left|\frac{x^4}{4}\right|_0^4 + 16\left|\frac{x^2}{2}\right|_4^8 = 448$$

7.2 CHANGE OF ORDER OF INTEGRATION

In a double integral with variable limits, the change of order of integration changes the limit of integration. While doing so, sometimes it is required to split up the region of integration and the given integral is expressed as the sum of a number of double integrals with changed limits. To fix up the new limits, it is always advisable to draw a rough sketch of the region of integration.

The change of order of integration quite often facilitates the evaluation of a double integral. The following examples will make these ideas clear.

Example 7.4. *By changing the order of integration of* $\int_0^\infty\int_0^\infty e^{-xy}\sin px\,dxdy$, *show that*

$$\int_0^\infty \frac{\sin px}{x}\,dx = \frac{\pi}{2}. \qquad \textit{(U.P.T.U., 2004)}$$

Solution. $\int_0^\infty\int_0^\infty e^{-xy}\sin px\,dxdy = \int_0^\infty\left(\int_0^\infty e^{-xy}\sin px\,dx\right)dy$

$$= \int_0^\infty \left| -\frac{e^{-xy}}{p^2+y^2}(p\cos px + y\sin px)\right|_0^\infty dy$$

$$= \int_0^\infty \frac{p}{p^2+y^2}\,dy = \left|\tan^{-1}\left(\frac{y}{p}\right)\right|_0^\infty = \frac{\pi}{2} \qquad ...(i)$$

On changing the order of integration, we have

$$\int_0^\infty \int_0^\infty e^{-xy}\sin px\,dx\,dy = \int_0^\infty \sin px\left\{\int_0^\infty e^{-xy}\,dy\right\}dx$$

$$= \int_0^\infty \sin px \left|\frac{e^{-xy}}{-x}\right|_0^\infty dx = \int_0^\infty \frac{\sin px}{x}\,dx \qquad ...(ii)$$

Thus from (*i*) and (*ii*), we have $\int_0^\infty \frac{\sin px}{x}\,dx = \frac{\pi}{2}$.

Example 7.5. *Change the order of integration in the integral*

$$I = \int_{-a}^{a}\int_0^{\sqrt{(a^2-y^2)}} f(x,y)\,dx\,dy.$$

Solution. Here the elementary strip is parallel to x-axis (such as PQ) and extends from $x = 0$ to $x = \sqrt{(a^2-y^2)}$ (*i.e.*, to the circle $x^2+y^2=a^2$) and this strip slides from $y = -a$ to $y = a$. This shaded semi-circular area is, therefore, the region of integration (Fig. 7.6).

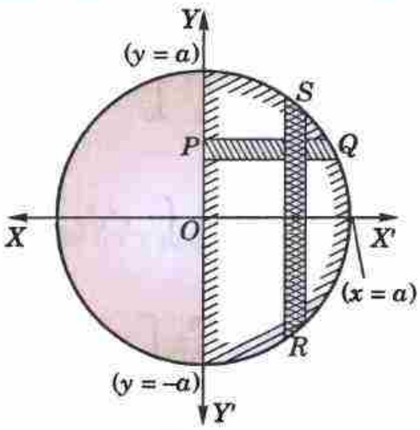


Fig. 7.6

On changing the order of integration, we first integrate w.r.t. y along a vertical strip RS which extends from R [$y = -\sqrt{(a^2-y^2)}$] to S [$y = \sqrt{(a^2-y^2)}$]. To cover the given region, we then integrate w.r.t. x from $x = 0$ to $x = a$.

Thus
$$I = \int_0^a dx \int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} f(x,y)\,dy$$

or
$$= \int_0^a \int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} f(x,y)\,dydx.$$

Example 7.6. *Evaluate* $\int_0^1 \int_{e^x}^{e} dydx/\log y$ *by changing the order of integration.*

Solution. Here the integration is first w.r.t. y from P on $y = e^x$ to Q on the line $y = e$. Then the integration is w.r.t. x from $x = 0$ to $x = 1$, giving the shaded region ABC (Fig. 7.7).

On changing the order of integration, we first integrate w.r.t. x from R on $x = 0$ to S on $x = \log y$ and then w.r.t. y from $y = 1$ to $y = e$.

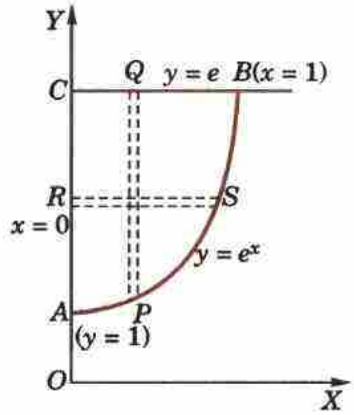


Fig. 7.7

Thus
$$\int_0^1 \int_{e^x}^{e} \frac{dydx}{\log y} = \int_1^e \int_0^{\log y} \frac{dxdy}{\log y}$$

$$= \int_1^e \frac{dy}{\log y}\left| x \right|_0^{\log y} = \int_1^e dy = \left| y \right|_1^e = e - 1.$$

Example 7.7. *Change the order of integration in* $I = \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dydx$ *and hence evaluate.*

(Nagpur, 2009 ; P.T.U., 2009 S)

Solution. Here integration is first w.r.t. y and P on the parabola $x^2 = 4ay$ to Q on the parabola $y^2 = 4ax$ and then w.r.t. x from $x = 0$ to $x = 4a$ giving the shaded region of integration (Fig. 7.8).

On changing the order of integration, we first integrate w.r.t. x from R to S, then w.r.t. y from $y = 0$ to $y = 4a$

$$\therefore \quad I = \int_0^{4a} \int_{y^2/4a}^{2\sqrt{ay}} dxdy = \int_0^{4a} dy \left| x \right|_{y^2/4a}^{2\sqrt{ay}} = \int_0^{4a} (2\sqrt{ay} - y^2/4a)\, dy$$

$$= \left| 2\sqrt{a}\, \frac{y^{3/2}}{3/2} - \frac{y^3}{12a} \right|_0^{4a} = \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}.$$

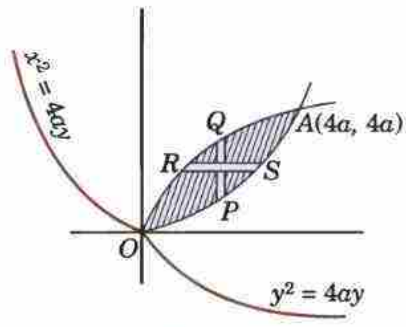


Fig. 7.8

Example 7.8. *Change the order of integration and hence evaluate*

$$I = \int_0^a \int_{\sqrt{ax}}^a \frac{y^2\, dxdy}{\sqrt{(y^4 - a^2x^2)}}$$

(*S.V.T.U., 2006 S*)

Solution. Here integration is first w.r.t. y from P on the parabola $y^2 = ax$ to Q on the line $y = a$, then w.r.t. x from $x = 0$ to $x = a$, giving the shaded region OAB of integration (Fig. 7.9).

On changing the order of integration, we first integrate w.r.t. x from R to S, then w.r.t. y from $y = 0$ to $y = a$.

$$\therefore \quad I = \int_0^a \int_0^{y^2/a} \frac{y^2\, dy}{\sqrt{(y^4 - a^2x^2)}}\, dx = \frac{1}{a} \int_0^a \int_0^{y^2/a} y^2\, dy \frac{dx}{\sqrt{[(y^2/a)^2 - x^2]}}\, dx$$

$$= \frac{1}{a} \int_0^a y^2 dy \left| \sin^{-1}\left(\frac{xa}{y^2}\right) \right|_0^{y^2/a} = \frac{1}{a} \int_0^a y^2 dy\ [\sin^{-1}(1) - \sin^{-1}(0)]$$

$$= \frac{\pi}{2a} \int_0^a y^2 dy = \frac{\pi}{2a} \left| \frac{y^3}{3} \right|_0^a = \frac{\pi a^2}{6}.$$

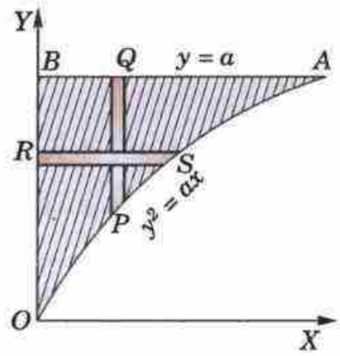


Fig. 7.9

Example 7.9. *Change the order of integration in* $I = \int_0^1 \int_{x^2}^{2-x} xy\, dxdy$ *and hence evaluate the same.*

(*Bhopal, 2008 ; V.T.U., 2008 ; S.V.T.U., 2007 ; P.T.U., 2005 ; U.P.T.U., 2005*)

Solution. Here the integration is first w.r.t. y along a vertical strip PQ which extends from P on the parabola $y = x^2$ to Q on the line $y = 2 - x$. Such a strip slides from $x = 0$ to $x = 1$, giving the region of integration as the curvilinear triangle OAB (shaded) in Fig. 7.10.

On changing the order of integration, we first integrate w.r.t. x along a horizontal strip $P'Q'$ and that requires the splitting up of the region OAB into two parts by the line AC ($y = 1$), *i.e.*, the curvilinear triangle OAC and the triangle ABC.

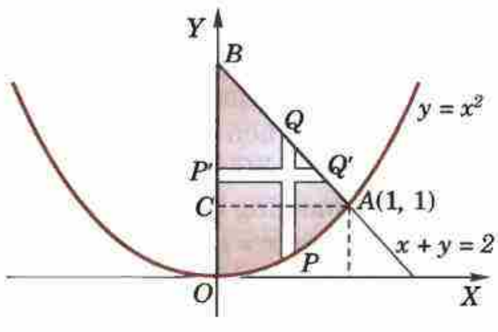


Fig. 7.10

For the region OAC, the limits of integration for x are from $x = 0$ to $x = \sqrt{y}$ and those for y are from $y = 0$ to $y = 1$. So the contribution to I from the region OAC is

$$I_1 = \int_0^1 dy \int_0^{\sqrt{y}} xy\, dx$$

For the region ABC, the limits of integration for x are from $x = 0$ to $x = 2 - y$ and those for y are from $y = 1$ to $y = 2$. So the contribution to I from the region ABC is

$$I_2 = \int_1^2 dy \int_0^{2-y} xy\, dx.$$

Hence, on reversing the order of integration,

$$I = \int_0^1 dy \int_0^{\sqrt{y}} xy\,dx + \int_1^2 dy \int_0^{2-y} xy\,dx$$

$$= \int_0^1 dy \left| \frac{x^2}{2} . y \right|_0^{\sqrt{y}} + \int_1^2 dy \left| \frac{x^2}{2} . y \right|_0^{2-y} = \frac{1}{2}\int_0^1 y^2 dy + \frac{1}{2}\int_1^2 y(2-y)^2\,dy = \frac{1}{6} + \frac{5}{24} = \frac{3}{8}.$$

Example 7.10. *Change the order of integration in* $I = \int_0^1 \int_x^{\sqrt{(2-x^2)}} \frac{x}{\sqrt{(x^2+y^2)}}\,dx$ *and hence evaluate it.*

(J.N.T.U., 2005 ; Rohtak, 2003)

Solution. Here the integration is first w.r.t. y along PQ which extends from P on the line $y = x$ to Q on the circle $y = \sqrt{(2 . x^2)}$. Then PQ slides from $y = 0$ to $y = 1$, giving the region of integration OAB as in Fig. 7.11.

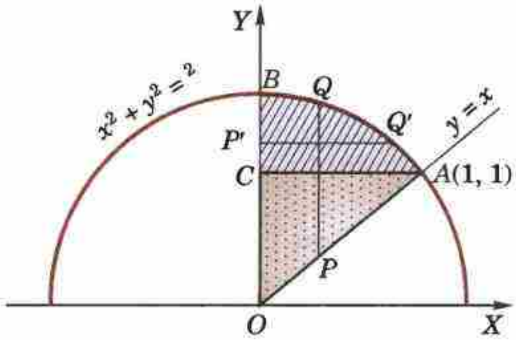


Fig. 7.11

On changing the order of integration, we first integrate w.r.t. x from P' to Q' and that requires splitting the region OAB into two parts OAC and ABC.

For the region OAC, the limits of integration for x are from $x = 0$ to $x = 1$ and those for y are from $y = 0$ to $y = 1$. So the contribution to I from the region OAC is

$$I_1 = \int_0^1 dy \int_0^y \frac{x}{\sqrt{(x^2+y^2)}}\,dx.$$

For the region ABC, the limits of integration for x are 0 to $\sqrt{(2-y^2)}$ and these for y are from 1 to $\sqrt{2}$. So the contribution to I from the region ABC is

$$I_2 = \int_1^{\sqrt{2}} dy \int_0^{\sqrt{(2-y^2)}} \frac{x}{\sqrt{(x^2+y^2)}}\,dx$$

Hence $$I = \int_0^1 \left| (x^2+y^2)^{1/2} \right|_0^y dy + \int_1^{\sqrt{2}} \left| (x^2+y^2)^{1/2} \right|_0^{\sqrt{(2-y^2)}} dy$$

$$= \int_0^1 (\sqrt{2}-1)\,y\,dy + \int_1^{\sqrt{2}} \sqrt{(2-y)}\,dy = \frac{1}{2}(\sqrt{2}-1) + \sqrt{2}\sqrt{(2-1)} - \frac{1}{2} = 1 - 1/\sqrt{2}.$$

7.3 DOUBLE INTEGRALS IN POLAR COORDINATES

To evaluate $\int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r, \theta)\,dr\,d\theta$, we first integrate w.r.t. r between limits $r = r_1$ and $r = r_2$ keeping θ fixed and the resulting expression is integrated w.r.t. θ from θ_1 to θ_2. In this integral, r_1, r_2 are functions of θ and θ_1, θ_2 are constants.

Figure 7.12 illustrates the process geometrically.

Here AB and CD are the curves $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$ bounded by the lines $\theta = \theta_1$ and $\theta = \theta_2$. PQ is a wedge of angular thickness $\delta\theta$.

Then $\int_{r_1}^{r_2} f(r, \theta)\,dr$ indicates that the integration is along PQ from P to Q while the integration w.r.t. θ corresponds to the turning of PQ from AC to BD.

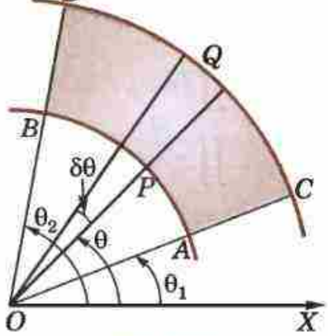


Fig. 7.12

Thus the whole region of integration is the area $ACDB$. The order of integration may be changed with appropriate changes in the limits.

Example 7.11. *Evaluate* $\iint r \sin\theta \, dr \, d\theta$ *over the cardioid* $r = a(1 - \cos\theta)$ *above the initial line.* *(Kerala, 2005)*

Solution. To integrate first w.r.t. r, the limits are from 0 $(r = 0)$ to P $[r = a(1 - \cos\theta)]$ and to cover the region of integration R, θ varies from 0 to π (Fig. 7.13).

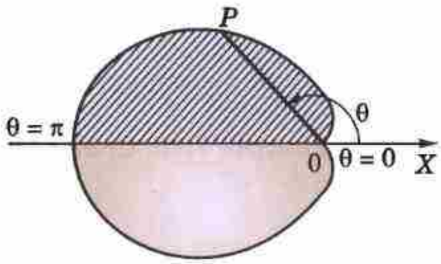


Fig. 7.13

$$\therefore \quad \iint_R r\sin\theta \, dr d\theta = \int_0^{\pi} \sin\theta \left[\int_0^{r = a(1-\cos\theta)} r dr \right] d\theta$$

$$= \int_0^{\pi} \sin\theta \, d\theta \left| \frac{r^2}{2} \right|_0^{a(1-\cos\theta)} = \frac{a^2}{2} \int_0^{\pi} (1 - \cos\theta)^2 \, . \sin\theta \, d\theta$$

$$= \frac{a^2}{2} \left| \frac{(1-\cos\theta)^3}{3} \right|_0^{\pi} = \frac{a^2}{2} \cdot \frac{8}{3} = \frac{4a^2}{3}.$$

Example 7.12. *Calculate* $\iint r^3 \, dr \, d\theta$ *over the area included between the circles* $r = 2\sin\theta$ *and* $r = 4\sin\theta$.

Solution. Given circles $r = 2\sin\theta$ ...(*i*)

and $r = 4\sin\theta$ ...(*ii*)

are shown in Fig. 7.14. The shaded area between these circles is the region of integration.

If we integrate first w.r.t. r, then its limits are from $P(r = 2\sin\theta)$ to $Q(r = 4\sin\theta)$ and to cover the whole region θ varies from 0 to π. Thus the required integral is

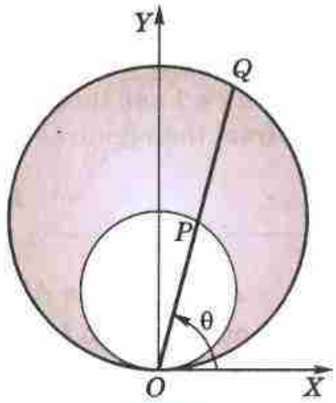


Fig. 7.14

$$I = \int_0^{\pi} d\theta \int_{2\sin\theta}^{4\sin\theta} r^3 \, dr = \int_0^{\pi} d\theta \left[\frac{r^4}{4} \right]_{2\sin\theta}^{4\sin\theta}$$

$$= 60 \int_0^{\pi} \sin^4\theta \, d\theta = 60 \times 2 \int_0^{\pi/2} \sin^4\theta \, d\theta = 120 \times \frac{3 . 1}{4 . 2} \cdot \frac{\pi}{2} = 22.5\,\pi.$$

PROBLEMS 7.1

Evaluate the following integrals (1–7) :

1. $\int_1^2 \int_1^3 xy^2 \, dxdy$.

2. $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2) \, dx \, dy$. *(V.T.U., 2000)*

3. $\int_0^1 \int_0^x e^{x/y} \, dxdy$. *(P.T.U., 2005)*

4. $\int_0^1 \int_0^{\sqrt{(1+x^2)}} \frac{dydx}{1 + x^2 + y^2}$ *(Rajasthan, 2005)*

5. $\iint xy \, dxdy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$. *(Rajasthan, 2006)*

6. $\iint (x + y)^2 dxdy$ over the area bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$. *(Kurukshetra, 2009 S ; U.P.T.U., 2004 S)*

7. $\iint xy(x + y) \, dxdy$ over the area between $y = x^2$ and $y = x$. *(V.T.U., 2010)*

Evaluate the following integrals by changing the order of integration (8–15) :

8. $\int_0^a \int_y^a \frac{xdxdy}{x^2 + y^2}$. *(Bhopal, 2008)*

9. $\int_0^3 \int_1^{\sqrt{(4-y)}} (x + y) \, dxdy$. *(V.T.U., 2005 ; Anna, 2003 S ; Delhi, 2002)*

10. $\int_0^1 \int_x^{\sqrt{(2-x^2)}} \frac{x\,dydx}{\sqrt{(x^2+y^2)}}$. (*P.T.U., 2010 ; Marathwada, 2008 ; U.P.T.U., 2006*)

11. $\int_0^{a/\sqrt{2}} \int_y^{\sqrt{(a^2-y^2)}} \log(x^2+y^2)\,dxdy\ (a>0)$.

12. $\int_0^1 \int_x^{\sqrt{x}} xy\,dydx$. (*V.T.U., 2010*)

13. $\int_0^a \int_{a-\sqrt{(a^2-y^2)}}^{a+\sqrt{(a^2-y^2)}} xy\,dx\,dy$. (*Anna, 2009*)

14. $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dydx$. (*Bhopal, 2009 ; S.V.T.U., 2009 ; V.T.U., 2007*)

15. $\int_0^\infty \int_0^x x e^{-x^2/y}\,dydx$. (*S.V.T.U., 2006 ; U.P.T.U., 2005 ; V.T.U., 2004*)

16. Sketch the region of integration of the following integrals and change the order of integrations,

(*i*) $\int_0^{2a} \int_{\sqrt{(2ax-x^2)}}^{\sqrt{(2ax)}} f(x)\,dxdy$ (*Rajasthan, 2006*) (*ii*) $\int_0^{ae^{\pi/4}} \int_{2\log(r/a)}^{\pi/2} f(r,\theta)\,r\,drd\theta$.

17. Show that $\iint_R r^2 \sin\theta\,drd\theta = 2a^2/3$, where R is the semi-circle $r = 2a\cos\theta$ above the initial line.

18. Evaluate $\iint \frac{r\,drd\theta}{\sqrt{a^2+r^2}}$ over one loop of the lemniscate $r^2 = a^2\cos 2\theta$. (*Rohtak, 2006 S ; P.T.U., 2005*)

19. Evaluate $\iint r^3\,drd\theta$ over the area bounded between the circles $r = 2\cos\theta$ and $r = 4\cos\theta$.

(*Anna, 2009 ; Madras, 2006*)

7.4 AREA ENCLOSED BY PLANE CURVES

(1) *Cartesian coordinates.*

Consider the area enclosed by the curves $y = f_1(x)$ and $y = f_2(x)$ and the ordinates $x = x_1$, $x = x_2$ [Fig. 7.15 (*a*)].

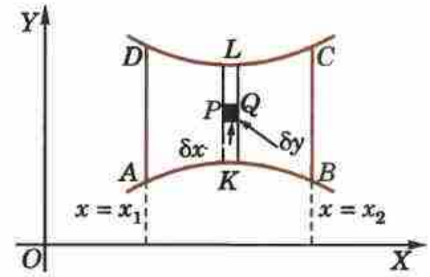


Fig. 7.15(*a*)

Divide this area into vertical strips of width δx. If $P(x, y)$, $Q(x+\delta x, y+\delta y)$ be two neighbouring points, then the area of the small rectangle $PQ = \delta x \delta y$.

$\therefore$ area of strip $KL = \underset{\delta y \to 0}{\text{Lt}} \Sigma\, \delta x\, \delta y$.

Since for all rectangles in this strip δx is the same and y varies from $y = f_1(x)$ to $y = f_2(x)$.

$$\therefore \quad \text{area of the strip } KL = \delta x \underset{\delta y \to 0}{\text{Lt}} \sum_{f_1(x)}^{f_2(x)} dy = \delta x \int_{f_1(x)}^{f_2(x)} dy.$$

Now adding up all such strips from $x = x_1$ to $x = x_2$, we get the area $ABCD$

$$= \underset{\delta x \to 0}{\text{Lt}} \sum_{x_1}^{x_2} \delta x \,.\, \int_{f_1(x)}^{f_2(x)} dy = \int_{x_1}^{x_2} dx \int_{f_1(x)}^{f_2(x)} dy = \int_{x_1}^{x_2} \int_{f_1(x)}^{f_2(x)} dxdy$$

Similarly, dividing the area $A'B'C'D$ [Fig. 7.15(*b*)] into horizontal strips of width δy, we get the area $A'B'C'D'$.

$$= \int_{y_1}^{y_2} \int_{f_1(y)}^{f_2(y)} dxdy$$

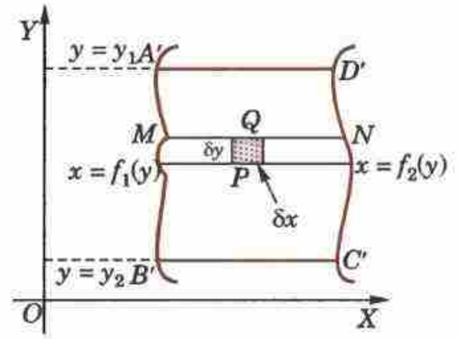


Fig. 7.15 (*b*)

(2) *Polar coordinates.*

Consider an area A enclosed by a curve whose equation is in polar coordinates.

Let $P(r, \theta)$, $Q(r+\delta r, \theta+\delta\theta)$ be two neighbouring points. Mark circular areas of radii r and $r + \delta r$ meeting OQ in R and OP (produced) in S (Fig. 7.16).

Since arc $PR = r\delta\theta$ and $PS = \delta r$.

$\therefore$ area of the curvilinear rectangle $PRQS$ is approximately $= PR \,.\, PS = r\delta\theta \,.\, \delta r$.

If the whole area is divided into such curvilinear rectangles, the sum $\Sigma\Sigma r\delta\theta\delta r$ taken for all these rectangles, gives in the limit the area A.

Hence $$A = \underset{\substack{\delta r \to 0 \\ \delta\theta \to 0}}{\text{Lt}} \Sigma\Sigma r\delta\theta\delta r = \iint r\,d\theta dr$$

where the limits are to be so chosen as to cover the entire area.

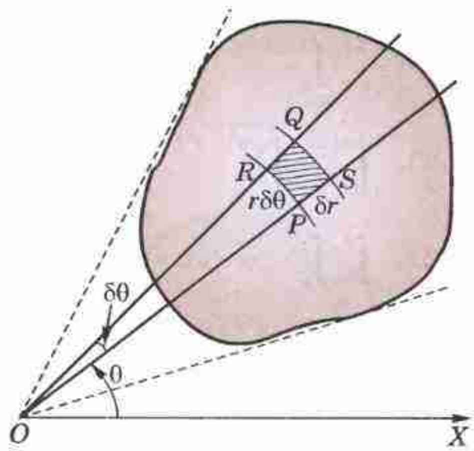


Fig. 7.16

Example 7.13. *Find the area of a plate in the form of a quadrant of the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$ *(V.T.U., 2001 ; Osmania, 2000 S)*

Solution. Dividing the area into vertical strips of width δx, y varies from $K(y = 0)$ to $L[y = b\sqrt{(1 - x^2/b^2)}]$ and then x varies from 0 to a (Fig. 7.17).

$\therefore$ required area

$$= \int_0^a dx \int_0^{b\sqrt{(1-x^2/a^2)}} dy = \int_0^a dx\,[y]_0^{b\sqrt{(1-x^2/a^2)}}$$

$$= \frac{b}{a}\int_0^a \sqrt{(a^2 - x^2)}\,dx = \pi ab/4.$$

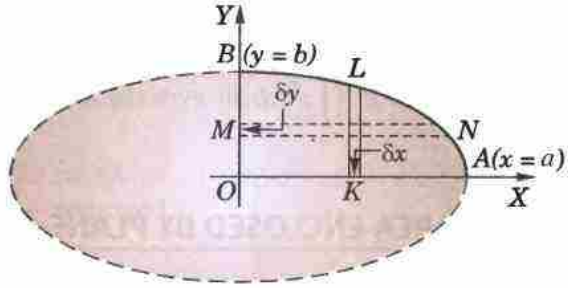


Fig. 7.17

Otherwise, dividing this area into horizontal strips of width δy, x varies from $M(x = 0)$ to $N[x = a\sqrt{(1 - y^2/b^2)}\,]$ and then y varies from 0 to b.

$$\therefore \text{ required area } = \int_0^b dy \int_0^{a\sqrt{(1-y^2/b^2)}} dx = \int_0^b dy\,[x]_0^{a\sqrt{(1-y^2/b^2)}}$$

$$= \frac{a}{b}\int_0^b \sqrt{(b^2 - y^2)}\,dy = \pi ab/4.$$

Obs. The change of the order of integration does not in any way affect the value of the area.

Example 7.14. *Show that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$.*

(Kerala, 2005 ; Rohtak, 2003)

Solution. Solving the equations $y^2 = 4ax$ and $x^2 = 4ay$, it is seen that the parabolas intersect at O (0, 0) and A ($4a$, $4a$). As such for the shaded area between these parabolas (Fig. 7.18) x varies from 0 to $4a$ and y varies from P to Q *i.e.*, from $y = x^2/4a$ to $y = 2\sqrt{(ax)}$. Hence the *required area*

$$= \int_0^{4a}\int_{x^2/4a}^{2\sqrt{(ax)}} dydx = \int_0^{4a} (2\sqrt{(ax)} - x^2/4a)\,dx$$

$$= \left| 2\sqrt{a}\,.\,\frac{2}{3}x^{3/2} - \frac{1}{4a}\cdot\frac{x^3}{3}\right|_0^{4a} = \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2.$$

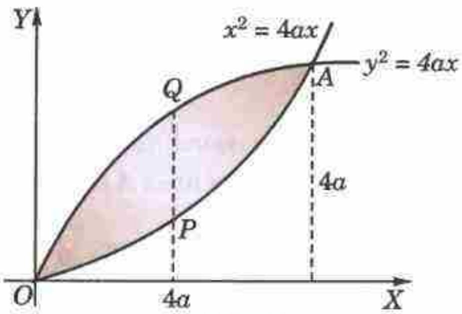


Fig. 7.18

Example 7.15. *Calculate the area included between the curve $r = a(\sec\theta + \cos\theta)$ and its asymptote.*

Solution. The curve is symmetrical about the initial line and has an asymptote $r = a\sec\theta$ (Fig. 7.19).

Draw any line OP cutting the curve at P and its asymptote at P'. Along this line, θ is constant and r varies from $a\sec\theta$ at P' to $a(\sec\theta + \cos\theta)$ at P. Then to get the upper half of the area, θ varies from 0 to $\pi/2$.

$$\therefore \quad \text{required area} = 2\int_0^{\pi/2}\int_{a\sec\theta}^{a(\sec\theta+\cos\theta)} r\,dr d\theta$$

$$= 2\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{a\sec\theta}^{a(\sec\theta+\cos\theta)} d\theta$$

$$= a^2\int_0^{\pi/2}(2+\cos^2\theta)\,d\theta = 5\pi a^2/4.$$

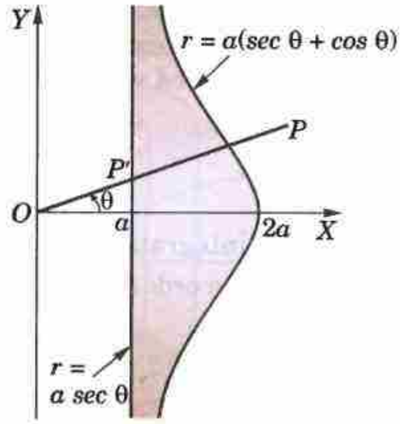


Fig. 7.19

Example 7.16. *Find the area lying inside the cardioid $r = a(1 + \cos\theta)$ and outside the circle $r = a$.*

Solution. In Fig. 7.20, $ABODA$ represents the cardioid $r = a(1 + \cos\theta)$ and $CBA'DC$ is the circle $r = a$.

Required area (shaded) = 2 (area $ABCA$)

$$= 2\int_0^{\pi/2}\int_{r=OP'}^{r=OP} r\,d\theta\,dr = 2\int_0^{\pi/2}\int_a^{a(1+\cos\theta)}(r dr)\,d\theta$$

$$= 2\int_0^{\pi/2}\left|\frac{r^2}{2}\right|_a^{a(1+\cos\theta)} d\theta = a^2\int_0^{\pi/2}[(1+\cos\theta)^2 - 1]\,d\theta$$

$$= a^2\int_0^{\pi/2}(\cos^2\theta + 2\cos\theta)\,d\theta = a^2\left(\frac{1}{2}\cdot\frac{\pi}{2}+2\right) = \frac{a^2}{4}(\pi+8).$$

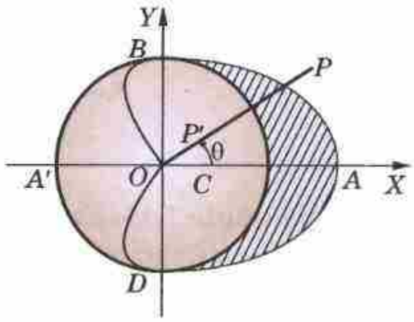


Fig. 7.20

PROBLEMS 7.2

1. Find, by double integration, the area lying between the parabola $y = 4x - x^2$ and the line $y = x$.
2. Find the area lying between the parabola $y = x^2$ and the line $x + y - 2 = 0$. *(Anna, 2009)*
3. By double integration, find the whole area of the curve $a^2x^2 = y^3(2a - y)$. *(U.P.T.U., 2001)*
4. Find, by double integration, the area enclosed by the curves $y = 3x/(x^2 + 2)$ and $4y = x^2$. *(J.N.T.U., 2005)*
5. Find, by double integration, the area of the lemniscate $r^2 = a^2\cos 2\theta$. *(Madras, 2000 S)*
6. Find, by double integration, the area lying inside the circle $r = a\sin\theta$ and outside the cardioid $r = a(1 - \cos\theta)$. *(Anna 2009 ; Mumbai, 2006)*
7. Find the area lying inside the cardioid $r = 1 + \cos\theta$ and outside the parabola $r(1 + \cos\theta) = 1$.
8. Find the area common to the circles $r = a\cos\theta$, $r = a\sin\theta$ by double integration. *(Mumbai, 2007)*

7.5 TRIPLE INTEGRALS

Consider a function $f(x, y, z)$ defined at every point of the 3-dimensional finite region V. Divide V into n elementary volumes $\delta V_1, \delta V_2, \ldots, \delta V_n$. Let (x_r, y_r, z_r) be any point within the rth sub-division δV_r. Consider the sum

$$\sum_{r=1}^{\infty} f(x_r, y_r, z_r)\delta V_r .$$

The limit of this sum, if it exists, as $n \to \infty$ and $\delta V_r \to 0$ is called the *triple integral of f(x, y, z) over the region V* and is denoted by

$$\iiint f(x, y, z)\,dV.$$

For purposes of evaluation, it can also be expressed as the repeated integral

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x, y, z)\,dxdydz.$$

If x_1, x_2 are constants ; y_1, y_2 are either constants or functions of x and z_1, z_2 are either constants or functions of x and y, then this integral is evaluated as follows :

First $f(x, y, z)$ is integrated w.r.t. z between the limits z_1 and z_2 keeping x and y fixed. The resulting expression is integrated w.r.t. y between the limits y_1 and y_2 keeping x constant. The result just obtained is finally integrated w.r.t. x from x_1 to x_2.

Thus $$I = \boxed{\int_{x_1}^{x_2} \boxed{\int_{y_1(x)}^{y_2(x)} \boxed{\int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)dz} dy} dx}$$

where the integration is carried out from the innermost rectangle to the outermost rectangle.

The order of integration may be different for different types of limits.

Example 7.17. *Evaluate* $\int_{-1}^{1} \int_{0}^{z} \int_{x-z}^{x+z} (x + y + z)\, dxdydz.$ *(J.N.T.U., 2006 ; Cochin, 2005)*

Solution. Integrating first w.r.t. y keeping x and z constant, we have

$$I = \int_{-1}^{1} \int_{0}^{z} \left| xy + \frac{y^2}{2} + yz \right|_{x-z}^{x+z} dx\, dz = \int_{-1}^{1} \int_{0}^{z} \left[(x+z)(2z) + \frac{1}{2} 4xz \right] dx\, dz$$

$$= 2 \int_{-1}^{1} \left| \frac{x^2 z}{2} + z^2 x + \frac{x^2}{2} z \right|_0^z dz = 2 \int_{-1}^{1} \left(\frac{z^3}{2} + z^3 + \frac{z^3}{2} \right) dz = 4 \left| \frac{z^4}{4} \right|_{-1}^{1} = 0.$$

Example 7.18. *Evaluate* $\int_0^1 \int_0^{\sqrt{(1-x^2)}} \int_0^{\sqrt{(1-x^2-y^2)}} xyz\, dxdydz.$ *(V.T.U., 2003 S)*

Solution. We have

$$I = \int_0^1 x \left[\int_0^{\sqrt{(1-x^2)}} y \left\{ \int_0^{\sqrt{(1-x^2-y^2)}} z\, dz \right\} dy \right] dx = \int_0^1 x \left\{ \int_0^{\sqrt{(1-x^2)}} y \,.\left| \frac{z^2}{2} \right|_0^{\sqrt{(1-x^2-y^2)}} dy \right\} dx$$

$$= \int_0^1 x \left\{ \int_0^{\sqrt{(1-x^2)}} y \,.\, \frac{1}{2}(1 - x^2 - y^2)\, dy \right\} dx = \frac{1}{2} \int_0^1 x \left| (1-x^2)\frac{y^2}{2} - \frac{y^4}{4} \right|_0^{\sqrt{(1-x^2)}} dx$$

$$= \frac{1}{8} \int_0^1 [(1-x^2)^2 \,.\, 2x - (1-x^2)^4 \,.\, x]\, dx = \frac{1}{8} \int_0^1 (x - 2x^3 + x^5)\, dx$$

$$= \frac{1}{8} \left| \frac{x^2}{2} - \frac{2x^4}{4} + \frac{x^6}{6} \right|_0^1 = \frac{1}{8}\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6} \right) = \frac{1}{48}.$$

PROBLEMS 7.3

Evaluate the following integrals :

1. $\int_0^a \int_0^b \int_0^c (x^2 + y^2 + z^2)\, dx\, dy\, dz.$ *(Anna, 2009)*
2. $\int_{-c}^{c} \int_{-b}^{b} \int_{-a}^{a} (x^2 + y^2 + z^2)\, dx\, dy\, dz$ *(S.V.T.U., 2009 ; V.T.U. 2000)*
3. $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\, dz\, dx\, dy$ *(Nagpur, 2009)*
4. $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx.$ *(V.T.U., 2010 ; Kurukshetra, 2009 S ; J.N.T.U., 2005)*
5. $\int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^{x+y+z}\, dx\, dy\, dz.$ *(Bhopal, 2008)*
6. $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z\, dz\, dx\, dy.$ *(S.V.T.U., 2008 ; Rohtak, 2005)*
7. $\int_0^{\pi/2} \int_0^{a \sin\theta} \int_0^{\frac{a^2 - r^2}{a}} r dz\, dr\, d\theta.$ *(V.T.U., 2009)*

7.6 VOLUMES OF SOLIDS

(1) Volumes as double integrals. Consider a surface $z = f(x, y)$. Let the orthogonal projection on XY-plane of its portion S' be the area S (Fig. 7.21).

Divide S into elementary rectangles of area $\delta x\ \delta y$ by drawing lines parallel to X and Y-axes. With each of these rectangles as base, erect a prism having its length parallel to OZ.

$\therefore$ volume of this prism between S and the given surface $z = f(x, y)$ is $z\delta x\delta y$.

Hence the volume of the solid cylinder on S as base, bounded by the given surface with generators parallel to the Z-axis.

$$= \underset{\substack{\delta x \to 0 \\ \delta y \to 0}}{\text{Lt}} \Sigma\Sigma\, z\delta x\delta y$$

$$= \iint z\, dx\, dy \quad \text{or} \quad \iint f(x, y)\, dx\, dy$$

where the integration is carried over the area S.

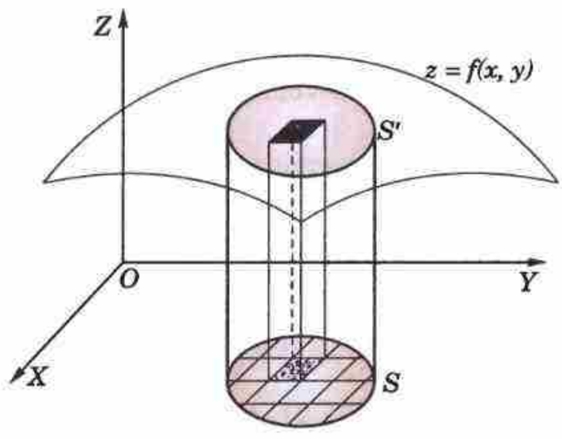


Fig. 7.21

Obs. While using polar coordinates, divide S into elements of area $r\delta\theta\ \delta r$.

$\therefore$ replacing $dxdy$ by $r\delta\theta\delta r$, we get the required volume $= \iint zr\, d\theta\, dr$.

Example 7.19. *Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.*
(S.V.T.U., 2007 ; Cochin, 2005 ; Madras, 2000 S)

Solution. From Fig. 7.22, it is self-evident that $z = 4 - y$ is to be integrated over the circle $x^2 + y^2 = 4$ in the XY-plane. To cover the shaded half of this circle, x varies from 0 to $\sqrt{(4 - y^2)}$ and y varies from -2 to 2.

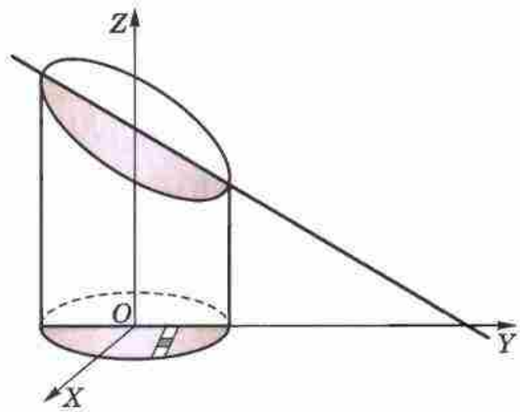


Fig. 7.22

$\therefore$ Required volume

$$= 2\int_{-2}^{2}\int_{0}^{\sqrt{(4-y^2)}} z\, dx\, dy = 2\int_{-2}^{2}\int_{0}^{\sqrt{(4-y^2)}} (4-y)\, dx\, dy$$

$$= 2\int_{-2}^{2} (4-y)\,[x]_0^{\sqrt{(4-y^2)}}\, dy = 2\int_{-2}^{2} (4-y)\sqrt{(4-y^2)}\, dy$$

$$= 2\int_{-2}^{2} 4\sqrt{(4-y^2)}\, dy - 2\int_{-2}^{2} y\sqrt{(4-y^2)}\, dy$$

$$= 8\int_{-2}^{2} \sqrt{(4-y^2)}\, dy$$ [The second term vanishes as the integrand is an odd function.]

$$= 8\left|\frac{y\sqrt{(4-y)^2}}{2} + \frac{4}{2}\sin^{-1}\frac{y}{2}\right|_{-2}^{2} = 16\pi.$$

(2) Volume as triple integral

Divide the given solid by planes parallel to the coordinate planes into rectangular parallelopipeds of volume $\delta x\ \delta y\ \delta z$ (Fig. 7.23).

$\therefore$ the total volume $= \underset{\substack{\delta x \to 0 \\ \delta y \to 0 \\ \delta z \to 0}}{\text{Lt}} \Sigma\Sigma\Sigma\, \delta x\ \delta y\ \delta z$

$$= \iiint dx\, dy\, dz$$

with appropriate limits of integration.

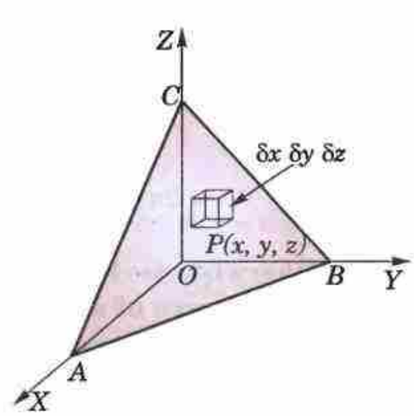


Fig. 7.23

Example 7.20. *Calculate the volume of the solid bounded by the planes $x = 0$, $y = 0$, $x + y + z = a$ and $z = 0$.* (P.T.U., 2009)

Solution. Volume required $= \int_0^a \int_0^{a-x} \int_0^{a-x-y} dz\, dy\, dx$

$$= \int_0^a \int_0^{a-x} (a - x - y)\, dy\, dx = \int_0^a \left| (a-x)y - \frac{y^2}{2} \right|_0^{a-x} dx$$

$$= \int_0^a \left\{ (a-x)^2 - \frac{(a-x)^2}{2} \right\} dx = \frac{1}{2}\int_0^a (a-x)^2\, dx = \frac{1}{2}\left| -\frac{(a-x)^3}{3} \right|_0^a = \frac{a^3}{6}.$$

Example 7.21. *Find the volume of the ellipsoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

(Anna, 2009 ; P.T.U., 2006 ; Kottayam, 2005)

Solution. Let $OABC$ be the positive octant of the given ellipsoid which is bounded by the planes OAB $(z = 0)$, OBC $(x = 0)$, OCA $(y = 0)$ and the surface ABC, *i.e.*,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Divide this region R into rectangular parallellopipeds of volume $\delta x \delta y \delta z$. Consider such an element at $P(x, y, z)$. (Fig. 7.24)

$\therefore$ the required volume $= 8 \iiint_R dx\, dy\, dz$.

In this region R,

(*i*) z varies from 0 to MN where

$$MN = c\sqrt{(1 - x^2/a^2 - y^2/b^2)}.$$

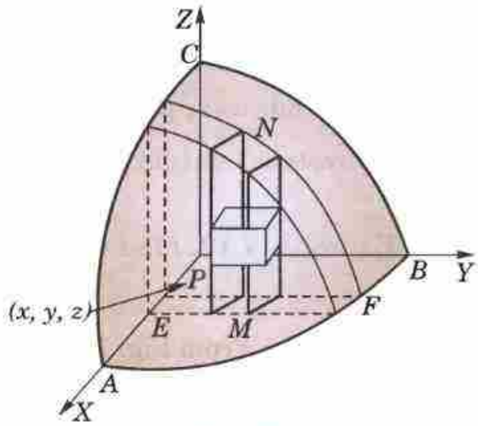


Fig. 7.24

(*ii*) y varies from 0 to EF, where $EF = b\sqrt{(1 - x^2/a^2)}$ from the equation of the ellipse OAB, *i.e.*,

$$x^2/a^2 + y^2/b^2 = 1.$$

(*iii*) x varies from 0 to $OA = a$.

Hence the volume of the whole ellipsoid

$$= 8\int_0^a \int_0^{b\sqrt{(1-x^2/a^2)}} \int_0^{c\sqrt{(1-x^2/a^2-y^2/b^2)}} dx\, dy\, dz = 8\int_0^a dx \int_0^{b\sqrt{(1-x^2/a^2)}} dy \,\Big|\, z \,\Big|_0^{c\sqrt{(1-x^2/a^2-y^2/b^2)}}$$

$$= 8c\int_0^a dx \int_0^{b\sqrt{(1-x^2/a^2)}} \sqrt{(1 - x^2/a^2 - y^2/b^2)}\, dy$$

$$= \frac{8c}{b}\int_0^a dx \int_0^{\rho} \sqrt{(\rho^2 - y^2)}\, dy \quad \text{when } \rho = b\sqrt{1 - x^2/a^2}.$$

$$= \frac{8c}{b}\int_0^a dx \left[\frac{y\sqrt{(\rho^2 - y^2)}}{2} + \frac{\rho^2}{2}\sin^{-1}\frac{y}{\rho} \right]_0^{\rho} = \frac{8c}{b}\int_0^a \frac{b^2}{2}\left(1 - \frac{x^2}{a^2}\right)\frac{\pi}{2}\, dx$$

$$= 2\pi bc \int_0^a \left(1 - \frac{x^2}{a^2}\right) dx = 2\pi bc \left| x - \frac{x^3}{3a^2} \right|_0^a = \frac{4\pi abc}{3}.$$

Otherwise. See Problem 27 page 292.

(3) Volumes of solids of revolution

Consider an elementary area $\delta x \delta y$ at the point $P(x, y)$ of a plane area A. (Fig. 7.25)

As this elementary area revolves about x-axis, we get a ring of volume

$$= \pi[(y + \delta y)^2 - y^2]\, \delta x = 2\pi y\, \delta x\, \delta y,$$

nearly to the first powers of δy.

Hence the total volume of the solid formed by the revolution of the area A about x-axis.

$$= \iint_A 2\pi y \, dxdy.$$

In polar coordinates, the above formula for the volume becomes

$$\iint_A 2\pi r \sin\theta \,.\, r d\theta dr, \text{ i.e. } \iint_A 2\pi r^2 \sin\theta \, d\theta dr$$

Similarly, the volume of the solid formed by the revolution of the area A about y-axis $= \iint_A 2\pi x \, dx \, dy$.

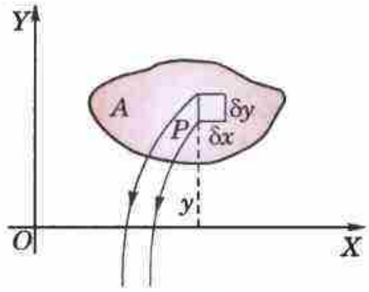


Fig. 7.25

Example 7.22. *Calculate by double integration, the volume generated by the revolution of the cardioid $r = a(1 - \cos\theta)$ about its axis.*

Solution. Required volume

$$= \int_0^{\pi} \int_0^{a(1-\cos\theta)} 2\pi r^2 \sin\theta \, dr \, d\theta$$

$$= 2\pi \int_0^{\pi} \left| \frac{r^3}{3} \right|_0^{a(1-\cos\theta)} \sin\theta \, d\theta$$

$$= \frac{2\pi a^3}{3} \int_0^{\pi} (1 - \cos\theta)^3 \,.\, \sin\theta \, d\theta = \frac{2\pi a^3}{3} \left| \frac{(1-\cos\theta)^4}{4} \right|_0^{\pi} = \frac{8\pi a^3}{3}.$$

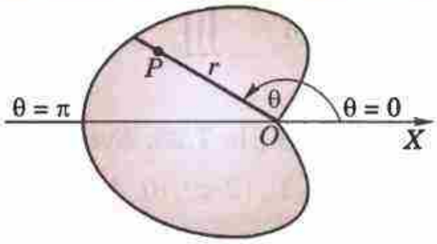


Fig. 7.26

7.7 CHANGE OF VARIABLES

An appropriate choice of co-ordinates quite often facilitates the evaluation of a double or a triple integral. By changing the variables, a given integral can be transformed into a simpler integral involving the new variables.

(1) **In a double integral,** let the variables x, y be changed to the new variables u, v by the transformation.

$$x = \phi(u, v), \; y = \psi(u, v)$$

where $\phi(u, v)$ and $\psi(u, v)$ are continuous and have continuous first order derivatives in some region R'_{uv} in the uv-plane which corresponds to the region R_{xy} in the xy-plane. Then

$$\iint_{R_{xy}} f(x, y) \, dxdy = \iint_{R'_{uv}} f[\phi(u, v), \psi(u, v)] \, |J| \, dudv \qquad ...(1)$$

where $$J = \frac{\partial(x, y)}{\partial(u, v)} \; (\neq 0)$$

is the *Jacobian of transformation** from (x, y) to (u, v) coordinates.

(2) **For triple integrals,** the formula corresponding to (1) is

$$\iiint_{R_{xyz}} f(x, y, z) \, dxdydz = \iiint_{R'_{uvw}} f[x(u, v, w), y(u, v, w), z(u, v, w)] \, |J| \, dudvdw$$

where $$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} \; (\neq 0)$$

is the *Jacobian of transformation* from (x, y, z) to (u, v, w) coordinates.

Particular cases :

(*i*) ***To change cartesian coordinates (x, y) to polar coordinates (r, θ),*** *we have* $x = r\cos\theta, y = r\sin\theta$ *and*

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = r \qquad \text{[Ex. 5.25, p. 216]}$$

$$\therefore \quad \iint_{R_{xy}} f(x, y) \, dx \, dy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta) \,.\, r \, dr \, d\theta.$$

* See footnote page 215.

(*ii*) ***To change rectangular coordinates (x, y, z) to cylindrical coordinates*** (ρ, ϕ, z) *— Fig. 8.27, we have*

$$x = \rho \cos \phi, y = \rho \sin \phi, z = z$$

and $$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho \qquad \text{[Ex. 5.25]}$$

Then $$\iiint_{R_{xyz}} f(x, y, z)\, dxdydz = \iiint_{R'_{\rho\phi z}} f(\rho \cos \phi, \rho \sin \phi, z) \,.\, \rho d\rho d\phi dz.$$

(*iii*) ***To change rectangular coordinates (x, y, z) to spherical polar coordinates (r,*** θ, ϕ***)****—Fig. 8.28, we have*

$$x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta$$

and $$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin \theta \qquad \text{[Ex. 5.25]}$$

Then $$\iiint_{R_{xyz}} f(x,y, z)\, dxdydz = \iiint_{R'_{r\theta\phi}} f(r \sin \theta \cos \phi, r \sin \theta, \sin \phi, r \cos \theta) \,.\, r^2 \sin \theta \, dr d\theta d\phi$$

Example 7.23. *Evaluate* $\iint_R (x + y)^2 \, dxdy$, *where R is the parallelogram in the xy-plane with vertices (1, 0), (3, 1), (2, 2), (0, 1) using the transformation u = x + y and v = x − 2y.* (*U.P.T.U., 2004*)

Solution. The region R, *i.e.*, parallelogram $ABCD$ in the xy-plane becomes the region R', *i.e.*, rectangle $A'B'C'D'$ in the uv-plane as shown in Fig. 7.27, by taking

$$u = x + y \quad \text{and} \quad v = x - 2y \qquad ...(i)$$

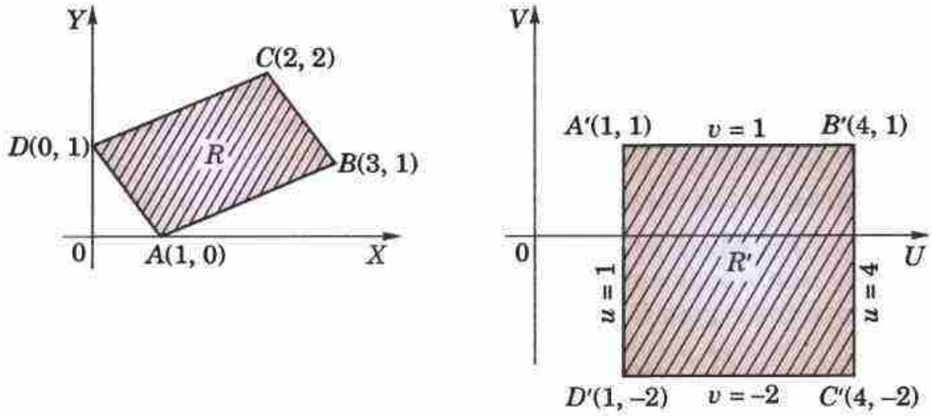


Fig. 7.27

From (*i*), we have $$x = \frac{1}{3}(2u + v), y = \frac{1}{3}(u - v)$$

$$\therefore \qquad \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} = -\frac{1}{3}$$

Hence, the given integral

$$= \iint_{R'} u^2 \,|\, J \,|\, dudv = \int_1^4 \int_{-2}^1 u^2 \,.\, \frac{1}{3} \,.\, dudv = \frac{1}{3} \left| \frac{u^3}{3} \right|_1^4 \,.\, |\, v \,|_{-2}^1 = 21.$$

Example 7.24. *Evaluate* $\iint_D xy\sqrt{(1 - x - y)}\, dx\, dy$ *where D is the region bounded by x = 0, y = 0 and x + y = 1 using the transformation x + y = u, y = uv.* (*Marathwada, 2008*)

Solution. We have $x = u - uv$, $y = uv$

$$\therefore \qquad J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 1-v & v \\ -u & u \end{vmatrix} = u.$$

Also when $x = 0$, $u = 0$, $v = 1$; when $y = 0$, $u = 0$, $v = 0$ and when $x + y = 1$, $u = 1$

$\therefore$ the limits of u are from 0 to 1 and limits of v are from 0 to 1.

Thus $$\iint_D xy\sqrt{(1-x-y)}\, dxdy = \int_0^1 \int_0^1 u(1-v)\, uv\, (1-u)^{1/2}\, |J|\, dudv$$

$$= \int_0^1 \int_0^1 u^3 (1-u)^{1/2}\, v(1-v)\, du\, dv$$

where $u = \sin^2 \theta$
$du = 2 \sin\theta \cos\theta\, d\theta$
$u = 0, \theta = 0$
$u = 1, \theta = \pi/2$

$$= \int_0^1 u^3 (1-u)^{1/2}\, du \times \int_0^1 v(1-v)\, dv$$

$$= \int_0^{\pi/2} \sin^6\theta \cos\theta \,.\, 2\sin\theta\cos d\theta \times \left| \frac{v^2}{2} - \frac{v^3}{3} \right|_0^1$$

$$= 2\int_0^{\pi/2} \sin^7\theta \cos^2\theta\, d\theta \left(\frac{1}{6}\right) = \frac{1}{3} \cdot \frac{6\,.\,1}{9\,.\,7\,.\,5\,.\,3} = \frac{2}{945}.$$

Example 7.25. *Evaluate* $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dxdy$ *by changing to polar coordinates.* (Anna, 2003)

Hence show that $\int_0^\infty e^{-x^2}\, dx = \sqrt{\pi/2}$. (Madras, 2003 ; U.P.T.U., 2003 ; J.N.T.U., 2000)

Solution. The region of integration being the first quadrant of the xy-plane, r varies from 0 to ∞ and θ varies from 0 to $\pi/2$. Hence,

$$I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dxdy = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2}\, rdrd\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2} \left\{ \int_0^\infty e^{-r^2} (-2r)\, dr \right\} d\theta = -\frac{1}{2}\int_0^{\pi/2} \left| e^{-r^2} \right|_0^\infty d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4}. \qquad ...(i)$$

Also $$I = \int_0^\infty e^{-x^2}\, dx \times \int_0^\infty e^{-y^2}\, dy = \left\{ \int_0^\infty e^{-x^2}\, dx \right\}^2 \qquad ...(ii)$$

Thus, from (*i*) and (*ii*), we have $\int_0^\infty e^{-x^2} = \sqrt{\pi/2}$. ...(*iii*)

Example 7.26. *Find the volume bounded by the paraboloid $x^2 + y^2 = az$, the cylinder $x^2 + y^2 = 2ay$ and the plane $z = 0$.*

Solution. The required volume is found by integrating $z = (x^2 + y^2)/a$ over the circle $x^2 + y^2 = 2ay$.

Changing to polar coordinates in the xy-plane, we have $x = r\cos\theta$, $y = r\sin\theta$ so that $z = r^2/a$ and the polar equation of the circle is $r = 2a\sin\theta$.

To cover this circle, r varies from 0 to $2a\sin\theta$ and θ varies from 0 to π. (Fig. 7.28)

Hence the required volume

$$= \int_0^\pi \int_0^{2a\sin\theta} z\,.\, r\, d\theta\, dr = \frac{1}{a}\int_0^\pi d\theta \int_0^{2a\sin\theta} r^3\, dr$$

$$= \frac{1}{a}\int_0^\pi d\theta \left| \frac{r^4}{4} \right|_0^{2a\sin\theta} = 4a^3 \int_0^\pi \sin^4\theta\, d\theta = \frac{3\pi a^3}{2}.$$

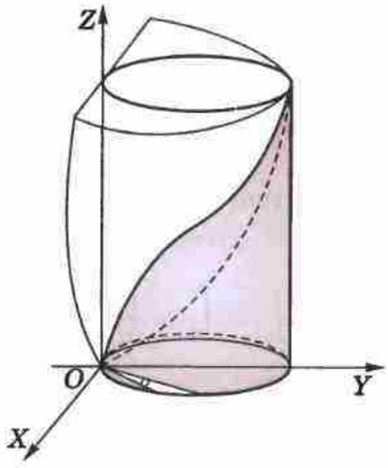


Fig. 7.28

Example 7.27. *Find, by triple integration, the volume of the sphere $x^2 + y^2 + z^2 = a^2$.*
(*Bhopal, 2009 ; Madras, 2006 ; V.T.U., 2003 S*)

Solution. Changing to polar spherical coordinates by putting

$$x = r \sin\theta \cos\phi,\ y = r \sin\theta \sin\phi,\ z = r\cos\theta$$

we have $dx\,dy\,dz = r^2 \sin\theta\, dr\, d\theta\, d\phi$.

Also the volume of the sphere is 8 times the volume of its portion in the positive octant for which r varies from 0 to a, θ varies from 0 to $\pi/2$ and ϕ varies from 0 to $\pi/2$.

$\therefore$ volume of the sphere

$$= 8 \int_0^a \int_0^{\pi/2} \int_0^{\pi/2} r^2 \sin\theta\, dr\, d\theta\, d\phi = 8 \int_0^a r^2\, dr \,.\int_0^{\pi/2} .\sin\theta\, d\theta \,.\int_0^{\pi/2} d\phi$$

$$= 8 \,.\left|\frac{r^3}{3}\right|_0^a .\left| -\cos\theta \right|_0^{\pi/2} .\frac{\pi}{2} = 4\pi \,.\frac{a^3}{3} .(-0+1) = \frac{4}{3}\pi a^3.$$

Example 7.28. *Find the volume of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ lying inside the cylinder $x^2 + y^2 = ay$.* (*Rohtak, 2003*)

Solution. The required volume is easily found by changing to cylindrical coordinates (ρ, ϕ, z). We therefore, have

$$x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = z$$

and
$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho.$$

Then the equation of the sphere becomes $\rho^2 + z^2 = a^2$ and that of cylinder becomes $\rho = a\sin\phi$.

The volume inside the cylinder bounded by the sphere is twice the volume shown shaded in the Fig. 7.29 for which z varies from 0 to $\sqrt{(a^2-\rho^2)}$, ρ varies from 0 to $a\sin\phi$ and ϕ varies from 0 to π.

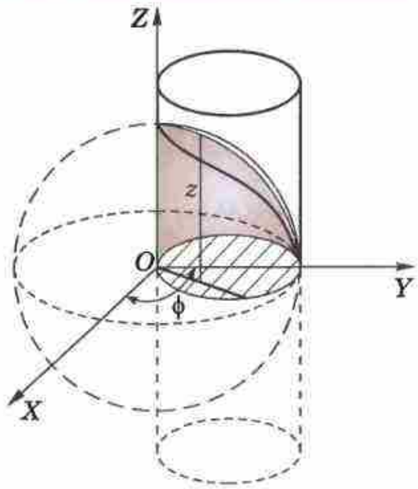


Fig. 7.29

Hence the required volume $= 2\int_0^{\pi} \int_0^{a\sin\phi} \int_0^{\sqrt{(a^2-\rho^2)}} \rho\, dz\, d\rho\, d\phi$

$$= 2\int_0^{\pi}\int_0^{a\sin\phi} \rho\sqrt{(a^2-\rho^2)}\, d\rho\, d\phi = 2\int_0^{\pi}\left| -\frac{1}{3}(a^2-\rho^2)^{3/2}\right|_0^{a\sin\phi} d\phi$$

$$= \frac{2a^3}{3}\int_0^{\pi}(1-\cos^3\phi)\, d\phi = \frac{2a^3}{9}(3\pi - 4).$$

Example 7.29. *Evaluate* $\displaystyle\int_0^1 \int_0^{\sqrt{(1-x^2)}} \int_{\sqrt{(x^2+y^2)}}^1 \frac{dz\, dy\, dx}{\sqrt{(x^2+y^2+z^2)}}$. (*V.T.U., 2008*)

Solution. We change to spherical polar coordinates (r, θ, ϕ), so that

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$$

and
$$J = r^2\sin\theta,\ x^2 + y^2 + z^2 = r^2.$$

The region of integration is common to the cone $z^2 = x^2 + y^2$ and the cylinder $x^2 + y^2 = 1$ bounded by the plane $z = 1$ in the positive octant (Fig. 7.30). Hence θ varies from 0 to $\pi/4$, r varies from 0 to $\sec\theta$ and ϕ varies from 0 to $\pi/2$.

$\therefore$ given integral becomes

$$\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sec\theta} \frac{1}{r} .r^2 \sin\theta\, dr\, d\theta\, d\phi = \int_0^{\pi/2} d\phi \int_0^{\pi/4} \left|\frac{r^2}{2}\right|_0^{\sec\theta} \sin\theta\, d\theta$$

$$= \frac{\pi}{2}\int_0^{\pi/4} \frac{\sec^2\theta}{2}\sin\theta\, d\theta = \frac{\pi}{4}\int_0^{\pi/4} \sec\theta\tan\theta\, d\theta = \frac{\pi}{4}\left|\sec\theta\right|_0^{\pi/4} = \frac{(\sqrt{2}-1)\pi}{4}.$$

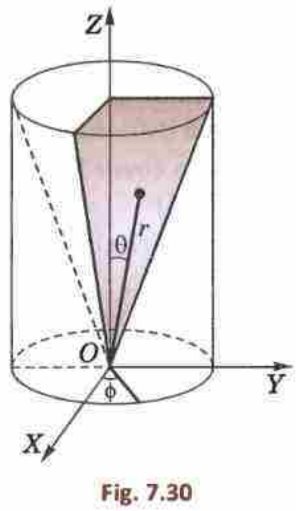


Fig. 7.30

Example 7.30. *Find the volume of the solid surrounded by the surface*

$$(x/a)^{2/3} + (y/b)^{2/3} + (z/c)^{2/3} = 1.$$ *(Hissar, 2005 S)*

Solution. Changing the variables, x, y, z to X, Y, Z where, $(x/a)^{1/3} = X$, $(y/b)^{1/3} = Y$, $(z/c)^{1/3} = Z$

i.e., $x = aX^3, y = bY^3, z = cZ^3$ so that $J = \partial (x, y, z)/\partial (X, Y, Z) = 27\ abc\ X^2Y^2Z^2$.

$$\therefore \text{ required volume} = \iiint dx\,dy\,dz = 27\,abc \iiint X^2Y^2Z^2\, dX\,dY dZ$$

taken throughout the sphere $X^2 + Y^2 + Z^2 = 1$. ...(*i*)

Now change X, Y, Z to spherical polar coordinates r, θ, ϕ so that $X = r \sin\theta \cos\phi$, $Y = r \sin\theta \sin\phi$, $Z = r\cos\theta$, and $\partial (X, Y, Z)/\partial (r, \theta, \phi) = r^2 \sin\theta$. To describe the positive octant of the sphere (*i*), r varies from 0 to 1, θ from 0 to $\pi/2$ and ϕ from 0 to $\pi/2$.

$$\therefore \text{ required volume} = 27abc \times 8 \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} r^2 \sin^2\theta \cos^2\phi \times r^2 \sin^2\theta \sin^2\phi \,.\, r^2 \cos^2\theta \,.\, r^2 \sin\theta\, dr d\theta d\phi$$

$$= 216\,abc \int_0^1 r^8\, dr \int_0^{\pi/2} \sin^4\theta \cos^2\theta\, d\theta \int_0^{\pi/2} \sin^2\phi \cos^2\phi\, d\phi = 4\pi\, abc/35.$$

PROBLEMS 7.4

Evaluate the following integrals by changing to polar co-ordinates :

1. $\int_0^1 \int_0^{\sqrt{(1-y^2)}} (x^2 + y^2)\, dy\, dx.$ *(P.T.U., 2010)*
2. $\int_0^2 \int_0^{\sqrt{(2x-x^2)}} \frac{x\,dx\,dy}{x^2+y^2}$ *(Anna, 2009)*
3. $\int_0^{4a} \int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\, dxdy$ *(Mumbai, 2006)*
4. $\iint xy(x^2 + y^2)^{n/2}\, dxdy$ over the positive quadrant of $x^2 + y^2 = 4$, supposing $n + 3 > 0$. *(S.V.T.U., 2007)*
5. $\iint \frac{dxdy}{(1+x^2+y^2)^2}$ over one loop of the lemniscate $(x^2 + y^2) = x^2 - y^2$. *(Mumbai, 2007)*
6. Transform the following to cartesian form and hence evaluate $\int_0^{\pi} \int_0^{a} r^3 \sin\theta \cos\theta\, dr d\theta$. *(P.T.U., 2005)*
7. $\iint y^2\, dx\, dy$ over the area outside $x^2 + y^2 - ax = 0$ and inside $x^2 + y^2 - 2ax = 0$. *(Mumbai, 2006)*
8. By using the transformation $x + y = u$, $y = uv$, show that $\int_0^1 \int_0^{1-x} e^{y/(x+y)}\, dydx = \frac{1}{2}(e-1)$. *(P.T.U., 2003)*
9. Transform $\int_0^{\pi/2} \int_0^{\pi/2} \sqrt{\frac{\sin\phi}{\sin\theta}}\, d\phi\, d\theta$ by the substitution $x = \sin\phi \cos\theta$, $y = \sin\phi \sin\theta$ and show that its value is π. *(U.P.T.U., 2001)*

Evaluate the following integrals by changing to spherical coordinates :

10. $\int_0^1 \int_0^{\sqrt{(1-x^2)}} \int_0^{\sqrt{(1-x^2-y^2)}} \frac{dx\,dy\,dz}{\sqrt{(1-x^2-y^2-z^2)}}$. *(V.T.U., 2006 ; Kottayam, 2005)*
11. $\iiint_V \frac{dx\,dy\,dz}{x^2+y^2+z^2}$ where V is the volume of the sphere $x^2 + y^2 + z^2 = a^2$. *(Anna, 2009)*
12. Evaluate $\iiint \frac{dx\,dy\,dz}{(1+x+y+z)^3}$ over the volume of the tetrahedron $x = 0, y = 0, z = 0, x + y + z = 1$. *(Mumbai, 2007)*
13. Show that $\iiint \frac{dx\,dy\,dz}{\sqrt{(a^2-x^2-y^2-z^2)}} = \frac{\pi^2 a^3}{8}$, the integral being extended for all the values of the variables for which the expression is real. *(U.T.U., 2010)*
14. $\iiint z^2\, dx\,dy\,dz$, taken over the volume bounded by the surfaces $x^2 + y^2 = a^2$, $x^2 + y^2 = z$ and $z = 0$.

15. Find the volume bounded by the xy-plane, the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 3$. *(I.S.M., 2001)*
16. Find the volume bounded by the xy-plane, the paraboloid $2z = x^2 + y^2$ and the cylinder $x^2 + y^2 = 4$. *(Raipur, 2005)*
17. Find the volume cut from the sphere $x^2 + y^2 + z^2 = a^2$ by the cone $x^2 + y^2 = z^2$.
18. Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$. *(S.V.T.U., 2006)*
19. Find the volume cut off from the cylinder $x^2 + y^2 = ax$ by the planes $z = 0$ and $z = x$. *(U.P.T.U., 2006)*
20. Find the volume enclosed by the cylinders $x^2 + y^2 = 2ax$ and $z^2 = 2ax$. *(Marathwada, 2008)*
21. Find the volume of the cylinder $x^2 + y^2 - 2ax = 0$, intercepted between the paraboloid $x^2 + y^2 = 2az$ and the xy-plane.
22. Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the hyperboloid $x^2 + y^2 - z^2 = 1$.
23. Find the volume of the region bounded by $z = x^2 + y^2$, $z = 0$, $x = -a$, $x = a$ and $y = -a$, $y = a$.
24. Prove, by using a double integral that the volume generated by the revolution of the cardioid $r = a(1 + \cos\theta)$ about its axis is $8\pi a^3/3$. *(V.T.U., 2000)*
25. Evaluate $\iiint (x + y + z)\, dx\, dy\, dz$ over the tetrahedron bounded by the planes $x = 0, y = 0, z = 0$ and $x + y + z = 1$. [See Fig. 7.34]
26. Find the volume of the tetrahedron bounded by the coordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. *(Burdwan, 2003)*
27. Work out example 7.21 by changing the variables.

7.8 AREA OF A CURVED SURFACE

Consider a point P of the surface $S : z = f(x, y)$. Let its projection on the xy-plane be the region A. Divide it into area elements by drawing lines parallel to the axes of X and Y. (Fig. 7.31).

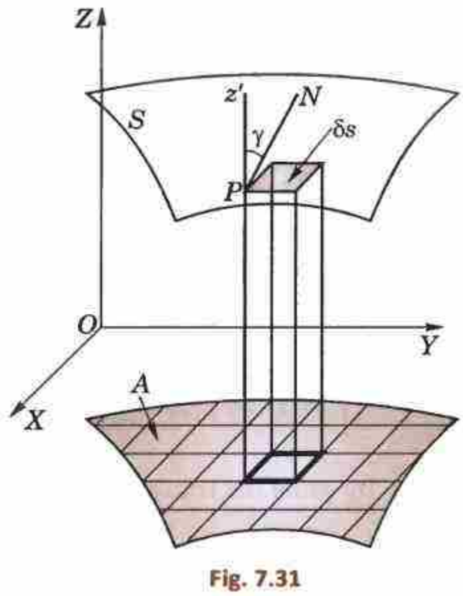


Fig. 7.31

On the element $\delta x \delta y$ as base, erect a cylinder having generators parallel to OZ and meeting the surface S in an element of area δS.

As $\delta x \delta y$ is the projection of δS on the xy-plane,

$\therefore$ $\delta x \delta y = \delta S \,.\, \cos\gamma$, where γ is the angle between the xy-plane and the tangent plane to S at P, *i.e.*, it is the angle between the Z-axis and the normal to S at $P (= \angle Z'PN)$.

Now since the direction cosines of the normal to the surface $F(x, y, z) = 0$ proportional to

$$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}.$$

$\therefore$ the direction cosines of the normal to $S[F = f(x, y) - z]$ are proportional to $-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}$, 1 and those of the z-axis are 0, 0, 1.

Hence $$\cos\gamma = \frac{1}{\sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}} \quad \therefore \quad \delta S = \frac{\delta x\, \delta y}{\cos\gamma} = \sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}\, \delta x\, \delta y$$

Hence $$S = \underset{\delta S \to 0}{\text{Lt}} \sum \delta S = \iint_A \sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}\, dxdy$$

Similarly, if B and C be the projections of S on the yz-and zx-planes respectively, then

$$S = \iint_B \sqrt{\left[\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 + 1\right]}\, dy\, dz$$

and $$S = \iint_C \sqrt{\left[\left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial x}\right)^2 + 1\right]}\, dz\, dx.$$

Example 7.31. *Find the area of the portion of the cylinder $x^2 + z^2 = 4$ lying inside the cylinder $x^2 + y^2 = 4$.*

Solution. Figure 7.32 shows one-eighth of the required area. Its projection on the xy-plane is a quadrant circle $x^2 + y^2 = 4$.

For the cylinder $\qquad x^2 + z^2 = 4, \qquad$ ...(*i*)

we have $$\frac{\partial z}{\partial x} = -\frac{x}{z}, \frac{\partial z}{\partial y} = 0.$$

so that $$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 = \frac{x^2 + z^2}{z^2} = \frac{4}{4 - x^2}.$$

Hence the required surface area = 8 (surface area of the upper portion of (*i*) lying within the cylinder $x^2 + y^2 = 4$ in the positive octant)

$$= 8 \int_0^2 \int_0^{\sqrt{(4-x^2)}} \frac{2}{\sqrt{(4 - x^2)}} \, dxdy = 16 \int_0^2 dx = 32 \text{ sq. units.}$$

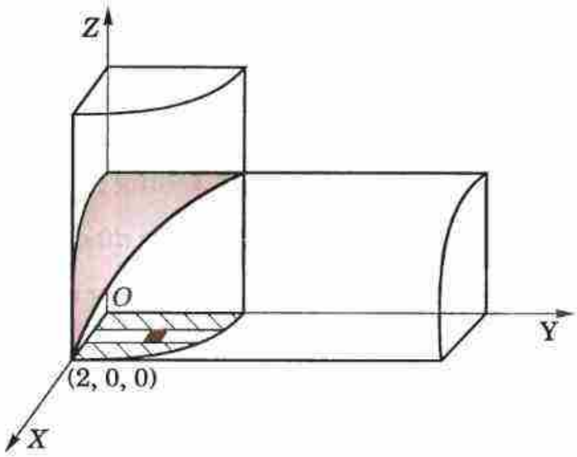


Fig. 7.32

Example 7.32. *Find the area of the portion of the sphere $x^2 + y^2 + z^2 = 9$ lying inside the cylinder $x^2 + y^2 = 3y$.*

Solution. Figure 7.33 shows one-fourth of the required area. Its projection on the xy-plane is the semi-circle $x^2 + y^2 = 3y$ bounded by the Y-axis.

For the sphere

$$x^2 + y^2 + z^2 = 9, \ \frac{\partial z}{\partial x} = -\frac{x}{z} \text{ and } \frac{\partial z}{\partial y} = -\frac{y}{z}$$

$$\therefore \qquad \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 = (x^2 + y^2 + z^2)/z^2$$

$$= \frac{9}{9 - x^2 - y^2} = \frac{9}{9 - r^2} \qquad \text{when} \quad x = r\cos\theta, y = r\sin\theta.$$

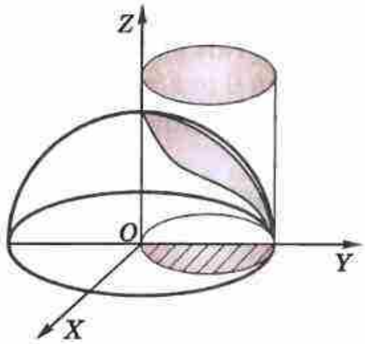


Fig. 7.33

Using polar coordinates, the required area is found by integrating $3/\sqrt{(9 - r^2)}$ over the semi-circle $r = 3\sin\theta$, for which r varies from 0 to $3\sin\theta$ and θ varies from 0 to $\pi/2$.

Hence the required surface area

$$= 4 \int_0^{\pi/2} \int_0^{3\sin\theta} \frac{3}{\sqrt{(9 - r^2)}} \, r d\theta \, dr = -6 \int_0^{\pi/2} \left| \frac{\sqrt{(9 - r^2)}}{1/2} \right|_0^{3\sin\theta} d\theta$$

$$= 36 \int_0^{\pi/2} (1 - \cos\theta) \, d\theta = 36 \left| \theta - \sin\theta \right|_0^{\pi/2} = 18(\pi - 2) \text{ sq. units.}$$

PROBLEMS 7.5

1. Show that the surface area of the sphere $x^2 + y^2 + z^2 = a^2$ is $4\pi a^2$.
2. Find the area of the portion of the cylinder $x^2 + y^2 = 4y$ lying inside the sphere $x^2 + y^2 + z^2 = 16$.
3. Find the area of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ lying inside the cylinder $x^2 + y^2 = ax$.
4. Find the area of the surface of the cone $x^2 + y^2 = z^2$ cut off by the surface of the cylinder $x^2 + y^2 = a^2$ above the xy-plane.
5. Compute the area of that part of the plane $x + y + z = 2a$ which lies in the first octant and is bounded by the cylinder $x^2 + y^2 = a^2$. (*Burdwan, 2003*)

7.9 CALCULATION OF MASS

(*a*) **For a plane lamina,** if the surface density at the point $P(x, y)$ be $\rho = f(x, y)$ then the elementary mass at $P = \rho\delta x\delta y$.

$$\therefore \quad \textit{total mass of the lamina} = \iint \rho \, dx \, dy \qquad ...(i)$$

with integrals embracing the whole area of the lamina.

In polar coordinates, taking $\rho = \phi(r, \theta)$ at the point $P(r, \theta)$,

$$\textit{total mass of the lamina} = \iint \rho r \, d\theta \, dr \qquad ...(ii)$$

(*b*) **For a solid,** if the density at the point $P(x, y, z)$ be $\rho = f(x, y, z)$, then

total mass of the solid $= \iiint \rho \, dx \, dy \, dz$ with appropriate limits of integration.

Example 7.33. *Find the mass of the tetrahedron bounded by the coordinates planes and the plane* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, *the variable density* $\rho = \mu xyz$. *(Rohtak, 2003 ; U.P.T.U., 2003)*

Solution. Elementary mass at $P = \mu xyz \,.\, \delta x \, \delta y \, \delta z$.

$$\therefore \quad \text{the whole mass} = \iiint \mu xyz \, dx \, dy \, dz,$$

the integrals embracing the whole volume $OABC$ (Fig. 7.34). The limits for z are from 0 to $z = c(1 - x/a - y/b)$.

The limits for y are from 0 to $y = b(1 - x/a)$ and limits for x are from 0 to a.

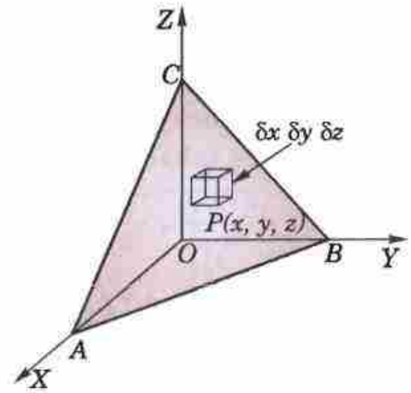


Fig. 7.34

Hence the required mass

$$= \int_0^a \int_0^{b(1-x/a)} \int_0^{c(1-x/a-y/b)} \mu xyz \, dz \, dy \, dx$$

$$= \mu \int_0^a \int_0^{b(1-x/a)} xy \left| z^2/2 \right|_0^{c(1-x/a-y/b)} dydz$$

$$= \mu \int_0^a \int_0^{b(1-x/a)} xy \,.\, \frac{c^2}{2}\left(1 - \frac{x}{a} - \frac{y}{b}\right)^2 dydx$$

$$= \frac{\mu c^2}{2} \int_0^a \int_0^{b(1-x/a)} x \,.\left[\left(1 - \frac{x}{a}\right)^2 y - 2\left(1 - \frac{x}{a}\right)\frac{y^2}{b} + \frac{y^3}{b^2}\right] dy \, dx$$

$$= \frac{\mu c^2}{2} \int_0^a x \left| \left(1 - \frac{x}{a}\right)^2 \frac{y^2}{2} - 2\left(1 - \frac{x}{a}\right)\frac{y^3}{3b} + \frac{y^4}{4b^2} \right|_0^{b(1-x/a)} dx$$

$$= \frac{\mu c^2}{2} \int_0^a x \left[\frac{b^2}{2}\left(1 - \frac{x}{a}\right)^4 - \frac{2b^2}{3}\left(1 - \frac{x}{a}\right)^4 + \frac{b^2}{4}\left(1 - \frac{x}{a}\right)^4\right] dx = \frac{\mu b^2 c^2}{24} \int_0^a x \, (1 - x/a)^4 \, dx = \frac{\mu a^2 b^2 c^2}{720}.$$

7.10 CENTRE OF GRAVITY

(*a*) **To find the C.G. $(\bar{x}, \bar{y})$ of a plane lamina,** take the element of mass $\rho\delta x\delta y$ at the point $P(x, y)$. Then

$$\bar{x} = \frac{\iint x\rho \, dxdy}{\iint \rho \, dxdy}, \; \bar{y} = \frac{\iint y \, \rho dxdy}{\iint \rho \, dxdy}$$ with integrals embracing the whole lamina.

While using polar coordinates, take the elementary mass as $\rho r\delta\theta\delta r$ at the point $P(r, \theta)$ so that $x = r\cos\theta$, $y = r\sin\theta$.

$$\therefore \qquad \bar{x} = \frac{\iint r\cos\theta \, \rho r \, d\theta dr}{\iint \rho \, r d\theta dr}, \; \bar{y} = \frac{\iint r\sin\theta \, \rho \, r d\theta dr}{\iint \rho \, r d\theta dr}$$

(*b*) **To find the C.G. $(\bar{x}, \bar{y}, \bar{z})$ of a solid,** take an element of mass $\rho\delta x\delta y\delta z$ enclosing the point $P(x, y, z)$. *Then*

$$\bar{x} = \frac{\iiint x\rho\, dxdydz}{\iiint \rho\, dxdydz},\ \bar{y} = \frac{\iiint y\rho\, dxdydz}{\iiint \rho\, dxdydz} \text{ and } \bar{z} = \frac{\iiint z\rho\, dxdydz}{\iiint \rho\, dxdydz}.$$

Example 7.34. *Find by double integration, the centre of gravity of the area of the cardioid $r = a(1 + \cos\theta)$.*

Solution. The cardioid being symmetrical about the initial line, its C.G. lies on OX, *i.e.*, $\bar{y} = 0$ (Fig. 7.35).

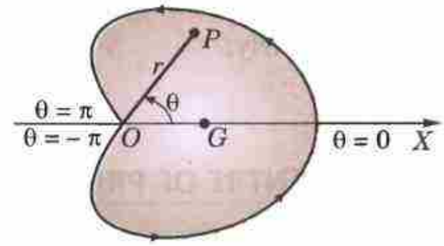


Fig. 7.35

$$\therefore \quad \bar{x} = \frac{\iint \rho\, r\cos\theta \,.\, r d\theta dr}{\iint \rho\, r d\theta dr} = \frac{\int_{-\pi}^{\pi}\int_0^{a(1+\cos\theta)} \cos\theta.\, r^2 dr\,.\,d\theta}{\int_{-\pi}^{\pi}\int_0^{a(1+\cos\theta)} r dr\,.\,d\theta}$$

$$= \frac{\int_{-\pi}^{\pi} \cos\theta \left| \frac{r^3}{3} \right|_0^{a(1+\cos\theta)} d\theta}{\int_{-\pi}^{\pi} \left| \frac{r^2}{2} \right|_0^{a(1+\cos\theta)} d\theta} = \frac{2a}{3}\cdot\frac{\int_{-\pi}^{\pi} \cos\theta\,(1+\cos\theta)^3\, d\theta}{\int_{-\pi}^{\pi} (1+\cos\theta)^2\, d\theta}$$

$$= \frac{2a}{3}\cdot\frac{2\,.\int_0^{\pi} (3\cos^2\theta + \cos^4\theta)\, d\theta}{2\,.\int_0^{\pi} (1+\cos^2\theta)\, d\theta} \qquad \left\{\because \int_{-\pi}^{\pi} \cos^n\theta\, d\theta = 2\int_0^{\pi} \cos^n\theta\, d\theta \text{ or } 0 \text{ according as } n \text{ in even or odd.}\right\}$$

$$= \frac{2a}{3}\cdot\frac{2\,.\int_0^{\pi/2} (3\cos^2\theta + \cos^4\theta)\, d\theta}{2\,.\int_0^{\pi/2} (1+\cos^2\theta)\, d\theta} \text{ (as the powers of } \cos\theta \text{ are even)} = \frac{2a}{3}\cdot\frac{3\,.\frac{1}{2}\cdot\frac{\pi}{2} + \frac{3\,.\,1}{4\,.\,2}\cdot\frac{\pi}{2}}{\frac{\pi}{2} + \frac{1}{2}\cdot\frac{\pi}{2}} = \frac{5a}{6}$$

Hence the C.G. of the cardioid is at $G(5a/6, 0)$.

Example 7.35. *Using double integration, find the C.G. of a lamina in the shape of a quadrant of the curve $(x/a)^{2/3} + (y/b)^{2/3} = 1$, the density being $\rho = kxy$, where k is a constant.*

Solution. Let $G\ (\bar{x}, \bar{y})$ be the C.G. of the lamina OAB (Fig. 7.36), so that

$$\bar{x} = \frac{\iint kxy\,.\,xdxdy}{\iint kxy\,.\,dxdy} = \frac{\iint x^2 y\, dxdy}{\iint xy\, dxdy}$$

where the integrals are taken over the area OAB so that y varies from 0 to y (to be found from the equation of the curve in terms of x) and then x varies from 0 to a.

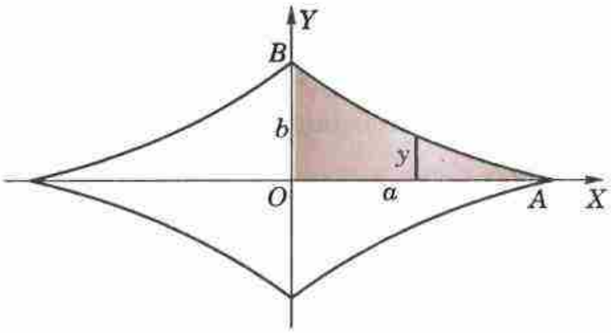


Fig. 7.36

Thus

$$\bar{x} = \frac{\int_0^a\int_0^y x^2 y\, dydx}{\int_0^a\int_0^y xy\, dydx} = \frac{\int_0^a x^2\,.\left| y^2/2 \right|_0^y dx}{\int_0^a x\,.\left| y^2/2 \right|_0^y dx} = \frac{\int_0^a x^2 y^2\, dx}{\int_0^a xy^2\, dx}$$

For any point on the curve, we have

$$x = a\cos^3\theta,\ y = b\sin^3\theta \text{ so that}$$
$$dx = -3a\cos^2\theta\sin\theta\, d\theta.$$

Also when $\quad x = 0, \theta = \pi/2$; when $x = a, \theta = 0$.

Hence $$\bar{x} = \frac{\int_{\pi/2}^{0} a^2 \cos^6 \theta . b^2 \sin^6 \theta . (-3a \cos^2 \theta \sin \theta)\, d\theta}{\int_{\pi/2}^{0} a \cos^3 \theta . b^2 \sin^6 \theta . (-3a \cos^2 \theta \sin \theta)\, d\theta}$$

$$= a \frac{\int_{0}^{\pi/2} \sin^7 \theta \cos^8 \theta\, d\theta}{\int_{0}^{\pi/2} \sin^7 \theta \cos^5 \theta\, d\theta} = \frac{128}{429} a$$

Similarly, $$\bar{y} = \frac{\int_0^a \int_0^y kxy . y\, dxdy}{\int_0^a \int_0^y kxy . dxdy} = \frac{128}{429} b.$$ Hence the required C.G. is $G\left(\frac{128}{429} a, \frac{128}{429} b\right)$.

7.11 CENTRE OF PRESSURE

Consider plane area A immersed vertically in a homogeneous liquid. Take the line of intersection of the given plane with the free surface of the liquid as the x-axis and any line lying in this plane and perpendicular to it downwards as the y-axis (Fig. 7.37).

If p be the pressure at the point $P(x, y)$ of the area A, then the pressure on an elementary area $\delta x \delta y$ at P is $p \delta x \delta y$ which is normal to the plane.

$\therefore$ the resultant pressure on $A = \iint p dxdy$.

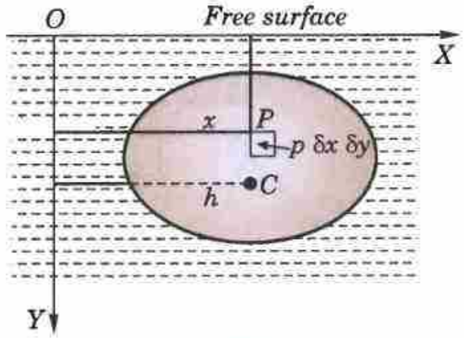


Fig. 7.37

If this resultant pressure acting at $C(h, k)$ is equivalent to pressure at various points such as $p\delta x \delta y$ distributed over the whole area A, then C is called the *centre of pressure*.

$\therefore$ taking the moment of the resultant pressure at C and the sum of the moments of the individual pressures such as $p\delta x\delta y$ at $P(x, y)$ about the y-axis, we get

$$h \iint p\, dxdy = \iint x . pdxdy, \quad i.e., h = \iint x . dxdy \Big/ \iint p dxdy$$

Similarly, taking moments about x-axis, we have

$$k = \iint y . pdxdy \Big/ \iint p dxdy$$ with integrals embracing the whole of the area A.

While using polar coordinates, replace x by $r \cos \theta$, y by $r \sin \theta$ and $dxdy$ by $r\, d\theta dr$ in the above formulae.

Example 7.36. *A horizontal boiler has a flat bottom and its ends are plane and semi-circular. If it is just full of water, show that the depth of the centre of pressure of either end is 0.7 × total depth approximately.*

Solution. Let the semi-circle $x^2 + y^2 = a^2$ represent an end of the given boiler (Fig. 7.38). By symmetry, its centre of pressure lies on OY.

If w be the weight of water per unit volume, then the pressure p at the point $P(x, y) = w(a - y)$.

$\therefore$ the height k of the C.P. above OX, is given by

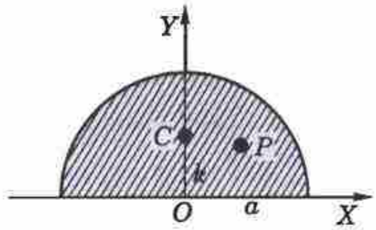


Fig. 7.38

$$k = \frac{\iint y \cdot pdxdy}{\iint pdxdy} = \frac{\int_{-a}^{a} \int_0^{\sqrt{(a^2 - x^2)}} w(a - y) y\, dy \cdot dx}{\int_{-a}^{a} \int_0^{\sqrt{(a^2 - x^2)}} w(a - y)\, dy \cdot dx}$$

$$= \frac{\int_{-a}^{a} \left| ay^2/2 - y^3/3 \right|_0^{\sqrt{(a^2 - x^2)}} dx}{\int_{-a}^{a} \left| ay - y^2/2 \right|_0^{\sqrt{(a^2 - x^2)}} dx} = \frac{\int_{-a}^{a} \left[\frac{a}{2}(a^2 - x^2) - \frac{1}{3}(a^2 - x^2)^{3/2}\right] dx}{\int_{-a}^{a} \left[a(a^2 - x^2)^{1/2} - \frac{1}{2}(a^2 - x^2)\right] dx}$$

Now put $x = a \sin\theta$, so that $dx = a\cos\theta\, d\theta$.

Also when $x = -a$, $\theta = -\pi/2$, and when $x = a$, $\theta = \pi/2$.

$$\therefore \qquad k = \frac{\int_{-\pi/2}^{\pi/2} \left[\frac{a^3}{2}\cos^2\theta - \frac{a^3}{3}\cos^3\theta\right] a\cos\theta\, d\theta}{\int_{-\pi/2}^{\pi/2} \left[a^2\cos\theta - \frac{a^2}{2}\cos^2\theta\right] a\cos\theta\, d\theta}$$

$$= \frac{a}{3}\cdot\frac{2\int_0^{\pi/2}(3\cos^3\theta - 2\cos^4\theta)\, d\theta}{2\int_0^{\pi/2}(2\cos^2\theta - \cos^3\theta)\, d\theta} = \frac{a}{4}\left(\frac{16-3\pi}{3\pi-4}\right) = 0.3a \text{ nearly.}$$

Hence, the depth of the C.P. $= a - k = 0.7a$ approximately.

PROBLEMS 7.6

1. A lamina is bounded by the curves $y = x^2 - 3x$ and $y = 2x$. If the density at any point is given by λxy, find by double integration, the mass of the lamina.
2. Find the mass of a lamina in the form of cardioid $r = a(1 + \cos\theta)$ whose density at any point varies as the square of its distance from the initial line.
3. Find the mass of a solid in the form of the positive octant of the sphere $x^2 + y^2 + z^2 = 9$, if the density at any point is $2xyz$.
4. Find the centroid of the area enclosed by the parabola $y^2 = 4ax$, the axis of x and its latus-rectum.
5. The density at any point (x, y) of a lamina is $\sigma(x + y)/a$ where σ and a are constants. The lamina is bounded by the lines $x = 0, y = 0, x = a, y = b$. Find the position of its centre of gravity.
6. Find the centroid of a loop of the lemniscate $r^2 = a^2\cos 2\theta$.
7. A plane in the form of a quadrant of the ellipse $(x/a)^2 + (y/b)^2 = 1$ is of small but varying thickness, the thickness at any point being proportional to the product of the distances of that point from the axes ; show that the coordinates of the centroid are $(8a/15, 8b/15)$. *(Nagpur, 2009)*
8. In a semi-circular disc bounded by a diameter OA, the density at any point varies as the distance from O ; find the position of the centre of gravity.
9. Find the centroid of the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$, the density at any point varying as its distance from the face $z = 0$.
10. Find $\bar{x}$ where $(\bar{x}, \bar{y}, \bar{z})$ is the centroid of the region R bounded by the parabolic cylinder $z = 4 - x^2$ and the planes $x = 0, y = 0, y = 6, z = 0$. (Assume that the density is constant).
11. If the density at any point of the solid octant of the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$ varies as xyz, find the coordinates of the C.G. of the solid. *(P.T.U., 2005)*
12. A horizontal boiler has a flat bottom and its ends consist of a square 1 metre wide surmounted by an isosceles triangle of height 0.5 metre. Determine the depth of the centre of pressure of either end when the boiler is just full.
13. A quadrant of a circle is just, immersed vertically in a heavy homogeneous liquid with one edge in the surface. Find the centre of pressure.
14. Find the depth of the centre of pressure of a square lamina immersed in the liquid, with one vertex in the surface and the diagonal vertical.
15. Find the centre of pressure of a triangular lamina immersed in a homogeneous liquid with one side in the free surface. *(P.T.U., 2003)*
16. A uniform semi-circular is lamina immersed in a fluid with its plane vertical and its boundary diameter on the free surface. If the density at any point of the fluid varies as the depth of the point below the free surface, find the position of the centre of pressure of the lamina.

7.12 (1) MOMENT OF INERTIA

If a particle of mass m of a body be at a distance r from a given line, then mr^2 is called the *moment of inertia of the particle about the given line* and the sum of similar expressions taken for all the particles of the body, *i.e.*, Σmr^2 is called the *moment of inertia of the body about the given line* (Fig. 7.39).

If M be the total mass of the body and we write its moment of inertia $= Mk^2$, then k is called the *radius of gyration* of the body about the axis.

(2) M.I. of plane lamina. Consider the elementary mass $\rho\delta x\delta y$ at the point $P(x, y)$ of a plane area A so that its M.I. about x-axis $= \rho\delta x\delta y\, y^2$.

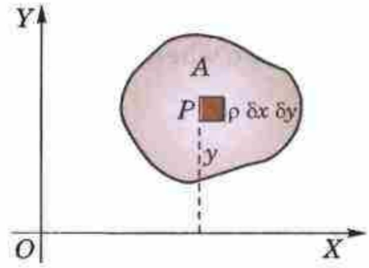


Fig. 7.39

$\therefore$ M.I. of the lamina about x-axis, *i.e.* $I_x = \iint_A \rho\, y^2\, dxdy$.

Similarly, M.I. of the lamina about y-axis' *i.e.*, $I_y = \iint_A \rho\, x^2\, dxdy$.

Also M.I. of the lamina about an axis perpendicular to the xy-plane, *i.e.*,

$$I_z = \iint_A \rho(x^2 + y^2)\, dxdy.$$

(3) M.I. of a solid. Consider an elementary mass $\rho\delta x\delta y\delta z$ enclosing a point $P(x, y, z)$ of a solid of volume V.

Distance of P from the x-axis $= \sqrt{(y^2 + z^2)}$.

$\therefore$ M.I. of this element about the x-axis $= \rho\, \delta x\delta y\delta z\, (y^2 + z^2)$.

Thus *M.I. of this solid about x-axis, i.e.,* $I_x = \iiint_V \rho(y^2 + z^2)\, dxdydz$.

Similarly, *its M.I. about y-axis, i.e.,* $I_y = \iiint_V \rho(z^2 + x^2)\, dxdydz$

and M.I. *about z-axis, i.e.,* $I_z = \iiint_V \rho(x^2 + y^2)\, dxdydz$.

(4) Sometimes we require the moment of inertia of a body about axes other than the principal axes. The following theorems prove useful for this purpose :

I. Theorem of perpendicular axis. *If the moment of inertia of a lamina about two perpendicular axes OX, OY in its plane are I_x and I_y, then the moment of inertia of the lamina about an axis OZ, perpendicular to it is given by $I_z = I_x + I_y$.*

Its proof follows from the relations giving I_x, I_y and I_z for a plane lamina [(2) above].

II. Steiner's theorem*. *If the moment of inertia of a body of mass M about an axis through its centre of gravity is I, then I′, moment of inertia about a parallel axis at a distance d from the first axis, is given by $I' = I + Md^2$.*

Its proof will be found in any text book on Dynamics of a Rigid Body.

Example 7.37. *Find the M.I. of the area bounded by the curve $r^2 = a^2 \cos 2\theta$ about its axis.*

Solution. Given curve is symmetrical about the pole and for half of the loop in the first quadrant θ varies from 0 to $\pi/4$ (Fig. 7.40).

Elementary area at $P(r, \theta) = r\, d\theta dr$.

If ρ be the surface density, then elementary mass

$$= \rho\, r\, d\theta dr \quad ...(i)$$

$\therefore$ its total mass $M = 4 \int_0^{\pi/4} \int_0^{a\sqrt{(\cos 2\theta)}} \rho\, r\, dr d\theta$

$$= 2\rho a^2 \int_0^{\pi/4} \cos 2\theta\, d\theta = \rho a^2 \quad ...(ii)$$

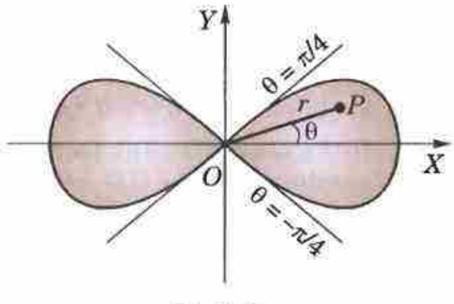


Fig. 7.40

Now M.I. of the elementary mass (i) about the x-axis.

$$= \rho r\, d\theta dr \cdot y^2 = \rho r d\theta\, dr\, (r \sin\theta)^2 = \rho r^3 \sin^2\theta\, dr d\theta$$

Hence the M.I. of the whole area

$$= 4 \int_0^{\pi/4} \int_0^{a\sqrt{(\cos 2\theta)}} \rho r^3 \sin^2\theta\, dr d\theta = 4\rho \int_0^{\pi/4} \sin^2\theta \cdot \Big| r^4/4 \Big|_0^{a\sqrt{(\cos 2\theta)}} d\theta$$

$$= \rho a^2 \int_0^{\pi/4} \cos^2 2\theta \cdot \sin^2\theta\, d\theta = \rho a^4 \int_0^{\pi/2} \cos^2\phi \cdot \sin^2 \frac{\phi}{2} \cdot \frac{d\phi}{2} \qquad \text{[Put } 2\theta = \phi,\ d\theta = d\phi/2]$$

$$= \frac{\rho a^4}{4} \int_0^{\pi/2} (\cos^2\phi - \cos^3\phi)\, d\phi = \frac{\rho a^4}{48}(3\pi - 8) = \frac{Ma^2}{48}(3\pi - 8). \qquad \text{[By } (ii)]$$

*Named after a Swiss geometrer *Jacob Steiner* (1796–1863) who was a professor at Berlin University.

Example 7.38. *Find the moment of inertia of a hollow sphere about a diameter, its external and internal radii being 5 metres and 4 metres.*

Solution. Let ρ be the density of the given hollow sphere. Then the M.I. about the diameter, *i.e.*, *x*-axis is

$$I_x = \iiint_V \rho(y^2 + z^2)\, dx\, dy\, dz$$

Changing to polar spherical coordinates, we get

$$I_x = \int_0^{2\pi}\int_0^{\pi}\int_4^5 \rho\{(r\sin\theta\sin\phi)^2 + (r\cos\theta)^2\}\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \rho\left\{\int_0^{2\pi}\sin^2\phi\, d\phi \cdot \int_0^{\pi}\sin^3\theta\, d\theta \left|\frac{r^5}{5}\right|_4^5 + \int_0^{2\pi} d\phi \int_0^{\pi}\cos^2\theta\sin\theta\, d\theta \cdot \left|\frac{r^5}{5}\right|_4^5\right\}$$

$$= \frac{8\pi\rho}{15}(5^5 - 4^5) = 1120.5 \text{ m.}$$

Example 7.39. *A solid body of density ρ is in the shape of the solid formed by revolution of the centroid $r = a(1 + \cos\theta)$ about the initial line. Show that its moment of inertia about a straight line through the pole perpendicular to the initial line is $\frac{352}{105}\pi\rho a^5$.* *(U.P.T.U., 2001)*

Solution. An elementary area $r d\theta dr$, when revolved about OX generates a circular ring of radius $LP = r\sin\theta$ (Fig. 7.41).

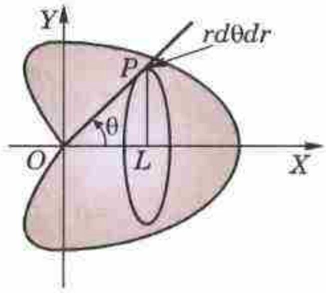


Fig. 7.41

M.I. of this ring about a diameter parallel to OY

$$= (2\pi r\sin\theta)(r d\theta dr)\rho \cdot \frac{(r\sin\theta)^2}{2}.$$

[∵ M.I. of a ring about a diameter = $Ma^2/2$.]

Now using Steiner's theorem, we have M.I. of the ring about OY = M.I. of the ring about a diameter LP parallel to OY + Mass of the ring $(OL)^2$ $(r\cos\theta)^2$

$$= 2\pi\rho r^4 \sin^3\theta\, d\theta dr + 2\pi r\sin\theta\,(r d\theta dr)(r\cos\theta)^2$$

Hence M.I. of the solid generated by revolution about OY

$$= \pi\rho\int_0^{\pi}\int_0^{r = a(1+\cos\theta)} (r^4\sin^3\theta + 2r^4\sin\theta\cos^2\theta)\, d\theta dr$$

$$= \pi\rho\int_0^{\pi}(\sin^3\theta + 2\sin\theta\cos^2\theta)\, d\theta \int_0^{a(1+\cos\theta)} r^4\, dr$$

$$= \frac{\pi\rho a^5}{5}\int_0^{\pi}\sin\theta\,(1+\cos^2\theta)(1+\cos\theta)^5\, d\theta \qquad [\text{Put } \theta = 2\phi]$$

$$= \frac{\pi\rho a^5}{5}\int_0^{\pi/2}\sin 2\phi\,(1+\cos^2 2\phi)(1+\cos 2\phi)^5\, 2d\phi$$

$$= \frac{\pi\rho a^5}{5}\int_0^{\pi/2} 2\sin\phi\cos\phi\,\{1 + (2\cos^2\phi - 1)^2\}(2\cos^2\phi)^5\, 2d\phi$$

$$= \frac{256\,\pi\rho a^5}{5}\int_0^{\pi/2}(\cos^{11}\phi - 2\cos^{13}\phi + 2\cos^{15}\phi)\sin\phi\, d\phi$$

$$= \frac{256\,\pi\rho a^5}{5}\left|-\frac{\cos^{12}\phi}{12} + \frac{2\cos^{14}\phi}{14} - \frac{2\cos^{16}\phi}{16}\right|_0^{\pi/2} = \frac{352\,\pi\rho a^5}{105}.$$

Example 7.40. *A hemisphere of radius R has a cylindrical hole of radius a drilled through it, the axis of the hole being along the radius normal to the plane face of the hemisphere. Find its radius of gyration about a diameter of this face.*

Solution. M.I. of the given solid about x-axis

$$= \iiint \rho(y^2 + z^2)\, dx\, dy\, dz$$

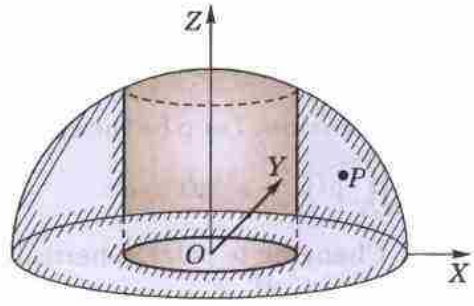


Fig. 7.42

The limits of integration for z are from 0 to $z = \sqrt{(R^2 - x^2 - y^2)}$ found from the equation of the sphere $x^2 + y^2 + z^2 = R^2$. The limits for x and y are to be such as to cover the shaded area A in the xy-plane between the concentric circles of radii a and R (Fig. 7.42).

Thus the required M.I. about x-axis

$$= \rho \iint_A \int_0^{\sqrt{(R^2 - x^2 - y^2)}} (y^2 + z^2)\, dz\, dx\, dy$$

$$= \rho \iint_A \left| y^2 z + z^3/3 \right|_0^{\sqrt{(R^2 - x^2 - y^2)}} dx\, dy = \rho \iint_A \left[y^2 (R^2 - x^2 - y^2)^{1/2} + \frac{1}{3}(R^2 - x^2 - y^2)^{3/2} \right] dx\, dy.$$

Now changing to polar coordinates, we have $x = r\cos\theta$, $y = r\sin\theta$ and $dxdy = rd\theta dr$.

Also to cover the area A, r varies from a to R and θ varies from 0 to 2π.

Hence the required M.I. about x-axis

$$= \rho \int_a^R \int_0^{2\pi} \left[r^2 \sin^2\theta \cdot (R^2 - r^2)^{1/2} + \frac{1}{3}(R^2 - r^2)^{3/2} \right] r d\theta dr$$

$$= \rho \int_a^R \int_0^{2\pi} \left[\frac{1}{2} r^2 (1 - \cos 2\theta) + \frac{1}{3}(R^2 - r^2) \right] d\theta \cdot r(R^2 - r^2)^{1/2}\, dr$$

$$= \rho \int_a^R \left| \frac{r^2}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + \frac{1}{3}(R^2 - r^2)\,\theta \right|_0^{2\pi} \cdot r(R^2 - r^2)^{1/2}\, dr$$

$$= \rho \int_a^R 2\pi \left(\frac{r^2}{2} + \frac{R^2 - r^2}{3} \right) . r(R^2 - r^2)^{1/2}\, dr$$

$$= \frac{\pi\rho}{3} \int_a^R (2R^2 + r^2)(R^2 - r^2)^{1/2} \cdot r dr \qquad \text{[Put } r^2 = t \text{ and } rdr = dt/2]$$

$$= \frac{\pi\rho}{6} \int_{a^2}^{R^2} (2R^2 + t)(R^2 - t)^{1/2}\, dt \qquad \text{[Integrate by parts]}$$

$$= \frac{\pi\rho}{9} \left[(2R^2 + a^2)(R^2 - a^2)^{3/2} + \frac{2}{5}(R^2 - a^2)^{5/2} \right] = \frac{2\pi\rho}{3}(R^2 - a^2)^{3/2} \times \frac{1}{10}(4R^2 + a^2)$$

$$\left[\because \text{ Mass} = \rho \int_0^{2\pi} \int_a^R \int_0^{\sqrt{(R^2 - r^2)}} dz \cdot r dr \cdot d\theta = \frac{2\pi\rho}{3}(R^2 - a^2)^{3/2} \right]$$

Hence, the radius of gyration $= [(4R^2 + a^2)/10]^{1/2}$.

7.13 (1) PRODUCT OF INERTIA

If a particle of mass m of a body be at distances x and y from two given perpendicular lines, then Σmxy is called the *product of inertia* of the body about the given lines.

Consider an elementary mass $\delta x \delta y \delta z$ enclosing the point $P(x, y, z)$ of solid of volume V. Then the product of inertia (P.I.) of this element about the axes of x and $y = \rho\delta x \delta y \delta z\, xy$.

$\therefore$ P.I. of the solid about x and y-axes, *i.e.*, $P_{xy} = \iiint_V \rho\, xy\, dxdydz$

Similarly, $P_{yz} = \iiint_V \rho\, yz\, dxdydz$ and $P_{zx} = \iiint_V \rho\, zx\, dxdydz$.

In particular, for a plane lamina of surface density ρ and covering a region A in the xy-plane,

$$P_{xy} = \iint_A \rho\, xy\, dxdy \text{ whereas } P_{yz} = P_{zx} = 0. \qquad [\because \ z = 0]$$

(2) Principal axes. *The principal axes* of a lamina at a given point are that pair of axes in its plane through the given point, about which the product of inertia of the lamina vanishes.

Let $P(x, y)$ be a point of the plane area A referred to rectangular axes OX, OY. Let (x', y') be the coordinates of P referred to another pair of rectangular axes OX', OY' in the same plane and inclined at an angle θ to the first pair (Fig. 7.43).

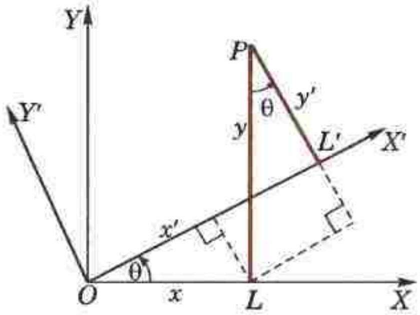


Fig. 7.43

Then $$x' = x\cos\theta + y\sin\theta$$
$$y' = y\cos\theta - x\sin\theta$$

If I_x, I_y be the moments of inetria of the area A about OX and OY and P_{xy} be its product of inertia about these axes, then

$$I_x = \iint_A \rho y^2\, dA,\ I_y = \iint_A \rho x^2\, dA,\ P_{xy} = \iint \rho\, xy\, dA.$$

$\therefore$ the product of inertia P'_{xy} about OX' and OY' is given by

$$P'_{xy} = \int_A \rho x'y'\, dA = \iint_A \rho(x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta)\, dA$$

$$= \sin\theta\cos\theta \iint_A \rho(y^2 - x^2)\, dA + (\cos^2\theta - \sin^2\theta) \iint_A \rho xy\, dA$$

$$= 1/2 \sin 2\theta \cdot (I_x - I_y) + \cos 2\theta\, P_{xy}.$$

Now OX', OY' will be the principal axes of the area A if P'_{xy} vanishes.

i.e., If $\quad 1/2 \sin 2\theta\, (I_x - I_y) + \cos 2\theta\, P_{xy} = 0$

i.e., If $\quad \mathbf{\tan 2\theta = 2P_{xy}/(I_y - I_x)}$.

This gives two values of θ differing by $\pi/2$.

Example 7.41. *Show that the principal axes at the node of a half-loop of the lemniscate $r^2 = a^2\cos 2\theta$ are inclined to the initial line at angles*

$$\frac{1}{2}\tan^{-1}\frac{1}{2} \text{ and } \frac{\pi}{2} + \frac{1}{2}\tan^{-1}\frac{1}{2}.$$

Solution. Let the element of mass at $P(r, \theta)$ be $\rho r d\theta dr$.

Then $$I_x = \rho \int_0^{\pi/4} \int_0^{a\sqrt{(\cos 2\theta)}} r^2 \sin^2\theta \cdot r d\theta dr$$ [See Fig. 7.40]

$$= \frac{\rho a^4}{4} \int_0^{\pi/4} \sin^2\theta \cos^2 2\theta\, d\theta = \frac{\rho a^4}{16}\left(\frac{\pi}{4} - \frac{2}{3}\right)$$

$$I_y = \rho \int_0^{\pi/4} \int_0^{a\sqrt{(\cos 2\theta)}} r^2 \cos^2\theta \cdot r\, d\theta dr = \frac{\rho a^4}{16}\left(\frac{\pi}{4} + \frac{2}{3}\right)$$

and $$P_{xy} = \rho \int_0^{\pi/4} \int_0^{a\sqrt{(\cos 2\theta)}} r^2 \sin\theta\cos\theta \cdot r d\theta dr = \frac{\rho a^4}{48}.$$

Hence the required direction of the principal axes at O are given by

$$\tan 2\theta = \frac{2P_{xy}}{I_y - I_x} = \frac{\rho a^4/24}{(\rho a^4/16) \times (4/3)} = \frac{1}{2}$$

or by $$\theta = \frac{1}{2}\tan^{-1}\frac{1}{2} \text{ and } \frac{\pi}{2} + \frac{1}{2}\tan^{-1}\frac{1}{2}.$$

PROBLEMS 7.7

1. Using double integrals, find the moment of inertia about the x-axis of the area enclosed by the lines $x = 0, y = 0, (x/a) + (y/b) = 1.$ (*P.T.U., 2005*)
2. Find the moment of inertia of a circular plate about a tangent.
3. Find the moment of inertia of the area $y = \sin x$ from $x = 0$ to $x = 2\pi$ about OX.

4. Find the moment of inertia of a quadrant of the ellipse $(x/a)^2 + (y/b)^2 = 1$ of mass M about the x-axis, if the density at a point is proportional to xy.
5. Find the moment of inertia about the initial line of the cardioid $r = a(1 + \cos\theta)$.
6. Find the moment of inertia of a uniform spherical ball of mass M and radius R about a diameter.
7. Find the moment of inertia of a solid right circular cylinder about (*i*) its axis (*P.T.U., 2006*) (*ii*) a diameter of the base.
8. Find the M.I. of a solid right circular cone having base-radius r and height h, about (*i*) its axis, (*ii*) an axis through the vertex and perpendicular to its axis, (*iii*) a diameter of its base.
9. Find the moment of inertia of a hollow sphere about a diameter, its external and internal radii being 51 metres and 49 metres.
10. Find the moment of inertia about z-axis of a homogeneous tetrahedron bounded by the planes $x = 0, y = 0, z = x + y$ and $z = 1$.
11. Find the moment of inertia of an octant of the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$, about the x-axis.
12. Find the product of inertia of a quadrant of the ellipse $(x/a)^2 + (y/b)^2 = 1$, about the coordinate axes.
13. Show that the principal axes at the origin of the triangle enclosed by $x = 0, y = 0, (x/a) + (y/b) = 1$ are inclined to the x-axis at angles α and $\alpha + \pi/2$, where $\alpha = \frac{1}{2}\tan^{-1}[ab/(a^2 - b^2)]$ (*U.P.T.U., 2002*)
14. The lengths AB and AD of the sides of a rectangle $ABCD$ are $2a$ and $2b$. Show that the inclination to AB of one of the principal axes at A is $\frac{1}{2}\tan^{-1}\left\{\frac{3ab}{2(a^2 - b^2)}\right\}$.

7.14 BETA FUNCTION

The beta function is defined as

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \begin{cases} m > 0 \\ n > 0 \end{cases} \quad ...(1)$$

Putting $x = 1 - y$ in (1), we get $\beta(m, n) = -\int_1^0 (1-y)^{m-1}\, y^{n-1}\,dy$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}\,dy = \beta(n, m)$$

Thus $\beta(m, n) = \beta(n, m)$ ...(2)

Putting $x = \sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\,d\theta$, (1) becomes

$$\beta(m, n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1}\, 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta \quad ...(3)$$

which is another form of $\beta(m, n)$.

This function is also *Euler's integral of the first kind**.

7.15 (1) GAMMA FUNCTION

The gamma function is defined as

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx \ (n > 0) \quad ...(i)$$

This integral is also known as *Euler's integral of the second kinds.* It defines a function of n for positive values of n.

*After an enormously creative Swiss mathematician *Leonhard Euler (1707–1783).* He studied under *John Bernoulli* and became a professor of mathematics in St. Petersburg, Russia. Even after becoming totally blind in 1771, he contributed to almost all branches of mathematics.

In particular, $\Gamma(1) = \int_0^\infty e^{-x}\,dx = \left| -e^{-x} \right|_0^\infty = 1.$ ...(ii)

(2) Reduction formula for Γ(n).

Since $\Gamma(n+1) = \int_0^\infty e^{-x} x^n\,dx$ [Integrating by parts] $= \left| -x^n e^{-x} \right|_0^\infty + n\int_0^\infty e^{-x} x^{n-1} dx$

$\therefore$ $\Gamma(n+1) = n\Gamma(n)$ ...(iii)

which is the reduction formula for $\Gamma(n)$. From this formula, it is clear that if $\Gamma(n)$ is known throughout a unit interval say : $1 < n \le 2$, then the values of $\Gamma(n)$ throughout the next unit interval $2 < n \le 3$ are found, from which the values of $\Gamma(n)$ for $3 < n \le 4$ are determined and so on. In this way, *the values of $\Gamma(n)$ for all positive values of $n > 1$ may be found by successive application of (iii).*

Also using (*iii*) in the form

$$\Gamma(n) = \frac{\Gamma(n+1)}{n} \qquad ...(iv)$$

We can define $\Gamma(n)$ for values of n for which the definition (1) fails. It gives the value of $\Gamma(n)$ for $0 < n \le 1$ in terms of the values of $\Gamma(n)$ for $1 < n \le 2$. Thus we can define $\Gamma(n)$ for all $n < 0$ provided its value for $1 < n \le 2$ is known. Also if $-1 < n < 0$, (4) gives $\Gamma(n)$ in terms of its values for $0 < n < 1$. Then we may find, $\Gamma(n)$ for $-2 < n < -1$ and so on.

Thus (i) and (iv) together give a complete definition of $\Gamma(n)$ for all values of n except when n is zero or a negative integer and its graph is as shown in Fig.7.44. The values of *Γ(n)* for *1 < n ≤ 2* are given in (Table I-Appendix 2) from which the values of *Γ(n)* for values of *n* outside the interval $1 < n \le 2$ ($n \ne 0, -1, -2, -3$ .....) may be found.

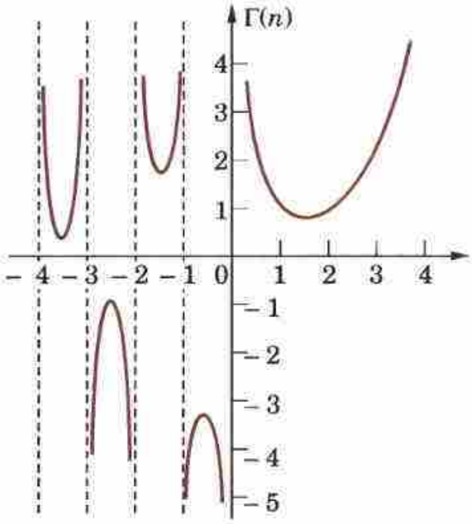


Fig. 7.44

(3) Value of Γ(n) in terms of factorial.

Using $\Gamma(n+1) = n\Gamma(n)$ successively, we get

$\Gamma(2) = 1 \times \Gamma(1) = 1\,!$

$\Gamma(3) = 2 \times \Gamma(2) = 2 \times 1 = 2\,!$

$\Gamma(4) = 3 \times \Gamma(3) = 3 \times 2\,! = 3\,!$

....

In general $\Gamma(n+1) = n\,!$ provided n is a positive integer ...(v)

Taking $n = 0$, it defines $0\,! = \Gamma(1) = 1$.

(4) Value of $\Gamma\left(\frac{1}{2}\right)$. We have

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty e^{-x} x^{-1/2}\,dx \qquad \text{[Put } x = y^2 \text{ so that } dx = 2y\,dy]$$

$$= 2\int_0^\infty e^{-y^2}\,dy \text{ which is also } = 2\int_0^\infty e^{-x^2}\,dx$$

$$\therefore \quad \left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta$$

$$= 4\cdot\frac{\pi}{2}\int_0^\infty e^{-r^2}\,r dr = 2\pi\left[\left(-\frac{1}{2}\right)e^{-r^2}\right]_0^\infty = \pi$$

whence $\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} = 1.772$ ...(vi) (*V.T.U., 2006*)

7.16 RELATION BETWEEN BETA AND GAMMA FUNCTIONS

$$\beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

We have $\Gamma(m) = \int_0^\infty e^{-t}\,t^{m-1}$ [Put $t = x^2$ so that $dt = 2x\,dx$

$$= 2\int_0^\infty e^{-x^2} x^{2m-1}\, dx \qquad ...(2)$$

Similarly, $\Gamma(n) = 2\int_0^\infty e^{-y^2} y^{2n-1}\, dy$

$\therefore$ $\Gamma(m)\Gamma(n) = 4\int_0^\infty e^{-x^2} x^{2m-1}\, dx \int_0^\infty e^{-y^2} y^{2n-1}\, dy$

$$= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1}\, dx\, dy \qquad ...(3)$$ [$\because$ the limits of integration are constant.]

Now change to polar coordinates by writing $x = r\cos\theta$, $y = r\sin\theta$ and $dx\, dy = rd\theta dr$. To cover the region in (3) which is the entire first quadrant, r varies from 0 to ∞ and θ from 0 to $\pi/2$. Thus (3) becomes

$$\Gamma(m)\Gamma(n) = 4\int_0^\infty \int_0^{\pi/2} e^{-r^2} r^{2(m+n)-1} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta dr$$

$$= \left[2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta\right] \times \left[2\int_0^\infty e^{-r^2} r^{2(m+n)-1}\, dr\right] \qquad ...(4)$$

But by (2), $2\int_0^\infty e^{-r^2} r^{2(m+n)-1}\, dr = \Gamma(m+n)$

and by (3) of § 7.14, $2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta = \beta(m, n)$.

Thus (4) gives $\Gamma(m)\Gamma(n) = \beta(m, n)\,\Gamma(m+n)$ *(U.T.U., 2010 ; Bhopal, 2009 ; V.T.U., 2008 S)*
whence follows (1) which is extremely useful for evaluating definite integrals in terms of gamma functions.

Cor. *Rule to evaluate* $\int_0^{\pi/2} \sin^p x \cos^q x\, dx$.

$$\int_0^{\pi/2} \sin^p x \cos^q x\, dx = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) \qquad \text{[By (3) of § 7.14]}$$

$$= \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)} \qquad ...(5)$$

In particular, when $q = 0$, and $p = n$, we have

$$\left.\begin{aligned} \int_0^{\pi/2} \sin^n x\, dx &= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \cdot \frac{\sqrt{\pi}}{2} \\ \text{Similarly,} \quad \int_0^{\pi/2} \cos^n x\, dx &= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \cdot \frac{\sqrt{\pi}}{2} \end{aligned}\right\} \qquad ...(6)$$

Example 7.42. *Show that*

(a) $\Gamma(n) = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy\ (n > 0)$. *(J.N.T.U., 2003 ; Madras, 2003 S)*

(b) $\beta(p, q) = \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\, dy$ *(V.T.U., 2003 ; Gauhati, 1999)*

$= \int_0^1 \frac{x^{p-1} + x^{q-1}}{(1+x)^{p+q}}\, dx$. *(V.T.U., 2008 ; Osmania, 2003 ; Rohtak, 2003)*

Solution. (a) $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx\ (n > 0)$

$$= \int_1^0 \left(\log \frac{1}{y}\right)^{n-1} y\left(-\frac{1}{y}dy\right) = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy.$$

Put $y = e^{-x}$
i.e., $x = \log(1/y)$
so that $dx = -(1/y)\,dy$

(*b*) $$\beta(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx$$

$$= \int_\infty^0 \frac{1}{(1+y)^{p-1}}\left(\frac{y}{1+y}\right)^{q-1} \frac{-1}{(1+y)^2}\,dy$$

Put $x = \dfrac{1}{1+y}$ *i.e.*, $y = \dfrac{1}{x} - 1$
so that $dx = \dfrac{-1}{(1+y)^2}\,dy$

$$= \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\,dy = \int_0^1 \frac{y^{q-1}}{(1+y)^{p+q}}\,dy + \int_1^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\,dy$$

Now substituting $y = 1/z$ in the second integral, we get

$$\int_1^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\,dy = \int_1^0 \frac{1}{z^{q-1}} \cdot \frac{1}{(1+1/z)^{p+q}}\left(-\frac{1}{z^2}\right)dz = \int_0^1 \frac{z^{p-1}}{(1+z)^{p+q}}\,dz.$$

Hence, $$\beta(p, q) = \int_0^1 \frac{y^{q-1}}{(1+y)^{p+q}}\,dy + \int_0^1 \frac{z^{p-1}}{(1+z)^{p+q}}\,dz = \int_0^1 \frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}}\,dx.$$

Example 7.43. *Express the following integrals in terms of gamma functions :*

(*a*) $\displaystyle\int_0^1 \frac{dx}{\sqrt{(1-x^4)}}$ (*b*) $\displaystyle\int_0^{\pi/2} \sqrt{(\tan\theta)}\,d\theta.$ (*Madras, 2006*)

(*c*) $\displaystyle\int_0^\infty \frac{x^c}{c^x}\,dx$ (*U.P.T.U., 2006*) (*d*) $\displaystyle\int_0^\infty a^{-bx^2}\,dx.$

(*e*) $\displaystyle\int_0^1 x^5\,[\log(1/x)]^3\,dx$ (*Madras, 2000*)

Solution. (*a*) $\displaystyle\int_0^1 \frac{dx}{\sqrt{(1-x^4)}}$

Put $x^2 = \sin\theta$, *i.e.*, $x = \sin^{1/2}\theta$
so that $dx = 1/2\sin^{-1/2}\theta\cos\theta\,d\theta$

$$= \int_0^{\pi/2} \frac{1}{2} \cdot \frac{\sin^{-1/2}\theta \,.\, \cos\theta\,d\theta}{\sqrt{(1-\sin^2\theta)}} = \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta\,d\theta = \frac{1}{2} \cdot \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(\dfrac{-\frac{1}{2}+1}{2}\right)}{\Gamma\left(\dfrac{-\frac{1}{2}+2}{2}\right)} = \frac{\sqrt{\pi}}{4}\,\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

(*b*) $$\int_0^{\pi/2} \sqrt{(\tan\theta)}\,d\theta = \int_0^{\pi/2} \sin^{1/2}\theta\cos^{-1/2}\theta\,d\theta$$

$$= \frac{\Gamma\left(\dfrac{\frac{1}{2}+1}{2}\right)\Gamma\left(\dfrac{-\frac{1}{2}+1}{2}\right)}{2\Gamma\left(\dfrac{\frac{1}{2}-\frac{1}{2}+2}{2}\right)} = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2\Gamma(1)} = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$

(*c*) $$\int_0^\infty \frac{x^c}{c^x}\,dx = \int_0^\infty \frac{x^c}{e^{x\log c}}\,dx \qquad [\because\ c^x = e^{\log c^x} = e^{x\log c}]$$

$$= \int_0^\infty e^{-x\log c}\,x^c\,dx \qquad [\text{Put } x\log c = t \text{ so that } dx = dt/\log c]$$

$$= \int_0^\infty e^{-t} \left(\frac{t}{\log c}\right)^c \frac{dt}{\log c} = \frac{1}{(\log c)^{c+1}} \int_0^\infty t^c e^{-t} dt = \Gamma(c+1)/(\log c)^{c+1}$$

(d) $\int_0^\infty a^{-bx^2} dx = \int_0^\infty e^{-bx^2 \log a} dx$ $\left[\begin{array}{l}\text{Put } (b \log a)\, x^2 = t \\ \text{so that } dx = dt/2\sqrt{(b \log a)}\end{array}\right.$

$$= \frac{1}{2\sqrt{(b \log a)}} \int_0^\infty e^{-t}\, t^{-1/2}\, dt = \frac{\Gamma\left(\frac{1}{2}\right)}{2\sqrt{(b \log a)}} = \frac{\sqrt{\pi}}{2\sqrt{(b \log a)}}$$

(e) $\int_0^1 x^4 [\log(1/x)]^3 dx = \frac{1}{625} \int_0^\infty e^{-t} \,.\, t^3\, dt$ $\left[\begin{array}{l}\text{Put } x = e^{-t/5} \text{ so that } \log(1/x) = t/5 \\ dx = -\frac{1}{5} e^{-t/5}\, dt\end{array}\right]$

$$= \frac{\Gamma(4)}{625} = \frac{6}{625}.$$

Example 7.44. *Evaluate* $\int_0^\infty e^{-ax} x^{m-1} \sin bx\, dx$ *in terms of Gamma function.* (U.P.T.U., 2003)

Solution. We have $\Gamma(m) = \int_0^\infty e^{-x} x^{m-1} dx$ [Put $x = ay$, $dx = a\,dy$]

$$= \int_0^\infty e^{-ay} a^m y^{m-1} dy \quad \text{or} \quad \int_0^\infty e^{-ay} y^{m-1} dy = \Gamma(m)/a^m. \quad ...(i)$$

Then $I = \int_0^\infty e^{-ax} x^{m-1} \sin bx\, dx = \int_0^\infty e^{-ax} x^{m-1}$ (Imaginary part of e^{ibx}) dx

$= \text{I.P. of} \int_0^\infty e^{-(a-ib)x} x^{m-1}\, dx$

$= \text{I.P. of } \{\Gamma(m)/(a-ib)^m$ [By (i)]

$= \text{I.P. of } \{\Gamma(m)/(r^m (\cos\theta - i \sin\theta)^m]$ where $a = r\cos\theta$, $b = r \sin\theta$

$= \text{I.P. of } \Gamma(m)/(r^m (\cos m\theta - i \sin m\theta)]$ (Using Demoivere's theorem §19.5)

$$= \text{I.P. of} \left\{\frac{\Gamma(m)\,.\,(\cos m\theta + i \sin m\theta)}{r^m (\cos m\theta + i \sin m\theta)(\cos m\theta - i \sin m\theta)}\right\}$$

$= \frac{\Gamma(m)}{r^m} \sin m\theta$ where $r\ \sqrt{(a^2+b^2)}$, $\theta = \tan^{-1} b/a$.

Example 7.45. *Prove that* $\int_0^1 \frac{x^2 dx}{\sqrt{(1-x^4)}} \times \int_0^1 \frac{dx}{\sqrt{(1+x^4)}} = \frac{\pi}{4\sqrt{2}}$

Solution. $\int_0^1 \frac{x^2 dx}{\sqrt{(1-x^4)}} \times \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta} \cdot \frac{\cos\theta}{2\sqrt{(\sin\theta)}} d\theta$ $\left[\text{Putting } x^2 = \sin\theta,\ dx = \frac{\cos\theta\, d\theta}{2\sqrt{(\sin\theta)}}\right.$

$$= \frac{1}{2} \int_0^{\pi/2} \sqrt{(\sin\theta)}\, d\theta = \frac{1}{4} \beta\left(\frac{3}{4}, \frac{1}{2}\right) = \frac{1}{4} \frac{\Gamma(3/4)\,\Gamma(1/2)}{\Gamma(5/4)} = \frac{\Gamma(3/4)\,\Gamma(1/2)}{\Gamma(1/4)}$$

$\int_0^1 \frac{dx}{\sqrt{(1+x^4)}} = \int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{2\sqrt{(\tan\theta)} \sec\theta}$ $\left[\text{Putting } x^2 = \tan\theta,\ dx = \frac{\sec^2\theta\, d\theta}{2\sqrt{(\tan\theta)}}\right.$

$= \frac{1}{\sqrt{2}} \int_0^{\pi/4} \frac{d\theta}{\sqrt{(\sin 2\theta)}} = \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \sin^{-1/2}\phi\, d\phi$ $\left[\text{Putting } 2\theta = \phi,\ d\theta = \frac{1}{2} d\phi\right.$

$$= \frac{1}{4\sqrt{2}} \beta\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{1}{4\sqrt{2}} \frac{\Gamma(1/4)\,\Gamma(1/2)}{\Gamma(3/4)}$$

$$\therefore \qquad \int_0^1 \frac{x^2\,dx}{\sqrt{(1-x^4)}} \times \int_0^1 \frac{dx}{\sqrt{(1+x^4)}} = \frac{1}{4\sqrt{2}}\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = \frac{\pi}{4\sqrt{2}}.$$

Example 7.46. *Prove that* (i) $\beta(m, 1/2) = 2^{2m-1}\,\beta(m, m)$ *(V.T.U., 2004)*

(ii) $\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m)$ *(Duplication Formula)*

(V.T.U., 2010 ; Kerala, M.E., 2005 ; Madras, 2003 S)

Solution. (i) We know that $\beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta$ ...(1)

Putting $n = \frac{1}{2}$, we have $\beta\left(m, \frac{1}{2}\right) = 2\int_0^{\pi/2}\sin^{2m-1}\theta\,d\theta$ ...(2)

Again putting $n = m$ in (i), we get $\beta(m, m) = 2\int_0^{\pi/2}(\sin\theta\cos\theta)^{2m-1}\,d\theta$

$$= \frac{1}{2^{2m-2}}\int_0^{\pi/2}\sin^{2m-1}2\theta\,d\theta$$

$$= \frac{1}{2^{2m-1}}\int_0^{\pi}\sin^{2m-1}\phi\,d\phi, \text{ putting } 2\theta = \phi$$

$$= \frac{1}{2^{2m-1}}\cdot 2\int_0^{\pi/2}\sin^{2m-1}\phi\,d\phi$$

or $$2^{2m-1}\beta(m, m) = 2\int_0^{\pi/2}\sin^{2m-1}\theta\,d\theta = \beta\left(m, \frac{1}{2}\right)$$ [by (2)

(ii) Rewriting the above result in terms of Γ functions, we get

$$2^{2m-1}\frac{\Gamma(m)\,\Gamma(m)}{\Gamma(m+m)} = \frac{\Gamma(m)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

or $$\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}\Gamma(2m)}{2^{2m-1}}.$$

Example 7.47. *Prove that*

(a) $\iint_D x^{l-1}\,y^{m-1}\,dx\,dy = \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)}h^{l+m}$ *where D is the domain* $x \ge 0$, $y \ge 0$ *and* $x + y \le h$.

(U.P.T.U., 2005)

(b) $\iiint_V x^{l-1}\,y^{n-1}\,z^{n-1}\,dxdydz = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}$

where V is the region $x \ge 0$, $y \ge 0$, $z \ge 0$ *and* $x + y + z \le 1$. *This important result is known as* **Dirichlet's integral*.**

Solution. (a) Putting $x/h = X$ and $y/h = Y$, we see that the given integral

$$= \iint_{D'}(hX)^{l-1}(hY)^{m-1}h^2\,dXdY \text{ where } D' \text{ is the domain } X \ge 0, Y \ge 0 \text{ and } X + Y \le 1.$$

$$= h^{l+m}\int_0^1\int_0^{1-X}X^{l-1}\,Y^{m-1}\,dY\,dX = h^{l+m}\int_0^1 X^{l-1}\left|\frac{Y^m}{m}\right|_0^{1-X}dX$$

$$= \frac{h^{l+m}}{m}\int_0^1 X^{l-1}(1-X)^m\,dX = \frac{h^{l+m}}{m}\beta(l, m+1) = \frac{h^{l+m}}{m}\cdot\frac{\Gamma(l)\,\Gamma(m+1)}{\Gamma(l+m+1)}$$

*Named after a German mathematician *Peter Gustav Lejeune Dirichlet* (1805–1859) who studied under Cauchy and succeeded Gauss at Gottingen. He is known for his contributions to Fourier series and number theory.

$$= h^{l+m}\,\frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)} \qquad \ldots(i)\ [\because\ \Gamma(m+1)/m = \Gamma(m)]$$

(*b*) Taking $y + z \le 1 - x (= h\text{: say})$, the triple integral

$$= \int_0^1\int_0^{1-x}\int_0^{1-x-y} x^{l-1}\, y^{m-1}\, z^{n-1}\, dzdydx$$

$$= \int_0^1 x^{l-1}\left[\int_0^h\int_0^{h-y} y^{m-1}\, z^{n-1}\, dzdy\right]dx = \int_0^1 x^{l-1}\,\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)}\,h^{m+n}\,dx \qquad \ldots \text{[By } (i)]$$

$$= \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)}\int_0^1 x^{l-1}\,(1-x)^{m+n}\,dx = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)}\,\beta(l, m+n+1)$$

$$= \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)}\cdot\frac{\Gamma(l)\,\Gamma(m+n+1)}{\Gamma(l+m+n+1)} = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}.$$

Example 7.48. *Evaluate the integral* $\iiint x^{l-1}\, y^{m-1}\, z^{n-1}\, dxdydz$ *where x, y, z are all positive with condition,* $(x/a)^p + (y/b)^q + (z/c)^r \le 1.$ *(U.P.T.U., 2005 S)*

Solution. Put $(x/a)^p = u$, *i.e.*, $x = au^{1/p}$ so that $dx = \dfrac{a}{p}\,u^{1/p-1}\,du$

$(y/b)^q = v$, *i.e.*, $y = bv^{1/q}$ so that $dy = \dfrac{b}{q}\,v^{1/q-1}\,dv$

and $(z/c)^r = w$, *i.e.*, $z = cw^{1/r}$ so that $dz = \dfrac{c}{r}\,w^{1/r-1}\,dw$

Then $\iiint x^{l-1}\, y^{m-1}\, z^{n-1}\, dxdydz$

$$= \iiint (au^{1/p})^{l-1}\,(bv^{1/q})^{m-1}\,(cw^{1/r})^{n-1}\left(\frac{a}{p}\right)u^{1/p-1}\left(\frac{b}{q}\right)v^{1/q-1}\left(\frac{c}{r}\right)w^{1/r-1}\,dudvdw$$

$$= \frac{a^l\, b^m\, c^n}{pqr}\iiint u^{l/p-1}\, v^{m/q-1}\, w^{n/r-1}\; du\, dv\, dw \text{ where } u+v+w \le 1.$$

$$= \frac{a^l\, b^m\, c^n}{pqr}\,\frac{\Gamma(l/p)\,\Gamma(m/q)\,\Gamma(n/r)}{\Gamma(l/p+m/q+n/r+1)} \qquad \text{[By Dirichlet's integral]}$$

Example 7.49. *The plane* $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c} = 1$ *meets the axes in A, B and C. Apply Dirichlet's integral to find the volume of the tetrahedron OABC. Also find its mass if the density at any point is kxyz.* *(U.P.T.U., 2004)*

Solution. Put $x/a = u$, $y/b = v$, $z/c = w$ then the tetrahedron *OABC* has $u \ge 0, v \ge 0, w \ge 0$ and $u + v + w \le 1$.

$\therefore$ volume of this tetrahedron $= \iiint_D dx\, dy\, dz$

$$= \iiint_{D'} abc\; du\, dv\, dw \qquad \left[\begin{array}{l} a\ dx = adu,\ dy = bdv,\ dz = cdw \\ \text{for } D' = u \ge 0, v \ge 0, w \ge 0\ \&\ u+v+w \le 1. \end{array}\right.$$

$$= abc\iiint_{D'} u^{1-1}\, v^{1-1}\, w^{1-1}\; du\, dv\, dw$$

$$= abc\,\frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(1+1+1+1)} = \frac{abc}{6} \qquad \text{[By Dirichlet's integral]}$$

$$\text{Mass} = \iiint kxyz dx\, dy\, dz = \iiint k\,(au)\,(bv)\,(cw)\,abc\; du\, dv\, dw$$

$$= ka^2b^2c^2\iiint u^{2-1}\, v^{2-1}\, w^{2-1}\; du\, dv\, dw$$

$$= ka^2b^2c^2\,\frac{\Gamma(2)\,\Gamma(2)\,\Gamma(2)}{\Gamma(2+2+2+1)}\, ka^2b^2c^2 \cdot \frac{1}{6!} = \frac{k}{720}\,a^2b^2c^2.$$

PROBLEMS 7.8

1. Compute :
   (i) $\Gamma(3.5)$ (*Assam, 1998*) (ii) $\Gamma(4.5)$
   (iii) $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$ (*S.V.T.U., 2009*) (iv) $\beta(2.5, 1.5)$ (v) $\beta\left(\frac{9}{2}, \frac{7}{2}\right)$. (*Andhra, 2000*)
2. Express the following integrals in terms of gamma functions :
   (i) $\int_0^\infty e^{-x^2}\,dx$ (ii) $\int_0^\infty x^{p-1}e^{-kx}\,dx (k > 0)$ (*Delhi, 2002 ; V.T.U., 2000*)
   (iii) $\int_0^\infty \sqrt{x}\,e^{-x^3}\,dx$ (*J.N.T.U., 2003*) (iv) $\int_0^\infty \frac{dx}{x^{p+1}.(x-1)^q} (-p < q < 1)$
3. Show that :
   (i) $\int_0^\infty \frac{x^4}{4^x}\,dx = \frac{\Gamma(5)}{(\log 4)^5}$ (*Marathwada, 2008*)
   (ii) $\int_0^{\pi/2} \sqrt{(\cot\theta)}\,d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$ (*Osmania, 2003 S ; V.T.U., 2001*)
   (iii) $\int_0^{\pi/2} [\sqrt{(\tan\theta)} + \sqrt{(\sec\theta)}]\,d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\left\{\Gamma\left(\frac{3}{4}\right) + \sqrt{\pi}/\Gamma\left(\frac{3}{4}\right)\right\}$ (*J.N.T.U., 2000*)
   (iv) $\int_0^{\pi/2} \sqrt{(\sin\theta)}\,d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{(\sin\theta)}} = \pi.$ (*V.T.U., 2007*)
4. Given $\int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \frac{\pi}{\sin n\pi}$, show that $\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$. (*S.V.T.U., 2008*)
   Hence evaluate $\int_0^\infty \frac{dy}{1+y^4}$. (*V.T.U., 2006 ; J.N.T.U., 2005*)
5. Prove that :
   (i) $\int_0^1 \frac{x\,dx}{\sqrt{(1-x^5)}} = \frac{1}{5}\beta\left(\frac{2}{5}, \frac{1}{2}\right)$ (*Raipur, 2006*) (ii) $\int_0^1 \frac{dx}{\sqrt{(1+x^4)}} = \frac{1}{4\sqrt{2}}\beta\left(\frac{1}{4}, \frac{1}{2}\right)$. (*V.T.U., 2003*)
   (iii) $\int_0^1 x^3(1-\sqrt{x})^5\,dx = 2\beta(8, 6).$ (*Marathwada, 2008 ; J.N.T.U., 2006*)
6. Show that (i) $\int_a^b (x-a)^{m-1}(b-x)^{n-1}\,dx = (b-a)^{m+n-1}\beta(m, n)$ (*P.T.U., 2010 ; Mumbai, 2005*)
   (ii) $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx = \frac{1}{a^n, b^m}\beta(m, n)$ (*Nagpur, 2009*) (iii) $\int_0^\infty \frac{x^{10}-x^{18}}{(1+x)^{30}}\,dx = 0$ (*Mumbai, 2005*)
   (iv) $\int_0^1 \frac{(1-x^4)^{3/4}}{(1+x^4)^2}\,dx = \frac{1}{2^{9/2}}\beta\left(\frac{7}{4}, \frac{1}{4}\right)$ (*Mumbai, 2007*)
7. Prove that $\int_0^1 x^m(\log x)^n\,dx = \frac{(-1)^n\,n!}{(m+1)^{n+1}}$, where n is a positive integer and $m > -1$. (*S.V.T.U., 2006*)
   Hence evaluate $\int_0^1 x(\log x)^3\,dx$. (*Nagpur, 2009*)
8. Show that $\int_0^1 y^{q-1}\left(\log\frac{1}{y}\right)^{p-1}dy = \frac{\Gamma(p)}{q^p}$, where $p > 0, q > 0$. (*Rohtak, 2006 S*)
9. Express $\int_0^1 x^m(1-x^n)^p\,dx$ in terms of gamma functions (*Marathwada, 2008*)
   Hence evaluate : (i) $\int_0^1 x(1-x^3)^{10}\,dx$. (*Bhopal, 2008*) (ii) $\int_0^1 \frac{dx}{\sqrt{(1-x^n)}}$ (*Anna, 2005*)

10. Prove that $\int_0^\infty \frac{dx}{(e^x + e^{-x})^n} = \frac{1}{4}\beta\left(\frac{n}{2}, \frac{n}{2}\right)$ and hence evaluate $\int_0^\infty \text{sech}^8\, x\, dx$.

11. Prove that $\beta\left(n + \frac{1}{2}, n + \frac{1}{2}\right) = \frac{\Gamma(n + 1/2)\sqrt{\pi}}{2^{2n}\,\Gamma(n+1)}$. Hence show that $2^n\,\Gamma(n + 1/2) = 1 \cdot 3 \cdot 5 \ldots (2n-1)\sqrt{\pi}$

(Mumbai, 2007)

12. Prove that :

(*i*) $\frac{\beta(m+1, n)}{m} = \frac{\beta(m, n+1)}{n} = \frac{\beta(m, n)}{m+n}$ (*ii*) $\beta(n, n) = \frac{\sqrt{\pi}\,\Gamma(n)}{2^{2n-1}\Gamma\left(n + \frac{1}{2}\right)}$

(*iii*) $\Gamma\left(n + \frac{1}{2}\right) = \frac{\Gamma(2n+1)\sqrt{\pi}}{2^{2n}\,.\,\Gamma(n+1)}$ (*iv*) $\beta(m + 1) + \beta(m, n + 1) = \beta(m, n)$

(Bhopal, 2008 ; J.N.T.U., 2006 ; Madras, 2003)

13. Show that $\iint x^{m-1}\, y^{n-1}\, dx\, dy$ over the positive quadrant of the ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \text{ is } \frac{a^m b^n}{2n}\beta\left(\frac{m}{2}, \frac{n}{2} + 1\right).$$

14. Show that the area in the first quadrant enclosed by the curve $(x/a)^\alpha + (y/b)^\beta = 1$, $\alpha > 0$, $\beta > 0$, is given by

$$\frac{ab}{\alpha + \beta}\,\frac{\Gamma(1/\alpha)\,\Gamma(1/\beta)}{\Gamma(1/\alpha + 1/\beta)}.$$

15. Find the mass of an octant of the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$, the density at any point being $\rho = kxyz$.

(U.P.T.U., 2002)

7.17 (1) ELLIPTIC INTEGRALS

In Applied Mathematics, we often come across integrals of the form $\int_0^1 e^{-x^2}\, dx$ or $\int_0^1 \sin x^2\, dx$ which cannot be evaluated by any of the standard methods of integration. In such cases, we may find the value to any desired degree of accuracy by expanding their integrands as power series. An important class of such integrals is the *elliptic integrals*.

Def. The integral $F(k, \phi) = \int_0^\phi \frac{dx}{\sqrt{(1 - k^2 \sin^2 x)}}\ (k^2 < 1)$ ...(*i*)

which is a function of the two variables k and ϕ, is called the *elliptic integral of the first kind with modulus* k and *amplitude* ϕ.

The integral $E(k, \phi) = \int_0^\phi \sqrt{(1 - k^2 \sin^2 x)}\, dx\ (k^2 < 1)$ ...(*ii*)

is called the *elliptic integral of the second kind* with *modulus k and amplitude ϕ.*

The name *elliptic integral* arose from its original application in finding the length of an elliptic arc (Fig.7.45), For instance, consider the ellipse

$$x = a\cos\phi, \qquad y = b\sin\phi, \qquad (a < b)$$

Then length of its arc

$$AP = \int_0^\phi \sqrt{\left[\left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2\right]}\, d\phi = \int_0^\phi \sqrt{[(-a\sin\phi)^2 + (b\cos\phi)^2]}\, d\phi$$

$$= \int_0^\phi \sqrt{[(b^2 + (a^2 - b^2)\sin^2\phi]}\, d\phi = b\int_0^\phi \sqrt{\left[1 - \left(1 - \frac{a^2}{b^2}\right)\sin^2\phi\right]}\, d\phi$$

$= bE(k, \phi)$ for $k^2 = 1 - a^2/b^2 \le 1$.

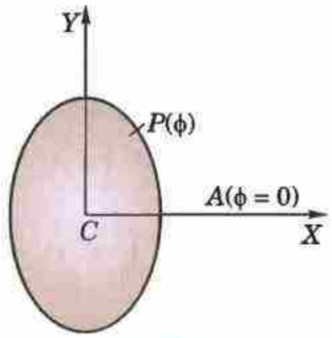


Fig. 7.45

Also the perimeter of the ellipse

$$= 4b \int_0^{\pi/2} \sqrt{(1 - k^2 \sin^2 \phi)}\, d\phi = 4bE(k, \pi/2).$$

This particular integral with upper limit $\phi = \pi/2$ is called the *complete elliptic integral of the second kind and is denoted by $E(k)$.*

Thus $$E(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 \phi)\, d\phi \; (k^2 < 1) \quad ...(iii)$$

Similarly, the *complete elliptic integral of first kind is*

$$K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{(1 - k^2 \sin^2 \phi)}} \; (k^2 < 1) \quad ...(iv)$$

To evaluate it, we expand the integral in the form

$$(1 - k^2 \sin^2 \phi)^{-1/2} = 1 + \frac{k^2}{2} \sin^2 \phi + \frac{3k^4}{4} \sin^4 \phi + ...$$

This series can be shown to be uniformly convergent for all k, and may, therefore, be integrated term by term [See § 9.19-II]. Then we have

$$K(k) = \int_0^{\pi/2} \left(1 + \frac{k^2}{2} \sin^2 \phi + \frac{3k^4}{8} \sin^4 \phi + \frac{5k^6}{16} \sin^6 \phi + ...\right) d\phi$$

$$= \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1.3}{2.4}\right)^2 k^4 + \left(\frac{1.3.5}{2.4.6.}\right)^2 k^6 + ...\right] \quad ...(v)$$

This series may be used to compute K for various values of k. In particular, if $k = \sin 10°$; we have

$$K = \frac{\pi}{2}(1 + 0.00754 + 0.00012 + ...) = 1.5828 \quad ...(vi)$$

In this way tables of the elliptic integrals are constructed. Values of $F(k, \phi)$ and $E(k, \phi)$ are readily available for $0 \le \phi \le \pi/2$, $0 < k < 1$. (*See Peirce's short tables*).

Example 7.50. *Express* $\int_0^{\pi/6} \frac{dx}{\sqrt{(\sin x)}}$ *in terms of elliptic integral.*

Solution. Put $\cos x = \cos^2 \phi$ and $dx = \dfrac{2 \cos \phi\, d\phi}{\sqrt{(1 + \cos^2 \phi)}}$

Then $$I = \int_0^{\pi/2} \frac{2 \cos^2 \phi}{\sqrt{(1 + \cos^2 \phi)}} d\phi = 2 \int_0^{\pi/2} \frac{(1 + \cos^2 \phi) - 1}{\sqrt{(1 + \cos^2 \phi)}} d\phi$$

$$= 2\left\{\int_0^{\pi/2} \sqrt{(1 + \cos^2 \phi)}\, d\phi - \int_0^{\pi/2} \frac{d\phi}{\sqrt{(1 + \cos^2 \phi)}}\right\} = 2\left\{\int_0^{\pi/2} \sqrt{(2 - \sin^2 \phi)}\, d\phi - \int_0^{\pi/2} \frac{d\phi}{\sqrt{(2 - \sin^2 \phi)}}\right\}$$

$$= 2\sqrt{2} \int_0^{\pi/2} \sqrt{(1 - 1/2 \sin^2 \phi)}\, d\phi - \sqrt{2} \int_0^{\pi/2} \frac{d\phi}{\sqrt{(1 - 1/2 \sin^2 \phi)}} = 2\sqrt{2}\, E\left(\frac{1}{\sqrt{2}}\right) - \sqrt{2}\, K\left(\frac{1}{\sqrt{2}}\right)$$

(2) **Elliptic functions.** By putting $\sin x = t$ and $\sin \phi = z$, (i) becomes

$$u = \int_0^z \frac{dt}{\sqrt{[(1 - t^2)(1 - k^2 t^2)]}} \; (k^2 < 1) \quad ...(vii)$$

This is known as *Jacobi's form of the elliptic integral of first kind** whereas (i) is the *Legendre's form*†.

If $k = 0$, (vii) gives $u = \sin^{-1} z$. By analogy, we denote (vii) $sn^{-1} z$ for a fixed non-zero value of k. This leads to the functions $sn\, u = z = \sin \phi$ and $cn\, u = \cos \phi$

which are called the *Jacobi's elliptic functions.*

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* See footnote p. 215.

† A French mathematician *Adrien Marie Legendre* (1752–1833) who made important contributions to number theory, special functions, calculus of variations and elliptic integrals.

The elliptic functions $sn\, u$ and $cn\, u$ are periodic with a period depending on k and an amplitude equal to unity. These behave somewhat like $\sin u$ and $\cos u$. For instance

$$sn(0) = 0,\ cn(1) = 1 \quad \text{and} \quad sn(-u) = -sn(u),\ cn(-u) = cn(u).$$

Example 7.51. *Show that* $\displaystyle\int_0^{a/2} \frac{dx}{\sqrt{(2ax - x^2)}\sqrt{(a^2 - x^2)}} = \frac{2}{3a} K\left(\frac{1}{3}\right)$.

Solution. Putting $\quad x = \dfrac{a}{2}(1 - \sin\theta),\ dx = -\dfrac{a}{2}\cos\theta\, d\theta,$

$$2ax - x^2 = \frac{a^2}{4}(1 - \sin\theta)(3 + \sin\theta) \text{ and } a^2 - x^2 = \frac{a^2}{4}(1 + \sin\theta)(3 - \sin\theta)$$

Also when $\quad x = 0,\ \theta = \pi/2$; when $x = a/2,\ \theta = 0$.

Thus the given integral

$$= \frac{4}{a^2}\int_{\pi/2}^{0} \frac{-(a/2)\cos\theta\, d\theta}{\sqrt{[(1 - \sin^2\theta)(2 - \sin^2\theta)]}} = \frac{2}{3a}\int_0^{\pi/2} \frac{d\theta}{\sqrt{[(1 - (1/3)^2\sin^2\theta]}} = \frac{2}{3a} K\left(\frac{1}{3}\right).$$

7.18 (1) ERROR FUNCTION OR PROBABILITY INTEGRAL

The error function or the probability integral is defined as

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\, dt.$$

This integral arises in the solution of certain partial differential equations of applied mathematics and occupies an important position in the probability theory.

The complementary error function erfc (x) is defined as $erfc(x) = 1 - erf(x)$.

(2) Properties : (*i*) $erf(-x) = -erf(x)$; $\quad$ (*ii*) $erf(0) = 0$

$$(iii)\ erf(\infty) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2} dt = \frac{2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} = 1 \qquad \text{[By } (iii)\text{, p. 289]}$$

This proves that the total area under the *Normal* or *Gaussian error function curve* is unity – § 26.16.

PROBLEMS 7.9

1. By means of the substitution $k \sin x = \sin z$, show that

   (*i*) $\displaystyle\int_0^\pi \frac{dx}{\sqrt{(1 - k^2\sin^2 x)}} = \frac{1}{k} F\left(\frac{1}{k}, \phi'\right)$;

   (*ii*) $\displaystyle\int_0^\phi \sqrt{(1 - k^2\sin^2 x)}\, dx = \left(\frac{1}{k} - k\right) F\left(\frac{1}{k}, \phi'\right) + kE\left(\frac{1}{k}, \phi'\right)$

   where $k > 1$ and $\phi' = \sin^{-1}(k \sin\phi)$.

   Express the following integrals in terms of elliptic integrals :

2. $\displaystyle\int_0^{\pi/2} \frac{dx}{\sqrt{(1 + 3\sin^2 x)}}$. (*Kerala, M.E., 2005*)
3. $\displaystyle\int_0^{\pi/2} \frac{dx}{\sqrt{(2 - \cos x)}}$.
4. $\displaystyle\int_0^{\pi/2} \sqrt{(\cos x)}\, dx$.
5. Expand $erf(x)$ in ascending powers of x. Hence evaluate $erf(0)$. (*P.T.U., 2009 S*)
6. Compute (*i*) $erf(0.3)$, (*ii*) $erf(0.5)$, correct to three decimal places.
7. Show that (*i*) $erf(x) + erf(-x) = 0$ $\quad$ (*ii*) $erfc(x) + erfc(-x) = 2$
8. Prove that

   (*i*) $\dfrac{d}{dx}[erf(ax)] = \dfrac{2a}{\sqrt{\pi}} e^{-a^2x^2}$ (*Osmania, 2003*) $\quad$ (*ii*) $\dfrac{d}{dx}[erfc(ax)] = -\dfrac{2a}{\sqrt{\pi}} e^{-a^2x^2}$.
9. Prove that $\displaystyle\int_0^\infty e^{-x^2 - 2ax}\, dx = \frac{\sqrt{\pi}}{2} e^{a^2}[1 - erf(0)]$

7.19 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 7.10

Fill up the blanks or choose the correct answer from the following problems :

1. $\int_0^2 \int_0^x (x+y)\, dx\, dy = \ldots\ldots$
2. $\int_0^1 \int_0^{1-x} dx\, dy \ldots\ldots$
3. $\int_0^1 e^{-x^2} dx = \ldots\ldots$
4. $\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \ldots\ldots$ (*V.T.U., 2010*)
5. $\Gamma(3.5) = \ldots\ldots$
6. The surface area of the sphere $x^2 + y^2 + z^2 + 2x - 4y + 8z - 2 = 0$ is ....
7. $\int_0^2 \int_1^3 \int_1^2 xy^2 z\, dzdydx = \ldots\ldots$
8. If $u = x + y$ and $v = x - 2y$, then the area-element $dxdy$ is replaced by .... $dudv$.
9. In terms of Beta function $\int_0^{\pi/2} \sin^7 \theta \sqrt{\cos \theta}\, d\theta = \ldots\ldots$
10. The value of $\beta(2, 1) + \beta(1, 2)$ is ....
11. $\int_0^1 \int_1^2 xy\, dy\, dx = \ldots\ldots$
12. Volume bounded by $x \geq 0, y \geq 0, z \geq 0$ and $x^2 + y^2 + z^2 = 1$ as a triple integral integral.
13. Value of $\int_0^1 \int_0^{x^2} xe^y\, dy\, dx$ is equal to

    (*a*) $e/2$ (*b*) $e - 1$ (*c*) $1 - e$ (*d*) $e/2 - 1$. (*Bhopal, 2008*)
14. $\iint x^2 y^3\, dxdy$ over the rectangle $0 \leq x \leq 1$ and $0 \leq y \leq 3$ is ....
15. $\int_0^\pi \int_0^{a \sin \theta} r\, dr\, d\theta = \ldots\ldots$
16. $\int_{x=0}^{x=3} \int_{y=0}^{y=1/x} ye^{xy}\, dxdy = \ldots\ldots$
17. $\int_0^{\pi/2} \int_0^\infty \frac{r\, dr\, d\theta}{(r^2 + a^2)} = \ldots\ldots$
18. $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)\, dx\, dy = \ldots\ldots$
19. To change cartesian coordinates (x, y, z) to spherical polar coordinate (r, θ, ϕ); $dx\, dy\, dz$ is replaced by ....
20. $\int_0^2 \int_0^{x^2} e^{y/x}\, dydx = \ldots\ldots$
21. $\iint (x+y)^2\, dxdy$ over the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is ....
22. $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\, dy}{1 + x^2 + y^2} = \ldots\ldots$
23. $\iint xy(x+y)\, dx\, dy$ over the area between $y + x^2$ and $y = x$, is ....
24. Value of $\int_0^1 \int_x^{\sqrt{x}} xy\, dx\, dy$ is

    (*a*) zero (*b*) $-1/24$ (*c*) $1/24$ (*d*) 24. (*V.T.U., 2010*)
25. $\iint dx\, dy$ over the area bounded by $x = 0, y = 0, x^2 + y^2 = 1$ and $5y = 3$, is ....
26. $\iint_R y\, dx\, dy$ where R is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$, is ....
27. $\iint (x^2 + y^2)\, dx\, dy$ in the positive quadrant for which $x + y \leq 1$, is ....
28. Area between the parabolas $y^2 = 4x$ and $x^2 = 4y$ is ....
29. Changing the order of integration in $\int_{-a}^a \int_0^{\sqrt{a^2 - y^2}} f(x, y)\, dx\, dy = \ldots\ldots$
30. $\Gamma(1/4)\Gamma(3/4) = \ldots\ldots$ (*V.T.U., 2011*)
31. $\beta(5/2, 7/2) = \ldots\ldots$
32. $\int_0^\infty \int_0^x xe^{-x^2/y}\, dydx = \ldots\ldots$
33. On changing to polar coordinates $\int_0^{2a} \int_0^{\sqrt{(2ax - x^2)}} dx\, dy$ becomes ....

34. A square lamina is immersed in the liquid with one vertex in the surface and the diagonal of length vertical. Its centre of pressure is at a depth ....
35. The centroid of the area enclosed by the parabola $y^2 = 4x$, x-axis and its latus-rectum is ....
36. The moment of inertia of a uniform spherical ball of mass 10 gm and radius 2 cm about a diameter is ....
37. M.I. of a solid right circular cone (base-radius r and height h) about its axis is ....

38. $erf_c(-x) - erf(x) =$ .... 39. $\int_0^1 \frac{x-1}{\log x}\, dx =$ .... 40. $\Gamma\left(\frac{3}{2}\right) =$ ....

41. Value of $\int_0^a \int_0^b \int_0^c x^2y^2z^2\, dx\, dy\, dz$ is

(a) $\frac{abc}{3}$ (b) $\frac{a^2b^2c^2}{27}$ (c) $\frac{a^3b^3c^3}{27}$ (d) $\frac{a^2b^2c^2}{9}$.

42. The integral $\int_0^1 \int_0^{\sqrt{(1-x^2)}} (x+y)\, dy\, dx$ after changing the order of integration.

(a) $\int_0^2 \int_0^{\sqrt{(1-y^2)}} (x+y)\, dx\, dy$ (b) $\int_0^1 \int_0^{\sqrt{(1-y^2)}} (x+y)\, dx\, dy$

(c) $\int_0^1 \int_0^{\sqrt{(1+y^2)}} (x+y)\, dx\, dy$ (d) $\int_0^{-1} \int_0^{\sqrt{(1-y^2)}} (x+y)\, dx\, dy$. (V.T.U., 2011)

CHAPTER 8

Vector Calculus and Its Applications

1. Differentiation of vectors. 2. Curves in space. 3. Velocity and acceleration, Tangential and normal acceleration, Relative velocity and acceleration. 4. Scalar and vector point functions—Vector operator del. 5. Del applied to scalar point functions—Gradient. 6. Del applied to vector point functions—Divergence and Curl. 7. Physical interpretations of div **F** and curl **F**. 8. Del applied twice to point functions. 9. Del applied to products of point functions. 10. Integration of vectors. 11. Line integral—Circulation—Work. 12. Surface integral—Flux. 13. Green's theorem in the plane. 14. Stoke's theorem. 15. Volume integral. 16. Divergence theorem. 17. Green's theorem. 18. Irrotational and Solenoidal fields. 19. Orthogonal curvilinear coordinates, Del applied to functions in orthogonal curvilinear coordinates. 20. Cylindrical coordinates. 21. Spherical polar coordinates. 22. Objective Type of Questions.

8.1 (1) DIFFERENTIATION OF VECTORS

If a vector **R** varies continuously as a scalar variable t changes, then **R** is said to be a function of t and is written as $\mathbf{R} = \mathbf{F}(t)$.

Just as in scalar calculus, *we define* **derivative** *of a vector function* $\mathbf{R} = \mathbf{F}(t)$ *as*

$$\underset{\delta t \to 0}{\text{Lt}} \frac{\mathbf{F}(t + \delta t) - \mathbf{F}(t)}{\delta t} \text{ and write it as } \frac{d\mathbf{R}}{dt} \text{ or } \frac{d\mathbf{F}}{dt} \text{ or } \mathbf{F}'(t).$$

(2) **General rules of differentiation** are similar to those of ordinary calculus *provided the order of factors in vector products is maintained.* Thus, if ϕ, **F**, **G**, **H** are scalar and vector functions of a scalar variable t, we have

(*i*) $\dfrac{d}{dt}(\mathbf{F} + \mathbf{G} - \mathbf{H}) = \dfrac{d\mathbf{F}}{dt} + \dfrac{d\mathbf{G}}{dt} - \dfrac{d\mathbf{H}}{dt}$

(*ii*) $\dfrac{d}{dt}(\mathbf{F}\phi) = \mathbf{F}\dfrac{d\phi}{dt} + \dfrac{d\mathbf{F}}{dt}\phi$

(*iii*) $\dfrac{d}{dt}(\mathbf{F} \cdot \mathbf{G}) = \mathbf{F} \cdot \dfrac{d\mathbf{G}}{dt} + \dfrac{d\mathbf{F}}{dt} \cdot \mathbf{G}$

(*iv*) $\dfrac{d}{dt}(\mathbf{F} \times \mathbf{G}) = \mathbf{F} \times \dfrac{d\mathbf{G}}{dt} + \dfrac{d\mathbf{F}}{dt} \times \mathbf{G}$

(*v*) $\dfrac{d}{dt}[\mathbf{FGH}] = \left[\dfrac{d\mathbf{F}}{dt}\mathbf{GH}\right] + \left[\mathbf{F}\dfrac{d\mathbf{G}}{dt}\mathbf{H}\right] + \left[\mathbf{FG}\dfrac{d\mathbf{H}}{dt}\right]$

(*vi*) $\dfrac{d}{dt}[(\mathbf{F} \times \mathbf{G}) \times \mathbf{H}] = \left(\dfrac{d\mathbf{F}}{dt} \times \mathbf{G}\right) \times \mathbf{H} + \left(\mathbf{F} \times \dfrac{d\mathbf{G}}{dt}\right) \times \mathbf{H} + (\mathbf{F} \times \mathbf{G}) \times \dfrac{d\mathbf{H}}{dt}$

As an illustration, let us prove (*iv*), while the others can be proved similarly :

$$\frac{d}{dt}(\mathbf{F} \times \mathbf{G}) = \underset{\delta t \to 0}{\text{Lt}} \frac{(\mathbf{F} + \delta\mathbf{F}) \times (\mathbf{G} + \delta\mathbf{G}) - \mathbf{F} \times \mathbf{G}}{\delta t} = \underset{\delta t \to 0}{\textbf{Lt}} \frac{\mathbf{F} \times \delta\mathbf{G} + \delta\mathbf{F} \times \mathbf{G} + \delta\mathbf{F} \times \delta\mathbf{G}}{\delta t}$$

$$= \underset{\delta t \to 0}{\text{Lt}} \left[\mathbf{F} \times \frac{\delta\mathbf{G}}{\delta t} + \frac{\delta\mathbf{F}}{\delta t} \times \mathbf{G} + \frac{\delta\mathbf{F}}{\delta t} \times \delta\mathbf{G}\right] = \mathbf{F} \times \frac{d\mathbf{G}}{dt} + \frac{d\mathbf{F}}{dt} \times G \qquad [\because \ \delta\mathbf{G} \to 0 \text{ as } \delta t \to 0]$$

Obs. 1. *If* $\mathbf{F}(t)$ *has a constant magnitude, then* $\mathbf{F} : \dfrac{d\mathbf{F}}{dt} = 0$

For $\mathbf{F}(t) \cdot \mathbf{F}(t) = [\mathbf{F}(t)]^2$ = constant

$$\therefore \qquad \mathbf{F} \cdot \frac{d\mathbf{F}}{dt} = 0, \quad i.e., \quad \frac{d\mathbf{F}}{dt} \perp \mathbf{F}.$$

Obs. 2. *If* $\mathbf{F}(t)$ *has constant (fixed) direction, then* $\mathbf{F} \times \frac{d\mathbf{F}}{dt} = \mathbf{0}$

Let $\mathbf{G}(t)$ be a unit vector in the direction of $\mathbf{F}(t)$ so that

$$\mathbf{F}(t) = f(t)\,\mathbf{G}(t) \quad \text{where} \quad f(t) = |\,\mathbf{F}(t)\,|.$$

$$\therefore \qquad \frac{d\mathbf{F}}{dt} = f\frac{d\mathbf{G}}{dt} + \frac{df}{dt}\mathbf{G} \quad \text{and} \quad \mathbf{F} \times \frac{d\mathbf{F}}{dt} = f\,\mathbf{G} \times \left[f\frac{d\mathbf{G}}{dt} + \frac{df}{dt}\mathbf{G}\right]$$

$$= f^2\,\mathbf{G} \times \frac{d\mathbf{G}}{dt} = \mathbf{0}. \qquad \text{[since } \mathbf{G} \text{ is constant, } d\mathbf{G}/dt = \mathbf{0}.]$$

Example 8.1. *If* $\mathbf{A} = 5t^2\mathbf{I} + t\mathbf{J} - t^3\mathbf{K}$, $\mathbf{B} = \sin t\mathbf{I} - \cos t\mathbf{J}$, *find (i)* $\frac{d}{dt}(\mathbf{A} \cdot \mathbf{B})$; *(ii)* $\frac{d}{dt}(\mathbf{A} \times \mathbf{B})$.

Solution. (*i*) $\frac{d}{dt}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot \frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt} \cdot \mathbf{B}$

$$= (5t^2\mathbf{I} + t\mathbf{J} - t^3\mathbf{K}) \cdot [\cos t\mathbf{I} - (-\sin t)\,\mathbf{J}] + (10t\mathbf{I} + \mathbf{J} - 3t^2\mathbf{K}) \cdot (\sin t\mathbf{I} - \cos t\mathbf{J})$$

$$= (5t^2 \cos t + t \sin t) + (10t \sin t - \cos t) = 5t^2 \cos t + 11t \sin t - \cos t.$$

(*ii*) $\frac{d}{dt}(\mathbf{A} \times \mathbf{B}) = \mathbf{A} \times \frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt} \times \mathbf{B}$

$$= (5t^2\mathbf{I} + t\mathbf{J} - t^3\mathbf{K}) \times (\cos t\mathbf{I} + \sin t\mathbf{J}) + (10\,t\mathbf{I} + \mathbf{J} - 3t^2\mathbf{K}) \times (\sin t\mathbf{I} - \cos t\mathbf{J})$$

$$= [5t^2 \sin t\mathbf{K} + r \cos t\,(-\mathbf{K}) - t^3 \cos t\mathbf{J} - t^3 \sin t\,(-\mathbf{I})]$$

$$+ [-10t \cos t\mathbf{K} + \sin t(-\mathbf{K}) - 3t^2 \sin t\mathbf{J} + 3t^2 \cos t\,(-\mathbf{I})]$$

$$= (t^3 \sin t - 3t^2 \cos t)\,\mathbf{I} - t^2(t \cos t + 3 \sin t)\,\mathbf{J} + [(5t^2 - 1) \sin t - 11t \cos t]\,\mathbf{K}.$$

8.2 CURVES IN SPACE

(1) Tangent. Let $\mathbf{R}(t) = x(t)\mathbf{I} + y(t)\mathbf{J} + z(t)\mathbf{K}$ be the position vector of a point P. Then as the scalar parameter t takes different values, the point P traces out a *curve in space* (Fig. 8.1). If the neighbouring point Q corresponds to $t + \delta t$, then $\delta\mathbf{R} = \mathbf{R}(t + \delta t) - \mathbf{R}(t)$ or $\delta\mathbf{R}/\delta t$ is directed along the chord PQ. As $\delta t \to 0$, Lt $\delta\mathbf{R}/\delta t$ becomes the tangent (vector) to the curve at P whenever it exists and is not zero.

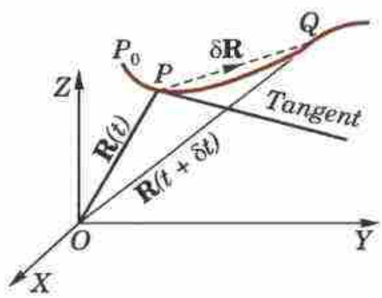

Fig. 8.1

Thus the vector $\mathbf{R}' = d\mathbf{R}/dt$ is a tangent to the space curve $\mathbf{R} = \mathbf{F}(t)$.

Let P_0 be a fixed point of this curve corresponding to $t = t_0$. If s be the length of the arc P_0P, then

$$\frac{\delta s}{\delta t} = \frac{\delta s}{|\,\delta\mathbf{R}\,|} \cdot \frac{|\,\delta\mathbf{R}\,|}{\delta t} = \frac{\text{arc } PQ}{\text{chord } PQ}\left|\frac{\delta\mathbf{R}}{\delta t}\right|$$

As $Q \to P$ along the curve QR *i.e.*, $\delta t \to 0$, then arc PQ/chord $PQ \to 1$ and

$$\frac{ds}{dt} = \left|\frac{d\mathbf{R}}{dt}\right| \quad \text{or} \quad |\,\mathbf{R}'(t)\,|.$$

If $\mathbf{R}'(t)$ is continuous, then arc P_0P is given by

$$s = \int_{t_0}^{t} |\,\mathbf{R}'\,|\,dt = \int_{t_0}^{t} \sqrt{(x')^2 + (y')^2 + (z')^2}\,dt$$

If we take s the parameter in place of t then the magnitude of the tangent vector, *i.e.*, $|\,d\mathbf{R}/ds\,| = 1$. Thus denoting the unit tangent vector by $\mathbf{T}$, we have

$$\mathbf{T} = \frac{d\mathbf{R}}{ds} \qquad \ldots(1)$$

(2) **Principal normal.** Since $\mathbf{T}$ is a unit vector, we have

$$d\mathbf{T}/ds \cdot \mathbf{T} = 0$$

i.e., $d\mathbf{T}/ds$ is perpendicular to $\mathbf{T}$. Or else $d\mathbf{T}/ds = 0$, in which case $\mathbf{T}$ is a constant vector w.r.t. the arc length s and so has a fixed direction, *i.e.*, the curve is a straight line.

If we denote a unit normal vector to the curve at P by $\mathbf{N}$ then $d\mathbf{T}/ds$ is in the direction of $\mathbf{N}$ which is known as the *principal normal* to the space curve at P. The plane of $\mathbf{T}$ and $\mathbf{N}$ is called the *osculating plane* of the curve at P (Fig. 8.2).

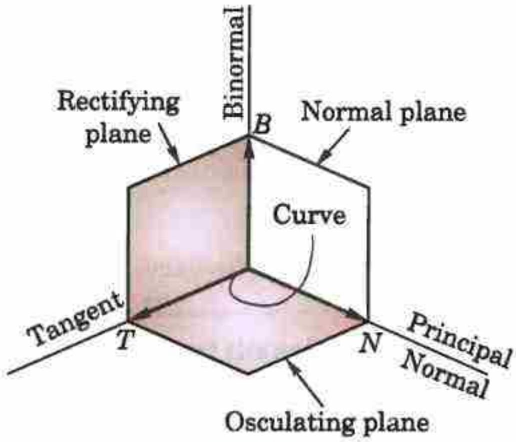

Fig. 8.2

(3) Binormal. A third unit vector $\mathbf{B}$ defined by $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, is called the *binormal at P*. Since $\mathbf{T}$ and $\mathbf{N}$ are unit vectors, $\mathbf{B}$ is also a unit vector perpendicular to both $\mathbf{T}$ and $\mathbf{N}$ and hence normal to the *osculating plane at P*.

Thus at each point P of a space curve there are three mutually perpendicular unit vectors, $\mathbf{T}, \mathbf{N}, \mathbf{B}$ which form a moving trihedral such that

$$\mathbf{T} = \mathbf{N} \times \mathbf{B}, \mathbf{N} = \mathbf{B} \times \mathbf{T}, \mathbf{B} = \mathbf{T} \times \mathbf{N} \quad ...(2)$$

This moving trihedral determines the following three fundamental planes at each point of the curve :

(*i*) The osculating plane containing $\mathbf{T}$ and $\mathbf{N}$

(*ii*) The normal plane containing $\mathbf{N}$ and $\mathbf{B}$

(*iii*) The rectifying plane containing $\mathbf{B}$ and $\mathbf{T}$.

(4) Curvature. The arc rate of turning of the tangent (*i.e.*, the magnitude of $d\mathbf{T}/ds$) is called the *curvature* of the curve and is denoted by k.

Since $d\mathbf{T}/ds$ is in the direction of the principal normal $\mathbf{N}$, therefore,

$$\frac{d\mathbf{T}}{ds} = k\mathbf{N} \quad ...(3)$$

(5) Torsion. Since $\mathbf{B}$ is a unit vector, we have $\dfrac{d\mathbf{B}}{ds} . \mathbf{B} = 0$

Also $\mathbf{B} . \mathbf{T} = 0$, therefore $\dfrac{d\mathbf{B}}{ds} . \mathbf{T} + \mathbf{B} . \dfrac{d\mathbf{T}}{ds} = 0.$

or $$\frac{d\mathbf{B}}{ds} . \mathbf{T} + \mathbf{B} . (k\mathbf{N}) = 0, \qquad i.e., \quad \frac{d\mathbf{B}}{ds} . \mathbf{T} = 0 \qquad [\because \mathbf{B} . \mathbf{N} = 0]$$

Hence $d\mathbf{B}/ds$ is perpendicular to both $\mathbf{B}$ and $\mathbf{T}$ and is, therefore, parallel to $\mathbf{N}$.

The arc rate of turning of the binormal (*i.e.*, the magnitude of $d\mathbf{B}/ds$) is called *torsion* of the curve and is denoted by τ. We may, therefore, write

$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N} \quad ...(4)$$

(The negative sign indicates that for $\tau > 0$, $d\mathbf{B}/ds$ has direction of $-\mathbf{N}$).

Finally to find $d\mathbf{N}/ds$, we differentiate $\mathbf{N} = \mathbf{B} \times \mathbf{T}$.

$\therefore$ $$\frac{d\mathbf{N}}{ds} = \frac{d\mathbf{B}}{ds} \times \mathbf{T} + \mathbf{B} \times \frac{d\mathbf{T}}{ds} = -\tau\mathbf{N} \times \mathbf{T} + \mathbf{B} \times k\mathbf{N}$$

Using the relation (2), it reduces to $\dfrac{d\mathbf{N}}{ds} = \tau\mathbf{B} - k\mathbf{T}$ ...(5)

The equations (3), (4) and (5) constitute the well-known *Frenet formulae** for space curves.

Obs. 1. $\rho = 1/k$ and $\sigma = 1/\tau$ are respectively called the radii of curvature and torsion.

2. For a plane curve $\tau = 0$.

Example 8.2. *Find the angle between the tangents to the curve* $\mathbf{R} = t^2\mathbf{I} + 2t\mathbf{J} - t^3\mathbf{K}$ *at the point* $t = \pm 1$. *(V.T.U., 2010)*

Solution. The tangent at any point 't' is given by

$$\frac{d\mathbf{R}}{dt} = 2t\mathbf{I} + 2\mathbf{J} - 3t^2\mathbf{K}$$

$\therefore$ the tangents $\mathbf{T}_1$, $\mathbf{T}_2$ at $t = 1$ and $t = -1$ are respectively given by

$$\mathbf{T}_1 = 2\mathbf{I} + 2\mathbf{J} - 3\mathbf{K} \, ; \, \mathbf{T}_2 = -2\mathbf{I} + 2\mathbf{J} - 3\mathbf{K},$$

* Named after a French mathematician *Jean-Frederic Frenet* (1816–1900).

Then the required $\angle\theta$ is given by $T_1 T_2 \cos\theta = \mathbf{T}_1 \,.\, \mathbf{T}_2 = 2(-2) + 2\,.\,2 + (-3)(-3)$

i.e., $\sqrt{17}\,\sqrt{17}\,\cos\theta = 9 \quad \therefore \quad \theta = \cos^{-1}(9/17).$

Example 8.3. *Find the curvature and torsion of the curve $x = a\cos t$, $y = a\sin t$, $z = bt$.*

(This curve is drawn on a circular cylinder cutting its generators at a constant angle and is known as a *circle helix*).

Solution. The vector equation of the curve is $\mathbf{R} = a\cos t\mathbf{I} + a\sin t\mathbf{J} + bt\mathbf{K}$

$\therefore \quad d\mathbf{R}/dt = -a\sin t\mathbf{I} + a\cos t\mathbf{J} + b\mathbf{K}$

Its arc length from P_0 $(t = 0)$ to any point $P(t)$ (Fig. 8.3) is given by

$$s = \int_0^t |d\mathbf{R}/dt|\,dt = \sqrt{(a^2+b^2)}t$$

$\therefore$ $$\frac{ds}{dt} = \sqrt{(a^2+b^2)}$$

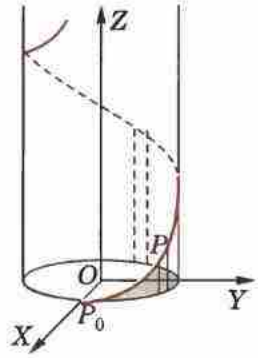

Fig. 8.3

Then $$\mathbf{T} = \frac{d\mathbf{R}}{ds} = \frac{d\mathbf{R}}{dt}\Big/\frac{ds}{dt} = \frac{-a\sin t\mathbf{I} + a\cos t\mathbf{J} + b\mathbf{K}}{\sqrt{(a^2+b^2)}}$$

and $$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dt}\Big/\frac{ds}{dt} = \frac{-a(\cos t\mathbf{I} + \sin t\mathbf{J})}{a^2+b^2}$$

Thus $$k = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{a}{a^2+b^2} \quad ...(i) \qquad \text{and} \quad \mathbf{N} = -(\cos t\mathbf{I} + \sin t\mathbf{J})$$

Also $$\mathbf{B} = \mathbf{T}\times\mathbf{N} = (b\sin t\mathbf{I} - b\cos t\mathbf{J} + a\mathbf{K})/\sqrt{(a^2+b^2)}$$

$\therefore$ $$\frac{d\mathbf{B}}{ds} = \frac{d\mathbf{B}}{dt}\Big/\frac{ds}{dt} = b(\cos t\mathbf{I} + \sin t\mathbf{J})/(a^2+b^2) = -\tau\mathbf{N} = \tau(\cos t\mathbf{I} + \sin t\mathbf{J})$$

Hence $$\tau = \frac{b}{a^2+b^2}. \qquad ...(ii)$$

PROBLEMS 8.1

1. Show that, if $\mathbf{R} = \mathbf{A}\sin\omega t + \mathbf{B}\cos\omega t$, where $\mathbf{A}, \mathbf{B}, \omega$ are constants, then (*i*) $\dfrac{d^2\mathbf{R}}{dt^2} = -\omega^2\mathbf{R}$ (*Bhopal, 2007 S*)

   (*ii*) $\mathbf{R}\times\dfrac{d\mathbf{R}}{dt} = -\omega\mathbf{A}\times\mathbf{B}$.
2. Given $\mathbf{R} = t^m\mathbf{A} + t^n\mathbf{B}$, where $\mathbf{A}, \mathbf{B}$ are constant vectors, show that, if $\mathbf{R}$ and $d^2\mathbf{R}/dt^2$ are parallel vectors, then $m + n = 1$, unless $m = n$.
3. If $\mathbf{P} = 5t^2\mathbf{I} + t^3\mathbf{J} - t\mathbf{K}$ and $\mathbf{Q} = 2\mathbf{I}\sin t - \mathbf{J}\cos t + 5t\mathbf{K}$, find (*i*) $\dfrac{d}{dt}(\mathbf{P}\,.\,\mathbf{Q})$; (*ii*) $\dfrac{d}{dt}(\mathbf{P}\times\mathbf{Q})$.
4. If $\dfrac{d\mathbf{U}}{dt} = \mathbf{W}\times\mathbf{U}$ and $\dfrac{d\mathbf{V}}{dt} = \mathbf{W}\times\mathbf{V}$, prove that $\dfrac{d}{dt}(\mathbf{U}\times\mathbf{V}) = \mathbf{W}\times(\mathbf{U}\times\mathbf{V})$. (*Mumbai, 2009*)
5. If $\mathbf{A} = x^2yz\mathbf{I} - 2xz^3\mathbf{J} + xz^2\mathbf{K}$ and $\mathbf{B} = 2z\mathbf{I} + y\mathbf{J} - x^2\mathbf{K}$, find $\dfrac{\partial^2}{\partial x\partial y}(\mathbf{A}\times\mathbf{B})$ at $(1, 0, -2)$.
6. If $\mathbf{R} = (a\cos t)\,\mathbf{I} + (a\sin t)\,\mathbf{J} + (at\tan\alpha)\,\mathbf{K}$, find the value of

   (*i*) $\left|\dfrac{d\mathbf{R}}{dt}\times\dfrac{d^2\mathbf{R}}{dt^2}\right|$ (*ii*) $\left|\dfrac{d\mathbf{R}}{dt}, \dfrac{d^2\mathbf{R}}{dt^2}, \dfrac{d^3\mathbf{R}}{dt^3}\right|$ (*Rohtak, 2005*)

   Also find the unit tangent vector at any point t of the curve.
7. Find the unit tangent vector at any point on the curve $x = t^2 + 2$, $y = 4t - 5$, $z = 2t^2 - 6t$, where t is any variable. Also determine the unit tangent vector at the point $t = 2$.
8. Find the equation of the tangent line to the curve $x = a\cos\theta$, $y = a\sin\theta$, $z = a\theta\tan\alpha$ at $\theta = \pi/4$.
9. Find the curvature of the (*i*) ellipse $\mathbf{R}(t) = a\cos t\mathbf{I} + b\sin t\mathbf{J}$; (*ii*) parabola $\mathbf{R}(t) = 2t\mathbf{I} + t^2\mathbf{J}$ at the point $t = 1$.

10. Find the education of the osculating plane and binormal to the curve
(*i*) $x = 2\cosh(t/2), y = 2\sinh(t/2), z = 2t$ at $t = 0$; (*ii*) $x = e^t \cos t, y = e^t \sin t, z = e^t$ at $t = 0$.
11. A circular helix is given by the equation $\mathbf{R}(t) = (2\cos t)\mathbf{I} + (2\sin t)\mathbf{J} + \mathbf{K}$. Find the curvature and torsion of the curve at any point and show that they are constant.
12. Show that for the curve $\mathbf{R} = a(3t - t^3)\mathbf{I} + 3at^2\mathbf{J} + a(3t + t^2)\mathbf{K}$, the curvature equals torsion.

8.3 (1) VELOCITY AND ACCELERATION

Let the position of a particle P at time t on a path (curve) C be $\mathbf{R}(t)$. At time $t + \delta t$, let the particle be at Q (Fig. 8.1), then $\delta\mathbf{R} = \mathbf{R}(t + \delta t) - \mathbf{R}(t)$ or $\delta\mathbf{R}/\delta t$ is directed along PQ. As $Q \to P$ along C, the line PQ becomes the tangent at P to C.

$$\therefore \qquad \underset{\delta t \to 0}{\text{Lt}} \frac{\delta\mathbf{R}}{\delta t} = \frac{d\mathbf{R}}{dt} = \mathbf{V}$$

is the tangent vector of C at P which is the *velocity* (vector) $\mathbf{V}$ of the motion and its magnitude is the *speed* $v = ds/dt$, where s is the arc length of P from a fixed point P_0 $(s = 0)$ on C.

The derivative of the velocity vector $\mathbf{V}(t)$ is called the *acceleration* (vector) $\mathbf{A}(t)$, which is given by

$$\mathbf{A}(t) = \frac{d\mathbf{V}}{dt} = \frac{d^2\mathbf{R}}{dt^2}.$$

(2) Tangential and normal accelerations. It is important to note that the magnitude of acceleration is not always the rate of change of $|\mathbf{V}|$ because $\mathbf{A}(t)$ is not always tangential to the path C. Infact

$$\mathbf{V}(t) = \frac{d\mathbf{R}}{dt} = \frac{d\mathbf{R}}{ds} \cdot \frac{ds}{dt}. \text{ where } d\mathbf{R}/ds \text{ is a unit tangent vector of } C.$$

$$\therefore \qquad \mathbf{A}(t) = \frac{d\mathbf{V}}{dt} = \frac{d}{dt}\left[\frac{ds}{dt} \cdot \frac{d\mathbf{R}}{ds}\right] = \frac{d^2s}{dt^2} \cdot \frac{d\mathbf{R}}{ds} + \left(\frac{ds}{dt}\right)^2 \frac{d^2\mathbf{R}}{ds^2}$$

Now since $d\mathbf{R}/dt \cdot d^2\mathbf{R}/dt^2 = 0$, $d^2\mathbf{R}/dt^2$ is perpendicular to $d\mathbf{R}/dt$. Hence the acceleration $\mathbf{A}(t)$ is comprised of (*i*) the tangential component $d^2s/dt^2 \cdot d\mathbf{R}/ds$, called the *tangential acceleration, and*

(*ii*) the normal component $(ds/dt)^2 \cdot d^2\mathbf{R}/ds^2$, called the *normal acceleration.*

Obs. The acceleration is the time rate change of $|\mathbf{V}| = ds/dt$, if the normal acceleration is zero, for then

$$|A| = \left|\frac{d^2s}{dt^2}\right| \cdot \left|\frac{d\mathbf{R}}{ds}\right| = \left|\frac{d^2s}{dt^2}\right|.$$

(3) Relative velocity and acceleration. Let two particles P_1 and P_2 moving along the curves C_1 and C_2 have position vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ at time t, (Fig. 8.4), so that $\mathbf{R} = \overrightarrow{P_1P_2} = \mathbf{R}_2 - \mathbf{R}_1$

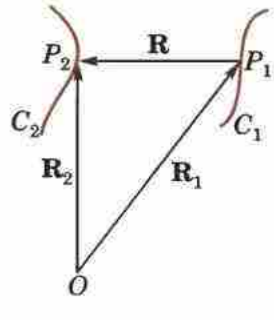

Fig. 8.4

Differentiating w.r.t. t, we get $\dfrac{d\mathbf{R}}{dt} = \dfrac{d\mathbf{R}_2}{dt} - \dfrac{d\mathbf{R}_1}{dt}$ ...(*iii*)

This defines the *relative velocity* (vector) of P_2 w.r.t. P_1 and states that the *velocity (vector) of P_2 relative to P_1 = velocity (vector) of P_2 – velocity (vector) of P_1.*

Again differentiating (*iii*), we have $\dfrac{d^2\mathbf{R}}{dt^2} = \dfrac{d^2\mathbf{R}_2}{dt^2} - \dfrac{d^2\mathbf{R}_1}{dt^2}$ ...(*iv*)

i.e., acceleration (vector) of P_2 relative to P_1 = acceleration (vector) of P_2 – acceleration (vector) of P_1.

Example 8.4. *A particle moves along the curve $x = t^3 + 1$, $y = t^2$, $z = 2t + 3$ where t is the time. Find the components of its velocity and acceleration at $t = 1$ in the direction* $\mathbf{I} + \mathbf{J} + 3\mathbf{K}$.

Solution. *Velocity* $= \dfrac{d\mathbf{R}}{dt} = \dfrac{d}{dt}[(t^3 + 1)\mathbf{I} + t^2\mathbf{J} + (2t + 3)\mathbf{K}]$

$= 3t^2\mathbf{I} + 2t\mathbf{J} + 2\mathbf{K} = 3\mathbf{I} + 2\mathbf{J} + 2\mathbf{K}$ at $t = 1$

and *acceleration* $= \dfrac{d^2\mathbf{R}}{dt^2} = 6t\mathbf{I} + 2\mathbf{J} + 0\mathbf{K} = 6\mathbf{I} + 2\mathbf{J}$ at $t = 1$.

Now unit vector in the direction of $\mathbf{I} + \mathbf{J} + 3\mathbf{K}$ is $\dfrac{\mathbf{I} + \mathbf{J} + 3\mathbf{K}}{\sqrt{(1^2 + 1^2 + 3^2)}} = \dfrac{1}{\sqrt{11}}\,(\mathbf{I} + \mathbf{J} + 3\mathbf{K})$.

$\therefore$ component of velocity at $t = 1$ in the direction $\mathbf{I} + \mathbf{J} + 3\mathbf{K}$

$$= \frac{(3\mathbf{I} + 2\mathbf{J} + 2\mathbf{K}) . (\mathbf{I} + \mathbf{J} + 3\mathbf{K})}{\sqrt{(11)}} = \frac{3 + 2 + 6}{\sqrt{(11)}} = \sqrt{(11)}$$

and component of acceleration at $t = 1$ in the direction

$$\mathbf{I} + \mathbf{J} + 3\mathbf{K} = (6\mathbf{I} + 2\mathbf{J}) . (\mathbf{I} + \mathbf{J} + 3\mathbf{K})/\sqrt{(11)} = \frac{6 + 2}{\sqrt{(11)}} = \frac{8}{\sqrt{(11)}}.$$

Example 8.5. *A particle moves along the curve* $\mathbf{R} = (t^3 - 4t)\,\mathbf{I} + (t^2 + 4t)\mathbf{J} + (8t^2 - 3t^3)\mathbf{K}$ *where t denotes time. Find the magnitudes of acceleration along the tangent and normal at time t = 2.* (V.T.U., 2003 S)

Solution. Velocity $\dfrac{d\mathbf{R}}{dt} = (3t^2 - 4)\mathbf{I} + (2t + 4)\mathbf{J} + (16t - 9t^2)\mathbf{K}$

and acceleration $\dfrac{d^2\mathbf{R}}{dt^2} = 6t\mathbf{I} + 2\mathbf{J} + (16 - 18t)\mathbf{K}$

$\therefore$ at $t = 2$, velocity $\mathbf{V} = 8\mathbf{I} + 8\mathbf{J} - 4\mathrm{K}$ and acceleration $\mathbf{A} = 12\mathbf{I} + 2\mathbf{J} - 20\mathbf{K}$.

Since the velocity is along the tangent to the curve, therefore, the component of **A** along the tangent

$$= \mathbf{A} . \frac{\mathbf{V}}{|\mathbf{V}|} = (12\mathbf{I} + 2\mathbf{J} - 20\mathbf{K}) . \frac{8\mathbf{I} + 8\mathbf{J} - 4\mathbf{K}}{\sqrt{(64 + 64 + 16)}}$$

$$= \frac{12 \times 8 + 2 \times 8 + (-20) \times (-4)}{12} = 16.$$

Now the component of **A** along the normal

$= |\, \mathbf{A}$ – Resolved part of **A** along the tangent $|$

$$= \left| 12\mathbf{I} + 2\mathbf{J} - 20\mathbf{K} - 16\,\frac{8\mathbf{I} + 8\mathbf{J} - 4\mathbf{K}}{12} \right| = \frac{1}{3}\,|\, 4\mathbf{I} - 26\mathbf{J} - 44\mathbf{K} \,| = 2\sqrt{73}.$$

Example 8.6. *The position vector of a particle at time t is* $\mathbf{R} = \cos(t - 1)\,\mathbf{I} + \sinh(t - 1)\,\mathbf{J} + \alpha t^3\mathbf{K}$. *Find the condition imposed on α by requiring that at time t = 1, the acceleration is normal to the position vector.*

Solution. *Velocity* $= \dfrac{d\mathbf{R}}{dt} = -\sin(t - 1)\,\mathbf{I} + \cosh(t - 1)\,\mathbf{J} + 3\alpha t^2\mathbf{K}$

Acceleration $= \dfrac{d^2\mathbf{R}}{dt^2} = -\cos(t - 1)\,\mathbf{I} + \sinh(t - 1)\,\mathbf{J} + 6\alpha t\mathbf{K} = -\mathbf{I} + 6\alpha\mathbf{K}$ at $t = 1$.

Also $\mathbf{R} = \mathbf{I} + \alpha\mathbf{K}$ at $t = 1$.

If **R** and acceleration at $t = 1$ are normal, then their scalar product is zero.

$\therefore$ $(-\mathbf{I} + 6\alpha\mathbf{K}) . (\mathbf{I} + \alpha\mathbf{K}) = 0$ or $-1 + 6\alpha^2 = 0$

or $\alpha^2 = 1/6$ or $\alpha = 1/\sqrt{6}$.

Example 8.7. *Find the radial and transverse acceleration of a particle moving in a plane curve.* (Kurukshetra, 2006 ; Rajasthan, 2006)

Solution. At any time t, let the position vector of the moving particle $P(r, \theta)$ be **R** (Fig. 8.5) so that

$$\mathbf{R} = r\hat{\mathbf{R}} = r(\cos\theta\mathbf{I} + \sin\theta\mathbf{J})$$

$\therefore$ its velocity $\mathbf{V} = \dfrac{d\mathbf{R}}{dt} = \dfrac{dr}{dt}\hat{\mathbf{R}} + r\dfrac{d\hat{\mathbf{R}}}{dt}$ ...(*i*)

As $\hat{\mathbf{R}} = \cos\theta\mathbf{I} + \sin\theta\mathrm{J}$

and $\dfrac{d\hat{\mathbf{R}}}{dt} = (-\sin\theta\mathbf{I} + \cos\theta\mathbf{J})\,\dfrac{d\theta}{dt}$

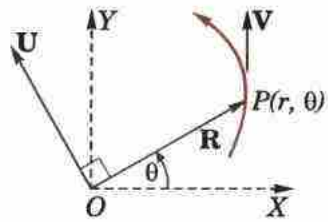

Fig. 8.5

$$\therefore \qquad \frac{d\hat{\mathbf{R}}}{dt} \perp \hat{\mathbf{R}} \text{ and } \left|\frac{d\hat{\mathbf{R}}}{dt}\right| = \frac{d\theta}{dt}, \textit{ i.e.}, \text{ if } \mathbf{U} \text{ is a unit vector } \perp \mathbf{R}, \text{ then}$$

$$\frac{d\hat{\mathbf{R}}}{dt} = \frac{d\theta}{dt}\,\mathbf{U}$$

$\therefore$ (*i*) becomes, $$\mathbf{V} = \frac{dr}{dt}\hat{\mathbf{R}} + r\frac{d\theta}{dt}\mathbf{U} \qquad ...(ii)$$

Thus the radial and transverse components of the velocity are dr/dt and $r\, d\theta/dt$.

Also $$\mathbf{A} = \frac{d\mathbf{V}}{dt} = \frac{d^2r}{dt^2}\hat{\mathbf{R}} + \frac{dr}{dt}\frac{d\hat{\mathbf{R}}}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{U} + r\frac{d^2\theta}{dt^2}\mathbf{U} + r\frac{d\theta}{dt}\frac{d\mathbf{U}}{dt}$$

$$= \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{\mathbf{R}} + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\mathbf{U} \qquad \left[\because\ \mathbf{U} = -\sin\theta\mathbf{I} + \cos\theta\mathbf{J} \text{ gives } \frac{d\mathbf{U}}{dt} = -\frac{d\theta}{dt}\hat{\mathbf{R}}\right]$$

Thus the radial and transverse components of the acceleration are

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \text{ and } 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}.$$

Example 8.8. *A person going eastwards with a velocity of 4 km per hour, finds that the wind appears to blow directly from the north. He doubles his speed and the wind seems to come from north-east. Find the actual velocity of the wind.*

Solution. Let the actual velocity of the wind be $x\mathbf{I} + y\mathbf{J}$, where **I**, **J** represent velocities of 1 km per hour towards the east and north respectively. As the person is going eastwards with a velocity of 4 km per hour, his actual velocity is $4\mathbf{I}$.

Then the velocity of the wind relative to the man is $(x\mathbf{I} + y\mathbf{J}) - 4\mathbf{I}$, which is parallel to $-\mathbf{J}$, as it appears to blow from the north. Hence $x = 4$. ...(*i*)

When the velocity of the person becomes $8\mathbf{I}$, the velocity of the wind relative to man is $(x\mathbf{I} + y\mathbf{J}) - 8\mathbf{I}$. But this is parallel to $-(\mathbf{I} + \mathbf{J})$.

$\therefore$ $(x - 8)/y = 1$, which by (*i*) gives $y = -4$.

Hence the actual velocity of the wind is $4(\mathbf{I} - \mathbf{J})$, *i.e.*, $4\sqrt{2}$ km. per hour towards the south-east.

PROBLEMS 8.2

1. A particle moves along a curve $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$, where t is the time variable. Determine its velocity and acceleration vectors and also the magnitudes of velocity and acceleration at $t = 0$.
   (*P.T.U., 2003 ; V.T.U., 2003 S*)
2. The position vector of a particle at time t is $\mathbf{R} = \cos(t-1)\,\mathbf{I} + \sinh(t-1)\,\mathbf{J} + at^3\mathbf{K}$. Find the condition imposed on a by requiring that at time $t = 1$, the acceleration is normal to the position vector.
3. A particle moves on the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$, where t is the time. Find the components of velocity and acceleration at time $t = 1$ in the direction $\mathbf{I} - 3\mathbf{J} + 2\mathbf{K}$. (*V.T.U., 2008*)
4. A particle moves so that its position vector is given by $\mathbf{R} = \mathbf{I}\cos\omega t + \mathbf{J}\sin\omega t$. Show that the velocity **V** of the particle is perpendicular to **R** and $\mathbf{R} \times \mathbf{V}$ is a constant vector.
5. A particle (position vector **R**) is moving in a circle with constant angular velocity ω. Show by vector methods, that the acceleration is equal to $-\omega^2\mathbf{R}$.
6. (*a*) Find the tangential and normal accelerations of a point moving in a plane curve. (*Rajasthan, 2005*)
   (*b*) The position vector of a moving particle at a time t is $\mathbf{R} = 3\cos t\mathbf{I} + 3\sin t\mathbf{J} + 4t\mathbf{K}$. Find the tangent and normal components of its acceleration at $t = 1$. (*Marathwada, 2008*)
7. The velocity of a boat relative to water is represented by $3\mathbf{I} + 4\mathbf{J}$ and that of wate relative to earth is $\mathbf{I} - 3\mathbf{J}$. What is the velocity of the boat relative to the earth if **I** and **J** represent one km an hour east and north respectively.
8. A vessel A is sailing with a velocity of 11 knots per hour in the direction S.E. and a second vessel B is sailing with a velocity of 13 knots per hour in a direction 30°E of N. Find the velocity of A relative to B.
9. A person travelling towards the north-east with a velocity of 6 km per hour finds that the wind appears to blow from the north, but when he doubles his speed it seems to come from a direction inclined at an angle $\tan^{-1} 2$ to the north of east. Show that the actual velocity of the wind is $3\sqrt{2}$ km per hour towards the east.

8.4 SCALAR AND VECTOR POINT FUNCTIONS

(1) *If to each point P(**R**) of a region E in space there corresponds a definite scalar denoted by f(**R**), then f(**R**) is called a* **scalar point function** *in E. The region E so defined is called a* **scalar field.**

The temperature at any instant, density of a body and potential due to gravitational matter are all examples of scalar point functions.

(2) *If to each point P(**R**) of a region E in space there corresponds a definite vector denoted by **F**(**R**), then it is called the* **vector point function** *in E. The region E so defined is called a* **vector field.**

The velocity of a moving fluid at any instant, the gravitational intensity of force are examples of vector point functions.

Differentiation of vector point functions follows the same rules as those of ordinary calculus. Thus if $\mathbf{F}(x, y, z)$ be a vector point function, then

$$\frac{d\mathbf{F}}{dt} = \frac{\partial \mathbf{F}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{F}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{F}}{\partial z}\frac{dz}{dt}$$ (See (*iii*) p. 203]

and $$d\mathbf{F} = \frac{\partial \mathbf{F}}{\partial x}dx + \frac{\partial \mathbf{F}}{\partial y}dy + \frac{\partial \mathbf{F}}{\partial z}dz = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy + \frac{\partial}{\partial z}dz\right)\mathbf{F} \quad ...(i)$$

(3) Vector operator del. The operator on the right side of the equation (*i*) is in the form of a scalar product of $\mathbf{I}\frac{\partial}{\partial x} + \mathbf{J}\frac{\partial}{\partial y} + \mathbf{K}\frac{\partial}{\partial z}$ and $\mathbf{I}dx + \mathbf{J}dy + \mathbf{K}dz$.

If ∇ *(read as del) be defined by the equation* $$\nabla = \mathbf{I}\frac{\partial}{\partial x} + \mathbf{J}\frac{\partial}{\partial y} + \mathbf{K}\frac{\partial}{\partial z} \quad ...(ii)$$

then (*i*) may be written as $d\mathbf{F} = (\nabla \,.\, d\mathbf{R})\,\mathbf{F}$ for when $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$, $d\mathbf{R} = \mathbf{I}dx + \mathbf{J}dy + \mathbf{K}dz$.

8.5 DEL APPLIED TO SCALAR POINT FUNCTIONS—GRADIENT

(1) Def. *The vector function ∇f is defined as the gradient of the scalar point function f and is written as grad f.*

Thus $$\text{grad } f = \nabla f = \mathbf{I}\frac{\partial f}{\partial x} + \mathbf{J}\frac{\partial f}{\partial y} + \mathbf{K}\frac{\partial f}{\partial z}$$

(2) Geometrical interpretation. Consider the scalar point function $f(\mathbf{R})$, where $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$.

If a surface $f(x, y, z) = c$ be drawn through any point $P(\mathbf{R})$ such that at each point on it, the function has the same value as at P, then such a surface is called a *level surface* of the function f through P, *e.g.*, equipotential or isothermal surface (Fig. 8.6).

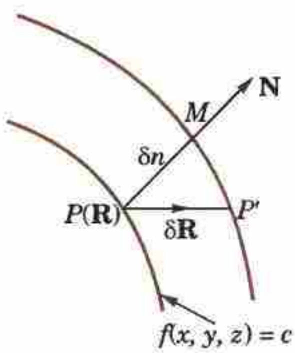

Fig. 8.6

Let $P'(\mathbf{R} + \delta\mathbf{R})$ be a point on a neighbouring level surface $f + \delta f$. Then

$$\nabla f \,.\, \delta\mathbf{R} = \left[\mathbf{I}\frac{\partial f}{\partial x} + \mathbf{J}\frac{\partial f}{\partial y} + \mathbf{K}\frac{\partial f}{\partial z}\right] . (\mathbf{I}\delta x + \mathbf{J}\delta y + \mathbf{K}\delta z)$$

$$= \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z = \delta f.$$

Now if P' lies on the same level surface as P, then $\delta f = 0$, *i.e.*, $\nabla f \,.\, \delta\mathbf{R} = 0$. This means that ∇f is perpendicular to every $\delta\mathbf{R}$ lying on this surface. Thus ∇f *is normal to the surface* $f(x, y, z) = c$.

$$\therefore \qquad \nabla f = |\nabla f|\,\mathbf{N}$$

where $\mathbf{N}$ is a unit vector normal to this surface. If the perpendicular distance PM between the surfaces through P and P' be δn, then the rate of change of f normal to the surface through P

$$= \frac{\partial f}{\partial n} = \underset{\delta n \to 0}{\text{Lt}} \frac{\delta f}{\delta n} = \underset{\delta n \to 0}{\text{Lt}} \nabla f \,.\, \frac{\delta\mathbf{R}}{\delta n}$$

$$= |\nabla f| \underset{\delta n \to 0}{\text{Lt}} \frac{\mathbf{N}\,.\,\delta\mathbf{R}}{\delta n} = |\nabla f|. \qquad [\because \ \mathbf{N}\,.\,\delta\mathbf{R} = |\delta\mathbf{R}|\cos\theta = \delta n]$$

Hence the magnitude of $\nabla f = \partial f/\partial n$.

Thus *grad f is a vector normal to the surface f = constant and has a magnitude equal to the rate of change of f along this normal.*

(3) Directional derivative. If δr denotes the length PP' and $\mathbf{N}'$ is a unit vector in the direction PP', then the limiting value of $\delta f/\delta r$ as $\delta r \to 0$ (*i.e.*, $\partial f/\partial r$) is known as the *directional derivative of f at P along the direction PP'.*

Since $\qquad \delta r = \delta n/\cos\alpha = \delta n/\mathbf{N}\,.\,\mathbf{N}'$

$$\therefore \qquad \frac{\partial f}{\partial r} = \underset{\delta r \to 0}{\mathrm{Lt}} \left[\mathbf{N}\,.\,\mathbf{N}'\,\frac{\delta f}{\delta n}\right] = \mathbf{N}'\,.\,\frac{\partial f}{\partial n}\,\mathbf{N} = \mathbf{N}'\,.\,\nabla f$$

Thus the directional derivation of f in the direction of $\mathbf{N}'$ is the resolved part of ∇f in the direction $\mathbf{N}'$.

Since $\qquad \nabla f\,.\,\mathbf{N}' = |\,\nabla f\,|\,\cos\alpha \le |\,\nabla f\,|$

It follows that ∇f *gives the maximum rate of change of f.*

Example 8.9. *Prove that $\nabla r^n = nr^{n-2}\,\mathbf{R}$, where $\mathbf{R} + x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$.*
(Bhopal, 2007 ; Anna, 2003 S ; V.T.U., 2000)

Solution. We have $\quad f(x, y, z) = r^n = (x^2 + y^2 + z^2)^{n/2}$

$$\therefore \qquad \frac{\partial f}{\partial x} = \frac{n}{2}(x^2 + y^2 + z^2)^{n/2-1}.\,2x = nxr^{n-2}. \text{ Similarly, } \frac{\partial f}{\partial y} = nyr^{n-2} \text{ and } \frac{\partial f}{\partial z} = nzr^{n-2}$$

$$\text{Thus} \qquad \nabla r^n = \mathbf{I}\frac{\partial f}{\partial x} + \mathbf{J}\frac{\partial f}{\partial y} + \mathbf{K}\frac{\partial f}{\partial z} = nr^{n-2}(x\mathbf{I} + y\mathbf{J} + z\mathbf{K}) = nr^{n-2}\,\mathbf{R}.$$

Otherwise : The level surfaces for f = constant, *i.e.*, r^n = constant are concentric spheres with centre O and hence unit normal $\mathbf{N}$ to the level surface through P is along the radius $\mathbf{R}$

i.e., $\qquad \mathbf{N} = \hat{\mathbf{R}}$.

$$\therefore \qquad \nabla f = \frac{\partial f}{\partial n}\,.\,\mathbf{N} = \frac{df}{dr}\hat{\mathbf{R}} = nr^{n-1}\,\hat{\mathbf{R}} \qquad [\because \; f = r^n]$$
$$= nr^{n-1}(\mathbf{R}/r) = nr^{n-2}\mathbf{R}.$$

Example 8.10. *If $\nabla u = 2r^4\,\mathbf{R}$, find u.* *(Mumbai, 2008)*

Solution. We have $\nabla u = 2(x^2 + y^2 + z^2)^2\,\mathbf{R} \qquad [\because \; r = \sqrt{(x^2 + y^2 + z^2)}\,]$

$$= 2(x^2 + y^2 + z^2)^2\,(x\mathbf{I} + y\mathbf{J} + z\mathbf{K}) \qquad ...(i)$$

$$\text{But} \qquad \nabla u = \frac{\partial u}{\partial x}\mathbf{I} + \frac{\partial u}{\partial y}\mathbf{J} + \frac{\partial u}{\partial z}\mathbf{K} \qquad ...(ii)$$

Comparing (*i*) and (*ii*), we get

$$\frac{\partial u}{\partial x} = 2x(x^2 + y^2 + z^2)^2,\; \frac{\partial u}{\partial y} = 2y(x^2 + y^2 + z^2)^2\,;\; \frac{\partial u}{\partial z} = 2z(x^2 + y^2 + z^2)$$

$$\text{Also} \qquad du(x, y, z) = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 2(x^2 + y^2 + z^2)^2\,(xdx + ydy + zdz)$$

$$= 2t^2\,.\,\frac{dt}{2}, \text{ taking } x^2 + y^2 + z^2 = t \quad \text{and} \quad 2(xdx + ydy + zdz) = dt$$

Integrating both sides, $u = \int t^2\,dt + c = \dfrac{1}{3}t^3 + c = \dfrac{1}{3}(x^2 + y^2 + z^2)^3 + c$

$$\text{Hence} \qquad u = \frac{1}{3}r^{3/2} + c.$$

Example 8.11. *If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = yz + zx + xy$, prove that grad u, grad v and grad w are coplanar.* *(U.T.U., 2010 ; U.P.T.U., 2002)*

Solution. $\qquad \text{grad } u = \left(\mathbf{I}\dfrac{\partial}{\partial x} + \mathbf{J}\dfrac{\partial}{\partial y} + \mathbf{K}\dfrac{\partial}{\partial z}\right)(x + y + z) = \mathbf{I} + \mathbf{J} + \mathbf{K}$

$$\text{grad } v = 2x\mathbf{I} + 2y\mathbf{J} + 2z\mathbf{K},\; \text{grad } w = (y + z)\,\mathbf{I} + (z + x)\,\mathbf{J} + (z + x)\,\mathbf{K}$$

We know that three vectors are coplanar if their scalar triple product is zero.

Here [grad u, grad v, grad w]

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 1 \\ x+y+z & y+z+x & z+x+y \\ y+z & z+x & x+y \end{vmatrix} \qquad [\text{Operate } R_2 + R_3]$$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix} = 0.$$

Hence grad u, grad v and grad w are coplanar.

Example 8.12. *Find a unit vector normal to the surface $xy^3z^2 = 4$ at the point $(-1, -1, 2)$.* *(Mumbai, 2008)*

Solution. A vector normal to the given surface is $\nabla(xy^3z^2)$

$$= \mathbf{I}\frac{\partial}{\partial x}(xy^3z^2) + \mathbf{J}\frac{\partial}{\partial y}(xy^3z^2) + \mathbf{K}\frac{\partial}{\partial z}(xy^3z^2) = \mathbf{I}(y^3z^2) + \mathbf{J}(3xy^2z^2) + \mathbf{K}(2xy^3z)$$

$$= -4\mathbf{I} - 12\mathbf{J} + 4\mathbf{K} \text{ at the point } (-1, -1, 2).$$

Hence the desired unit normal to the surface

$$= \frac{-4\mathbf{I} - 12\mathbf{J} + 4\mathbf{K}}{\sqrt{[(-4)^2 + (-12)^2 + 4^2]}} = -\frac{1}{\sqrt{11}}(\mathbf{I} + 3\mathbf{J} - \mathbf{K}).$$

Example 8.13. *Find the directional derivative of $f(x, y, z) = xy^3 + yz^3$ at the point $(2, -1, 1)$ in the direction of vector* $\mathbf{I} + 2\mathbf{J} + 2\mathbf{K}$. *(Bhopal, 2008 ; Kurukshetra, 2006 ; Rohtak, 2003)*

Solution. Here $\nabla f = \mathbf{I}(y^2) + \mathbf{J}(2xy + z^3) + \mathbf{K}(3yz^2) = \mathbf{I} - 3\mathbf{J} - 3\mathbf{K}$ at the point $(2, -1, 1)$.

$\therefore$ directional derivative of f in the direction $\mathbf{I} + 2\mathbf{J} + 2\mathbf{K}$

$$= (\mathbf{I} - 3\mathbf{J} - 3\mathbf{K}) \cdot \frac{\mathbf{I} + 2\mathbf{J} + 2\mathbf{K}}{\sqrt{(1^2 + 2^2 + 2^2)}} = (1 \cdot 1 - 3 \cdot 2 - 3 \cdot 2)/3 = -3\frac{2}{3}.$$

Example 8.14. *Find the directional derivative of $f = x^2 - y^2 + 2z^2$ at the point $P(1, 2, 3)$ in the direction of the line PQ where Q is the point $(5, 0, 4)$. Also calculate the magnitude of the maximum directional derivative.*

Solution. We have $\nabla f = \left(\mathbf{I}\frac{\partial}{\partial x} + \mathbf{J}\frac{\partial}{\partial y} + \mathbf{K}\frac{\partial}{\partial z}\right)(x^2 - y^2 + 2z^2) = 2x\mathbf{I} - 2y\mathbf{J} + 4z\mathbf{K}$

$$= 2\mathbf{I} - 4\mathbf{J} + 12\mathbf{K} \text{ at } P(1, 2, 3)$$

Also $\quad \vec{PQ} = \vec{OQ} - \vec{OP} = (5\mathbf{I} + 0\mathbf{J} + 4\mathbf{K}) - (\mathbf{I} + 2\mathbf{J} + 3\mathbf{K}) = 4\mathbf{I} - 2\mathbf{J} + \mathbf{K} = \mathbf{A}$ (say)

$\therefore$ unit vector of $\mathbf{A} = \hat{A} = \dfrac{\mathbf{A}}{a} = \dfrac{4\mathbf{I} - 2\mathbf{J} + \mathbf{K}}{\sqrt{(16 + 4 + 1)}} = \dfrac{4\mathbf{I} + 2\mathbf{J} + \mathbf{K}}{\sqrt{21}}$

Thus the directional derivative of f in the direction of $\vec{PQ}$

$$\nabla f \cdot \hat{A} = (2\mathbf{I} - 4\mathbf{J} + 12\mathbf{K}) \cdot (4\mathbf{I} - 2\mathbf{J} + \mathbf{K})/\sqrt{21}$$

$$= (8 + 8 + 12)/\sqrt{21} = 28/\sqrt{21}$$

The directional derivative of its maximum in the direction of the normal to the surface *i.e.*, in the direction of ∇f.

Hence maximum value of this directional derivative

$$= |\nabla f| = |\,2\mathbf{I} - 4\mathbf{J} + 12\mathbf{K}\,| = (4 + 16 + 144) = \sqrt{164}.$$

Example 8.15. *Find the directional derivative of $\phi = 5x^2y - 5y^2z + 2.5z^2x$ at the point $P(1, 1, 1)$ in the direction of the line $\frac{x-1}{2} = \frac{y-3}{-2} = z$.* (*Bhopal, 2008 ; U.P.T.U., 2004*)

Solution. We have
$$\nabla\phi = \mathbf{I}\frac{\partial\phi}{\partial x} + \mathbf{J}\frac{\partial\phi}{\partial y} + \mathbf{K}\frac{\partial\phi}{\partial z}$$
$$= (10xy + 2.5z^2)\,\mathbf{I} + (5x^2 - 10yz)\,\mathbf{J} + (-5y^2 + 5zx)\,\mathbf{K}$$
$$= 12.5\mathbf{I} - 5\mathbf{J} \text{ at } P(1, 1, 1)$$

Also direction of the given line is $\hat{A} = \dfrac{2\mathbf{I} - 2\mathbf{J} + \mathbf{K}}{3}$

Hence the required directional derivative
$$= \nabla\phi \,.\, \hat{A} = (12.5\mathbf{I} - 5\mathbf{J}) \,.\, (2\mathbf{I} - 2\mathbf{J} + \mathbf{K})/3 = (25 + 10)/3 = 11\frac{2}{3}.$$

Example 8.16. *Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$.* (*V.T.U., 2010 ; Kottayam, 2005 ; U.P.T.U., 2003*)

Solution. Let $\quad f_1 = x^2 + y^2 + z^2 - 9 = 0$ and $f_2 = x^2 + y^2 - z - 3 = 0$

Then $\quad N_1 = \nabla f_1$ at $(2, -1, 2) = (2x\mathbf{I} + 2y\mathbf{J} + 2z\mathbf{K})$ at $(2, -1, 2) = 4\mathbf{I} - 2\mathbf{J} + 4\mathbf{K}$

and $\quad N_2 = \nabla f_2$ at $(2, -1, 2) = (2x\mathbf{I} + 2y\mathbf{J} - \mathbf{K})$ at $(2, -1, 2) = 4\mathbf{I} - 2\mathbf{J} - \mathbf{K}$

Since the angle θ between the two surfaces at a point is the angle between their normals at that point and $\mathbf{N}_1, \mathbf{N}_2$ are the normals at $(2, -1, 2)$ to the given surfaces, therefore
$$\cos\theta = \frac{\mathbf{N}_1 \,.\, \mathbf{N}_2}{n_1 n_2} = \frac{(4I - 2J + 4K) \,.\, (4I - 2J - K)}{\sqrt{(16+4+16)}\,\sqrt{(16+4+1)}}$$
$$= \frac{4(4) + (-2)(-2) + 4(-1)}{6\sqrt{21}} = \frac{16}{6\sqrt{21}}$$

Hence the required angle $\theta = \cos^{-1}\left(\dfrac{8}{3\sqrt{21}}\right)$.

Example 8.17. *Find the values of a and b such that the surface $ax^2 - byz = (a + 2)\,x$ and $4x^2y + z^3 = 4$ cut orthogonally at $(1, -1, 2)$.* (*Madras, 2004*)

Solution. Let $\quad f_1 = ax^2 - byz - (a + 2)\,x = 0 \quad$ ...(*i*)

and $\quad f_2 = 4x^2y + z^3 - 4 = 0 \quad$ ...(*ii*)

Then $\quad \nabla f_1 = (2ax - a - 2)\mathbf{I} - 4z\mathbf{J} - by\mathbf{K} = (a - 2)\,\mathbf{I} - 2b\mathbf{J} + b\mathbf{K}$ at $(1, -1, 2)$.

$\quad \nabla f_2 = 8xy\mathbf{I} + 4x^2\mathbf{J} + 3z^2\mathbf{K} = -8\mathbf{I} + 4\mathbf{J} + 12\mathbf{K}$ at $(1, -1, 2)$.

The surfaces (*i*) and (*ii*) will cut orthogonally if $\nabla f_1 \,.\, \nabla f_2 = 0$, *i.e.*, $-8\,(a - 2) - 8b + 12b = 0$

or $\quad -2a + b + 4 = 0 \quad$ ...(*iii*)

Also since the point $(1, -1, 2)$ lies on (*i*) and (*ii*),

∴ $\quad a + 2b - (a + 2) = 0 \quad$ or $\quad b = 1$

From (*iii*), $\quad -2a + 5 = 0 \quad$ or $\quad a = 5/2$.

Hence $\quad a = 5/2$ and $\quad b = 1$.

PROBLEMS 8.3

1. (*a*) Find $\nabla\phi$, if $\phi = \log(x^2 + y^2 + z^2)$. (*b*) Show that grad $(1/r) = -\mathbf{R}/r^3$.
2. Find a unit vector normal to the surface $x^3 + y^3 + 3xyz = 3$ at the point $(1, 2, -1)$. (*P.T.U., 1999*)
3. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point $(1, -2, 1)$ in the direction of the vector $2\mathbf{I} - \mathbf{J} - 2\mathbf{K}$. (*V.T.U., 2007 ; Rohtak 2006 S ; J.N.T.U., 2006 ; U.P.T.U., 2006*)
4. What is the directional derivative of $\phi = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of the normal to the surface $x \log z - y^2 = -4$ at $(-1, 2, 1)$? (*S.V.T.U., 2009*)

5. Find the values of constants a, b, c so that the directional derivative of $p = axy^2 + byz + cz^2x^3$ at $(1, 2, -1)$ has a maximum magnitude 64 in the direction parallel to the z-axis. *(Rajasthan, 2006)*
6. Find the directional derivative of $\phi = x^4 + y^4 + z^4$ at the point $A(1, -2, 1)$ in the direction AB where B is $(2, 6, -1)$. Also find the maximum directional derivative of ϕ at $(1, -2, 1)$. *(Mumbai, 2009)*
7. If the directional derivative of $\phi = ax^2y + by^2z + cz^2x$ at the point $(1, 1, 1)$ has maximum magnitude 15 in the direction parallel to the line $\frac{x-1}{2} = \frac{y-3}{-2} = z$, find the values of a, b and c. *(U.P.T.U., 2002)*
8. In what direction from $(3, 1, -2)$ is the directional derivative of $\phi = x^2y^2z^4$ maximum ? Find also the magnitude of this maximum. *(Rohtak, 2003)*
9. What is the greatest rate of increase of $u = xyz^2$ at the point $(1, 0, 3)$? *(Bhopal, 2008)*
10. The temperature of points in space is given by $T(x, y, z) = x^2 + y^2 - z$. A mosquito located at $(1, 1, 2)$ desires to fly in such a direction that it will get warm as soon as possible. In what direction should it move ?
11. Calculate the angle between the normals to the surface $xy = z^2$ at the points $(4, 1, 2)$ and $(3, 3, -3)$.
12. Find the angle between the tangent planes to the surfaces $x \log z = y^2 - 1$, $x^2y = 2 - z$ at the point $(1, 1, 1)$. *(Hissar, 2005 S ; J.N.T.U., 2003)*
13. Find the values of a and b so that the surface $5x^2 - 2yz - 9z = 0$ may cut the surface $ax^2 + by^3 = 4$ orthogonally at $(1, -1, 2)$. *(Nagpur, 2009)*
14. If f and $\mathbf{G}$ are point functions, prove that the components of the latter normal and tangential to the surface $f = 0$ are
$$\frac{(\mathbf{G} \cdot \nabla f)\nabla f}{(\nabla f)^2} \text{ and } \frac{\nabla f \times (\mathbf{G} \times \nabla f)}{(\nabla f)^2}$$ *[Cf. Ex. 3.24]*

8.6 DEL APPLIED TO VECTOR POINT FUNCTIONS

(1) Divergence. *The divergence of a continuously differentiable vector point function* $\mathbf{F}$ *is denoted by div* $\mathbf{F}$ *and is defined by the equation*

$$\textbf{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \mathbf{I} \cdot \frac{\partial \mathbf{F}}{\partial x} + \mathbf{J} \cdot \frac{\partial \mathbf{F}}{\partial y} + \mathbf{K} \cdot \frac{\partial \mathbf{F}}{\partial z}$$

If $\mathbf{F} = f\mathbf{I} + \phi\mathbf{J} + \psi\mathbf{K}$

then
$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \left(\mathbf{I}\frac{\partial}{\partial x} + \mathbf{J}\frac{\partial}{\partial y} + \mathbf{K}\frac{\partial}{\partial z}\right) \cdot (f\mathbf{I} + \phi\mathbf{J} + \psi\mathbf{K}) = \frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial z}$$

(2) Curl. *The curl of a continuously differentiable vector point function* $\mathbf{F}$ *is defined by the equation*

$$\textbf{curl } \mathbf{F} = \nabla \times \mathbf{F} = \mathbf{I} \times \frac{\partial \mathbf{F}}{\partial x} + \mathbf{J} \times \frac{\partial \mathbf{F}}{\partial y} + \mathbf{K} \times \frac{\partial \mathbf{F}}{\partial z}$$

If $\mathbf{F} = f\mathbf{I} + \phi\mathbf{J} + \psi\mathbf{K}$ then $\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \left(\mathbf{I}\frac{\partial}{\partial x} + \mathbf{J}\frac{\partial}{\partial y} + \mathbf{K}\frac{\partial}{\partial z}\right) \times (f\mathbf{I} + \phi\mathbf{J} + \psi\mathbf{K})$

$$= \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & \phi & \psi \end{vmatrix} = \mathbf{I}\left(\frac{\partial \psi}{\partial y} - \frac{\partial \phi}{\partial z}\right) + \mathbf{J}\left(\frac{\partial f}{\partial z} - \frac{\partial \psi}{\partial x}\right) + \mathbf{K}\left(\frac{\partial \phi}{\partial x} - \frac{\partial f}{\partial y}\right).$$

Example 8.18. *If* $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$, *show that*
(i) $\nabla \cdot \mathbf{R} = 3$ *(ii)* $\nabla \times \mathbf{R} = 0$. *(V.T.U. 2008 ; P.T.U., 2006 ; U.P.T.U., 2006)*

Solution. *(i)* $\nabla \cdot \mathbf{R} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3.$

(ii)
$$\nabla \times \mathbf{R} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \mathbf{I}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) - \mathbf{J}\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right) + \mathbf{K}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$$
$$= \mathbf{I}(0 - 0) - \mathbf{J}(0 - 0) + \mathbf{K}(0 - 0) = \mathbf{0}.$$

[Remember : *div* $\mathbf{R} = \mathbf{3}$; *curl* $\mathbf{R} = \mathbf{0}$**]**

Example 8.19. *Find div* **F** *and curl* **F**, *where* $\mathbf{F} = grad\,(x^3 + y^3 + z^3 - 3xyz)$.

(*V.T.U., 2008 ; Kurukshetra, 2006 ; Burdwan, 2003*)

Solution. If $u = x^3 + y^3 + z^3 - 3xyz$, then

$$\mathbf{F} = \nabla u = \mathbf{I}\frac{\partial u}{\partial x} + \mathbf{J}\frac{\partial u}{\partial y} + \mathbf{K}\frac{\partial u}{\partial z} = \mathbf{I}(3x^2 - 3yz) + \mathbf{J}(3y^2 - 3zx) + \mathbf{K}(3z^2 - 3xy)$$

$$\therefore \qquad \text{div}\,\mathbf{F} = \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3zx) + \frac{\partial}{\partial z}(3z^2 - 3xy) = 6(x + y + z)$$

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3(x^2 - yz) & 3(y^2 - zx) & 3(z^2 - xy) \end{vmatrix}$$

$$= \mathbf{I}(-3x + 3x) - \mathbf{J}(-3y + 3y) + \mathbf{K}(-3z + 3z) = \mathbf{0}.$$

8.7 (1) PHYSICAL INTERPRETATION OF DIVERGENCE

Consider the motion of the fluid having velocity $\mathbf{V} = v_x\mathbf{I} + v_y\mathbf{J} + v_z\mathbf{K}$ at a point $P(x, y, z)$. Consider a small parallelopiped with edges δx, δy, δz parallel to the axes in the mass of fluid, with one of its corners at P (Fig. 8.7).

$\therefore$ the amount of fluid entering the face PB' in unit time $= v_y\delta z\delta x$ and the amount of fluid leaving the face $P'B$ in unit time

$$= v_{y+\delta y}\delta z\delta x = \left(v_y + \frac{\partial v_y}{\partial y}\delta y\right)\delta z\delta x \quad \text{nearly}$$

$\therefore$ the net decrease of the amount of fluid due to flow across these two faces $= \dfrac{\partial v_y}{\partial y}\delta x\delta y\delta z$.

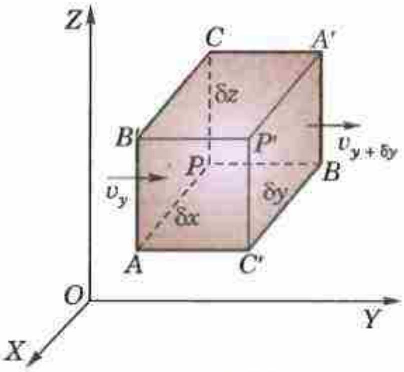

Fig. 8.7

Finding similarly the contributions of other two pairs of faces, we have the total decrease of amount of fluid inside the parallelopiped per unit time $= \left(\dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} + \dfrac{\partial v_z}{\partial z}\right)\delta x\delta y\delta z$.

Thus the rate of loss of fluid per unit volume

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \text{div}\,\mathbf{V}.$$

Hence *div* **V** *gives the rate at which fluid is originating at a point per unit volume.*

Similarly, if V represents an electric flux, div **V** is the amount of flux which diverges per unit volume in unit time. If **V** represents heat flux, div **V** is the rate at which heat is issuing from a point per unit volume. In general, the divergence of a vector point function representing any physical quantity gives at each point, the rate per unit volume at which the physical quantity is issuing from that point. This explains the justification for the name *divergence of a vector point function.*

If the fluid is incompressible, there can be no gain or loss in the volume element. Hence div $\mathbf{V} = 0$, which is known in Hydrodynamics as the **equation of continuity** for incompressible fluids.

Def. *If the flux entering any element of space is the same as that leaving it, i.e., div* $\mathbf{V} = 0$ *everywhere then such a point function is called a* **solenoidal vector function.**

(2) Physical interpretation of curl. Consider the motion of a rigid body rotating about a fixed axis through O. If Ω be its angular velocity, then the velocity **V** of any particle $P(\mathbf{R})$ of the body is given by $\mathbf{V} = \Omega \times \mathbf{R}$. [See p. 91]

If $\qquad \Omega = \omega_1\mathbf{I} + \omega_2\mathbf{J} + \omega_3\mathbf{K}$ and $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$

then
$$\mathbf{V} = \Omega \times \mathbf{R} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = \mathbf{I}(\omega_2 z - \omega_3 y) + \mathbf{J}(\omega_3 x - \omega_1 z) + \mathbf{C}(\omega_1 y - \omega_2 x)$$

$$\therefore \qquad \text{curl } \mathbf{V} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y, & \omega_3 x - \omega_1 z, & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= \mathbf{I}(\omega_1 + \omega_1) + \mathbf{J}(\omega_2 + \omega_2) + \mathbf{K}(\omega_3 + \omega_3) \qquad [\because \ \omega_1, \omega_2, \omega_3 \text{ are constants.}]$$

$$= 2(\omega_1 \mathbf{I} + \omega_2 \mathbf{J} + \omega_3 \mathbf{K}) = 2\Omega. \quad \text{Hence} \quad \Omega = \frac{1}{2} \text{ curl } \mathbf{V}$$

Thus the angular velocity of rotation at any point is equal to half the curl of the velocity vector which justifies the name *rotation* used for curl.

In general, the curl of any vector point function gives the measure of the angular velocity at any point of the vector field.

Def. *Any motion in which the curl of the velocity vector is zero is said to be* **irrotational**, *otherwise* **rotational.**

Example 8.20. *Prove that div* $(r^n \mathbf{R}) = (n + 3)\, r^n$. *Hence show that* $\mathbf{R}/r^3$ *is solenoidal.*
(V.T.U., 2006 ; U.P.T.U., 2006 ; P.T.U., 2005)

Solution. We have $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ and $r = \sqrt{(x^2 + y^2 + z^2)}$

$$\therefore \qquad div\ (r^n \mathbf{R}) = \nabla \cdot (x^2 + y^2 + z^2)^{n/2} (x\mathbf{I} + y\mathbf{J} + z\mathbf{K})$$

$$= \frac{\partial}{\partial x}[x(x^2 + y^2 + z^2)^{n/2}] + \frac{\partial}{\partial y}[y(x^2 + y^2 + z^2)^{n/2}] + \frac{\partial}{\partial z}[z(x^2 + y^2 + z^2)^{n/2}]$$

$$= \Sigma \left\{ 1 \,.\, (x^2 + y^2 + z^2)^{n/2} + x \,.\, \frac{n}{2}(x^2 + y^2 + z^2)^{\frac{n}{2} - 1} \,.\, 2x \right\}$$

$$= \Sigma r^n + n\ \Sigma x^2 (x^2 + y^2 + z^2)^{\frac{n}{2} - 1} = 3r^n + nr^2 \,.\, r^{n-2}$$

Thus $div\ (r^n \mathbf{R}) = (n + 3)\, r^n$

When $n = -3$, $div\ (\mathbf{R}/r^3) = 0$ *i.e.*, $\mathbf{R}/r^3$ is solenoidal.

Example 8.21. *Show that* $r^\alpha \mathbf{R}$ *is any irrotational vector for any value of* α *but is solenoidal if* $\alpha + 3 = 0$ *where* $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ *and r is the magnitude of* $\mathbf{R}$. *(V.T.U., 2006 ; Kottayam, 2005)*

Solution. Let $\mathbf{A} = r^\alpha \mathbf{R} = (x^2 + y^2 + z^2)^{\alpha/2} (x\mathbf{I} + y\mathbf{I} + z\mathbf{K}) = \Sigma x\ (x^2 + y^2 + z^2)^{\alpha/2}\ \mathbf{I}$

$$\therefore \qquad \text{curl } \mathbf{A} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x(x^2 + y^2 + z^2)^{\alpha/2} & y(x^2 + y^2 + z^2)^{\alpha/2} & z(x^2 + y^2 + z^2)^{\alpha/2} \end{vmatrix}$$

$$= \Sigma\, \mathbf{I} \left\{ \frac{\alpha z}{2}(x^2 + y^2 + z^2)^{\alpha/2 - 1}\,(2y) - \frac{\alpha y}{2}(x^2 + y^2 + z^2)^{\alpha/2 - 1} \,.\, 2z \right\} = 0$$

Hence $\mathbf{A}$ is irrotational for any value of α.

But $\qquad div\ \mathbf{A} = \nabla \cdot (r^\alpha\, \mathbf{R}) = (\alpha + 3)\, r^\alpha$

which is zero for $\alpha + 3 = 0$, *i.e.*, $\mathbf{A}$ is solenoidal if $\alpha + 3 = 0$.

8.8 DEL APPLIED TWICE TO POINT FUNCTIONS

∇f and $\nabla \times \mathbf{F}$ being vector point functions, we can form their divergence and curl whereas $\nabla . \mathbf{F}$ being a scalar point function, we can have its gradients only. Thus we have the following five formulae :

(1) $div\ grad\ f = \nabla^2 f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$

(2) $curl\ grad\ f = \nabla \times \nabla f = \mathbf{0}$

(3) $div\ curl\ \mathbf{F} = \nabla \,.\, \nabla \times \mathbf{F} = 0$

(4) *curl curl* $\mathbf{F}$ = *grad div* $\mathbf{F} - \nabla^2\mathbf{F}$, *i.e.*, $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla . \mathbf{F}) - \nabla^2\mathbf{F}$

(5) *grad div* $\mathbf{F}$ = *curl curl* $\mathbf{F} + \nabla^2\mathbf{F}$, *i.e.*, $\nabla(\nabla . \mathbf{F}) = \nabla \times (\nabla \times \mathbf{F}) + \nabla^2\mathbf{F}$.

Proofs. **(1)** $\nabla^2 f = \nabla . \nabla f = \nabla . \left(\mathbf{I}\frac{\partial f}{\partial x} + \mathbf{J}\frac{\partial f}{\partial y} + \mathbf{K}\frac{\partial f}{\partial z}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f$$

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the *Laplacian operator* and $\nabla^2 f = 0$ is called the *Laplace's equation.*

(2) $$\nabla \times \nabla f = \nabla \times \left(\mathbf{I}\frac{\partial f}{\partial x} + \mathbf{J}\frac{\partial f}{\partial y} + \mathbf{K}\frac{\partial f}{\partial z}\right) = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \Sigma\mathbf{I}\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right) = \mathbf{0}$$ *(V.T.U., 2007)*

(3) $$\nabla \cdot \nabla \times \mathbf{F} = \left(\Sigma\mathbf{I}\frac{\partial}{\partial x}\right) \cdot \left(\mathbf{I} \times \frac{\partial \mathbf{F}}{\partial x} + \mathbf{J} \times \frac{\partial \mathbf{F}}{\partial y} + \mathbf{K} \times \frac{\partial \mathbf{F}}{\partial z}\right)$$

$$= \Sigma\mathbf{I} \cdot \left(\mathbf{I} \times \frac{\partial^2 \mathbf{F}}{\partial x^2} + \mathbf{J} \times \frac{\partial^2 \mathbf{F}}{\partial x \partial y} + \mathbf{K} \times \frac{\partial^2 \mathbf{F}}{\partial x \partial z}\right)$$

$$= \Sigma\left(\mathbf{I} \times \mathbf{I} \cdot \frac{\partial^2 \mathbf{F}}{\partial x^2} + \mathbf{I} \times \mathbf{J} \cdot \frac{\partial^2 \mathbf{F}}{\partial x \partial y} + \mathbf{I} \times \mathbf{K} \cdot \frac{\partial^2 \mathbf{F}}{\partial x \partial z}\right) = \Sigma\left(\mathbf{K} \cdot \frac{\partial^2 \mathbf{F}}{\partial x \partial y} - \mathbf{J} \cdot \frac{\partial^2 \mathbf{F}}{\partial x \partial z}\right) = 0.$$

(4) $$\nabla \times (\nabla \times \mathbf{F}) = \left(\Sigma\mathbf{I}\frac{\partial}{\partial x}\right) \times \left(\mathbf{I} \times \frac{\partial \mathbf{F}}{\partial x} + \mathbf{J} \times \frac{\partial \mathbf{F}}{\partial y} + \mathbf{K} \times \frac{\partial \mathbf{F}}{\partial z}\right)$$

$$= \Sigma\mathbf{I} \times \left(\mathbf{I} \times \frac{\partial^2 \mathbf{F}}{\partial x^2} + \mathbf{J} \times \frac{\partial^2 \mathbf{F}}{\partial x \partial y} + \mathbf{K} \times \frac{\partial^2 \mathbf{F}}{\partial x \partial z}\right)$$

$$= \Sigma\left[\left\{\left(\mathbf{I} \cdot \frac{\partial^2 \mathbf{F}}{\partial x^2}\right)\mathbf{I} - (\mathbf{I} \cdot \mathbf{I})\frac{\partial^2 \mathbf{F}}{\partial x^2}\right\} + \left\{\left(\mathbf{I} \cdot \frac{\partial^2 \mathbf{F}}{\partial x \partial y}\right)\mathbf{J} - (\mathbf{I} \cdot \mathbf{J})\frac{\partial^2 \mathbf{F}}{\partial x \partial y}\right\} + \left\{\left(\mathbf{I} \cdot \frac{\partial^2 \mathbf{F}}{\partial x \partial z}\right)\mathbf{K} - (\mathbf{I} \cdot \mathbf{K})\frac{\partial^2 \mathbf{F}}{\partial x \partial z}\right\}\right]$$

$$= \Sigma\left[\left(\mathbf{I} \cdot \frac{\partial^2 \mathbf{F}}{\partial x^2}\right)\mathbf{I} + \left(\mathbf{I} \cdot \frac{\partial^2 \mathbf{F}}{\partial x \partial y}\right)\mathbf{J} + \left(\mathbf{I} \cdot \frac{\partial^2 \mathbf{F}}{\partial x \partial z}\right)\mathbf{K}\right] - \Sigma\frac{\partial^2 \mathbf{F}}{\partial x^2}$$

$$= \Sigma\mathbf{I}\frac{\partial}{\partial x}\left(\mathbf{I} \cdot \frac{\partial \mathbf{F}}{\partial x} + \mathbf{J} \cdot \frac{\partial \mathbf{F}}{\partial y} + \mathbf{K} \cdot \frac{\partial \mathbf{F}}{\partial z}\right) - \Sigma\frac{\partial^2 \mathbf{F}}{\partial x^2} = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}.$$ *(Madras, 2006)*

(5) is just another way of writing (4) above.

Obs. *Interpretation of* ∇ *as a vector according to rules of vector products leads to correct results so far so the repeated application of* ∇ *is concerned.*

e.g., **1.** $\nabla . \nabla f = \nabla^2 f$ $(\because \nabla . \nabla = \nabla^2)$

2. $\nabla \times \nabla f = \mathbf{0}$ $(\because \nabla \times \nabla = \mathbf{0})$

3. $\nabla . \nabla \times \mathbf{F} = \mathbf{0}$ $(\because [\nabla\nabla\mathbf{F}] = 0)$

4. $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla . \mathbf{F}) - \nabla^2\mathbf{F}$ by expanding it as a vector triple product.

8.9 DEL APPLIED TO PRODUCTS OF POINT FUNCTIONS

To prove that

(1) grad $(fg) = f$ (grad g) + g(grad f) *i.e.* $\nabla(fg) = f\nabla g + g\nabla f$.

(2) *div* $(f\mathbf{G})$ = (grad f) . $\mathbf{G} + f$ (div $\mathbf{G}$) *i.e.* $\nabla(f\mathbf{G}) = \nabla f . \mathbf{G} + f\nabla . \mathbf{G}$

(3) *curl* $(f\mathbf{G})$ = (*grad* f) × $\mathbf{G} + f$ (*curl* $\mathbf{G}$) *i.e.* $\nabla \times (f\mathbf{G}) = \nabla f \times \mathbf{G} + f\nabla \times \mathbf{G}$

(4) *grad* $(\mathbf{F}.\mathbf{G}) = (\mathbf{F} . \nabla)\mathbf{G} + (\mathbf{G} . \nabla)\mathbf{F} + \mathbf{F} \times$ *curl* $\mathbf{G} + \mathbf{G} \times$ *curl* $\mathbf{F}$

i.e., $\nabla(\mathbf{F} . \mathbf{G}) = (\mathbf{F} . \nabla)\mathbf{G} + (\mathbf{G} . \nabla)\mathbf{F} + \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F})$

(5) *div* $(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\,.\,(curl\ \mathbf{F}) - \mathbf{F}\,.\,(curl\ \mathbf{G})$ *i.e.,* $\nabla\,.\,(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\,.\,(\nabla\times\mathbf{F}) - \mathbf{F}\,.\,(\nabla\times\mathbf{G})$

(6) *curl* $(\mathbf{F}\times\mathbf{G}) = \mathbf{F}\,(div\ \mathbf{G}) - \mathbf{G}\,(div\ \mathbf{F}) + (\mathbf{G}\,.\,\nabla)\,\mathbf{F} - (\mathbf{F}\,.\,\nabla)\,\mathbf{G}$

i.e., $\nabla\times(\mathbf{F}\times\mathbf{G}) = \mathbf{F}(\nabla\,.\,\mathbf{G}) - \mathbf{G}\,(\nabla\,.\,\mathbf{F}) + (\mathbf{G}\,.\,\nabla)\,\mathbf{F} - (\mathbf{F}\,.\,\nabla)\,\mathbf{G}$

Proofs **(1)** $\nabla\,(fg) = \Sigma\mathbf{I}\,.\,\dfrac{\partial}{\partial x}(fg) = \Sigma\mathbf{I}\left(f\dfrac{\partial g}{\partial x} + g\dfrac{\partial f}{\partial x}\right)$

$$= f\,\Sigma\mathbf{I}\,\frac{\partial g}{\partial x} + g\,\Sigma\mathbf{I}\,\frac{\partial f}{\partial x} = f\,\nabla g + g\nabla f$$

(2) $\nabla\,.\,(f\,\mathbf{G}) = \Sigma\mathbf{I}\,.\,\dfrac{\partial}{\partial x}(f\,\mathbf{G}) = \Sigma\mathbf{I}\,.\left(\dfrac{\partial f}{\partial x}\mathbf{G} + f\,\dfrac{\partial \mathbf{G}}{\partial x}\right)$

$$= \left(\Sigma\mathbf{I}\,\frac{\partial f}{\partial x}\right).\,\mathbf{G} + f\left(\Sigma\mathbf{I}\,.\,\frac{\partial \mathbf{G}}{\partial x}\right) = \nabla f\,.\,\mathbf{G} + f\,\nabla\,.\,\mathbf{G}$$ *(V.T.U., 2011)*

(3) $\nabla\times(f\,\mathbf{G}) = \Sigma\mathbf{I}\times\dfrac{\partial}{\partial x}(f\,\mathbf{G}) = \Sigma\mathbf{I}\times\left(f\,\dfrac{\partial \mathbf{G}}{\partial x} + \dfrac{\partial f}{\partial x}\,\mathbf{G}\right)$

$$= f\,\Sigma\mathbf{I}\times f\,\frac{\partial \mathbf{G}}{\partial x} + \Sigma\mathbf{I}\,\frac{\partial f}{\partial x}\times\mathbf{G} = f\,\nabla\times\mathbf{G} + \nabla f\times\mathbf{G}$$ *(V.T.U. 2008)*

(4) $\nabla(\mathbf{F}\,.\,\mathbf{G}) = \Sigma\mathbf{I}\,\dfrac{\partial}{\partial x}(\mathbf{F}.\,\mathbf{G}) = \Sigma\mathbf{I}\left(\dfrac{\partial \mathbf{F}}{\partial x}\,.\,\mathbf{G} + \mathbf{F}\,.\,\dfrac{\partial \mathbf{G}}{\partial x}\right) = \Sigma\mathbf{I}\,\dfrac{\partial \mathbf{F}}{\partial x}\,.\,\mathbf{G} + \Sigma\mathbf{I}\left(\mathbf{F}\,.\,\dfrac{\partial \mathbf{G}}{\partial x}\right)$ ...(*i*)

Now $\mathbf{G}\times\left(\mathbf{I}\times\dfrac{\partial \mathbf{F}}{\partial x}\right) = \left(\mathbf{G}\,.\,\dfrac{\partial \mathbf{F}}{\partial x}\right)\mathbf{I} - (\mathbf{G}\,.\,\mathbf{I})\,\dfrac{\partial \mathbf{F}}{\partial x}$

or $\left(\mathbf{G}\,.\,\dfrac{\partial \mathbf{F}}{\partial x}\right)\mathbf{I} = \mathbf{G}\times\left(\mathbf{I}\times\dfrac{\partial \mathbf{F}}{\partial x}\right) + (\mathbf{G}\,.\,\mathbf{I})\,\dfrac{\partial \mathbf{F}}{\partial x}$

$\therefore$ $\Sigma\left(\mathbf{G}\,.\,\dfrac{\partial \mathbf{F}}{\partial x}\right)\mathbf{I} = \mathbf{G}\times\Sigma\mathbf{I}\times\dfrac{\partial \mathbf{F}}{\partial x} + \Sigma\,(\mathbf{G}\,.\,\mathbf{I})\,\dfrac{\partial \mathbf{F}}{\partial x} = \mathbf{G}\times(\nabla\times\mathbf{F}) + (\mathbf{G}\,.\,\nabla)\mathbf{F}$ ...(*ii*)

Interchanging **F** and **G**, $\Sigma\left(\mathbf{F}\,.\,\dfrac{\partial \mathbf{G}}{\partial x}\right)\mathbf{I} = \mathbf{F}\times(\nabla\times\mathbf{G}) + (\mathbf{F}\,.\,\nabla)\,\mathbf{G}$ ...(*iii*)

Substituting in (*i*) from (*ii*) and (*iii*), we get

$$\nabla(\mathbf{F}\,.\,\mathbf{G}) = (\mathbf{F}\,.\,\nabla)\mathbf{G} + (\mathbf{G}\,.\,\nabla)\mathbf{F} + \mathbf{F}\times(\nabla\times\mathbf{G}) + \mathbf{G}\times(\nabla\times\mathbf{F})$$

(5) $\nabla\,.\,(\mathbf{F}\times\mathbf{G}) = \Sigma\mathbf{I}\,.\,\dfrac{\partial}{\partial x}(\mathbf{F}\times\mathbf{G}) = \Sigma\mathbf{I}\left(\dfrac{\partial \mathbf{F}}{\partial x}\times\mathbf{G} + F\times\dfrac{\partial \mathbf{G}}{\partial x}\right) = \Sigma\mathbf{I}\,.\,\dfrac{\partial \mathbf{F}}{\partial x}\times\mathbf{G} - \Sigma\mathbf{I}\,.\left(\dfrac{\partial \mathbf{G}}{\partial x}\times\mathbf{F}\right)$

$$= \Sigma\left(I\times\frac{\partial \mathbf{F}}{\partial x}\right).\,\mathbf{G} - \Sigma\left(I\times\frac{\partial \mathbf{G}}{\partial x}\right).\,\mathbf{F}$$ $[\because\ \mathbf{A}\,.\,(\mathbf{B}\times\mathbf{C}) = (A\times B)\,.\,C]$

$$= \mathbf{G}\,.\,(\nabla\times\mathbf{F}) - \mathbf{F}\,.\,(\nabla\times\mathbf{G})$$

(6) $\nabla\times(\mathbf{F}\times\mathbf{G}) = \Sigma\mathbf{I}\times\dfrac{\partial}{\partial x}(\mathbf{F}\times\mathbf{G}) = \Sigma\mathbf{I}\times\left(\dfrac{\partial \mathbf{F}}{\partial x}\times\mathbf{G} + \mathbf{F}\times\dfrac{\partial \mathbf{G}}{\partial x}\right)$

$$= \Sigma\left[(\mathbf{I}\,.\,\mathbf{G})\,\frac{\partial \mathbf{F}}{\partial x} - \left(\mathbf{I}\,.\,\frac{\partial \mathbf{F}}{\partial x}\right)\mathbf{G}\right] + \Sigma\left[\left(\mathbf{I}\,.\,\frac{\partial \mathbf{G}}{\partial x}\right)\mathbf{F} - (\mathbf{I}\,.\,\mathbf{F})\,\frac{\partial \mathbf{G}}{\partial x}\right]$$

$$= \Sigma\,(\mathbf{G}\,.\,\mathbf{I})\,\frac{\partial \mathbf{F}}{\partial x} - \mathbf{G}\Sigma\mathbf{I}\,.\,\frac{\partial \mathbf{F}}{\partial x} + \mathbf{F}\Sigma\mathbf{I}\,.\,\frac{\partial \mathbf{G}}{\partial x} - \Sigma(\mathbf{F}\,.\,\mathbf{I})\,\frac{\partial \mathbf{G}}{\partial x}$$

$$= \mathbf{F}\left(\Sigma\mathbf{I}\,.\,\frac{\partial \mathbf{G}}{\partial x}\right) - \mathbf{G}\Sigma\left(\mathbf{I}\,.\,\frac{\partial \mathbf{F}}{\partial x}\right) + \Sigma\,(\mathbf{G}\,.\,\mathbf{I})\,\frac{\partial \mathbf{F}}{\partial x} - \Sigma\,(\mathbf{F}\,.\,\mathbf{I})\,\frac{\partial \mathbf{G}}{\partial x}$$

$$= \mathbf{F}\,(\nabla\,.\,\mathbf{G}) - \mathbf{G}(\nabla\,.\,\mathbf{F}) + (\mathbf{G}\,.\,\nabla)\,\mathbf{F} - (\mathbf{F}\,.\,\nabla)\,\mathbf{G}$$

Rule to reproduce the above formulae easily :

(*i*) *Treating each of the factors as constants separately, expresss the results of* ∇*-operation as a sum of the two terms.*

(*ii*) *Transform each of the two terms, noting that* ∇ *always appears before a function and keeping in mind whether the result of operation is a scalar or a vector.* To carry out the simplification, we may sometimes, employ the properties of triple products.

(*iii*) *Restore the change of treating the functions as constants.*

Let us illustrate the application of this rule to (2), (4) and (6) above :

(2) $\nabla \cdot (f\mathbf{G}) = \nabla \cdot (f_c \mathbf{G} + f\mathbf{G}_c) = f_c \nabla \cdot \mathbf{G} + \mathbf{G}_c \cdot \nabla f = f \nabla \cdot \mathbf{G} + \mathbf{G} \cdot \nabla f$

(4) $\nabla(\mathbf{F} \cdot \mathbf{G}) = \nabla(\mathbf{F}_c \cdot \mathbf{G}) + \nabla(\mathbf{F} \cdot \mathbf{G}_c)$
$= [\mathbf{F}_c \times (\nabla \times \mathbf{G}) + (\mathbf{F}_c \cdot \nabla)\mathbf{G}] + [\mathbf{G}_c \times (\nabla \times \mathbf{F}) + (\mathbf{G}_c \cdot \nabla)\mathbf{F}]$
$= \mathbf{F} \times (\nabla \times \mathbf{G}) + (\mathbf{F} \cdot \nabla)\mathbf{G} + \mathbf{G} \times (\nabla \times \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F}$

(6) $\nabla \times (\mathbf{F} \times \mathbf{G}) = \nabla \times (\mathbf{F}_c \times \mathbf{G}) + \nabla \times (\mathbf{F} \times \mathbf{G}_c) = [\nabla \cdot \mathbf{G}\mathbf{F}_c - (\mathbf{F}_c \cdot \nabla)\mathbf{G}] + (\mathbf{G}_c \cdot \nabla)\mathbf{F} - \nabla \cdot \mathbf{F}\mathbf{G}_c]$
$= \mathbf{F}(\nabla \cdot \mathbf{G}) - (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} - \mathbf{G}(\nabla \cdot \mathbf{F}).$

Example 8.22. *Show that $\nabla^2(r^n) = n(n+1)r^{n-2}$* *(S.V.T.U., 2006 ; J.N.T.U., 2006 ; U.P.T.U., 2005)*

Solution. $\nabla^2 r^n = \nabla \cdot (\nabla r^n)$

$$= \nabla \cdot \left(nr^{n-1}\frac{\mathbf{R}}{r}\right) = n\nabla \cdot (r^{n-2}\mathbf{R}) = n[(\nabla r^{n-2}) \cdot \mathbf{R} + r^{n-2}(\nabla \cdot \mathbf{R})] \quad \text{[By § 8.9 (2)]}$$

$$= n\left[(n-2)r^{n-3}\frac{\mathbf{R}}{r} \cdot \mathbf{R} + r^{n-2}(3)\right] \quad \text{[Using Ex. 8.18 (}i\text{)]}$$

$$= n[(n-2)r^{n-4}(r^2) + 3r^{n-2}] = n(n+1)\,r^{n-2} \quad [\because \ \mathbf{R} \cdot \mathbf{R} = r^2]$$

Otherwise : $\nabla^2(r^n) = \dfrac{\partial^2(r^n)}{\partial x^2} + \dfrac{\partial^2(r^n)}{\partial y^2} + \dfrac{\partial^2(r^n)}{\partial z^2}$ [By § 8.8 (1)] ...(*i*)

Now $\dfrac{\partial(r^n)}{\partial x} = nr^{n-1}\dfrac{\partial r}{\partial x} = nr^{n-1}\dfrac{x}{r} = nr^{n-2}x$ $[\because \ r^2 = x^2 + y^2 + z^2]$

$$\therefore \quad \frac{\partial^2(r^n)}{\partial x^2} = n\left[r^{n-2} + (n-2)r^{n-3}\frac{\partial r}{\partial x}x\right] = n\left[r^{n-2} + (n-2)r^{n-3}\frac{x}{r}x\right]$$

$$= n\left[r^{n-2} + (n-2)r^{n-4}x^2\right] \quad ...(ii)$$

Similarly, $\dfrac{\partial^2(r^n)}{\partial y^2} = n\left[r^{n-2} + (n-2)r^{n-4}y^2\right]$ ...(*iii*)

$$\frac{\partial^2(r^n)}{\partial z^2} = n\left[r^{n-2} + (n-2)r^{n-4}z^2\right] \quad ...(iv)$$

Adding (*ii*), (*iii*) and (*iv*), (*i*) gives

$$\nabla^2(r^n) = n\,[3r^{n-2} + (n-2)r^{n-4}(x^2 + y^2 + z^2)]$$
$$= n\,[3r^{n-2} + (n-2)r^{n-4}r^2] = n(n+1)r^{n-2}.$$

In particular $\nabla^2(1/r) = 0$. *(U.P.T.U., 2003 ; P.T.U., 2003)*

Example 8.23. *If $u\mathbf{F} = \nabla v$, where u, v are scalar fields and $\mathbf{F}$ is a vector field, show that $\mathbf{F}$. curl $\mathbf{F} = 0$.*

Solution. Since $\mathbf{F} = \dfrac{1}{u}\nabla v \quad \therefore \quad \text{curl } \mathbf{F} = \nabla \times \left(\dfrac{1}{u}\nabla v\right)$

or $$\text{curl } \mathbf{F} = \nabla\frac{1}{u} \times \nabla v + \frac{1}{u}\nabla \times (\nabla v) \quad \text{[By § 8.9 (3)]}$$

$$= \nabla\frac{1}{u} \times \nabla v \quad [\because \ \nabla \times \nabla v = 0$$

Hence $\mathbf{F} \cdot \text{curl } \mathbf{F} = \dfrac{1}{u}\nabla v \cdot \left(\nabla\dfrac{1}{u} \times \nabla v\right) = 0$, for it is a scalar triple product in which two factors are equal.

Example 8.24. *If r and $\mathbf{R}$ have their usual meanings and $\mathbf{A}$ is a constant vector, prove that* $\nabla \times \left(\dfrac{\mathbf{A} \times \mathbf{R}}{r^n}\right) = \dfrac{2-n}{r^n}\mathbf{A} + \dfrac{n(\mathbf{A} \cdot \mathbf{R})}{r^{n+2}}\mathbf{R}.$ *(Mumbai, 2009 ; Kurukshetra, 2006 ; J.N.T.U., 2005)*

Solution. $\nabla \times [r^{-n}(\mathbf{A} \times \mathbf{R})] = r^{-n}[\nabla \times (\mathbf{A} \times \mathbf{R})] + \nabla r^{-n} \times (\mathbf{A} \times \mathbf{R})]$ [By § 8.9 (3)]

$$= r^{-n}[(\nabla \cdot \mathbf{R})\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{R}] + (-nr^{-(n+1)}\mathbf{R}/r) \times (\mathbf{A} \times \mathbf{R})$$

$$= r^{-n}(3\mathbf{A}-\mathbf{A}) - nr^{-(n+2)}\,\mathbf{R}\times(\mathbf{A}\times\mathbf{R}) \qquad [\because\ \nabla\,.\,\mathbf{R}=3,\ (\mathbf{A}\,.\,\nabla)\,\mathbf{R}=\mathbf{A}]$$

$$= 2\mathbf{A}r^{-n} - nr^{-(n+2)}\,[(\mathbf{R}\,.\,\mathbf{R})\,\mathbf{A} - (\mathbf{A}\,.\,\mathbf{R})\,\mathbf{R}]$$

$$= \frac{2\mathbf{A}}{r^n} - \frac{n}{r^{n+2}}\,[r^2\mathbf{A} - (\mathbf{A}\,.\,\mathbf{R})\,\mathbf{R}] = \frac{2-n}{r^n}\,\mathbf{A} + \frac{n(\mathbf{A}\,.\,\mathbf{R})}{r^{n+2}}\,\mathbf{R}.$$

Example 8.25. *If r is the distance of a point (x, y, z) from the origin, prove that curl* $\left(\mathbf{K}\times grad\frac{1}{r}\right) + grad\left(\mathbf{K}\,.\,grad\frac{1}{r}\right) = 0$, *where* **K** *is the unit vector in the direction OZ.* (U.P.T.U., 2001)

Solution. $$grad\,\frac{1}{r} = \left(\mathbf{I}\frac{\partial}{\partial x}+\mathbf{J}\frac{\partial}{\partial y}+\mathbf{K}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{-1/2} \qquad [\because\ r=\sqrt{(x^2+y^2+z^2)}\,]$$

$$= -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\,(2x\mathbf{I}+2y\mathbf{J}+2z\mathbf{K})$$

$$= -(x^2+y^2+z^2)^{-3/2}\,(x\mathbf{I}+y\mathbf{J}+z\mathbf{K})$$

$$curl\left(\mathbf{K}\times grad\frac{1}{r}\right) = \nabla\times[-(x^2+y^2+z^2)^{-3/2}\,(x\mathbf{J}-y\mathbf{I})]$$

$$= \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y/(x^2+y^2+z^2)^{3/2} & -x/(x^2+y^2+z^2)^{3/2} & 0 \end{vmatrix}$$

$$= \mathbf{I}\,\frac{\partial}{\partial z}\left\{\frac{x}{(x^2+y^2+z^2)^{3/2}}\right\} + \mathbf{J}\,\frac{\partial}{\partial z}\left\{\frac{y}{(x^2+y^2+z^2)^{3/2}}\right\}$$

$$- \mathbf{K}\left\{\frac{\partial}{\partial x}\left[\frac{x}{(x^2+y^2+z^2)^{3/2}}\right] + \frac{\partial}{\partial y}\left[\frac{y}{(x^2+y^2+z^2)^{3/2}}\right]\right\}$$

$$= \frac{-3xz\mathbf{I} - 3yz\mathbf{J} + (x^2+y^2-2z^2)\mathbf{K}}{(x^2+y^2+z^2)^{5/2}} \qquad ...(i)$$

$$\text{grad}\left(\mathbf{K}\,.\,\text{grad}\,\frac{1}{r}\right) = \nabla\left\{-\mathbf{K}\,.\,\frac{(x\mathbf{I}+y\mathbf{J}+z\mathbf{K})}{(x^2+y^2+z^2)^{3/2}}\right\}$$

$$= \left(\mathbf{I}\frac{\partial}{\partial x}+\mathbf{J}\frac{\partial}{\partial y}+\mathbf{K}\frac{\partial}{\partial z}\right)\left\{\frac{-z}{(x^2+y^2+z^2)^{3/2}}\right\}$$

$$= \frac{3xz\mathbf{I}}{(x^2+y^2+z^2)^{5/2}} + \frac{3yz\mathbf{J}}{(x^2+y^2+z^2)^{5/2}} + \frac{(3z^2-x^2-y^2-z^2)\mathbf{K}}{(x^2+y^2+z^2)^{5/2}}$$

$$= \frac{3xz\mathbf{I} + 3yz\mathbf{J} - (x^2+y^2-2z^2)\mathbf{K}}{(x^2+y^2+z^2)^{5/2}} \qquad ...(ii)$$

Adding (i) and (ii), we get

$$\text{curl}\left(\mathbf{K}\times\text{grad}\,\frac{1}{r}\right) + \text{grad}\left(\mathbf{K}\,.\,\text{grad}\,\frac{1}{r}\right) = \mathbf{0}.$$

Example 8.26. *In electromagnetic theory, we have* $\nabla\,.\,\mathbf{D}=\rho$, $\nabla\,.\,\mathbf{H}=0$, $\nabla\times D = -\dfrac{1}{c}\dfrac{\partial\mathbf{H}}{\partial t}$, $\nabla\times\mathbf{H} = \dfrac{1}{c}\left(\rho\mathbf{V}+\dfrac{\partial\mathbf{D}}{\partial t}\right)$. *Prove that*

(i) $\nabla^2\mathbf{D} - \dfrac{1}{c^2}\dfrac{\partial^2\mathbf{D}}{\partial t^2} = \nabla\rho + \dfrac{1}{c^2}\dfrac{\partial}{\partial t}(\rho\mathbf{V})$ $\qquad$ (ii) $\nabla^2\mathbf{H} - \dfrac{1}{c^2}\dfrac{\partial^2\mathbf{H}}{\partial t^2} = -\dfrac{1}{c}\,\nabla\times\rho\mathbf{V}$

Solution. (*i*) We have $\frac{1}{c^2}\left\{\frac{\partial^2 \mathbf{D}}{\partial t^2}+\frac{\partial}{\partial t}(\rho\mathbf{V})\right\}=\frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{D}}{\partial t}+\rho\mathbf{V}\right)$

$$=\frac{1}{c}\frac{\partial}{\partial t}(\nabla\times\mathbf{H})=\frac{1}{c}\nabla\times\frac{\partial \mathbf{H}}{\partial t} \qquad \left[\because \nabla\times\mathbf{H}=\frac{1}{c}\left(\rho\mathbf{V}+\frac{\partial\mathbf{D}}{\partial t}\right)\right]$$

$$=-\nabla\times(\nabla\times\mathbf{D}) \qquad \left[\because \nabla\times\mathbf{D}=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}\right]$$

$$=-[\nabla(\nabla\cdot\mathbf{D})-\nabla^2\mathbf{D}] \qquad \text{[Using § 8.8 (4)]}$$

$$=-\nabla\rho+\nabla^2\mathbf{D}$$

Hence $\nabla^2\mathbf{D}-\frac{1}{c^2}\frac{\partial^2\mathbf{D}}{\partial t^2}=\nabla\rho+\frac{1}{c^2}\frac{\partial}{\partial t}(\rho\mathbf{V})$

(*ii*) L.H.S. $=\nabla^2\mathrm{H}-\frac{1}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2}=\nabla^2\mathbf{H}+\frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}\right)$

$$=\nabla^2\mathbf{H}+\frac{1}{c}\frac{\partial}{\partial t}(\nabla\times\mathbf{D}) \qquad \left[\because \nabla\times\mathbf{D}=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}\right]$$

$$=\nabla^2\mathbf{H}+\frac{1}{c}\left(\nabla\times\frac{\partial\mathbf{D}}{\partial t}\right) \qquad \left[\because \nabla\times\mathbf{H}=\frac{1}{c}\left(\rho V+\frac{\partial\mathbf{D}}{\partial t}\right)\right]$$

$$=\nabla^2\mathbf{H}+\nabla\times\left(\nabla\times\mathbf{H}-\frac{1}{C}\rho\mathbf{V}\right)=\nabla^2\mathbf{H}+\nabla\times(\nabla\times\mathbf{H})-\frac{1}{c}\nabla\times(\rho\mathbf{V})$$

$$=\nabla^2\mathbf{H}+\nabla(\nabla\cdot\mathbf{H})-\nabla^2\mathbf{H}-\frac{1}{c}\nabla\times(\rho\mathbf{V}), \qquad \text{[Using § 8.9 (4)]}$$

$$=\nabla(\nabla\cdot\mathbf{H})-\frac{1}{c}\nabla\times(\rho\mathbf{V}) \qquad [\because \nabla\cdot\mathbf{H}=0]$$

$$=-\frac{1}{c}\nabla\times\rho V=\text{R.H.S.}$$

PROBLEMS 8.4

1. Evaluate div **F** and curl **F** at the point (1, 2, 3) given (*i*) $\mathbf{F}=x^2yz\mathbf{I}+xy^2z\mathbf{J}+xyz^2\mathbf{K}$. (*B.P.T.U., 2005*)
   (*ii*) $\mathbf{F}=3x^2\mathbf{I}+5xy^2\mathbf{J}+5xyz^3\mathbf{K}$. (*S.V.T.U., 2009*)
   (*iii*) $\mathbf{F}=\text{grad }[x^3y+y^3z+z^3x-x^2y^2z^2]$ (*V.T.U., 2007*)
2. If $\mathbf{V}=(x\mathbf{I}+y\mathbf{J}+z\mathbf{K})/\sqrt{(x^2+y^2+z^2)}$, show that $\nabla.\mathbf{V}=2/\sqrt{(x^2+y^2+z^2)}$ and $\nabla\times\mathbf{V}=\mathbf{0}$. (*Osmania, 2002*)
3. If $\mathbf{F}=(x+y+1)\mathbf{I}+\mathbf{J}-(x+y)\mathbf{K}$, show that **F**. curl **F** = 0. (*V.T.U., 2000 S*)
4. Find the value of a if the vector $(ax^2y+yz)\mathbf{I}+(xy^2-xz^2)\mathbf{J}+(2xyz-2x^2y^2)\mathbf{K}$ has zero divergence. Find the curl of the above vector which has zero divergence.
5. Show that each of following vectors are solenoidal :
   (*i*) $(-x^2+yz)\mathbf{I}+(4y-z^2x)\mathbf{J}+(2xz-4z)\mathbf{K}$ (*Delhi, 2002*)
   (*ii*) $3y^4z^2\mathbf{I}+4x^3z^2\mathbf{J}+3x^2y^2\mathbf{K}$ (*iii*) $\nabla\phi\times\nabla\psi$.
6. If **A** and **B** are irrotational, prove that $\mathbf{A}\times\mathbf{B}$ is solenoidal. (*Madras, 2003 ; V.T.U., 2001*)
7. If $u=x^2+y^2+z^2$ and $\mathbf{V}=x\mathbf{I}+y\mathbf{J}+z\mathbf{K}$, show that div $(u\mathbf{V})=5u$.
8. If $\mathbf{R}=x\mathbf{I}+y\mathbf{J}+z\mathbf{K}$ and $r\neq 0$, show that (*i*) $\nabla/(1/r^2)=-2\mathbf{R}/r^4$; $\nabla\cdot(\mathbf{R}/r^2)=1/r^2$
   (*ii*) div $(r^n\mathbf{R})=(n+3)r^n$; curl $(r^n\mathbf{R})=\mathbf{0}$ (*P.T.U., 2006 ; Kottayam, 2005*)
   (*iii*) *grad* $\left(\text{div}\frac{\mathbf{R}}{r}\right)=-\frac{2\mathbf{R}}{r^3}$. (*V.T.U., 2010 S*)
9. If $\mathbf{V}_1$ and $\mathbf{V}_2$ be the vectors joining the fixed points (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively to a variable point (x, y, z), prove that
   (*i*) div $(\mathbf{V}_1\times\mathbf{V}_2)=0$, (*ii*) grad $(\mathbf{V}_1\cdot\mathbf{V}_2)=\mathbf{V}_1+\mathbf{V}_2$,
   (*iii*) curl $(\mathbf{V}_1\times\mathbf{V}_2)=2(\mathbf{V}_1-\mathbf{V}_2)$

10. Show that (*i*) $\nabla \cdot \left[\frac{f(r)}{r}\mathbf{R}\right] = \frac{1}{r^2}\frac{d}{dr}[r^2 f(r)]$ (*Mumbai, 2008*)

(*ii*) $\nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r)$ (*U.T.U., 2010 ; Bhopal, 2008 ; S.V.T.U., 2008 ; V.T.U., 2006*)

(*iii*) $\nabla \cdot (\phi\nabla\psi - \psi\nabla\phi) = \phi\nabla^2\psi - \psi\nabla^2\phi$.

11. If **A** is a constant vector and $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$, prove that

(*i*) grad $(\mathbf{A} \cdot \mathbf{R}) = \mathbf{A}$ (*Delhi, 2002*) (*ii*) div $(\mathbf{A} \times \mathbf{R}) = 0$ (*Burdwan, 2003*)

(*iii*) curl $(\mathbf{A} \times \mathbf{R}) = 2\mathbf{A}$ (*V.T.U., 2010 S*) (*iv*) curl $[(\mathbf{A} \cdot \mathbf{R})\mathbf{R}] = \mathbf{A} \times \mathbf{R}$. (*Kurukshetra, 2009 S*)

12. Prove that (*i*) $\nabla\mathbf{A}^2 = 2(\mathbf{A} \cdot \nabla)\mathbf{A} + 2\mathbf{A} \times (\nabla \times \mathbf{A})$, where **A** is a constant vector.

(*ii*) $\nabla \times (\mathbf{R} \times \mathbf{U}) = \mathbf{R}(\nabla \cdot \mathbf{U}) - 2\mathbf{U} - (\mathbf{R} \cdot \nabla)\mathbf{U}$.

13. Calculate (*i*) curl (grad f), given $f(x, y, z) = x^2 + y^2 - z$. (*B.P.T.U., 2006*)

(*ii*) curl (curl A) given $A = x^2y\mathbf{I} + y^2z\mathbf{J} + z^2y\mathbf{K}$ (*V.T.U., 2003*)

14. (*a*) If $f = (x^2 + y^2 + z^2)^{-n}$, find div grad f and determine n if div grad $f = 0$. (*S.V.T.U., 2009 ; J.N.T.U. 2003*)

(*b*) Show that div (grad r^n) $= n(n+1)r^{n-2}$ where $r^2 = x^2 + y^2 + z^2$. (*Bhopal, 2008 ; U.P.T.U., 2003*)

15. For a solenoidal vector **F**, show that curl curl curl curl $\mathbf{F} = \nabla^4\mathbf{F}$.

16. If $u = x^2yz$, $v = xy - 3z^2$, find (*i*) $\nabla(\nabla u \cdot \nabla v)$; (*ii*) $\nabla \cdot (\nabla u \times \nabla v)$.

17. Find the directional derivative of $\nabla \cdot (\nabla\phi)$ at the point $(1, -2, 1)$ in the direction of the normal to the surface $xy^2z = 3x + z^2$, where $\phi = 2x^3y^2z^4$. (*Raipur, 2005*)

18. Prove that $\mathbf{A} \cdot \nabla\left(\mathbf{B} \cdot \nabla\frac{1}{r}\right) = \frac{3(\mathbf{A} \cdot \mathbf{R})(\mathbf{B} \cdot \mathbf{R})}{r^5} - \frac{\mathbf{A} \cdot \mathbf{B}}{r^3}$ where **A** and **B** are constant vectors.

8.10 INTEGRATION OF VECTORS

If two vector functions $\mathbf{F}(t)$ and $\mathbf{G}(t)$ be such that

$$\frac{d\mathbf{G}(t)}{dt} = \mathbf{F}(t),$$

then $\mathbf{G}(t)$ is called an integral of $\mathbf{F}(t)$ with respect to the scalar variable t and we write

$$\int \mathbf{F}(t)\, dt = \mathbf{G}(t).$$

If **C** be an arbitrary constant vector, we have

$$\mathbf{F}(t) = \frac{d\mathbf{G}(t)}{dt} = \frac{d}{dt}[\mathbf{G}(t) + \mathbf{C}] \quad \text{then} \quad \int \mathbf{F}(t)\, dt = \mathbf{G}(t) + \mathbf{C}$$

This is called the *indefinite integral of* $\mathbf{F}(t)$ and its *definite integral is*

$$\int_a^b \mathbf{F}(t)\, dt = [\mathbf{G}(t) + \mathbf{C}]_a^b = \mathbf{G}(b) - \mathbf{G}(a).$$

Example 8.27. *Given* $\mathbf{R}(t) = 3t^2\mathbf{I} + t\mathbf{J} - t^3\mathbf{K}$, *evaluate* $\int_0^1 (R \times d^2\mathbf{R}/dt^2)\, dt$.

Solution. $$\frac{d}{dt}\left(\mathbf{R} \times \frac{d\mathbf{R}}{dt}\right) = \frac{d\mathbf{R}}{dt} \times \frac{d\mathbf{R}}{dt} + \mathbf{R} \times \frac{d^2\mathbf{R}}{dt^2} = \mathbf{R} \times \frac{d^2\mathbf{R}}{dt^2}$$

$\therefore$
$$\int \left(\mathbf{R} \times \frac{d^2\mathbf{R}}{dt^2}\right) dt = \mathbf{R} \times \frac{d\mathbf{R}}{dt}$$

$$= (3t^2\mathbf{I} + t\mathbf{J} - t^3\mathbf{K}) \times (6t\mathbf{I} + \mathbf{J} - 3t^2\mathbf{K})$$

$$= \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ 3t^2 & t & -t^3 \\ 6t & 1 & -3t^2 \end{vmatrix} = -2t^3\mathbf{I} + 3t^4\mathbf{J} - 3t^2\mathbf{K}$$

Thus
$$\int_0^1 \left(\mathbf{R} \times \frac{d^2\mathbf{R}}{dt^2}\right) dt = \left| -2t^3\mathbf{I} + 3t^4\mathbf{J} - 3t^2\mathbf{K} \right|_0^1$$

$$= -2\mathbf{I} + 3\mathbf{J} - 3\mathbf{K}$$

PROBLEMS 8.5

1. Given $\mathbf{F}(t) = (5t^2 - 3t)\mathbf{I} + 6t^3\mathbf{J} - 7t\,\mathbf{K}$, evaluate $\int_{t=2}^{t=4} \mathbf{F}(t)\,dt$.
2. If $\frac{d^2\mathbf{P}}{dt^2} = 6t\mathbf{I} - 12t^2\mathbf{J} + 4\cos t\mathbf{K}$, find $\mathbf{P}$, Given that $\frac{d\mathbf{P}}{dt} = -1 - 3\mathbf{K}$ and $\mathbf{P} = 2\mathbf{I} + \mathbf{J}$ when $t = 0$.
3. The acceleration of a particle at any time $t \geq 0$ is given by $12\cos 2t\mathbf{I} - 8\sin 2t\mathbf{J} + 16t\mathbf{K}$, the velocity and acceleration are initially zero. Find the velocity and displacement at any time.
4. If $\mathbf{R}(t) = \begin{cases} 2\mathbf{I} - \mathbf{J} + 2\mathbf{K} & \text{when } t = 1 \\ 3\mathbf{I} - 2\mathbf{J} + 4\mathbf{K} & \text{when } t = 2, \end{cases}$

   show that $\int_1^2 \left(\mathbf{R} \cdot \frac{d\mathbf{R}}{dt}\right) dt = 10.$

8.11 (1) LINE INTEGRAL

Consider a continuous vector function $\mathbf{F}(\mathbf{R})$ which is defined at each point of curve C in space. Divide C into n parts at the points $A = P_0, P_1, \ldots, P_{i-1}, P_i \ldots, P_n = B$ (Fig. 8.8). Let their position vectors be $\mathbf{R}_0, \mathbf{R}_1, \ldots, \mathbf{R}_{i-1}, \mathbf{R}_i, \ldots, \mathbf{R}_n$. Let $\mathbf{U}_i$, be the position vector of any point on the arc $P_{i-1}P_i$.

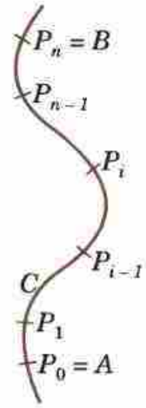

Fig. 8.8

Now consider the sum $S = \sum_{i=0}^{n} \mathbf{F}(\mathbf{U}_i) \cdot \delta\mathbf{R}_i$, *where* $d\mathbf{R}_i = \mathbf{R}_i - \mathbf{R}_{i-1}$.

The limit of this sum as $n \to \infty$ *in such a way that* $|\delta\mathbf{R}_i| \to 0$, *provided it exists, is called the* **tangential line integral** of $\mathbf{F}(\mathbf{R})$ *along* C and is symbolically written as

$$\int_C \mathbf{F}(\mathbf{R}) \cdot d\mathbf{R} \quad \text{or} \quad \int_C \mathbf{F} \cdot \frac{d\mathbf{R}}{dt}\,dt.$$

When the path of integration is a closed curve, this fact is denoted by using $\oint$ in place of $\int$.

If $\qquad \mathbf{F}(\mathbf{R}) = \mathbf{I}f(x, y, z) + \mathbf{J}\phi(x, y, z) + \mathbf{K}\psi(x, y, z)$

and $\qquad d\mathbf{R} = \mathbf{I}dx + \mathbf{J}dy + \mathbf{K}dz$

then $\qquad \int_C \mathbf{F}(\mathbf{R}) \cdot d\mathbf{R} = \int_C (f\,dx + \phi dy + \psi dz).$

Two other types of line integrals are $\int_C \mathbf{F} \times d\mathbf{R}$ and $\int_C f d\mathbf{R}$ which are both vectors.

(2) Circulation. *If* $\mathbf{F}$ *represents the velocity of a fluid particle then the line integral* $\int_C \mathbf{F} \cdot d\mathbf{R}$ *is called the* **circulation of F** *around the curve. When the circulation of* $\mathbf{F}$ *around every closed curve in a region E vanishes,* $\mathbf{F}$ *is said to be* **irrotational** *in E.*

(3) Work. If $\mathbf{F}$ represents the force acting on a particle moving along an arc AB then the work done during the small displacement $\delta\mathbf{R} = \mathbf{F} \cdot \delta\mathbf{R}$.

$\therefore$ *the total work done by* $\mathbf{F}$ *during the displacement from A to B is given by the line integral* $\int_A^B \mathbf{F} \cdot d\mathbf{R}$.

Example 8.28. *If* $\mathbf{F} = 3xy\mathbf{I} - y^2\mathbf{J}$, *evaluate* $\int \mathbf{F} \cdot d\mathbf{R}$, *where C is the curve in the xy-plane* $y = 2x^2$ *from (0, 0) to (1, 2).* *(V.T.U., 2010)*

Solution. Since the particle moves in the xy-plane ($z = 0$), we take $\mathbf{R} = x\mathbf{I} + y\mathbf{J}$. Then $\int_C \mathbf{F} \cdot d\mathbf{R}$, where C is the parabola $y = 2x^2$

$$= \int_C (3xy\mathbf{I} - y^2\mathbf{J}) \cdot (dx\mathbf{I} + dy\mathbf{J}) = \int_C (3xydx - y^2dy) \qquad \ldots(i)$$

Substituting $y = 2x^2$, where x goes from 0 to 1, (i) becomes

$$= \int_{x=0}^{1} [3x(2x^2)\, dx - (2x^2)^2\, d(2x^2)] = \int_0^1 (6x^3 - 16x^5)\, dx = -7/6.$$

Otherwise, let $x = t$ in $y = 2x^2$. Then the parametric equation of C are $x = t$, $y = 2t^2$. The points (0, 0) and (1, 2) correspond to $t = 0$ and $t = 1$ respectively. Then (i) becomes

$$= \int_{t=0}^{1} [3t(2t^2)\, dt - (2t^2)^2 d(2t^2)] = \int_0^1 (6t^3 - 16t^5)\, dt = -7/6.$$

Example 8.29. *A vector field is given by* $\mathbf{F} = \sin y\mathbf{I} + x(1 + \cos y)\,\mathbf{J}$. *Evaluate the line integral over a circular path given by* $x^2 + y^2 = a^2$, $z = 0$. *(Rohtak, 2006 S ; P.T.U., 2003)*

Solution. As the particle moves in xy-plane $(z = 0)$, let $\mathbf{R} = x\mathbf{I} + y\mathbf{J}$ so that $d\mathbf{R} = dx\mathbf{I} + dy\mathbf{J}$. Also the circular path is $x = a\cos t$, $y = a \sin t$, $z = 0$ where t varies from 0 to 2π.

$$\therefore \quad \oint_c \mathbf{F} \cdot d\mathbf{R} = \oint_c [\sin y\mathbf{I} + x(1 + \cos y)\mathbf{J}] \cdot (dx\mathbf{I} + dy\mathbf{J})$$

$$= \oint_c [\sin y\, dx + x(1 + \cos y)\, dy] = \oint_c [(\sin y\, dx + x\cos y\, dy) + xdy]$$

$$= \oint [d(x \sin y) + x\, dy] = \int_0^{2\pi} [d\{a \cos t \sin (a \sin t)\} + a^2 \cos^2 t\, dt]$$

$$= \left| a \cos t \sin (a \sin t) \right|_0^{2\pi} + \frac{a^2}{2}\int_0^{2\pi} (1 + \cos 2t)\, dt = \frac{a^2}{2}\left| t + \frac{\sin 2t}{2} \right|_0^{2\pi} = \pi a^2.$$

Example 8.30. *Find the work done in moving a particle in the force field* $\mathbf{F} = 3x^2\mathbf{I} + (2xz - y)\,\mathbf{J} + z\mathbf{K}$, *along*

(a) the straight line from (0, 0, 0) to (2, 1, 3). *(S.V.T.U., 2007 ; J.N.T.U., 2002)*

(b) the curve defined by $x^2 = 4y$, $3x^3 = 8z$ *from* $x = 0$ *to* $x = 2$. *(Delhi, 2002)*

Solution. $$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_C [3x^2\mathbf{I} + (2xz - y)\mathbf{J} + z\mathbf{K}] \cdot (dx\mathbf{I} + dy\mathbf{J} + dz\mathbf{K})$$

$$= \int_C [3x^2 dx + (2xz - y)dy + zdz] \qquad \ldots(i)$$

(a) The equations of the straight line from (0, 0, 0) to (2, 1, 3) are $x/2 = y/1 = z/3 = t$ (say)

$\therefore$ $x = 2t$, $y = t$, $z = 3t$ are its parametric equations. The points (0, 0, 0) and (2, 1, 3) correspond to $t = 0$ and $t = 1$, respectively

$$\therefore \quad \text{work done} = \int_C \mathbf{F} \cdot d\mathbf{R} = \int_0^1 [3(2t)^2\, 2dt + \{(4t)(3t) - t\}dt + (3t)\, 3dt]$$

$$= \int_0^1 (36t^2 + 8t)\, dt = 16.$$

(b) Let $x = t$ in $x^2 = 4y$, $3x^3 = 8z$. Then the parametric equations of C are $x = t$, $y = t^2/4$, $z = 3t^3/8$ and t varies from 0 to 2.

$$\therefore \quad \text{work done} = \int_C \mathbf{F} \cdot d\mathbf{R} = \int_0^2 \left[3t^2\, dt + \left\{2t\left(\frac{3t^3}{8}\right) - \frac{t^2}{4}\right\} d\left(\frac{t^2}{4}\right) + \frac{3t^3}{8}\, d\left(\frac{3t^2}{8}\right)\right]$$

$$= \int_0^2 \left(3t^2 - \frac{t^3}{8} + \frac{51}{64}t^5\right) dt = \left| t^3 - \frac{t^4}{32} + \frac{17}{128}t^6 \right|_0^2 = 16.$$

PROBLEMS 8.6

1. Evaluate the line integral $\int_C [(x^2 + xy)dx + (x^2 + y^2)dy]$ where C is the square formed by the lines $y = \pm 1$ and $x = \pm 1$. *(Delhi, 2002)*

2. If $\mathbf{F} = (5xy - 6x^2)\mathbf{I} + (2y - 4x)\mathbf{J}$, evaluate $\int_C \mathbf{F} \cdot d\mathbf{R}$ along the curve C in the xy-plane, $y = x^3$ from the point (1, 1) to (2, 8). *(J.N.T.U., 2006)*
3. Compute the line integral $\int_C (y^2 dx - x^2 dy)$ about the triangle whose vertices are (1, 0), (0, 1) and (– 1, 0).
4. If $A = (3x^2 + 6y)\mathbf{I} - 14yz\mathbf{J} + 20xz^2\mathbf{K}$, evaluate $\int \mathbf{A} \cdot d\mathbf{R}$ from (0, 0, 0) to (1, 1, 1) along the path $x = t, y = t^2, z = t^3$. *(V.T.U., 2001)*
5. Evaluate $\int_C (xy + z^2)\, ds$ where C is the arc of the helix $x = \cos t, y = \sin t, z = t$ which joins the points (1, 0, 0) and $(-1, 0, \pi)$.
6. Find the total work done by the force $\mathbf{F} = 3xy\mathbf{I} - y\mathbf{J} + 2zx\mathbf{K}$ in moving a particle around the circle $x^2 + y^2 = 4$. *(V.T.U., 2010)*
7. Find the total work done in moving a particle in a force field given by $\mathbf{F} = 3xy\mathbf{I} - 5z\mathbf{J} + 10x\mathbf{K}$ along the curve $x = t^2 + 1, y = 2t^2, z = t^3$ from $t = 1$ to $t = 2$. *(Bhopal, 2008)*
8. Using the line integral, compute the work done by the force $\mathbf{F} = (2y + 3)\mathbf{I} + xz\mathbf{J} + (yz - x)\mathbf{K}$ when it moves a particle from the point (0, 0, 0) to the point (2, 1, 1) along the curve $x = 2t^2, y = t, z = t^3$. *(Madras, 2000)*
9. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{R}$, where $\mathbf{F} = [2z, x, -y]$ and C is $\mathbf{R} = [\cos t, \sin t, 2t]$ from (1, 0, 0) to $(1, 0, 4\pi)$. *(B.P.T.U., 2006)*
10. If $\mathbf{F} = 2y\mathbf{I} - z\mathbf{J} + x\mathbf{K}$, evaluate $\int_C \mathbf{F} \times d\mathbf{R}$ along the curve $x = \cos t, y = \sin t, z = 2\cos t$ from $t = 0$ to $t = \pi/2$.

8.12 (1) SURFACES

As seen in § 5.8, a surface S may be represented by $F(x, y, z) = 0$.

The *parametric representation of* S is of the form $\mathbf{R}(u, v) = x(u, v)\,\mathbf{I} + y(u, v)\,\mathbf{J} + z(u, v)\mathbf{K}$ and the continuous functions $u = \phi(t)$ and $v = \psi(t)$ of a real parameter t represent a curve C on this surface S.

For example, the parametric representation of the circular cylinder $x^2 + y^2 = a^2, -1 \le z \le 1$, (radius a and height 2), is

$$\mathbf{R}(u, v) = a \cos u\mathbf{I} + a \sin u\mathbf{J} + v\mathbf{K}$$

where the parameters u and v vary in the rectangle $0 \le u \le 2\pi$ and $-1 \le v \le 1$. Also $u = t, v = bt$ represent a *circular helix* (Fig. 8.3) on this circular cylinder. The equation of the circular helix is $\mathbf{R} = a \cos t\mathbf{I} + a \sin t\mathbf{J} + bt\mathbf{K}$.

Differentiating $\mathbf{R} = \mathbf{R}(u, v)$, w.r.t. t, we get $\dfrac{d\mathbf{R}}{dt} = \dfrac{\partial \mathbf{R}}{\partial u} \cdot \dfrac{du}{dt} + \dfrac{\partial \mathbf{R}}{\partial v} \cdot \dfrac{dv}{dt}$

The vectors $\partial\mathbf{R}/\partial u$ and $\partial\mathbf{R}/\partial v$ are tangential to S at P and determine the tangent plane of S at P. $\mathbf{N} = \partial\mathbf{R}/\partial u \times \partial\mathbf{R}/\partial v\ (\neq 0)$ gives a normal vector $\mathbf{N}$ of S at P.

Def. *If S has a unique normal at each of its points whose direction depends continuously on the points of S, then the surface S is called a* **smooth surface**. If S is not smooth but can be divided into finitely many smooth portions, then it is called a **piecewise smooth surface.**

For instance, the surface of a sphere is *smooth* while the surface of a cube is *piecewise smooth.*

Def. *A surface S is said to be* **orientable or two sided** *if the positive normal direction at any point P of S can be continued in a unique and continuous way to the entire surface.* If the positive direction of the normal is reversed as we move around a curve on S passing through P, then the surface is **non-orientable** (*i.e.*, **one-sided**).

An example of a non-orientable surface is the *Mobius strip**. If we take a long rectangular strip of paper and giving a half-twist join the shorter sides so that the two points A and the two points B in Fig. 8.9 coincide, then the surface generated is non-orientable. Such a surface is a model of a Mobius strip.

Fig. 8.9

(2) Surface integral. Consider a continuous function $\mathbf{F}(\mathbf{R})$ and a surface S. Divide S into a finite number of sub-surfaces. Let the surface element surrounding any point $P(\mathbf{R})$ be $\delta\mathbf{S}$ which can be regarded as a vector ; its magnitude being the area and its direction that of the outward normal to the element.

*Named after a German mathematician *August Ferdinand Möbius (1790–1868)* who was a student of Gauss and professor of astronomy at Leipzig. His important contributions are in projective geometry, theory of surfaces and mechanics.

Consider the sum $\Sigma \mathbf{F}(\mathbf{R}) \,.\, \delta\mathbf{S}$, *where the summation extends over all the sub-surfaces. The limit of this sum as the number of sub-surfaces tends to infinity and the area of each sub-surface tends to zero, is called the* **normal surface integral** *of* $\mathbf{F}(\mathbf{R})$ *over S and is denoted by*

$$\int_S \mathbf{F} \,.\, d\mathbf{S} \quad \text{or} \quad \int_S \mathbf{F} \,.\, \mathbf{N} ds \text{ where } \mathbf{N} \text{ is a unit outward normal at } P \text{ to } S.$$

Other types of surface integrals are $\int_S \mathbf{F} \times d\mathbf{S}$ and $\int_S f\, d\mathbf{S}$ which are both vectors.

Notation : Only one integrals sign is used when there is one differential (say $d\mathbf{R}$ or $d\mathbf{S}$) and two (or three) signs when there are two (or three) differentials.

(3) **Flux across a surface.** *If* $\mathbf{F}$ *represent the velocity of a fluid particle then the total outward flux of* $\mathbf{F}$ *across a closed surface S is the surface integral* $\int_S \mathbf{F} \,.\, d\mathbf{S}$.

When the flux of $\mathbf{F}$ *across every closed surface S in a region E vanishes,* $\mathbf{F}$ *is said to be a* **solenoidal vector point function** *in E.*

It may be noted that $\mathbf{F}$ could equally well be taken as any other physical quantity *e.g.*, gravitational force, electric force and magnetic force.

Example 8.31. *Evaluate* $\int_S \mathbf{F} \,.\, \mathbf{N} ds$ *where* $\mathbf{F} = 2x^2y\mathbf{I} - y^2\mathbf{J} + 4xz^2\mathbf{K}$ *and S is the closed surface of the region in the first octant bounded by the cylinder* $y^2 + z^2 = 9$ *and the planes* $x = 0, x = 2, y = 0$ *and* $z = 0$.

Solution. The given closed surface S is piecewise smooth and is comprised of S_1 – the rectangular face $OAEB$ in xy-plane ; S_2–the rectangular face $OADC$ in xz-plane ; S_3–the circular quadrant ABC in yz-plane, S_4–the circular quadrant AED and S_5–the curved surface $BCDE$ of the cylinder in the first octant (Fig. 8.10).

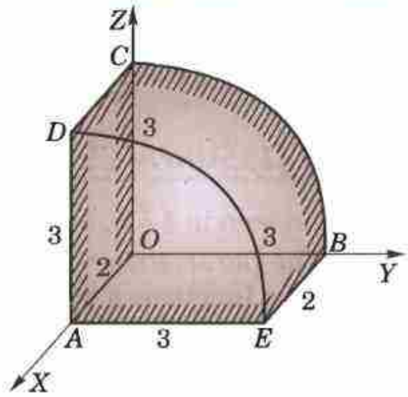

Fig. 8.10

$$\therefore \quad \int_S \mathbf{F} \,.\, \mathbf{N} ds = \int_{S_1} \mathbf{F} \,.\, \mathbf{N} ds + \int_{S_2} \mathbf{F} \,.\, \mathbf{N} ds + \int_{S_3} \mathbf{F} \,.\, \mathbf{N} ds + \int_{S_4} \mathbf{F} \,.\, \mathbf{N} ds + \int_{S_5} \mathbf{F} \,.\, \mathbf{N} ds \qquad ...(i)$$

Now $\int_{S_1} \mathbf{F} \,.\, \mathbf{N} ds = \int_{S_1} (2x^2y\mathbf{I} - y^2\mathbf{J} + 4xz^2\mathbf{K}) \,.\, (-\mathbf{K})\, ds$

$$= -4\int_{S_1} xz^2\, ds = 0 \qquad [\because \; z = 0 \text{ in the } xy\text{–plane}]$$

Similarly, $\int_{S_2} \mathbf{F} \,.\, \mathbf{N} ds = 0$ and $\int_{S_3} \mathbf{F} \,.\, \mathbf{N} ds = 0$

$$\int_{S_4} \mathbf{F} \,.\, \mathbf{N} ds = \int_{S_4} (2x^2y\mathbf{I} - y^2\,\mathbf{J} + 4xz^2\,\mathbf{K}) \,.\, \mathbf{I}\, ds$$

$$= \int_{S_4} 2x^2y\, ds = \int_0^3 \int_0^{\sqrt{9-z^2}} 8y\, dydz = 4\int_0^3 (9 - z^2)\, dz = 72$$

To find $\mathbf{N}$ in S_5, we note that $\nabla(y^2 + z^2) = 2y\mathbf{J} + 2z\mathbf{K}$

$$\therefore \qquad \mathbf{N} = \frac{2y\mathbf{J} + 2z\mathbf{K}}{\sqrt{(4y^2 + 4z^2)}} = \frac{y\mathbf{J} + z\mathbf{K}}{3} \qquad [\because \; y^2 + z^2 = 9]$$

and $\quad |\,\mathbf{N} \,.\, \mathbf{K}\,| = z/3 \qquad$ so that $ds = dxdy/(z/3)$

Thus
$$\int_{S_5} \mathbf{F} \,.\, \mathbf{N}\, ds = \int_0^2 \int_0^3 \frac{(-y^3 + 4xz^3)}{3} \,.\, dydx/(z/3) = \int_0^2 \int_0^3 \left(\frac{-y^3}{z} + 4xz^2\right) dydx$$

$$\left[\begin{array}{l}\text{Put } y = 3\sin\theta,\, z = 3\cos\theta \\ \therefore \;\; dy = 3\cos\theta\, d\theta\end{array}\right]$$

$$= \int_0^2 \int_0^{\pi/2} \left[\frac{-27\sin^3\theta}{3\cos\theta} + 4x(9\cos^2\theta\right] 3\cos\theta\, d\theta dx = \int_0^2 \left[-27 \times \frac{2}{3} + 108x \times \frac{2}{3}\right] dx = 108$$

Hence (*i*) gives $\int_S \mathbf{F} \,.\, \mathbf{N}\, ds = 0 + 0 + 0 + 72 + 108 = 180.$

PROBLEMS 8.7

1. If velocity vector is $\mathbf{F} = y\mathbf{I} + 2\mathbf{J} + xz\mathbf{K}$ m/sec., show that the flux of water through the parabolic cylinder $y = x^2$, $0 \le x \le 3$, $0 \le z \le 2$ is 69 m^3/sec.
2. Evaluate $\int_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = x\mathbf{I} + (z^2 - zx)\mathbf{J} - xy\mathbf{K}$ and S is the triangular surface with vertices (2, 0, 0), (0, 2, 0) and (0, 0, 4).
3. Evaluate $\int_S \mathbf{F} \cdot \mathbf{N}\, ds$ where $\mathbf{F} = 6z\mathbf{I} - 4\mathbf{J} + y\mathbf{K}$ and S is the portion of the plane $2x + 3y + 6z = 12$ in the first octant.
4. If $\mathbf{F} = 2y\mathbf{I} - 3\mathbf{J} + x^2\mathbf{K}$ and S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$ and $z = 6$, show that $\int_S \mathbf{F} \cdot \mathbf{N}\, ds = 132$.

8.13 GREEN'S THEOREM IN THE PLANE*

If $\phi(x, y)$, $\psi(x, y)$, ϕ_y and ψ_x be continuous in a region E of the xy-plane bounded by a closed curve C, then

$$\int_C (\phi dx + \psi dy) = \iint_E \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dxdy \qquad ...(1)$$

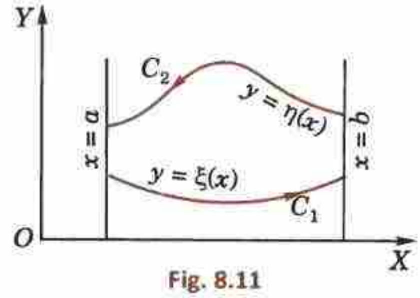

Fig. 8.11

Consider the region E bounded by a single closed curve C which is cut by any line parallel to the axes at the most in two points.

Let E be bounded by $x = a$, $y = \xi(x)$, $x = b$ and $y = \eta(x)$, where $\eta \ge \xi$, so that C is divided into curves C_1 and C_2 (Fig. 8.11).

$$\therefore \quad \iint_E \frac{\partial \phi}{\partial y} dxdy = \int_a^b dx \left[\int_\xi^\eta \frac{\partial \phi}{\partial y} dy\right] = \int_a^b dx \mid \phi \mid_\xi^\eta$$

$$= \int_a^b [\phi(x, \eta) - \phi(x, \xi)]\, dx = -\int_{C_2} \phi(x, y)\, dx - \int_{C_1} \phi(x, y)\, dx$$

$$= -\int_C \phi(x, y)\, dx \qquad ...(2)$$

Similarly, it can be shown that

$$\iint_E \frac{\partial \psi}{\partial x} dxdy = \int_C \psi(x, y)\, dy \qquad ...(3)$$

On subtracting (2) from (3), we get (1).

This result can be extended to regions which may be divided into a finite number of sub-regions such that the boundary of each is cut at the most in two points by any line parallel to either axis. Applying (1) to each of these sub-regions and adding the results, the surface integrals combine into an integral over the whole region ; the line integrals over the common boundaries cancel (for each is covered twice but in opposite directions), whereas the remaining line integrals combine into the line integral over the external curve C.

Obs. This theorem converts a line integral around a closed curve into a double integral and is a special case of Stoke's theorem. (See Cor. p. 342)

Example 8.32. *Verify Green's theorem for $\int_C [(xy + y^2)\, dx + x^2 dy]$, where C is bounded by $y = x$ and $y = x^2$.*

(V.T.U., 2011 ; S.V.T.U., 2009 ; Rohtak, 2003)

Solution. Here $\qquad \phi = xy + y^2$ and $\psi = x^2$

$$\therefore \quad \int_C (\phi\, dx + \psi\, dy) = \int_{C_1} + \int_{C_2}$$

*Named after the English mathematician *George Green* (1793–1841) who taught at Cambridge and is known for his work on potential theory in connection with waves, vibrations, elasticity, electricity and magnetism.

Along C_1, $y = x^2$ and x varies from 0 to 1 (Fig. 8.12)

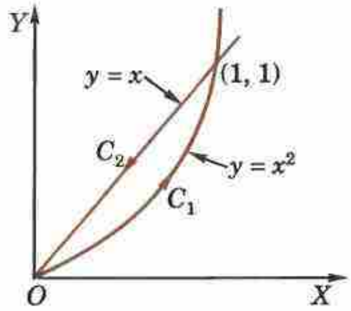

Fig. 8.12

$$\therefore \quad \int_{C_1} = \int_0^1 [\{x(x)^2 + (x^2)^2\}] \, dx + x^2 d(x^2)]$$

$$= \int_0^1 (3x^3 + x^4)\, dx = \frac{19}{20}$$

Along C_2, $y = x$ and x varies from 1 to 0.

$$\therefore \quad \int_{C_2} = \int_1^0 [\{x(x) + (x)^2\}\, dx + x^2\, d(x)] = \int_1^0 3x^2\, dx = -1.$$

Thus $\displaystyle \int_C (\phi\, dx + \psi\, dy) = \frac{19}{20} - 1 = -\frac{1}{20}$ ...(*i*)

Also $\displaystyle \iint_E \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dxdy = \iint_E \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy + y^2)\right] dxdy$

$$= \int_0^1 \int_{x^2}^{x} (2x - x - 2y)\, dydx = \int_0^1 \left[xy - y^2\right]_{x^2}^{x} dx = \int_0^1 (x^4 - x^3)\, dx = -\frac{1}{20} \quad ...(ii)$$

Hence, Green theorem is verified from the equality of (*i*) and (*ii*).

Example 8.33. *If C is a simple closed curve in the xy-plane not enclosing the origin, show that*

$$\int_C \mathbf{F} \cdot d\mathbf{R} = 0, \quad \text{where} \quad \mathbf{F} = \frac{y\mathbf{I} - x\mathbf{J}}{x^2 + y^2}$$ (*P.T.U., 2005*)

Solution. $\displaystyle \int_C \mathbf{F} \cdot d\mathbf{R} = \int_C \frac{y\mathbf{I} - x\mathbf{J}}{x^2 + y^2} (dx\mathbf{I} + dy\mathbf{J})$ $[\because \ \mathbf{R} = x\mathbf{I} + y\mathbf{J}]$

$$= \int_C \frac{ydx - xdy}{x^2 + y^2} = \int_C (\phi dx + \psi dy) \text{ where } \phi = \frac{y}{x^2 + y^2}, \psi = \frac{-x}{x^2 y^2}$$

$$= \iint_S \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dxdy \quad \text{[By Green's theorem]}$$

$$= \iint_S \left[\frac{-(x^2 + y^2) + x(2x)}{(x^2 + y^2)^2} - \frac{(x^2 + y^2) - y(2y)}{(x^2 + y^2)^2}\right] dxdy$$

$$= \iint \left[\frac{x^2 - y^2}{(x^2 + y^2)^2} - \frac{x^2 - y^2}{(x^2 + y^2)^2}\right] dxdy = 0.$$

Example 8.34. *Using Green's theorem, evaluate* $\int_C [(y - \sin x)\, dx + \cos x\, dy]$ *where C is the plane triangle enclosed by the lines* $y = 0$, $x = \pi/2$ *and* $y = \frac{2}{\pi} x$. (*J.N.T.U., 2005 ; Anna, 2003*)

Solution. Here $\phi = y - \sin x$ and $\psi = \cos x$.

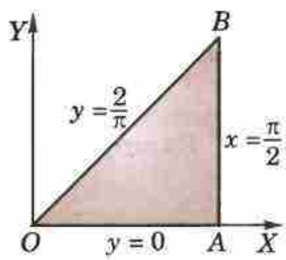

Fig. 8.13

By Green's theorem $\int_C [(y - \sin x)\, dx + \cos x\, dy]$

$$= \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy$$

$$= \int_{x=0}^{x=\pi/2} \int_{y=0}^{y=2x/\pi} (-\sin x - 1)\, dy\, dx = -\int_0^{\pi/2} (\sin x + 1) \left| \, y \, \right|_0^{2x/\pi} dx$$

$$= -\frac{2}{\pi} \int_0^{\pi/2} x(\sin x + 1)\, dx = -\frac{2}{\pi}\left\{\left| \, x(-\cos x + x) \, \right|_0^{\pi/2} - \int_0^{\pi/2} 1 \,.\, (-\cos x + x)\, dx\right\}$$

$$= -\frac{2}{\pi}\left\{\frac{\pi^2}{4} - \left| -\sin x + \frac{x^2}{2} \right|_0^{\pi/2}\right\} = -\frac{\pi}{2} + \frac{2}{\pi}\left(-1 + \frac{\pi^2}{8}\right) = -\left(\frac{\pi}{4} + \frac{2}{\pi}\right)$$

Example 8.35. *Apply Green's theorem to evaluate* $\int_C [(2x^2 - y^2)dx + (x^2 + y^2)dy]$, *where C is the boundary of the area enclosed by the x-axis and the upper-half of the circle* $x^2 + y^2 = a^2$. *(U.P.T.U., 2005)*

Solution. By Green's theorem

$$\int_C [(2x^2 - y^2)dx + (x^2 + y^2)dy]$$

$$= \iint_A \left[\frac{\partial}{\partial x}(x^2 + y^2) - \frac{\partial}{\partial y}(2x^2 - y^2)\right] dxdy$$

$$= 2\iint_A (x + y)\, dxdy, \text{ where } A \text{ is the region of Fig. 8.14}$$

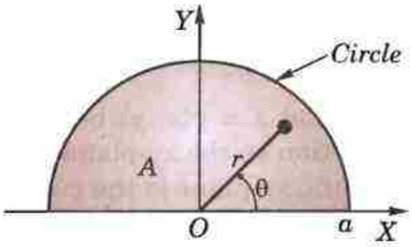

Fig. 8.14

$$= 2\int_0^a \int_0^\pi r\ (\cos\theta + \sin\theta) \cdot r d\theta\, dr$$

[Changing to polar coordinates (r, θ), r varies from 0 to a and θ varies from 0 to π]

$$= 2\int_0^a r^2\, dr \int_0^\pi (\cos\theta + \sin\theta)\, d\theta = 2 \cdot \frac{a^3}{3} \cdot (1 + 1) = \frac{4a^3}{3}.$$

PROBLEMS 8.8

1. Verify Green's theorem for $\int_C [(3x - 8y^2)\, dx + (4y - 6xy)\, dy]$ where C is the boundary of the region bounded by $x = 0, y = 0$ and $x + y = 1$. *(Nagpur, 2008 ; Kerala, 2005 ; Anna, 2003 S)*
2. Verify Green's theorem for $\int_C [(x^2 - \cosh y)dx + (y + \sin x)dy]$ where C is the rectangle with vertices $(0, 0), (\pi, 0), (\pi, 1), (0, 1)$. *(Nagpur, 2009 ; P.T.U., 2006)*
3. Verify Green's theorem for $\int (x^2 y dx + x^2 dy)$ where C is the boundary described counter clockwise of triangle with vertices $(0, 0), (1, 0), (1, 1)$. *(U.T.U., 2010)*
4. Apply Green's theorem to prove that the area enclosed by a plane curve is $\frac{1}{2}\int_C (xdy - ydx)$.

   Hence find the area of an ellipse whose semi-major and semi-minor axes are of lengths a and b. *(Kerala, 2005 ; V.T.U., 2000 S)*
5. Find the area of a circle of radius a using Green's theorem. *(Madras, 2003)*
6. Evaluate $\int_C [(x^2 + xy)dx + (x^2 + y^2)dy]$, where C is the square formed by the lines $x = \pm 1, y = \pm 1$. *(S.V.T.U., 2008 ; Marathwada, 2008)*
7. Evaluate $\int_C [(x^2 - 2xy)dx + (x^2 y + 3)dy]$, around the boundary of the region defined by $y^2 = 8x$ and $x = 2$.
8. Evaluate by Green's theorem $\int_C \mathbf{F} \cdot d\mathbf{R}$ where $\mathbf{F} = -xy(x\mathbf{I} - y\mathbf{J})$ and C is $r = a(1 + \cos\theta)$. *(Mumbai, 2006)*

8.14 STOKE'S THEOREM* (*Relation between line and surface integrals*)

If S be an open surface bounded by a closed curve C and $\mathbf{F} = f_1\mathbf{I} + f_2\mathbf{J} + f_3\mathbf{K}$ *be any continuously differentiable vector point function, then*

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_S curl\ \mathbf{F} \cdot \mathbf{N} ds$$

where $\mathbf{N} = \cos\alpha\mathbf{I} + \cos\beta\mathbf{J} + \cos\gamma\mathbf{K}$ *is a unit external normal at any point of S.*

* Named after an Irish mathematician *Sir George Gabriel Stokes* (1819–1903) who became professor in Cambridge. His important contributions are to infinite series, geodesy and theory of viscous fluids.

Writing $d\mathbf{R} = dx\mathbf{I} + dy\mathbf{J} + dz\mathbf{K}$, it may be reduced to the form

$$\int_C (f_1dx + f_2dy + f_3dz) = \int_S \left[\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\cos\beta + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\cos\gamma\right] ds \quad ...(1)$$

Let us first prove that

$$\oint_C f_1 dx = \int_S \left(\frac{\partial f_1}{\partial z}\cos\beta - \frac{\partial f_1}{\partial y}\cos\gamma\right) ds \quad ...(2)$$

Let $z = g(x, y)$ be the equation of the surface S whose projection on the xy-plane is the region E. Then the projection of C on the xy-plane is the curve C' enclosing region E.

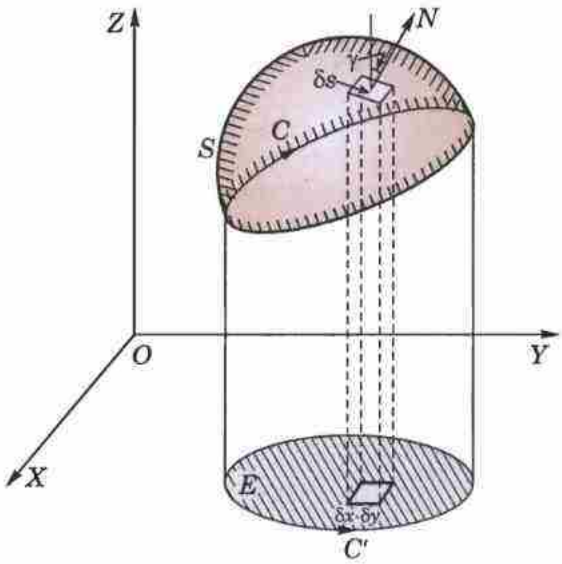

Fig. 8.15

$$\therefore \quad \int_C f_1(x, y, z)dx = \int_{C'} f_1\{x, y, g(x, y)\}dx$$

$$= -\iint_E \frac{\partial}{\partial y} f_1(x, y, g)\,dxdy, \text{ by Green's theorem}$$

$$= -\iint_E \left(\frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial z}\frac{\partial g}{\partial y}\right)dxdy \quad ...(3)$$

The direction cosines of the normal to the surface $z = g(x, y)$ are given by

$$\frac{\cos\alpha}{-\partial g/\partial x} = \frac{\cos\beta}{-\partial g/\partial y} = \frac{\cos\gamma}{1} \quad \text{(See p. 219)} \quad ...(4)$$

Moreover

$dxdy$ = projection of ds on the xy-plane
$= ds\cos\gamma$, *i.e.*, $ds = dxdy/\cos\gamma$.

$\therefore$ right side of (2)

$$= \iint_E \left(\frac{\partial f_1}{\partial z}\frac{\cos\beta}{\cos\gamma} - \frac{\partial f_1}{\partial y}\right)dxdy = -\iint \left(\frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial z}\cdot\frac{\partial g}{\partial y}\right)dxdy \quad \left[\frac{\cos\beta}{\cos\gamma} = -\frac{\partial g}{\partial y} \text{ by (4)}\right]$$

= Left side of (2), by (3).

Thus we have proved (2). Similarly, we can prove the other corresponding relations for f_2 and f_3. Adding these three results, we get (1).

Cor. Green's theorem in a plane as a special case of Stokes theorem. Let $\mathbf{F} = \phi\mathbf{I} + \psi\mathbf{J}$ be a vector function which is continuously differentiable in a region S of the xy-plane bounded by a closed curve C. Then

$$\int_C \mathbf{F}\,.\,dR = \int_C (\phi\mathbf{I} + \psi\mathbf{J})\cdot(dx\,\mathbf{I} + dy\,\mathbf{J}) = \int_C (\phi\,dx + \psi\,dy)$$

and

$$\text{curl}\,\mathbf{F}\cdot\mathbf{N} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \partial/\partial x & \partial/\partial y & 0 \\ \phi & \psi & 0 \end{vmatrix}\cdot\mathbf{K} = \frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}$$

Hence *Stoke's theorem* takes the form $\int_C (\phi\,dx + \psi\,dy) = \int_C \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right) dxdy$ which is *Green's theorem* in a plane.

Example 8.36. *Verify Stoke's theorem for $\mathbf{F} = (x^2 + y^2)\mathbf{I} - 2xy\mathbf{J}$ taken around the rectangle bounded by the lines $x = \pm a$, $y = 0$, $y = b$.* *(Bhopal, 2008 S ; V.T.U., 2007 ; J.N.T.U., 2003 ; U.P.T.U., 2003)*

Solution. Let $ABCD$ be the given rectangle as shown in Fig. 8.16.

$$\int_{ABCD} \mathbf{F}\cdot d\mathbf{R} = \int_{AB}\mathbf{F}\cdot d\mathbf{R} + \int_{BC}\mathbf{F}\cdot d\mathbf{R} + \int_{CD}\mathbf{F}\cdot d\mathbf{R} + \int_{DA}\mathbf{F}\cdot d\mathbf{R}$$

and $\quad \mathbf{F}\cdot d\mathbf{R} = [(x^2 + y^2)\mathbf{I} - 2xy\mathbf{J}]\cdot(\mathbf{I}dx + \mathbf{J}dy) = (x^2 + y^2)dx - 2xydy$

Along AB, $x = a$ (*i.e.*, $dx = 0$) and y varies from 0 to b.

$$\therefore \quad \int_{AB}\mathbf{F}\cdot d\mathbf{R} = -2a\int_0^b y\,dy = -2a\cdot\frac{b^2}{2} = -ab^2.$$

Similarly, $$\int_{BC} \mathbf{F}\cdot d\mathbf{R} = \int_a^{-a} (x^2+b^2)dx = -\frac{2a^3}{3} - 2ab^2 \cdot$$

$$\int_{CD} \mathbf{F}\cdot d\mathbf{R} = 2a \int_b^0 y\,dy = -ab^2$$

and $$\int_{DA} \mathbf{F}\cdot d\mathbf{R} = \int_{-a}^{a} x^2\,dx = \frac{2a^3}{3}\cdot$$

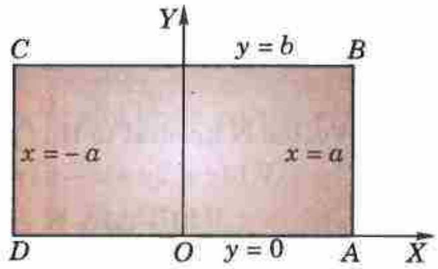

Fig. 8.16

Thus $$\int_{ABCD} \mathbf{F}\cdot d\mathbf{R} = -4ab^2 \qquad ...(i)$$

Also since curl $\mathbf{F} = -4\mathbf{K}y$

$$\therefore \quad \int_S \text{curl } \mathbf{F}\cdot\mathbf{N}\,ds = \int_0^b \int_{-a}^{a} -4\mathbf{K}y\cdot\mathbf{K}\,dxdy = -4\int_0^b \int_{-a}^{a} y\,dxdy$$

$$= -4\int_0^b |x|_{-a}^{a}\, y\,dy = -8a\left|\frac{y^2}{2}\right|_0^b = -4ab^2 \qquad ...(ii)$$

Hence Stoke's theorem is verified from the equality of (*i*) and (*ii*).

Example 8.37. *Verify Stoke's theorem for the vector field $F = (2x-y)\,\mathbf{I} - yz^2\,\mathbf{J} - y^2z\,\mathbf{K}$ over the upper half surface of $x^2+y^2+z^2 = 1$, bounded by its projection on the xy-plane.*

(*Bhopal, 2008 ; Madras, 2006 ; S.V.T.U., 2006*)

Solution. The projection of the upper half of given sphere on the xy-plane ($z = 0$) is the circle $c[x^2+y^2 = 1]$ (Fig. 8.17).

$$\oint_c \mathbf{F}.\,d\mathbf{R} = \oint_c [(2x-y)dx - yz^2\,dy - y^2\,zdz] = \oint_c (2x-y)dx \qquad [z = 0 \text{ in the } xy\text{-plane}]$$

$$= \int_{\theta=0}^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta\,d\theta) \qquad [\text{Putting } x = \cos\theta,\ y = \sin\theta]$$

$$= \int_0^{2\pi} (-\sin 2\theta + \sin^2\theta)\,d\theta = \times 0 + 4\int_0^{\pi/2} \sin^2\theta\,d\theta = \pi. \qquad ...(i)$$

Now $$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= (-2yz + 2yz)\,\mathbf{I} + 0\,\mathbf{J} + \mathbf{K} = \mathbf{K}$$

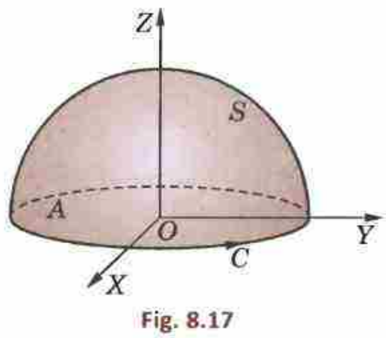

Fig. 8.17

$$\therefore \quad \int \text{curl } \mathbf{F}.\,Nds = \int_S K.\mathbf{N}\,ds = \int_A \mathbf{K}.\mathbf{N}\frac{dxdy}{|\mathbf{N}.\mathbf{K}|}$$

where A is the projection of S on xy-plane and $ds = dxdy/\mathbf{N}.\mathbf{K}$

$$= \int_A dx\,dy = \text{area of circle } C = \pi \qquad ...(ii)$$

Hence, the Stokes theorem is verified from the equality of (*i*) and (*ii*).

Example 8.38. *Uses Stoke's theorem evaluate $\int_C [(x+y)dx + (2x-z)dy + (y+z)dz]$ where C is the boundary of the triangle with vertices (2, 0, 0), (0, 3, 0) and (0, 0, 6).*

(*Nagpur, 2009 ; Kurukshetra, 2009 S ; Kerala, 2005*)

Solution. Here $\mathbf{F} = (x+y)\,\mathbf{I} + (2x-z)\,\mathbf{J} + (y+z)\,\mathbf{K}$

$$\therefore \quad \text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix} = 2\mathbf{I} + \mathbf{K}$$

Also equation of the plane through A, B, C (Fig. 8.18) is

$$\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=1 \text{ or } 3x+2y+z=6$$

Vector $\mathbf{N}$ normal to this plane is

$$\nabla(3x+2y+z-6)=3\mathbf{I}+2\mathbf{J}+\mathbf{K}$$

$$\therefore \quad \hat{\mathbf{N}}=\frac{3\mathbf{I}+2\mathbf{J}+\mathbf{K}}{\sqrt{(9+4+1)}}=\frac{1}{\sqrt{14}}(3\mathbf{I}+2\mathbf{J}+\mathbf{K})$$

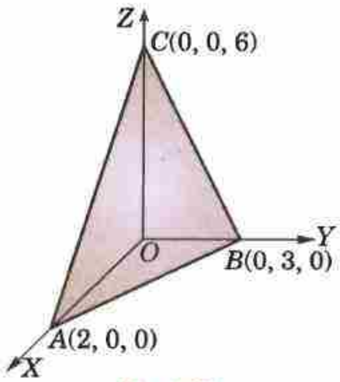

Fig. 8.18

Hence $\int_C [(x+y)dx+(2x-z)dy+(y+z)dz]=\int_C \mathbf{F}\cdot d\mathbf{R}$

$$=\int_S \text{curl } \mathbf{F}\cdot\hat{\mathbf{N}}\, ds \qquad \text{where } S \text{ is the triangle } ABC$$

$$=\int_S (2\mathbf{I}+\mathbf{K})\cdot\left(\frac{3\mathbf{I}+2\mathbf{J}+\mathbf{K}}{\sqrt{14}}\right)ds=\frac{1}{\sqrt{14}}(6+1)\int_S ds$$

$$=\frac{7}{\sqrt{14}}(\text{Area of } \Delta ABC)=\frac{7}{\sqrt{14}}\cdot 3\sqrt{14}=21.$$

Example 8.39. *If $\mathbf{F}=3y\mathbf{I}-xz\mathbf{J}+yz^2\mathbf{K}$ and S is the surface of the paraboloid $2z=x^2+y^2$ bounded by $z=2$, evaluate $\iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{S}$ using Stoke's theorem.*

Solution. By Stokes theorem, $I=\iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{S}=\oint_C \mathbf{F}\cdot d\mathbf{R}$

where S is the surface $2z=x^2+y^2$ bounded by $z=2$.

$$\therefore \quad I=\oint_C \mathbf{F}\cdot d\mathbf{R}=\oint_C (3yI-xzJ+yz^2K)\cdot(dxI+dyJ+dzK)$$

$$=\oint_C (3ydx-xzdy+yz^2dz)$$

$\because \; S\equiv x^2+y^2=4, z=2$
$\therefore$ Put $x=2\cos\theta, y=2\sin\theta$
$C\equiv x^2+y^2=4, \theta=0$ to 2π.

$$=\int_0^{2\pi}[6\sin\theta(-2\cos\theta\, d\theta)-4\cos\theta(2\cos\theta\, d\theta)+8\sin\theta(0)]$$

$$=-4\int_0^{2\pi}(12\sin^2\theta+8\cos^2\theta)\,d\theta$$

$$=-4\left(12\cdot\frac{1}{2}\frac{\pi}{2}+8\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right)=-20\pi.$$

Example 8.40. *Apply Stoke's theorem to evaluate $\int_C (ydx+zdy+xdz)$ where C is the curve of intersection of $x^2+y^2+z^2=a^2$ and $x+z=a$.* (Bhopal, 2008)

Solution. The curve C is evidently a circle lying in the plane $x+z=a$, and having $A(a,0,0)$, $B(0,0,a)$ as the extremities of the diameter (Fig. 8.19).

$$\therefore \quad \int_C (y\,dx+z\,dy+x\,dz)=\int_C (y\mathbf{I}+z\mathbf{J}+x\mathbf{K})\cdot d\mathbf{R}$$

$$=\int_S \text{curl}(y\mathbf{I}+z\mathbf{J}+x\mathbf{K})\cdot\mathbf{N}ds$$

where S is the circle on AB as diameter and $\mathbf{N}=\frac{1}{\sqrt{2}}\mathbf{I}+\frac{1}{\sqrt{2}}\mathbf{K}$

$$=\int_S -(\mathbf{I}+\mathbf{J}+\mathbf{K})\cdot\left(\frac{1}{\sqrt{2}}\mathbf{I}+\frac{1}{\sqrt{2}}\mathbf{K}\right)ds$$

$$=-\frac{2}{\sqrt{2}}\int_S ds=-\frac{2}{\sqrt{2}}\pi\left(\frac{a}{\sqrt{2}}\right)^2=-\frac{\pi a^2}{\sqrt{2}}.$$

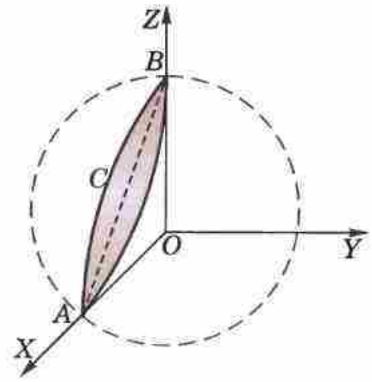

Fig. 8.19

Example 8.41. *If S be any closed surface, prove that* $\int_S$ *curl* $\mathbf{F} . d\mathbf{S} = 0$.

Solution. Cut open the surface S by any plane and let S_1, S_2 denote its upper and lower portions. Let C be the common curve bounding both these portions.

$$\therefore \qquad \int_S \text{curl } \mathbf{F} . d\mathbf{S} = \int_{S_1} \text{curl } \mathbf{F} . d\mathbf{S} + \int_{S_2} \text{curl } \mathbf{F} . d\mathbf{S} = \int_C \mathbf{F} . d\mathbf{R} - \int_C \mathbf{F} . d\mathbf{R} = 0,$$

on applying Stoke's theorem. The second integral is negative because it is traversed in a direction opposite to that of the first.

PROBLEMS 8.9

1. Verify Stoke's theorem for the vector field (*i*) $\mathbf{F} = (x^2 - y^2)\mathbf{I} + 2xy\mathbf{J}$ over the box bounded by the planes $x = 0, x = a$; $y = 0, y = b; z = 0, z = c$; if the face $z = 0$ is cut. *(B.P.T.U., 2006 ; Delhi, 2002)*
(*ii*) $\mathbf{F} = (z^2, 5x, 0)$ and $S : 0 \le x \le 1, 0 \le y \le 1, z = 1$.
2. Verify Stoke's theorem for a vector field defined by $\mathbf{F} = -y^3\mathbf{I} + x^3\mathbf{J}$, in the region $x^2 + y^2 \le 1, z = 0$.
3. Evaluate $\int_c \mathbf{F} . d\mathbf{R}$ where $\mathbf{F} = (x^2 + y^2)\mathbf{I} - 2xy\,\mathbf{J}$ and c is the rectangle in the xy-plane bounded by $y = 0, x = a, y = b$, $x = 0$. *(Mumbai, 2007)*
4. Verity Stoke's theorem for $\mathbf{F} = (y - z + 2)\mathbf{I} + (yz + 4)\mathbf{J} - xz\mathbf{K}$ where S is the surface of the cube $x = 0, y = 0, z = 0$, $x = 2, y = 2, z = 2$ above the xy-plane. *(Andhra, 2000)*
5. Evaluate $\int_C \mathbf{F} . d\mathbf{R}$ where $\mathbf{F} = y\mathbf{I} + xz^3\mathbf{J} - zy^3\mathbf{K}$, C is the circle $x^2 + y^2 = 4, z = 1.5$.
6. Evaluate by Stoke's theorem $\oint_C (yz\,dz + zx\,dy + zx\,dz)$ where C is the curve $x^2 + y^2 = 1, z = y^2$. *(J.N.T.U., 2005)*
7. If S be the surface of the sphere $x^2 + y^2 + z^2 = 1$, prove that $\int_S \text{curl } \mathbf{F} . d\mathbf{S} = 0$. *(J.N.T.U., 1999)*
8. Prove that $\int_C A \times \mathbf{R} . d\mathbf{R} = 2\mathbf{A} . \int d\mathbf{S}$, $\mathbf{A}$ being any constant vector, and deduce that $\oint_C \mathbf{R} \times d\mathbf{R}$ is twice the vector area of the surface enclosed by C.
9. If ϕ is a scalar point function, use Stoke's theorem to prove that (*i*) curl (grad ϕ) = 0. (*ii*) div curl $\mathbf{F} = 0$. *(Kerala, 2005)*
10. Evaluate $\oint_C (\sin z\,dx - \cos x\,dy + \sin y\,dz)$ where C is the boundary of the rectangle $0 \le x \le \pi$. $0 \le y \le 1, z = 3$. *(Rohtak, 2005)*
11. Use Stoke's theorem to evaluate $(\nabla \times \mathbf{F}) . \mathbf{N}\,ds$, where $\mathbf{F} = y\mathbf{I} + (x - 2xz)\,\mathbf{J} - xy\mathbf{K}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy-plane. *(Kottayam, 2005)*
12. Evaluate $\int \nabla \times \mathbf{V} . d\mathbf{S}$ over the surface of the paraboloid $z = 1 - x^2 - y^2, z \ge 0$ where $\mathbf{V} = y\mathbf{I} + z\mathbf{J} + x\mathbf{K}$.

8.15 VOLUME INTEGRAL

Consider a continuous vector function $\mathbf{F}(\mathbf{R})$ and surface S enclosing the region E. Divide E into finite number of sub-regions $E_1, E_2, ..., E_n$. Let δv_i be the volume of the sub-region E_i enclosing any point whose position vector is $\mathbf{R}_i$.

Consider the sum $$\mathbf{V} = \sum_{i=1}^{n} \mathbf{F}(\mathbf{R}_i)\,\delta v_i$$

The limit of this sum as $n \to \infty$ in such a way that $\delta v_i \to 0$, is called the volume integral of $\mathbf{F}(\mathbf{R})$ *over E and is symbolically written as* $\int_E \mathbf{F}\,dv$.

If $\mathbf{F}(\mathbf{R}) = f(x, y, z)\mathbf{I} + \phi(x, y, z)\mathbf{J} + \psi(x, y, z)\mathbf{K}$ so that $\delta v = \delta x \delta y \delta z$, then

$$\int_E \mathbf{F}\,dv = \mathbf{I} \iiint_E f dxdydz + \mathbf{J} \iiint_E \phi dxdydz + \mathbf{K} \iiint_E \psi dxdydz.$$

8.16 GAUSS DIVERGENCE THEOREM* (*Relation between surface and volume integrals*)

If **F** is a continuously differentiable vector function in the region E bounded by the closed surface S, then

$$\int_S \mathbf{F} \cdot \mathbf{N} ds = \int_E div\, \mathbf{F}\, dv$$

where **N** *is the unit external normal vector.*

If $\mathbf{F}(\mathbf{R}) = f(x, y, z)\mathbf{I} + \phi(x, y, z)\mathbf{J} + \psi(x, y, z)\mathbf{K}$

then it is required to prove that

$$\iint_S (f dydz + \phi dzdx + \psi dxdy)$$

$$= \iiint_E \left(\frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial z} \right) dxdydz \quad ...(1)$$

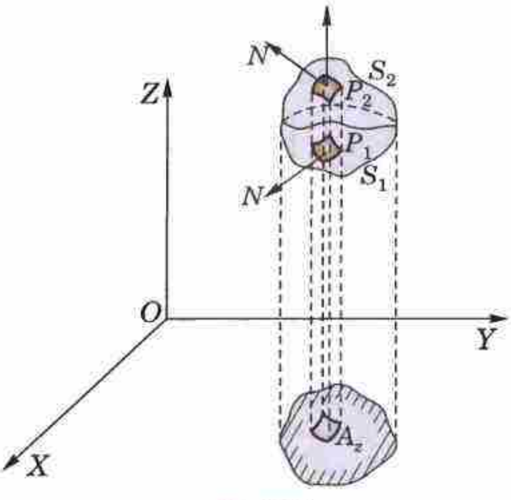

Fig. 8.20

Firstly consider such a surface S that a line parallel to z-axis cuts it in two points; say $P_1(x, y, z_1)$ *and* $P_2(x, y, z_2)$ $(z_1 \le z_2)$ (Fig. 8.20).

If S projects into the area A_z on the xy-plane, then

$$\iiint_E \frac{\partial \psi}{\partial z}\, dxdydz = \iint_{A_z} dxdy \int_{z_1}^{z_2} \frac{\partial \psi}{\partial z}\, dz$$

$$= \iint_{A_z} [\Psi(x, y, z_2) - \Psi(x, y, z_1)]\, dxdy = \iint_{A_z} \Psi(x, y, z_2)\, dxdy - \iint_{A_z} \Psi(x, y, z_1)\, dxdy \quad ...(2)$$

Let S_1, S_2 be the lower and upper parts of the surface S corresponding to the points P_1 and P_2 respectively and **N** be the unit external normal vector at any point of S. As the external normal at any point of S_2 makes an acute angle with the positive direction of z-axis and that at any point of S_1 an obtuse angle, therefore

$$\iint_{A_z} \Psi(x, y, z_2)\, dxdy = \int_{S_2} \psi \mathbf{N} \cdot \mathbf{K} ds \quad ...(3)$$

$$\iint_{A_z} \Psi(x, y, z_1)\, dxdy = -\int_{S_1} \psi \mathbf{N} \cdot \mathbf{K} ds \quad ...(4)$$

Using (3) and (4), (2) now becomes

$$\iiint_E \frac{\partial \psi}{\partial z} dxdydz = \int_{S_2} \psi \mathbf{N} \cdot \mathbf{K} ds + \int_{S_1} \psi \mathbf{N} \cdot \mathbf{K} ds = \int_S \psi \mathbf{N} \cdot \mathbf{K} ds \quad ...(5)$$

Similarly, we have

$$\iiint_E \frac{\partial f}{\partial x} dxdydz = \int_S f \mathbf{N} \cdot \mathbf{I} ds \quad ...(6)$$

$$\iiint_E \frac{\partial \phi}{\partial y} dxdydz = \int_S \phi \mathbf{N} \cdot \mathbf{J} ds \quad ...(7)$$

Addition of (5), (6) and (7) gives

$$\iiint_E \left(\frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial z} \right) dxdydz = \int_S (f\mathbf{I} + \phi\mathbf{J} + \psi\mathbf{K}) \cdot \mathbf{N} ds \text{ which is same as (1).}$$

Secondly, consider a general region E. Assume that it can be split up into a finite number of sub-regions each of which is met by a line parallel to any axis in only two points. Applying (1) to each of these sub-regions and adding the results, the volume integrals will combine to give the volume integral over the whole region E. Also the surface integrals over the common boundaries of two sub-regions cancel because each occurs twice and having corresponding normals in opposite directions whereas the remaining surface integrals combine to give the surface integral over the entire surface S.

Finally consider a region E bounded by two closed surfaces S_1, S_2 (S_1 *being within* S_2). Noting that outward normal at points of S_1 is directed inwards (*i.e.*, away from S_2) and introducing an additional surface cutting S_1, S_2 so that all parts of E are bounded by a single closed surface, the truth of the theorem follows as before. Thus theorem also holds for regions enclosed by several surfaces.

Hence the theorem is completely established.

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*See footnote p. 37.

Example 8.42. *Verify Divergence theorem for* $\mathbf{F} = (x^2 - yz)\mathbf{I} + (y^2 - zx)\mathbf{J} + (z^2 - xy)\mathbf{K}$ *taken over the rectangular parallelepiped* $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$. *(Rohtak, 2006 S ; Madras, 2000 S)*

Solution. As $\quad \text{div}\,\mathbf{F} = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)$

$$= 2(x + y + z)$$

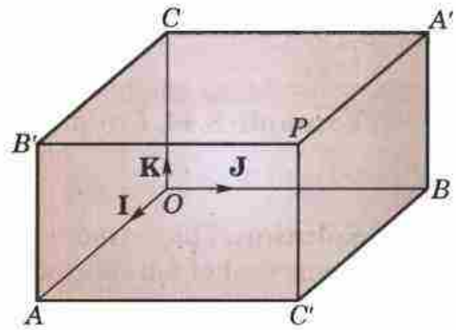

Fig. 8.21

$$\therefore \quad \int_R \text{div}\,\mathbf{F}\,dv = 2\int_0^c \int_0^b \int_0^a (x + y + z)\,dxdydz$$

$$= 2\int_0^c dz \int_0^b dy\left(\frac{a^2}{2} + ya + za\right)$$

$$= 2\int_0^c dz\left(\frac{a^2}{2}b + \frac{ab^2}{2} + abz\right)$$

$$= 2\left(\frac{a^2 b}{2}c + \frac{ab^2}{2}c + ab\frac{c^2}{2}\right)$$

$$= abc\,(a + b + c) \qquad \ldots(i)$$

Also $\quad \int_S \mathbf{F}\,.\,\mathbf{N}ds = \int_{S_1} \mathbf{F}\,.\,\mathbf{N}ds + \int_{S_2} \mathbf{F}\,.\,\mathbf{N}ds + \ldots + \int_{S_6} \mathbf{F}\,.\,\mathbf{N}ds$

where S_1 in the face $OAC'B$, S_2 the face $CB'PA'$, S_3 the face $OBA'C$, S_4 the face $AC'PB'$, S_5 the face $OCB'A$ and S_6 the face $BAP'C'$ (Fig. 8.21).

Now $\quad \int_{S_1} \mathbf{F}\,.\,\mathbf{N}ds = \int_{S_1} \mathbf{F}\,.\,(-\mathbf{K})\,ds = -\int_0^b \int_0^a (0 - xy)\,dxdy = \frac{a^2 b^2}{4}$

$$\int_{S_2} \mathbf{F}\,.\,\mathbf{N}ds = \int_{S_2} \mathbf{F}\,.\,\mathbf{K}ds = \int_0^b \int_0^a (c^2 - xy)\,dxdy = abc^2 - \frac{a^2 b^2}{4}$$

Similarly, $\quad \int_{S_3} \mathbf{F}\,.\,\mathbf{N}ds = \frac{b^2 c^2}{4}, \int_{S_4} \mathbf{F}\,.\,\mathbf{N}ds = a^2 bc - \frac{b^2 c^2}{4},$

$$\int_{S_5} \mathbf{F}\,.\,\mathbf{N}ds = \frac{c^2 a^2}{4} \text{ and } \int_{S_6} \mathbf{F}\,.\,\mathbf{N}ds = ab^2 c - \frac{c^2 a^2}{4}$$

Thus $\quad \int_S \mathbf{F}\,.\,\mathbf{N}ds = abc(a + b + c) \qquad \ldots(ii)$

Hence the theorem is verified from the equality of (i) and (ii).

Example 8.43. *Evaluate* $\int_S \mathbf{F}\,.\,d\mathbf{s}$ *where* $\mathbf{F} = 4x\mathbf{I} - 2y^2\mathbf{J} + z^2\mathbf{K}$ *and S is the surface bounding the region* $x^2 + y^2 = 4$, $z = 0$ *and* $z = 3$. *(S.V.T.U., 2007 S ; Mumbai, 2006 ; J.N.T.U., 2006)*

Solution. By divergence theorem,

$$\int_S \mathbf{F}\,.\,d\mathbf{s} = \int_V div\,\mathbf{F}\,dv$$

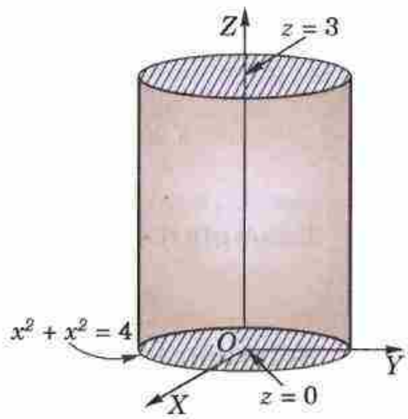

Fig. 8.22

$$= \int_V \left[\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)\right]dv$$

$$= \iiint_V ((4 - 4y + 2z)\,dxdydz$$

$$= \int_{-2}^{2} \int_{\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} \int_0^3 (4 - 4y + 2z)\,dzdydx$$

$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} \left|4z - 4yz + z^2\right|_0^3 dydx$$

$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} (12 - 12y + 9)\,dydx$$

$$= \int_{-2}^{2} \left| 21y - 6y^2 \right|_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} dx$$

$$= 42 \int_{-2}^{2} \sqrt{(4-x^2)}\, dx = 84 \int_{0}^{2} \sqrt{(4-x^2)}\, dx = 84 \left| \frac{x\sqrt{(4-x^2)}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2} \right|_0^2 = 84\pi.$$

Example 8.44. *Evaluate* $\int_S (yz\mathbf{I} + zx\mathbf{J} + xy\mathbf{K}) \cdot d\mathbf{S}$ *where S is the surface of the sphere* $x^2 + y^2 + z^2 = a^2$ *in the first octant.* *(U.P.T.U., 2004 S)*

Solution. The surface of the region V: $OABC$ is piecewise smooth (Fig. 8.23) and is comprised of four surfaces (*i*) S_1 – circular quadrant OBC in the yz-plane,

(*ii*) S_2 – circular quadrant OCA in the zx-plane,

(*iii*) S_3 – circular quadrant OAB in the xy-plane,

and (*iv*) S–surface ABC of the sphere in the first octant.

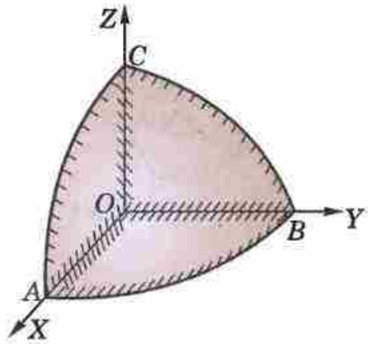

Fig. 8.23

Also $\mathbf{F} = yz\mathbf{I} + zx\mathbf{J} + xy\mathbf{K}$

By Divergence theorem,

$$\int_V div\, \mathbf{F}\, dv = \int_{S_1} \mathbf{F} \cdot d\mathbf{S} + \int_{S_2} \mathbf{F} \cdot d\mathbf{S} + \int_{S_3} \mathbf{F} \cdot d\mathbf{S} + \int_S \mathbf{F} \cdot d\mathbf{S} \quad ...(1)$$

Now $$div\, \mathbf{F} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(zx) + \frac{\partial}{\partial z}(xy) = 0.$$

For the surface S_1, $x = 0$

$$\therefore \quad \int_{S_1} \mathbf{F} \cdot d\mathbf{S} = \int_0^a \int_0^{\sqrt{(a^2-y^2)}} (yz\mathbf{I}) \cdot (-\, dydz\mathbf{I}) = - \int_0^a \int_0^{\sqrt{(a^2-y^2)}} yzdydz = = -\frac{a^4}{8}$$

Thus (1) becomes $0 = -\dfrac{3a^4}{8} + \int_S \mathbf{F} \cdot d\mathbf{S}$ whence $\int_S \mathbf{F} \cdot d\mathbf{S} = 3a^4/8$.

Example 8.45. *Apply divergence theorem to evaluate* $\int (lx^2 + my^2 + nz^2)\, ds$ *taken over the sphere* $(x-a)^2 + (y-b)^2 + (z-c)^2 = \rho^2$; *l, m, n being the direction cosines of the external normal to the sphere.*

Solution. The parametric equations of the sphere are $x = a + \rho \sin\theta \cos\phi$, $y = b + \rho \sin\theta \sin\phi$, $z = c + \rho\cos\theta$ and to cover the whole sphere, r varies from 0 to ρ, θ varies from 0 to π and ϕ from 0 to 2π.

$$\therefore \quad \int_S (lx^2 + my^2 + nz^2)\, ds = \int_S (x^2\mathbf{I} + y^2\mathbf{J} + z^2\mathbf{K}) \cdot \mathbf{N} ds$$

$$= \int_V \text{div}\,(x^2\mathbf{I} + y^2\mathbf{J} + z^2\mathbf{K}) dv = 2 \int_V (x + y + z) dv$$

$$= 2 \int_0^{2\pi} \int_0^{\pi} \int_0^{\rho} [(a + b + c) + \rho(\sin\theta\cos\phi + \sin\theta\ \sin\phi + \cos\theta)] \times \rho^2 \sin\theta\, dr d\theta d\phi$$

$$= 2(a + b + c)\ \frac{\rho^3}{3} \left| -\cos\theta \right|_0^{\pi} \cdot 2\pi = \frac{8\pi}{3}(a + b + c)\,\rho^3.$$

Example 8.46. *Evaluate* $\int_S (a^2x^2 + b^2y^2 + c^2z^2)^{-1/2}\, dS$, *where S is the surface of the ellipsoid*

$$ax^2 + by^2 + cz^2 = 1.$$

Solution. Taking $\phi = ax^2 + by^2 + cz^2 - 1 = 0$, $\nabla\phi = 2ax\mathbf{I} + 2by\mathbf{J} + 2cz\mathbf{K}$

$\therefore$ Unit vector normal to the ellipsoid $= \hat{N} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{ax\mathbf{I} + by\mathbf{J} + cz\mathbf{K}}{\sqrt{(a^2x^2 + b^2y^2 + c^2z^2)}}$

Since $\mathbf{F} \cdot \hat{\mathbf{N}} = (a^2x^2 + b^2y^2 + c^2z^2)^{-1/2}$, $\quad \therefore \mathbf{F} \cdot (ax\mathbf{I} + by\mathbf{J} + cz\mathbf{K}) = 1$

Obviously $\mathbf{F} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ $[\because ax^2 + by^2 + cz^2 = 1$

$\therefore$ By Divergence theorem,

$$\int_S \mathbf{F} \,.\, d\mathbf{S} = \int_V div\, \mathbf{F} dv = \int_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) \right] dv = 3 \int_V dv = 3V$$

$$= 3 \,.\, \frac{4\pi}{3} \frac{1}{\sqrt{(abc)}} = \frac{4\pi}{\sqrt{(abc)}}. \qquad \left[\because \text{ Vol. of ellipsoid} = \frac{4\pi}{3}\frac{1}{\sqrt{(abc)}}\right]$$

Example 8.47. *If the position vector of any point (x, y, z) within a closed surface S, be* **R** *measured from an origin O, then show that*

$$\iint_S \frac{\mathbf{R}}{r^3} \,.\, \mathbf{N}\, ds = \begin{cases} 0, \text{ if } O \text{ lies outside } S \\ 4\pi, \text{ if } O \text{ lies inside } S \end{cases}$$

Solution. (*a*) *When O is outside S.* Here $\mathbf{F} = \mathbf{R}/r^3$ is continuously differentiable throughout the volume V enclosed by S. Hence by Divergence theorem, we have

$$\iint_S \frac{\mathbf{R}}{r^3} \,.\, \mathbf{N} ds = \iiint_V \text{div}\left(\frac{\mathbf{R}}{r^3}\right) dV = 0 \qquad \left[\because \text{div}\left(\frac{\mathbf{R}}{r^3}\right) = 0\right]$$

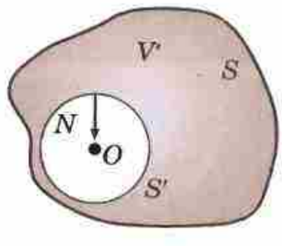

Fig. 8.24

(*b*) *When O is inside S.* Hence $F = \mathbf{R}/r^3$ has a point of discontinuity at O and as such Divergence theorem cannot be applied to the region V enclosed by S. To remove this point of discontinuity, we enclose O by a small sphere S' of radius ρ.

Now $\mathbf{F}$ is continuously differentiable throughout the region V' enclosed between S and S'. Therefore applying Divergence theorem to region V', we get

$$\iint_S \frac{\mathbf{R}}{r^3} \,.\, \mathbf{N}\, ds + \iint_{S'} \frac{\mathbf{R}}{r^3} \,.\, \mathbf{N}\, ds' = \iiint_{V'} \text{div}\left(\frac{\mathbf{R}}{r^3}\right) dV' = 0 \qquad \left[\because \text{div}\left(\frac{\mathbf{R}}{r^3}\right) = 0\right]$$

$$\therefore \qquad \iint_S \frac{\mathbf{R}}{r^3} \,.\, \mathbf{N}\, ds = -\iint_{S'} \frac{\mathbf{R}}{r^3} \,.\, \mathbf{N}\, ds' \qquad ...(i)$$

Now the outward normal $\mathbf{N}$ on the sphere S' is directed towards the centre O. Therefore $\mathbf{N} = -\mathbf{R}/\rho$ on S' (Fig. 8.24).

$$\therefore \qquad -\iint_{S'} \frac{\mathbf{R}}{r^3} \,.\, \mathbf{N}\, ds' = -\iint_{S'} \frac{\mathbf{R}}{\rho^3} . \left(-\frac{\mathbf{R}}{\rho}\right) ds' \qquad [\because \text{ on } S', r = \rho]$$

$$= \iint_{S'} \frac{r^2}{\rho^4} ds' = \iint_{S'} \frac{\rho^2}{\rho^4} ds' = \frac{1}{\rho^2} \iint_{S'} ds' = \frac{1}{\rho^2} \,.\, 4\pi\rho^2 = 4\pi$$

Hence from (*i*), $\displaystyle\iint_S \frac{\mathbf{R}}{r^3} \,.\, \mathbf{N}\, ds = 4\pi.$

8.17 GREEN'S THEOREM*

If ϕ and ψ are scalar point functions possessing continuous derivatives of first and second orders, then

$$\int_E (\phi\nabla^2\psi - \psi\nabla^2\phi) dv = \int_S \left(\phi \frac{\partial\psi}{\partial n} - \psi \frac{\partial\psi}{\partial n}\right) ds \qquad ...(1)$$

where $\partial/\partial n$ denotes differentiation in the direction of the external normal to the bounding surface S enclosing the region E.

Applying Divergence theorem : $\int_S \mathbf{F} \,.\, \mathbf{N} ds = \int_S \nabla \,.\, \mathbf{F} dv$ to the function $\phi\nabla\psi$, we get

$$\int_S \phi\nabla\psi \,.\, \mathbf{N} ds = \int_E \nabla \,.\, (\phi\nabla\psi) dv = \int_E (\nabla\phi \,.\, \nabla\psi + \phi\nabla^2\psi) dv \qquad \text{[By (2) page 329]}$$

$$= \int_E \nabla\phi \,.\, \nabla\psi dv + \int_E \phi\nabla^2\psi dv \qquad ...(2)$$

*See footnote p. 339.

Interchanging ϕ and ψ, (*ii*) gives

$$\int_S \psi\nabla\phi \,.\, \mathbf{N}ds = \int_E \nabla\psi \,.\, \nabla\phi dv + \int_E \psi\nabla^2\phi dv \qquad \text{...(3)}$$

Subtracting (3) from (2), we have $\int_S (\phi\nabla\psi - \psi\nabla\phi) \,.\, \mathbf{N}ds = \int_E (\phi\nabla^2\psi - \psi\nabla^2\phi)dv$

But $\nabla\psi \,.\, \mathbf{N} = \partial\psi/\partial n$ the directional derivative of ψ along the external normal at any point of S. Hence

$$\int_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)ds = \int_E (\phi\nabla^2\psi - \psi\nabla^2\phi)dv$$ which is the required result (1).

Obs. Harmonic function. *A scalar point function ϕ satisfying the Laplace's equation $\nabla^2\phi = 0$ at every point of a region E, is called a harmonic function in E.*

If ϕ and ϕ be both harmonic functions in E, (1) gives

$$\int_S \phi\frac{\partial\psi}{\partial n}ds = \int_S \psi\frac{\partial\phi}{\partial n}ds$$ which is known as **Green's reciprocal theorem.**

PROBLEMS 8.10

1. Verify divergence theorem for $\mathbf{F}$ taken over the cube bounded by $x = 0, x = 1$; $y = 0, y = 1$; $z = 0, z = 1$ where
(*i*) $\mathbf{F} = 4xz\mathbf{I} - y^2\mathbf{J} + yz\mathbf{K}$ (*Madras, 2006*) (*ii*) $x^2\mathbf{I} + z\mathbf{J} + yz\mathbf{K}$ (*Bhopal, 2008*)
2. Verify Gauss divergence theorem for the function $\mathbf{F} = y\mathbf{I} + x\mathbf{J} + z^2\mathbf{K}$ over the cylindrical region bounded by $x^2 + y^2 = 9$, $z = 0$ and $z = 2$.
3. Using divergence theorem, prove that
(*i*) $\int_S \mathbf{R} \,.\, d\mathbf{S} = 3V$ (*ii*) $\int_S \nabla r^2 \,.\, d\mathbf{S} = 6V$ (*U.P.T.U., 2003*)
where S is any closed surface enclosing a volume V and $r^2 = x^2 + y^2 + z^2$.
4. Using divergence theorem, evaluate $\int_S \mathbf{R} \,.\, \mathbf{N}ds$ where S is the surface of the sphere $x^2 + y^2 + z^2 = 9$.
5. If S is any closed surface enclosing a volume V and $\mathbf{F} = ax\mathbf{I} + by\mathbf{J} + cz\mathbf{K}$, prove that
$$\int_S \mathbf{F} \,.\, \mathbf{N}ds = (a + b + c)V$$ (*Madras, 2003*)
6. For any closed surface S, prove that $\int [x(y - z)\mathbf{I} + y(z - x)\mathbf{J} + z(x - y)\mathbf{K}] \,.\, ds = 0$.
7. Use divergence theorem to evaluate $\int_S \mathbf{F} \,.\, d\mathbf{S}$, where
(*i*) $\mathbf{F} = x^3\mathbf{I} + y^3\mathbf{J} + z^3\mathbf{K}$, and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$. (*V.T.U., 2008 ; P.T.U., 2005*)
(*ii*) $\mathbf{F} = [e^x, e^y, e^z]$ and S is the surface of the cube $|x| \le 1$, $|y| \le 1$, $|z| \le 1$. (*B.P.T.U., 2005*)
8. Evaluate $\iint (xdydz + ydzdx + zdxdy)$ over the surface of a sphere of radius a. (*Kurukshetra, 2008 S*)
9. Evaluate $\int_S \mathbf{F} \,.\, d\mathbf{S}$ where $F = y^2z^2\mathbf{I} + z^2x^2\mathbf{J} + x^2y^2\mathbf{K}$ and S is the upper part of the sphere $x^2 + y^2 + z^2 = a^2$ above XOY plane.
10. By transforming to triple integral, evaluate $\iint_S (x^3dydz + x^2ydzdx + x^2zdxdy)$ where S is the closed surface consisting of the cylinder $x^2 + y^2 = a^2$ and the circular discs $z = 0$ and $z = b$. (*Burdwan, 2003*)
11. Evaluate $\int_S (\nabla \times \mathbf{F}) \,.\, d\mathbf{S}$, where S is the surface of the paraboloid $x^2 + y^2 + z = 4$ above the xy-plane, and $\mathbf{F} = (x^2 + y - 4)\mathbf{I} + 3xy\mathbf{J} + (2xz + z^2)\mathbf{K}$.
12. If $\mathbf{F} = (2x^2 - 3z)\mathbf{I} - 2xy\mathbf{J} - 4x\mathbf{K}$, then evaluate $\iiint_V \nabla \,.\, \mathbf{F}\, dv$, where V is bounded by $x = y = z = 0$ and $2x + 2y + z = 4$. (*Bhopal, 2008*)
13. If $\mathbf{F} = \text{grad}\, \phi$ and $\nabla^2\phi = -4\pi\rho$, prove that $\int_S \mathbf{F} \,.\, \mathbf{N}ds = -4\pi\rho \int_V dV$ where the symbol have their usual meanings.

8.18 (1) IRROTATIONAL FIELDS

An irrotational field **F** *is characterised by any one of the following conditions :*

(*i*) $\Delta \times \mathbf{F} = \mathbf{0}$. (*ii*) *Circulation* $\int \mathbf{F} \cdot d\mathbf{R}$ *along every closed surface is zero.*

(*iii*) $\mathbf{F} = \Delta\phi$, *if the domain is simply connected.**

If $\nabla \times \mathbf{F} = \mathbf{0}$, then by Stoke's theorem,

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \mathbf{0},$$ *i.e.*, the circulation along every closed surface is zero.

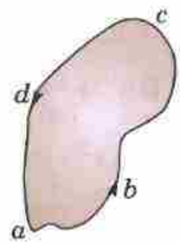

Fig. 8.25

Again since $\nabla \times \nabla\phi = 0$

$\therefore$ in an irrotational field for which $\Delta \times \mathbf{F} = 0$, the vector **F** can always be expressed as the gradient of a scalar function ϕ provided the domain is simply connected. Thus

$$\mathbf{F} = \nabla\phi.$$

Such a scalar function ϕ is called the *potential*. In a rotational field, **F** cannot be expressed as the gradient of a scalar potential.

Obs. 1. *In an irrotational field, the line integral* **F** *between two points is independent of the path of integration and is equal to the potential difference between these points.*

If *aced* be any closed contour in an irrotational field F (Fig. 8.25), then

$$\int_{abcd} \mathbf{F} \cdot d\mathbf{R} = \int_{abc} \mathbf{F} \cdot d\mathbf{R} + \int_{cda} \mathbf{F} \cdot d\mathbf{R} = 0$$

or $$\int_{abc} \mathbf{F} \cdot d\mathbf{R} = \int_{abc} \mathbf{F} \cdot d\mathbf{R}$$

i.e. the value of the line integral is independent of the path joining the end points.

Further, substituting $\mathbf{F} = \nabla\phi$, we have

$$\int_a^c \mathbf{F} \cdot d\mathbf{R} = \int_a^c \nabla\phi \cdot d\mathbf{R} = \int_a^c \left(\mathbf{I}\frac{\partial\phi}{\partial x} + \mathbf{J}\frac{\partial\phi}{\partial y} + \mathbf{K}\frac{\partial\phi}{\partial z} \right) \cdot (\mathbf{I}dx + \mathbf{J}dy + \mathbf{K}dz)$$

$$= \int_a^c \left(\frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz \right) = \int_a^c d\phi = \phi_c - \phi_a .$$

Obs. 2. If **F** is a vector force acting on a particle, then $\oint_C \mathbf{F} \cdot d\mathbf{R}$ represents the work done in moving the particle around a closed path. [See p. 328]

When $\oint_C \mathbf{F} \cdot d\mathbf{R} = 0$, *the field is said to be* **conservative,** *i.e.*, no work is done in displacement from a point a to another point in the field and back to a and the mechanical energy is conserved.

Thus *every irrotational field is conservative.*

Obs. 3. *The well-known equations of the Poisson and Laplace hold good for every irrotational field.*

Suppose $\nabla \cdot \mathbf{F} = f(x, y, z)$. Then $\nabla \cdot \nabla\phi = f(x, y, z)$ *i.e.*, $\nabla^2\phi = f(x, y, z)$ ...(*i*)

which is known as *Poisson's equation*. Its solutions for electrostatic fields enable us to determine the potential ϕ as a function of the charge distribution $f(x, y, z)$.

If $f(x, y, z) = 0$ then (*i*) reduces to $\nabla^2\phi = 0$ which is the *Laplace's equation*. The solutions of this equation are of great importance in modern engineering and physics, some of which we'll study in § 18.11 and 18.12.

(2) **Solenoidal fields.** *A solenoidal field* **F** *is characterised by any one of the following conditions :*

(*i*) $\nabla \cdot \mathbf{F} = 0$. (*ii*) *flux* $\int \mathbf{F} \cdot \mathbf{N}\, ds$ *across every closed surface is zero.* (*iii*) $\mathbf{F} = \nabla \times \mathbf{V}$.

If $\nabla \cdot \mathbf{F} = 0$ then by the Divergence theorem,

$$\int_S \mathbf{F} \cdot \mathbf{N} ds = \int_V \nabla \cdot \mathbf{F} dv = 0,$$ *i.e.*, the flux across every closed surface is zero.

Again since $\nabla \cdot \nabla \times \mathbf{V} = 0$.

$\therefore$ in a solenoidal field for which $\nabla \cdot \mathbf{F} = 0$, the vector **F** can always be expressed as the curl of a vector function **V**; thus $\mathbf{F} = \nabla \times \mathbf{V}$.

*A domain D is said to be *simply connected* if every closed curve in D can be shrunk to any point within D.

Example 8.48. *A vector field is given by* $\mathbf{F} = (x^2 - y^2 + x)\mathbf{I} - (2xy + y)\mathbf{J}$. *Show that the field is irrotational and find its scalar potential. Hence evaluate the line integral from (1, 2) to (2, 1).*

Solution. Since $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - y^2 + x & -(2xy + y) & 0 \end{vmatrix} = \mathbf{0}$

∴ this field is *irrotational* and the vector **F** can be expressed as the gradient of a scalar potential,

i.e., $(x^2 - y^2 + x)\mathbf{I} - (2xy + y)\mathbf{J} = \nabla\phi = \dfrac{\partial\phi}{\partial x}\mathbf{I} + \dfrac{\partial\phi}{\partial y}\mathbf{J}$

whence $\partial\phi/\partial x = x^2 - y^2 + x$ ...(*i*)

$\partial\phi/\partial y = -(2xy + y)$ ...(*ii*)

Integrating (*i*) w.r.t. x, keeping y constant, we get $\phi = \dfrac{x^3}{3} - y^2x + \dfrac{x^2}{2} + f(y)$ ...(*iii*)

Similarly integrating (*ii*) w.r.t. y, keeping x constant, we obtain $\phi = -xy^2 - \dfrac{y^2}{2} + g(x)$ ...(*iv*)

Equating (*iii*) and (*iv*), we get $\dfrac{x^3}{3} - y^2x + \dfrac{x^2}{2} + f(y) = -xy^2 - \dfrac{y^2}{2} + g(x)$

∴ $f(y) = -\dfrac{y^2}{2}$ and $g(x) = \dfrac{x^3}{3} + \dfrac{x^2}{2}$

Hence $\phi = \dfrac{x^3}{3} - xy^2 + \dfrac{x^2}{2} - \dfrac{y^2}{2}$

Since the field is irrotational,

∴ $\int \mathbf{F} \cdot d\mathbf{R}$ from (1, 2) to (2, 1) $= \phi_{1,2} - \phi_{2,1} = \left(\dfrac{1}{3} - 1 \times 4 + \dfrac{1}{2} - \dfrac{4}{2}\right) - \left(\dfrac{8}{3} - 2 \times 1 + \dfrac{4}{2} - \dfrac{1}{2}\right) = -7\dfrac{1}{3}$.

Example 8.49. *A fluid motion is given by* $\mathbf{V} = (y + z)\mathbf{I} + (z + x)\mathbf{J} + (x + y)\mathbf{K}$.
(*a*) *Is this motion irrotational? If so, find the velocity potential.* (*U.P.T.U., 2004*)
(*b*) *Is the motion possible for an incompressible fluid?*

Solution. We have $\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y + z & z + x & x + y \end{vmatrix} = \mathbf{I}(1 - 1) - \mathbf{J}(1 - 1) + \mathbf{K}(1 - 1) = \mathbf{0}.$

∴ this motion is irrotational and if ϕ is the velocity potential then $\mathbf{V} = \nabla\phi$. [§ 20.6]

i.e., $(y + z)\mathbf{I} + (z + x)\mathbf{J} + (x + y)\mathbf{K} = \dfrac{\partial\phi}{\partial x}\mathbf{I} + \dfrac{\partial\phi}{\partial y}\mathbf{J} + \dfrac{\partial\phi}{\partial z}\mathbf{K}$

∴ $\dfrac{\partial\phi}{\partial x} = y + z, \dfrac{\partial\phi}{\partial y} = z + x, \dfrac{\partial\phi}{\partial z} = x + y$

Integrating these, we get $\phi = (y + z)x + f_1(y, z)$ ...(*i*)

$\phi = (z + x)y + f_2(z, x)$ ...(*ii*)

and $\phi = (x + y)z + f_3(x, y)$ ...(*iii*)

Equality of (*i*), (*ii*) and (*iii*), requires that

$$f_1(y, z) = yz, f_2(z, x) = zx, f_3(x, y) = xy.$$

Hence $\phi = yz + zx + xy$.

(*b*) The fluid motion is possible if **V** satisfies the equation of continuity which for an incompressible fluid is $\nabla \cdot \mathbf{V} = 0$. [See § 8.7 (1)

Here $$\nabla \cdot \mathbf{V} = \frac{\partial}{\partial x}(y+z) + \frac{\partial}{\partial y} = (z+x) + \frac{\partial}{\partial z}(x+y) = 0.$$

Hence, the fluid motion is possible.

Example 8.50. *Find whether* $\int_C [2xyz^2 dx + (x^2z^2 + z \cos yz)\, dy + (2x^2yz + y \cos yz)\, dz]$ *is independent of the path joining* $(0, \pi/2, 1)$ *and* $(1, 0, 1)$. *If so, evaluate this line integral.*

Solution. The line integral of **F** is independent of path of integration if $\nabla \times \mathbf{F} = \mathbf{0}$.

$$= \int_C [2xyz^2\,\mathbf{I} + (x^2z^2 + z\cos yz)\mathbf{J} + (2x^2yz + y\cos yz)\mathbf{K}] \cdot (\mathbf{I}dx + \mathbf{J}dy + \mathbf{K}dz) = \int_C \mathbf{F} \cdot d\mathbf{R}$$

and
$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xyz^2 & x^2z^2 + z\cos yz & 2x^2yz + y\cos yz \end{vmatrix}$$
$$= \mathbf{I}\,[2x^2z + \cos yz - yz\sin yz - (2x^2z + \cos yz - yz\sin yz)]$$
$$- \mathbf{J}[4xyz - 4xyz] + \mathbf{K}[2xz^2 - 2xz^2] = \mathbf{0}$$

$\therefore$ the given integral is independent of the path C.

Now let $\mathbf{F} = \nabla\phi$

i.e., $$(2xyz^2)\mathbf{I} + (x^2z^2 + z\cos yz)\mathbf{J} + (2x^2yz + y\cos yz)\,\mathbf{K} = \mathbf{I}\frac{\partial\phi}{\partial x} + \mathbf{J}\frac{\partial\phi}{\partial y} + \mathbf{K}\frac{\partial\phi}{\partial z}$$

$\therefore$ $$2xyz^2 = \frac{\partial\phi}{\partial x},\ x^2z^2 + z\cos yz = \frac{\partial\phi}{\partial y},\ 2x^2yz + y\cos yz = \frac{\partial\phi}{\partial z}$$

Integrating first w.r.t. x partially, we get
$$\phi = x^2y^2z^2 + \Psi_1(y, z) \qquad ...(i)$$
Integrating second w.r.t. y partially, we get
$$\phi = x^2yz^2 + \sin yz + \Psi_2(z, x) \qquad ...(ii)$$
Integrating third w.r.t. z partially, we get
$$\phi = x^2yz^2 + \sin yz + \Psi_3(x, y) \qquad ...(iii)$$
Comparing (*i*), (*ii*), (*iii*), we have
$$\Psi_1(y, z) = \text{terms in } \phi \text{ independent of } x = \sin yz$$
$$\Psi_2(z, x) = \text{terms in } \phi \text{ independent of } y = 0$$
$$\Psi_3(z, x) = \text{terms in } \phi \text{ independent of } z = 0$$
Thus $$\phi = x^2yz^2 + \sin yz$$

Hence the value of the given integral $= |\,\phi\,|_{(0,\pi/2,1)}^{(1,0,1)}$
$$= (0 + 0) - (0 + \sin \pi/2) = -1.$$

Example 8.51. *Determine whether* $\mathbf{F} = (y^2\cos x + z^3)\mathbf{I} + (2y\sin x - 4)\mathbf{J} + (3xz^2 + 2)\mathbf{K}$ *is a conservative vector field? If so find the scalar potential* ϕ. *Also compute the work done in moving the particle from* $(0, 1, -1)$ *to* $(\pi/2, -1, 2)$. *(Mumbai, 2006)*

Solution. **F** is a conservative vector field when curl $\mathbf{F} = \mathbf{0}$. Here

$$Curl = F = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2 + 2 \end{vmatrix}$$
$$= \mathbf{I}(0 - 0) - \mathbf{J}(3z^2 - 3z^2) + \mathbf{K}(2y\cos x - 2y\cos x) = \mathbf{0}$$

$\therefore$ **F** is a conservative field.

Now let $\mathbf{F} = \nabla\phi$

i.e.,
$$(y^2 \cos x + z^3)\,\mathbf{I} + (2y \sin x - 4)\,\mathbf{J} + (3xz^2 + 2)\,\mathbf{K} = \mathbf{I}\frac{\partial\phi}{\partial x} + \mathbf{J}\frac{\partial\phi}{\partial y} + \mathbf{K}\frac{\partial\phi}{\partial z}$$

$$\therefore \qquad y^2 \cos x + z^3 = \frac{\partial\phi}{\partial x},\ 2y \sin x - 4 = \frac{\partial\phi}{\partial y},\ 3xz^2 + 2 = \frac{\partial\phi}{\partial z}$$

Integrating first w.r.t. x partially, we get

$$\phi = y^2 \sin x + xz^3 + \psi_1(y, z) \qquad \text{...}(i)$$

Integrating second w.r.t. y partially, we get

$$\phi = y^2 \sin x - 4y + \Psi_2(z, x) \qquad \text{...}(ii)$$

Integrating third w.r.t. z partially, we obtain

$$\phi = xz^3 + 2z + \Psi_3(x, y) \qquad \text{...}(iii)$$

Comparing (i), (ii), (iii), we get

$$\Psi_1(y, z) = \text{terms in } \phi \text{ independent of } x = -4y + 2z$$
$$\Psi_2(z, x) = \text{terms in } \phi \text{ independent of } y = xz^3 + 2z$$
$$\Psi_3(z, x) = \text{terms in } \phi \text{ independent of } z = y^2 \sin x - 4y$$

Thus $\qquad \phi = xz^3 + y^2 \sin x - 4y + 2z$

In a conservative field, the work done $= \phi_B - \phi_A$

$$= \phi\left(\frac{\pi}{2}, -1, 2\right) - \phi(0, 1, -1)$$
$$= (4\pi + 1 + 4 + 4) - (-4 - 2) = 4\pi + 15.$$

PROBLEMS 8.11

1. If ϕ is a solution of the Laplace equation, prove that $\nabla\phi$ is both solenoidal and irrotational.
2. Show that the vector field defined by $\mathbf{F} = (x^2 + xy^2)\mathbf{I} + (y^2 + x^2y)\mathbf{J}$ is conservative and find the scalar potential. Hence evaluate $\int \mathbf{F} \cdot d\mathbf{R}$ from (0, 1) to (1, 2).
3. Find the work done by the variable force $\mathbf{F} = 2y\mathbf{I} + xy\mathbf{J}$ on a particle when it is displaced from the origin to the point $\mathbf{R} = 4\mathbf{I} + 2\mathbf{J}$ along the parabola $y^2 = x$.
4. Show that the vector field given by $\mathbf{A} = 3x^2y\mathbf{I} + (x^3 - 2yz^2)\mathbf{J} + (3z^2 - 2y^2z)\mathbf{K}$ is irrotational but not solenoidal. Also find $\phi(x, y, z)$ such that $\nabla\phi = \mathbf{A}$.
5. Show that the following vectors are irrotational and find the scalar potential in each case :
   (*i*) $(x^2 - yz)\mathbf{I} + (y^2 - zx)\mathbf{J} + (z^2 - xy)\mathbf{K}$
   (*ii*) $2xy\mathbf{I} + (x^2 + 2yz)\mathbf{J} + (y^2 + 1)\mathbf{K}$ (*V.T.U., 2007*)
   (*iii*) $(6xy + z^3)\mathbf{I} + (3x^2 - z)\mathbf{J} + (3xz^2 - y)\mathbf{K}$ (*Raipur, 2005 ; V.T.U., 2003 S*)
   (*iv*) $(2xy^2 + yz)\mathbf{I} + (2x^2y + xz + 2yz^2)\mathbf{J} + (2y^2z + xy)\mathbf{K}$. (*V.T.U., 2010*)
6. Fluid motion is given by $\mathbf{V} = ax\mathbf{I} + ay\mathbf{J} - 2az\mathbf{K}$.
   (*i*) Is it possible to find out the velocity potential ? If so, find it.
   (*ii*) Is the motion possible for an incompressible fluid ?
7. Show that the vector field defined by $\mathbf{F} = (y \sin z - \sin x)\mathbf{I} + (x \sin z + 2yz)\mathbf{J} + (xy \cos z + y^2)\mathbf{K}$ is irrotational and find its velocity potential. (*Kottayam, 2005*)
8. Show that $\mathbf{F} = (2xy + z^3)\mathbf{I} + x^2\mathbf{J} + 3xz^2\mathbf{K}$ is a conservative vector field and find a function ϕ such that $\mathbf{F} = \nabla\phi$. Also find the work done in moving an object in this field from (1, – 2, 1) to (3, 1, 4). (*Nagpur, 2009*)
9. If $\mathbf{F} = (x + y + az)\mathbf{I} + (bx + 2y - z)\mathbf{J} + (x + cy + 2z)\mathbf{K}$, find a, b, c such that curl $\mathbf{F} = \mathbf{0}$, then find ϕ such that $\mathbf{F} = \nabla\phi$. (*V.T.U., 2000*)
10. Find the constant a so that **V** is a conservative vector field, where
$$\mathbf{V} = (axy - z^3)\mathbf{I} + (a - 2)x^2\mathbf{J} + (1 - a)xz^2\mathbf{K}.$$
Calculate its scalar potential and work done in moving a particle from (1, 2, – 3) to (1, – 4, 2) in the field. (*Mumbai, 2006 ; Rajasthan, 2006*)

8.19 (1) ORTHOGONAL CURVILINEAR COORDINATES

Let the rectangular coordinates (x, y, z) of any point be expressed as functions of u, v, w so that

$$x = x(u, v, w), y = y(u, v, w), z = z(u, v, w) \quad ...(1)$$

Suppose that (1) can be solved for u, v, w in terms of x, y, z, so that

$$u = u(x, y, z), v = v(x, y, z), w = w(x, y, z) \quad ...(2)$$

We assume that the functions in (1) and (2) are single-valued and have continuous partial derivatives so that the correspondence between (x, y, z) and (u, v, w) is unique. Then (u, v, w) are called *curvilinear coordinates* of (x, y, z).

Each of u, v, w has a level surface through an arbitrary point. The surfaces $u = u_0, v = v_0, w = w_0$ are called *coordinate surfaces* through $P(u_0, v_0, w_0)$. Each pair of these coordinate surfaces intersect in curves called the *coordinate curves*. The curve of intersection of $u = u_0$ and $v = v_0$ will be called the w-curve, for only w changes along this curve. Similarly we define u and v-curves.

In vector notation, (1) can be written as $\mathbf{R} = x(u, v, w)\mathbf{I} + y(u, v, w)\mathbf{J} + z(u, v, w)\mathbf{K}$

$$\therefore \quad d\mathbf{R} = \frac{\partial \mathbf{R}}{\partial u} du + \frac{\partial \mathbf{R}}{\partial v} dv + \frac{\partial \mathbf{R}}{\partial w} dw \quad ...(3)$$

Then $\partial\mathbf{R}/\partial u$ is a tangent vector to the u-curve at P. If $\mathbf{T}_u$ is a unit vector at P in this direction, then $\partial\mathbf{R}/\partial u = h_1\mathbf{T}_v$ where $h_1 = |\partial\mathbf{R}/\partial u|$.

Similarly if $\mathbf{T}_v$ and $\mathbf{T}_w$ be unit tangent vectors to v- and w-curves at P, then

$$\frac{\partial \mathbf{R}}{\partial v} = h_2\mathbf{T}_v \text{ and } \frac{\partial \mathbf{R}}{\partial w} = h_3\mathbf{T}_w$$

where $h_2 = |\partial\mathbf{R}/\partial v|$ and $h_3 = |\partial\mathbf{R}/\partial w|$. [$h_1, h_2, h_3$ are called scalar factors.]

Then (3) can be written as

$$d\mathbf{R} = h_1 du\, \mathbf{T}_u + h_2 dv\, \mathbf{T}_v + h_3\, dw\, \mathbf{T}_w \quad ...(4)$$

Since ∇u is normal to the surface $u = u_0$ at P, therefore, a unit vector in this direction is given by $\mathbf{N}_u = \dfrac{\nabla u}{|\nabla u|}$.

Similarly, the unit vectors $\mathbf{N}_v = \dfrac{\nabla v}{|\nabla v|}$ and $\mathbf{N}_w = \dfrac{\nabla w}{|\nabla w|}$ are normal to the surfaces $v = v_0$ and $w = w_0$ at P respectively. Thus at each point P of a curvilinear coordinate system there exist two triads of unit vectors : $\mathbf{T}_u, \mathbf{T}_v, \mathbf{T}_w$ tangents to u, v, w-curves and $\mathbf{N}_u, \mathbf{N}_v, \mathbf{N}_w$ normals to the co-ordinates surfaces (Fig. 8.26).

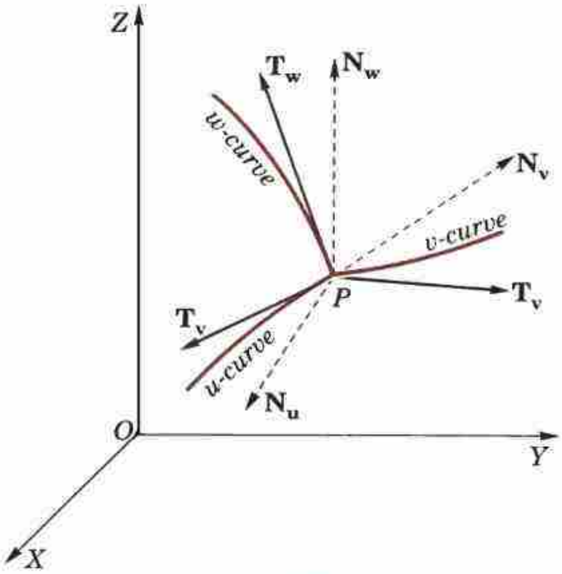

Fig. 8.26

In particular, when the coordinate surfaces intersect a right angles, the three coordinate curves are also mutually orthogonal and u, v, w are called the *orthogonal curvilinear coordinates*. In this case $\mathbf{T}_u, \mathbf{T}_v, \mathbf{T}_w$ and $\mathbf{N}_u, \mathbf{N}_v, \mathbf{N}_w$ are mutually perpendicular unit vector triads and hence become identical. Henceforth, we shall refer to orthogonal curvilinear coordinates only.

Multiplying (3) scalarly by ∇u, we get

$$\nabla u \,.\, d\mathbf{R} = du = \left(\nabla u \,.\, \frac{\partial \mathbf{R}}{\partial u}\right) du + \left(\nabla u \,.\, \frac{\partial \mathbf{R}}{\partial v}\right) dv + \left(\nabla u \,.\, \frac{\partial \mathbf{R}}{\partial w}\right) dw$$

whence $$\nabla u \,.\, \frac{\partial \mathbf{R}}{\partial u} = 1, \nabla u \,.\, \frac{\partial \mathbf{R}}{\partial v} = 0, \nabla u \,.\, \frac{\partial \mathbf{R}}{\partial w} = 0$$

Similarly, $$\nabla v \,.\, \frac{\partial \mathbf{R}}{\partial u} = 0, \nabla v \,.\, \frac{\partial \mathbf{R}}{\partial v} = 1, \nabla v \,.\, \frac{\partial \mathbf{R}}{\partial w} = 0$$

and $$\nabla w \,.\, \frac{\partial \mathbf{R}}{\partial u} = 0, \nabla w \,.\, \frac{\partial \mathbf{R}}{\partial v} = 0, \nabla w \,.\, \frac{\partial \mathbf{R}}{\partial w} = 1.$$

These relations show that the sets $\partial\mathbf{R}/\partial u, \partial\mathbf{R}/\partial v, \partial\mathbf{R}/\partial w$ and $\nabla u, \nabla v, \nabla w$ constitute reciprocal system of vectors.

$$\therefore \qquad \nabla u = \frac{\dfrac{\partial \mathbf{R}}{\partial v} \times \dfrac{\partial \mathbf{R}}{\partial w}}{\left[\dfrac{\partial \mathbf{R}}{\partial u} \cdot \dfrac{\partial \mathbf{R}}{\partial v} \times \dfrac{\partial \mathbf{R}}{\partial w}\right]} = \frac{(h_2 \mathbf{T_v}) \times (h_3 \mathbf{T_w})}{[(h_1 \mathbf{T_u}) \cdot (h_2 \mathbf{T_v}) \times (h_3 T_\mathbf{w})]}$$

$$= \frac{h_2 h_3 \ \mathbf{T}_v \times \mathbf{T}_w}{h_1 h_2 h_3 [\mathbf{T}_u \mathbf{T}_v \mathbf{T}_w]} = \frac{\mathbf{T}_u}{h_1} \qquad [\because \ \mathbf{T}_u \mathbf{T}_v \mathbf{T}_w = 1]$$

or $\qquad \mathbf{T}_v = h_1 \nabla u$

Similarly $\qquad \mathbf{T}_v = h_2 \nabla v$ and $\mathbf{T}_w = h_3 \nabla w$ $\qquad$ ...(5)

Also $\qquad = \mathbf{T}_v \times \mathbf{T}_w = h_2 h_3 \nabla v \times \nabla w$

Similarly $\qquad \mathbf{T}_v = h_3 h_1 \nabla w \times \nabla u$ and $\mathbf{T}_w = h_1 h_2 \nabla u \times \nabla v$ $\qquad$ ...(6)

Arc, area and volume elements

(*i*) *Arc element*. The element of arc length ds is determined from (4).

$$\therefore \qquad ds^2 = d\mathbf{R} \cdot d\mathbf{R} = h_1^2 du^2 + h_2^2 dv^2 + h_3^2 dw^2 \qquad \text{...(7)}$$

The arc length ds_1 along u-curve at P is $h_1 du$ for v and w are constants. Therefore the vector arc element along the u-curve is $d\mathbf{u} = h_1 du \mathbf{T}_u$. Similarly vector arc elements along v and w curves at P are $d\mathbf{v} = h_2 dv\ \mathbf{T}_v$ and $dw = h_3 dw\ \mathbf{T}_w$. The arc element ds therefore corresponds to the length of the diagonal of the rectangular parallelopiped of Fig. 8.27.

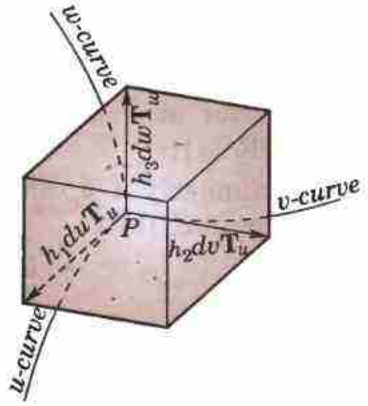

Fig. 8.27

(*ii*) *Area elements*. The area of the parallelogram formed by $d\mathbf{u}$ and $d\mathbf{v}$ is called the area element on the uv surface which is perpendicular to w-curve and we denote it by dS_w. Hence, $dS_w = |du \times dv| = h_1 h_2 du dv$. Similarly, $dS_u = h_2 h_3 dv dw$, $dS_v = h_3 h_1 dw du$.

(*iii*) *Volume element* is the volume of the parallelopiped formed by $d\mathbf{u}$, $d\mathbf{v}$, $d\mathbf{w}$.

$$\therefore \qquad dV = [h_1 du \mathbf{T}_u) \cdot (h_2 dv \mathbf{T}_v) \times (h_3 dw \mathbf{T}_w)]$$

$$= h_1 h_2 h_3 \, du dv dw \qquad \text{...(8)} \quad [\because [\mathbf{T}_u \mathbf{T}_v \mathbf{T}_w] = 1$$

This can also be written as

$$dV = \frac{\partial \mathbf{R}}{\partial u} \cdot \frac{\partial \mathbf{R}}{\partial v} \times \frac{\partial \mathbf{R}}{\partial w} \ du dv dw = \frac{\partial(x, y, z)}{\partial(u, v, w)} du dv dw \qquad \text{...(9)}$$

where $\partial(x, y, z)/\partial(u, v, w)$ is called the *Jacobian of the transformation* from (x, y, z) to (u, v, w) coordinates.

(2) Del applied to Functions in Orthogonal Curvilinear coordinates

To prove that

(1) $\nabla f = \dfrac{\mathbf{T}_u}{h_1} \dfrac{\partial f}{\partial u} + \dfrac{\mathbf{T}_v}{h_2} \dfrac{\partial f}{\partial v} + \dfrac{\mathbf{T}_w}{h_3} \dfrac{\partial f}{\partial w}$

(2) $\nabla \cdot \mathbf{F} = \dfrac{1}{h_1 h_2 h_3} \left[\dfrac{\partial}{\partial u}(h_2 h_3 f_1) + \dfrac{\partial}{\partial v}(h_3 h_1 f_2) + \dfrac{\partial}{\partial w}(h_1 h_2 f_3) \right]$

(3) $\nabla \times \mathbf{F} = \begin{vmatrix} \dfrac{\mathbf{T}_u}{h_2 h_3} & \dfrac{\mathbf{T}_v}{h_3 h_1} & \dfrac{\mathbf{T}_w}{h_1 h_2} \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ h_1 f_1 & h_2 f_2 & h_3 f_3 \end{vmatrix}$ where $\mathbf{F} = f_1 \mathbf{T}_u + f_2 \mathbf{T}_v + f_3 \mathbf{T}_w$.

(1) Let $f(u, v, w)$ be any scalar point function in terms of u, v, w, the orthogonal curvilinear coordinates. Taking u, v, w as functions of x, y, z, we have

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} \qquad \text{...}(i)$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y} \qquad \text{...}(ii)$$

and

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z} \qquad \text{...}(iii)$$

Multiplying (*i*) by **I**, (*ii*) by **J**, (*iii*) by **K** and adding, we have

$$\nabla f = \frac{\partial f}{\partial u}\nabla u + \frac{\partial f}{\partial v}\nabla v + \frac{\partial f}{\partial w}\nabla w \qquad ...(iv)$$

$$= \frac{\mathbf{T}_u}{h_1}\frac{\partial f}{\partial u} + \frac{\mathbf{T}_v}{h_2}\frac{\partial f}{\partial v} + \frac{\mathbf{T}_w}{h_3}\frac{\partial f}{\partial w} \qquad \text{[By (5) p. 356]}$$

which is the required result.

(2) Let $\mathbf{F}(u, v, w)$ be a vector point function such that

$$\mathbf{F} = f_1\mathbf{T}_u + f_2\mathbf{T}_v + f_3\mathbf{T}_w = \Sigma f_1 h_2 h_3 \nabla v \times \nabla w \qquad \text{[By (6) p. 356]}$$

$$\therefore \quad \nabla \cdot \mathbf{F} = \Sigma\nabla \cdot \{(f_1h_2h_3)(\nabla v \times \nabla w)\}$$

$$= \Sigma[(f_1h_2h_3)\,\nabla \cdot (\nabla v \times \nabla w) + (\nabla v \times \nabla w)\,\nabla(f_1h_2h_3)] \qquad ...(v)$$

Now $\nabla \cdot (\nabla v \times \nabla w) = \nabla w \cdot \nabla \times (\nabla v) - \nabla v \cdot \nabla \times (\nabla w) = 0$ [By (5) p. 330]

and

$$\nabla(f_1h_2h_3) = \frac{\partial(f_1h_2h_3)}{\partial u}\nabla u + \frac{\partial(f_1h_2h_3)}{\partial v}\nabla v + \frac{\partial(f_1h_2h_3)}{\partial w}\nabla w \qquad \text{[By (iv) above]}$$

$\therefore$ (*v*) now becomes

$$\nabla \cdot \mathbf{F} = \sum(\nabla v \times \nabla w) \cdot \left\{\frac{\partial(f_1h_2h_3)}{\partial u}\nabla u + \frac{\partial(f_1h_2h_3)}{\partial v}\nabla v + \frac{\partial(f_1h_2h_3)}{\partial w}\nabla w\right\}$$

$$= [\nabla u, \nabla v, \nabla w] \sum \frac{\partial(f_1h_2h_3)}{\partial u} = \frac{1}{h_1h_2h_3}\sum \frac{\partial(f_1h_2h_3)}{\partial u}$$ which is the required result.

Cor. Laplacian. $\nabla^2 f = \nabla \cdot (\nabla f)$

$$= \nabla \cdot \left(\frac{\mathbf{T}_u}{h_1}\frac{\partial f}{\partial u} + \frac{\mathbf{T}_v}{h_2}\frac{\partial f}{\partial v} + \frac{\mathbf{T}_w}{h_3}\frac{\partial f}{\partial w}\right) = \frac{1}{h_1h_2h_3}\sum \frac{\partial}{\partial u}\left(\frac{1}{h_1}\frac{\partial f}{\partial u}h_2h_3\right)$$

(3) Let $\mathbf{F}(u, v, w)$ be a vector point function such that

$$\mathbf{F} = f_1\mathbf{T}_u + f_2\mathbf{T}_v + f_3\mathbf{T}_w = f_1h_1\nabla u + f_2h_2\nabla v + f_3h_3\nabla w \qquad \text{[By (5) p. 356]}$$

$$\nabla \times \mathbf{F} = \Sigma\nabla \times (f_1h_1\nabla u) \qquad \text{[Using (3) p. 329]}$$

$$= \Sigma[\nabla(f_1h_1) \times \nabla u + f_1h_1 \cdot \nabla \times \nabla u] = \sum\left[\frac{\partial(f_1h_1)}{\partial u}\nabla u + \frac{\partial(f_1h_1)}{\partial v}\nabla v + \frac{\partial(f_1h_1)}{\partial w}\nabla w\right] \times \nabla u$$

$$= \sum\left[\frac{\partial(f_1h_1)}{\partial v}\nabla v \times \nabla u + \frac{\partial(f_1h_1)}{\partial w}\nabla w \times \nabla u\right]$$

$$= \sum\left[\frac{\partial(f_1h_1)}{\partial v}\left(-\frac{\mathbf{T}_u \times \mathbf{T}_v}{h_1h_2}\right) + \frac{\partial(f_1h_1)}{\partial w}\left(\frac{\mathbf{T}_w \times \mathbf{T}_u}{h_3h_1}\right)\right]$$

$$= -\frac{\partial(f_1h_1)}{\partial v}\frac{\mathbf{T}_w}{h_1h_2} + \frac{\partial(f_1h_1)}{\partial w}\frac{\mathbf{T}_v}{h_3h_1} - \frac{\partial(f_2h_2)}{\partial w}\frac{\mathbf{T}_u}{h_2h_3} + \frac{\partial(f_2h_2)}{\partial u}\frac{\mathbf{T}_w}{h_1h_2} - \frac{\partial(f_3h_3)}{\partial u}\frac{\mathbf{T}_v}{h_3h_1} + \frac{\partial(f_3h_3)}{\partial v}\frac{\mathbf{T}_u}{h_2h_3}$$

$$= \frac{\mathbf{T}_u}{h_2h_3}\left[\frac{\partial(f_3h_3)}{\partial v} - \frac{\partial(f_2h_2)}{\partial w}\right]$$ + two similar terms, whence follows the required result.

TWO SPECIAL CURVILINEAR SYSTEMS

8.20 (1) CYLINDRICAL COORDINATES

Any point $P(x, y, z)$ whose projection on the xy-plane is $Q(x, y)$ has the *cylindrical coordinates* (ρ, ϕ, z), where $\rho = OQ$, $\phi = \angle XOQ$ and $z = QP$.

The level surfaces $\rho = \rho_0$, $\phi = \phi_0$, $z = z_0$ are respectively cylinders about the Z-axis; planes through the Z-axis and planes perpendicular to the Z-axis.

The coordinate curves for ρ are rays perpendicular to the Z-axis; for ϕ, horizontal circles with centres on the Z-axis; for z, lines parallel to the Z-axis.

From Fig. 8.28, we have

$$x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = z$$

(*i*) *Arc element*.

$\therefore \qquad (ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 = (d\rho)^2 + \rho^2 (d\phi)^2 + (dz)^2$

so that the scale factors are $h_1 = 1, h_2 = \rho, h_3 = 1$.

(*ii*) *Area elements* $dS_\rho = \rho d\phi dz$, $dS_\phi = dz d\rho$, $dS_z = \rho d\rho d\phi$ where dS_ρ is the area element $\perp$ to ρ-direction, etc.

(*iii*) *Volume element* $dV = \rho d\rho d\phi dz$.

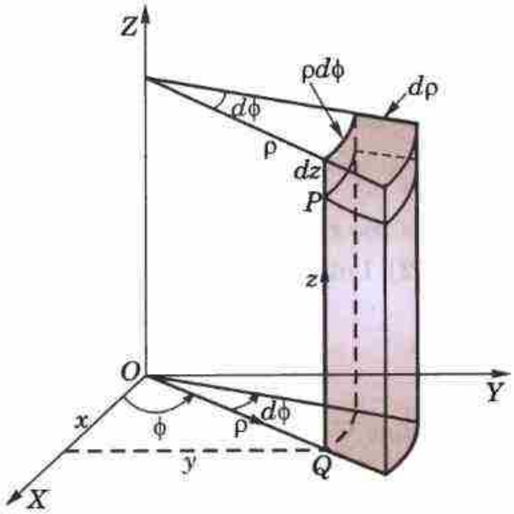

Fig. 8.28

(2) Cylindrical co-ordinate system is orthogonal

At any point P, we have $x = \rho \cos\phi$, $y = \rho \sin\phi$, $z = z$,

so that $\qquad \mathbf{R} = \rho\cos\phi\mathbf{I} + \rho\sin\phi\mathbf{J} + z\mathbf{K}$

If $\mathbf{T}_\rho$, $\mathbf{T}_\phi$, $\mathbf{T}_z$ be the unit vectors at P in the directions of the tangents to the ρ, ϕ, z-curves respectively, then

$$\mathbf{T}_\rho = \frac{\partial\mathbf{R}/\partial\rho}{|\partial\mathbf{R}/\partial\rho|} = \frac{\cos\phi\mathbf{I} + \sin\phi\mathbf{J}}{\sqrt{(\cos^2\phi + \sin^2\phi)}} = \cos\phi\mathbf{I} + \sin\phi\mathbf{J}$$

$$\mathbf{T}_\phi = \frac{\partial\mathbf{R}/\partial\phi}{|\partial\mathbf{R}/\partial\phi|} = \frac{-\rho\sin\phi\mathbf{I} + \rho\cos\phi\mathbf{J}}{\sqrt{[(-\rho\sin\phi)^2 + (\rho\cos\phi)^2]}} = -\sin\phi\mathbf{I} + \cos\phi\mathbf{J}$$

and $$\mathbf{T}_z = \frac{\partial\mathbf{R}/\partial z}{|\partial\mathbf{R}/\partial z|} = \mathbf{K}$$

Now $\mathbf{T}_\rho \cdot \mathbf{T}_\phi = (\cos\phi\mathbf{I} + \sin\phi\mathbf{J}) \cdot (-\sin\phi\mathbf{I} + \cos\phi\mathbf{J}) = -\cos\phi\sin\phi + \sin\phi\cos\phi = 0$,

$\mathbf{T}_\phi \cdot \mathbf{T}_z = (-\sin\phi\mathbf{I} + \cos\phi\mathbf{J}) \cdot \mathbf{K} = 0$, and $\mathbf{T}_z \cdot \mathbf{T}_\rho = \mathbf{K} \cdot (\cos\phi\mathbf{I} + \sin\phi\mathbf{J}) = 0$.

Hence the cylindrical coordinate system is orthogonal.

Also $\quad \mathbf{T}_\rho \times \mathbf{T}_\phi = (\cos\phi\mathbf{I} + \sin\phi\mathbf{J}) \times (-\sin\phi\mathbf{I} + \cos\phi\mathbf{J}) = (\cos^2\phi + \sin^2\phi)\,\mathbf{I} \times \mathbf{J} = \mathbf{K} = T_z$

$\mathbf{T}_\phi \times \mathbf{T}_z = (-\sin\phi\mathbf{I} + \cos\phi\mathbf{J}) \times \mathbf{K} = \sin\phi\mathbf{J} + \cos\phi\mathbf{I} = \mathbf{T}_\rho$

$\mathbf{T}_z \times \mathbf{T}_\rho = \mathbf{K} \times (\cos\phi\mathbf{I} + \sin\phi\mathbf{J}) = \cos\phi\mathbf{J} - \sin\phi\mathbf{I} = \mathbf{T}_\phi$

These conditions satisfied by T_ρ, T_ϕ, and T_z, show that the *cylindrical coordinates system is a right handed orthogonal coordinate system*. (*V.T.U., 2008*)

(3) Del applied to functions in Cylindrical coordinates

We have $u = \rho$, $v = \phi$, $w = z$ and $h_1 = 1$, $h_2 = \rho$, $h_3 = 1$.

Let T_ρ, T_ϕ, T_z be the unit vectors in the directions of the tangents to the ρ, ϕ, z curves.

(*i*) *Expression for grad f*.

Since $$\nabla f = \frac{\mathbf{T}_u}{h_1}\frac{\partial f}{\partial u} + \frac{\mathbf{T}_v}{h_2}\frac{\partial f}{\partial v} + \frac{\mathbf{T}w}{h_3}\frac{\partial f}{\partial w}$$

$$\therefore \qquad \nabla f = \frac{\partial f}{\partial\rho}\mathbf{T}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial\phi}\mathbf{T}_\phi + \frac{\partial f}{\partial z}\mathbf{T}_z$$

(*ii*) *Expression for div* $\mathbf{F}$ *where* $\mathbf{F} = f_1\mathbf{T}_u + f_2\mathbf{T}_v + f_3\mathbf{T}_w$

Since $$\nabla \cdot \mathbf{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u}(h_2h_3f_1) + \frac{\partial f}{\partial v}(h_3h_1f_2) + \frac{\partial}{\partial w}(h_1h_2f_3)\right]$$

$$\therefore \qquad \nabla \cdot \mathbf{F} = \frac{1}{\rho}\left\{\frac{\partial}{\partial\rho}(\rho f_1) + \frac{\partial f_2}{\partial\phi} + \frac{\partial}{\partial z}(\rho f_3)\right\}$$

(*iii*) *Expression for curl* $\mathbf{F}$ *where* $\mathbf{F} = f_1\mathbf{T}_u + f_2\mathbf{T}_v + f_3\mathbf{T}_w$

Since $$\nabla \times \mathbf{F} = \begin{vmatrix} \dfrac{\mathbf{T}_u}{h_2h_3} & \dfrac{\mathbf{T}_v}{h_3h_1} & \dfrac{\mathbf{T}_w}{h_1h_2} \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ h_1f_1 & h_2f_2 & h_3f_3 \end{vmatrix} = \begin{vmatrix} \mathbf{T}_\rho/\rho & \mathbf{T}_\phi & \mathbf{T}_z/\rho \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z} \\ f_1 & \rho f_2 & f_3 \end{vmatrix}$$

$$= \mathbf{T}_\rho\left(\frac{1}{\rho}\frac{\partial f_3}{\partial\phi} - \frac{\partial f_2}{\partial z}\right) + \mathbf{T}_\phi\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial\rho}\right) + \mathbf{T}_z\left(\frac{\partial f_2}{\partial\rho} - \frac{1}{\rho}\frac{\partial f_1}{\partial\phi}\right)$$

(iv) Expression for $\nabla^2 f$

Since $$\nabla^2 f = \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial u}\left(\frac{1}{h_1}\frac{\partial f}{\partial u} h_2 h_3\right) + \frac{\partial}{\partial v}\left(\frac{1}{h_2}\frac{\partial f}{\partial v} h_3 h_1\right) + \frac{\partial}{\partial w}\left(\frac{1}{h_3}\frac{\partial f}{\partial w} h_1 h_2\right)\right\}$$

$\therefore$ $$\nabla^2 f = \frac{1}{\rho}\left\{\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\rho}\frac{\partial f}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(\rho\frac{\partial f}{\partial z}\right)\right\} = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}.$$

Example 8.52. *Express the vector* $z\mathbf{I} - 2x\mathbf{J} + y\mathbf{K}$ *in cylindrical coordinates.* (V.T.U., 2010)

Solution. We have $x = \rho \cos \phi$, $y = \rho \sin \phi$ and $z = z$.

so that $$\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K} = \rho \cos \phi \mathbf{I} + \rho \sin \phi \mathbf{J} + z\mathbf{K}$$

If $\mathbf{T}_\rho$, $\mathbf{T}_\phi$, $\mathbf{T}_z$ be the unit vectors along the tangents to ρ, ϕ and z curves respectively, then

$$\mathbf{T}_\rho = \frac{\partial \mathbf{R}/\partial \rho}{|\partial \mathbf{R}/\partial \rho|} = \frac{\cos \phi \mathbf{I} + \sin \phi \mathbf{J}}{\sqrt{(\cos^2 \phi + \sin^2 \phi)}} = \cos \phi \mathbf{I} + \sin \phi \mathbf{J}$$

$$\mathbf{T}_\phi = \frac{\partial \mathbf{R}/\partial \phi}{|\partial \mathbf{R}/\partial \phi|} = \frac{-\rho \sin \phi \mathbf{I} + \rho \cos \phi \mathbf{J}}{\sqrt{[(-\rho \sin \phi)^2 + (\rho \cos \phi)^2]}} = -\sin \phi \mathbf{I} + \cos \phi \mathbf{J}$$

$$\mathbf{T}_z = \frac{\partial \mathbf{R}/\partial z}{|\partial \mathbf{R}/\partial z|} = \mathbf{K}$$

Let the expression for $\mathbf{F} = z\mathbf{I} - 2x\mathbf{J} + y\mathbf{K}$ in cylindrical coordinates be

$$\mathbf{F} = f_1 \mathbf{T}_\rho + f_2 \mathbf{T}_\phi + f_3 \mathbf{T}_z \qquad ...(i)$$

Then $$f_1 = \mathbf{F} \cdot \mathbf{T}_\rho = z \cos \phi - 2x \sin \phi$$
$$f_2 = \mathbf{F} \cdot \mathbf{T}_\phi = -z \sin \phi - 2x \cos \phi$$
$$f_3 = \mathbf{F} \cdot \mathbf{T}_z = y$$

Substituting the values of f_1, f_2, f_3 in (i), we get

$$\mathbf{F} = (z \cos \phi - 2x \sin \phi)\, \mathbf{T}_\rho - (z \sin \phi + 2x \cos \phi)\, \mathbf{T}_\phi + y\mathbf{T}_z$$
$$= (z \cos \phi - \rho \sin 2\phi)\, \mathbf{T}_\rho - (z \sin \phi + 2\rho \cos^2 \phi)\, \mathbf{T}_\phi + \rho \sin \phi\, \mathbf{T}_z$$

Example 8.53. *Show that* $\nabla (\log \rho)$ *and* $\nabla \phi$, $\rho \neq 0$, $\phi \neq 0$ *are solenoidal vectors.*

Solution. (i) $f = \log \rho$ is a function of ρ only. We have to prove that $\nabla \cdot (\nabla f)$, *i.e.*, $\nabla^2 f = 0$

$$\nabla^2 f = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} = \frac{\partial^2}{\partial \rho^2}(\log \rho) + \frac{1}{\rho}\frac{\partial (\log \rho)}{\partial \rho} + 0 + 0 = -\frac{1}{\rho^2} + \frac{1}{\rho^2} = 0$$

Hence $\nabla(\log \rho)$ is a solenoidal vector.

(ii) $f = \nabla \phi$ is a function of ϕ only. We have to show that $\nabla \cdot (\nabla f)$, *i.e.*, $\nabla^2 f = 0$.

$$\nabla^2 f = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} = 0 + 0 + \frac{1}{\rho^2}\frac{\partial^2 \phi}{\partial \phi^2} + 0 = 0.$$

Hence the result.

8.21 (1) SPHERICAL POLAR COORDINATES

Let $P(x, y, z)$ be any point whose projection on the XY-plane is $Q(x, y)$. Then the spherical polar co-ordinates of P are (r, θ, ϕ) such that $r = OP$, $\theta = \angle ZOP$ and $\phi = \angle XOQ$.

The level surfaces $r = r_0$, $\theta = \theta_0$, $\phi = \phi_0$ are respectively spheres about O, cones about the Z-axis with vertex at O and planes through the Z-axis.

The co-ordinate curves for r are rays from the origin; for θ, vertical circles with centre at O (called *meridians*); for ϕ, horizontal circles with centres on the Z-axis

From Fig. 8.29, we have

$$x = OQ \cos \phi = OP \cos (90° - \theta) \cos \phi = r \sin \theta \cos \phi,$$
$$y = OQ \sin \phi = r \sin \theta \sin \phi;\ z = r \cos \theta.$$

(i) *Arc element*

$$\therefore \quad (ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 = (dr)^2 + r^2\,(d\theta)^2 + (r \sin\theta)^2\,(d\phi)^2$$

so that the scale factors are

$$h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta.$$

(ii) *Area elements*

$$dS_r = r^2\ \sin\theta\, d\theta d\phi,\ dS_\theta = r\sin\theta d\phi dr,\ dS_\phi = r dr d\theta$$

where dS_r is the area element perpendicular to the r-direction, etc.

(iii) *Volume element* $dV = r^2 \sin\theta\, dr d\theta d\phi$.

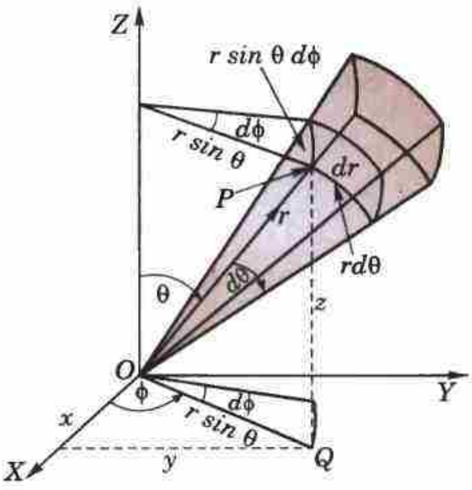

Fig. 8.29

(2) Spherical polar coordinate system is orthogonal

At any point P, we have $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$,

so that $\quad \mathbf{R} = r\sin\theta\cos\phi\mathbf{I} + r\sin\theta\sin\phi\mathbf{J} + r\cos\theta\mathbf{K}$

If $\mathbf{T}_r, \mathbf{T}_\theta, \mathbf{T}_\phi$ be the unit vectors at P in the directions of the tangents to the r, θ, ϕ-curves respectively, then

$$\mathbf{T}_r = \frac{\partial\mathbf{R}/\partial r}{|\partial\mathbf{R}/\partial r|} = \frac{\sin\theta\cos\phi\mathbf{I} + \sin\theta\sin\phi\mathbf{J} + \cos\theta\mathbf{K}}{\sqrt{(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta)}}$$

$$= \sin\theta\cos\phi\mathbf{I} + \sin\theta\sin\phi\mathbf{J} + \cos\theta\mathbf{K}$$

$$\mathbf{T}_\theta = \frac{\partial\mathbf{R}/\partial\theta}{|\partial\mathbf{R}/\partial\theta|} = \frac{r\cos\theta\cos\phi\mathbf{I} + r\cos\theta\sin\phi\mathbf{J} - r\sin\theta\mathbf{K}}{r\sqrt{(\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta)}}$$

$$= \cos\theta\cos\phi\mathbf{I} + \cos\theta\sin\phi\mathbf{J} - \sin\theta\mathbf{K}$$

and
$$\mathbf{T}_\phi = \frac{\partial\mathbf{R}/\partial\phi}{|\partial\mathbf{R}/\partial\phi|} = \frac{-r\sin\theta\sin\phi\mathbf{I} + r\sin\theta\cos\phi\mathbf{J}}{r\sin\theta} = -\sin\phi\mathbf{I} + \cos\phi\mathbf{J}$$

Now
$$\mathbf{T}_r \cdot \mathbf{T}_\theta = \sin\theta\cos\theta\cos^2\phi + \sin\theta\cos\theta\sin^2\phi - \sin\theta\cos\theta = 0$$
$$\mathbf{T}_\theta \cdot \mathbf{T}_\phi = -\cos\theta\cos\phi\sin\phi + \cos\theta\sin\phi\cos\phi = 0$$
$$\mathbf{T}_\phi \cdot \mathbf{T}_r = -\sin\theta\cos\phi\sin\phi + \sin\theta\sin\phi\cos\phi = 0$$

Also
$$\mathbf{T}_r \times \mathbf{T}_\theta = \sin\theta\cos\phi\cos\theta\sin\phi\mathbf{K} + \sin^2\theta\cos\phi\mathbf{J} - \sin\theta\sin\phi\cos\theta\cos\phi\mathbf{K}$$
$$- \sin^2\theta\sin\phi\mathbf{I} + \cos^2\theta\cos\phi\mathbf{J} - \cos^2\theta\sin\phi\mathbf{I}$$
$$= -\sin\phi\mathbf{I} + \cos\phi\mathbf{J} = \mathbf{T}_\phi$$
$$\mathbf{T}_\theta \times \mathbf{T}_\phi = \cos\theta\cos^2\phi\mathbf{K} + \sin^2\phi\cos\theta\mathbf{K} + \sin\theta\sin\phi\mathbf{J} + \sin\theta\cos\phi\mathbf{I} = \mathbf{T}_r$$

and
$$\mathbf{T}_\phi \times \mathbf{T}_r = -\sin\theta\sin^2\phi\mathbf{K} + \sin\phi\cos\theta\mathbf{J} - \sin\theta\cos^2\phi\mathbf{K} + \cos\phi\cos\theta\mathbf{I} = \mathbf{T}_\theta$$

The above conditions satisfied by $\mathbf{T}_r$, $\mathbf{T}_\theta$, and $\mathbf{T}_\phi$ show that *the spherical polar coordinate system is a right handed orthogonal coordinate system.* *(V.T.U., 2008)*

(3) Del applied to functions in spherical polar coordinates

We have $u = r,\ v = \theta,\ w = \phi$ and $h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta$.

Let $\mathbf{T}_r, \mathbf{T}_\theta, \mathbf{T}_\phi$ be the unit vectors in the directions of the tangents to the r, θ, ϕ-curves.

(i) *Expression for grad f*

Since
$$\nabla f = \frac{\mathbf{T}_u}{h_1}\frac{\partial f}{\partial u} + \frac{\mathbf{T}_v}{h_2}\frac{\partial f}{\partial v} + \frac{\mathbf{T}_w}{h_3}\frac{\partial f}{\partial w}$$

$$\therefore \quad \nabla f = \frac{\partial f}{\partial r}\mathbf{T}_r + \frac{1}{r}\frac{\partial f}{\partial\theta}\mathbf{T}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\mathbf{T}_\phi$$

(ii) *Expression for div* $\mathbf{F}$ *where* $\mathbf{F} = f_1\mathbf{T}_u + f_2\mathbf{T}_v + f_3\mathbf{T}_w$

Since
$$\nabla \cdot \mathbf{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u}(h_2h_3f_1) + \frac{\partial}{\partial v}(h_3h_1f_2) + \frac{\partial}{\partial w}(h_1h_2f_3)\right]$$

$$\therefore \quad \nabla \cdot \mathbf{F} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta\, f_1) + \frac{\partial}{\partial\theta}(r\sin\theta\, f_2) + \frac{\partial}{\partial\phi}(rf_3)\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(f_1 r^2) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(f_2\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial f_3}{\partial\phi}$$

(iii) Expression for curl $\mathbf{F}$ *where* $\mathbf{F} = f_1\mathbf{T}_u + f_2\mathbf{T}_v + f_3\mathbf{T}_w$

Since
$$\nabla \times \mathbf{F} = \begin{vmatrix} \dfrac{\mathbf{T}_u}{h_2h_3} & \dfrac{\mathbf{T}_v}{h_3h_1} & \dfrac{\mathbf{T}_w}{h_1h_2} \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ h_1f_1 & h_2f_2 & h_3f_3 \end{vmatrix}$$

$\therefore$
$$\nabla \times \mathbf{F} = \begin{vmatrix} \dfrac{\mathbf{T}_r}{r^2 \sin\theta} & \dfrac{\mathbf{T}_\theta}{r\sin\theta} & \dfrac{\mathbf{T}\phi}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ f_1 & rf_2 & r\sin\theta f_3 \end{vmatrix}$$

$$= \frac{\mathbf{T}_r}{r^2\sin\theta}\left\{\frac{\partial}{\partial\theta}(r\sin\theta f_3) - \frac{\partial}{\partial\phi}(rf_2)\right\} - \frac{\mathbf{T}_\theta}{r\sin\theta}\left\{\frac{\partial}{\partial r}(r\sin\theta f_3) - \frac{\partial f_1}{\partial\phi}\right\} + \frac{\mathbf{T}_\phi}{r}\left\{\frac{\partial}{\partial r}(rf_2) - \frac{\partial f_1}{\partial\theta}\right\}$$

$$= \frac{\mathbf{T}_r}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}(f_3\sin\theta) - \frac{\partial f_2}{\partial\phi}\right\} + \frac{\mathbf{T}_\theta}{r}\left\{\frac{1}{\sin\theta}\frac{\partial f_1}{\partial\phi} - \frac{\partial}{\partial r}(rf_3)\right\} + \frac{\mathbf{T}_\phi}{r}\left\{\frac{\partial}{\partial r}(rf_2) - \frac{\partial f_1}{\partial\theta}\right\}$$

(iv) Expression for $\nabla^2 f$.

Since
$$\nabla^2 f = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_3h_1}{h_2}\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial w}\right)\right\}$$

$\therefore$
$$\nabla^2 f = \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{r\sin\theta}{r}\frac{\partial f}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{r}{r\sin\theta}\frac{\partial f}{\partial\phi}\right)\right\}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

$$= \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial f}{\partial\theta}.$$

Example 8.54. *Express the vector field* $2y\mathbf{I} - z\mathbf{J} + 3x\mathbf{K}$ *in spherical polar coordinate system.*

Solution. We have $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

so that $\mathbf{R} = r\sin\theta\cos\phi\mathbf{I} + r\sin\theta\sin\phi\mathbf{J} + r\cos\theta\mathbf{K}$.

If $\mathbf{T}_r$, $\mathbf{T}_\theta$, $\mathbf{T}_\phi$ be the unit vectors along the tangents to r, θ, ϕ, curves respectively, then

$$\mathbf{T}_r = \frac{\partial\mathbf{R}/\partial r}{|\partial\mathbf{R}/\partial r|} = \frac{\sin\theta\cos\phi\mathbf{I} + \sin\theta\sin\phi\mathbf{J} + \cos\theta\mathbf{K}}{\sqrt{[(\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + \cos^2\theta]}}$$
$$= \sin\theta\cos\phi\mathbf{I} + \sin\theta\sin\phi\mathbf{J} + \cos\theta\mathbf{K}$$

$$\mathbf{T}_\theta = \frac{\partial\mathbf{R}/\partial\theta}{|\partial\mathbf{R}/\partial\theta|} = \frac{r\cos\theta\cos\phi\mathbf{I} + r\cos\theta\sin\phi\mathbf{J} - r\sin\theta\mathbf{K}}{\sqrt{[(r\cos\theta\cos\phi)^2 + (r\cos\theta\sin\phi)^2 + (-r\sin\theta)^2]}}$$
$$= \cos\theta\cos\phi\mathbf{I} + \cos\theta\sin\phi\mathbf{J} - \sin\theta\mathbf{K}$$

$$\mathbf{T}_\phi = \frac{\partial\mathbf{R}/\partial\phi}{|\partial\mathbf{R}/\partial\phi|} = \frac{-r\sin\theta\sin\phi\mathbf{I} + r\sin\theta\cos\phi\mathbf{J}}{\sqrt{[(-r\sin\theta\sin\phi)^2 + (r\sin\theta\cos\phi)^2]}} = -\sin\phi\mathbf{I} + \cos\phi\mathbf{J}.$$

Let the expression for $\mathbf{F} = 2y\mathbf{I} - z\mathbf{J} + 3x\mathbf{K}$ in spherical polar coordinates be

$$\mathbf{F} = f_1\mathbf{T}_r + f_2\mathbf{T}_\theta + f_3\mathbf{T}_\phi \qquad ...(i)$$

Then $f_1 = \mathbf{F} . \mathbf{T}_r = (2r\sin\theta\sin\phi\mathbf{I} - r\cos\theta\mathbf{J} + 3r\sin\theta\cos\phi\mathbf{K}) . (\sin\theta\cos\phi\mathbf{I} + \sin\theta\sin\phi\mathbf{J} + \cos\theta\mathbf{K})$

$= 2r\sin^2\theta\sin\phi\cos\phi - r\sin\theta\cos\theta\sin\phi + 3r\sin\theta\cos\theta\cos\phi$

$$f_2 = \mathbf{F} \cdot \mathbf{T}_\theta = (2r \sin\theta \sin\phi \mathbf{I} - r\cos\theta \mathbf{J} + 3r \sin\theta\cos\phi\mathbf{K}) \cdot (\cos\theta\cos\phi\mathbf{I} + \cos\theta\sin\phi\mathbf{J} - \sin\theta\mathbf{K})$$
$$= 2r\sin\theta\cos\theta\sin\phi\cos\phi - r\cos^2\theta\sin\phi - 3r\sin^2\theta\cos\phi.$$

and
$$f_3 = \mathbf{F} \cdot \mathbf{T}_\phi = (2r\sin\theta\sin\phi\mathbf{K} - r\cos\theta\mathbf{J} + 3r\sin\theta\cos\phi\mathbf{K}) . (-\sin\phi\mathbf{I} + \cos\phi\mathbf{J})$$
$$= -2r\sin\theta\sin^2\phi - r\cos\theta\cos\phi$$

Substituting the values of f_1, f_2, f_3 in (*i*), we get the desired expression.

Example 8.55. *Prove that* $\nabla(\cos\theta) \times \nabla\phi = \nabla(1/r).\ r \neq 0.$

Solution. In spherical polar coordinates,

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{T}_r + \frac{1}{r}\frac{\partial f}{\partial\theta}\mathbf{T}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\mathbf{T}_\phi$$

$$\therefore \quad \nabla(\cos\theta) = \frac{1}{r}\frac{\partial}{\partial\theta}(\cos\theta)\mathbf{T}_\theta = -\frac{1}{r}\sin\theta\,\mathbf{T}_\theta \qquad ...(i)$$

$$\nabla\phi = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\phi)\mathbf{T}_\phi = \frac{1}{r\sin\theta}\mathbf{T}_\phi \qquad ...(ii)$$

and
$$\nabla\left(\frac{1}{r}\right) = \frac{\partial}{\partial r}(r^{-1})\mathbf{T}_r = -\frac{1}{r^2}\mathbf{T}_r$$

Now from (*i*) and (*ii*), we get

$$\nabla(\cos\theta) \times \nabla\phi = -\frac{1}{r^2}\mathbf{T}_\theta \times \mathbf{T}_\phi = -\frac{1}{r^2}\mathbf{T}_r = \nabla\left(\frac{1}{r}\right).$$

Example 8.56. *If* $F = r^2\cos\theta\ \mathbf{T}_r - \frac{1}{r}\mathbf{T}_\theta + \frac{1}{r\sin\theta}\mathbf{T}_\phi$, *find the value of* $\mathbf{F} \times$ *curl* $\mathbf{F}$.

Solution. In spherical coordinates,

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{T}_r/r^2\sin\theta & \mathbf{T}_\theta/r\sin\theta & \mathbf{T}_\phi/r \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ f_1 & rf_2 & r\sin\theta f_3 \end{vmatrix}$$

Here $f_1 = r^2\cos\theta, f_2 = -1/r, f_3 = 1/r\sin\theta$.

$$\therefore \quad \text{curl}\,\mathbf{F} = \frac{2}{r^2\sin\theta}\begin{vmatrix} \mathbf{T}_r & r\mathbf{T}_\theta & r\sin\theta\,\mathbf{T}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ r^2\cos\theta & -1 & 1 \end{vmatrix} = r\sin\theta\mathbf{T}_\phi$$

$$\therefore \quad \mathbf{F} \times \text{curl}\,\mathbf{F} = \left(r^2\cos\theta\mathbf{T}_r - \frac{1}{r}\mathbf{T}_\theta + \frac{1}{r\sin\theta}\mathbf{T}_\phi\right) \times (r\sin\theta\mathbf{T}_\phi) = -(r^3\sin\theta\cos\theta\mathbf{T}_\theta + \sin\theta\mathbf{T}_r).$$

PROBLEMS 8.12

1. Express the following vectors in cylindrical coordinates
   (*i*) $2y\mathbf{I} - z\mathbf{J} + 3x\mathbf{K}$ (*ii*) $2x\mathbf{I} - 3y^2\mathbf{J} + zx\mathbf{K}$ (*V.T.U., 2009*)
2. Express the following vectors in spherical polar coordinates
   (*i*) $x\mathbf{I} + 2y\mathbf{J} + yz\mathbf{K}$ (*ii*) $xy\mathbf{I} + yz\mathbf{J} + zx\mathbf{K}$
3. Evaluate $\nabla\phi = xyz$ in cylindrical coordinates.
4. Show that $\nabla(r/\sin\theta) \times \nabla\theta = \nabla\phi$.
5. Prove that $\mathbf{V} = \dfrac{\cos\theta}{r^3}(\mathbf{T}_r/\sin\theta - \mathbf{T}_\theta/\cos\theta + r^4\mathbf{T}_\phi)$ is solenoidal.
6. Show that (*i*) $\nabla^2(\log r) = 1/r^2$ (*ii*) $\nabla \times [(\cos\theta)(\nabla\phi)] = \nabla(1/r)$.

7. Prove that $\mathbf{V} = \rho z \sin 2\phi \left[\mathbf{T}_\rho + \cot 2\phi \mathbf{T}_\phi + \frac{\rho}{2z}\mathbf{T}_z\right]$ is irrotational.
8. If u, v, w are orthogonal curvilinear coordinates with h_1, h_2, h_3 as scale factors, prove that

$$\left[\frac{\partial \mathbf{R}}{\partial u}, \frac{\partial \mathbf{R}}{\partial v}, \frac{\partial \mathbf{R}}{\partial w}\right] = \frac{1}{[\nabla u, \nabla v, \nabla w]} = h_1 h_2 h_3.$$

8.22 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 8.13

Fill up the blanks or choose the correct answer from the following problems :

1. A unit tangent vector to the surface $x = t, y = t^2, z = t^3$ at $t = 1$ is ....
2. The equation of the normal to the surface $2x^2 + y^2 + 2z = 3$ at $(2, 1, -3)$ is ....
3. If $u = u(x, y)$ and $v = v(x, y)$, then the area-element $dudv$ is related to the area-element $dxdy$ by the relation ....
4. If $A = 2x^2\mathbf{I} - 3yz\mathbf{J} + xz^2\,\mathbf{K}$, then $\nabla.\,\mathbf{A}$ = ...
5. div curl $\mathbf{F}$ = ...
6. Area bounded by a simple closed curve C is ....
7. If S is a closed surface enclosing a volume V and if $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$, then

$$\int_S \mathbf{R}\,.\,\mathbf{N}\,ds = \text{.......}$$

8. div $\mathbf{R}$ = ...; curl $\mathbf{R}$ = ...
9. If $\mathbf{A}$ is such that $\nabla \times \mathbf{A} = 0$, then $\mathbf{A}$ is called ....
10. If $\nabla\,.\,\mathbf{F} = 3$, then $\int_S \mathbf{F}\,.\,\mathbf{N}\,ds$ where S is a surface of a unit sphere, is ....
11. If $\nabla\,.\,\mathbf{F} = 0$, then $\mathbf{F}$ is called.......
12. The directional derivative of $\phi(x, y, z) = x^2yz + 4xz^2$ at the point $(1, -2, -1)$ in the direction PQ where $P = (1, 2, -1)$ and $Q = (-1, 2, 3)$ is ....
13. If $u = x^2yz$, $v = xy - 3z^2$, then $\nabla\,.\,(\nabla u \times \nabla v)$ =.......
14. curl $(xy\mathbf{I} + yz\mathbf{J} + zx\mathbf{K})$ = ...
15. If $\mathbf{F} = f_1\mathbf{I} + f_2\mathbf{J} + f_3\mathbf{K}$, then $\nabla\,.\,\mathbf{F}$ = ...
16. If $\mathbf{F}$ is a conservative force field then curl $\mathbf{F}$ is ....
17. If $\phi = 3x^2y - y^3z^2$, grad ϕ at the point $(1, -2, -1)$ is ....
18. curl $(x\mathbf{I} + y\mathbf{J} + z\mathbf{K})$ = ...
19. Workdone by a particle along the square formed by the lines $y = \pm 1$ and $x = \pm 1$ under the force
$\mathbf{F} = (x^2 + xy)\mathbf{I} + (x^2 + y^2)\mathbf{J}$ is.......
20. Curl (grad ϕ) = ...
21. If $\mathbf{A}$ is a constant vector, then $div\ (\mathbf{A} \times \mathbf{R})$ = ...
22. If $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$, then $\nabla \log r$ = ... ; $\nabla(r^n)$ = ...
23. A level surface is defined as ....
24. Unit normal vector to the surface $z = 2xy$ at the point $(2, 1, 4)$ is ....
25. If the directional derivative of $f = ax + by + cz$ at $(1, 1, 1)$ has maximum magnitude 4 in direction parallel to x-axis, then the values of a, b, c are ....
26. Maximum value of the directional derivative of $\phi = x^2 - 2y^2 + 4z^2$ at the point $(1, 1, -1)$ is ....
27. If $r^2 = x^2 + y^2 + z^2$, then $\nabla\,.\,(\mathbf{R}/r)$ = ...
28. Directional derivative of $f = xyz$ at the point $(1, -1, -2)$ in the direction of the vector $2\mathbf{I} - 2\mathbf{J} + \mathbf{K}$ is ....
29. If $\mathbf{V} = x^2\mathbf{I} + xye^x\mathbf{J} + \sin z\mathbf{K}$, then $\nabla\,.\,(\nabla \times F)$ = ...
30. If $f = \tan^{-1}(y/x)$ then div (grad f) is equal to
(*a*) 1 (*b*) -1 (*c*) 0 (*d*) 2.
31. The value of curl (grad f), where $f = 2x^2 - 3y^2 + 4z^2$ is
(*a*) $4x - 6y + 8z$, (*b*) $4x\mathbf{I} - 6y\mathbf{J} + 8z\mathbf{K}$ (*c*) 0 (*d*) 3.

32. The value of $\int \text{grad}\,(x + y - z)\, d\mathbf{R}$ from (0, 1, –1) to (1, 2, 0) is

(*a*) 0 (*b*) 3 (*c*) – 1 (*d*) not obtainable.

33. If $\mathbf{F} = ax\mathbf{I} + by\mathbf{J} + cz\mathbf{K}$, then $\int_S \mathbf{F} \cdot d\mathbf{S}$, S being the surface of a unit sphere, is

(*a*) $(4/3)\,\pi\,(a + b + c)^2$ (*b*) 0 (*c*) $4\pi/3\,(a + b + c)$ (*d*) none of these.

34. A necessary and sufficient condition that the line integral $\int_L \mathbf{F} \cdot d\mathbf{R}$ for every closed C vanishes, is

(*a*) curl $\mathbf{F} = 0$ (*b*) div $\mathbf{F} = 0$ (*c*) curl $\mathbf{F} \neq 0$ (*d*) div $\mathbf{F} \neq 0$.

35. The value of $\iint_S (yzdydz + zxdz\,dx + xy\,dxdy)$, where S is the surface of unit sphere $x^2 + y^2 + z^2 = 1$ is

(*a*) 0 (*b*) 4π (*c*) $4\pi/3$ (*d*) 10π.

36. If $u = x^2 + y^2 + z^2$ and $\mathbf{V} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$, then $\nabla\,(u\mathbf{V}) =$ ....

37. For any scalar function ψ, $\nabla \times \nabla\,\psi =$ ....

38. $\int_C \mathbf{F} \cdot d\mathbf{R}$ is independent of the path joining any two points if and only if it is ....

39. The value of the line integral $\int_C (y^2dx + x^2dy)$ where C is the boundary of the square $-1 \le x \le 1, -1 \le y \le 1$ is

(*a*) 0 (*b*) $2(x + y)$ (*c*) 4 (*d*) 4/3. (*V.T.U., 2010*)

40. If $\mathbf{V}$ is the instantaneous velocity vector of the moving fluid at a point P, then div $\mathbf{V}$ represents ....

41. The spherical coordinate system is

(*a*) Orthogonal (*b*) Coplanar (*c*) Non-coplanar (*d*) Not orthogonal. (*V.T.U., 2010*)

42. Physical interpretation of $\nabla\,\phi$ is that ....

43. The magnitude of the vector drawn perpendicular to the surface $x^2 + 2y^2 + z^2 = 7$ at the point (1, – 1, 2) is

(*a*) 2/3 (*b*) 3/2 (*c*) 3 (*d*) 6.

44. The value of λ so that the vector $(x + 3y)\mathbf{I} + (y - 2z)\mathbf{J} + (x + \lambda z)\mathbf{K}$ is a solenoidal vector, is

(*a*) – 2 (*b*) 3 (*c*) 1 (*d*) none of these.

45. The work done by the force $\mathbf{F} = yz\mathbf{I} + zx\mathbf{J} + xy\mathbf{K}$, in moving a particle from the point (1, 1, 1) to the point (3, 3, 2) along the path c is

(*a*) 17 (*b*) 10 (*c*) 0 (*d*) cannot be found.

46. Value of $\int_c (y^2dx + x^2dy)$ where c is the boundary of the square $-1 \le x \le 1, -1 \le y \le 1$, is

(*a*) 4 (*b*) 0 (*c*) $2(x + y)$ (*d*) 4/3.

47. The directional derivative of $f(x, y) = (x^2 - y^2)/xy$ at (1, 1) is zero along a ray making an angle with the positive direction of x-axis :

(*a*) 45° (*b*) 60° (*c*) 135° (*d*) none of these.

48. The vector $\mathbf{V} = e^x \sin y\mathbf{I} + e^x \cos y\mathbf{J}$, is

(*a*) solenoidal (*b*) irrotational (*c*) rotational.

49. If $u = 1/r$ where $r^2 = x^2 + y^2$, then $\nabla^2 u = 0$. (True or False)

50. $\mathbf{F} = (x + 3y)\,\mathbf{I} + (z - 3y)\mathbf{J} + (x + 2z)\mathbf{K}$ is a solenoidal vector function. (True or False)

51. $\mathbf{F} = yz\mathbf{I} + zx\mathbf{J} + xy\mathbf{K}$ is irrotational. (True or False)

CHAPTER 9

Infinite Series

9.1 INTRODUCTION

Infinite series occur so frequently in all types of problems that the necessity of studying their convergence or divergence is very important. Unless a series employed in an investigation is convergent, it may lead to absurd conclusions. Hence it is essential that the students of engineering begin by acquiring an intelligent grasp of this subject.

9.2 SEQUENCES

(1) An ordered set of real numbers, $a_1, a_2, a_3, \ldots, a_n$ is called a *sequence* and is denoted by (a_n). If the number of terms is unlimited, then the sequence is said to be an *infinite sequence* and a_n is its *general term*.

For instance (*i*) $1, 3, 5, 7, \ldots, (2n - 1), \ldots$, (*ii*) $1, 1/2, 1/3, \ldots, 1/n, \ldots$,

(*iii*) $1, -1, 1, -1, \ldots, (-1)^{n-1}, \ldots$ are infinite sequences.

(2) Limit. A sequence is said to tend to a limit l, if for every $\varepsilon > 0$, a value N of n can be found such that $|a_n - l| < \varepsilon$ for $n \geq N$.

We then write $\underset{n \to \infty}{\text{Lt}} (a_n) = l$ or simply $(a_n) \to l$ as $n \to \infty$.

(3) Convergence. *If a sequence (a_n) has a finite limit, it is called a* **convergent sequence.** If (a_n) is not convergent, it is said to be **divergent.**

In the above examples, (*ii*) is convergent, while (*i*) and (*iii*) are divergent.

(4) Bounded sequence. A sequence (a_n) is said to be bounded, if there exists a number k such that $a_n < k$ for every n.

(5) Monotonic sequence. The sequence (a_n) is said to increase steadily or to decrease steadily according as $a_{n+1} \geq a_n$ or $a_{n+1} \leq a_n$, for all values of n. Both increasing and decreasing sequences are called *monotonic sequences.*

A monotonic sequence always tends to a limit, finite or infinite. Thus, *a sequence which is monotonic and bounded is* **convergent.**

(6) Convergence, Divergence and Oscillation. If $\underset{n \to \infty}{\text{Lt}} (a_n) = l$ is finite and unique then the sequence is said to be *convergent*.

If $\underset{n \to \infty}{\text{Lt}} (a_n)$ is infinite ($\pm \infty$), the sequence is said to be *divergent*.

If $\underset{n \to \infty}{\text{Lt}} (a_n)$ is not unique, then (a_n) is said to be *oscillatory*.

Example 9.1. *Examine the following sequences for convergence :*

(*i*) $a_n = \dfrac{n^2 - 2n}{3n^2 + n}$ (*ii*) $a_n = 2^n$ (*iii*) $a_n = 3 + (-1)^n$.

Solution. (*i*) $\underset{n \to \infty}{\text{Lt}} \left(\dfrac{n^2 - 2n}{3n^2 + n} \right) = \underset{n \to \infty}{\text{Lt}} \dfrac{1 - 2/n}{3 + 1/n} = 1/3$ which is finite and unique. Hence the sequence (a_n) is convergent.

(*ii*) $\underset{n \to \infty}{\text{Lt}} (2^n) = \infty$. Hence the sequence (a_n) is divergent.

(*iii*) $\underset{n \to \infty}{\text{Lt}} [3 + (-1)^n] = 3 + 1 = 4$ when n is even

$= 3 - 1 = 2$, when n is odd

i.e., this sequence doesn't have a unique limit. Hence it oscillates.

PROBLEMS 9.1

Examine the convergence of the following sequences :

1. $a_n = \dfrac{3n - 1}{1 + 2n}$
2. $a_n = 1 + 2/n$
3. $a_n = [n + (-1)^n]^{-1}$
4. $a_n = \sin n$
5. $a_n = 1/2n$
6. $a_n = 1 + (-1)^n/n$
7. $\left(\dfrac{n}{n - 1} \right)^2$
8. $a_n = 2n$.

9.3 SERIES

(1) Def. *If $u_1, u_2, u_3, \ldots, u_n, \ldots$ be an infinite sequence of real numbers, then*

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots \infty$$

is called an infinite series. An infinite series is denoted by Σu_n and the sum of its first n terms is denoted by s_n.

(2) Convergence, divergence and oscillation of a series.

Consider the infinite series $\Sigma u_n = u_1 + u_2 + u_3 + \ldots + u_n + \ldots \infty$

and let the sum of the first n terms be $s_n = u_1 + u_2 + u_3 + \ldots + u_n$

Clearly, s_n is a function of n and as n increases indefinitely three possibilities arise :

(*i*) If s_n tends to a finite limit as $n \to \infty$, the series Σu_n is said to be *convergent*.

(*ii*) If s_n tends to $\pm \infty$ as $n \to \infty$, the series Σu_n is said to be *divergent*.

(*iii*) If s_n does not tend to a unique limit as $n \to \infty$, then the series Σu_n is said to be *oscillatory* or *non-convergent*.

Example 9.2. *Examine for convergence the series (i) $1 + 2 + 3 + \ldots + n + \ldots \infty$.*
(ii) $5 - 4 - 1 + 5 - 4 - 1 + 5 - 4 - 1 + \ldots \infty$

Solution. (*i*) Here $s_n = 1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$

$\therefore$ $\underset{n \to \infty}{\text{Lt}} s_n = \dfrac{1}{2} \underset{n \to \infty}{\text{Lt}} n(n+1) \to \infty$. Hence this series is *divergent*.

(*ii*) Here $s_n = 5 - 4 - 1 + 5 - 4 - 1 + 5 - 4 - 1 + \ldots n$ terms

$= 0, 5$ or 1 according as the number of terms is $3m, 3m + 1, 3m + 2$.

Clearly in this case, s_n does not tend to a unique limit. Hence the series is *oscillatory*.

Examples 9.3. Geometric series. *Show that the series* $1 + r + r^2 + r^3 + ... \infty$ *(i) converges if* $|r| < 1$, *(ii) diverges if* $r \geq 1$, *and (iii) oscillates if* $r \leq -1$.

Solution. Let $s_n = 1 + r + r^2 + ... + r^{n-1}$

Case I. When $|r| < 1$, $\underset{n \to \infty}{\text{Lt}} \ r^n = 0$.

Also $$s_n = \frac{1-r^n}{1-r} = \frac{1}{1-r} - \frac{r^n}{1-r} \text{ so that } \underset{n \to \infty}{\text{Lt}} \ s_n = \frac{1}{1-r}$$

$\therefore$ the series is convergent.

Case II. (*i*) When $r > 1$, $\underset{n \to \infty}{\text{Lt}} \ r^n \to \infty$.

Also $$s_n = \frac{r^n - 1}{r-1} = \frac{r^n}{r-1} - \frac{1}{r-1} \text{ so that } \underset{n \to \infty}{\text{Lt}} \ s_n \to \infty$$

$\therefore$ the series is divergent.

(*ii*) When $r = 1$, then $s_n = 1 + 1 + 1..... + 1 = n$

and $\underset{n \to \infty}{\text{Lt}} \ s_n \to \infty$ $\therefore$ The series is divergent.

Case III. (*i*) When $r = -1$, then the series becomes $1 - 1 + 1 - 1 + 1 - 1$ ... which is an oscillatory series.

(*ii*) When $r < -1$, let $r = -\rho$ so that $\rho > 1$. Then $r^n = (-1)^n \rho^n$

and $$s_n = \frac{1-r^n}{1-r} = \frac{1-(-1)^n \rho^n}{1+\rho} \text{ as } \underset{n \to \infty}{\text{Lt}} \ \rho^n \to \infty.$$

$\therefore$ $\underset{n \to \infty}{\text{Lt}} \ s_n \to -\infty$ or $+\infty$ according as n is even or odd. Hence the series oscillates.

PROBLEMS 9.2

Examine the following series for convergence :

1. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ..., \infty$.
2. $1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \frac{1}{3^4} - ... \infty$.
3. $6 - 10 + 4 + 6 - 10 + 4 + 6 - 10 + 4 + ... \infty$.
4. $\frac{1}{1 \cdot 2} + \frac{1}{2.3} + \frac{1}{3.4} + ... \infty$. (*V.T.U., 2006*)
5. A ball is dropped from a height h metres. Each time the ball hits the ground, it rebounds a distance r times the distance fallen where $0 < r < 1$. If $h = 3$ metres and $r = 2/3$, find the total distance travelled by the ball.

9.4 GENERAL PROPERTIES OF SERIES

The truth of the following properties is self-evident and these may be regarded as axioms :

1. *The convergence or divergence of an infinite series remains unaffected by the addition or removal of a finite number of its terms* ; for the sum of these terms being the finite quantity does not on addition or removal alter the nature of its sum.

2. *If a series in which all the terms are positive is convergent, the series remains convergent even when some or all of its terms are negative* ; for the sum is clearly the greatest when all the terms are positive.

3. *The convergence or divergence of an infinite series remains unaffected by multiplying each term by a finite number.*

9.5 SERIES OF POSITIVE TERMS

1. *An infinite series in which all the terms after some particular terms are positive, is a positive term series. e.g.*, $-7 - 5 - 2 + 2 + 7 + 13 + 20 + ...$ is a positive term series as all its terms after the third are positive.

2. *A series of positive terms either converges or diverges to* $+\infty$; for the sum of its first n terms, omitting the negative terms, tends to either a finite limit or $+\infty$.

3. **Necessary condition for convergence.** *If a positive term series Σu_n is convergent, then $\underset{n \to \infty}{Lt}\ u_n = 0$.*

(P.T.U., 2009)

Let $s_n = u_1 + u_2 + u_3 + \dots + u_n$. Since Σu_n is given to be convergent.

$\therefore$ $\underset{n \to \infty}{\text{Lt}}\ s_n =$ a finite quantity k (say). Also $\underset{n \to \infty}{\text{Lt}}\ s_{n-1} = k$

But $u_n = s_n - s_{n-1}$ $\therefore$ $\underset{n \to \infty}{\text{Lt}}\ u_n = \underset{n \to \infty}{\text{Lt}}\ (s_n - s_{n-1}) = 0.$

Hence the result.

Obs. 1. *It is important to note that the converse of this result is not true.*

Consider, for instance, the series $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} + \dots \infty$

Since the term go on descending,

$$\therefore \quad s_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \dots + \frac{1}{\sqrt{n}} = \frac{n}{\sqrt{n}} \quad i.e., \ \sqrt{n}$$

$$\therefore \quad \underset{n \to \infty}{\text{Lt}}\ s_n = \underset{n \to \infty}{\text{Lt}}\ \sqrt{n} \to \infty$$

Thus the series is divergent even though $\underset{n \to \infty}{\text{Lt}}\ u_n = \underset{n \to \infty}{\text{Lt}}\ \frac{1}{\sqrt{n}} = 0$

Hence $\underset{n \to \infty}{Lt}\ u_n = 0$ is a necessary but not sufficient condition for convergence of Σn_n.

Obs. The above result leads to a simple test for divergence :

If $\underset{n \to \infty}{Lt}\ u_n \neq 0$, the series Σu_n must be divergent.

9.6 COMPARISON TESTS

I. If two positive term series Σu_n and Σv_n be such that

(i) Σv_n converges, *(ii) $u_n \leq v_n$ for all values of n, then Σu_n also converges.*

Proof. Since Σv_n is convergent,

$$\therefore \quad \underset{n \to \infty}{\text{Lt}}\ (v_1 + v_2 + v_3 + \dots + v_n) = \text{a finite quantity } k \text{ (say)}$$

Also since $u_1 \leq v_1, u_2 \leq v_2, \dots, u_n \leq v_n$

$\therefore$ Adding, $u_1 + u_2 + \dots + u_n \leq v + v_2 + \dots + v_n$

$$\therefore \quad \underset{n \to \infty}{\text{Lt}}\ (u_1 + u_2 + \dots + u_n) \leq \underset{n \to \infty}{\text{Lt}}\ (v_1 + v_2 + \dots + v_n) = k.$$

Hence the series Σu_n also converges.

Obs. If, however, the relation $u_n \leq v_n$ holds for values of n greater than a fixed number m, then the first m terms of both the series can be ignored without affecting their convergence or divergence.

II. If two positive term series Σu_n and Σv_n be such that :

(i) Σv_n diverges, *(ii) $u_n \geq v_n$ for all values of n, then Σu_n also diverges.*

Its proof is similar to that of Test *I*.

III. **Limit form**

If two positive term series Σu_n and Σv_n be such that

$$\underset{n \to \infty}{Lt}\ \frac{u_n}{v_n} = \textit{finite quantity } (\neq 0), \textit{ then } \Sigma u_n \textit{ and } \Sigma v_n \textit{ converge or diverge together.}$$

Proof. Since $\underset{n \to \infty}{\text{Lt}}\ \frac{u_n}{v_n} = l$, a finite number $(\neq 0)$

By definition of a limit, there exists a positive number ε, however small, such that

$$\left| \frac{u_n}{v_n} - l \right| < \varepsilon \qquad \text{for } n \geq m$$

or $$-\varepsilon < \frac{u_n}{v_n} - l < \varepsilon \qquad \text{for } n \geq m$$

or $$l - \varepsilon < \frac{u_n}{v_n} < l + \varepsilon \qquad \text{for } n \geq m$$

Omitting the first m terms of both the series, we have

$$l - \varepsilon < \frac{u_n}{v_n} < l + \varepsilon \qquad \text{for all } n \qquad ...(1)$$

Case I. When Σv_n is convergent, then

$$\underset{n \to \infty}{\text{Lt}} (v_1 + v_2 + ... + v_n) = k, \text{ a finite number} \qquad ...(2)$$

Also from (1), $\frac{u_n}{v_n} < l + \varepsilon$, *i.e.*, $u_n < (l + \varepsilon)v_n$ for all n.

$$\therefore \quad \underset{n \to \infty}{\text{Lt}} (u_1 + u_2 + ... + u_n) < (l + \varepsilon) \underset{n \to \infty}{\text{Lt}} (v_1 + v_2 + ... + v_n) = (l + \varepsilon)k \qquad \text{[By (2)]}$$

Hence Σu_n is also convergent.

Case II. When Σv_n is divergent, then

$$\underset{n \to \infty}{\text{Lt}} (v_1 + v_2 + ... + v_n) \to \infty \qquad ...(3)$$

Also from (1) $l - \varepsilon < \frac{u_n}{v_n}$ or $u_n > (l - \varepsilon)v_n$ for all n

$$\therefore \quad \underset{n \to \infty}{\text{Lt}} (u_1 + u_2 + ... + u_n) > (l - \varepsilon) \underset{n \to \infty}{\text{Lt}} (v_1 + v_2 + ... + v_n) \to \infty \qquad \text{[By (3)]}$$

Hence Σu_n is also divergent.

9.7 INTEGRAL TEST

A positive term series $f(1) + f(2) + ... + f(n) + ...$, where $f(n)$ decreases as n increases, converges or diverges according as the integral

$$\int_1^\infty f(x)\,dx \qquad ...(1) \text{ is finite or infinite.}$$

The area under the curve $y = f(x)$, between any two ordinates lies between the set of inscribed and escribed rectangles formed by ordinates at $x = 1, 2, 3, ...$ as in Fig. 9.1. Then

$$f(1) + f(2) + ... + f(n) \geq \int_1^{n+1} f(x)\,dx \geq f(2) + f(3) + ... + f(n + 1)$$

or $$s_n \geq \int_1^{n+1} f(x)\,dx \geq s_{n+1} - f(1)$$

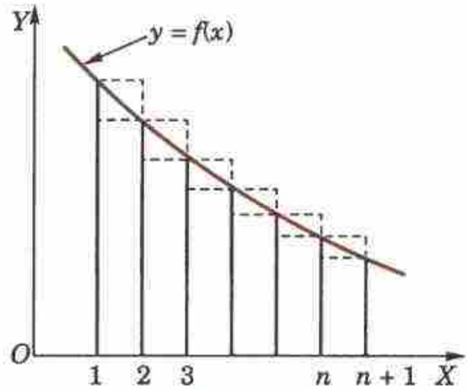

Fig. 9.1

Taking limits as $n \to \infty$, we find from the second inequality that $\text{Lt } s_{n+1} \leq \int_1^\infty f(x)\,dx + f(1)$.

Hence if integral (1) is finite, so is Lt s_{n+1}. Similarly, from the first inequality, we see that if the integral (1) is infinite, so is Lt s_n. But the given series either converges or diverges to $+ \infty$, *i.e.*, Lt s_n is either finite or infinite as $n \to \infty$.

Hence the result follows.

Example 9.4. Test for Comparison. *Show that the p-series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + ... \infty$$

(i) converges for $p > 1$ (ii) diverges for $p \leq 1$. *(P.T.U., 2009 ; V.T.U., 2006 ; Rohtak, 2003)*

Solution. By the above test, this series will converge or diverge according as $\int_1^\infty \frac{dx}{x^p}$ is finite or infinite.

If $p \neq 1$, $$\int_1^\infty \frac{dx}{x^p} = \underset{m\to\infty}{\text{Lt}} \int_1^m \frac{dx}{x^p} = \underset{m\to\infty}{\text{Lt}} \left(\frac{m^{1-p}-1}{1-p}\right)$$

$$= \frac{1}{p-1}, \text{ i.e. finite} \quad \text{for } p > 1$$

$$\to \infty \quad \text{for } p < 1$$

If $p = 1$, $\int_1^\infty \frac{dx}{x} = \int_1^\infty \log x \to \infty$, this proves the result.

Obs. Application of comparison tests. Of all the above tests the 'limit form' is the most useful. To apply this comparison test to a given series Σu_n, the auxiliary series Σv_n must be so chosen that Lt (u_n/v_n) is non-zero and finite. To do this, we take v_n equal to that term of u_n which is of the highest degree in $1/n$ and the convergence or divergence of v_n is known with the help of the above series.

Example 9.5. *Test for convergence the series*

(*i*) $\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \ldots \infty$ (*P.T.U., 2009*)

(*ii*) $\frac{1}{4.7.10} + \frac{4}{7.10.13} + \frac{9}{10.13.16} + \ldots \infty$ (*V.T.U., 2010*)

(*iii*) $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \ldots \infty$.

Solution. (*i*) We have $u_n = \frac{2n-1}{n(n+1)(n+2)} = \frac{1}{n^2} \frac{2-1/n}{(1+1/n)(1+2/n)}$

Take $v_n = 1/n^2$; then

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{2-1/n}{(1+1/n)(1+2/n)} = \frac{2-0}{(1+0)(1+0)}$$

$$= 2, \text{ which is finite and non-zero}$$

$\therefore$ both Σu_n and Σv_n converge or diverge together.
But $\Sigma v_n = \Sigma 1/n^2$ is known to be convergent.
Hence Σu_n is also convergent.

(*ii*) Here $$u_n = \frac{n^2}{(3n+1)(3n+4)(3n+7)} = \frac{1}{n\left(3+\frac{1}{n}\right)\left(3+\frac{4}{n}\right)\left(3+\frac{7}{n}\right)}$$

Taking $v_n = \frac{1}{n}$, we find that

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{1}{\left(3+\frac{1}{n}\right)\left(3+\frac{4}{n}\right)\left(3+\frac{7}{n}\right)} = \frac{1}{27} \neq 0$$

Now since Σv_n is divergent, therefore Σu_n is also divergent.

(*iii*) Here $u_n = \frac{n^n}{(n+1)n+1} = \frac{1}{n+1} \cdot \left(\frac{n}{n+1}\right)^n$, ignoring the first term.

Taking $v_n = 1/n$, we have

$$\underset{n\to\infty}{\text{Lt}} \left(\frac{u_n}{v_n}\right) = \underset{n\to\infty}{\text{Lt}} \frac{n}{n+1} \cdot \underset{n\to\infty}{\text{Lt}} \left(\frac{n}{n+1}\right)^n$$

$$= \underset{n\to\infty}{\text{Lt}} \left(\frac{1}{1+1/n}\right) \cdot \underset{n\to\infty}{\text{Lt}} \frac{1}{(1+1/n)^n} = 1 \cdot \frac{1}{e} \neq 0$$

Now since Σv_n is divergent, therefore Σu_n is also divergent.

Example 9.6. *Test the convergence of the series :*

(*i*) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{(n+1)}}$ (*V.T.U., 2008*) (*ii*) $\sum_{n=1}^{\infty} \frac{1}{x^n + x^{-n}}$ (*iii*) $\sum_{n=1}^{\infty} \sqrt{\frac{3^n-1}{2^n+1}}$ (*V.T.U., 2000 S*)

Solution. (*i*) We have $u_n = \frac{\sqrt{(n+1)}-\sqrt{n}}{[\sqrt{(n+1)}+\sqrt{n}][\sqrt{(n+1)}-\sqrt{n}]} = \sqrt{(n+1)}-\sqrt{n}$

$= \sqrt{n}\,[(1+1/n)^{1/2}-1]$ (Expanding by Binomial Theorem)

$$= \sqrt{n}\left\{\left(1+\frac{1}{2n}-\frac{1}{8n^2}+\dots\right)-1\right\} = \sqrt{n}\left(\frac{1}{2n}-\frac{1}{8n^2}+\dots\right) = \frac{1}{\sqrt{n}}\left(\frac{1}{2}-\frac{1}{8n}+\dots\right)$$

Taking $v_n = 1/\sqrt{n}$, we have

$$\underset{u\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}}\left(\frac{1}{2}-\frac{1}{8n}+\dots\right) = \frac{1}{2}, \text{ which is finite and non-zero.}$$

$\therefore$ both Σu_n and Σv_n converge or diverge together.

But $\Sigma v_n = \Sigma 1/\sqrt{n}$ is known to be divergent. Hence Σu_n is also divergent.

(*ii*) *When* $x < 1$, comparing the given series Σu_n with $\Sigma v_n = \Sigma x^n$,

we get $$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}}\left(\frac{1}{x^n+x^{-n}}\cdot\frac{1}{x^n}\right) = \underset{n\to\infty}{\text{Lt}} \frac{1}{x^{2n}+1} = 1 \qquad [\because\ x^{2n}\to 0 \text{ as } n\to\infty]$$

But Σv_n is convergent, so Σu_n is also convergent.

When $x > 1$, comparing Σu_n with $\Sigma w_n = \Sigma x^{-n}$, we get

$$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{w_n} = \underset{n\to\infty}{\text{Lt}}\left(\frac{1}{x^n+x^{-n}}\cdot x^n\right) = \underset{n\to\infty}{\text{Lt}} \frac{1}{1+x^{-2n}} = 1. \qquad [\because\ x^{-2n}\to 0 \text{ as } n\to\infty]$$

But Σw_n is convergent, so Σu_n is also convergent.

When $x = 1$, $\Sigma u_n = \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\dots\infty$ which is divergent.

Hence, Σu_n converges for $x < 1$ and $x > 1$ but diverges for $x = 1$.

(*iii*) Here $$u_n = \sqrt{\frac{3^n-1}{2^n+1}} = \left(\frac{3}{2}\right)^{n/2}\sqrt{\left(\frac{1-1/3^n}{1+1/2^n}\right)}$$

Taking $$v_n = \left(\frac{3}{2}\right)^{n/2}, \text{ we get}$$

$$\underset{n\to\infty}{\text{Lt}}\left(\frac{u_n}{v_n}\right) = \underset{n\to\infty}{\text{Lt}}\sqrt{\left(\frac{1-1/3^n}{1+1/2^n}\right)} = 1 \neq 0$$

Also since $\Sigma v_n = r^n$ where $r = \sqrt{3/2}$ is a geometric series having $r > 1$, is divergent.

$\therefore$ Σu_n is also divergent.

Example 9.7. *Determine the nature of the series :*

(*i*) $\frac{\sqrt{2}-1}{3^3-1}+\frac{\sqrt{3}-1}{4^3-1}+\frac{\sqrt{4}-1}{5^3-1}+\dots\infty$ (*ii*) $\sum \frac{1}{n}\sin\frac{1}{n}$

(*iii*) $\sum_{1}^{\infty} \frac{(\log n)^2}{n^{3/2}}$ (*iv*) $\sum_{2}^{\infty} \frac{1}{n(\log n)\,p}\ (p > 0)$ (*P.T.U., 2010*)

Solution. (*i*) We have $u_n = \frac{\sqrt{(n+1)}-1}{(n+2)^3-1} = \frac{\sqrt{n}[(1+1/n)-1/\sqrt{n}]}{n^3[(1+2/n)^3-1/n^3]}$

Taking $v_n = \frac{1}{n^{5/2}}$, we find that

$$\underset{n \to \infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n \to \infty}{\text{Lt}} \frac{[\sqrt{(1+1/n)} - 1/\sqrt{n}]}{[(1+2/n)^3 - 1/n^3]} = 1 \neq 0$$

Since Σv_n is convergent, therefore Σu_n is also convergent.

(*ii*) Here $$u_n = \frac{1}{n} \sin \frac{1}{n} = \frac{1}{n}\left[\frac{1}{n} - \frac{1}{3!\,n^3} + \frac{1}{5!\,n^5} - \ldots\right] = \frac{1}{n^2}\left[1 - \frac{1}{3!\,n^2} + \frac{1}{5!\,n^4} - \ldots\right]$$

Taking $v_n = \frac{1}{n^2}$, we have

$$\underset{n \to \infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n \to \infty}{\text{Lt}} \left[1 - \frac{1}{3!\,n^2} + \frac{1}{5!\,n^4} \ldots\right] = 1 \neq 0$$

Since Σv_n is convergent, therefore Σu_n is also convergent.

(*iii*) We have $\underset{n \to \infty}{\text{Lt}} \frac{(\log n)^2}{n^{1/4}} = 0$, *i.e.*, $\frac{(\log n)^2}{n^{1/4}} < 1$ or $(\log n)^2 < n^{1/4}$

$$\therefore \quad u_n = \frac{(\log n)^2}{n^{3/2}} < \frac{n^{1/4}}{n^{3/2}} = \frac{1}{n^{5/4}}$$

Since $\Sigma 1/n^{5/4}$ converges by p-series. $(\because \ p = 5/4 > 1)$

Hence by comparison test, Σu_n also converges.

(*iv*) Let $$f(n) = \frac{1}{n(\log n)^p} \text{ so that } f(x) = \frac{(\log x)^{-p}}{x}$$

$$\therefore \quad f'(x) = \frac{-p}{x}(\log x)^{-p-1} \cdot \frac{1}{x} + (\log x)^{-p} \cdot \left(-\frac{1}{x^2}\right) = -\frac{1}{x^2}\left\{\frac{p}{(\log x)^{p+1}} + \frac{1}{(\log x)^p}\right\} < 0$$

i.e., $f(x)$ is a decreasing function.

Also $$\int_2^\infty f(x)\,dx = \int_2^\infty \frac{dx}{x(\log x)^p} = \left|\frac{(\log x)^{-p+1}}{-p+1}\right|_2^\infty$$

If $p > 1$, then $p - 1 = k$ (say) > 0

$$\therefore \quad \int_2^\infty f(x)\,dx = \left|\frac{(\log x)^{-k}}{-k}\right|_2^\infty = \frac{1}{k}\,[0 + (\log 2)^{-k}] \text{ which is finite}$$

Thus by integral test, *the given series converges for* $p > 1$.

If $p < 1$, then $1 - p > 0$ and $(\log x)^{1-p} \to \infty$ as $x \to \infty$.

$$\therefore \quad \int_2^\infty f(x)\,dx \to \infty.$$

Thus *the given series diverges for* $p < 1$.

If $p = 1$, then $$\int_2^\infty f(x)\,dx = \int_2^\infty \frac{dx}{x \log x} = |\log(\log x)|_2^\infty \to \infty$$

Thus *the given series diverges for* $p = 1$.

PROBLEMS 9.3

Test the following series for convergence :

1. $1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \frac{1}{3^4} - \ldots \infty$ (*J.N.T.U., 2000*)
2. $\frac{3}{5} + \frac{4}{5^2} + \frac{3}{5^3} + \frac{4}{5^4} + \ldots \infty$
3. $\frac{1}{1.2} + \frac{2}{3.4} + \frac{3}{5.6} + \ldots \infty$ (*Cochin, 2001*)
4. $\frac{1}{1.3} + \frac{2}{3.5} + \frac{3}{5.7} + \ldots \infty$ (*P.T.U., 2009*)

5. $1+\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+\ldots\infty$

6. $\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\frac{5}{4^p}+\ldots\infty$

7. $\frac{1}{1^2}+\frac{1+2}{1^2+2^2}+\frac{1+2+3}{1^2+2^2+3^2}+\ldots\infty$

8. $\frac{1}{1.3.5}+\frac{2}{3.5.7}+\frac{3}{5.7.9}+\ldots\infty$ (V.T.U., 2009 S)

9. $\frac{3}{1}+\frac{4}{8}+\frac{5}{27}+\frac{6}{64}+\ldots\infty$

10. $\sum\frac{\sqrt{n}}{n^2+1}$ (Osmania, 2000 S)

11. $\sum_{n=0}^{\infty}\frac{2n^3+5}{4n^5+1}$

12. $\sum\frac{(n+1)(n+2)}{n^2\sqrt{n}}$ (J.N.T.U., 2006 S)

13. $\sum_{n=1}^{\infty}[\sqrt{(n^2+1)}-n]$ (V.T.U., 2010 ; P.T.U., 2009)

14. $\sum[\sqrt[3]{(n^3+1)}-n]$ (P.T.U., 2007 ; Rohtak 2003)

15. $\sum[\sqrt{(n^4+1)}-\sqrt{(n^4-1)}]$

16. $\sum\frac{1}{\sqrt{n}}\sin\frac{1}{n}$

17. $\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}\log n}$

18. $\sum_{n=1}^{\infty}\frac{\sqrt{(n+1)}-1}{(n+2)^3-1}$ (J.N.T.U., 2003)

9.8 COMPARISON OF RATIOS

If Σu_n and Σv_n be two positive term series, then Σu_n converges if (i) Σv_n converges, and (ii) from and after some particular term,

$$\frac{u_{n+1}}{u_n}<\frac{u_{n+1}}{v_n}$$

Let the two series beginning from the particular term be $u_1+u_2+u_3+\ldots$ and $v_1+v_2+v_3+\ldots$

If $\frac{u_2}{u_1}<\frac{v_2}{v_1},\frac{u_3}{u_2}<\frac{v_3}{v_2},\ldots$

then $$u_1+u_2+u_3+\ldots=u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_1}+\ldots\right)$$

$$=u_1\left(1+\frac{u_2}{u_1}+\frac{u_2}{u_1}\cdot\frac{u_3}{u_2}+\ldots\right)<u_1\left(1+\frac{v_2}{v_1}+\frac{v_2}{v_1}\cdot\frac{v_3}{v_2}+\ldots\right)<\frac{u_1}{v_1}(v_1+v_2+v_3+\ldots).$$

Hence, if Σv_n converges, Σu_n also converges.

Obs. A more convenient form of the above test to apply is as follows :

Σu_n converges if (i) Σv_n converges and (ii) from and after a particular term $\frac{u_n}{u_{n+1}}>\frac{v_n}{v_{n+1}}$.

Similarly, Σu_n diverges, if (i) Σv_n diverges and (ii) from and after a particular term $\frac{u_n}{u_{n+1}}<\frac{v_n}{v_{n+1}}$.

9.9 D'ALEMBERT'S RATIO TEST*

In a positive term series Σu_n, if

$$\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\lambda,$$ *then the series converges for $\lambda<1$ and diverges for $\lambda>1$.*

Case I. *When* $\underset{n\to\infty}{Lt}\frac{u_{n+1}}{u_n}=\lambda<1$.

*Called after the French mathematician *Jean le-Rond d'Alembert* (1717–1783), who also made important contributions to mechanics.

By definition of a limit, we can find a positive number r (< 1) such that $\frac{u_{n+1}}{u_n} < r$ for all $n > m$

Leaving out the first m terms, let the series be $u_1 + u_2 + u_3 + \ldots$

so that $\frac{u_2}{u_1} < r, \frac{u_3}{u_2} < r, \frac{u_4}{u_3} < r, \ldots$ and so on. Then $u_1 + u_2 + u_3 + \ldots \infty$

$$= u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \frac{u_4}{u_3}\cdot\frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \ldots \infty\right) < u_1(1 + r + r^2 + r^3 + \ldots \infty)$$

$= \frac{u_1}{1-r}$, which is finite quantity. Hence Σu_n is convergent. $[\because \; r < 1]$

Case II. *When* $\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \lambda > 1$

By definition of limit, we can find m, such that $\frac{u_{n+1}}{u_n} \geq 1$ for all $n \geq m$.

Leaving out the first m terms, let the series be

$$u_1 + u_2 + u_3 + \ldots \text{ so that } \frac{u_2}{u_1} \geq 1, \quad \frac{u_3}{u_2} \geq 1, \quad \frac{u_4}{u_3} \geq 1 \text{ and so on.}$$

$$\therefore \quad u_1 + u_2 + u_3 + u_4 + \ldots + u_n = u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \ldots\right)$$

$$\geq u_1(1 + 1 + 1 + \ldots \text{ to } n \text{ terms}) = nu_1$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} (u_1 + u_2 + \ldots + u_n)$$

$\geq \underset{n\to\infty}{\text{Lt}} (nu_1)$, which tends to infinity. Hence Σu_n is divergent.

Obs. 1. *Ratio test fails when $\lambda = 1$. Consider, for instance, the series $\Sigma u_n = \Sigma 1/n^p$.*

Here $$\lambda = \underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{\text{Lt}} \left[\frac{1}{(n+1)^p}\cdot\frac{n^p}{1}\right] = \underset{n\to\infty}{\text{Lt}} \frac{1}{(1+1/n)^p} = 1.$$

Then for all values of p, $\lambda = 1$; whereas $\Sigma 1/n^p$ converges for $p > 1$ and diverges for $p < 1$.

Hence $\lambda = 1$ both for convergence and divergence of Σu_n, which is absurd.

Obs. 2. *It is important to note that this test makes no reference to the magnitude of u_{n+1}/u_n but concerns only with the limit of this ratio.*

For instance in the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} + \ldots$, the ratio $\frac{u_{n+1}}{u_n} = \frac{n}{n+1} < 1$ for all finite values of n, but tends to unity as $n \to \infty$. Hence the Ratio test fails although this series is divergent.

Practical form of Ratio test. Taking reciprocals, the ratio test can be stated as follows :

In the positive term series Σu_n, if $\underset{n\to\infty}{Lt} \frac{u_n}{u_{n+1}} = k$, then the series converges for $k > 1$ and diverges for $k < 1$ but fails for $k = 1$.

Example. 9.8. *Test for convergence the series*

(*i*) $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \ldots \infty.$ *(P.T.U., 2005 ; V.T.U., 2003 ; I.S.M., 2001)*

(*ii*) $1 + \frac{2}{5}x + \frac{6}{9}x^2 + \frac{14}{17}x^3 + \ldots + \frac{2^n - 2}{2^n + 1}x^{n-1} + \ldots (x > 0).$ *(P.T.U., 2009 ; V.T.U., 2004)*

Solution. (*i*) We have $u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}}$ and $u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{(n+1)}}$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{x^{2n-2}}{(n+1)\sqrt{n}} \cdot \frac{(n+2)\sqrt{(n+1)}}{x^{2n}}$$

$$= \underset{n\to\infty}{\text{Lt}} \left[\frac{n+2}{n+1}\left(\frac{n+1}{n}\right)^{1/2}\right] x^{-2} = \underset{n\to\infty}{\text{Lt}} \left[\frac{1+2/n}{1+1/n} \cdot \sqrt{(1+1/n)}\right] x^{-2} = x^{-2}.$$

Hence Σu_n converges if $x^{-2} > 1$, *i.e.*, for $x^2 < 1$ and diverges for $x^2 > 1$.

If $x^2 = 1$, then, $u_n = \dfrac{1}{(n+1)\sqrt{n}} = \dfrac{1}{n^{3/2}} \cdot \dfrac{1}{1+1/n}$

Taking $v_n = \dfrac{1}{n^{3/2}}$, we get $\underset{n\to\infty}{\text{Lt}} \dfrac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \dfrac{1}{1+1/n} = 1$, a finite quantity.

$\therefore$ Both Σu_n and Σv_n converge or diverge together. But $\Sigma v_n = \sum \dfrac{1}{n^{3/2}}$ is a convergent series.

$\therefore$ Σu_n is also convergent. Hence the given series converges if $x^2 \le 1$ and diverges if $x^2 > 1$.

(*ii*) Here
$$\frac{u_n}{u_{n+1}} = \frac{2^n - 2}{2^n + 1} x^{n-1} \cdot \frac{2^{n+1}+1}{2^{n+1}-2} \frac{1}{x^n} = \frac{1-\dfrac{2}{2^n}}{1+\dfrac{1}{2^n}} \cdot \frac{2+\dfrac{1}{2^n}}{2-\dfrac{2}{2^n}} \frac{1}{x}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{1-0}{1+0} \cdot \frac{2+0}{2-0} \frac{1}{x} = \frac{1}{x}$$

Thus by Ratio test, Σu_n converges for $x^{-1} > 1$ *i.e.*, for $x < 1$ diverges for $x > 1$. But it fails for $x = 1$.

When $x = 1$,
$$\underset{n\to\infty}{\text{Lt}} u_n = \underset{n\to\infty}{\text{Lt}} \frac{2^n - 2}{2^n + 1} = \underset{n\to\infty}{\text{Lt}} \frac{1-\dfrac{2}{2^n}}{1+\dfrac{1}{2^n}} = 1 \ne 0$$

$\therefore$ Σu_n diverges for $x = 1$. Hence the given series converges for $x < 1$ and diverges for $x \ge 1$.

Example 9.9. *Discuss the convergence of the series*

(*i*) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(n^n)^2}$ (*P.T.U., 2010*) (*ii*) $1 + \dfrac{2!}{2^2} + \dfrac{3!}{3^3} + \dfrac{4!}{4^4} + \ldots \infty$ (*V.T.U., 2008 S*)

Solution. (*i*) We have $u_n = \dfrac{n!}{(n^n)^2}$ and $u_{n+1} = \dfrac{(n+1)!}{[(n+1)^{n+1}]^2}$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{n!}{n^{2n}} \times \frac{(n+1)^{2(n+1)}}{(n+1)!}$$

$$= \underset{n\to\infty}{\text{Lt}} \frac{(n+1)^{2n+1}}{n^{2n}} = \underset{n\to\infty}{\text{Lt}} \left(\frac{n+1}{n}\right)^{2n} .(n+1)$$

$$= \underset{n\to\infty}{\text{Lt}} \left[\left(1+\frac{1}{n}\right)^n\right]^2 .(n+1) = e.\underset{n\to\infty}{\text{Lt}} (n+1) \to \infty$$

Hence the given series is convergent.

(*ii*) Given series is $\Sigma u_n = \displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$. Here $\dfrac{u_n}{u_{n+1}} = \dfrac{n!}{n^n} \cdot \dfrac{(n+1)^{n+1}}{(n+1)!} = \dfrac{(n+1)^n}{n^n} = \left(1+\dfrac{1}{n}\right)^n$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \left(1+\frac{1}{n}\right)^n = e,$$ which is > 1. Hence the given series is convergent.

Example 9.10. *Examine the convergence of the series :*

(*i*) $\frac{x}{1+x}+\frac{x^2}{1+x^2}+\frac{x^3}{1+x^3}+\ldots\infty$

(*ii*) $1+\frac{a+1}{b+1}+\frac{(a+1)(2a+1)}{(b+1)(2b+1)}+\frac{(a+1)(2a+1)(3a+1)}{(b+1)(2b+1)(3b+1)}+\ldots\infty$

Solution. (*i*) Here $\quad u_n=\frac{x^n}{1+x^n}$ and $u_{n+1}=\frac{x^{n+1}}{1+x^{n+1}}$

$$\therefore \qquad \underset{n\to\infty}{\text{Lt}}\frac{u_n}{u_{n+1}}=\underset{n\to\infty}{\text{Lt}}\left(\frac{x^n}{x^{n+1}}\cdot\frac{1+x^{n+1}}{1+x^n}\right)=\underset{n\to\infty}{\text{Lt}}\left(\frac{1+x^{n+1}}{x+x^{n+1}}\right)$$

$$=\frac{1}{x},\text{ if } x<1. \qquad [\because\ x^{n+1}\to 0 \text{ as } n\to\infty]$$

Also $$\underset{n\to\infty}{\text{Lt}}\frac{u_n}{u_{n+1}}=\underset{n\to\infty}{\text{Lt}}\left(\frac{1+1/x^{n+1}}{1+x/x^{n+1}}\right)=1 \text{ if } x>1.$$

$\therefore$ by *Ratio test*, Σu_n converges for $x<1$ and fails for $x\geq 1$.

When $x=1$, $\Sigma u_n=\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots+\infty$, which is divergent.

Hence the given series converges for $x<1$ and diverges for $x\geq 1$.

(*ii*) Neglecting the first term, we have

$$u_{n+1}=u_n\cdot\frac{n_{a+1}}{n_{b+1}}$$

$$\therefore \qquad \underset{n\to\infty}{\text{Lt}}\frac{u_n}{u_{n+1}}=\underset{n\to\infty}{\text{Lt}}\frac{n_{b+1}}{n_{a+1}}=\underset{n\to\infty}{\text{Lt}}\frac{b+1/n}{a+1/n}=\frac{b}{a}.$$

By *Ratio test*, Σu_n converges for $b/a>1$ or $a<b$, and diverges for $a>b$.

When $a=b$, the series becomes $1+1+1+\ldots\infty$, which is divergent.

Hence the given series converges for $0<a<b$ and diverges for $0<b\leq a$.

PROBLEMS 9.4

Test for convergence the following series :

1. $x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots\infty.$

2. $\sqrt{\frac{1}{2}}x+\sqrt{\frac{2}{3}}x^2+\sqrt{\frac{3}{4}}x^3+\ldots\infty$

3. $1+\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\ldots+\frac{x^n}{n^2+1}\ldots\infty$

4. $\sum_{n=2}^{\infty}\frac{x^n}{n(n-1)(n-2)}$ (*J.N.T.U., 2006*)

5. $1+\frac{2^p}{2!}+\frac{3^p}{3!}+\frac{4^p}{4!}+\ldots\infty$ (*Kurukshetra, 2005*)

6. $\sum_{n=1}^{\infty}\left(\frac{n^2}{2^n}+\frac{1}{n^2}\right)$ (*Rohtak, 2005*)

7. $\sum_{n=1}^{\infty}\frac{n!\,3^n}{n^n}$ (*Kerala, 2005*)

8. $\sum_{n=1}^{\infty}\frac{n^3+a}{2^n+a}$

9. $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{\sqrt{(n^2+1)}}x^n$ (*P.T.U., 2006*)

10. $\sum_{1}^{\infty}\frac{n^3-n+1}{n!}$ (*Madras, 2000*)

11. $\frac{2}{3.4}+\frac{2.4}{3.5.6}+\frac{2.4.6}{3.5.7.8}+\ldots$ (*V.T.U., 2010*)

12. $\left(\frac{1}{3}\right)^2+\left(\frac{1.2}{3.5}\right)^2+\left(\frac{1.2.3}{3.5.7}\right)^2+\ldots$

13. $1+\frac{1^2 . 2^2}{1 . 3 . 5}+\frac{1^2 . 2^2 . 3^2}{1 . 3 . 5 . 7 . 9}+\ldots \infty$ *(Delhi, 2002)*

14. $\frac{4}{18}+\frac{4 . 12}{18 . 27}+\frac{4 . 12 . 20}{18 . 27 . 36}+\ldots \infty$ *(Madras, 2000)*

15. $\frac{1}{1^p}+\frac{x}{3^p}+\frac{x^2}{5^p}+\ldots+\frac{x^{n-1}}{(2n-1)^p}+\ldots \infty$ *(J.N.T.U., 2006)*

16. $\sum_{n=1}^{\infty} \frac{3 . 6 . 9 \ldots 3n}{4 . 7 . 10 \ldots (3n+1)} \cdot \frac{5^n}{3n+2}$ *(V.T.U., 2004)*

17. $1+\frac{1+\alpha}{1+\beta}+\frac{(1+\alpha)(1+2\alpha)}{(1+\beta)(1+2\beta)}+\frac{(1+\alpha)(1+2\alpha)(1+3\alpha)}{(1+\beta)(1+2\beta)(1+3\beta)}+\ldots$

9.10 FURTHER TESTS OF CONVERGENCE

When the Ratio test fails, we apply the following tests :

(1) Raabe's test*. *In the positive term series* Σu_n, if $\underset{n \to \infty}{\text{Lt}}\, n\left(\frac{u_n}{u_{n+1}}-1\right)=k$,

then the series converges for $k > 1$ and diverges for $k < 1$, but the test fails for $k = 1$.

When $k > 1$, choose a number p such that $k > p > 1$, and compare Σu_n with the series $\sum \frac{1}{n^p}$ which is convergent since $p > 1$.

$\therefore$ Σu_n will converge, if from and after some term,

$$\frac{u_n}{u_{n+1}} > \frac{(n+1)^p}{n^p} \text{ or } \left(1+\frac{1}{n}\right)^p \qquad \text{or if, } \frac{u_n}{u_{n+1}} > 1+\frac{p}{n}+\frac{p(p-1)}{2n^2}+\ldots$$

or if, $$n\left(\frac{u_n}{u_{n+1}}-1\right) > p+\frac{p(p-1)}{2n}+\ldots \quad \text{or if, } \underset{n \to \infty}{\text{Lt}}\, n\left(\frac{u_n}{u_{n+1}}-1\right) > \underset{n \to \infty}{\text{Lt}}\left[p+\frac{p(p-1)}{2n}+\ldots\right]$$

i.e., if $k > p$, which is true. Hence Σu_n is convergent.

The other case when $k < 1$ can be proved similarly.

(2) Logarithmic test. *In the positive term series Σu_n if $\underset{n \to \infty}{Lt}\left(n \log \frac{u_n}{u_{n+1}}\right)=k$,*

then the series converges for $k > 1$, and diverges for $k < 1$, but the test fails for $k = 1$.

Its proof is similar to that of Raabe's test.

Obs. 1. *Logarithmic test is a substitute for Raabe's test and should be applied when* **either** *n occurs as an exponent in u_n / u_{n+1},* **or** *evaluation of $\underset{n \to \infty}{Lt}$ becomes easier on taking logarithm of u_n / u_{n+1}.*

Obs. 2. If u_n/u_{n+1} does not involve n as an exponent or a logarithm, the series Σu_n diverges.

Example 9.11. *Test for convergence the series*

(*i*) $\sum \frac{4 . 7 \ldots (3n+1)}{1 . 2 \ldots n} x^n$ *(V.T.U., 2009 ; P.T.U., 2006 S)* (*ii*) $\sum \frac{(n!)^2}{(2n)!} x^{2n}$.

Solution. (*i*) Here $\frac{u_n}{u_{n+1}}=\frac{4 . 7 \ldots (3n+1)}{1 . 2 \ldots n} x^n \div \frac{4 . 7 \ldots (3n+4)}{1 . 2 \ldots (n+1)} x^{n+1}=\frac{n+1}{3n+4} \cdot \frac{1}{x}=\left[\frac{1+1/n}{3+4/n}\right]\frac{1}{x}$

$$\therefore \qquad \underset{n \to \infty}{\text{Lt}} \frac{u_n}{u_{n+1}}=\frac{1}{3x}.$$

*Called after the Swiss mathematician *Joseph Ludwig Raabe* (1801–1859).

Thus by *Ratio test*, the series converges for $\frac{1}{3x} > 1$, *i.e.*, for $x < \frac{1}{3}$ and diverges for $x > \frac{1}{3}$. But it fails for $x = \frac{1}{3}$. $\therefore$ Let us try the *Raabe's test.*

Now $$\frac{u_n}{u_{n+1}} = \left(1+\frac{1}{n}\right)\left(1+\frac{4}{3n}\right)^{-1}$$ [*Expand by Binomial Theorem*]

$$= \left(1+\frac{1}{n}\right)\left(1-\frac{4}{3n}+\frac{16}{9n^2}-\dots\right) = 1-\frac{1}{3n}+\frac{4}{9n^2}+\dots$$

$$\therefore \quad n\left(\frac{u_n}{u_{n+1}}-1\right) = -\frac{1}{3}+\frac{4}{9n}+\dots \quad \therefore \quad \underset{n\to\infty}{\text{Lt}}\, n\left(\frac{u_n}{u_{n+1}}-1\right) = -\frac{1}{3} \text{ which } < 1.$$

Thus by *Raabe's test,* the series diverges.

Hence the given series converges for $x < \frac{1}{3}$ and diverges for $x \geq \frac{1}{3}$.

(*ii*) Here $$\frac{u_n}{u_{n+1}} = \left(\frac{n\,!}{(n+1)\,!}\right)^2 \frac{[2(n+1)]\,!}{(2n)\,!}\cdot\frac{x^{2n}}{x^{2(n+1)}} = \frac{(2n+1)(2n+2)}{(n+1)^2}\frac{1}{x^2} = \frac{2(2n+1)}{n+1}\frac{1}{x^2}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}} \frac{2(2+1/n)}{1+1/n}\cdot\frac{1}{x^2} = \frac{4}{x^2}$$

Thus by *Ratio Test,* the series converges for $x^2 < 4$ and diverges for $x^2 > 4$. But fails for $x^2 = 4$.

When $x^2 = 4$, $$n\left(\frac{u_n}{u_{n+1}}-1\right) = n\left(\frac{2n+1}{2n+2}-1\right) = -\frac{n}{2n+2}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\, n\left(\frac{u_n}{u_{n+1}}-1\right) = -\frac{1}{2} < 1$$

Thus by *Raabe's test*, the series diverges.

Hence the given series converges for $x^2 < 4$ and diverges for $x^2 \geq 4$.

Example 9.12. *Discuss the convergence of the series*

$$x + \frac{2^2x^2}{2\,!} + \frac{3^3x^3}{3\,!} + \frac{4^4x^4}{4\,!} + \frac{5^5x^5}{5\,!} + \dots\infty$$ (*P.T.U., 2008 ; Cochin, 2005 ; Rohtak, 2003*)

Solution. Here $$\frac{u_n}{u_{n+1}} = \frac{n^nx^n}{n\,!} \div \frac{(n+1)^{n+1}\,x^{n+1}}{(n+1)\,!} = \frac{n^n}{(n+1)^n\,x} = \frac{1}{(1+1/n)^n}\cdot\frac{1}{x}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{1}{ex}.$$

Thus by Ratio test, the series converges for $x < 1/e$ and diverges for $x > 1/e$. But it fails for $x = 1/e$. Let us try the *log-test*.

Now $$\frac{u_n}{u_{n+1}} = \frac{e}{(1+1/n)^n}$$

$$\therefore \quad \log\frac{u_n}{u_{n+1}} = \log_e e - n\log\left(1+\frac{1}{n}\right) = 1 - n\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\dots\right) = \frac{1}{2n}-\frac{1}{3n^2}+\dots$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\left(n\log\frac{u_n}{u_{n+1}}\right) = \frac{1}{2},$$ which < 1. Thus by the *log-test*, the series diverges.

Hence the given series converges for $x < 1/e$ and diverges for $x \geq 1/e$.

Example 9.13. *Discuss the convergence of the* ***hypergeometric series***

$$1 + \frac{\alpha\,.\,\beta}{1\,.\,\gamma}x + \frac{\alpha(\alpha+1)\,\beta(\beta+1)}{1\,.\,2\,.\,\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\,\beta(\beta+1)(\beta+2)}{1\,.\,2\,.\,3\,.\,\gamma(\gamma+1)(\gamma+2)}x^3 + \dots\infty.$$ (*Kurukshetra, 2005*)

Solution. Neglecting the first term, we have

$$u_{n+1} = u_n \frac{(\alpha + n)(\beta + n)}{(n+1)(\gamma + n)} x$$

$$\therefore \quad \underset{n \to \infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \underset{n \to \infty}{\text{Lt}} \frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)} \cdot \frac{1}{x} = \underset{n \to \infty}{\text{Lt}} \frac{(1+1/n)(1+\gamma/n)}{(1+\alpha/n)(1+\beta/n)} \cdot \frac{1}{x} = \frac{1}{x}$$

$\therefore$ by *Ratio test,* the series converges for $1/x > 1$, *i.e.*, for $x < 1$, and diverges for $x > 1$. But it fails for $x = 1$.

$\therefore$ let us try the *Raabe's test.*

$$\underset{n \to \infty}{\text{Lt}}\, n\left(\frac{u_n}{u_{n+1}} - 1\right) = \underset{n \to \infty}{\text{Lt}}\, n\left\{\frac{(n+1)(n+\gamma)}{(n+\alpha)(n+\beta)} - 1\right\} = \underset{n \to \infty}{\text{Lt}}\, n\left\{\frac{n(1+\gamma-\alpha-\beta)+\gamma-\alpha\beta}{n^2+n(\alpha+\beta)+\alpha\beta}\right\}$$

$$= \underset{n \to \infty}{\text{Lt}} \left\{\frac{(1+\gamma-\alpha-\beta)+(\gamma-\alpha\beta)\dfrac{1}{n}}{1+(\alpha+\beta)\dfrac{1}{n}+\alpha\beta\cdot\dfrac{1}{n^2}}\right\} = 1+\gamma-\alpha-\beta$$

Thus the series converges for $1+\gamma-\alpha-\beta > 1$, *i.e.*, for $\gamma > \alpha+\beta$ and diverges for $\gamma < \alpha+\beta$. But it fails for $\gamma = \alpha+\beta$. Since u_n/u_{n+1} does not involve n as an exponent or a logarithm, the series $\sum u_n$ diverges for $\gamma = \alpha + \beta$.

Hence the series converges for $x < 1$ and diverges for $x > 1$. When $x = 1$, the series converges for $\gamma > \alpha + \beta$ and diverges for $\gamma \le \alpha + \beta$.

PROBLEMS 9.5

Test the following series for convergence :

1. $\dfrac{x}{1.2} + \dfrac{x^2}{3.4} + \dfrac{x^3}{5.6} + \dfrac{x^4}{7.8} + \ldots \infty \ (x > 0)$ *(Mumbai, 2009)*
2. $\dfrac{x}{1.2} + \dfrac{x^2}{2.3} + \dfrac{x^3}{3.4} + \dfrac{x^4}{4.5} + \ldots \infty$ *(V.T.U., 2008 ; J.N.T.U., 2003)*
3. $1 + \dfrac{1}{2}x + \dfrac{1.3}{2.4}x^2 + \dfrac{1.3.5}{2.4.6}x^3 + \ldots \infty \ (x > 0)$ *(Raipur, 2005)*
4. $1 + \dfrac{2}{3}x + \dfrac{2.3}{3.5}x^2 + \dfrac{2.3.4}{3.5.7}x^3 + \ldots \infty$ *(V.T.U., 2009 S)*
5. $1 + \dfrac{x}{2} + \dfrac{2!}{3^2}x^2 + \dfrac{3!}{4^3}x^3 + \dfrac{4!}{5^4}x^4 + \ldots \infty$
6. $1 + \dfrac{3}{7}x + \dfrac{3.6}{7.10}x^2 + \dfrac{3.6.9}{7.10.13}x^3 + \dfrac{3.6.9.12}{7.10.13.16}x^4 + \ldots \infty$
7. $\dfrac{x}{1} + \dfrac{1}{2}\cdot\dfrac{x^3}{3} + \dfrac{1.3}{2.4}\cdot\dfrac{x^5}{5} + \dfrac{1.3.5}{2.4.6}\cdot\dfrac{x^7}{7} + \ldots \infty.\ (x > 0)$ *(V.T.U., 2007 ; Raipur, 2005)*
8. $1 + \dfrac{1}{2}\cdot\dfrac{x^2}{4} + \dfrac{1.3.5}{2.4.6}\cdot\dfrac{x^4}{8} + \dfrac{1.3.5.7.9}{2.4.6.8.10}\cdot\dfrac{x^6}{12} + \ldots \infty$ *(Rohtak, 2006 S ; Roorkee, 2000)*
9. $1 + \dfrac{(1!)^2}{2!}x^2 + \dfrac{(2!)^2}{4!}x^4 + \dfrac{(3!)^2}{6!}x^6 + \ldots \infty \ (x > 0)$
10. $\dfrac{1^2}{4^2} + \dfrac{1^2.5^2}{4^2.8^2} + \dfrac{1^2.5^2.9^2}{4^2.8^2.12^2} + \ldots \infty$
11. $\dfrac{a+x}{1!} + \dfrac{(a+2x)^2}{2!} + \dfrac{(a+3x)^3}{3!} + \ldots \infty$
12. $x^2(\log 2)^q + x^3(\log 3)^q + x^4(\log 4)^q + \ldots \infty$
13. $\dfrac{1}{1^2} + \dfrac{1+2}{1^2+2^2} + \dfrac{1+2+3}{1^2+2^2+3^2} + \ldots$ *(V.T.U., 2000)*
14. $1 + \dfrac{a}{b}x + \dfrac{a(a+1)}{b(b+1)}x^2 + \dfrac{a(a+1)(a+2)}{b(b+1)(b+2)}x^3 + \ldots \infty \ (a, b > 0, x > 0).$

9.11 CAUCHY'S ROOT TEST*

In a positive series Σu_n, if $\underset{n\to\infty}{Lt} (u_n)^{1/n} = \lambda$,

then the series converges for $\lambda < 1$, and diverges for $\lambda > 1$.

Case I. When $\underset{n\to\infty}{\text{Lt}} (u_n)^{1/n} = \lambda < 1$.

By definition of a limit, we can find a positive number r ($\lambda < r < 1$) such that

$$(u_n)^{1/n} < r \text{ for all } n > m, \quad \text{or} \quad u_n < r^n \text{ for all } n > m.$$

Since $r < 1$, the geometric series Σr^n is convergent. Hence, by comparison test, Σu_n is also convergent.

Case II. When $\text{Lt}\,(u_n)^{1/n} = \lambda > 1$.

By definition of a limit, we can find a number m, such that

$$(u_n)^{1/n} > 1 \text{ for all } n > m, \quad \text{or} \quad u_n > 1 \text{ for all } n > m.$$

Omitting the first m terms, let the series be $u_1 + u_2 + u_3 + \ldots$. so that $u_1 > 1$, $u_2 > 1$, $u_3 > 1$ and so on.

$\therefore \qquad u_1 + u_2 + u_3 + \ldots + u_n > n \quad \text{and} \quad \underset{n\to\infty}{\text{Lt}} (u_1 + u_2 + \ldots + u_n) \to \infty$

Hence the series Σu_n is divergent.

Obs. *Cauchy's root test fails when $\lambda = 1$.*

Example 9.14. *Test for convergence the series*

(i) $\Sigma \dfrac{n^3}{3^n}$ (ii) $\Sigma (\log n)^{-2n}$ (iii) $\Sigma (1 + 1/\sqrt{n})^{-n^{3/2}}$ (P.T.U., 2009 ; Kurukshetra, 2005)

Solution. (*i*) We have $u_n = n^3/3^n$.

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} (u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}} \left(\frac{n^{3/n}}{3}\right) = \underset{n\to\infty}{\text{Lt}} \frac{(n^{1/n})^3}{3} = \frac{1}{3} \ (< 1) \qquad \left[\because \underset{n\to\infty}{\text{Lt}} n^{1/n} = 1\right]$$

Hence the given series converges by Cauchy's root test.

(*ii*) Here $\qquad u_n = (\log n)^{-2n}$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} (u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}} (\log n)^{-2} = 0 \ (< 1) \qquad [\because \underset{n\to\infty}{\text{Lt}} \log n = 0]$$

Hence, by Cauchy's root test, the given series converges.

(*iii*) Here $\qquad u_n = (1 + 1/\sqrt{n})^{-n^{3/2}}$

$$\therefore \quad (u_n)^{1/n} = \left[\frac{1}{(1+1/\sqrt{n})^{n^{3/2}}}\right]^{1/n} = \frac{1}{(1+1/\sqrt{n})^{\sqrt{n}}}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} (u_n)^{1/n} = \underset{n\to\infty}{\text{Lt}} \frac{1}{(1+1/\sqrt{n})^{\sqrt{n}}} = \frac{1}{e}$$, which is < 1. Hence the given series is convergent.

Example 9.15. *Discuss the nature of the following series :*

(i) $\dfrac{1}{2} + \dfrac{2}{3}x + \left(\dfrac{3}{4}\right)^2 x^2 + \left(\dfrac{4}{5}\right)^3 x^3 + \ldots \infty \ (x > 0)$ (J.N.T.U., 2006)

(ii) $\Sigma \dfrac{(n+1)^n \, x^n}{x^{n+1}}$

(iii) $\left(\dfrac{2^2}{1^2} - \dfrac{2}{1}\right)^{-1} + \left(\dfrac{3^3}{2^3} - \dfrac{3}{2}\right)^{-2} + \left(\dfrac{4^4}{3^4} - \dfrac{4}{3}\right)^{-3} + \ldots \infty$ (V.T.U., 2006)

*See footnote p. 144.

Solution. (*i*) After leaving the first term, we find that $u_n = \left(\frac{n+1}{n+2}\right)^n x^n$, so that

$$\underset{n \to \infty}{\text{Lt}} (u_n)^{1/n} = \underset{n \to \infty}{\text{Lt}} \left(\frac{1+1/n}{1+2/n}\right) x = x$$

$\therefore$ By Cauchy's root test, the given series converges for $x < 1$ and diverges for $x > 1$.

When $x = 1$, $\quad u_n = \left(\frac{n+1}{n+2}\right)^n = \frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}}\left(1+\frac{1}{n+1}\right)$

$\therefore \quad \underset{n \to \infty}{\text{Lt}} u_n = \frac{1}{e} \neq 0$. Since u_n does not tend to zero, Σu_n is divergent.

Thus the given series converges for $x < 1$ and diverges for $x \geq 1$.

(*ii*) Here $\quad (u_n^{1/n}) = \frac{n+1}{n^{1+1/n}} x$

$$\therefore \quad \underset{n \to \infty}{\text{Lt}} (u_n)^{1/n} = \underset{n \to \infty}{\text{Lt}} \frac{n+1}{n} \cdot \frac{1}{n^{1/n}} x = \underset{n \to \infty}{\text{Lt}} \left(1+\frac{1}{n}\right)\left(\frac{1}{n^{1/n}}\right) x = x \qquad \left[\because \underset{n \to \infty}{\text{Lt}} n/n = 1\right]$$

$\therefore$ The given series converges for $x < 1$ and diverges for $x > 1$.

When $x = 1$, $\quad u_n = \frac{(n+1)^n}{n^{n+1}} = \frac{1}{n}\left(1+\frac{1}{n}\right)^n$

Taking $\quad v_n = \frac{1}{n}, \ \underset{n \to \infty}{\text{Lt}} \left(\frac{u_n}{v_n}\right) = \underset{n \to \infty}{\text{Lt}} \left(1+\frac{1}{n}\right)^n = e \neq 0$ and finite.

$\therefore$ By comparison test both Σu_n and Σv_n behave alike.

But $\Sigma v_n = \Sigma \frac{1}{n}$ is divergent ($\because \ p = 1$). $\therefore$ Σu_n also diverges. Hence the given series converges for $x < 1$ and diverges for $x \geq 1$.

(*iii*) Here $\quad u_n = \left[\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{n+1}{n}\right]^{-n}$

$$\therefore \quad (u_n)^{1/n} = \left(\frac{n+1}{n}\right)^{-1}\left[\left(\frac{n+1}{n}\right)^n - 1\right]^{-1} = \left(1+\frac{1}{n}\right)^{-1}\left[\left(1+\frac{1}{n}\right)^n - 1\right]^{-1}$$

$$\therefore \quad \underset{n \to \infty}{\text{Lt}} (u_n)^{1/n} = 1 \cdot (e-1)^{-1} = \frac{1}{e-1} 9 < 1 \qquad [\because \ e > 1]$$

Thus the given series converges.

PROBLEMS 9.6

Discuss the convergence of the following series :

1. $\sum \frac{1}{n^n}$

2. $\sum \frac{1}{(\log n)^n}$ (*P.T.U., 2005*)

3. $\sum \left(\frac{n}{n+1}\right)^{n^2}$ (*P.T.U., 2010*)

4. $1 + \frac{x}{2} + \frac{x^2}{3^2} + \frac{x^3}{4^3} + \ldots + (x > 0)$

5. $\sum \left(\frac{n+2}{n+3}\right)^n x^n$

6. $\sum \frac{[(2n+1)\,x]^n}{n^{n+1}}$, $x > 0$

7. $\frac{3}{4}x + \left(\frac{4}{5}\right)^2 x^2 + \left(\frac{5}{6}\right)^3 x^3 + \ldots - \infty \ (x > 0)$ (*V.T.U., 2007*)

9.12 ALTERNATING SERIES

(1) Def. *A series in which the terms are alternately positive or negative is called an alternating series.*

(2) Leibnitz's series. *An alternating series* $u_1 - u_2 + u_3 - u_4 + \dots$ *converges if (i) each term is numerically less than its preceding term, and (ii)* $\underset{n \to \infty}{Lt}\ u_n = 0$.

If $\underset{n \to \infty}{Lt}\ u_n \neq 0$, *the given series is oscillatory.*

The given series is $\quad u_1 - u_2 + u_3 - u_4 + \dots$

Suppose $\quad u_1 > u_2 > u_3 > u_4 \dots > u_{n+1} \dots \qquad \dots(1)$

and $\quad \underset{n \to \infty}{\text{Lt}}\ u_n = 0 \qquad \dots(2)$

Consider the sum of $2n$ terms. It can be written as

$$s_{2n} = (u_1 - u_2) + (u_3 - u_4) + \dots + (u_{2n-1} - u_{2n}) \qquad \dots(3)$$

or as

$$s_{2n} = u_1 - (u_2 - u_3) - (u_4 - u_5) \dots - u_{2n} \qquad \dots(4)$$

By virtue of (1), the expressions within the brackets in (3) and (4) are all positive.

$\therefore$ It follows from (3) that s_{2n} is positive and increases with n.

Also from (4), we note that s_{2n} always remains less than u_1.

Hence s_{2n} must tend to a finite limit.

Moreover $\quad \underset{n \to \infty}{\text{Lt}}\ s_{2n+1} = \underset{n \to \infty}{\text{Lt}}\ (s_{2n} + u_{2n+1}) = \underset{n \to \infty}{\text{Lt}}\ s_{2n} + 0 \qquad$ [by (2)

Thus $\underset{n \to \infty}{\text{Lt}}\ s_n$ tends to the same finite limit whether n is even or odd.

Hence the given series is convergent.

When $\quad \underset{n \to \infty}{\text{Lt}}\ u_n \neq 0,\ \underset{n \to \infty}{\text{Lt}}\ s_{2n} \neq \underset{n \to \infty}{\text{Lt}}\ s_{2n+1}$. $\quad \therefore$ The given series is oscillatory.

Example 9.16. *Discuss the convergence of the series*

(i) $1 - \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} - \dfrac{1}{\sqrt{4}} + \dots \qquad$ *(ii)* $\dfrac{5}{2} - \dfrac{7}{4} + \dfrac{9}{6} - \dfrac{11}{8} + \dots$

(iii) $\dfrac{1}{\log 2} - \dfrac{1}{\log 3} + \dfrac{1}{\log 4} - \dfrac{1}{\log 5} + \dots \qquad$ *(P.T.U., 2010)*

Solution. (*i*) The terms of the given series are alternately positive and negative ; each term is numerically less than its preceding term $\left[\because\ u_n - u_{n-1} = \dfrac{1}{\sqrt{n}} - \dfrac{1}{\sqrt{n-1}} < 0\right]$

Also $\underset{n \to \infty}{\text{Lt}}\ u_n = \underset{n \to \infty}{\text{Lt}}\ (1/\sqrt{n}) = 0$. Hence by Leibnitz's rule, the given series is convergent.

(*ii*) The terms of the given series are alternately positive and negative and

$$u_n - u_{n-1} = \frac{2n+3}{2n} - \frac{2n+1}{2n-2} = \frac{-6}{4n(n-1)} < 0 \text{ for } n > 1.$$

i.e., $\quad u_n < u_{n-1}$ for $n > 1$. Also $\underset{n \to \infty}{\text{Lt}}\ u_n = \underset{n \to \infty}{\text{Lt}}\ \dfrac{2n+3}{2n} = 1 \neq 0$

Hence by Leibnitz's rule, the given series is oscillatory.

(*iii*) The terms of the given series are alternately positive and negative.

Also $n + 2 > n + 1$, *i.e.*, $\log(n+2) > \log(n+1)$

i.e., $\quad \dfrac{1}{\log(n+2)} < \dfrac{1}{\log(n+1)}$, *i.e.*, $u_{n+1} < u_n$.

and $\quad \underset{n \to \infty}{\text{Lt}}\ u_n = \underset{n \to \infty}{\text{Lt}}\ \dfrac{1}{\log(n+1)} = 0$

Hence the given series is convergent.

Example 9.17. *Examine the character of the series*

$$(i)\ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, n}{2n-1}. \qquad (ii)\ \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\, x^n}{n(n-1)},\ 0 < x < 1.$$

Solution. (*i*) The terms of the given series are alternately positive and negative ; each term is numerically less than its preceding term.

$$\left[\because\ u_n - u_{n-1} = \frac{n}{2n-1} - \frac{n-1}{2n-3} = \frac{-1}{(2n-1)(n-3)} < 0\right.$$

But $\quad \underset{n\to\infty}{\text{Lt}}\ u_n = \underset{n\to\infty}{\text{Lt}} \frac{n}{2n-1} = \underset{n\to\infty}{\text{Lt}} \frac{1}{2-1/n} = \frac{1}{2}$ which is not zero.

Hence the given series is oscillatory.

(*ii*) The terms of the given series are alternately positive and negative

$$u_n - u_{n-1} = \frac{x^n}{n(n-1)} - \frac{x^{n-1}}{(n-1)(n-2)} = \frac{x^{n-1}[(n-2)x - n]}{n\,(n-1)\,(n-2)} < 0 \quad \text{for} \quad n \geq 2, \qquad (\because\ 0 < x < 1)$$

i.e., $\quad u_n < u_{n-1}$ for $n \geq 2$. Also $\underset{n\to\infty}{\text{Lt}}\ u_n = \underset{n\to\infty}{\text{Lt}} \frac{x^n}{n(n-1)} = 0 \qquad (\because\ 0 < x < 1)$

Hence the given series is convergent.

PROBLEMS 9.7

Discuss the convergence of the following series :

1. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \infty.$ (*P.T.U., 2009*)
2. $1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \ldots \infty.$ (*V.T.U., 2010*)
3. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ (*Delhi, 2002*)
4. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}.$
5. $\frac{1}{1\cdot 2} - \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} - \frac{1}{7\cdot 8} + \ldots \infty$ (*Osmania, 2003*)
6. $\frac{1}{6} - \frac{2}{11} + \frac{3}{16} - \frac{4}{21} + \frac{5}{26} - \ldots \infty.$
7. $1 - 2x + 3x^2 - 4x^3 + \ldots + \infty,\ \left(x < \frac{1}{2}\right).$ (*Cochin, 2005*)
8. $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1}.$
9. $\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \frac{x^4}{1+x^4} + \ldots \infty\ (0 < x < 1).$ (*V.T.U., 2004 ; Delhi, 2002*)
10. $\left(\frac{1}{2} - \frac{1}{\log 2}\right) - \left(\frac{1}{2} - \frac{1}{\log 3}\right) + \left(\frac{1}{2} - \frac{1}{\log 4}\right) - \left(\frac{1}{2} - \frac{1}{\log 5}\right) + \ldots \infty.$

9.13 SERIES OF POSITIVE AND NEGATIVE TERMS

The series of positive terms and the alternating series are special types of these series with arbitrary signs.

Def. **(1)** *If the series of arbitrary terms* $u_1 + u_2 + u_3 + \ldots + u_n + \ldots$
be such that the series $\quad |\,u_1\,| + |\,u_2\,| + |\,u_3\,| + \ldots + |\,u_n\,| + \ldots$
is convergent, then the series Σu_n *is said to be* **absolutely convergent.**

(2) *If* $\Sigma\ |\,u_n\,|$ *is divergent but* Σu_n *is convergent, then* Σu_n *is said to be* **conditionally convergent.**

For instance, the series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \ldots$ *is absolutely convergent,* since the series

$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots$ is known to be convergent.

Again, since the alternating series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$ is convergent, and the series of absolute values $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$ is divergent, so the original series is conditionally convergent.

Obs. 1. *An absolutely convergent series is necessarily convergent but not conversely.*

Let Σu be an absolutely convergent series.

Clearly $u_1 + u_2 + u_3 + \dots + u_n + \dots$

$\le |u_1| + |u_2| + |u_3| + \dots + |u_n| + \dots$ which is known to be convergent.

Hence the series Σu_n is also convergent.

Obs. 2. As the series $\Sigma |u_n|$ is of positive terms, the tests already established for positive term series can be applied to examine Σu_n for its absolute convergence. For instance, Ratio test can be restated as follows :

The series Σu_n *is absolutely convergent if* $\underset{n \to \infty}{\text{Lt}} \frac{|u_{n+1}|}{|u_n|} < 1$,

and is divergent if $\underset{n \to \infty}{\text{Lt}} \frac{|u_{n+1}|}{|u_n|} > 1$. This test fails when the limit is unity.

Example 9.18. *Examine the following series for convergence :*

(*i*) $1 + \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} - \frac{1}{8^2} + \dots \infty$ (*V.T.U., 2006*)

(*ii*) $\frac{1}{2^3} - \frac{1}{3^3}(1+2) + \frac{1}{4^3}(1+2+3) - \frac{1}{5^3}(1+2+3+4) + \dots \infty$.

Solution. (*i*) The series of absolute terms is $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \dots$ which is, evidently convergent.

$\therefore$ the given series is absolutely convergent and hence it is convergent.

(*ii*) Here
$$u_n = (-1)^{n-1} \frac{(1+2+3+\dots+n)}{(n+1)^3}$$
$$= (-1)^{n-1} \frac{n(n+1)}{2(n+1)^3} = (-1)^{n-1} \frac{n}{2(n+1)^2} = (-1)^{n-1} a_n \text{ (Say).}$$

Then
$$a_n - a_{n+1} = \frac{1}{2}\left[\frac{n}{(n+1)^2} - \frac{n+1}{(n+2)^2}\right] = \frac{1}{2} \frac{n^2+n-1}{(n+1)^2 (n+2)^2} > 0.$$

i.e.,
$$a_{n+1} < a_n. \text{ Also } \underset{n \to \infty}{\text{Lt}} a_n = \frac{1}{2} \underset{n \to \infty}{\text{Lt}} \frac{n}{(n+1)^2} = 0.$$

Thus by Leibnitz's rule, Σa_n and therefore Σu_n is convergent.

Also
$$|u_n| = \frac{1}{2} \frac{n}{n^2+1}. \text{ Taking } v_n = \frac{1}{n}, \text{ we note that}$$
$$\underset{n \to \infty}{\text{Lt}} \frac{|u_n|}{v_n} = \frac{1}{2} \underset{n \to \infty}{\text{Lt}} \frac{n^2}{n^2+1} = \frac{1}{2} \ne 0$$

Since Σv_n is divergent, therefore $\Sigma |u_n|$ is also divergent.

i.e., Σu_n is convergent but $\Sigma |u_n|$ is divergent.

Thus the given series Σu_n is conditionally convergent.

Example 9.19. *Test whether the following series are absolutely convergent or not ?*

(*i*) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$ (*ii*) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\log n)^2}$.

Solution. (*i*) Given series is $\Sigma u_n = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \infty$

This is an alternating series of which terms go on decreasing and $\underset{n \to \infty}{\text{Lt}}\ u_n = \underset{n \to \infty}{\text{Lt}} \dfrac{1}{2n-1} = 0$

$\therefore$ by Leibnitz's rule, Σu_n converges.

The series of absolute terms is $1 + \dfrac{1}{3} + \dfrac{1}{5} + \dfrac{1}{7} + \ldots \infty$

Here $u_n = \dfrac{1}{2n-1}$. Taking $v_n = \dfrac{1}{n}$, we have

$$\underset{n \to \infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n \to \infty}{\text{Lt}} \left(\frac{n}{2n-1}\right) = \underset{n \to \infty}{\text{Lt}} \left(\frac{1}{2 - \dfrac{1}{n}}\right) = \frac{1}{2} \neq 0 \text{ and finite.}$$

$\therefore$ by Comparison test, Σu_n diverges [$\because$ Σv_n diverges.

Hence the given series converges and the series of absolute terms diverges, therefore the given series converges conditionally.

(*ii*) The terms of given series are alternately positive and negative. Also each term is numerically less than the preceding term and $\underset{n \to \infty}{\text{Lt}} |u_n| = \underset{n \to \infty}{\text{Lt}} [1/n(\log n)^2] = 0.$

$\therefore$ by Leibnitz's rule, the given series converges.

Also $$\int_2^\infty \frac{dx}{x(\log x)^2} = \left| -\frac{1}{\log x} \right|_2^\infty = \frac{1}{\log 2} = 0 \text{ and finite.}$$

i.e., the series of absolute terms converges.

Hence, the given series converges absolutely.

9.14 POWER SERIES

(1) Def. *A series of the form* $a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots$ ...(*i*)
where the a's are independent of x, is called a **power series** *in x.* Such a series may converge for some or all values of x.

(2) Interval of convergence

In the power series (*i*), $u_n = a_nx^n$.

$$\therefore \quad \underset{n \to \infty}{\text{Lt}} \frac{u_{n+1}}{u_n} = \underset{n \to \infty}{\text{Lt}} \frac{a_{n+1}x^{n+1}}{a_nx^n} = \underset{n \to \infty}{\text{Lt}} \left(\frac{a_{n+1}}{a_n}\right) . x$$

If $\underset{n \to \infty}{\text{Lt}} \left(\dfrac{a_{n+1}}{a_n}\right) = l$, then by Ratio test, the series (*i*) converges, when lx is numerically less than 1, *i.e.*, when $|x| < 1/l$ and diverges for other values.

Thus the power series (*i*) has an interval $-1/l < x < 1/l$ within which it converges and diverges for values of x outside this interval. Such an interval is called the *interval of convergence of the power series.*

Example 9.20. *State the values of x for which the following series converge :*

(*i*) $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dfrac{x^5}{5} - \ldots \infty$, (*ii*) $\dfrac{1}{1-x} + \dfrac{1}{2(1-x)^2} + \dfrac{1}{3(1-x)^3} + \ldots \infty.$

Solution. (*i*) Here $u_n = (-1)^{n-1} \dfrac{x^n}{n}$ and $u_{n+1} = (-1)^n \dfrac{x^{n+1}}{n+1}$

$$\therefore \quad \frac{u_{n+1}}{u_n} = -\frac{n}{n+1}x \quad \text{and} \quad \underset{n \to \infty}{\text{Lt}} \left|\frac{u_{n+1}}{u_n}\right| = \left(\underset{n \to \infty}{\text{Lt}} \frac{1}{1+1/n}\right) |x| = |x|$$

$\therefore$ by Ratio test the given series converges for $|x| < 1$ and diverges for $|x| > 1$.

Let us examine the series for $x = \pm 1$.

For x = 1, the series reduces to $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \dots$

which is an alternating series and is convergent.

For x = – 1, the series becomes $-\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots\right)$

which is a divergent series as can be seen by comparison with p-series when $p = 1$.

Hence the given series converges for $-1 < x \le 1$.

(*ii*) Here $$u_n = \frac{1}{n(1-x)^n}$$

$$\underset{n\to\infty}{\text{Lt}}\left|\frac{u_{n+1}}{u_n}\right| = \underset{n\to\infty}{\text{Lt}}\left|\frac{1}{(n+1)(1-x)^{n+1}}\cdot n(1-x)^n\right| = \left|\frac{1}{1-x}\right|\underset{n\to\infty}{\text{Lt}}\frac{n}{n+1} = \left|\frac{1}{1-x}\right|$$

By Ratio test, Σu_n converges for $\left|\frac{1}{1-x}\right| < 1$, *i.e.*, $|1 - x| > 1$

i.e., for $-1 > 1 - x > 1$ or $x < 0$ and $x > 2$.

Let us examine the series for $x = 0$ and $x = 2$.

For x = 0, the given series becomes $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n}$ which is a divergent harmonic series.

For x = 2, the given series becomes $-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots + \frac{(-1)^n}{n} + \dots$

It is an alternating series which is convergent by Leibnitz's rule

$$[\because\ u_n < u_{n-1} \text{ for all } n \text{ and } \underset{n\to\infty}{\text{Lt}}\, u_n = 0.]$$

Hence the given series converges for $x < 0$ and $x \ge 2$.

Example 9.21. *Test the series* $\frac{x}{\sqrt{3}} - \frac{x^2}{\sqrt{5}} + \frac{x^3}{\sqrt{7}} - \dots$ *for absolute convergence and conditional convergence.* *(V.T.U., 2010)*

Solution. We have $u_n = (-1)^{n-1}\frac{x^n}{\sqrt{(2n+1)}}$ and $u_{n+1} = \frac{(-1)^n x^{n+1}}{\sqrt{(2n+3)}}$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\left|\frac{u_{n+1}}{u_n}\right| = \underset{n\to\infty}{\text{Lt}}\left|\frac{\frac{(-1)^n x^{n+1}}{\sqrt{(2n+3)}}\cdot\sqrt{(2n+1)}}{(-1)^{n-1}x^n}\right| = \underset{n\to\infty}{\text{Lt}}\left|(-1)\sqrt{\left(\frac{2n+1}{2n+3}\right)}x\right|$$

$$= \underset{n\to\infty}{\text{Lt}}\left|\sqrt{\left(\frac{2+1/n}{2+3/n}\right)}x\right| = |x|$$

Hence *the given series is absolutely convergent for* $|x| < 1$ *and is divergent for* $|x| > 1$ and the test fails for $|x| = 1$.

For x = 1, $$u_n = \frac{(-1)^{n-1}}{\sqrt{(2n+1)}}. \text{ Since } 2n+1 < 2n+3 \text{ or } (2n+1)^{-1/2} > (2n+3)^{-1/2}$$

i.e., $$u_n > u_{n+1}. \text{ Also } \underset{n\to\infty}{\text{Lt}}\, u_n = \underset{n\to\infty}{\text{Lt}}\frac{1}{\sqrt{(2n+1)}} = 0.$$

$\therefore$ the series is convergent by Leibnitz's test.

But $$\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} + \dots \text{ has } u_n = \frac{1}{\sqrt{(2n+1)}} = \frac{1}{\sqrt{n}}\cdot\frac{1}{\sqrt{(2+1/n)}}$$

On comparing it with $v_n = \dfrac{1}{\sqrt{n}}$, Σu_n is divergent.

Hence the *given series is conditionally convergent for x = 1.*

For x = – 1, the series becomes $-\left(\dfrac{1}{\sqrt{3}} + \dfrac{1}{\sqrt{5}} + \dfrac{1}{\sqrt{7}} + \ldots\right)$

But we have seen that the series $\dfrac{1}{\sqrt{3}} + \dfrac{1}{\sqrt{5}} + \dfrac{1}{\sqrt{7}} + \ldots$ is divergent.

Hence, *the given series is divergent when x = – 1.*

9.15 (1) CONVERGENCE OF EXPONENTIAL SERIES

The series $1 + x + \dfrac{x^2}{2!} + \ldots + \dfrac{x^n}{n!} + \ldots \infty$ *is convergent for all values of x.* (*J.N.T.U., 2006*)

Here $$\underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{\text{Lt}} \left[\frac{x^n}{n!} \div \frac{x^{n-1}}{(n-1)!}\right] = \underset{n\to\infty}{\text{Lt}} \frac{x}{n} = 0$$

Hence the series converges, whatever be the value of x.

(2) Convergence of logarithmic series

The series $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \ldots + (-1)^n \dfrac{x^n}{n} + \ldots \infty$ *is convergent for* $-1 < x \le 1$.

Here $$\underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{\text{Lt}} \frac{(-1)^{n+1}x^{n+1}}{n+1} \cdot \frac{n}{(-1)^n x^n} = -x \underset{n\to\infty}{\text{Lt}} \frac{n}{n+1} = -x \underset{n\to\infty}{\text{Lt}} \left\{\frac{1}{1+1/n}\right\} = -x.$$

Hence the series converges for $|x| < 1$ and diverges for $|x| > 1$.

When $x = 1$, the series being $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$, is convergent.

When $x = -1$, the series being $-\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\right)$, is divergent.

Hence the series converges for $-1 < x \le 1$.

(3) Convergence of binomial series

The series $1 + nx + \dfrac{n(n-1)}{2!} x^2 + \ldots + \dfrac{n(n-1)\ldots(n-r+1)}{r!} x^r + \ldots \infty$

converges for $|x| < 1$.

Here $$u_r = \frac{n(n-1)\ldots(n-r)}{(r-1)!} x^{r-1} \text{ and } u_{r+1} = \frac{n(n-1)\ldots(n-r+1)}{r!} x^r$$

$$\therefore \quad \underset{r\to\infty}{\text{Lt}} \frac{u_{r+1}}{u_r} = \underset{r\to\infty}{\text{Lt}} \frac{n-r+1}{r} x = \underset{r\to\infty}{\text{Lt}} \left(\frac{n+1}{r} - 1\right) x = -x \text{ for } r > n+1.$$

Hence, the series converges for $|x| < 1$.

PROBLEMS 9.8

1. Test the following series for conditional convergence : (*i*) $\sum \dfrac{(-1)^{n-1}}{\sqrt{n}}$ (*ii*) $\sum \dfrac{(-1)^{n-1} n}{n^2+1}$.
2. Prove that the series $\dfrac{\sin x}{1^3} - \dfrac{\sin 2x}{2^3} + \dfrac{\sin 3x}{3^3} - \ldots$ converges absolutely. (*Rohtak, 2006 S*)
3. Test the following series for conditional convergence :

   (*i*) $1 - \dfrac{1}{2^p} + \dfrac{1}{3^p} - \dfrac{1}{4^p} + \ldots \infty$ (*ii*) $1 - \dfrac{1}{2} + \dfrac{1.3}{2.4} - \dfrac{1.3.5}{2.4.6} + \ldots \infty$

4. Discuss the absolute convergence of (*i*) $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$ (*Hissar, 2005 S*)

(*ii*) $x - \frac{x^2}{4} + \frac{x^3}{9} - \frac{x^4}{16} + \ldots \infty$ (*iii*) $\frac{1}{\sqrt{(1^3+1)}} - \frac{1}{\sqrt{(2^3+1)}}x + \frac{1}{\sqrt{(3^3+1)}}x^2 - \ldots \infty$

5. Find the nature of the series $\frac{x}{1 \cdot 2} - \frac{x^2}{2 \cdot 3} + \frac{x^3}{3 \cdot 4} - \frac{x^4}{4 \cdot 5} + \ldots \infty$ (*V.T.U., 2009*)

6. For what values of x are the following series convergent :

(*i*) $x - \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} - \frac{x^4}{\sqrt{4}} + \ldots \infty$ (*P.T.U., 2009 S ; V.T.U., 2008*)

(*ii*) $x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \ldots \infty$

7. Find the radius of convergence of the series $\sum \frac{n!}{n^n} x^n$. (*Calicut, 2005*)

8. Prove that $\frac{1}{a} + \frac{1}{a+1} - \frac{1}{a+2} + \frac{1}{a+3} + \frac{1}{a+4} - \frac{1}{a+5} + \ldots$ is a divergent series.

9. Test the series $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}}$ for

(*i*) absolute convergence and (*ii*) conditional convergence. (*V.T.U., 2007 ; Rohtak, 2005*)

9.16 PROCEDURE FOR TESTING A SERIES FOR CONVERGENCE

First see whether the given series is

(*i*) a series with terms alternately positive and negative ;

(*ii*) a series of positive terms excluding power series ;

or (*iii*) a power series.

For alternating series (*i*), apply the Leibnitz's rule (§ 9.12).

For series (*ii*), first find u_n and if possible evaluate Lt u_n. If Lt $u_n \neq 0$, the series is divergent. If Lt $u_n = 0$, compare Σu_n with $\Sigma 1/n^p$ and apply the comparison tests (§ 9.6).

If the comparison tests are not applicable, apply the Ratio test (§ 9.9). If Lt $u_n/u_{n+1} = 1$, *i.e.*, the ratio test fails, apply Raabe's teat (§ 9.10). If Raabe's test fails for a similar reason, apply Logarithmic test (§ 9.10). If this also fails, apply Cauchy's root test (§ 9.11).

For the power series (*iii*), apply the Ratio test as in § 9.14. If the Ratio test fails, examine the series as in case (*ii*) above.

PROBLEMS 9.9

Test the convergence of the following series :

1. $\sum_{n=1}^{\infty} \frac{2^n - 2}{2^n + 1} \cdot x^{n-1} (x > 0)$. (*Osmania, 1999*)

2. $\sum \left(\frac{1}{\sqrt{n}} - \sqrt{\frac{n}{n+1}} \right)$.

3. $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \ldots$

4. $\sum_{n=1}^{\infty} \sqrt{\left\{ \frac{2^n+1}{3^n+1} \right\}}$

5. $\frac{1}{1+\sqrt{2}} + \frac{2}{1+2\sqrt{3}} + \frac{3}{1+3\sqrt{4}} + \ldots \infty$.

6. $\frac{x}{1+\sqrt{1}} + \frac{x^2}{2+\sqrt{2}} + \frac{x^3}{3+\sqrt{3}} + \ldots \infty$.

7. $1 + \frac{2^2}{3^2}x + \frac{2^2 . 4^2}{3^2 . 5^2}x^2 + \frac{2^2 . 4^2 . 6^2}{3^2 . 5^2 . 7^2}x^3 + \ldots \infty$.

8. $\sum_{1}^{\infty} \frac{n x^n}{(n+1)(n+2)} (x > 0)$.

9. $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^n$.

10. $\sum_{n=1}^{\infty} \frac{x^n}{(2n-1)^2 2^n}$.

11. $\sum_{0}^{\infty} \frac{(3x+5)^n}{(n+1)!}$.

12. $\sum_{n=1}^{\infty} \frac{(x+2)^n}{3^n n}$.

13. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\log n}$.

14. $\sum \frac{(-1)^{n-1} \sin nx}{n^3}$.

15. $\frac{\sqrt{2}-1}{3^3-1} + \frac{\sqrt{3}-1}{4^3-1} + \frac{\sqrt{4}-1}{5^3-1} + \dots \infty$. *(V.T.U., 2003)*

16. $\sum_{n=2}^{\infty} \frac{1}{(n \log n)(\log \log n)^p}$.

9.17 UNIFORM CONVERGENCE

Let $$u_1(x) + u_2(x) + \dots \infty = \sum_{n=1}^{\infty} u_n(x) \quad \dots(1)$$

be an infinite series of functions each of which is defined in the interval (a, b). Let $s_n(x)$ be the sum of its first n terms, *i.e.*, $s_n(x) = u_1(x) + u_2(x) + \dots + u_n(x)$

At some point $x = x_1$, if $\underset{n \to \infty}{\text{Lt}}\, s_n(x_1) = s(x_1)$,

then the series (1) is said to converge to sum $s(x_1)$ at that point. This means at $x = x_1$ given a positive number ε, we can find a number N such that $| s(x_1) - s_n(x_1) | < \varepsilon$ for $n > N$ ...(2)

Evidently N will depend on ε but generally it will also depend on x_1. Now if we keep the same ε but take some other value x_2 of x for which (1) is convergent, then we may have to change N for the inequality (2) to hold. If we wish to approximate the sum $s(x)$ of the series by its partial sums $s_n(x)$, we shall require different partial sums at different points of the interval and the problem will become quite complicated. If, however, we choose an N which is independent of the values of x, the problem becomes simpler. Then the partial sum $s_n(x)$, $(n > N)$ approximates to $s(x)$ for all values of x in the interval (a, b) and ε is uniform throughout this interval. Thus we have

Definition. *The series $\sum u_n(x)$ is said to be uniformly convergent in the interval (a, b), if for a given $\varepsilon > 0$, a number N can be found* **independent of x,** *such that for every x in the interval (a, b),*

$$| s(x) - s_n(x) | < \varepsilon \text{ for all } n > N.$$

Example 9.21. *Examine the geometric series $1 + x + x^2 + \dots + x^{n-1} + \dots \infty$ for uniform convergence in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$.*

Solution. We have $$s_n(x) = 1 + x + x^2 + \dots + x^{n-1} = \frac{1-x^n}{1-x}.$$

and $$s(x) = \underset{n \to \infty}{\text{Lt}} \frac{1-x^n}{1-x} = \frac{1}{1-x} \text{ for } | x | < 1$$

$\therefore$ $$| s(x) - s_n(x) | = \left| \frac{x^n}{1-x} \right| = \frac{|x^n|}{1-x} = \frac{|x|^n}{1-x} \text{ which will be } < \varepsilon, \text{ if } | x |^n < \varepsilon(1-x).$$

Choose N such that $| x |^N = \varepsilon(1-x)$

or $$N = \log[\varepsilon(1-x)] / \log | x | \quad \dots(i)$$

Evidently N increases with the increase of $| x |$ and in the interval $-\frac{1}{2} \le x \le \frac{1}{2}$, it assumes a maximum value $N' = \log(\varepsilon/2)/\log \frac{1}{2}$ at $x = \frac{1}{2}$ for a given ε.

Thus $| s(x) - s_n(x) | < \varepsilon$ for all $n \ge N'$ for every value of x in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Hence the geometric series converges uniformly in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Obs. *The geometric series though convergent in the interval $(-1, 1)$, is not uniformly convergent in this interval,* since we cannot find a fixed number N for every x in this interval

($\because$ N given by $(i) \to \infty$ as $| x | \to 1$).

9.18 WEIERSTRASS'S M-TEST*

A series $\Sigma u_n(x)$ is uniformly convergent in an interval (a, b), if there exists a convergent series ΣM_n of positive constants such that $| u_n(x) | \le M_n$ for all values of x in (a, b).

Since ΣM_n is convergent, therefore, for a given $\varepsilon > 0$, we can find a number N, such that $| s - s_n | < \varepsilon$ for every $n > N$,

where $\quad s = M_1 + M_2 + ... + M_n + M_{n+1} + ...$ and $s_n = M_1 + M_2 + ... + M_n$

This implies that $| M_{n+1} + M_{n+2} + ... | < \varepsilon$ for every $n > N$.

Since $\quad | u_n(x) | \le M_n$

$$\therefore \quad | u_{n+1}(x) | + u_{n+2}(x) + ... | \le | u_{n+1}(x) | + | u_{n+2}(x) | + ...$$
$$\le M_{n+1} + M_{n+2} + ... < \varepsilon \text{ for every } n > N.$$

i.e., $\quad | s(x) - s_n(x) | < \varepsilon$ for every $n > N$, where $s(x)$ is the sum of the series $\Sigma u_n(x)$.

Since N does not depend on x, the series $\Sigma u_n(x)$ converges uniformly in (a, b).

Obs. *$\Sigma u_n(x)$ is also absolutely convergent for every x, since $| u_n(x) | \le M_n$.*

Example 9.22. *Show that the following series converges uniformly in any interval :*

(i) $\sum \dfrac{\cos nx}{n^p}$ *(Andhra, 1999)* *(ii)* $\sum \dfrac{1}{n^3 + n^4x^2}$.

Solution. (*i*) $\left| \dfrac{\cos nx}{n^p} \right| = \dfrac{| \cos nx |}{n^p} \le \dfrac{1}{n^p}$ $(= M_n)$ for all values of x.

Since $\quad \sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \dfrac{1}{n^p}$ converges for $p > 1$,

$\therefore$ By M-test, the given series converges uniformly for all real values of x and $p > 1$.

(*ii*) For all values of x, $n^3 + n^4x^2 > n^3$

$\therefore \quad \left| \dfrac{1}{n^3 + n^4x^2} \right| < \dfrac{1}{n^3}$ $(= M_n)$. But ΣM_n being p-series with $p > 1$, is convergent.

$\therefore$ By M-test, the given series converges uniformly in any interval.

Example 9.23. *Examine the following series for uniform convergence :*

(i) $\sum_{n=1}^{\infty} \dfrac{\sin (nx + x^2)}{n(n+2)}$ *(P.T.U., 2009)* *(ii)* $\sum_{n=1}^{\infty} \dfrac{1}{n^p + n^q x^2}$ *(P.T.U., 2005 S)*

Solution. (*i*) $\left| \dfrac{\sin (nx + x^2)}{n(n+2)} \right| = \dfrac{| \sin (nx + x^2) |}{n^2 + 2n} \le \dfrac{1}{n^2}$ $(= M_n)$ for all real x.

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \dfrac{1}{n^2}$ is convergent, therefore, by M-test, the given series is uniformly convergent for all real values of x.

(*ii*) For all real values of x, $x^2 \ge 0$, *i.e.*, $n^q x^2 \ge 0$

i.e., $\quad n^p + n^q x^2 \ge n^p \quad$ or $\quad \dfrac{1}{n^p + n^q x^2} \le \dfrac{1}{n^p}$ $(= M_n)$

Since $\quad \sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \dfrac{1}{n^p}$ is convergent for $p > 1$,

$\therefore$ by M-test, the given series is uniformly convergent for all real values of x and $p > 1$.

* Named after the great German mathematician *Karl Weierstrass* (1815–1897) who made basic contributions to Calculus, Approximation theory, Differential geometry and Calculus of variations. He was also one of the founders of Complex analysis.

9.19 PROPERTIES OF UNIFORMLY CONVERGENT SERIES

I. If the series $\Sigma u_n(x)$ converges uniformly to sum $s(x)$ in the interval (a, b) and each of the functions $u_n(x)$ is continuous in this interval, then the sum $s(x)$ is also continuous in (a, b).

II. If the series $\Sigma u_n(x)$ converges uniformly in the interval (a, b) and each of the functions $u_n(x)$ is continuous in this interval, then the series can be integrated term by term

i.e.,
$$\int_a^b [u_1(x) + u_2(x) + \ldots]\, dx = \int_a^b u_1(x)\, dx + \int_a^b u_2(x)\, dx + \ldots$$

III. If $\Sigma u_n(x)$ is a convergent series having continuous derivatives of its terms, and the series $\Sigma u_n(x)$ converges uniformly, then the series can be differentiated term by term

$$\frac{d}{dx}[u_1(x) + u_2(x) + \ldots] = u_1'(x) + u_2'(x) + \ldots.$$

Example 9.24. *Prove that* $\int_0^1 \left(\sum \frac{x^n}{n^2}\right) dx = \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$.

Solution. $|x^n| \le 1$ for $0 \le x \le 1$

$\therefore$ $\left|\frac{x^n}{n^2}\right| \le \frac{1}{n^2}$ $(= M_n)$ for $0 \le x \le 1$. But ΣM_n is a convergent series.

$\therefore$ by M-test, the series $\Sigma(x^n/n^2)$ is uniformly convergent in $0 \le x \le 1$. Also x^n/n^2 is continuous in this interval.

$\therefore$ the series $\Sigma(x^n/n^2)$ can be integrated term by term in the interval $0 \le x \le 1$.

i.e.,
$$\int_0^1 \left(\sum \frac{x^n}{n^2}\right) dx = \sum \left(\int_0^1 \frac{x^n}{n^2}\, dx\right) = \sum \left(\frac{1}{n^2}\int_0^1 x^n\, dx\right) = \sum \frac{1}{n^2(n+1)}.$$

Imp. Obs. *There is no relation between absolute and uniform convergence.* In fact, a series may converge absolutely but not uniformly while another series may converge uniformly but not absolutely.

For instance, the series

$\frac{1}{x^2+1} - \frac{1}{x^2+2} + \frac{1}{x^2+3} - \ldots$ can be seen to converge uniformly but not absolutely, while the series

$x^2 + \frac{x^2}{x^2+1} + \frac{x^2}{(x^2+1)^2} + \frac{x^2}{(x^2+1)^3} + \ldots$ can be shown to converge absolutely but not uniformly.

PROBLEMS 9.10

Test for uniform convergence the series :

1. $\sum_{n=1}^{\infty} \frac{x^n}{n^{3/2}}$.

2. $\sum \frac{\cos nx}{2^n}$.

3. $\frac{\sin x}{1^2} + \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} + \frac{\sin 4x}{4^2} + \ldots \infty$. *(P.T.U., 2003 ; Andhra, 2000)*

4. $\sin x - \frac{\sin 2x}{2\sqrt{2}} + \frac{\sin 3x}{3\sqrt{3}} - \frac{\sin 4x}{4\sqrt{4}} + \ldots \infty$.

5. $\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots \infty$.

6. $\frac{ax}{2} + \frac{a^2x^2}{5} + \frac{a^3x^3}{10} + \ldots + \frac{a^nx^n}{n^2+1} + \ldots \infty$.

7. Show that the series $\Sigma r^n \sin n\theta$ and $\Sigma r^n \cos n\theta$ converge uniformly for all real values of θ if $0 < r < 1$.

8. Show that $\frac{1}{1+x^2} - \frac{1}{2+x^2} + \frac{1}{3+x^2} - \frac{1}{4+x^2} + \ldots$ converges uniformly in the interval $x \ge 0$ but not absolutely.

9. Prove that $\sum \frac{x}{n(1+nx^2)}$ is uniformly convergent for all real values of x.

10. Examine the following series for uniform convergence :

(i) $\sum_{n=1}^{\infty} \frac{1}{n^4 + n^3 x^2}$ (ii) $\sum_{n=1}^{\infty} \frac{\cos(x^2 + n^2 x)}{n(n^2+2)}$.

11. Show that

(i) $\int_0^1 \left(\sum \frac{\sin x}{x}\right) dx = 1 - \frac{1}{3.3!} + \frac{1}{5.5!} - \frac{1}{7.7!} + \dots \infty$; (ii) $\int_0^{\pi} \left(\sum \frac{\sin n\theta}{n^3}\right) d\theta = 2 \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$.

9.20 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 9.11

Choose the correct answer or fill up the blanks in each of the following problems :

1. The series $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots$ converges if

(a) $p > 0$ (b) $p < 1$ (c) $p > 1$ (d) $p \le 1$.

2. The series $\sum_{n=0}^{\infty} (2x)^n$ converges if

(a) $-1 \le x \le 1$ (b) $-\frac{1}{2} < x < \frac{1}{2}$ (c) $-2 < x < 2$ (d) $-\frac{1}{2} \le x \le \frac{1}{2}$.

3. The series $\frac{2}{1^2} - \frac{3}{2^2} + \frac{4}{3^2} - \frac{5}{4^2} + \dots$ is

(a) conditionally convergent (b) absolutely convergent
(c) divergent (d) none of the above.

4. Which one of the following series is *not* convergent ?

(a) $\frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \dots \infty$ (b) $1\frac{1}{2} - 1\frac{1}{3} + 1\frac{1}{4} - 1\frac{1}{5} + \dots \infty$

(c) $\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \dots \infty$ (d) $x + x^2 + x^3 + x^4 + \dots \infty$ where $|x| < 1$.

5. The sum of the alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ is

(a) zero (b) infinite (c) log 2
(d) not defined as the series is not convergent.

6. Let Σu_n be a series of positive terms. Given that Σu_n is convergent and also

$\underset{n \to \infty}{\text{Lt}} \frac{u_{n+1}}{u_n}$ exists, then the said limit is

(a) necessarily equal to 1 (b) necessarily greater than 1
(c) may be equal to 1 or less than 1 (d) necessarily less than 1.

7. $\sum \left(1 + \frac{1}{n}\right)^{-n^2}$ is

(a) convergent (b) oscillatory (c) divergent.

8. $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \dots$ is

(a) oscillatory (b) conditionally convergent
(c) divergent (d) absolutely convergent.

9. $1+\frac{1}{2^2}-\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}-\frac{1}{7^2}-\frac{1}{8^2}+\ldots\infty$ is

(a) conditionally convergent (b) convergent
(c) oscillatory (d) divergent.

10. $\int_0^1 \left(\sum \frac{x^n}{n^2}\right) dx =$

(a) $\sum_{n=0}^{\infty} \frac{1}{n(n+1)}$ (b) $\sum_{n=1}^{\infty} \frac{1}{n^2(n-1)}$ (c) $\sum_{n=0}^{\infty} \frac{1}{n(n-1)}$ (d) $\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$.

11. If Σu_n is a convergent series of positive terms, then $\underset{n\to\infty}{\text{Lt}}\ u_n$ is

(a) 1 (b) $\pm$ 1 (c) 0 (d) 0. (V.T.U., 2010)

12. Geometric series $1+x+x^2+\ldots+x^{n-1}+\ldots\infty$

(a) converges in the interval .... (b) converges uniformly in the interval ....

13. The series $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\infty$ converges in the interval ....

14. If $\underset{n\to\infty}{\text{Lt}}\ \frac{u_n}{u_{n+1}}=k$, then Σu_n converges for k ....

15. A sequence (a_n) is said to be bounded, if there exists a number k such that for every n, a_n is ....

16. The series $2-5+3+2-5+3-5+\ldots\infty$ is .... (Convergent etc.)

17. The series $1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\infty$ converges for ....

18. If $\underset{n\to\infty}{\text{Lt}}\ n\left\{\frac{u_n}{u_{n+1}}-1\right\}=k$, then Σu_n diverges for k ....

19. A sequence which is monotonic and bounded is ....

20. The series $\frac{1}{1.2}+\frac{2}{3.4}+\frac{3}{5.6}+\ldots\infty$ is .... (Convergent etc.)

21. The series $\frac{2^p}{1^q}+\frac{3^p}{2^q}+\frac{4^p}{3^q}+\ldots\infty$ converges for ....

22. The series $\sqrt{\frac{1}{4}}+\sqrt{\frac{2}{6}}+\sqrt{\frac{3}{8}}+\ldots\infty$ is .... (Convergent etc.)

23. The series $\sqrt{\left(\frac{2^n-1}{3^n-1}\right)}$ is ... (Convergent etc.)

24. The series $1-\frac{1}{2}(x-2)+\frac{1}{4}(x-2)^2\ldots+\left(-\frac{1}{2}\right)^n(x-2)^n+\ldots\infty$ converges in the interval ......

25. Is the series $\sum_{n=1}^{\infty}\frac{8\tan^{-1}n}{1+n^2}$ convergent ?

26. The exponential series $1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+\ldots\infty$ is absolutely convergent. (True/False)

27. The series $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{n(n+1)}+\ldots\infty$, is .... (Convergent/divergent/oscillatory)

28. Is the series $\Sigma\, n \tan 1/n$ convergent ?

29. The series $\sum \frac{1}{nx^n}$ converges for x ......

30. The series $\sum_{n=1}^{\infty}\frac{x^n}{n^3}$ converges uniformly when x lies in the interval ....

31. Every absolutely convergent series is necessarily
(*a*) divergent (*b*) convergent
(*c*) conditionally convergent (*d*) none of these. (*V.T.U., 2009*)

32. The convergence of the series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ is tested by
(*a*) Ratio test (*b*) Raabe's test (*c*) Leibnitz's (*d*) Cauchy root test. (*V.T.U., 2009*)

33. The series $\sum \frac{x^n}{(n+1)^n}, x > 0$ is
(*a*) divergent (*b*) convergent (*c*) oscillatory (*d*) none of these. (*V.T.U., 2010*)

34. $\sum \sin\left(\frac{1}{n}\right)$ is
(*a*) convergent (*b*) divergent (*c*) oscillatory (*d*) none of these.

35. $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$ is convergent. (True or False)

CHAPTER 10

Fourier Series

1. Introduction. 2. Euler's Formulae. 3. Conditions for a Fourier expansion. 4. Functions having points of discontinuity. 5. Change of interval. 6. Odd and even function—Expansions of odd or even periodic functions. 7. Half-range series. 8. Typical wave-forms. 9. Parseval's formula. 10. Complex form of F-series. 11. Practical Harmonic Analysis. 12. Objective Type of Questions.

10.1 INTRODUCTION

In many engineering problems, especially in the study of periodic phenomenae* in conduction of heat, electro-dynamics and acoustics, it is necessary to express a function in a series of sines and cosines. Most of the single-valued functions which occur in applied mathematics can be expressed in the form.

$$\tfrac{1}{2}a_0\dagger + a_1 \cos x + a_2 \cos 2x + \ldots\ldots\dagger$$
$$+ b_1 \sin x + b_2 \sin 2x + \ldots\ldots$$

within a desired range of values of the variable. Such a series is known as the **Fourier series$**.

10.2 EULER'S FORMULAE

The Fourier series for the function f(x) in the interval $\alpha < x < \alpha + 2\pi$ *is given by*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx \end{aligned}\right\} \quad \ldots(1)$$

These values of a_0, a_n, b_n are known as *Euler's formulae***.

***Periodic functions.** If at equal intervals of abscissa x, the value of each ordinate $f(x)$ repeats itself, *i.e.*, $f(x) = f(x + a)$, for all x, then $y = f(x)$ is called a *periodic function* having **period** α, *e.g.*, $\sin x$, $\cos x$ are periodic functions having a period 2π.

† To write $a_0/2$ instead of a_0 is a conventional device to be able to get more symmetric formulae for the coefficients.

$ Named after the French mathematician and physicist *Jacques Fourier* (1768–1830) who was first to use Fourier series in his memorable work *'Theorie Analytique de la Chaleur'* in which he developed the theory of heat conduction. These series had a deep influence in the further development of mathematics and mathematical physics.

**See footnote p. 205.

To establish these formulae, the following definite integrals will be required :

1. $\int_{\alpha}^{\alpha+2\pi} \cos nx\, dx = \left| \frac{\sin nx}{n} \right|_{\alpha}^{\alpha+2\pi} = 0$ $(n \neq 0)$

2. $\int_{\alpha}^{\alpha+2\pi} \sin nx\, dx = -\left| \frac{\cos nx}{n} \right|_{\alpha}^{\alpha+2\pi} = 0$ $(n \neq 0)$

3. $\int_{\alpha}^{\alpha+2\pi} \cos mx \cos nx\, dx$ $(n \neq 0)$

$$= \frac{1}{2}\int_{\alpha}^{\alpha+2\pi} [\cos(m+n)x + \cos(m-n)x]\, dx$$

$$= \frac{1}{2}\left| \frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n} \right|_{\alpha}^{\alpha+2\pi} = 0 \qquad (m \neq n)$$

4. $\int_{\alpha}^{\alpha+2\pi} \cos^2 nx\, dx = \left| \frac{x}{2} + \frac{\sin 2nx}{4n} \right|_{\alpha}^{\alpha+2\pi} = \pi$ $(n \neq 0)$

5. $\int_{\alpha}^{\alpha+2\pi} \sin mx \cos nx\, dx = -\frac{1}{2}\left[\frac{\cos(m-n)x}{m-n} + \frac{\cos(m+n)x}{m+n} \right] = 0$ $(m \neq n)$

6. $\int_{\alpha}^{\alpha+2\pi} \sin nx \cos nx\, dx = \left| \frac{\sin^2 nx}{2n} \right|_{\alpha}^{\alpha+2\pi} = 0$ $(n \neq 0)$

7. $\int_{\alpha}^{\alpha+2\pi} \sin mx \sin nx\, dx = \frac{1}{2}\left| \frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right|_{\alpha}^{\alpha+2\pi} = 0$ $(m \neq n)$

8. $\int_{\alpha}^{\alpha+2\pi} \sin^2 nx\, dx = \left| \frac{x}{2} - \frac{\sin 2nx}{4n} \right|_{\alpha}^{\alpha+2\pi} = \pi.$ $(n \neq 0)$

Proof. Let $f(x)$ be represented in the interval $(\alpha, \alpha + 2\pi)$ by the Fourier series :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad ...(i)$$

To find the coefficients a_0, a_n, b_n, we assume that the series (i) can be integrated term by term from $x = \alpha$ to $x = \alpha + 2\pi$.

To find a_0, integrate both sides of (i) from $x = \alpha$ to $x = \alpha + 2\pi$. Then

$$\int_{\alpha}^{\alpha+2\pi} f(x)\, dx = \frac{1}{2} a_0 \int_{\alpha}^{\alpha+2\pi} dx + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{n=1}^{\infty} a_n \cos nx \right) dx + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{n=1}^{\infty} b_n \sin nx \right) dx$$

$$= \frac{1}{2} a_0(\alpha + 2\pi - \alpha) + 0 + 0 = a_0\pi \qquad \text{[By integrals (1) and (2) above]}$$

Hence $$a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\, dx.$$

To find a_n, multiply each side of (i) by $\cos nx$ and integrate from $x = \alpha$ to $x = \alpha + 2\pi$. Then

$$\int_{\alpha}^{\alpha+2\pi} f(x) \cos nx\, dx = \frac{1}{2} a_0 \int_{\alpha}^{\alpha+2\pi} \cos nx\, dx + \int_{\alpha}^{\alpha+2\pi} \left(\sum_{n=1}^{\infty} a_n \cos nx \right) \cos nx\, dx$$

$$+ \int_{\alpha}^{\alpha+2\pi} \left(\sum_{n=1}^{\infty} b_n \sin nx \right) \cos nx\, dx$$

$$= 0 + \pi a_n + 0 \qquad \text{[By integrals (1), (3), (4), (5) and (6)]}$$

Hence $$a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x) \cos nx\, dx \cdot$$

To find b_n, multiply each side of (i) by $\sin nx$ and integrate from $x = \alpha$ to $x = \alpha + 2\pi$. Then

$$\int_\alpha^{\alpha+2\pi} f(x)\sin nx\,dx = \frac{1}{2}a_0\int_\alpha^{\alpha+2\pi}\sin nx\,dx + \int_\alpha^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} a_n\cos nx\right)\sin nx\,dx$$

$$+\int_\alpha^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} b_n\sin nx\right)\sin nx\,dx$$

$$= 0 + 0 + \pi b_n \qquad \text{[By integrals (2), (5), (6), (7) and (8)]}$$

Hence $$b_n = \frac{1}{\pi}\int_\alpha^{\alpha+2\pi} f(x)\sin nx\,dx.$$

Cor. 1. Making $\alpha = 0$, *the interval becomes* $0 < x < 2\pi$, and the formulae (I) reduce to

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \end{aligned}\right\} \qquad \text{...(II)}$$

Cor. 2. Putting $\alpha = -\pi$, *the interval becomes* $-\pi < x < \pi$ and the formulae (I) take the form :

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \end{aligned}\right\} \qquad \text{...(III)}$$

Example 10.1. *Obtain the Fourier series for* $f(x) = e^{-x}$ *in the interval* $0 < x < 2\pi$. *(S.V.T.U., 2007)*

Solution. Let $$e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx \qquad \text{...}(i)$$

Then $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx = \frac{1}{\pi}\Big| -e^{-x}\Big|_0^{2\pi} = \frac{1-e^{-2\pi}}{\pi}$$

and $$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi(n^2+1)}\Big| e^{-x}(-\cos nx + n\sin nx)\Big|_0^{2\pi} = \left(\frac{1-e^{-2\pi}}{\pi}\right)\cdot\frac{1}{n^2+1}$$

$\therefore$ $$a_1 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\frac{1}{2},\ a_2 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\cdot\frac{1}{5} \text{ etc.}$$

Finally, $$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$= \frac{1}{\pi(n^2+1)}\Big| e^{-x}(-\sin nx - n\cos nx)\Big|_0^{2\pi} = \left(\frac{1-e^{-2\pi}}{\pi}\right)\cdot\frac{n}{n^2+1}$$

$\therefore$ $$b_1 = \frac{1-e^{-2\pi}}{\pi}\cdot\frac{1}{2},\ b_2 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\cdot\frac{2}{5} \text{ etc.}$$

Substituting the values of a_0, a_n, b_n in (i), we get

$$e^{-x} = \frac{1-e^{-2\pi}}{\pi}\left\{\frac{1}{2} + \left(\frac{1}{2}\cos x + \frac{1}{5}\cos 2x + \frac{1}{10}\cos 3x + \ldots\right) + \left(\frac{1}{2}\sin x + \frac{2}{5}\sin 2x + \frac{3}{10}\sin 3x + \ldots\right)\right\}.$$

Example 10.2. *Find a Fourier series to represent $x - x^2$ from $x = -\pi$ to $x = \pi$.*
(V.T.U., 2011 ; Madras, 2006)

Solution. Let $$x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad ...(i)$$

Then $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\, dx = \frac{1}{\pi}\left| \frac{x^2}{2} - \frac{x^3}{3} \right|_{-\pi}^{\pi} = -\frac{2\pi^2}{3}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2) \cos nx\, dx^*$$

$$= \frac{1}{\pi}\left[(x - x^2)\frac{\sin nx}{n} - (1 - 2x) \times \left(-\frac{\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{-4(-1)^n}{n^2} \qquad [\because \cos n\pi = (-1)^n]$$

$\therefore$ $a_1 = 4/1^2, a_2 = -4/2^2, a_3 = 4/3^2, a_4 = -4/4^2$ etc.

Finally, $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2) \sin nx\, dx$$

$$= \frac{1}{\pi}\left[(x - x^2)\left(-\frac{\cos nx}{n}\right) - (1 - 2x) \times \left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi} = -2(-1)^n/n$$

$\therefore$ $b_1 = 2/1, b_2 = -2/2, b_3 = 2/3, b_4 = -2/4$ etc.

Substituting the values of a's and b's in (i), we get

$$x - x^2 = -\frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \dots\right]$$
$$+ 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \dots\right]$$

Obs. Putting $x = 0$, we find another interesting series $0 = -\frac{\pi^2}{3} + 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots\right)$

i.e., $$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}.$$ *(V.T.U., 2011)*

Note. In the above example, we have used the results $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$

Also $\sin\left(n + \frac{1}{2}\right)\pi = (-1)^n$ and $\cos\left(n + \frac{1}{2}\right)\pi = 0$. *The reader should remember these results.*

Example 10.3. *Expand $f(x) = x \sin x$ as a Fourier series in the interval $0 < x < 2\pi$.*
(S.V.T.U., 2009 ; Bhopal, 2009 ; Rohtak, 2006)

Solution. Let $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad ...(i)$$

Then $$a_0 = \frac{1}{\pi}\int_0^{2\pi} x \sin x\, dx = \frac{1}{\pi}\left| x(-\cos x) - 1.(-\sin x)\right|_0^{2\pi} = -2.$$

---

* Apply the *general rule of integration by parts* which states that if u, v be two functions of x and dashes denote differentiations and suffixes integrations w.r.t. x, then

$$\int uv\, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \dots$$

In other words : *Integral of the product of two functions*

= 1st function × integral of 2nd – go on differentiating 1st, integrating 2nd signs alternately +ve and –ve.

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos nx \, dx = \frac{1}{2\pi}\int_0^{2\pi} x\,(2\cos nx \sin x)\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\,[\sin(n+1)x - \sin(n-1)x]\, dx$$

$$= \frac{1}{2\pi}\left[x\left\{\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - 1\left\{-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left\{-\frac{\cos 2(n+1)\pi}{n+1} + \frac{\cos 2(n-1)\pi}{n-1}\right\}\right] = \frac{2}{n^2-1}.\ (n \neq 1)$$

When $n = 1$, $a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x\, dx = \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x\, dx$

$$= \frac{1}{2\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - 1.\left(-\frac{\sin 2x}{4}\right)\right]_0^{2\pi} = -\frac{1}{2}.$$

Finally, $b_n = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx\, dx = \frac{1}{2\pi}\int_0^{2\pi} x\,[\cos(n-1)x - \cos(n+1)x]dx$

$$= \frac{1}{2\pi}\left[x\left\{\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)}{n+1}\right\} - 1.\left\{-\frac{\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\cos 2(n-1)\pi}{(n-1)^2} - \frac{\cos 2(n+1)\pi}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right] = 0\ (n \neq 1)$$

When $n = 1$, $b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin x\, dx = \frac{1}{2\pi}\int_0^{2\pi} x\,(1 - \cos 2x)\, dx$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - 1.\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi} = \pi$$

Substituting the values of a's and b's, in (i), we get

$$x \sin x = -1 + \pi \sin x - \frac{1}{2}\cos x + \frac{2}{2^2-1}\cos 2x + \frac{2}{3^2-1}\cos 3x + \ldots$$

Example 10.4. *Expand $f(x) = \sqrt{(1-\cos x)}$, $0 < x < 2\pi$ in a Fourier series. Hence evaluate*

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots\ldots$$ *(Mumbai, 2006 ; J.N.T.U., 2006)*

Solution. We have $f(x) = \sqrt{(1-\cos x)} = \sqrt{2}\sin x/2$.

Let $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \ldots(i)$$

Then $$a_0 = \frac{1}{\pi}\int_0^{2\pi}\sqrt{2}\sin x/2\, dx = \frac{\sqrt{2}}{\pi}\left| -2\cos\frac{\pi}{2}\right|_0^{2\pi} = \frac{4\sqrt{2}}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi}\sqrt{2}\sin\frac{x}{2}\cos nx\, dx = \frac{\sqrt{2}}{2\pi}\int_0^{2\pi} 2\cos nx \sin x/2\, dx$$

$$= \frac{1}{\sqrt{2}\pi}\int_0^{2\pi}\left[\sin\left(n+\frac{1}{2}\right)x - \sin\left(n-\frac{1}{2}\right)x\right]dx$$

$$= \frac{1}{\sqrt{2}\,\pi}\left| -\frac{2}{2n+1}\cos\left(\frac{2n+1}{2}\right) + \frac{2}{2n-1}\cos\frac{2n-1}{2}x\right|_0^{2\pi}$$

$$= \frac{2}{\sqrt{2}\,\pi}\left\{-\frac{1}{2n+1}[\cos(2n+1)\pi - 1] + \frac{1}{2n-1}[\cos(2n-1)\pi - 1]\right\}$$

$$= \frac{\sqrt{2}}{\pi}\left(\frac{2}{2n+1} - \frac{2}{2n-1}\right) = -\frac{4\sqrt{2}}{\pi(4n^2-1)} \qquad [\because \cos(2n+1)\pi = \cos(2n-1)\pi = -1]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin\frac{x}{2}\sin nx\, dx = \frac{\sqrt{2}}{2\pi}\int_0^{2\pi} 2\sin nx \sin x/2\, dx$$

$$= \frac{1}{\sqrt{2}\pi}\int_0^{2\pi}\left[\cos\left(n-\frac{1}{2}\right)x - \cos\left(n+\frac{1}{2}\right)x\right]dx$$

$$= \frac{1}{\sqrt{2}\pi}\left|\frac{2}{2n-1}\sin\left(\frac{2n-1}{2}\right)x - \frac{2}{2n+1}\sin\left(\frac{2n+1}{2}\right)x\right|_0^{2\pi}$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{1}{2n-1}\{\sin(2n-1)\pi - 0\} - \frac{1}{2n+1}\{\sin(2n+1)\pi - 0\}\right] = 0$$

Substituting the values of a's and b's in (i), we get

$$\sqrt{(1-\cos x)} = \frac{2\sqrt{2}}{\pi} - \sum_{n=1}^{\infty}\frac{4\sqrt{2}}{(4n^2-1)\pi}\cos nx$$

When $x = 0$, we have

$$0 = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)} \qquad i.e., \quad \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots\ldots \infty = \frac{1}{2}.$$

PROBLEMS 10.1

1. Obtain a Fourier series to represent e^{-ax} from $x = -\pi$ to $x = \pi$. Hence derive series for $\pi/\sinh \pi$.

2. Prove that $x^2 = \dfrac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\dfrac{\cos nx}{n^2}$, $-\pi < x < \pi$. *(P.T.U., 2009 ; Bhopal, 2008 ; B.P.T.U., 2006)*

   Hence show that (i) $\sum \dfrac{1}{n^2} = \dfrac{\pi^2}{6}$. *(Anna, 2009 ; P.T.U., 2009 ; Osmania, 2003)*

   (ii) $\sum \dfrac{1}{(2n-1)^2} = \dfrac{\pi^2}{8}$ $\qquad (iii)$ $\dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \ldots = \dfrac{\pi^2}{12}$ *(S.V.T.U., 2008)*

   (iv) $\sum \dfrac{1}{n^4} = \dfrac{\pi^4}{90}$. *(Bhopal, 2008)*

3. If $f(x) = \left(\dfrac{\pi - x}{2}\right)^2$ in the range 0 to 2π, show that $f(x) = \dfrac{\pi^2}{12} + \sum_{n=1}^{\infty}\dfrac{\cos nx}{n^2}$. *(Delhi, 2002 ; Madras, 2000)*

4. Prove that in the range $-\pi < x < \pi$, $\cosh ax = \dfrac{2a^2}{\pi}\sinh a\pi\left[\dfrac{1}{2a^2} + \sum_{n=1}^{\infty}\dfrac{(-1)^n}{n^2+a^2}\cos nx\right]$.

5. $f(x) = x + x^2$ for $-\pi < x < \pi$ and $f(x) = \pi^2$ for $x = \pm\pi$. Expand $f(x)$ in Fourier series. *(Kurukshetra, 2005 ; U.P.T.U., 2003)*

   Hence show that $x + x^2 = \dfrac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n\left\{\dfrac{4}{n^2}\cos nx - \dfrac{2}{n}\sin nx\right\}$

   and $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots\ldots \dfrac{\pi^2}{8}$ *(V.T.U., 2008)*

10.3 CONDITIONS FOR A FOURIER EXPANSION

The reader must not be misled by the belief that the Fourier expansion of $f(x)$ in each case shall be valid. The above discussion has merely shown that if $f(x)$ has an expansion, then the coefficients are given by Euler's formulae. The problems concerning the possibility of expressing a function by Fourier series and convergence of this series are many and cumbersome. Such questions should be left to the curiosity of a pure-mathematician. However, almost all engineering applications are covered by the following well-known **Dirichlet's conditions***:

Any function f(x) can be developed as a Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ *where* a_0, a_n, b_n *are constants, provided :*

(i) $f(x)$ *is periodic, single-valued and finite;*

(ii) $f(x)$ *has a finite number of discontinuties in any one period;*

(iii) $f(x)$ *has at the most a finite number of maxima and minima.* (*Anna, 2009 ; P.T.U., 2009*)

In fact the problem of expressing any function $f(x)$ as a Fourier series depends upon the evaluation of the integrals.

$$\frac{1}{\pi}\int f(x)\cos nx\, dx \,;\, \frac{1}{\pi}\int f(x)\sin nx\, dx$$

within the limits $(0, 2\pi)$, $(-\pi, \pi)$ or $(\alpha, \alpha + 2\pi)$ according as $f(x)$ is defined for every value of x in $(0, 2\pi)$, $(-\pi, \pi)$ or $(\alpha, \alpha + 2\pi)$.

PROBLEMS 10.2

State giving reasons whether the following functions can be expanded in Fourier series in the interval $-\pi \le x \le \pi$.

1. $\operatorname{cosec} x$
2. $\sin 1/x$ (*P.T.U., 2002*)
3. $f(x) = (m + 1)/m,\ \pi/(m + 1) < | x | \le \pi/m,\ m = 1, 2, 3, \ldots \infty$.

10.4 FUNCTIONS HAVING POINTS OF DISCONTINUITY

In deriving the Euler's formulae for a_0, a_n, b_n, it was assumed that $f(x)$ was continuous. Instead a function may have a finite number of points of finite discontinuity *i.e.*, its graph may consist of a finite number of different curves given by different equations. Even then such a function is expressible as a Fourier series.

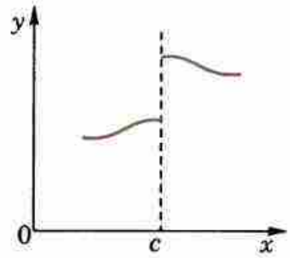

Fig. 10.1

For instance, if in the interval $(\alpha, \alpha + 2\pi)$, $f(x)$ is defined by

$$f(x) = \phi(x),\ \alpha < x < c.$$
$$= \psi(x),\ c < x < \alpha + 2\pi,$$ *i.e.*, c is the point of discontinuity, then

$$a_0 = \frac{1}{\pi}\left[\int_\alpha^c \phi(x)\, dx + \int_c^{\alpha+2\pi} \psi(x)\, dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_\alpha^c \phi(x)\cos nx\, dx + \int_c^{\alpha+2\pi} \psi(x)\cos nx\, dx\right]$$

and
$$b_n = \frac{1}{\pi}\left[\int_\alpha^c \phi(x)\sin nx\, dx + \int_c^{\alpha+2\pi} \psi(x)\sin nx\, dx\right]$$

At a point of finite discontinuity $x = c$, there is a finite jump in the graph of function (Fig. 10.1). Both the limit on the left [*i.e.*, $f(c - 0)$] and the limit on the right [*i.e.*, $f(c + 0)$] exist and are different. At such a point, Fourier series gives the value of $f(x)$ as the arithmetic mean of these two limits,

i.e., at $x = c$, $\quad f(x) = \frac{1}{2}[f(c-0) + f(c+0)]$.

Example 10.5. *Find the Fourier series expansion for* $f(x)$, *if*

$$f(x) = -\pi,\ -\pi < x < 0$$
$$x,\ 0 < x < \pi.$$ (*Bhopal, 2008 S*)

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$. (*Kottayam, 2005*)

*See footnote p. 307.

Solution. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ ...(*i*)

Then $$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\, dx + \int_0^{\pi} x\, dx\right] = \frac{1}{\pi}\left[-\pi \left| x \right|_{-\pi}^{0} + \left| x^2/2 \right|_0^{\pi}\right] = \frac{1}{\pi}\left(-\pi^2 + \frac{\pi^2}{2}\right) = -\frac{\pi}{2};$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi) \cos nx\, dx + \int_0^{\pi} x \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[-\pi \left| \frac{\sin nx}{n} \right|_{-\pi}^{0} + \left| \frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{1}{n^2} \cos n\pi - \frac{1}{n^2}\right] = \frac{1}{\pi n^2} (\cos n\pi - 1)$$

$\therefore$ $$a_1 = \frac{-2}{\pi . 1^2},\ a_2 = 0,\ a_3 = -\frac{2}{\pi . 3^2},\ a_4 = 0,\ a_5 = -\frac{2}{\pi . 5^2} \text{ etc.}$$

Finally, $$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi) \sin nx\, dx + \int_0^{\pi} x \sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\left| \frac{\pi \cos nx}{n} \right|_{-\pi}^{0} + \left| -x\frac{\cos nx}{n} + \frac{\sin nx}{n^2} \right|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1 - \cos n\pi) - \frac{\pi}{n} \cos n\pi\right] = \frac{1}{n}(1 - 2 \cos n\pi)$$

$\therefore$ $$b_1 = 3,\ b_2 = -\frac{1}{2},\ b_3 = 1,\ b_4 = -\frac{1}{4}, \text{ etc.}$$

Hence substituting the values of a's and b's in (*i*), we get

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right) + 3 \sin x - \frac{\sin 2x}{2} + \frac{3 \sin 3x}{3} - \frac{\sin 4x}{4} + \ldots \quad \ldots(ii)$$

which is the required result.

Putting $x = 0$ in (*ii*), we obtain $f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots \infty\right)$ ...(*iii*)

Now $f(x)$ is discontinuous at $x = 0$. As a matter of fact

$f(0 - 0) = -\pi$ and $f(0 + 0) = 0 \quad \therefore \quad f(0) = \frac{1}{2} [f(0 - 0) + f(0 + 0)] = -\pi/2.$

Hence (*iii*) takes the form $-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right]$ whence follows the result.

Example 10.6. *If* $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ \sin x, & 0 \le x \le \pi \end{cases}$, *prove that* $f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1}$.

Hence show that $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots - \infty = \frac{1}{4}(\pi - 2)$ (Bhopal, 2008 ; Mumbai, 2005 S ; Rohtak, 2005)

Solution. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

Then $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\, dx + \int_0^{\pi} \sin x\, dx\right] = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 . dx + \int_0^{\pi} \sin x \cos nx\, dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi} [\sin (n+1)x - \sin (n-1) x]\, dx = \frac{1}{2\pi}\left[-\frac{\cos (n+1)x}{n+1} + \frac{\cos (n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] \qquad (n \neq 1)$$

$$= \frac{1}{2\pi}\left\{\frac{1-(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}-1}{n-1}\right\} = 0, \text{ when } n \text{ is odd}$$

$$= -\frac{2}{\pi(n^2-1)}, \text{ when } n \text{ is even.}$$

When $n = 1$, $a_1 = \dfrac{1}{\pi}\displaystyle\int_0^\pi \sin x \cos x\, dx = \frac{1}{2\pi}\int_0^\pi \sin 2x\, dx = \frac{1}{2\pi}\left|-\frac{\cos 2x}{2}\right|_0^\pi = 0$

Finally, $\quad b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,.\,dx + \int_0^\pi \sin x \sin nx\, dx\right]$

$$= \frac{1}{2\pi}\int_0^\pi [\cos \overline{n-1}x - \cos \overline{n+1}x]\, dx = \frac{1}{2\pi}\left[\frac{\sin \overline{n-1}x}{n-1} - \frac{\sin \overline{n+1}x}{n+1}\right]_0^\pi = 0 \quad (n \neq 1)$$

When $n = 1$, $b_1 = \dfrac{1}{\pi}\displaystyle\int_0^\pi \sin x \sin x\, dx = \frac{1}{2\pi}\int_0^\pi (1-\cos 2x)\, dx = \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^\pi = \frac{1}{2}$

Hence $\quad f(x) = \dfrac{1}{\pi} - \dfrac{2}{\pi}\left[\dfrac{\cos 2x}{2^2-1} + \dfrac{\cos 4x}{4^2-1} + \dfrac{\cos 6x}{6^2-1} + \ldots\right] + \dfrac{1}{2}\sin x \qquad \ldots(i)$

Putting $x = \dfrac{\pi}{2}$ in (*i*), we get $1 = \dfrac{1}{\pi} - \dfrac{2}{\pi}\left(-\dfrac{1}{1.3} + \dfrac{1}{3.5} - \dfrac{1}{5.7} + \ldots \infty\right) + \dfrac{1}{2}$

Whence $\dfrac{1}{1.3} - \dfrac{1}{3.5} + \dfrac{1}{5.7} - \ldots \infty = \dfrac{1}{4}(\pi - 2).$

Example 10.7. *Find the Fourier series for the function*

$$f(t) = \begin{cases} -1 & \text{for } -\pi < t < -\pi/2 \\ 0 & \text{for } -\pi/2 < t < \pi/2 \\ 1 & \text{for } \pi/2 < t < \pi \end{cases}$$

Solution. Let $\quad f(t) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt \qquad \ldots(i)$

Then $\quad a_0 = \dfrac{1}{\pi}\left\{\displaystyle\int_{-\pi}^{-\pi/2}(-1)\,dt + \int_{-\pi/2}^{\pi/2}(0)\,dt + \int_{\pi/2}^{\pi}(1)\,dt\right\}$

$$= \frac{1}{\pi}\left\{\left|-x\right|_{-\pi}^{-\pi/2} + \left|x\right|_{\pi/2}^{\pi}\right\} = \frac{1}{\pi}(\pi/2 - \pi + \pi - \pi/2) = 0$$

$$a_n = \frac{1}{\pi}\left\{\int_{-\pi}^{-\pi/2}(-1)\cos nt\,dt + \int_{-\pi/2}^{\pi/2}(0)\cos nt\,dt + \int_{\pi/2}^{\pi}(1)\cos nt\,dt\right\}$$

$$= \frac{1}{\pi}\left\{\left|-\frac{\sin nt}{n}\right|_{-\pi}^{-\pi/2} + \left|\frac{\sin nt}{n}\right|_{\pi/2}^{\pi}\right\} = \frac{1}{n\pi}\left(\frac{\sin n\pi}{2} - \frac{\sin n\pi}{2}\right) = 0$$

and $\quad b_n = \dfrac{1}{\pi}\left\{\displaystyle\int_{-\pi}^{-\pi/2}(-1)\sin nt\,dt + \int_{-\pi/2}^{\pi/2}(0)\sin nt\,dt + \int_{\pi/2}^{\pi}(1)\sin nt\,dt\right\}$

$$= \frac{1}{\pi}\left\{\left|\frac{\cos nt}{n}\right|_{-\pi}^{-\pi/2} + \left|-\frac{\cos nt}{n}\right|_{\pi/2}^{\pi}\right\} = \frac{2}{n\pi}\left(\cos\frac{n\pi}{2} - \cos n\pi\right)$$

$\therefore \qquad b_1 = \frac{2}{\pi}, b_2 = -\frac{2}{\pi}, b_3 = \frac{2}{3\pi}$ etc.

Hence substituting the values of a's and b's in (i), we get $f(t) = \frac{2}{\pi}\left(\sin t - \sin 2t + \frac{1}{3}\sin 3t + \ldots\right)$.

PROBLEMS 10.3

1. Find the Fourier series to represent the function $f(x)$ given by
$$f(x) = x \text{ for } 0 \le x \le \pi, \text{ and } = 2\pi - x \text{ for } \pi \le x \le 2\pi.$$ *(S.V.T.U., 2008 ; B.P.T.U., 2005 S)*

   Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \infty = \frac{\pi^2}{8}$. *(Madras 2000 S ; V.T.U., 2000 S)*

2. An alternating current after passing through a rectifier has the form
$$i = I_0 \sin x \quad \text{for } 0 \le x \le \pi$$
$$= 0 \quad \text{for } \pi \le x \le 2\pi$$
where I_0 is the maximum current and the period is 2π (Fig. 10.2). Express i as a Fourier series and evaluate
$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots \infty$$ *(V.T.U., 2007 ; Calicut, 2005)*

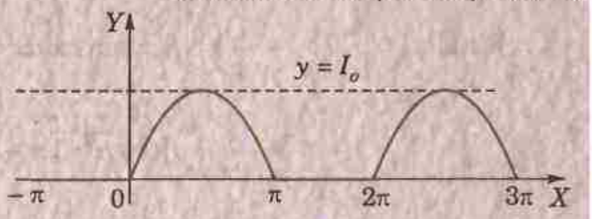

Fig. 10.2

3. Draw the graph of the function $f(x) = 0, -\pi < x < 0$
$$= x^2, 0 < x < \pi.$$
If $f(2\pi + x) = f(x)$, obtain Fourier series of $f(x)$.

4. Find the Fourier series of the following function :
$$f(x) = x^2, \quad 0 \le x \le \pi,$$ *(Mumbai, 2009)*
$$= -x^2, \quad -\pi \le x \le 0.$$ *(Hissar, 2007)*

5. Find a Fourier series for the function defined by
$$f(x) = \begin{cases} -1, \text{ for } -\pi < x < 0 \\ 0, \text{ for } x = 0 \\ 1, \text{ for } 0 < x < \pi \end{cases}$$
Hence prove that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \infty = \frac{\pi}{4}$. *(U.P.T.U., 2005)*

10.5 CHANGE OF INTERVAL

In many engineering problems, the period of the function required to be expanded is not 2π but some other interval, say : $2c$. In order to apply the foregoing discussion to functions of period $2c$, this interval must be converted to the length 2π. This involves only a proportional change in the scale.

Consider the periodic function $f(x)$ defined in $(\alpha, \alpha + 2c)$. To change the problem to period 2π

put $\qquad z = \pi x/c \quad$ or $\quad x = cz/\pi \qquad$ ...(1)

so that when $\qquad x = \alpha, \qquad z = \alpha\pi/c = \beta$ (say)

when $\qquad x = \alpha + 2c, \qquad z = (\alpha + 2c)\pi/c = \beta + 2\pi$.

Thus the function $f(x)$ of period $2c$ in $(\alpha, \alpha + 2c)$ is transformed to the function $f(cz/\pi)$ [$= F(z)$ say] of period 2π in $(\beta, \beta + 2\pi)$. Hence $f(cz/\pi)$ can be expressed as the Fourier series

$$f\left(\frac{cz}{\pi}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nz + \sum_{n=1}^{\infty} b_n \sin nz \qquad ...(2)$$

where
$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} f\left(\frac{cz}{\pi}\right) dz \\ a_n &= \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} f\left(\frac{cz}{\pi}\right) \cos nz\, dz \\ b_n &= \frac{1}{\pi}\int_{\beta}^{\beta+2\pi} f\left(\frac{cz}{\pi}\right) \sin nz\, dz \end{aligned}\right\} \qquad ...(3)$$

Making the inverse substitutions $z = \pi x/c$, $dz = (\pi/c)\, dx$ in (2) and (3) *the Fourier expansion of $f(x)$ in the interval $(\alpha, \alpha + 2c)$ is given by*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}$$

where

$$\left.\begin{aligned} a_0 &= \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\,dx \\ a_n &= \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\cos\frac{n\pi x}{c}\,dx \\ b_n &= \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\sin\frac{n\pi x}{c}\,dx \end{aligned}\right\} \qquad ...(4)$$

Cor. Putting $\alpha = 0$ in (4), we get the results for the interval $(0, 2c)$ and putting $\alpha = -c$ in (4), we get results for the interval $(-c, c)$.

Example 10.8. *Expand $f(x) = e^{-x}$ as a Fourier series in the interval $(-l, l)$.*
(Kerala, 2005 ; V.T.U., 2004)

Solution. The required series is of the form

$$e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad ...(i)$$

Then $$a_0 = \frac{1}{l}\int_{-l}^{l} e^{-x}\,dx = \frac{1}{l}\left|-e^{-x}\right|_{-l}^{l} = \frac{1}{l}(e^{l} - e^{-l}) = \frac{2\sinh l}{l}$$

and $$a_n = \frac{1}{l}\int_{-l}^{l} e^{x}\cos\frac{n\pi x}{l}\,dx \qquad \left[\because \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$$

$$= \frac{1}{l}\left|\frac{e^{-x}}{1+(n\pi/l)^2}\left(-\cos\frac{n\pi x}{l} + \frac{n\pi}{l}\sin\frac{n\pi x}{l}\right)\right|_{-l}^{l} = \frac{2l(-1)^n\sinh l}{l^2+(n\pi)^2} \qquad [\because \cos n\pi = (-1)^n$$

$$\therefore \qquad a_1 = \frac{-2l\sinh l}{l^2+\pi^2},\; a_2 = \frac{2l\sinh l}{l^2+2^2\pi^2},\; a_3 = \frac{2l\sinh l}{l^2+3^2\pi^2} \text{ etc.}$$

Finally, $$b_n = \frac{1}{l}\int_{-l}^{l} e^{-x}\sin\frac{n\pi x}{l}\,dx \qquad \left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

$$= \frac{1}{l}\left|\frac{e^{-x}}{1+(n\pi/l)^2}\left(-\sin\frac{n\pi x}{l} - \frac{n\pi}{l}\cos\frac{n\pi x}{l}\right)\right|_{-l}^{l} = \frac{2n\pi(-1)^n\sinh l}{l^2+(n\pi)^2}$$

$$\therefore \qquad b_1 = \frac{-2\pi\sinh l}{l^2+\pi^2},\; b_2 = \frac{4\pi\sinh l}{l^2+2^2\pi^2},\; b_2 = \frac{-6\pi\sinh l}{l^2+3^2\pi^2} \text{ etc.}$$

Substituting the values of a's and b's in (i), we get

$$e^{-x} = \sinh l\left\{\frac{1}{l} - 2l\left(\frac{1}{l^2+\pi^2}\cos\frac{\pi x}{l} - \frac{1}{l^2+2^2\pi^2}\cos\frac{2\pi x}{l} + \frac{1}{l^2+3^2\pi^2}\cos\frac{3\pi x}{l} - ...\right)\right.$$
$$\left. - 2\pi\left(\frac{1}{l^2+\pi^2}\sin\frac{n\pi}{l} - \frac{2}{l^2+2^2\pi^2}\sin\frac{2\pi x}{l} + \frac{3}{l^2+3^2\pi^2}\sin\frac{3\pi x}{l} - ...\right)\right\}$$

Example 10.9. *Find the Fourier series expansion of $f(x) = 2x - x^2$ in $(0, 3)$ and hence deduce that* $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ... - \infty = \frac{\pi}{12}$. *(Mumbai, 2005)*

Solution. The required series is of the form

$$2x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x \quad \text{where } l = 3/2. \qquad ...(i)$$

Then $$a_0 = \frac{1}{l}\int_0^{2l} (2x - x^2)dx = \frac{2}{3}\left| x^2 - \frac{x^3}{3}\right|_0^3 = 0$$

$$a_n = \frac{1}{l}\int_0^{2l} (2x - x^2)\cos\frac{n\pi x}{l}dx = \frac{2}{3}\int_0^3 (2x - x^2)\cos\frac{2n\pi x}{3}dx$$

$$= \frac{2}{3}\left[(2x - x^2)\frac{\sin 2n\pi x/3}{2n\pi/3} - (2 - 2x)\frac{-\cos 2n\pi x/3}{(2n\pi/3)^2} + (-2)\frac{-\sin 2n\pi x/3}{(2n\pi/3)^3}\right]_0^3$$

$$= \frac{2}{3}\cdot\frac{9}{4n^2\pi^2}[(2-6)\cos 2n\pi - 2] = -\frac{9}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} (2x - x^2)\sin\frac{n\pi x}{l}dx = \frac{2}{3}\int_0^3 (2x - x^2)\sin\frac{2n\pi x}{3}dx$$

$$= \frac{2}{3}\left[(2x - x^2)\frac{-\cos 2n\pi x/3}{2n\pi/3} - (2 - 2x)\frac{-\sin 2n\pi x/3}{(2n\pi/3)^2} + (-2)\frac{\cos 2n\pi x/3}{(2n\pi/3)^3}\right]_0^3$$

$$= \frac{2}{3}\left\{-\frac{6}{n^2\pi^2}\cos 2n\pi - \frac{27}{4n^3\pi^3}(\cos 2n\pi - 1)\right\} = \frac{3}{n\pi}$$

Substituting the values of a_0, a_n, b_n in (i), we get

$$2x - x^2 = -\sum_{n=1}^{\infty}\frac{9}{n^2\pi^2}\cos\frac{2n\pi x}{3} + \sum_{n=1}^{\infty}\frac{3}{n\pi}\sin\frac{2n\pi x}{3}$$

Putting $x = 3/2$, we get

$$3 - \frac{9}{4} = -\sum_{n=1}^{\infty}\frac{9}{n^2\pi^2}\cos n\pi \quad \text{or} \quad -\sum_{n=1}^{\infty}\frac{\cos n\pi}{n^2} = \frac{\pi^2}{9}\cdot\frac{3}{4}$$

or $$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \dots \infty = \frac{\pi^2}{12}.$$

Example 10.10. *Obtain Fourier series for the function*

$$f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2 - x), & 1 \le x \le 2 \end{cases}$$ *(V.T.U., 2011 ; Bhopal, 2008 ; Mumbai, 2007)*

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \infty = \frac{\pi^2}{8}$.

Solution. The required series is of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

Then $$a_0 = \int_0^1 \pi x dx + \int_1^2 \pi(2 - x)dx = \pi\left|\frac{x^2}{2}\right|_0^1 + \pi\left|2x - \frac{x^2}{2}\right|_1^2 = \pi\left(\frac{1}{2}\right) + \pi\left\{(4-2) - \left(2 - \frac{1}{2}\right)\right\} = \pi$$

$$a_n = \int_0^1 \pi x \cos n\pi x dx + \int_1^2 \pi(2 - x)\cos n\pi x dx$$

$$= \left|\pi x\cdot\frac{\sin n\pi x}{n\pi} - \pi\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_0^1 + \left|\pi(2 - x)\frac{\sin n\pi x}{n\pi} - (-\pi)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_1^2$$

$$= \left(\frac{\cos n\pi}{n^2\pi} - \frac{1}{n^2\pi^2}\right) - \left(\frac{\cos 2n\pi}{n^2\pi} - \frac{\cos n\pi}{n^2\pi}\right) = \frac{2}{n^2\pi}[(-1)^n - 1]$$

$$= 0 \text{ when } n \text{ is even ; } -\frac{4}{n^2\pi} \text{ when } n \text{ is odd.}$$

$$b_n = \int_0^1 \pi x \sin n\pi x \, dx + \int_1^2 \pi(2-x) \sin n\pi x \, dx$$

$$= \left| \pi x \left(-\frac{\cos n\pi x}{n\pi}\right) - \pi\left(-\frac{\sin n\pi x}{n^2\pi^2}\right) \right|_0^1 + \left| \pi(2-x)\left(\frac{-\cos n\pi x}{n\pi}\right) - (-\pi)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right) \right|_1^2$$

$$= \left(-\frac{\cos n\pi}{n}\right) + \left(\frac{\cos n\pi}{n}\right) = 0$$

Hence $\quad f(x) = \dfrac{\pi}{2} - \dfrac{4}{\pi}\left(\dfrac{\cos \pi x}{1^2} + \dfrac{\cos 3\pi x}{3^2} + \dfrac{\cos 5\pi x}{5^2} + \dots \infty\right)$

Putting $x = 2$, $0 = \dfrac{\pi}{2} - \dfrac{4}{\pi}\left(\dfrac{\cos 2\pi}{1^2} + \dfrac{\cos 6\pi}{3^2} + \dfrac{\cos 10\pi}{5^2} + \dots \infty\right)$

Whence $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \dots \infty = \dfrac{\pi^2}{8}$.

Example 10.11. *Find the Fourier series for*

$$f(t) = 0, \; -2 < t < -1$$
$$= 1 + t, \; -1 < t < 0$$
$$= 1 - t, \; 0 < t < 1$$
$$= 0, \quad 1 < t < 2.$$

Solution. Let $\quad f(t) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{2} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{2}$ ...(*i*)

[$\because$ $2c = 2 - (-2)$ so that $c = 2$]

Then $\quad a_0 = \dfrac{1}{2}\left\{\displaystyle\int_{-2}^{-1} (0)\,dt + \int_{-1}^{0} (1+t)\,dt + \int_0^1 (1-t)\,dt + \int_1^2 (0)\,dt\right\} = \dfrac{1}{2}\left\{\left| t + \dfrac{t^2}{2}\right|_{-1}^{0} + \left| t - \dfrac{t^2}{2}\right|_0^1\right\}$

$$= \frac{1}{2}\left\{-\left(-1+\frac{1}{2}\right) + \left(1 - \frac{1}{2}\right)\right\} = \frac{1}{2}$$

$$a_n = \frac{1}{2}\left\{\int_{-1}^{0} (1+t) \cos \frac{n\pi t}{2} dt + \int_0^1 (1-t) \cos \frac{n\pi t}{2} dt\right\}$$ [Integrate by parts]

$$= \frac{1}{2}\left\{\left| (1+t)\left(\sin \frac{n\pi t}{2}\right)\frac{2}{n\pi} - (1)\left(-\cos\frac{n\pi t}{2}\right)\frac{4}{n^2\pi^2}\right|_{-1}^{0}\right.$$
$$\left. + \left| (1-t)\left(\sin\frac{n\pi t}{2}\right)\frac{2}{n\pi} - (-1)\left(-\cos\frac{n\pi t}{2}\right)\frac{4}{n^2\pi^2}\right|_0^1\right\}$$

$$= \frac{4}{n^2\pi^2}(1 - \cos n\pi/2)$$

$$b_n = \frac{1}{2}\left\{\int_{-1}^{0} (1+t)\sin\frac{n\pi t}{2} dt + \int_0^1 (1-t)\sin\frac{n\pi t}{2} dt\right\}$$

$$= \frac{1}{2}\left\{\left| (1+t)\left(-\cos\frac{n\pi t}{2}\right)\frac{2}{n\pi} - 1\left(-\sin\frac{n\pi t}{2}\right)\frac{4}{n^2\pi^2}\right|_{-1}^{0}\right.$$
$$\left. + \left| (1-t)\left(-\cos\frac{n\pi t}{2}\right)\frac{2}{n\pi} - (-1)\left(-\sin\frac{n\pi t}{2}\right)\frac{4}{n^2\pi^2}\right|_0^1\right\}$$

$$= \frac{1}{2}\left\{\left(\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right)-\left(\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right\} = 0$$

Substituting the values of a's and b's in (i), we get

$$f(t) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left(1-\cos\frac{n\pi}{2}\right)\cos\frac{n\pi t}{2}.$$

PROBLEMS 10.4

1. Obtain the Fourier series for $f(x) = \pi x$ in $0 \le x \le 2$.
2. (i) Find the Fourier series to represent x^2 in the interval $(0, a)$. *(Mumbai, 2009)*
   (ii) Find a Fourier series for $f(t) = 1 - t^2$ when $-1 \le t \le 1$. *(Mumbai, 2006)*
3. If $f(x) = 2x - x^2$ in $0 \le x \le 2$, show that $f(x) = \frac{2}{3} - \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\cos n\pi x$. *(V.T.U., 2006)*
4. Find the Fourier series for $f(x) = \begin{cases} x & \text{in } 0 \le x \le 3 \\ 6-x & \text{in } 3 \le x \le 6 \end{cases}$ *(Anna, 2008)*
5. A sinusoidal voltage $E \sin \omega t$ is passed through a half-wave rectifier which clips the negative portion of the wave. Develop the resulting periodic function
   $U(t) = 0$ when $-T/2 < t < 0$
   $= E \sin \omega t$ when $0 < t < T/2$,
   and $T = 2\pi/\omega$, in a Fourier series. *(Calicut, 1999)*
6. Find the Fourier series of the function $f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ 0, & x = 1 \\ \pi(x-2), & 1 < x < 2 \end{cases}$

   Hence show that $\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ *(Mumbai, 2008)*

10.6 (1) EVEN AND ODD FUNCTIONS

A function $f(x)$ is said to be **even** *if $f(-x) = f(x)$,*

e.g., $\cos x$, $\sec x$, x^2 are all even functions. Graphically an even function is symmetrical about the y-axis.

A function $f(x)$ is said to be **odd** *if $f(-x) = -f(x)$,*

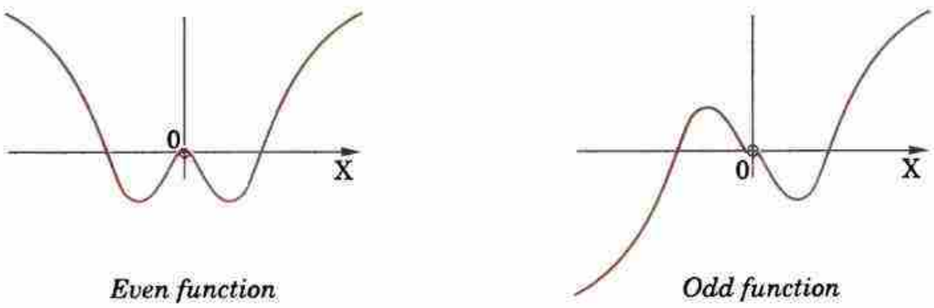

Fig. 10.3

e.g. $\sin x$, $\tan x$, x^3 are odd functions. Graphically, an odd function is symmetrical about the origin.

We shall be using the following property of definite integrals in the next paragraph :

$$\int_{-c}^{c} \mathbf{f(x)\,dx} = \mathbf{2}\int_{0}^{c} \mathbf{f(x)\,dx}, \text{ when } f(x) \text{ is an even function.}$$
$$= \mathbf{0}, \text{ when } f(x) \text{ is an odd function.}$$

(2) Expansions of even or odd periodic functions. We know that a periodic function $f(x)$ defined in $(-c, c)$ can be represented by the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c},$$

where $$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)dx,\ a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}dx,\ b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}dx.$$

Case I. *When f(x) is an even function* $a_0 = \frac{1}{c}\int_{-c}^{c} f(x)dx = \frac{2}{c}\int_{0}^{c} f(x)dx.$

Since $f(x)\cos\frac{n\pi x}{c}$ is also an even function,

$$\therefore \quad a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}dx = \frac{2}{c}\int_{0}^{c} f(x)\cos\frac{n\pi x}{c}dx$$

Again since $f(x)\sin\frac{n\pi x}{c}$ is an odd function, $\therefore \quad b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}dx = 0.$

Hence, *if a periodic function f (x) is even, its Fourier expansion contains only cosine terms, and*

$$\left.\begin{aligned} a_0 &= \frac{2}{c}\int_{0}^{c} f(x)dx \\ a_n &= \frac{2}{c}\int_{0}^{c} f(x)\cos\frac{n\pi x}{c}dx \end{aligned}\right\} \quad ...(1)$$

Case II. *When f(x) is an odd function,* $a_0 = \frac{1}{c}\int_{-c}^{c} f(x)dx = 0,$

Since $\cos\frac{n\pi x}{c}$ is an even function, therefore, $f(x)\cos\frac{n\pi x}{c}$ is an odd function.

$$\therefore \quad a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c} = 0$$

Again since $\sin\frac{n\pi x}{c}$ is an odd function, therefore, $f(x)\sin\frac{n\pi x}{c}$ is an even function.

$$\therefore \quad b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}dx = \frac{2}{c}\int_{0}^{c} f(x)\sin\frac{n\pi x}{c}dx$$

Thus, *if a periodic function f (x) is odd, its Fourier expansion contains only sine terms and*

$$b_n = \frac{2}{c}\int_{0}^{c} f(x)\sin\frac{n\pi x}{c}dx \quad ...(2)$$

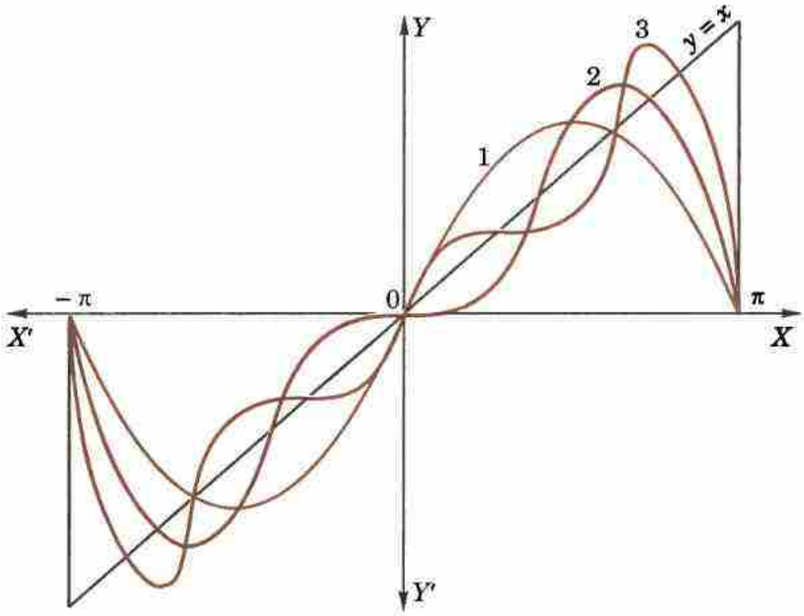

Fig. 10.4

Example 10.12. *Express $f(x) = x/2$ as a Fourier series in the interval $-\pi < x < \pi$.* (J.N.T.U., 2006)

Solution. Since $f(-x) = -x/2 = -f(x).$

$\therefore$ $f(x)$ is an odd function and hence $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

where $$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx = \frac{2}{\pi}\int_0^{\pi} \frac{x}{2} \sin nx\, dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi} = -\frac{\cos n\pi}{n}.$$

$\therefore$ $b_1 = 1/1, b_2 = -1/2, b_3 = 1/3, b_4 = -1/4$, etc.

Hence the series is $x/2 = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots$ ...(*i*)

Obs. The graphs of $y = 2\sin x$, $y = 2(\sin x - \frac{1}{2}\sin 2x)$ and $y = 2(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x)$ are shown in Fig. 10.4, by the curves 1, 2 and 3 respectively. These illustrate the manner in which the successive approximations to the series (*i*) approach more and more closely to $y = x$ for all values of x in $-\pi < x < \pi$, but not for $x = \pm\pi$.

As the series has a period 2π, it represents the discontinuous function, called *saw-toothed waveform*, shown in Fig. 10.5. It is important to note that the given function $y = x$ is continuous and each term of the series (*i*) is continuous, but the function represented by the series (*i*) has finite discontinuities at $x = \pm\pi, \pm 3\pi, \pm 5\pi$ etc.

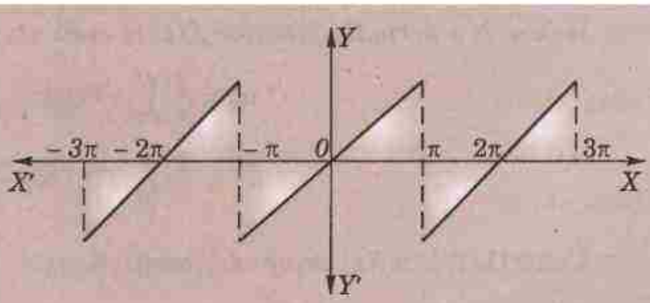

Fig. 10.5

Example 10.13. *Find a Fourier series to represent x^2 in the interval $(-l, l)$.* (*S.V.T.U., 2008*)

Solution. Since $f(x) = x^2$ is an even function in $(-l, l)$,

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \qquad ...(i)$$

Then $$a_0 = \frac{2}{l}\int_0^l x^2\, dx = \frac{2}{l}\left|\frac{x^3}{3}\right|_0^l = \frac{2l^2}{3}$$

$$a_n = \int_0^l x^2 \cos\frac{n\pi x}{l}\, dx$$ [*See* footnote p. 398]

$$= \frac{2}{l}\left[x^2\left(\frac{\sin n\pi x/l}{n\pi/l}\right) - 2x\left(-\frac{\cos n\pi x/l}{n^2\pi^2/l^2}\right) + 2\left(-\frac{\sin n\pi x/l}{n^3\pi^3/l^3}\right)\right]_0^l$$

$$= 4l^2(-1)^n/n^2\pi^2 \qquad [\because \cos n\pi = (-1)^n]$$

$\therefore$ $a_1 = -4l^2/\pi^2, a_2 = 4l^2/2^2\pi^2, a_3 = -4l^2/3^2\pi^2, a_4 = 4l^2/4^2\pi^2$ etc.

Substituting these values in (*i*), we get

$$x^2 = \frac{l^2}{3} - \frac{4l^2}{\pi^2}\left(\frac{\cos \pi x/l}{1^2} - \frac{\cos 2\pi x/l}{2^2} + \frac{\cos 3\pi x/l}{3^2} - \frac{\cos 4\pi x/l}{4^2} + \ldots\right)$$

which is the required Fourier series.

Example 10.14. *If $f(x) = |\cos x|$, expand $f(x)$ as a Fourier series in the interval $(-\pi, \pi)$.*

Solution. As $f(-x) = |\cos(-x)| = |\cos x| = f(x)$, $|\cos x|$ is an even function.

$$\therefore \quad f(x) = \frac{a_0}{2} + \Sigma a_n \cos nx$$

where $$a_0 = \frac{2}{\pi}\int_0^{\pi} |\cos x|\, dx = \frac{2}{\pi}\int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^{\pi} (-\cos x)\, dx$$

$[\because \cos x$ is $-$ve when $\pi/2 < x < \pi]$

$$= \frac{2}{\pi}\left\{\left| \sin x \right|_0^{\pi/2} - \left| \sin x \right|_{\pi/2}^{\pi}\right\} = \frac{2}{\pi}[(1-0)-(0-1)] = \frac{4}{\pi}$$

and $$a_n = \frac{2}{\pi}\int_0^{\pi} |\cos x| \cos nx \, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x \cos nx \, dx + \int_{\pi/2}^{\pi} (-\cos x) \cos nx \, dx\right]$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi/2} [\cos (n+1)x + \cos (n-1)x] \, dx - \int_{\pi/2}^{\pi} [\cos (n+1)x + \cos (n-1)x] \, dx\right\}$$

$$= \frac{1}{\pi}\left\{\left| \frac{\sin (n+1)x}{n+1} + \frac{\sin (n-1)x}{n-1} \right|_0^{\pi/2} - \left| \frac{\sin (n+1)x}{n+1} + \frac{\sin (n-1)x}{n-1} \right|_{\pi/2}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left[\left\{\frac{\sin (n+1)\pi/2}{n+1} + \frac{\sin (n-1)\pi/2}{n-1}\right\} + \left\{\frac{\sin (n+1)\pi/2}{n+1} + \frac{\sin (n-1)\pi/2}{n-1}\right\}\right]$$

$$= \frac{2}{\pi}\left(\frac{\cos n\pi/2}{n+1} - \frac{\cos n\pi/2}{n-1}\right) = \frac{-4 \cos n\pi/2}{\pi(n^2-1)} \; (n \neq 1)$$

In particular $a_1 = \frac{2}{\pi}\left[\int_0^{\pi/2} \cos^2 x \, dx - \int_{\pi/2}^{\pi} \cos^2 x \, dx\right] = 0$

Hence $|\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\left\{\frac{1}{3}\cos 2x - \frac{1}{15}\cos 4x + \ldots\right\}.$

Example 10.15. *Obtain Fourier series for the function f(x) given by*

$$f(x) = 1 + 2x/\pi, \qquad -\pi \le x \le 0,$$
$$= 1 - 2x/\pi, \qquad 0 \le x \le \pi.$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$. *(V.T.U., 2010 ; Mumbai, 2007)*

Solution. Since $f(-x) = 1 - \frac{2x}{\pi}$ in $(-\pi, 0) = f(x)$ in $(0, \pi)$

and $f(-x) = 1 + \frac{2x}{\pi}$ in $(0, \pi) = f(x)$ in $(-\pi, 0)$

$\therefore$ $f(x)$ is an even function in $(-\pi, \pi)$. This is also clear from its graph $A'BA$ (Fig. 10.6) which is symmetrical about the y-axis.

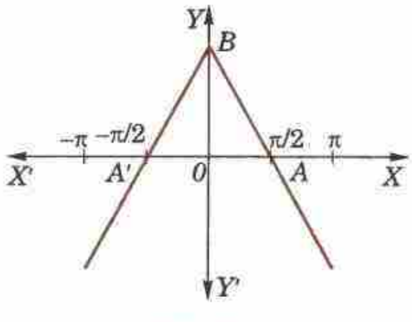

Fig. 10.6

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad \ldots(i)$$

where $$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)dx = \frac{2}{\pi}\left(x - \frac{x^2}{\pi}\right)_0^{\pi} = 0$$

and $$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \, dx = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)\cos nx \, dx$$

$$= \frac{2}{\pi}\left|\left(1 - \frac{2x}{\pi}\right)\frac{\sin nx}{n} - \left(-\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi} = \frac{2}{\pi}\left(-\frac{2\cos n\pi}{\pi n^2} + \frac{2}{\pi n^2}\right) = \frac{4}{n^2\pi^2}[1 - (-1)^n]$$

$\therefore$ $a_1 = 8/\pi^2, a^3 = 8/3^2\pi^2, a_5 = 8/5^2\pi^2, \ldots$

and $a_2 = a_4 = a_6 = \ldots = 0.$

Thus substituting the values of a's in (i), we get

$$f(x) = \frac{8}{\pi^2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right] \qquad \ldots(ii)$$

as the required Fourier expansion

Putting $x = 0$ in (*ii*), we get $1 = f(0) = \dfrac{8}{\pi^2}\left(\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots\right)$

whence follows the desired result.

PROBLEMS 10.5

1. Obtain the Fourier series expansion of $f(x) = x^2$ in $(0, a)$. Hence show that

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$ (*Mumbai, 2009 ; S.V.T.U., 2008*)

2. Show that for $-\pi < x < \pi$, $\sin ax = \dfrac{2\sin a\pi}{\pi}\left(\dfrac{\sin x}{1^2 - a^2} - \dfrac{2\sin 2x}{2^2 - a^2} + \dfrac{3\sin 3x}{3^2 - a^2} - \ldots\right)$

3. Expand the function $f(x) = x \sin x$ as a Fourier series in the interval $-\pi \le x \le \pi$. (*V.T.U., 2008 ; Anna, 2003*)

   Deduce that $\dfrac{1}{1.3} - \dfrac{1}{3.5} + \dfrac{1}{5.7} - \dfrac{1}{7.9} + \ldots = \dfrac{1}{4}(\pi - 2)$. (*U.P.T.U., 2005*)

4. Prove that in the interval $-\pi < x < \pi$, $x \cos x = -\dfrac{1}{2}\sin x + 2\displaystyle\sum_{n=2}^{\infty} \frac{n(-1)^n}{n^2 - 1}\sin nx$. (*S.V.T.U., 2009*)

5. For a function $f(x)$ defined by $f(x) = |x|, -\pi < x < \pi$, obtain a Fourier series. (*Bhopal, 2007 ; V.T.U., 2004*)

   Deduce that $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \dfrac{1}{7^2} + \ldots = \dfrac{\pi^2}{8}$. (*S.V.T.U., 2009 ; Kerala, 2005 ; P.T.U., 2005*)

6. Find the Fourier series to represent the function

   (*i*) $f(x) = |\sin x|, -\pi < x < \pi$. (*Mumbai, 2008*)

   (*ii*) $f(x) = |\cos(\pi x/l)|$ in the interval $(-1, 1)$. (*P.T.U., 2009 S*)

7. Given $f(x) = \begin{cases} -x + 1 \text{ for } -\pi \le x \le 0, \\ x + 1 \text{ for } 0 \le x \le \pi. \end{cases}$

   Is the function even or odd? Find the Fourier series for $f(x)$ and deduce the value of

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

8. Find the Fourier series of the periodic function $f(x)$: $f(x) = -k$ when $-\pi < x < 0$ and $f(x) = k$ when $0 < x < \pi$, and $f(x + 2\pi) = f(x)$. Sketch the graph of $f(x)$ and the two partial sums. (See Fig. 10.7)

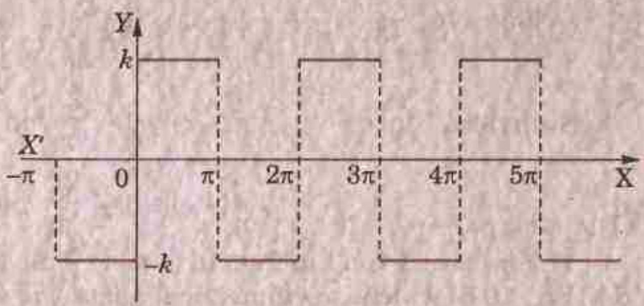

Fig. 10.7

   Deduce that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n - 1} = \frac{\pi}{4}$. (*Rohtak, 2005*)

9. A function is defined as follows :

$$f(x) = -x \text{ when } -\pi < x \le 0 = x \quad \text{when } 0 < x < \pi.$$

   Show that $f(x) = \dfrac{\pi}{2} - \dfrac{4}{\pi}\left[\dfrac{1}{1^2}\cos x + \dfrac{1}{3^2}\cos 3x + \dfrac{1}{5^2}\cos 5x + \ldots\right]$

   Deduce that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

10.7 HALF RANGE SERIES

Many a time is is required to obtain a Fourier expansion of a function $f(x)$ for the range $(0, c)$ which is half the period of the Fourier series. As it is immaterial whatever the function may be outside the range $0 < x < c$, we extend the function to cover the range $-c < x < c$ so that the new function may be odd or even. The Fourier expansion of such a function of half the period, therefore, consists of sine or cosine terms only. In such cases the graphs for the values of x in $(0, c)$ are the same but outside $(0, c)$ are different for odd or even functions. That is why we get different forms of series for the same function as is clear from the examples 10.16 and 10.17.

Sine series. If it be required to expand $f(x)$ as a sine series in $0 < x < c$; then we extend the function reflecting it in the origin, so that $f(x) = -f(-x)$.

Then the extended function is odd in $(-c, c)$ and the expansion will give the desired Fourier sine series :

$$\left.\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c} \\ \text{where} \quad b_n &= \frac{2}{c}\int_0^c f(x) \sin \frac{n\pi x}{c}\, dx \end{aligned}\right\} \quad ...(1)$$

Cosine series. If it be required to express $f(x)$ as a cosine series in $0 < x < c$, we extend the function reflecting it in the y-axis, so that $f(-x) = f(x)$.

Then the extended function is even in $(-c, c)$ and its expansion will give the required Fourier cosine series :

$$\left.\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{c} \\ \text{where} \quad a_0 &= \frac{2}{c}\int_0^c f(x)\, dx \\ \text{and} \quad a_n &= \frac{2}{c}\int_0^c f(x) \cos \frac{n\pi x}{c}\, dx \end{aligned}\right\} \quad ...(2)$$

Example 10.16. *Express $f(x) = x$ as a half-range sine series in $0 < x < 2$.* *(U.P.T.U., 2004)*

Solution. The graph of $f(x) = x$ in $0 < x < 2$ is the line OA. Let us extend the function $f(x)$ in the interval $-2 < x < 0$ (shown by the line BO) so that the new function is symmetrical about the origin and, therefore, *represents an odd function in* $(-2, 2)$ (*Fig. 10.8*)

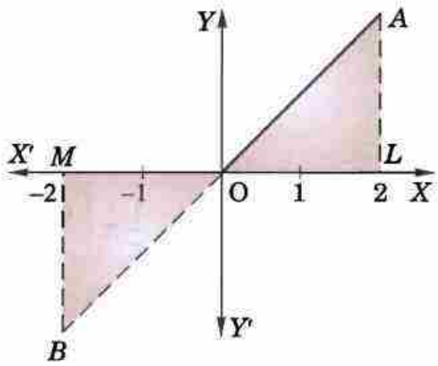

Fig. 10.8

Hence the Fourier series for $f(x)$ over the full period $(-2, 2)$ will contain only sine terms given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2}$$

where $$b_n = \frac{2}{2}\int_0^2 f(x) \sin \frac{n\pi x}{2}\, dx = \int_0^2 x \sin \frac{n\pi x}{2}\, dx$$

$$= \left| -\frac{2x}{n\pi} \cos \frac{n\pi x}{2} + \frac{4}{n^2\pi^2} \sin \frac{n\pi x}{2} \right|_0^2 = -\frac{4(-1)^n}{n\pi}$$

Thus $b_1 = 4/\pi$, $b_2 = -4/2\pi$, $b_3 = 4/3\pi$, $b_4 = -4/4\pi$ etc.

Hence the Fourier sine series for $f(x)$ over the half-range $(0, 2)$ is

$$f(x) = \frac{4}{\pi}\left(\sin \frac{\pi x}{2} - \frac{1}{2} \sin \frac{2\pi x}{2} + \frac{1}{3} \sin \frac{3\pi x}{2} - \frac{1}{4} \sin \frac{4\pi x}{2} + ... \right).$$

Example 10.17. *Express $f(x) = x$ as a half-range cosine series in $0 < x < 2$.*
(S.V.T.U., 2009 ; Bhopal, 2007 ; Mumbai, 2006)

Solution. The graph of $f(x) = x$ in $(0, 2)$ is the line OA. Let us extend the function $f(x)$ in the interval $(-2, 0)$ shown by the line OB' so that the new function is symmetrical about the y-axis and, therefore, *represents an even function in* $(-2, 2)$. (Fig. 10.9)

Hence the Fourier series for $f(x)$ over the full period $(-2, 2)$ will contain only cosine terms given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2}$$

where $a_0 = \frac{2}{2}\int_0^2 f(x)dx = \int_0^2 x\,dx = 2$

and $a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}dx = \int_0^2 x\cos\frac{n\pi x}{2}dx$

$$= \left|\frac{2x}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right|_0^2 = \frac{4}{n^2\pi^2}[(-1)^n - 1]$$

Thus $a_1 = -8/\pi^2, a_2 = 0, a_3 = -8/3^2\pi^2, a_4 = 0, a_5 = -8/5^2\pi^2$ etc.

Hence the desired Fourier series for $f(x)$ over the half-range (0, 2) is

$$f(x) = 1 - \frac{8}{\pi^2}\left[\frac{\cos\pi x/2}{1^2} + \frac{\cos 3\pi x/2}{3^2} + \frac{\cos 5\pi x/2}{5^2} + \ldots\right]$$

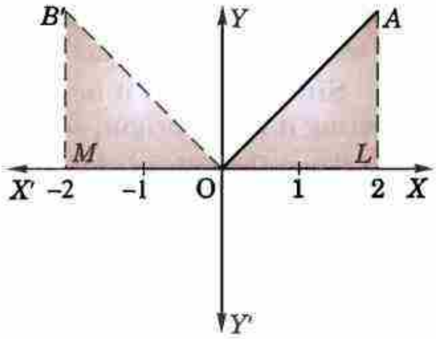

Fig. 10.9

Important Obs. It must be clearly understood that we expand a function in $0 < x < c$ as a series of sines or cosines, merely looking upon it as an odd or even function of period $2c$. It hardly matters whether the function is odd or even or neither.

Example 10.18. *Obtain the Fourier expansion of $x \sin x$ as a cosine series in $(0, \pi)$.* (V.T.U., 2003 ; U.P.T.U., 2002)

Hence show that $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots \infty = \frac{\pi - 2}{4}$. (Anna, 2001)

Solution. Let $x\sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

Then $a_0 = \frac{2}{\pi}\int_0^{\pi} x\sin x\,dx = \frac{2}{\pi}|x(-\cos x) - 1(-\sin x)|_0^{\pi} = 2$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} x(\sin\overline{n+1}\,x - \sin\overline{n-1}\,x)\,dx$$

$$= \frac{1}{\pi}\left[x\left\{\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - 1.\left\{\frac{-\sin(n+1)x}{(n+1)^2} - \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\pi\left\{\frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1}\right\}(n \neq 1).$$

When $n = 1$, $a_1 = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx$

$$= \frac{1}{\pi}\left|x\left(\frac{-\cos 2x}{2}\right) - 1\left(\frac{-\sin 2x}{2}\right)\right|_0^{\pi} = \frac{1}{\pi}\left(-\frac{\pi\cos 2\pi}{2}\right) = -\frac{1}{2}.$$

Hence $x\sin x = 1 - \frac{1}{2}\cos x - 2\left\{\frac{\cos 2x}{1.3} - \frac{\cos 3x}{3.5} + \frac{\cos 4x}{5.7} - \ldots\infty\right\}$

Putting $x = \pi/2$, we obtain $\pi/2 = 1 + 2\left\{\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots\infty\right\}$

Hence $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots\infty = \frac{\pi - 2}{4}$.

Example 10.19. *Obtain a half range cosine series for*

$$f(x) = \begin{cases} kx, & 0 \le x \le l/2 \\ k(l - x), & l/2 \le x \le l. \end{cases}$$ (Bhopal, 2008 ; V.T.U., 2008)

Deduce the sum of the series $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\infty$ (Rohtak, 2006 ; U.P.T.U., 2003)

Solution. Let the half-range cosine series be

$$f(x) = \frac{a_0}{2} + \sum_{n+1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

Then $$a_0 = \frac{2}{l}\left\{\int_0^{l/2} kx\,dx + \int_{l/2}^{l} k(l-x)\right\}dx = \frac{2k}{l}\left\{\left|\frac{x^2}{2}\right|_0^{l/2} - \left|\frac{(l-x)^2}{2}\right|_{l/2}^{l}\right\}$$

$$= \frac{2k}{l}\cdot\frac{1}{2}\left\{\frac{l^2}{4} - \left(0 - \frac{l^2}{4}\right)\right\} = \frac{kl}{2}$$

$$a_n = \frac{2}{l}\left\{\int_0^{l/2} kx\cos\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} k(l-x)\cos\frac{n\pi x}{l}\,dx\right\}$$

$$= \frac{2k}{l}\left|x\left(\frac{\sin n\pi x/l}{n\pi/l}\right) - 1\left\{-\cos\frac{n\pi x/l}{(n\pi/l)^2}\right\}\right|_0^{l/2}$$

$$+ \frac{2k}{l}\left|\left\{\frac{(l-x)\sin n\pi x/l}{n\pi/l}\right\} - (-1)\left(\frac{-\cos n\pi x/l}{(n\pi/l)^2}\right)\right|_{l/2}^{l}$$

$$= \frac{2k}{l}\left[\left(\frac{l^2}{2n\pi}\cdot\sin\frac{n\pi}{2}\right) + \frac{l^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - \cos 0\right)\right] + \frac{2k}{l}\left[\left(\frac{l}{n\pi}\left(-\frac{l}{2}\sin\frac{n\pi}{2}\right)\right.\right.$$

$$\left.\left. - \frac{l^2}{n^2\pi^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)\right)\right]$$

$$= \frac{2k}{l}\cdot\frac{l^2}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right] = \frac{2kl}{n^2\pi^2}\left\{2\cos\frac{n\pi}{2} - 1 - (-1)^n\right\}$$

Hence the required Fourier series is

$$f(x) = \frac{kl}{4} - \frac{8kl}{\pi^2}\left[\frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{6^2}\cos\frac{6\pi x}{l} + \frac{1}{10^2}\cos\frac{10\pi x}{l} + \ldots\right]$$

Putting $x = l$, we get

$$0 = \frac{kl}{4} - \frac{8kl}{\pi^2}\left(\frac{1}{2^2} + \frac{1}{6^2} + \frac{1}{10^2} + \ldots\infty\right)$$

Thus $$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\infty = \frac{\pi^2}{8}.$$

Example 10.20. *Expand* $f(x) = \frac{1}{4} - x$, *if* $0 < x < \frac{1}{2}$,

$= x - \frac{3}{4}$, *if* $\frac{1}{2} < x < 1$,

as the Fourier series of sine terms. *(V.T.U., 2011; Andhra, 2000)*

Solution. Let $f(x)$ represent an odd function in $(-1, 1)$ so that $f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$

where $$b_n = \frac{2}{1}\int_0^1 f(x)\sin n\pi x\,dx$$

$$= 2\left[\int_0^{\frac{1}{2}}\left(\frac{1}{4} - x\right)\sin n\pi x\,dx + \int_{\frac{1}{2}}^{1}\left(x - \frac{3}{4}\right)\sin n\pi x\,dx\right]$$

$$= 2\left|-\left(\frac{1}{4} - x\right)\frac{\cos n\pi x}{n\pi} - \frac{\sin n\pi x}{n^2\pi^2}\right|_0^{\frac{1}{2}} + 2\left|-\left(x - \frac{3}{4}\right)\frac{\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2}\right|_{\frac{1}{2}}^{1}$$

$$= 2\left[\frac{1}{4n\pi}\cos\frac{n\pi}{2} + \frac{1}{4n\pi} - \frac{\sin n\pi/2}{n^2\pi^2}\right] + 2\left[-\frac{1}{4n\pi}\cos n\pi - \frac{1}{4n\pi}\cos\frac{n\pi}{2} - \frac{\sin n\pi/2}{n^2\pi^2}\right]$$

$$= \frac{1}{2n\pi}[1 - (-1)^n] - \frac{4\sin n\pi/2}{n^2\pi^2}$$

Thus $$b_1 = \frac{1}{\pi} - \frac{4}{\pi^2};\ b_2 = 0$$
$$b_3 = \frac{1}{3\pi} + \frac{4}{3^2\pi^2};\ b_4 = 0$$
$$b_5 = \frac{1}{5\pi} - \frac{4}{5^2\pi^2};\ b_6 = 0 \text{ etc.}$$

Hence $$f(x) = \left(\frac{1}{\pi} - \frac{4}{\pi^2}\right)\sin \pi x + \left(\frac{1}{3\pi} + \frac{4}{3^2\pi^2}\right)\sin 3\pi x + \left(\frac{1}{5\pi} - \frac{4}{5^2\pi^2}\right)\sin 5\pi x + \cdots$$

PROBLEMS 10.6

1. Show that a constant c can be expanded in an infinite series $\frac{4c}{\pi}\left\{\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\right\}$ in the range $0 < x < \pi$. *(Marathwada, 2008 ; Kerala, 2005)*
2. Obtain cosine and sine series for $f(x) = x$ in the interval $0 \le x \le \pi$. Hence show that
$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}.$$ *(Osmania, 2003 S)*
3. Find the half-range cosine series for the function $f(x) = x^2$ in the range $0 \le x \le \pi$. *(B.P.T.U., 2005 ; Kottayam, 2005)*
4. Find the Fourier cosine series of the function $f(x) = \pi - x$ in $0 < x < \pi$. Hence show that
$$\sum_{r=0}^{\infty} \frac{1}{(2r+1)^2} = \frac{\pi^2}{8}$$ *(West Bengal, 2004)*
5. Find the half-range cosine series for the function $f(x) = (x - 1)^2$ in the interval $0 < x < 1$. *(V.T.U., 2010 ; J.N.T.U., 2006)*
Hence show that $\pi^2 = 8\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right)$ *(Anna, 2003)*
6. Find the half-range sine series for the function $f(t) = t - t^2$, $0 < t < 1$.
7. Represent $f(x) = \sin(\sin(\pi x/l))$, $0 < x < l$ by a half-range cosine series. *(Mumbai, 2009)*
8. Find the half range sine series for $f(x) = x \cos x$ in $(0, \pi)$. *(Anna, 2008 S)*
9. Obtain the half-range sine series for e^x in $0 < x < 1$.
10. Find the half range Fourier sine series of $f(x) = x(\pi - x)$, $0 \le x \le \pi$ and hence deduce that
(*i*) $\sum \frac{1}{n^4} = \frac{\pi^4}{90}$ *(Anna, 2009)* (*ii*) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} = \frac{\pi^6}{960}$ *(Mumbai, 2005)*
11. If $f(x) = x$, $0 < x < \pi/2$
$= \pi - x$, $\pi/2 < x < \pi$,
show that (*i*) $f(x) = \frac{4}{\pi}\left[\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \ldots\right]$ *(Mumbai, 2008 ; S.V.T.U., 2008 ; V.T.U., 2004)*
(*ii*) $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{12}\cos 2x + \frac{1}{3^2}\cos 6x + \frac{1}{5^2}\cos 10x + \ldots\right]$ *(V.T.U., 2011)*
12. Find the half-range cosine series expansion of the function $f(x) = \begin{cases} 0, & 0 \le x \le l/2 \\ l - x, & l/2 \le x \le l \end{cases}$. *(P.T.U., 2010)*
13. If $f(x) = \sin x$ for $0 \le x \le \pi/4$
$= \cos x$ for $\pi/4 \le x \le \pi/2$, expand $f(x)$ in a series of sines.
14. For the function defined by the graph OAB in Fig. 10.10, find the half-range Fourier sine series.

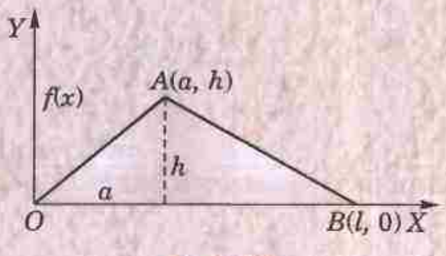

Fig. 10.10

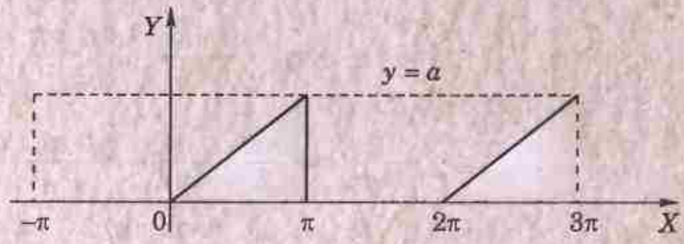

Fig 10.11

10.8 TYPICAL WAVEFORMS

We give below six typical waveforms usually met with in communication engineering :

(1) *Square waveform* (Fig. 10.7) is an extension of the function of Problem 8, page 412.

(2) *Saw-toothed waveform* (Fig. 10.5) is an extension of the function in Ex. 10.12, page 409.

(3) *Modified saw-toothed waveform* (Fig. 10.11) is extension of the function

$$f(x) = 0, \quad -\pi < x \le 0$$
$$= x, \quad 0 \le x < \pi,$$

Its Fourier expansion is

$$f(x) = \frac{a}{4} - \frac{2a}{\pi^2}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + ...\right) + \frac{\alpha}{\pi}\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + ...\right)$$

(4) *Triangular waveform* (Fig. 10-6) is an extension of the function of Ex. 10.15, page 411.

(5) *Half-wave rectifier* (Fig. 10.2) is an extension of the function of Problem 2, page 412.

(6) *Full-wave rectifier* (Fig. 10.12) is an extension of the function $f(x) = a \sin x, 0 \le x \le \pi$. Its Fourier expansion is

$$f(x) = \frac{4a}{\pi}\left\{\frac{1}{2} - \frac{1}{1.3}\cos 2x - \frac{1}{3.5}\cos 4x - \frac{1}{5.7}\cos 6x - ...\right\}$$

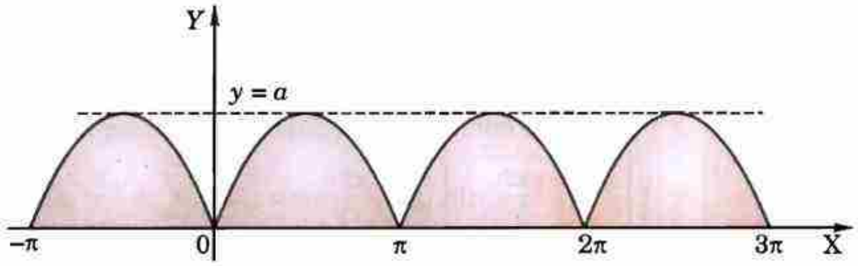

Fig. 10.12

10.9 (1) PARSEVAL'S FORMULA*

To prove that $\int_{-l}^{l} [f(x)]^2\, dx = l\left\{\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right\}$,

provided the Fourier series for f(x) converges uniformly in (– l, l).

The Fourier series for $f(x)$ in $(-l, l)$ is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$ ...(1)

Multiplying both sides of (1) by $f(x)$ and integrating term by term from $-l$ to l [which is justified as the series (1) is uniformly convergent –p. 389], we get

$$\int_{-l}^{l} [f(x)]^2\, dx = \frac{a_0}{2}\int_{-l}^{l} f(x)\, dx + \sum_{n=1}^{\infty}\left\{a_n \int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\, dx + b_n \int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\, dx\right\} \quad ...(2)$$

Now $\int_{-l}^{l} f(x)\, dx = la_0$,

$\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\, dx = la_n$ and $\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\, dx = lb_n$, by (4) of p. 405

$\therefore$ (2) takes the form $\int_{-l}^{l} [f(x)]^2\, dx = l\left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right\}$ ...(3)

which is the desired *Parseval's formula*. *(Mumbai, 2005 S)*

*Named after the French mathematician *Marc Antoine Parseval* (1755–1836).

Cor. 1. *If* $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$ *in* $(0, 2l)$, then

$$\int_0^{2l} [f(x)]^2 \, dx = l \left\{ \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right\} \qquad \text{...(4)}$$

Cor. 2. *If the half-range cosine series is (0, l) for f(x) is*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right), \text{ then}$$

$$\int_0^{l} [f(x)]^2 \, dx = \frac{l}{2} \left(\frac{a_0^2}{2} + a_1^2 + a_2^2 + a_3^2 + \dots \infty \right) \qquad \text{...(5)}$$

Cor. 3. *If the half-range sine series in (0, l) for f (x) is* $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$, then

$$\int_0^{l} [f(x)]^2 = \frac{l}{2} (b_1^2 + b_2^2 + b_3^2 + \dots \infty) \qquad \text{...(6)}$$

(2) Root mean square (rms) value. *The root mean square value of the function f (x) over an interval (a, b) is defined as*

$$[f(x)]_{\text{rms}} = \sqrt{\left\{ \frac{\int_a^b [f(x)]^2 \, dx}{b-a} \right\}} \qquad \text{...(7)}$$

The use of root mean square value of a periodic function is frequently made in the theory of mechanical vibrations and in electric circuit theory. The r.m.s. value is also known as the effective value of the function.

Example 10.21. *Obtain the Fourier series for* $y = x^2$ *in* $-\pi < x < \pi$. *Using the two values of y, show that*

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots = \frac{\pi^4}{90}$$

Solution. Let $\quad y = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

We have $\quad a_0 = 2\,\frac{n^2}{3}, a_n = \frac{4}{n^2}(-1)^n, b_n = 0$ for all n (See problem 2, p. 400)

If $\bar{y}$ be the r.m.s. value of y in $(-\pi, \pi)$, then

$$(\bar{y})^2 = \frac{\pi}{2\pi} \left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right] \qquad \text{[By (3) and (7) §10.9]}$$

$$= \frac{1}{4}\left(\frac{2\pi^2}{3}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty} \left[\frac{16}{n^4}(-1)^{2n} + 0\right] = \frac{\pi^4}{9} + 8\sum_{1}^{\infty} \frac{1}{n^4}$$

Also by definition,

$$(\bar{y})^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} y^2 \, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^4 \, dx = \frac{\pi^4}{5}$$

Equating the two values of $(\bar{y})^2$, we get

$$\frac{\pi^4}{9} + 8\sum_{1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{5} \text{ i.e., } \sum_{1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

PROBLEMS 10.7

1. By using the sine series for $f(x) = 1$ in $0 < x < \pi$, show that $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$

2. Prove that in $0 < x < l$, $x = \frac{l}{2} - \frac{4l}{\pi^2}\left(\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + \ldots\right)$

   and deduce that $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots = \frac{\pi^4}{96}$.

3. If $\frac{a_0}{2} + \sum a_n \cos\frac{n\pi x}{l}$ is the half-range cosine series of $f(x)$ of period $2l$ in $(0, l)$, then show that the mean square value of $f(x)$ in $(0, l)$ is $\frac{l}{2}\left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2\right\}$

   Use this result to evaluate $1^{-4} + 3^{-4} + 5^{-4} + \ldots\ldots$ from the half-range cosine series of the function $f(x)$ of period 4 defined in $(0, 2)$ by

$$f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ \pi(2 - x), & 1 < x < 2 \end{cases}.$$

10.10 COMPLEX FORM OF FOURIER SERIES

The Fourier series of a periodic function $f(x)$ of period $2l$, is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right) \qquad \ldots(1)$$

Since $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$,

therefore, we can express (1) as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n\left(\frac{e^{in\pi x/l} + e^{-in\pi x/l}}{2}\right) + b_n\left(\frac{e^{in\pi x/l} - e^{-in\pi x/l}}{2i}\right)\right\}$$

$$= c_0 + \sum_{n=1}^{\infty}\left\{c_n e^{in\pi x/l} + c_{-n} e^{-in\pi x/l}\right\} \qquad \ldots(2)$$

where $c_0 = \frac{1}{2}a_0$, $c_n = \frac{1}{2}(a_n - ib_n)$, $c_{-n} = \frac{1}{2}(a_n + ib_n)$

Now
$$c_n = \frac{1}{2l}\left\{\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx - i\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx\right\}$$

$$= \frac{1}{2l}\int_{-l}^{l} f(x)\left(\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right)dx = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}\,dx$$

and
$$c_{-n} = \frac{1}{2l}\int_{-l}^{l} f(x)\left(\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right)dx = \frac{1}{2l}\int_{-l}^{l} f(x)e^{in\pi x/l}\,dx$$

Combining these, we have $c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}\,dx$

where $n = 0, \pm 1, \pm 2, \pm 3, \ldots\ldots$ ...(3)

Then the series (2) can be compactly written as :

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}$$

which is the *complex form of Fourier series* and its coefficients are given by (3).

Obs. The complex form of a Fourier series is especially useful in problems on electrical circuits having impressed periodic voltage.

Example 10.22. *Find the complex form of the Fourier series of $f(x) = e^{-x}$ in $-1 \le x \le 1$.*

(Mumbai, 2005 S ; Madras, 2000 S)

Solution. We have $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x}$ $(\because\ l = 1)$

where
$$c_n = \frac{1}{2}\int_{-1}^{1} e^{-x} \cdot e^{-in\pi x}\,dx = \frac{1}{2}\int_{-1}^{1} e^{-(1+in\pi)x}\,dx = \frac{1}{2}\left|\frac{e^{-(1+in\pi)x}}{-(1+in\pi)}\right|_{-1}^{1} = \frac{e^{1+in\pi} - e^{-(1+in\pi)}}{2(1+in\pi)}$$

$$= \frac{e(\cos n\pi + i\sin n\pi) - e^{-1}(\cos n\pi - i\sin n\pi)}{2(1+in\pi)} = \frac{e - e^{-1}}{2}(-1)^n \cdot \frac{1 - in\pi}{1 + n^2\pi^2}$$

$$= \frac{(-1)^n(1 - in\pi)\sinh 1}{1 + n^2\pi^2}$$

Hence
$$e^{-x} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n(1 - in\pi)}{1 + n^2\pi^2}\sinh 1 \cdot e^{in\pi x}.$$

PROBLEMS 10.8

Find the complex form of the Fourier series of the following periodic functions :

1. $f(x) = e^{ax}, -l < x < l$. *(Madras, 2003)*
2. $f(t) = \sin t, 0 < t < \pi$
3. $f(x) = \cos ax, -\pi < x < \pi$ *(Anna, 2009 ; Mumbai, 2009)*
4. $f(x) = \cosh 3x + \sinh 3x$ in $(-3, 3)$. *(Mumbai, 2008)*
5. $f(x) = \begin{cases} 0 \text{ when } 0 < x < l \\ a \text{ when } l < x < 2l \end{cases}$

10.11 PRACTICAL HARMONIC ANALYSIS

We have discussed at length, the problem of expanding $y = f(x)$ as Fourier series :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \quad ...(1)$$

where
$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \end{aligned}\right\} \quad ...(2)$$

So far, the function has always been defined by an explicit function of an independent variable. In practice, however, the function is often given not by a formula but by a graph or by a table of corresponding values. In such cases, the integrals in (2) cannot be evaluated and instead, the following alternative forms of (2) are employed.

Since the mean value of a function $y = f(x)$ over the range (a, b) is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

$\therefore$ the equations (2) give,

$$\left.\begin{aligned} \mathbf{a_0} &= 2 \times \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \mathbf{2[mean\ value\ of\ f(x)\ in\ (0, 2\pi)]} \\ \mathbf{a}_n &= 2 \times \frac{1}{2\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \mathbf{2[mean\ value\ of\ f(x)\ cos\ nx\ in\ (0, 2\pi)]} \\ \mathbf{b}_n &= 2 \times \frac{1}{2\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \mathbf{2[mean\ value\ of\ f(x)\ sin\ nx\ in\ (0, 2\pi)]} \end{aligned}\right\} \quad ...(3)$$

There are numerous other methods of finding the value of a_0, a_n, b_n which constitute the field of **harmonic analysis.**

In (1), *the term* $(a_1 \cos x + b_1 \sin x)$ *is called the* **fundamental or first harmonic,** *the term* $(a_2 \cos 2x + b_2 \sin 2x)$ **the second harmonic** *and so on.*

Example 10.23. *The displacement y of a part of a mechanism is tabulated with corresponding angular movement x° of the crank. Express y as a Fourier series neglecting the harmonic above the third :*

$x°$	0	30	60	90	120	150	180	210	240	270	300	330
y	1.80	1.10	0.30	0.16	1.50	1.30	2.16	1.25	1.30	1.52	1.76	2.00

Solution. Let the Fourier series upto the third harmonic representing y in $(0, 2\pi)$ be

$$y = \tfrac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x \qquad ...(i)$$

To evaluate the coefficients, we form the following table.

$x°$	$\sin x$	$\cos x$	$\sin 2x$	$\cos 2x$	$\sin 3x$	$\cos 3x$	y	$y \sin x$	$y \cos x$	$y \sin 2x$	$y \cos 2x$	$y \sin 3x$	$y \cos 3x$
0	0	1	0	1	0	1	1.80	0.00	1.80	0.00	1.80	0.00	1.80
30	0.50	0.87	0.87	0.50	1	0	1.10	0.55	0.96	0.96	0.55	1.10	0.00
60	0.87	0.50	0.87	–0.50	0	–1	0.30	0.26	0.15	0.26	–0.15	0.00	–0.30
90	1.00	0	0	–1.00	–1	0	0.16	0.16	0.00	0.00	–0.16	–0.16	0.00
120	0.87	–0.50	–0.87	–0.50	0	1	0.50	0.43	–0.25	–0.43	–0.25	0.00	0.50
150	0.50	–0.87	–0.87	–0.50	1	0	1.30	0.65	–1.13	–1.13	0.65	1.30	0.00
180	0	–1.00	0	1.00	0	–1	2.16	0.00	–2.16	–0.00	2.16	0.00	–2.16
210	–0.50	–0.87	0.87	0.50	–1	0	1.25	–0.63	–1.09	1.09	0.63	–1.25	0.00
240	–0.87	–0.50	0.87	–0.50	0	1	1.30	–1.13	–0.65	1.13	–0.65	0.00	1.30
270	–1.00	0	0	–1.00	1	0	1.52	–1.52	0.00	0.00	–1.52	1.52	0.00
300	–0.87	0.50	–0.87	–0.50	0	–1	1.76	–1.53	0.88	–1.53	–0.88	0.00	–1.76
330	–0.50	0.87	–0.87	0.50	–1	0	2.00	–1.00	1.74	–1.74	1.00	–2.00	0.00
						$\Sigma =$	15.15	–3.76	0.25	–1.39	3.18	0.51	–0.62

$$\therefore \qquad a_0 = 2 \cdot \frac{\Sigma y}{12} = \frac{15.15}{6} = 2.53 \; ; \; a_1 = \frac{1}{6}\Sigma y \cos x = \frac{0.25}{6} = 0.04$$

$$a_2 = \frac{1}{6}\Sigma y \cos 2x = \frac{3.18}{6} = 0.53 \; ; \; a_3 = \frac{1}{6}\Sigma y \cos 3x = \frac{-0.62}{6} = -0.1$$

$$b_1 = \frac{1}{6}\Sigma y \sin x = \frac{-3.76}{6} = -0.63 \; ;$$

$$b_2 = \frac{1}{6}\Sigma y \sin 2x = \frac{-1.39}{6} = -0.23$$

$$b_3 = \frac{1}{6}\Sigma y \sin 3x = \frac{0.51}{6} = 0.085$$

Substituting the values of a's and b's in (i), we get

$$y = 1.26 + 0.04 \cos x + 0.53 \cos 2x - 0.1 \cos 3x - 0.63 \sin x - 0.23 \sin 2x + 0.085 \sin 3x.$$

Example 10.24. *The following table gives the variations of periodic current over a period.*

t sec	0	$T/6$	$T/3$	$T/2$	$2T/3$	$5T/6$	T
A amp.	1.98	1.30	1.05	1.30	– 0.88	– 0.25	1.98

Show that there is a direct current part of 0.75 amp in the variable current and obtain the amplitude of the first harmonic. (V.T.U., 2010 ; S.V.T.U., 2009)

Solution. Here length of the interval is T, *i.e.* $C = T/2$ (§ 10.5)

Then $$A = \frac{a_0}{2} + a_1 \cos \frac{2\pi t}{T} + b_1 \sin \frac{2\pi t}{T} + a_2 \cos \frac{4\pi t}{T} + b_2 \sin \frac{4\pi t}{T} + ...$$

The desired values are tabulated as follows :

t	$2\pi t/T$	$\cos 2\pi t/T$	$\sin 2\pi t/T$	A	$A \cos 2\pi t/T$	$A \sin 2\pi t/T$
0	0	1.0	0.000	1.98	1.980	0.000
$T/6$	$\pi/3$	0.5	0.866	1.30	0.650	1.126
$T/3$	$2\pi/3$	–0.5	0.866	1.05	–0.525	0.909
$T/2$	π	–1.0	0.000	1.30	–1.300	0.000
$2T/3$	$4\pi/3$	–0.5	–0.866	–0.88	0.440	0.762
$5T/6$	$5\pi/3$	0.5	–0.866	–0.25	–0.125	0.217
			$\Sigma =$	4.5	1.12	3.014

$$\therefore \quad a_0 = 2 \cdot \frac{1}{6} \Sigma A = \frac{1}{3}(4.5) = 1.5$$

$$a_1 = 2 \cdot \frac{1}{6} \Sigma A \cos \frac{2\pi t}{T} = \frac{1}{3}(1.12) = 0.373$$

$$b_1 = 2 \cdot \frac{1}{6} \Sigma A \sin \frac{2\pi t}{T} = \frac{1}{3}(3.014) = 1.005$$

Thus the direct current part in the variable current = $a_0/2 = 0.75$ and amplitude of the first harmonic

$$= \sqrt{(a_1^2 + b_1^2)} = \sqrt{[(0.373)^2 + (1.005)^2]} = 1.072$$

Example 10.25. *Obtain the first three coefficients in the Fourier cosine series for y, where y is given in the following table :*

x :	0	1	2	3	4	5
y :	4	8	15	7	6	2

(V.T.U., 2009 ; V.T.U., 2006 ; J.N.T.U., 2004 S)

Solution. Taking the interval as 60°, we have

$\theta =$	0°	60°	120°	180°	240°	300°
$x =$	0	1	2	3	4	5
$y =$	4	8	15	7	6	2

$\therefore$ Fourier cosine series in the intervals $(0, 2\pi)$ is

$$y = \frac{a_0}{2} + a_1 \cos\theta + a_2 \cos 2\theta + a_3 \cos 3\theta + \ldots$$

$\theta°$	$\cos\theta$	$\cos 2\theta$	$\cos 3\theta$	y	$y\cos\theta$	$y\cos 2\theta$	$y\cos 3\theta$
0°	1	1	1	4	4	4	4
60°	$\frac{1}{2}$	$-\frac{1}{2}$	–1	8	4	–4	–8
120°	$-\frac{1}{2}$	$-\frac{1}{2}$	1	15	–7.5	–7.5	15
180°	– 1	1	–1	7	–7	7	–7
240°	$-\frac{1}{2}$	$-\frac{1}{2}$	1	6	–3	–3	6
300°	$\frac{1}{2}$	$-\frac{1}{2}$	–1	2	1	–1	–2
			$\Sigma =$	42	– 8.5	–4.5	8

Hence $$a_0 = 2 \cdot \frac{42}{6} = 14,\ a_1 = 2\left(\frac{-8.5}{6}\right) = -2.8,\ a_2 = 2\left(\frac{-4.5}{6}\right) = -1.5,$$

$$a_3 = 2\left(\frac{8}{6}\right) = 2.7.$$

Example 10.26. *The turning moment T is given for a series of values of the crank angle θ° = 75°*

$\theta°$:	0	30	60	90	120	150	180
T :	0	5224	8097	7850	5499	2626	0

Obtain the first four terms in a series of sines to represent T and calculate T for θ = 75°.

Solution. Let the Fourier sine series to represent T in (0, 180) be

$T = b_1 \sin\theta + b_2 \sin 2\theta + b_3 \sin 3\theta + b_4 \sin 4\theta + \ldots$

To evaluate the coefficients, we form the following table :

θ°	T	sin θ	sin 2θ	sin 3θ	sin 4θ
0	0	0	0	0	0
30	5224	0.500	0.866	1	0.866
60	8097	0.866	0.866	0	–0.866
90	7850	1.000	0	–1	0
120	5499	0.866	–0.866	0	0.866
150	2626	0.500	–0.866	1	–0.866

$$\therefore \qquad b_1 = \frac{2}{6}\Sigma y \sin\theta = \frac{1}{3}\{(5224 + 2626)\,0.5 + (8097 + 5499)\,0.866 + 7850\} = 7850$$

$$b_2 = \frac{2}{6}\Sigma y \sin 2\theta = \frac{1}{3}\{(5224 + 8097)\,0.866 + (5499 + 2626)(-0.866)\} = 1500$$

$$b_3 = \frac{2}{6}\Sigma y \sin 3\theta = \frac{1}{3}\{5224 - 7850 + 2626\} = 0.$$

$$b_4 = \frac{2}{6}\Sigma y \sin 4\theta = \frac{1}{3}\{(5224 + 5499)(0.866) + (8097 + 2626)(-0.866)\} = 0$$

Hence $\qquad T = 785° \sin\theta + 150° \sin 2\theta$

For $\theta = 75°$, $T = 7850 \sin 75° + 1500 \sin 150°$

$= 7850° (0.9659) + 1500 (0.5) = 8332.$

PROBLEMS 10.9

1. The following values of y give the displacement in inches of a certain machine part for the rotation x of the flywheel. Expand y in terms of a Fourier series :

x :	0	π/6	2π/6	3π/6	4π/6	5π/6
y :	0	9.2	14.4	17.8	17.3	11.7

2. Compute the first two harmonics of the Fourier series of $f(x)$ given in the following table :

x :	0	π/3	2π/3	π	4π/3	5π/3	2π
$f(x)$:	1.0	1.4	1.9	1.7	1.5	1.2	1.0

(*Anna, 2009*)

3. Obtain the constant term and the coefficients of the first sine and cosine terms in the Fourier expansion of y as given in the following table :

x :	0	1	2	3	4	5
y :	9	18	24	28	26	20

(*V.T.U., 2011 ; Anna, 2005 S*)

4. In a machine the displacement y of a given point is given for a certain angle θ as follows :

θ° :	0	30	60	90	120	150	180	210	240	270	300	330
y :	7.9	8.0	7.2	5.6	3.6	1.7	0.5	0.2	0.9	2.5	4.7	6.8

Find the coefficient of sin 2θ in the Fourier series representing the above variation.

5. Determine the first two harmonics of the Fourier series for the following values :

$x°$:	30	60	90	120	150	180	210	240	270	300	330	360
y :	2.34	3.01	3.68	4.15	3.69	2.20	0.83	0.51	0.88	1.09	1.19	1.64

(*Madras, 2006 ; Cochin, 2005*)

6. The turning moment T on the crankshaft of a steam engine for the crank angle θ degrees is given as follows :

θ :	0	15	30	45	60	75	90	105	120	135	150	165	180
T :	0	2.7	5.2	7.0	8.1	8.3	7.9	6.8	5.5	4.1	2.6	1.2	0

Expand T in a series of sines upto the fourth harmonics.

10.12 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 10.10

Fill up the blanks or choose the correct answer in each of the following problems :

1. The period of $\cos 3x$ is $x =$ ......
2. If $x = c$ is a point of discontinuity then the Fourier series of $f(x)$ at $x = c$ gives $f(x) =$ ......
3. A function $f(x)$ defined for $0 < x < 1$ can be extended to an odd periodic function in ......
4. The mathematical function representing the following graph is ......
5. Fourier expansion of an odd function has only ...... terms.
6. Formulae for evaluation of Fourier coefficients for a given set of points (x_i, y_i) : $i = 0, 1, 2, ... n$ are......
7. If $f(x) = x^4$ in $(-1, 1)$, then the Fourier coefficient $b_n =$ ......
8. The period of a constant function is ......
9. If $f(t) = \begin{cases} -1, & -1 < t < 0 \\ 1, & 0 < t < 1 \end{cases}$, then $f(t)$ is an ......
10. Fourier expansion of an even function $f(x)$ in $(-\pi, \pi)$ has only ...... terms.
11. If $f(x) = \begin{cases} -x, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$, then $f(x)$ is an ...... function in $(-\pi, \pi)$.
12. The smallest period of the function $\sin\left(\dfrac{2n\pi x}{k}\right)$ is ......
13. In the Fourier series expansion of $f(x) = |\sin x|$ in $(-\pi, \pi)$, the value of $b_n =$ ......
14. In the Fourier series for $f(x) = x$ in $(-\pi \le x \le \pi)$, the ...... terms are absent.
15. If $f(x)$ is an even function in $(-l, l)$, then the value of $b_n =$ ......
16. If $f(x) = x^2$ in $-2 < x < 2$, $f(x + 4) = f(x)$, then a_n is ......
17. If $f(x)$ is a periodic function with period $2T$, then the value of the Fourier coefficient $b_n =$ ......
18. Dirichlet conditions for the expansion of a function as a Fourier series in the interval $c_1 \le x \le c_2$ are ......
19. If $f(x) = x \sin x$ in $(-\pi, \pi)$, then the value of $b_n =$ ......
20. The formulae for finding the half range cosine series for the function $f(x)$ in $(0, l)$ are ......
21. The half-range sine series for 1 in $(0, \pi)$, is ......
22. Period of $|\sin t|$ is ......
23. The value of bn in the Fourier series of $f(x) = |x|$ in $(-\pi, \pi) =$ ......
24. If $f(x)$ is defined in $(0, l)$ then the period of $f(x)$ to expand it as a half range sine series is ......
25. The complex form of Fourier series for e^{-x} in $(-1, 1)$ is ......
26. $f(x)$ is an odd function in $(-\pi, \pi)$, then the graph of $f(x)$ is symmetric about the x-axis. (True or False)
27. $f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi, \end{cases}$ then $f(0) =$ ......
28. If $f(x) = \begin{cases} \pi x & \text{in } 0 \le x \le 1 \\ \pi(2 - x) & \text{in } 1 \le x \le 2, \end{cases}$ then it is ...... function. (odd or even)
29. If $f(x)$ is an odd function in $(-l, l)$, then the values of a_0 and a_n are......
30. The root mean square value of $f(t) = 3 \sin 2t + 4 \cos 2t$ over the range $0 \le t \le \pi$ is...... (*Nagpur, 2009*)
31. In the Fourier series expansion of the function

$$f(x) = \begin{cases} -(\pi + x), & -\pi < x < 0 \\ -(\pi - x), & 0 < x < \pi, \end{cases} \quad \text{the value of } b_n \text{ is ......}$$ (*P.T.U., 2010*)

32. Let $f(x)$ be defined in $(0, 2\pi)$ by

$$f(t) = \begin{cases} \dfrac{1 + \cos x}{\pi - x}, & 0 < x < \pi \\ \cos x, & \pi < x < 2\pi, \end{cases} \quad f(x) + 2\pi = f(x). \text{ The value of } f(\pi) \text{ is ......}$$ (*Anna, 2009*)

33. The mean value of $f(x) \cos nx$ in $(0, 2\pi) =$ ......

34. Using sine series for $f(x) = 1$ in $0 < x < \pi$, show that $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots \infty =$ .....

35. Fourier series representing $f(x) = |x|$ in $-\pi < x < \pi$, is ......

36. Fourier series of $f(x) = \cos^4 x$ in $(0, 2\pi)$ is ......

37. If $f(x) = x^2 + x$ in $(0, l)$, then the even extension of $f(x)$ in $(-l, 0)$ is ......

38. If $f(x) = x(l - x)$ in $(0, l)$, then the extension of $f(x)$ in $(l, 2l)$ so as to get sine series is ......

39. A function $f(x)$ defined in $(-\pi, \pi)$ can be expanded into Fourier series containing both sine and cosine terms. (True or False)

40. The function $f(x) = \begin{cases} 1 - x \text{ in } & -\pi < x < 0 \\ 1 + x \text{ in } & 0 < x < \pi, \end{cases}$ is an odd function. (True or False)

41. If $f(x) = x^2$ in $(-\pi, \pi)$, then the Fourier series of $f(x)$ contains only sine terms. (True or False)

CHAPTER 11

Differential Equations of First Order

11.1 DEFINITIONS

(1) *A* **differential equation** *is an equation which involves differential coefficients or differentials.*

Thus (*i*) $e^x\,dx + e^y\,dy = 0$ (*ii*) $\dfrac{d^2x}{dt^2} + n^2x = 0$

(*iii*) $y = x\dfrac{dy}{dx} + \dfrac{x}{dy/dx}$ (*iv*) $\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2} \Big/ \dfrac{d^2y}{dx^2} = c$

(*v*) $\dfrac{dx}{dt} - wy = a\cos pt,\ \dfrac{dy}{dt} + wx = a\sin pt$ (*vi*) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u$

(*vii*) $\dfrac{\partial^2 y}{\partial t^2} = c^2\dfrac{\partial^2 y}{\partial x^2}$ are all examples of differential equations.

(2) *An* **ordinary differential equation** *is that in which all the differential coefficients have reference to a single independent variable.* Thus the equations (*i*) to (*v*) are all ordinary differential equations.

A **partial differential equation** *is that in which there are two or more independent variables and partial differential coefficients with respect to any of them.* Thus the equations (*vi*) and (*vii*) are partial differential equations.

(3) *The* **order** *of a differential equation is the order of the highest derivative appearing in it.*

The **degree** *of a differential equation is the degree of the highest derivative occurring in it, after the equation has been expressed in a form free from radicals and fractions as far as the derivatives are concerned.*

Thus, from the examples above,

(*i*) is of the first order and first degree ; (*ii*) is of the second order and first degree ;

(*iii*) written as $y\dfrac{dy}{dx} = x\left(\dfrac{dy}{dx}\right)^2 + x$ is clearly of the first order but of second degree ;

and (*iv*) written as $\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^3 = c^2\left(\dfrac{d^2y}{dx^2}\right)^2$ is of the second order and second degree.

11.2 PRACTICAL APPROACH TO DIFFERENTIAL EQUATIONS

Differential equations arise from many problems in oscillations of mechanical and electrical systems, bending of beams, conduction of heat, velocity of chemical reactions etc., and as such play a very important role in all modern scientific and engineering studies.

The approach of an engineering student to the study of differential equations has got to be practical unlike that of a student of mathematics, who is only interested in solving the differential equations without knowing as to how the differential equations are formed and how their solutions are physically interpreted.

Thus for an applied mathematician, the study of a differential equation consists of three phases :

(*i*) *formulation of differential equation from the given physical situation, called* **modelling.**

(*ii*) *solutions of this differential equation, evaluating the arbitrary constants from the given conditions,* and

(*iii*) *physical interpretation of the solution.*

11.3 FORMATION OF A DIFFERENTIAL EQUATION

An ordinary differential equation is formed in an attempt to eliminate certain arbitrary constant from a relation in the variables and constants. It will, however, be seen later that the partial differential equations may be formed by the elimination of either arbitrary constants or arbitrary functions. In applied mathematics, every geometrical or physical problem when translated into mathematical symbols gives rise to a differential equation.

Example 11.1. *Form the differential equation of simple harmonic motion given by $x = A\cos(nt + \alpha)$.*

Solution. To eliminate the constants A and α differentiating it twice, we have

$$\frac{dx}{dt} = -nA\sin(nt+\alpha) \quad \text{and} \quad \frac{d^2x}{dt^2} = -n^2A\cos(nt+\alpha) = -n^2x$$

Thus $$\frac{d^2x}{dt^2} + n^2x = 0$$

is the desired differential equation which states that the acceleration varies as the distance from the origin.

Example 11.2. *Obtain the differential equation of all circles of radius a and centre (h, k).* *(Andhra, 1999)*

Solution. Such a circle is $(x-h)^2 + (y-k)^2 = a^2$ ...(*i*)

where h and k, the coordinates of the centre, and a are the constants.

Differential it twice, we have

$$x - h + (y-k)\frac{dy}{dx} = 0 \quad \text{and} \quad 1 + (y-k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

Then $$y - k = -\frac{1 + (dy/dx)^2}{d^2y/dx^2}$$

and $$x - h = -(y-k)\,dy/dx = \frac{\dfrac{dy}{dx}\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]}{d^2y/dx^2}$$

Substituting these in (*i*) and simplifying, we get $[1 + (dy/dx)^2]^3 = a^2(d^2y/dx^2)^2$ ...(*ii*)

as the required differential equation

Writing (*ii*) in the form $\dfrac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2} = a,$

it states that the radius of curvature of a circle at any point is constant.

Example 11.3. *Obtain the differential equation of the coaxial circles of the system $x^2 + y^2 + 2ax + c^2 = 0$ where c is a constant and a is a variable.* *(J.N.T.U., 2003)*

Solution. We have $x^2 + y^2 + 2ax + c^2 = 0$ ...(*i*)

Differentiating w.r.t. x, $2x + 2y dy/dx + 2a = 0$

or $$2a = -2\left(x + y\frac{dy}{dx}\right)$$

Substituting in (*i*), $x^2 + y^2 - 2(x + y\, dy/dx)x + c^2 = 0$

or $$2xy\, dy/dx = y^2 - x^2 + c^2$$

which is the required differential equation.

11.4 (1) SOLUTION OF A DIFFERENTIAL EQUATION

A solution (or integral) of a differential equation is a relation between the variables which satisfies the given differential equation.

For example, $$x = A \cos (nt + \alpha) \quad ...(1)$$

is a solution of $$\frac{d^2x}{dt^2} + n^2x = 0 \text{ [Example 11.1]} \quad ...(2)$$

The **general** (*or* **complete**) **solution** *of a differential equation is that in which the number of arbitrary constants is equal to the order of the differential equation.* Thus (1) is a general solution (2) as the number of arbitrary constants (A, α) is the same as the order of (2).

A **particular solution** *is that which can be obtained from the general solution by giving particular values to the arbitrary constants.*

For example, $$x = A \cos (nt + \pi/4)$$

is the particular solution of the equation (2) as it can be derived from the general solution (1) by putting $\alpha = \pi/4$.

A differential equation may sometimes have an additional solution which cannot be obtained from the general solution by assigning a particular value to the arbitrary constant. Such a solution is called a **singular solution** and is not of much engineering interest.

Linearly independent solution. *Two solutions $y_1(x)$ and $y_2(x)$ of the differential equation*

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\, y = 0 \quad ...(3)$$

are said to be linearly independent if $c_1y_1 + c_2y_2 = 0$ such that $c_1 = 0$ and $c_2 = 0$

If c_1 and c_2 are not both zero, then the two solutions y_1 and y_2 are said to be *linearly dependent.*

If $y_1(x)$ and $y_2(x)$ any two solutions of (3), *then their linear combination $c_1y_1 + c_2y_2$ where c_1 and c_2 are constants, is also a solution of* (3).

Example 11.4. *Find the differential equation whose set of independent solutions is $[e^x, xe^x]$.*

Solution. Let the general solution of the required differential equation be $y = c_1e^x + c_2xe^x$ ...(*i*)

Differentiating (*i*) w.r.t. x, we get

$$y_1 = c_1e^x + c_2(e^x + xe^x)$$

$$\therefore \quad y - y_1 = c_2e^x \quad ...(ii)$$

Again differentiating (*ii*) w.r.t. x, we obtain

$$y_1 - y_2 = c_2e^x \quad ...(iii)$$

Subtracting (*iii*) from (*ii*), we get

$$y - y_1 - (y_1 - y_2) = 0 \quad \text{or} \quad y - 2y_1 + y_2 = 0$$

which is the desired differential equation.

(2) Geometrical meaning of a differential equation. Consider any differential equation of the first order and first degree

$$\frac{dy}{dx} = f(x, y) \quad ...(1)$$

If $P(x, y)$ be any point, then (1) can be regarded as an equation giving the value of dy/dx (= m) when the values of x and y are known (Fig. 11.1). Let the value of m at the point $A_0(x_0, y_0)$ derived from (1) be m_0. Take a neighbouring

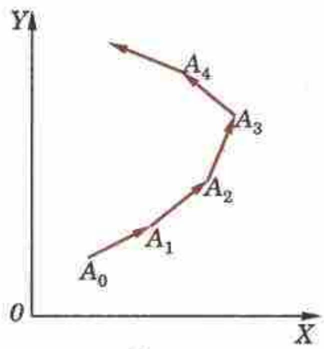

Fig. 11.1

point $A_1(x_1, y_1)$ such that the slope of A_0A_1 is m_0. Let the corresponding value of m at A_1 be m_1. Similarly take a neighbouring point $A_2(x_2, y_2)$ such that the slope of A_1A_2 is m_1 and so on.

If the successive points A_0, A_1, A_2, A_3 ... are chosen very near one another, the broken curve $A_0A_1A_2A_3$ ... approximates to a smooth curve $C[y = \phi(x)]$ which is a solution of (1) associated with the initial point $A_0\ (x_0, y_0)$. Clearly the slope of the tangent to C at any point and the coordinates of that point satisfy (1).

A different choice of the initial point will, in general, give a different curve with the same property. *The equation of each such curve is thus a* **particular solution** *of the differential equation (1). The equation of the whole family of such curves is the general solution of* (1). The slope of the tangent at any point of each member of this family and the co-ordinates of that point satisfy (1).

Such a simple geometric interpretation of the solutions of a second (or higher) order differential equation is not available.

PROBLEMS 11.1

Form the differential equations from the following equations :

1. $y = ax^3 + bx^2$.
2. $y = C_1 \cos 2x + C_2 \sin 2x$ (*Bhopal, 2008*)
3. $xy = Ae^x + Be^{-x} + x^2$. (*U.P.T.U., 2005*)
4. $y = e^x (A \cos x + B \sin x)$. (*P.T.U., 2003*)
5. $y = ae^{2x} + be^{-3x} + ce^x$.

Find the differential equations of :

6. A family of circles passing through the origin and having centres on the x-axis. (*J.N.T.U., 2006*)
7. All circles of radius 5, with their centres on the y-axis.
8. All parabolas with x-axis as the axis and $(a, 0)$ as focus.
9. If $y_1(x) = \sin 2x$ and $y_2(x) = \cos 2x$ are two solutions of $y'' + 4y = 0$, show that $y_1(x)$ and $y_2(x)$ are linearly independent solutions.
10. Determine the differential equation whose set of independent solutions is $[e^x, xe^x, x^2 e^x]$ (*U.P.T.U., 2002*)
11. Obtain the differential equation of the family of parabolas $y = x^2 + c$ and sketch those members of the family which pass through $(0, 0)$, $(1, 1)$, $(0, 1)$ and $(1, -1)$ respectively.

11.5 EQUATIONS OF THE FIRST ORDER AND FIRST DEGREE

It is not possible to solve such equations in general. We shall, however, discuss some special methods of solution which are applied to the following types of equations :

(*i*) Equations where variables are separable, (*ii*) Homogeneous equations,
(*iii*) Linear equations, (*iv*) Exact equations.

In other cases, the particular solution may be determined numerically (Chapter 31).

11.6 VARIABLES SEPARABLE

If in an equation it is possible to collect all functions of x and dx on one side and all the functions of y and dy on the other side, then the *variables are said to be separable.* Thus the general form of such an equation is $f(y)\, dy = \phi(x)\, dx$

Integrating both sides, we get $\int f(y)\, dy = \int \phi(x)\, dx + c$ as its solution.

Example 11.5. *Solve* $dy/dx = \dfrac{x(2 \log x + 1)}{\sin y + y \cos y}$. (*V.T.U., 2008*)

Solution. Given equation is $x\,(2 \log x + 1)\, dx = (\sin y + y \cos y)\, dy$

Integrating both sides, $2 \int (\log x \,.\, x + x)\, dx = \int \sin y\, dy + \int y \cos y\, dy + c$

or
$$2\left[\left(\log x \,.\, \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2}\, dx\right) + \frac{x^2}{2}\right] = -\cos y + \left[y \sin y - \int \sin y \,.\, 1\, dy + c\right]$$

or $$2x^2 \log x - \frac{x^2}{2} + \frac{x^2}{2} = -\cos y + y \sin y + \cos y + c$$

Hence the solution is $2x^2 \log x - y \sin y = c$.

Example 11.6. *Solve* $\frac{dy}{dx} = e^{3x-2y} + x^2 e^{-2y}$.

Solution. Given equation is $\frac{dy}{dx} = e^{-2y}(e^{3x} + x^2)$ or $e^{2y}\, dy = (e^{3x} + x^2)\, dx$

Integrating both sides, $\int e^{2y}\, dy = \int (e^{3x} + x^2)\, dx + c$

or $$\frac{e^{2y}}{2} = \frac{e^{3x}}{3} + \frac{x^3}{3} + c \quad \text{or} \quad 3e^{2y} = 2(e^{3x} + x^3) + 6c.$$

Example 11.7. *Solve* $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$. *(V.T.U., 2005)*

Solution. Putting $x + y = t$ so that $dy/dx = dt/dx - 1$

The given equation becomes $\frac{dt}{dx} - 1 = \sin t + \cos t$

or $$dt/dx = 1 + \sin t + \cos t$$

Integrating both sides, we get $dx = \int \frac{dt}{1 + \sin t + \cos t} + c.$

or $$x = \int \frac{2d\theta}{1 + \sin 2\theta + \cos 2\theta} + c \qquad [\text{Putting } t = 2\theta]$$

$$= \int \frac{2d\theta}{2\cos^2\theta + 2\sin\theta\cos\theta} + c = \int \frac{\sec^2\theta}{1 + \tan\theta}\, d\theta + c$$

$$= \log(1 + \tan\theta) + c$$

Hence the solution is $x = \log\left[1 + \tan \frac{1}{2}(x+y)\right] + c$.

Example 11.8. *Solve* $dy/dx = (4x + y + 1)^2$, *if* $y(0) = 1$.

Solution. Putting $4x + y + 1 = t$, we get $\frac{dy}{dx} = \frac{dt}{dx} - 4$.

$\therefore$ the given equation becomes $\frac{dt}{dx} - 4 = t^2$ or $\frac{dt}{dx} = 4 + t^2$

Integrating both sides, we get $\int \frac{dt}{4 + t^2} = \int dx + c$

or $$\frac{1}{2}\tan^{-1}\frac{t}{2} = x + c \quad \text{or} \quad \frac{1}{2}\tan^{-1}\left[\frac{1}{2}(4x + y + 1)\right] = x + c.$$

or $$4x + y + 1 = 2\tan 2(x + c)$$

When $x = 0, y = 1 \quad \therefore \quad \frac{1}{2}\tan^{-1}(1) = c$ *i.e.* $c = \pi/8$.

Hence the solution is $4x + y + 1 = 2\tan(2x + \pi/4)$.

Example 11.9. *Solve* $\frac{y}{x}\frac{dy}{dx} + \frac{x^2 + y^2 - 1}{2(x^2 + y^2) + 1} = 0$. *(V.T.U., 2003)*

Solution. Putting $x^2 + y^2 = t$, we get $2x + 2y\frac{dy}{dx} = \frac{dt}{dx}$ or $\frac{y}{x}\frac{dy}{dx} = \frac{1}{2x}\frac{dt}{dx} - 1$.

Therefore the given equation becomes $\frac{1}{2x}\frac{dt}{dx} - 1 + \frac{t-1}{2t+1} = 0$

or $$\frac{1}{2x}\frac{dt}{dx} = 1 - \frac{t-1}{2t+1} = \frac{t+2}{2t+1} \quad \text{or} \quad 2x\,dx = \frac{2t+1}{t+2}dt$$

or $$2x\,dx = \left(2 - \frac{3}{t+2}\right)dt$$

Integrating, we get $$x^2 = 2t - 3\log(t+2) + c$$

or $$x^2 + 2y^2 - 3\log(x^2+y^2+2) + c = 0 \qquad [\because\ t = x^2 + y^2]$$

which is the required solution.

PROBLEMS 11.2

Solve the following differential equations :

1. $y\sqrt{(1-x^2)}\,dy + x\sqrt{(1-y^2)}\,dx = 0.$
2. $(x^2 - yx^2)\frac{dy}{dx} + y^2 + xy^2 = 0.$
3. $\sec^2 x \tan y\,dx + \sec^2 y \tan x\,dy = 0.$ (*P.T.U., 2003*)
4. $\frac{y}{x}\frac{dy}{dx} = \sqrt{(1+x^2+y^2+x^2y^2)}.$ (*V.T.U., 2011*)
5. $e^x \tan y\,dx + (1-e^x)\sec^2 y\,dy = 0.$ (*V.T.U., 2009*)
6. $\frac{dy}{dx} = xe^{y-x^2}$, if $y = 0$ when $x = 0$. (*J.N.T.U., 2006*)
7. $x\frac{dy}{dx} + \cot y = 0$ if $y = \pi/4$ when $x = \sqrt{2}$.
8. $(xy^2 + x)\,dx + (yx^2 + y)\,dy = 0.$
9. $\frac{dy}{dx} = e^{2x-3y} + 4x^2e^{-3y}.$
10. $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right).$
11. $(x+1)\frac{dy}{dx} + 1 = 2e^{-y}.$ (*Madras, 2000 S*)
12. $(x-y)^2\frac{dy}{dx} = a^2.$
13. $(x+y+1)^2\frac{dy}{dx} = 1.$ (*Kurukshetra, 2005*)
14. $\sin^{-1}(dy/dx) = x + y$ (*V.T.U., 2010*)
15. $\frac{dy}{dx} = \cos(x+y+1)$ (*V.T.U., 2003*)
16. $\frac{dy}{dx} - x\tan(y-x) = 1.$
17. $x^4\frac{dy}{dx} + x^3y + \operatorname{cosec}(xy) = 0.$

11.7 HOMOGENEOUS EQUATIONS

are of the form $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$

where $f(x,y)$ and $\phi(x,y)$ are homogeneous functions of the same degree in x and y (see page 205).

To solve a homogeneous equation (*i*) Put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$,

(*ii*) Separate the variables v and x, and integrate.

Example 11.10. *Solve* $(x^2 - y^2)\,dx - xy\,dy = 0.$

Solution. Given equation is $\frac{dy}{dx} = \frac{x^2-y^2}{xy}$ which is homogeneous in x and y. ...(*i*)

Put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$. $\therefore$ (*i*) becomes $v + x\frac{dv}{dx} = \frac{1-v^2}{v}$

or $$x\frac{dv}{dx} = \frac{1-v^2}{v} - v = \frac{1-2v^2}{v}.$$

Separating the variables, $\dfrac{v}{1-2v^2}\,dv = \dfrac{dx}{x}$

Integrating both sides, $\displaystyle\int \frac{v\,dv}{1-2v^2} = \int \frac{dx}{x} + c$

or $$-\frac{1}{4}\int \frac{-4v}{1-2v^2}\,dv = \int \frac{dx}{x} + c \quad \text{or} \quad -\frac{1}{4}\log(1-2v^2) = \log x + c$$

or $$4\log x + \log(1-2v^2) = -4c \quad \text{or} \quad \log x^4(1-2v^2) = -4c \qquad [\text{Put } v = y/x]$$

or $$x^4(1-2y^2/x^2) = e^{-3c} = c'$$

Hence the required solution is $x^2(x^2-2y^2) = c'$.

Example 11.11. *Solve* $(x \tan y/x - y \sec^2 y/x)\,dx - x \sec^2 y/x\,dy = 0.$ *(V.T.U., 2006)*

Solution. The given equation may be rewritten as

$$\frac{dy}{dx} = \left(\frac{y}{x}\sec^2\frac{y}{x} - \tan\frac{y}{x}\right)\cos^2 y/x \qquad ...(i)$$

vy

which is a homogeneous equation. Putting $y = vx$, (i) becomes $v + x\dfrac{dv}{dx} = (v\sec^2 v - \tan v)\cos^2 v$

or $$x\frac{dv}{dx} = v - \tan v \cos^2 v - v$$

Separating the variables $\dfrac{\sec^2 v}{\tan v}\,dv = -\dfrac{dx}{x}$

Integrating both sides $\log \tan v = -\log x + \log c$

or $$x \tan v = c \quad \text{or} \quad x \tan y/x = c.$$

Example 11.12. *Solve* $(1 + e^{x/y})\,dx + e^{x/y}(1 - x/y)\,dy = 0.$

(P.T.U., 2006 ; Rajasthan, 2005 ; V.T.U., 2003)

Solution. The given equation may be rewritten as

$$\frac{dx}{dy} = -\frac{e^{x/y}(1-x/y)}{1+e^{x/y}} \qquad ...(i)$$

which is a homogeneous equation. Putting $x = vy$ so that (i) becomes

$$v + y\frac{dv}{dy} = -\frac{e^v(1-v)}{1+e^v} \quad \text{or} \quad y\frac{dv}{dy} = -\frac{e^v(1-v)}{1+e^v} - v = -\frac{v+e^v}{1+e^v}$$

Separating the variables, we get

$$-\frac{dy}{y} = \frac{1+e^v}{v+e^v}\,dv = \frac{d(v+e^v)}{v+e^v}$$

Integrating both sides, $-\log y = \log(v + e^v) + c$

or $$y(v + e^v) = e^{-c} \quad \text{or} \quad x + ye^{x/y} = c' \quad (\text{say})$$

which is the required solution.

PROBLEMS 11.3

Solve the following differential equations :

1. $(x^2 - y^2)\,dx = 2xy\,dy$
2. $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0.$ *(Bhopal, 2008)*
3. $x^2y\,dx - (x^3 + y^3)\,dy = 0.$ *(V.T.U., 2010)*
4. $y\,dx - x\,dy = \sqrt{x^2 + y^2}\,dx.$ *(Raipur, 2005)*
5. $y^2 + x^2\dfrac{dy}{dx} = xy\dfrac{dy}{dx}.$
6. $(3xy - 2ay^2)\,dx + (x^2 - 2axy)\,dy = 0.$ *(S.V.T.U., 2009)*

[Equations solvable like homogeneous equations : When a differential equation contains y/x a number of times, solve it like a homogeneous equation by putting $y/x = v$].

7. $\frac{dy}{dx} = \frac{y}{x} + \sin\frac{y}{x}$. (V.T.U., 2000 S)
8. $ye^{x/y}\,dx = (xe^{x/y} + y^2)\,dy$. (V.T.U., 2006)
9. $xy(\log x/y)\,dx + [y^2 - x^2\log(x/y)]\,dy = 0$.
10. $x\,dx + \sin^2(y/x)(ydx - xdy) = 0$.
11. $x\cos\frac{y}{x}(ydx + xdy) = y\sin\frac{y}{x}(xdy - ydx)$.

11.8 EQUATIONS REDUCIBLE TO HOMOGENEOUS FORM

The equations of the form $\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}$ ...(1)

can be reduced to the homogeneous form as follows :

Case I. When $\frac{a}{a'} \neq \frac{b}{b'}$

Putting $x = X + h,\ y = Y + k$, (h, k being constants)

so that $dx = dX,\ dy = dY$, (1) becomes

$$\frac{dY}{dX} = \frac{aX + bY + (ah + bk + c)}{a'X + b'Y + (a'h + b'k + c')} \qquad ...(2)$$

Choose h, k so that (2) may become homogeneous.

Put $ah + bk + c = 0$, and $a'h + b'k + c' = 0$

so that $$\frac{h}{bc' - b'c} = \frac{k}{ca' - c'a} = \frac{1}{ab' - ba'}$$

or $$h = \frac{bc' - b'c}{ab' - b'a},\ k = \frac{ca' - c'a}{ab' - ba'} \qquad ...(3)$$

Thus when $ab' - ba' \neq 0$, (2) becomes $\frac{dY}{dX} = \frac{aX + bY}{a'X + b'Y}$ which is homogeneous in X, Y and can be solved by putting $Y = vX$.

Case II. When $\frac{a}{a'} = \frac{b}{b'}$.

i.e., $ab' - b'a = 0$, the above method fails as h and k become infinite or indeterminate.

Now $\frac{a}{a'} = \frac{b}{b'} = \frac{1}{m}$ (say)

$\therefore$ $a' = am,\ b' = bm$ and (1) becomes

$$\frac{dy}{dx} = \frac{(ax + by) + c}{m(ax + by) + c'} \qquad ...(4)$$

Put $ax + by = t$, so that $a + b\frac{dy}{dx} = \frac{dt}{dx}$

or $\frac{dy}{dx} = \frac{1}{b}\left(\frac{dt}{dx} - a\right)$ $\therefore$ (4) becomes $\frac{1}{b}\left(\frac{dt}{dx} - a\right) = \frac{t + c}{mt + c'}$

or $$\frac{dt}{dx} = a + \frac{bt + bc}{mt + c'} = \frac{(am + b)t + ac' + bc}{mt + c'}$$

so that the variables are separable. In this solution, putting $t = ax + by$, we get the required solution of (1).

Example 11.13. *Solve* $\frac{dy}{dx} = \frac{y + x - 2}{y - x - 4}$. *(Raipur, 2005)*

Solution. Given equation is $\frac{dy}{dx} = \frac{y + x - 2}{y - x - 4}$ $\left[\text{Case } \frac{a}{a'} \neq \frac{b}{b'}\right]$ ...(*i*)

Putting $x = X + h$, $y = Y + k$, (h, k being constants) so that $dx = dX$, $dy = dY$, (i) becomes

$$\frac{dY}{dX} = \frac{Y + X + (k + h - 2)}{Y - X + (k - h - 4)} \quad ...(ii)$$

Put $k + h - 2 = 0$ and $k - h - 4 = 0$ so that $h = -1$, $k = 3$.

$\therefore$ (ii) becomes $\dfrac{dY}{dX} = \dfrac{Y + X}{Y - X}$ which is homogeneous in X and Y. ...(iii)

$\therefore$ put $Y = vX$, then $\dfrac{dY}{dX} = v + X\dfrac{dv}{dX}$

$\therefore$ (iii) becomes $\quad v + X\dfrac{dv}{dX} = \dfrac{v+1}{v-1}$ or $X\dfrac{dv}{dX} = \dfrac{v+1}{v-1} - v = \dfrac{1 + 2v - v^2}{v - 1}$

or $$\frac{v - 1}{1 + 2v - v^2}\,dv = \frac{dX}{X}.$$

Integrating both sides, $-\dfrac{1}{2}\displaystyle\int \dfrac{2 - 2v}{1 + 2v - v^2}\,dv = \int \dfrac{dX}{X} + c.$

or $$-\frac{1}{2}\log(1 + 2v - v^2) = \log X + c$$

or $$\log\left(1 + \frac{2Y}{X} - \frac{Y^2}{X^2}\right) + \log X^2 = -2c$$

or $$\log(X^2 + 2XY - Y^2) = -2c \quad \text{or} \quad X^2 + 2XY - Y^2 = e^{-2c} = c' \quad ...(iv)$$

Putting $X = x - h = x + 1$, $Y = y - k = y - 3$, (iv) becomes

$$(x + 1)^2 + 2(x + 1)(y - 3) - (y - 3)^2 = c'$$

or $\quad x^2 + 2xy - y^2 - 4x + 8y - 14 = c'$ which is the required solution.

Example 11.14. *Solve* $(3y + 2x + 4)\,dx - (4x + 6y + 5)\,dy = 0.$ *(Madras, 2000 S)*

Solution. Given equation is $\dfrac{dy}{dx} = \dfrac{(2x + 3y) + 4}{2(2x + 3y) + 5}$ ...(i)

Putting $2x + 3y = t$ so that $2 + 3\dfrac{dy}{dx} = \dfrac{dt}{dx}$ $\quad\therefore$ (i) becomes $\dfrac{1}{3}\left(\dfrac{dt}{dx} - 2\right) = \dfrac{t + 4}{2t + 5}$

or $$\frac{dt}{dx} = 2 + \frac{3t + 12}{2t + 5} = \frac{7t + 22}{2t + 5} \quad \text{or} \quad \frac{2t + 5}{7t + 22}\,dt = dx$$

Integrating both sides, $\displaystyle\int \frac{2t + 5}{7t + 22}\,dt = \int dx + c$

or $$\int\left(\frac{2}{7} - \frac{9}{7}\cdot\frac{1}{7t + 22}\right)dt = x + c \quad \text{or} \quad \frac{2}{7}t - \frac{9}{49}\log(7t + 22) = x + c$$

Putting $t = 2x + 3y$, we have $14(2x + 3y) - 9\log(14x + 21y + 22) = 49x + 49c$

or $\quad 21x - 42y + 9\log(14x + 21y + 22) = c'$ which is the required solution.

PROBLEMS 11.4

Solve the following differential equations :

1. $(x - y - 2)\,dx + (x - 2y - 3)\,dy = 0.$ *(Rajasthan, 2006)*
2. $(2x + y - 3)\,dy = (x + 2y - 3)\,dx.$ *(V.T.U., 2009 S ; Madras, 2000)*
3. $(2x + 5y + 1)\,dx - (5x + 2y - 1)\,dy = 0.$ *(J.N.T.U., 2000)*
4. $\dfrac{dy}{dx} + \dfrac{ax + hy + g}{hx + by + f} = 0.$
5. $\dfrac{dy}{dx} = \dfrac{x + y + 1}{2x + 2y + 3}.$
6. $(4x - 6y - 1)\,dx + (3y - 2x - 2)\,dy = 0.$
7. $(x + 2y)(dx - dy) = dx + dy.$ *(Bhopal, 2002 S ; V.T.U., 2001)*

11.9 LINEAR EQUATIONS

A differential equation is said to be linear if the dependent variable and its differential coefficients occur only in the first degree and not multiplied together.

Thus **the standard form of a linear equation of the first order,** *commonly known as* **Leibnitz's linear equation,*** *is*

$$\frac{\mathbf{dy}}{\mathbf{dx}} + \mathbf{Py} = \mathbf{Q} \qquad \textit{where, P, Q are the functions of x.} \qquad ...(1)$$

To solve the equation, multiply both sides by $e^{\int P dx}$ so that we get

$$\frac{dy}{dx} \cdot e^{\int P dx} + y\,(e^{\int P dx} P) = Q e^{\int P dx} \quad i.e., \quad \frac{d}{dx}(y e^{\int P dx}) = Q e^{\int P dx}$$

Integrating both sides, we get $y e^{\int P dx} = \int Q e^{\int P dx}\, dx + c$ as the required solution.

Obs. The factor $e^{\int P dx}$ on multiplying by which the left-hand side of (1) becomes the differential coefficient of a single function, is called the **integrating factor (I.F.)** of the linear equation (1).

It is important to remember that **I.F.** $= \mathbf{e}^{\int \mathbf{P dx}}$

and the solution is y **(I.F.)** $= \int \mathbf{Q\,(I.F.)\,dx + c}$.

Example 11.15. *Solve* $(x + 1)\dfrac{dy}{dx} - y\, e^{3x}(x + 1)^2$.

Solution. Dividing throughout by $(x + 1)$, given equation becomes

$$\frac{dy}{dx} - \frac{y}{x+1} = e^{3x}(x + 1) \text{ which is Leibnitz's equation.} \qquad ...(i)$$

Here $\qquad P = -\dfrac{1}{x+1}$ and $\displaystyle\int P dx = -\int \frac{dx}{x+1} = -\log(x + 1) = \log(x + 1)^{-1}$

$\therefore \qquad$ I.F. $= e^{\int P dx} = e^{\log(x+1)^{-1}} = \dfrac{1}{x+1}$

Thus the solution of (1) is $y(\text{I.F.}) = \int [e^{3x}(x + 1)]\,(\text{I.F.})\, dx + c$

or $\qquad \dfrac{y}{x+1} = \displaystyle\int e^{3x} dx + c = \frac{1}{3}e^{3x} + c$ or $y = \left(\frac{1}{3}e^{3x} + c\right)(x + 1)$.

Example 11.16. *Solve* $\left(\dfrac{e^{-2\sqrt{x}}}{\sqrt{x}} - \dfrac{y}{\sqrt{x}}\right)\dfrac{dx}{dy} = 1.$

Solution. Given equation can be written as $\dfrac{dy}{dx} + \dfrac{y}{\sqrt{x}} = \dfrac{e^{-2\sqrt{x}}}{\sqrt{x}}$ $\qquad ...(i)$

$\therefore \qquad$ I.F. $= e^{\int x^{1/2} dx} = e^{2\sqrt{x}}$

Thus solution of (i) is y (I.F.) $= \displaystyle\int \frac{e^{-2\sqrt{x}}}{\sqrt{x}}\,(\text{I.F.})\, dx + c$

or $\qquad y e^{2\sqrt{x}} = \displaystyle\int \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \cdot e^{2\sqrt{x}}\, dx + c$

or $\qquad y e^{2\sqrt{x}} = \displaystyle\int x^{-1/2}\, dx + c$ or $y e^{2\sqrt{x}} = 2\sqrt{x} + c$.

* See footnote p. 139.

Example 11.17. *Solve* $3x(1-x^2)\,y^2\,\frac{dy}{dx} + (2x^2-1)\,y^3 = ax^3$ *(Rajasthan, 2006)*

Solution. Putting $y^3 = z$ and $3y^2\,\frac{dy}{dx} = \frac{dz}{dx}$, the given equation becomes

$$x(1-x^2)\,\frac{dz}{dx} + (2x^2-1)\,z = ax^3, \quad \text{or} \quad \frac{dz}{dx} + \frac{2x^2-1}{x-x^3}\,z = \frac{ax^3}{x-x^3} \qquad ...(i)$$

which is Leibnitz's equation in z

$$\therefore \qquad \text{I.F.} = \exp\left(\int \frac{2x^2-1}{x-x^3}\,dx\right)$$

Now $\int \frac{2x^2-1}{x-x^3}\,dx = \int\left(-\frac{1}{x} - \frac{1}{2}\,\frac{1}{1+x} + \frac{1}{2}\cdot\frac{1}{1-x}\right)dx = -\log x - \frac{1}{2}\log(1+x) - \frac{1}{2}\log(1-x)$

$$= -\log\left[x\sqrt{(1-x^2)}\right]$$

$$\therefore \qquad \text{I.F.} = e^{-\log[x\sqrt{(1-x^2)}]} = [x\sqrt{(1-x^2)}]^{-1}$$

Thus the solution of (i) is

$$z(\text{I.F.}) = \int \frac{ax^3}{x-x^3}\,(\text{I.F.})\,dx + c$$

or

$$\frac{z}{[x\sqrt{(1-x^2)}]} = a\int \frac{x^3}{x(1-x^2)}\cdot\frac{1}{x\sqrt{(1-x^2)}}\,dx + c = a\int x(1-x^2)^{-3/2}\,dx$$

$$= -\frac{a}{2}\int(-2x)(1-x^2)^{-3/2}\,dx + c = a\,(1-x^2)^{-1/2} + c$$

Hence the solution of the given equation is

$$y^3 = ax + cx\,\sqrt{(1-x^2)}. \qquad [\because \; z = y^3]$$

Example 11.18. *Solve* $y\,(\log y)\,dx + (x - \log y)\,dy = 0.$ *(U.P.T.U., 2000)*

Solution. We have $\frac{dx}{dy} + \frac{x}{y\log y} = \frac{1}{y}$ ... (i)

which is a Leibnitz's equation in x

$$\therefore \qquad \text{I.F.} = e^{\int \frac{1}{y\log y}dy} = e^{\log(\log y)} = \log y$$

Thus the solution of (i) is $x\,(\text{I.F.}) = \int \frac{1}{y}(\text{I.F.})\,dy + c$

$$x\log y = \int \frac{1}{y}\log y\,dy + c = \frac{1}{2}(\log y)^2 + c$$

i.e.,

$$x = \frac{1}{2}\log y + c\,(\log y)^{-1}.$$

Example 11.19. *Solve* $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy.$ *(Bhopal, 2008 ; V.T.U., 2008 ; U.P.T.U., 2005)*

Solution. This equation contains y^2 and $\tan^{-1} y$ and is, therefore, not a linear in y; but since only x occurs, it can be written as

$$(1+y^2)\,\frac{dx}{dy} = \tan^{-1}y - x \quad \text{or} \quad \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

which is a Leibnitz's equation in x.

$$\therefore \qquad \text{I.F.} = e^{\int P\,dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

Thus the solution is $x\,(\text{I.F.}) = \int \frac{\tan^{-1}y}{1+y^2}\,(\text{I.F.})\,dy + c$

or $$xe^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} \cdot e^{\tan^{-1} y} dy + c$$ $\left[\begin{array}{l}\text{Put } \tan^{-1} y = t \\ \therefore \ \dfrac{dy}{1+y^2} = dt\end{array}\right]$

$$= \int te^t \, dt + c = t \cdot e^t - \int 1 \cdot e^t \, dt + c$$ (Integrating by parts)

$$= t \cdot e^t - e^t + c = (\tan^{-1} y - 1) e^{\tan^{-1} y} + c$$

or $$x = \tan^{-1} y - 1 + ce^{-\tan^{-1} y}.$$

Example 11.20. *Solve $r \sin \theta \, d\theta + (r^3 - 2r^2 \cos \theta + \cos \theta) \, dr = 0$.*

Solution. Given equation can be rewritten as

$$\sin \theta \frac{d\theta}{dr} + \frac{1}{r}(1 - 2r^2) \cos \theta = -r^2 \qquad ...(i)$$

Put $\cos \theta = y$ so that $-\sin \theta \, d\theta/dr = dy/dr$

Then (i) becomes $-\dfrac{dy}{dr} + \left(\dfrac{1}{r} - 2r\right) y = -r^2$ or $\dfrac{dy}{dr} + \left(2r - \dfrac{1}{r}\right) y = r^2$

which is a Leibnitz's equation $\therefore$ I.F. $= e^{\int (2r - 1/r) dr} = e^{r^2 - \log r} = \dfrac{1}{r} e^{r^2}$

Thus its solution is $y \left(\dfrac{1}{r} e^{r^2}\right) = \int r^2 \cdot e^{r^2} \cdot \dfrac{1}{r} \, dr + c$

or $$y e^{r^2} / r = \frac{1}{2} \int e^{r^2} 2r \, dr + c = \frac{1}{2} e^{r^2} + c$$

or $$2e^{r^2} \cos \theta = re^{r^2} + 2cr \quad \text{or} \quad r(1 + 2ce^{-r^2}) = 2 \cos \theta.$$

PROBLEMS 11.5

Solve the following differential equations :

1. $\cos^2 x \dfrac{dy}{dx} + y = \tan x$.
2. $x \log x \dfrac{dy}{dx} + y = \log x^2$. *(V.T.U., 2011)*
3. $2y' \cos x + 4y \sin x = \sin 2x$, given $y = 0$ when $x = \pi/3$. *(V.T.U., 2003)*
4. $\cosh x \dfrac{dy}{dx} + y \sinh x = 2 \cosh^2 x \sinh x$. *(J.N.T.U., 2003)*
5. $(1 - x^2) \dfrac{dy}{dx} - xy = 1$ *(V.T.U., 2010)*
6. $(1 - x^2) \dfrac{dy}{dx} + 2xy = x \sqrt{(1 - x^2)}$ *(Nagpur, 2009)*
7. $\dfrac{dy}{dx} = -\dfrac{x + y \cos x}{1 + \sin x}$.
8. $dr + (2r \cot \theta + \sin 2\theta) \, d\theta = 0$. *(J.N.T.U., 2003)*
9. $\dfrac{dy}{dx} + 2xy = 2e^{-x^2}$ *(P.T.U., 2005)*
10. $(x + 2y^3) \dfrac{dy}{dx} = y$. *(Marathwada, 2008)*
11. $\sqrt{(1 - y^2)} \, dx = (\sin^{-1} y - x) \, dy$.
12. $ye^y \, dx = (y^3 + 2xe^y) \, dy$.
13. $(1 + y^2) \, dx + (x - e^{-\tan^{-1} y}) \, dy = 0$. *(V.T.U., 2006)*
14. $e^{-y} \sec^2 y \, dy = dx + x \, dy$.

11.10 BERNOULLI'S EQUATION

The equation $\dfrac{dy}{dx} + Py = Qy^n$ ...(1)

where P, Q are functions of x, is reducible to the Leibnitz's linear equation and is usually called the Bernoulli's equation.*

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*Named after the Swiss mathematician *Jacob Bernoulli* (1654–1705) who is known for his basic work in probability and elasticity theory. He was professor at Basel and had amongst his students his youngest brother *Johann Bernoulli* (1667–1748) and his nephew *Niklaus Bernoulli* (1687–1759). Johann is known for his basic contributions to Calculus while Niklaus had profound influence on the development of Infinite series and probability. His son *Daniel Bernoulli* (1700–1782) is known for his contributions to kinetic theory of gases and fluid flow.

To solve (1), divide both sides by y^n, so that $y^{-n}\dfrac{dy}{dx} + Py^{1-n} = Q$ ...(2)

Put $y^{1-n} = z$ so that $(1-n)\,y^{-n}\dfrac{dy}{dx} = \dfrac{dz}{dx}$.

$\therefore$ (2) becomes $\dfrac{1}{1-n}\dfrac{dz}{dx} + Pz = Q$ or $\dfrac{dz}{dx} + P(1-n)\,z = Q(1-n)$,

which is Leibnitz's linear in z and can be solved easily.

Example 11.21. *Solve* $x\dfrac{dy}{dx} + y = x^3y^6$.

Solution. Dividing throughout by xy^6, $y^{-6}\dfrac{dy}{dx} + \dfrac{y^{-5}}{x} = x^2$ ...(i)

Put $y^{-5} = z$, so that $-5y^{-6}\dfrac{dy}{dx} = \dfrac{dz}{dx}$ $\therefore$ (i) becomes $-\dfrac{1}{5}\dfrac{dz}{dx} + \dfrac{z}{x} = x^2$

or $$\frac{dz}{dx} - \frac{5}{x}z = -5x^2 \text{ which is Leibnitz's linear in } z. \qquad ...(ii)$$

$$\text{I.F.} = e^{-\int (5/x)dx} = e^{-5\log x} = e^{\log x^{-5}} = x^{-5}$$

$\therefore$ the solution of (ii) is $z\,(\text{I.F.}) = \int(-5x^2)(\text{I.F.})\,dx + c$ or $zx^{-5} = \int(-5x^2)\,x^{-5}\,dx + c$

or $$y^{-5}x^{-5} = -5.\,\frac{x^{-2}}{-2} + c \qquad [\because\ z = y^{-5}]$$

Dividing throughout by $y^{-5}x^{-5}$, $1 = (2.5 + cx^2)\,x^3y^5$ which is the required solution.

Example 11.22. *Solve* $xy\,(1 + xy^2)\dfrac{dy}{dx} = 1.$ *(Nagpur, 2009)*

Solution. Rewriting the given equation as

$$\frac{dx}{dy} - yx = y^3x^2$$

and dividing by x^2, we have

$$x^{-2}\frac{dx}{dy} - yx^{-1} = y^3 \qquad ...(i)$$

Putting $x^{-1} = z$ so that $-x^{-2}\dfrac{dx}{dy} = \dfrac{dz}{dy}$ (i) becomes

$$\frac{dz}{dy} + yz = -y^3 \text{ which is Leibnitz's linear in } z.$$

Here $\quad \text{I.F.} = e^{\int y\,dy} = e^{y^2/2}$

$\therefore$ the solution is $z\,(\text{I.F.}) = \int(-y^3)(\text{I.F.})\,dy + c$

or $$ze^{y^2/2} = -\int y^2 .\, e^{\frac{1}{2}y^2} .\, y\,dy + c \qquad \left|\begin{array}{l}\text{Put } \frac{1}{2}y^2 = t \\ \text{so that } y\,dy = dt\end{array}\right.$$

$$= -2\int t\,.\,e^t\,dt + c \qquad \text{[Integrate by parts]}$$

$$= -2\,[t\,.\,e^t - \int 1\,.\,e^t\,dt] + c = -2\,[te^t - e^t] + c = (2-y^2)\,e^{y^2/2} + c$$

or $$z = (2-y^2) + ce^{-\frac{1}{2}y^2} \quad \text{or} \quad 1/x = (2-y^2) + ce^{-\frac{1}{2}y^2}.$$

Note. *General equation reducible to Leibnitz's linear is* $f'(y)\dfrac{dy}{dx} + Pf(y) = Q$ ...(A)

where P, Q are functions of x. To solve it, put $f(y) = z$.

Example 11.23. *Solve* $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y.$ *(V.T.U., 2011 ; Marathwada, 2008 ; J.N.T.U., 2005)*

Solution. Dividing throughout by $\cos^2 y$, $\sec^2 y \frac{dy}{dx} + 2x \frac{\sin y \cos y}{\cos^2 y} = x^3$

or $$\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3 \text{ which is of the form (A) above.} \quad ...(i)$$

$\therefore$ put $\tan y = z$ so that $\sec^2 y \frac{dy}{dx} = \frac{dz}{dx}$ $\quad\therefore\quad$ (i) becomes $\frac{dz}{dx} + 2xz = x^3$.

This is Leibnitz's linear equation in z. $\quad\therefore\quad$ I.F. $= e^{\int 2x\,dx} = e^{x^2}$

$\therefore$ the solution is $ze^{x^2} = \int e^{x^2} x^3 dx + c = \frac{1}{2}(x^2 - 1)e^{x^2} + c.$

Replacing z by $\tan y$, we get $\tan y = \frac{1}{2}(x^2 - 1) + ce^{-x^2}$ which is the required solution.

Example 11.24. *Solve* $\frac{dz}{dx} + \left(\frac{z}{x}\right) \log z = \frac{z}{x} (\log z)^2.$

Solution. Dividing by z, the given equation becomes

$$\frac{1}{z}\frac{dz}{dx} + \frac{1}{x}\log z = \frac{1}{x}(\log z)^2 \quad ...(i)$$

Put $\log z = t$ so that $\quad \frac{1}{z}\frac{dz}{dx} = \frac{dt}{dx}$. $\quad\therefore\quad$ (i) becomes

$$\frac{dt}{dx} + \frac{t}{x} = \frac{t^2}{x} \quad \text{or} \quad \frac{1}{t^2}\frac{dt}{dx} + \frac{1}{x}\cdot\frac{1}{t} = \frac{1}{x} \quad ...(ii)$$

This being Bernoulli's equation, put $1/t = v$ so that (ii) reduces to

$$-\frac{dv}{dx} + \frac{v}{x} = \frac{1}{x} \quad \text{or} \quad \frac{dv}{dx} - \frac{1}{x}v = -\frac{1}{x}$$

This is Leibnitz's linear in v. $\quad\therefore\quad$ I.F. $= e^{-\int 1/x\,dx} = 1/x$

$\therefore$ the solution is $\quad v \cdot \frac{1}{x} = -\int \frac{1}{x}\cdot\frac{1}{x}dx + c = \frac{1}{x} + c$

Replacing v by $1/\log z$, we get $(x \log z)^{-1} = x^{-1} + c$ or $(\log z)^{-1} = 1 + cx$
which is the required solution.

PROBLEMS 11.6

Solve the following equations :

1. $\frac{dy}{dx} = y \tan x - y^2 \sec x.$ *(P.T.U., 2005)*
2. $r \sin\theta - \cos\theta \frac{dr}{d\theta} = r^2.$ *(V.T.U., 2005)*
3. $2xy' = 10x^3y^5 + y.$
4. $(x^3y^2 + xy)\,dx = dy.$ *(B.P.T.U., 2005)*
5. $\frac{dy}{dx} = \frac{x^2 + y^2 + 1}{2xy}.$ *(Bhillai, 2005)*
6. $x(x - y)\,dy + y^2\,dx = 0.$ *(I.S.M., 2001)*
7. $\frac{dy}{dx} - \frac{\tan y}{1 + x} = (1 + x)e^x \sec y.$ *(Bhopal, 2009)*
8. $e^y\left(\frac{dy}{dx} + 1\right) = e^x.$ *(V.T.U., 2009)*
9. $\sec^2 y \frac{dy}{dx} + x \tan y = x^3.$
10. $\tan y \frac{dy}{dx} + \tan x = \cos y \cos^2 x.$ *(Sambalpur, 2002)*
11. $\frac{dy}{dx} = \frac{y}{x - \sqrt{(xy)}}.$ *(V.T.U., 2011)*
12. $(y \log x - 2)\,y\,dx - x\,dy = 0.$ *(V.T.U., 2006)*

11.11 EXACT DIFFERENTIAL EQUATIONS

(1) Def. *A differential equation of the form* $\mathbf{M}(x, y)\, \boldsymbol{dx} + \mathbf{N}(x, y)\, \boldsymbol{dy} = 0$ *is said to be* ***exact*** *if its left hand member is the exact differential of some function* $u(x, y)$ *i.e.,* $du = Mdx + Ndy = 0$. Its solution, therefore, is $u(x, y) = c$.

(2) Theorem. The necessary and sufficient condition for the differential equation $Mdx + Ndy = 0$ to be exact is

$$\frac{\partial \mathbf{M}}{\partial \mathbf{y}} = \frac{\partial \mathbf{N}}{\partial \mathbf{x}}$$

Condition is necessary :

The equation $Mdx + Ndy = 0$ will be exact, if

$$Mdx + Ndy \equiv du \qquad \text{...(1)}$$

where u is some function of x any y.

But $$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \qquad \text{...(2)}$$

$\therefore$ equating coefficients of dx and dy in (1) and (2), we get $M = \dfrac{\partial u}{\partial x}$ and $N = \dfrac{\partial u}{\partial y}$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \text{ and } \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}.$$

But $$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} \qquad \text{(Assumption)}$$

$\therefore$ $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ which is the necessary condition for exactness.

Condition is sufficient : *i.e.*, if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, then $Mdx + Ndy = 0$ is exact.

Let $\int Mdx = u$, where y is supposed constant while performing integration.

Then $$\frac{\partial}{\partial x}\left(\int Mdx\right) = \frac{\partial u}{\partial x}, \quad i.e., \ M = \frac{\partial u}{\partial x} \qquad \left\{ \begin{array}{l} \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \text{ (given)} \\ \text{and } \dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial^2 u}{\partial x \partial y} \end{array} \right. \qquad \text{...(3)}$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \text{or} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

Interating both sides w.r.t. x (taking y as constant).

$$N = \frac{\partial u}{\partial y} + f(y), \text{ where } f(y) \text{ is a function of } y \text{ alone.} \qquad \text{...(4)}$$

$$\therefore \quad Mdx + Ndy = \frac{\partial u}{\partial x} dx + \left\{\frac{\partial u}{\partial y} + f(y)\right\} dy \qquad \text{[By (3) and (4)]}$$

$$= \left\{\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy\right\} + f(y)\, dy = du + f(y)\, dy = d\left[u + \int f(y) dy\right] \qquad \text{...(5)}$$

which shows that $Mdx + Ndy = 0$ is exact.

(3) Method of solution. By (5), the equation $Mdx + Ndy = 0$ becomes $d[u + \int f(y) dy] = 0$

Integrating $\quad u + \int f(y) dy = 0.$

But $\quad u = \int_{y \text{ constant}} Mdx$ and $f(y)$ = terms of N not containing x.

$\therefore$ *The solution of* $Mdx + Ndy = 0$ *is*

$$\int_{(y \text{ cons.})} \mathbf{M\, dx} + \int (\textbf{terms of N not containing x})\, \mathbf{dy} = \mathbf{c}$$

provided $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Example 11.25. *Solve* $(y^2 e^{xy^2} + 4x^3)\,dx + (2xy\,e^{xy^2} - 3y^2)\,dy = 0.$ (V.T.U., 2006)

Solution. Here $M = y^2 e^{xy^2} + 4x^3$ and $N = 2xy\,e^{xy^2} - 3y^2$

$$\therefore \quad \frac{\partial M}{\partial y} = 2y\,e^{xy^2} + y^2 e^{xy^2} \,.\, 2xy = \frac{\partial N}{\partial x}$$

Thus the equation is exact and its solution is

$$\int_{(y \text{ const.})} M dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$$

i.e., $$\int_{(y \text{ const.})} (y^2 e^{y^2 x} + 4x^3)\,dx + \int (-3y^2)\,dy = c \quad \text{or} \quad e^{xy^2} + x^4 - y^3 = c.$$

Example 11.26. *Solve* $\left\{y\left(1 + \frac{1}{x}\right) + \cos y\right\} dx + (x + \log x - x \sin y)\,dy = 0.$

(Marathwada, 2008 S ; V.T.U., 2006)

Solution. Here $M = y\,(1 + 1/x) + \cos y$ and $N = x + \log x - x \sin y$

$$\therefore \quad \frac{\partial M}{\partial y} = 1 + 1/x - \sin y = \frac{\partial N}{\partial x}$$

Then the equation is exact and its solution is

$$\int_{(y \text{ const})} M dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$$

$$\int_{(y \text{ const})} \left\{\left(1 + \frac{1}{x}\right) y + \cos y\right\} dx = c \quad \text{or} \quad (x + \log x)\,y + x \cos y = c.$$

Example 11.27. *Solve* $(1 + 2xy \cos x^2 - 2xy)\,dx + (\sin x^2 - x^2)\,dy = 0.$

Solution. Here $M = 1 + 2xy \cos x^2 - 2xy$ and $N = \sin x^2 - x^2$

$$\therefore \quad \frac{\partial M}{\partial y} = 2x \cos x^2 - 2x = \frac{\partial N}{\partial x}$$

Thus the equation is exact and its solution is

$$\int_{(y \text{ const})} M dx + \int (\text{terms of } N \text{ not containing } x) = c$$

i.e., $$\int_{(y \text{ const})} (1 + 2xy \cos x^2 - 2xy)\,dx = c \quad \text{or} \quad x + y\left[\int \cos x^2 \,.\, 2x dx - \int 2x\,dx\right] = c$$

or $$x + y \sin x^2 - yx^2 = c.$$

Example 11.28. *Solve* $\dfrac{dy}{dx} + \dfrac{y \cos x + \sin y + y}{\sin x + x \cos y + x} = 0.$ (Kurukshetra, 2005)

Solution. Given equation can be written as

$$(y \cos x + \sin y + y)\,dx + (\sin x + x \cos y + x)\,dy = 0.$$

Here $M = y \cos x + \sin y + y$ and $N = \sin x + x \cos y + x.$

$$\therefore \quad \frac{\partial M}{\partial y} = \cos x + \cos y + 1 = \frac{\partial N}{\partial x}.$$

Thus the equation is exact and its solution is

$$\int_{(y \text{ const.})} M dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$$

i.e., $$\int_{(y \text{ const.})} (y \cos x + \sin y + y)\,dx + \int (0)\,dx = c \quad \text{or} \quad y \sin x + (\sin y + y)\,x = c.$$

Example 11.29. *Solve* $(2x^2 + 3y^2 - 7)\, xdx - (3x^2 + 2y^2 - 8)\, ydy = 0$. *(U.P.T.U., 2005)*

Solution. Given equation can be written as

$$\frac{ydy}{xdx} = \frac{2x^2 + 3y^2 - 7}{3x^2 + 2y^2 - 8}$$

or $$\frac{ydy + xdx}{ydy - xdx} = \frac{5(x^2 + y^2 - 3)}{-x^2 + y^2 + 1}$$ [By componendo & dividendo]

or $$\frac{xdx + ydy}{x^2 + y^2 - 3} = 5 \cdot \frac{xdx - ydy}{x^2 - y^2 - 1}$$

Integrating both sides, we get

$$\int \frac{2xdx + 2ydy}{x^2 + y^2 - 3} = 5 \int \frac{2xdx - 2ydy}{x^2 - y^2 - 1} + c$$

or $$\log (x^2 + y^2 - 3) = 5 \log (x^2 - y^2 - 1) + \log c'$$ [Writing $c = \log c'$]

or $$x^2 + y^2 - 3 = c' (x^2 - y^2 - 1)^5$$

which is the required solution.

PROBLEMS 11.7

Solve the following equations :

1. $(x^2 - ay)\, dx = (ax - y^2)\, dy$.
2. $(x^2 + y^2 - a^2)\, xdx + (x^2 - y^2 - b^2)\, ydy = 0$ *(Kurukshetra, 2005)*
3. $(x^2 - 4xy - 2y^2)\, dx + (y^2 - 4xy - 2x^2)\, dy = 0$.
4. $(x^4 - 2xy^2 + y^4)\, dx - (2x^2y - 4xy^3 + \sin y)\, dy = 0$
5. $ye^{xy}dx + (xe^{xy} + 2y)\, dy = 0$
6. $(5x^4 + 3x^2y^2 - 2xy^3)\, dx + (2x^3y - 3x^2y^2 - 5y^4)\, dy = 0$ *(V.T.U., 2008)*
7. $(3x^2 + 6xy^2)\, dx + (6x^2y + 4y^3)\, dy = 0$
8. $\dfrac{2x}{y^3}\, dx + \dfrac{y^2 - 3x^2}{y^4}\, dy = 0$
9. $y \sin 2x\, dx - (1 + y^2 + \cos^2 x)\, dy = 0$
10. $(\sec x \tan x \tan y - e^x)\, dx + \sec x \sec^2 y\, dy = 0$ *(Marathwada, 2008)*
11. $(2xy + y - \tan y)\, dx + x^2 - x \tan^2 y + \sec^2 y)\, dy = 0$. *(Nagpur, 2009)*

11.12 EQUATIONS REDUCIBLE TO EXACT EQUATIONS

Sometimes a differential equation which is not exact, can be made so on multiplication by a suitable factor called an *integrating factor*. The rules for finding integrating factors of the equation $Mdx + Ndy = 0$ are as follows :

(1) I.F. found by inspection. In a number of cases, the integrating factor can be found after regrouping the terms of the equation and recognizing each group as being a part of an exact differential. In this connection the following integrable combinations prove quite useful :

$$xdy + ydx = d\,(xy)$$

$$\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right); \frac{xdy - ydx}{xy} = d\left[\log\left(\frac{y}{x}\right)\right]$$

$$\frac{xdy - ydx}{y^2} = -\, d\left(\frac{x}{y}\right); \frac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$$

$$\frac{xdy - ydx}{x^2 - y^2} = d\left(\frac{1}{2}\log\frac{x + y}{x - y}\right).$$

Example 11.30. *Solve* $y\,(2xy + e^x)\, dx = e^x\, dy$. *(Kurukshetra, 2005)*

Solution. It is easy to note that the terms $ye^x dx$ and $e^x dy$ should be put together.

$\therefore$ $$(ye^x dx - e^x dy) + 2xy^2\, dx = 0$$

Now we observe that the term $2xy^2\,dx$ should not involve y^2. This suggests that $1/y^2$ may be I.F. Multiplying throughout by $1/y^2$, it follows

$$\frac{ye^x\,dx - e^x\,dy}{y^2} + 2xdx = 0 \quad \text{or} \quad d\left(\frac{e^x}{y}\right) + 2xdx = 0$$

Integrating, we get $\dfrac{e^x}{y} + x^2 = c$ which is the required solution.

(2) I.F. of a homogeneous equation. *If $Mdx + Ndy = 0$ be a homogeneous equation in x and y, then $1/(Mx + Ny)$ is an integrating factor $(Mx + Ny \neq 0)$.*

Example 11.31. *Solve* $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0.$ *(Osmania, 2003 S)*

Solution. This equation is homogeneous in x and y.

$$\therefore \qquad \text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{(x^2y - 2xy^2)\,x - (x^3 - 3x^2y)y} = \frac{1}{x^2y^2}$$

Multiplying throughout by $1/x^2y^2$, the equation becomes

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0 \text{ which is exact.}$$

$\therefore$ the solution is $\displaystyle\int_{(y\text{ const})} Mdx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$ or $\dfrac{x}{y} - 2\log x + 3\log y = c.$

(3) I.F. for an equation of the type $f_1(xy)ydx + f_2(xy)xdy = 0$.

If the equation $Mdx + Ndy = 0$ be of this form, then $1/(Mx - Ny)$ is an integrating factor $(Mx - Ny \neq 0)$.

Example 11.32. *Solve* $(1 + xy)\,ydx + (1 - xy)\,xdy = 0.$ *(S.V.T.U., 2008)*

Solution. The given equation is of the form $f_1(xy)\,ydx + f_2(xy)\,xdy = 0$

Here $\qquad M = (1 + xy)\,y,\ N = (1 - xy)\,x.$

$$\therefore \qquad \text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{(1 + xy)yx - (1 - xy)xy} = \frac{1}{2x^2y^2}$$

Multiplying throughout by $1/2x^2y^2$, it becomes

$$\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0, \text{ which is an exact equation.}$$

$$\therefore \text{ the solution is } \int_{(y\text{ const})} Mdx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$$

or
$$\frac{1}{2y}\left(-\frac{1}{x}\right) + \frac{1}{2}\log x - \frac{1}{2}\log y = c \quad \text{or} \quad \log\frac{x}{y} - \frac{1}{xy} = c'.$$

(4) In the equation $Mdx + Ndy = 0$,

(a) if $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}$ *be a function of x only = f(x) say, then* $e^{\int f(x)dx}$ *is an integrating factor.*

(b) if $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}$ *be a function of y only = F(y) say, then* $e^{\int F(y)dy}$ *is an integrating factor.*

Example 11.33. *Solve* $(xy^2 - e^{1/x^3})\,dx - x^2\,ydy = 0.$ *(S.V.T.U., 2009 ; Mumbai, 2007)*

Solution. Here $M = xy^2 - e^{1/x^3}$ and $N = -x^2y$

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{2xy - (-2xy)}{-x^2y} = -\frac{4}{x} \text{ which is a function of } x \text{ only.}$$

$\therefore$ $$\text{I.F.} = e^{\int \frac{-4}{x} dx} = e^{-4 \log x} = x^{-4}$$

Multiplying throughout by x^{-4}, we get $\left(\dfrac{y^2}{x^3} - \dfrac{1}{4^4} e^{1/x^3}\right) dx - \dfrac{y}{x^2}\, dy = 0$

which is an exact equation.

$\therefore$ the solution is $\int_{(y \text{ const})} (Mdx) + \int (\text{terms of } N \text{ not containing } x)\, dy = c.$

or $$\int \left(\frac{y^2}{x^3} - \frac{1}{x^4} e^{1/x^3}\right) dx + 0 = c$$

or $$-\frac{y^2 x^{-2}}{2} + \frac{1}{3} \int e^{x^{-3}} (-3x^{-4})\, dx = c \text{ or } \frac{1}{3} e^{x^{-3}} - \frac{1}{2} \frac{y^2}{x^2} = c.$$

Otherwise it can be solved as a Bernoulli's equation (§ 11.10)

Example 11.34. *Solve* $(xy^3 + y)\, dx + 2\, (x^2y^2 + x + y^4)\, dy = 0.$

Solution. Here $M = xy^3 + y,\ N = 2(x^2y^2 + x + y^4)$

$\therefore$ $$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y(xy^2+1)} (4xy^2 + 2 - 3xy^2 - 1) = \frac{1}{y},$$ which is a function of y alone.

$\therefore$ $$\text{I.F.} = e^{\int 1/y\, dy} = e^{\log y} = y$$

Multiplying throughout by y, it becomes $(xy^4 + y^2)\, dx + (2x^2y^3 + 2xy + 2y^5)\, dy = 0$, which is an exact equation.

$\therefore$ its solution is $\int_{(y \text{ const})} (Mdx) + \int (\text{terms of } N \text{ not containing } x)\, dy = 0$

or $$\int_{(y \text{ const})} (xy^4 + y^2) dx + \int 2y^5\, dy = c \quad \text{or} \quad \frac{1}{2} x^2y^4 + xy^2 + \frac{1}{3} y^6 = c.$$

Example 11.35. *Solve* $(y \log y)\, dx + (x - \log y)\, dy = 0$ *(U.P.T.U., 2004)*

Solution. Here $M = y \log y$ and $N = x - \log y$

$\therefore$ $$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y \log y} (1 - \log y - 1) = -\frac{1}{y},$$ which is a function of y alone.

$\therefore$ $$\text{I.F.} = e^{-\int \frac{1}{y} dy} = e^{-\log y} = \frac{1}{y}$$

Multiplying the given equation throughout by $1/y$, it becomes

$$\log y\, dx + \frac{1}{y}\, (x - \log y)\, dy = 0$$

which is an exact equation $\left[\because \dfrac{\partial}{\partial y} (\log y) = \dfrac{\partial}{\partial x}\left(\dfrac{x - \log y}{y}\right)\right]$

$\therefore$ its solution is $\int_{(y \text{ const})} (Mdx) + \int (\text{terms of } N \text{ not containing } x)\, dy = c$

or $$\log y \int dx + \int \left(\frac{-\log y}{y}\right) dy = c \qquad \text{or} \quad x \log y - \frac{1}{2} (\log y)^2 = c.$$

(5) For the equation of the type

$$x^a y^b\, (my dx + nx dy) + x^{a'} y^{b'}\, (m' y dx + n' x dy) = 0,$$

an integrating factor is $x^h y^k$

where $$\frac{a+h+1}{m} = \frac{b+k+1}{n},\ \frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}.$$

Example 11.36. *Solve* $y(xy + 2x^2y^3)\,dx + x\,(xy - x^2y^2)\,dy = 0.$ *(Hissar, 2005 ; Kurukshetra, 2005)*

Solution. Rewriting the equation as $xy\,(ydx + xdy) + x^2y^2\,(2ydx - xdy) = 0$ and comparing with

$$x^ay^b\,(mydx + nxdy) + x^{a'}y^{b'}\,(m'ydx + n'xdy) = 0,$$

we have $a = b = 1$, $m = n = 1$; $a' = b' = 2$, $m' = 2$, $n' = -1$.

$$\therefore \qquad \text{I.F.} = x^hy^k.$$

where
$$\frac{a+h+1}{m} = \frac{b+k+1}{n},\ \frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}$$

i.e.
$$\frac{1+h+1}{1} = \frac{1+k+1}{1},\ \frac{2+h+1}{2} = \frac{2+k+1}{-1}$$

or
$$h - k = 0,\ h + 2k + 9 = 0$$

Solving these, we get $h = k = -3$. $\therefore$ I.F. $= 1/x^3y^3$.

Multiplying throughout by $1/x^3y^3$, it becomes

$$\left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right)dx = 0, \text{ which is an exact equation.}$$

$\therefore$ The solution is $\int_{(y\ const)} Mdx + \int(\text{terms of } N \text{ not containing } x)\,dy = c$

or
$$\frac{1}{y}\left(-\frac{1}{x}\right) + 2\log x - \log y = c \quad \text{or} \quad 2\log x - \log y - 1/xy = c.$$

PROBLEMS 11.8

Solve the following equations :

1. $xdy - ydx + a(x^2 + y^2)\,dx = 0.$
2. $xdx + ydy = \dfrac{a^2(xdy - ydx)}{x^2 + y^2}.$ *(U.P.T.U., 2005)*
3. $ydx - xdy + \log x\,dx = 0.$
4. $\dfrac{dy}{dx} = \dfrac{x^3 + y^3}{xy^2}.$
5. $(x^3y^2 + x)\,dy + (x^2y^3 - y)\,dx = 0.$
6. $(x^2y^2 + xy + 1)ydx + (x^2y^2 - xy + 1)\,xdy = 0.$
7. $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0.$
8. $(4xy + 3y^2 - x)\,dx + x\,(x + 2y)dy = 0$ *(Mumbai, 2006)*
9. $x^4\dfrac{dy}{dx} + x^3y + \operatorname{cosec}(xy) = 0.$
10. $(y - xy^2)\,dx - (x + x^2y)\,dy = 0$ *(Mumbai, 2006)*
11. $ydx - xdy + 3x^2y^2e^{x^3}\,dx = 0.$ *(Kurukshetra, 2006)*
12. $(y^2 + 2x^2y)\,dx + (2x^3 - xy)dy = 0.$ *(Rajasthan, 2005)*
13. $2ydx + x(2\log x - y)\,dy = 0.$ *(P.T.U., 2005)*

11.13 EQUATIONS OF THE FIRST ORDER AND HIGHER DEGREE

As dy/dx will occur in higher degrees, it is convenient to denote dy/dx by p. Such equations are of the form $f(x, y, p) = 0$. Three cases arise for discussion :

Case. I. Equation solvable for p. A differential equation of the first order but of the nth degree is of the form

$$p^n + P_1p^{n-1} + P_2p^{n-2} + \ldots + P_n = 0 \qquad \ldots(1)$$

where $P_1, P_2, \ldots, P_n$ are functions of x and y.

Splitting up the left hand side of (1) into n linear factors, we have

$$[p - f_1(x, y)]\,[p - f_2(x, y)] \ldots [p - f_n(x, y)] = 0.$$

Equating each of the factors to zero,

$$p = f_1(x, y),\ p = f_2(x, y),\ \ldots,\ p = f_n(x, y)$$

Solving each of these equations of the first order and first degree, we get the solutions

$$F_1(x, y, c) = 0,\ F_2(x, y, c) = 0,\ \ldots,\ F_n(x, y, c) = 0.$$

These n solutions constitute the general solution of (1).

Otherwise, the general solution of (1) may be written as

$$F_1(x, y, c) \,.\, F_2(x, y, c) \,.......\, F_n(x, y, c) = 0.$$

Example 11.37. *Solve* $\dfrac{dy}{dx} - \dfrac{dx}{dy} = \dfrac{x}{y} - \dfrac{y}{x}$.

Solution. Given equation is $p - \dfrac{1}{p} = \dfrac{x}{y} - \dfrac{y}{x}$ where $p = \dfrac{dy}{dx}$ or $p^2 + p\left(\dfrac{y}{x} - \dfrac{x}{y}\right) - 1 = 0.$

Factorising $(p + y/x)(p - x/y) = 0.$

Thus we have $\quad p + y/x = 0 \quad ...(i) \quad$ and $\quad p - x/y = 0 \quad ...(ii)$

From (*i*), $\quad \dfrac{dy}{dx} + \dfrac{y}{x} = 0$ or $xdy + ydx = 0$

i.e., $\quad d(xy) = 0$. Integrating, $xy = c$.

From (*ii*), $\quad \dfrac{dy}{dx} - \dfrac{x}{y} = 0$ or $xdx - ydx = 0$

Integrating, $x^2 - y^2 = c$. Thus $xy = c$ or $x^2 - y^2 = c$, constitute the required solution.

Otherwise, combining these into one, the required solution can be written as

$$(xy - c)(x^2 - y^2 - c) = 0.$$

Example 11.38. *Solve* $p^2 + 2py \cot x = y^2$. *(Bhopal, 2008 ; Kerala, 2005)*

Solution. We have $p^2 + 2py \cot x + (y \cot x)^2 = y^2 + y^2 \cot^2 x$

or $\quad p + y \cot x = \pm\, y \operatorname{cosec} x$

i.e., $\quad p = y\,(-\cot x + \operatorname{cosec} x) \quad ...(i)$

or $\quad p = y\,(-\cot x - \operatorname{cosec} x) \quad ...(ii)$

From (*i*), $\dfrac{dy}{dx} = y(-\cot x + \operatorname{cosec} x)$ or $\dfrac{dy}{y} = (\operatorname{cosec} x - \cot x)\, dx$

Integrating, $\log y = \log \tan \dfrac{x}{2} - \log \sin x + \log c = \log \dfrac{c \tan x/2}{\sin x}$

or $\quad y = \dfrac{c}{2 \cos x^2/2}$ or $y\,(1 + \cos x) = c \quad ...(iii)$

From (*ii*), $\quad \dfrac{dy}{dx} = -\,y\,(\cot x + \operatorname{cosec} x)$ or $\dfrac{dy}{y} = -\,(\cot x + \operatorname{cosec} x)\, dx$

Integrating, $\log y = -\log \sin x - \log \tan \dfrac{x}{2} + \log c = \log \dfrac{c}{\sin x \tan \dfrac{x}{2}}$

or $\quad y = \dfrac{c}{2 \sin^2 \dfrac{x}{2}}$ or $y(1 - \cos x) = c \quad ...(iv)$

Thus combining (*iii*) and (*iv*), the required general solution is

$$y\,(1 \pm \cos x) = c.$$

PROBLEMS 11.9

Solve the following equations :

1. $y\left(\dfrac{dy}{dx}\right)^2 + (x - y)\dfrac{dy}{dx} - x = 0.$
2. $p(p + y) = x\,(x + y)$. (V.T.U., 2011)
3. $y = x\,[p + \sqrt{(1 + p^2)}]$.
4. $xy\left(\dfrac{dy}{dx}\right)^2 - (x^2 + y^2)\dfrac{dy}{dx} + xy = 0.$
5. $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0.$ (Madras, 2003)

Case II. Equations solvable for y. If the given equation, on solving for y, takes the form

$$y = f(x, p). \quad \text{...(1)}$$

then differentiation with respect to x gives an equation of the form

$$p = \frac{dy}{dx} = \phi\left(x, p, \frac{dp}{dx}\right).$$

Now it may be possible to solve this new differential equation in x and p.

Let its solution be $F(x, p, c) = 0$. ...(2)

The ellimination of p from (1) and (2) gives the required solution.

In case elimination of p is not possible, then we may solve (1) and (2) for x and y and obtain

$$x = F_1(p, c), y = F_2(p, c)$$

as the required solution, where p is the parameter.

Obs. This method is especially useful for equations which do not contain x.

Example 11.39. *Solve* $y - 2px = \tan^{-1}(xp^2)$.

Solution. Given equation is $y = 2px + \tan^{-1}(xp^2)$ ...(*i*)

Differentiating both sides with respect to x, $\dfrac{dy}{dx} = p = 2\left(p + x\dfrac{dp}{dx}\right) + \dfrac{p^2 + 2xp\dfrac{dp}{dx}}{1 + x^2p^4}$

or $$p + 2x\frac{dp}{dx} + \left(p + 2x\frac{dp}{dx}\right) \cdot \frac{p}{1 + x^2p^4} = 0 \text{ or } \left(p + 2x\frac{dp}{dx}\right)\left(1 + \frac{p}{1 + x^2p^4}\right) = 0$$

This gives $p + 2x\, dp/dx = 0$.

Separating the variables and integrating, we have $\displaystyle\int \frac{dx}{x} + 2\int \frac{dp}{p}$ = a constant

or $$\log x + 2 \log p = \log c \quad \text{or} \quad \log xp^2 = \log c$$

whence $$xp^2 = c \quad \text{or} \quad p = \sqrt{(c/x)} \quad \text{...(ii)}$$

Eliminating p from (*i*) and (*ii*), we get $y = 2\sqrt{(c/x)}\,x + \tan^{-1} c$

or $y = 2\sqrt{(cx)} + \tan^{-1} c$ which is the general solution of (*i*).

Obs. The significance of the factor $1 + p/(1 + x^2p^4) = 0$ which we didn't consider, will not be considered here as it concerns 'singular solution' of (*i*) whereas *we are interested only in finding general solution.*

Caution. Sometimes one is tempted to write (*ii*) as

$$\frac{dy}{dx} = \sqrt{\left(\frac{c}{x}\right)}$$

and integrating it to say that the required solution is $y = 2\sqrt{(cx)} + c'$. Such a reasoning is *incorrect*.

Example 11.40. *Solve* $y = 2px + p^n$. (*Bhopal, 2009*)

Solution. Given equation is $y = 2px + p^n$ ...(*i*)

Differentiating it with respect to x, we get

$$\frac{dy}{dx} = p = 2p + 2x\frac{dp}{dx} + np^{n-1}\frac{dp}{dx} \quad \text{or} \quad p\frac{dx}{dp} + 2x = -np^{n-1}$$

or $$\frac{dx}{dp} + \frac{2x}{p} = -np^{n-2} \quad \text{...(ii)}$$

This is Leibnitz's linear equation in x and p. Here I.F. $= e^{\int \frac{2}{p} dp} = e^{\log p^2} = p^2$

$\therefore$ the solution of (ii) is

$$x(\text{I.F.}) = \int(-np^{n-2})\,.\,(\text{I.F.})\,dp + c \quad \text{or} \quad xp^2 = -n\int p^n\,dp + c = -\frac{np^{n+1}}{n+1} + c$$

or $$x = cp^{-2} - \frac{np^{n-1}}{n+1} \qquad ...(iii)$$

Substituting this value of x in (i), we get $y = \dfrac{2c}{p} + \dfrac{1-n}{1+n}p^n$ ...(iv)

The equations (iii) and (iv) taken together, with parameter p, constitute the general solution (i).

Obs. In general, the equations of the form $y = xf(p) + \phi(p)$, known as *Lagrange's* equation, are solvable for y and lead to Leibnitz's equation in dx/dp.

PROBLEMS 11.10

Solve the following equations :

1. $y = x + a\tan^{-1}p$.
2. $y + px = x^4p^2$. (*S.V.T.U., 2007*)
3. $x^2\left(\dfrac{dy}{dx}\right)^4 + 2x\dfrac{dy}{dx} - y = 0$.
4. $xp^2 + x = 2yp$.
5. $y = xp^2 + p$.
6. $y = p\sin p + \cos p$.

Case III. **Equations solvable for x.** If the given equation on solving for x, takes the form

$$x = f(y, p) \qquad ...(1)$$

then differentiation with respect to y gives an equation of the form

$$\frac{1}{p} = \frac{dx}{dy} = \phi\left(y, p, \frac{dp}{dy}\right)$$

Now it may be possible to solve the new differential equation in y and p. Let its solution be $F(y, p, c) = 0$.

The elimination of p from (1) and (2) gives the required solution. In case the elimination is not feasible, (1) and (2) may be expressed in terms of p and p may be regarded as a parameter.

Obs. This method is especially useful for equations which do not contain y.

Example 11.41. *Solve* $y = 2px + y^2p^3$. (*Bhopal, 2008*)

Solution. Given equation, on solving for x, takes the form $x = \dfrac{y - y^2p^3}{2p}$

Differentiating with respect to y, $\dfrac{dx}{dy}\left(=\dfrac{1}{p}\right) = \dfrac{1}{2}\cdot\dfrac{p\left(1 - 2y\,.\,p^3 - y^2 3p^2\dfrac{dp}{dy}\right) - (y - y^2p^3)\dfrac{dp}{dy}}{p^2}$

or $$2p = p - 2yp^4 - 3y^2p^3\frac{dp}{dy} - y\frac{dp}{dy} + y^2p^3\frac{dp}{dy}$$

or $$p + 2yp^4 + 2y^2p^3\frac{dp}{dy} + y\frac{dp}{dy} = 0 \text{ or } p\,(1 + 2yp^3) + y\frac{dp}{dy}(1 + 2yp^3) = 0.$$

or $$\left(p + y\frac{dp}{dy}\right)(1 + 2py^3) = 0 \text{ This gives } p + y\frac{dp}{dy} = 0. \text{ or } \frac{d}{dy}(py) = 0.$$

Integrating $$py = c. \qquad ...(i)$$

Thus eliminating from the given equation and (i), we get $y = 2\dfrac{c}{y}x + \dfrac{c^3}{y^3}y^2$ or $y^2 = 2cx + c^3$

which is the required solution.

PROBLEMS 11.11

Solve the following equations :

1. $p^3 - 4xyp + 8y^2 = 0$. (*Kanpur, 1996*)
2. $p^3y + 2px = y$.
3. $x - yp = ap^2$. (*Andhra, 2000*)
4. $p = \tan\left(x - \dfrac{p}{1+p^2}\right)$. (*S.V.T.U., 2008*)

11.14 CLAIRAUT'S EQUATION*

An equation of the form $y = px + f(p)$ is known as Clairaut's equation ...(1)

Differentiating with respect to x, we have $p = p + x \dfrac{dp}{dx} + f'(p) \dfrac{dp}{dx}$

or $$[x + f'(p)] \frac{dp}{dx} = 0 \qquad \therefore \frac{dp}{dx} = 0, \text{ or } x + f'(p) = 0$$

$$\frac{dp}{dx} = 0, \text{ gives } p = c \qquad ...(2)$$

Thus eliminating p from (1) and (2), we get $y = cx + f(c)$ ...(3)

as the general solution of (1).

Hence the solution of the Clairaut's equation is obtained on replacing p by c.

Obs. If we eliminate p from $x + f'(p) = 0$ and (1), we get an equation involving no constant. *This is the* **singular solution** of (1) *which gives the envelope of the family of straight lines* (3).

To obtain the singular solution, we proceed as follows :

(*i*) Find the general solution by replacing p by c *i.e.*, (3)

(*ii*) Differentiate this w.r.t. c giving $x + f'(c) = 0$. ...(4)

(*iii*) Eliminate c from (3) and (4) which will be the *singular solution*.

Example 11.42. *Solve $p = \sin(y - xp)$. Also find its singular solutions.*

Solution. Given equation can be written as

$\sin^{-1} p = y - xp$ or $y = px + \sin^{-1} p$ which is the *Clairaut's equation*.

$\therefore$ its solution is $y = cx + \sin^{-1} c$.

To find the singular solution, differentiate (*i*) w.r.t. c giving

$$0 = x + \frac{1}{\sqrt{1-c^2}} \qquad ...(ii)$$

To eliminate c from (*i*) and (*ii*), we rewrite (*ii*) as

$$c = N(x^2 - 1)/x$$

Now substituting this value of c in (*i*), we get

$$y = N(x^2 - 1) + \sin^{-1}\{N(x^2 - 1)/x\}$$

which is the desired singular solution.

Obs. Equations reducible to Clairaut's form. Many equations of the first order but of higher degree can be easily reduced to the Clairaut's form by making suitable substitutions.

Example 11.43. *Solve $(px - y)(py + x) = a^2p$.* (*V.T.U., 2011 ; J.N.T.U., 2006*)

Solution. Put $x^2 = u$ and $y^2 = v$ so that $2x\,dx = du$ and $2y\,dy = dv$

$$\therefore \qquad p = \frac{dy}{dx} = \frac{dv}{y}\Big/\frac{du}{x} = \frac{x}{y}P, \text{ where } P = \frac{dv}{du}$$

*After the name of a youthful prodigy *Alexis Claude Clairaut* (1713–65) who first solved this equation. A French mathematician who is also known for his work in astronomy and geodesy.

Then the given equation becomes $\left(\frac{xP}{y}.x - y\right)\left(\frac{xP}{y}.y + x\right) = a^2 \frac{xP}{y}$

or $$(uP - v)(P + 1) = a^2P \text{ or } uP - v = \frac{a^2P}{P+1}$$

or $$v = uP - a^2P/(P + 1), \text{ which is Clairaut's form.}$$

$\therefore$ its solution is $v = uc - a^2c/(c + 1)$, *i.e.*, $y^2 = cx^2 - a^2c/(c + 1)$.

PROBLEMS 11.12

1. Find the general and singular solution of the equations :
   (*i*) $xp^2 - yp + a = 0$. (*J.N.T.U., 2006*) (*ii*) $p = \log(px - y)$.
   (*iii*) $y = px + \sqrt{(a^2p^2 + b^2)}$ (*W.B.T.U., 2005*) (*iv*) $\sin px \cos y = \cos px \sin y + p$ (*P.T.U., 2006*)

Solve the following equations :

2. $y + 2\left(\frac{dy}{dx}\right)^2 = (x+1)\frac{dy}{dx}$.
3. $(y - px)(p - 1) = p$.
4. $(x-a)\left(\frac{dy}{dx}\right)^2 + (x-y)\frac{dy}{dx} - y = 0$.
5. $x^2(y - px) = yp^2$.
6. $(px + y)^2 = py^2$.
7. $(px - y)(x + py) = 2p$.

11.15 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 11.13

Fill up the blanks or choose the correct answer in the following problems :

1. $y = cx - c^2$, is the general solution of the differential equation
   (*i*) $(y')^2 - xy' + y = 0$ (*ii*) $y'' = 0$ (*iii*) $y' = c$ (*iv*) $(y')^2 + xy' + y = 0$.
2. The differential equation having a basis for its solution as $\sinh 6x$ and $\cosh 6x$ is
   (*i*) $y'' + 36y = 0$ (*ii*) $y'' - 36y = 0$ (*iii*) $y'' + 6y = 0$ (*iv*) none of these.
3. The differential equation $(dx/dy)^2 + 5y^{1/3} = x$ is
   (*i*) linear of degree 3 (*ii*) non-linear of order 1 and degree 6
   (*iii*) non-linear of order 1 and degree 2.
4. The differential equation $ydx/dy + 1 = y$, $y(0) = 1$, has
   (*i*) a unique solution (*ii*) two solutions
   (*iii*) infinite number of solutions (*iv*) no solution
5. Solution of $(x^2 + y^2)\,dy = xy\,dx$ is ....
6. Solution of $(3x - 2y)\,dx = xdy$ is ....
7. Solution of $dy/dx - y = 2xy^2 e^{-x}$ is ....
8. The differential equation $(y^2e^{xy^2} + 6x)\,dx + (2xye^{xy^2} - 4y)\,dy = 0$ is
   (*i*) linear, homogeneous and exact (*ii*) non-linear, homogeneous and exact
   (*iv*) non-linear, non-homogeneous and exact (*iv*) non-linear, non-homogeneous and inexact.
9. Solution of $xdx + ydy + \frac{xdy - ydx}{x^2 + y^2}$ is ....
10. Solution of $dy/dx = \frac{x^3 + y^3}{xy^2}$ is ......
11. The differential equation $(x + x^8 + ay^2)\,dx + (y^8 - y + bxy)\,dy = 0$ is exact if
    (*i*) $b = 2a$ (*ii*) $a = b$ (*iii*) $a \neq 2b$ (*iv*) $a = 1, b = 3$.
12. Solution of $xy\,(1 + xy^2)\,dy = dx$ is ....
13. Solution of $xp^2 - yp + a = 0$ is ....
14. The differential equation $p = \log(px - y)$ has the solution ....
15. Solution of $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$ is ....

16. The order of the differential equation $(1 + y_1^2)^{3/2}/y_2 = c$ is ......

17. The general solution of $\frac{1}{x^2y^2}(xdy + ydx) = 0$ is ....

18. Integrating factor of the differential equation $\frac{dx}{dy} + \frac{3x}{y} = \frac{1}{y^2}$ is

(a) e^{y^3} (b) y^3 (c) x^3 (d) $-y^3$. (V.T.U., 2009)

19. Solution of the equation $\frac{dy}{dx} = \frac{y}{x} - \operatorname{cosec}\frac{y}{x}$ is

(a) $\cos(y/x) - \log x = c$ (b) $\cos(y/x) + \log x = c$
(c) $\cos^2(y/x) + \log x = c$ (d) $\cos^2(y/x) - \log x = c$. (V.T.U., 2010)

20. Solution of $x\sqrt{(1+x^2)} + y\sqrt{(1+y^2)}\ dy/dx = 0$ is ....

21. Solution of $dy/dx + y = 0$ given $y(0) = 5$ is ....

22. The substitution that transforms the equation $\frac{dy}{dx} = \frac{x+y+1}{2x+2y+3}$ to homogeneous form is ....

23. Integrating factor of $xy' + y = x^3y^6$ is ....

24. Solution of the exact differential equation $Mdx + Ndy = 0$ is ....

25. Solution of $(2x^3y^2 + x^4)\,dx + (x^4y + y^4)\,dy = 0$ is ....

26. The general solution of the differential equation $\frac{dy}{dx} + \frac{y}{x} = \tan 2x$ is ....

27. Degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^5 x^2y = 0$ is

(a) 2 (b) 0 (c) 3 (d) 5 (Bhopal, 2008)

28. Integrating factor of the differential equation $\frac{dy}{dx} + y\cos x = \frac{\sin 2x}{2}$ is

(a) $e^{\sin^2 x}$ (b) $e^{\sin^3 x}$ (c) $e^{\sin x}$ (d) $\sin x$

29. The differential equation of the family of circles with centre as origin is ...... (Nagarjuna, 2008)

30. Solution of $xe^{-x^2}\,dx + \sin y\,dy = 0$ is .... (Nagarjuna, 2008)

31. Solution of $p = \sin(y - xp)$ is

(a) $y = \frac{c}{x} + \sin^{-1}c$ (b) $y = cx + \sin c$ (c) $y = cx + \sin^{-1}c$ (d) $y = x + \sin^{-1}c$ (V.T.U., 2011)

32. Differential equation obtained by eliminating A and B form $y = A\cos x + B\sin x$ is $d^2y/dx^2 - y = 0$ (True or False)

33. $(x^3 - 3xy^2)\,dx + (y^3 - 2x^2y)\,dy = 0$ is an exact differential equation. (True or False)

CHAPTER 12

Applications of Differential Equations of First Order

1. Introduction. 2. Geometric applications. 3. Orthogonal trajectories. 4. Physical applications. 5. Simple electric circuits. 6. Newton's law of cooling. 7. Heat flow. 8. Rate of decay of radio-active materials. 9. Chemical reactions and solutions. 10. Objective Type of Questions.

12.1 INTRODUCTION

In this chapter, we shall consider only such practical problems which give rise to differential equations of the first order. The fundamental principles required for the formation of such differential equations are given in each case and are followed by illustrative examples.

12.2 GEOMETRIC APPLICATIONS

(*a*) *Cartesian coordinates.* Let $P(x, y)$ be any point on the curve $f(x, y) = 0$ (Fig. 12.1), then [as per 4.6 §(1) & 4.11(1) & (4)], we have

(*i*) slope of the tangent at $P(= \tan\psi) = dy/dx$

(*ii*) equation of the tangent at P is

$$Y - y = \frac{dy}{dx}(X - x)$$

so that its x-intercept $(= OT)$

$$= x - y \cdot dx/dy$$

and y-intercept $(= OT') = y - x \cdot dy/dx$

(*iii*) equation of the normal at P is $Y - y = -\dfrac{dx}{dy}(X - x)$

(*iv*) length of the tangent $(= PT) = y\sqrt{[1 + (dx/dy)^2]}$

(*v*) length of the normal $(= PN) = y\sqrt{[1 + (dy/dx)^2]}$

(*vi*) length of the sub-tangent $(= TM) = y \cdot dx/dy$

(*vii*) length of the sub-normal $(= MN) = y \cdot dy/dx$

(*viii*) $\dfrac{ds}{dx} = [1 + (dy/dx)^2]$; $\dfrac{ds}{dy} = \sqrt{[1 + (dx/dy)^2]}$

(*ix*) differential of the area $= ydx$ or xdy

(*x*) ρ, radius of curvature at $P = \dfrac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2}$

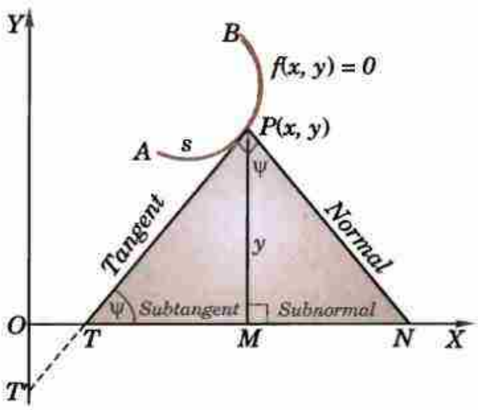


Fig. 12.1

(*b*) *Polar coordinates*. Let $P(r, \theta)$ be any point on the curve $r = f(\theta)$ (Fig. 12.2), then [as per § 4.7, 4.9 (2) & 4.11 (4)], we have

(*i*) $\psi = \theta + \phi$

(*ii*) $\tan \phi = r d\theta/dr$, $p = r \sin \phi$

(*iii*) $\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$

(*iv*) polar sub-tangent $(= OT) = r^2 d\theta/dr$

(*v*) polar sub-normal $(ON) = dr/d\theta$

(*vi*) $\dfrac{ds}{dr} = \sqrt{\left[1 + \left(r\dfrac{d\theta}{dr}\right)^2\right]}$; $\dfrac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right]}$

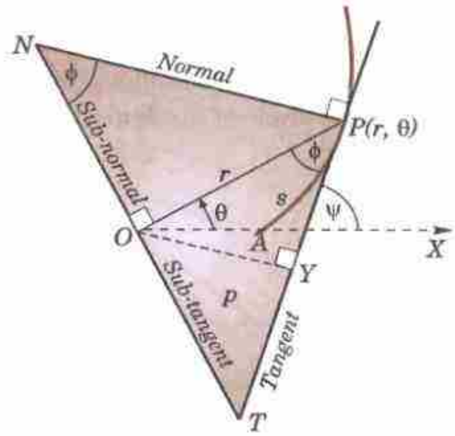


Fig. 12.2

Example 12.1. *Show that the curve in which the portion of the tangent included between the co-ordinates axes is bisected at the point of contact is a rectangular hyperbola.*

Solution. Let the tangent at any point $P(x, y)$ of a curve cut the axes at T and T' (Fig. 12.3).

We know that its x-intercept $(= OT) = x - y \cdot dx/dy$

and y-intercept $(= OT') = y - x \cdot dy/dx$

$\therefore$ the co-ordinates of T and T' are

$(x - y \cdot dx/dy, 0), (0, y - x\, dy/dx)$

Since P is the mid-point of TT'

$$\therefore \quad \frac{[x - y.dx/dy] + 0}{2} = x$$

or $x - y \cdot dx/dy = 2x$ or $x\,dy + y\,dx = 0$

or $d(xy) = 0$ Integrating, $xy = c$

which is the equation of a rectangular hyperbola, having x and y axes as its asymptotes.

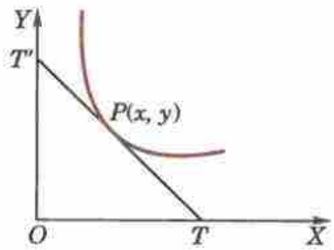


Fig. 12.3

Example 12.2. *Find the curve for which the normal makes equal angles with the radius vector and the initial line.*

Solution. Let PT and PN be the tangent and normal at $P(r, \theta)$ of the curve so that

$$\tan \phi = r\, d\theta/dr$$

By the condition of the problem,

$$\angle OPN = 90° - \phi = \angle ONP \text{ (Fig. 12.4).}$$

$$\therefore \quad \theta = \angle PON = 180° - (180° - 2\phi) = 2\phi$$

or $$\theta/2 = \phi \quad \therefore \quad \tan\frac{\theta}{2} = \tan\phi = r\frac{d\theta}{dr}.$$

Here the variables are separable.

$$\therefore \quad \frac{dr}{r} = \frac{\cos\theta/2}{\sin\theta/2}\, d\theta$$

Integrating both sides $\log r = 2 \log \sin \theta/2 + \log c$

or $$r = c \sin^2 \theta/2 = \frac{1}{2}\, c(1 - \cos\theta)$$

Thus the curve is the cardioid $r = a(1 - \cos\theta)$.

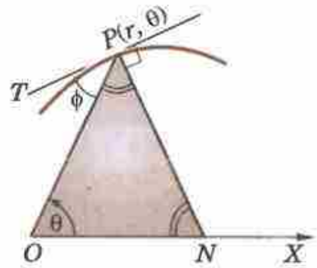


Fig. 12.4

Example 12.3. *Find the shape of a reflector such that light coming from a fixed source is reflected in parallel rays.*

Solution. Taking the fixed source of light as the origin and the X-axis parallel to the reflected rays; the reflector will be a surface generated by the revolution of a curve $f(x, y) = 0$ about X-axis (Fig. 12.5).

In the XY-plane, let PP' be the reflected ray, where P is the point (x, y) on the curve $f(x, y) = 0$.

If TPT' be the tangent at P, then

$\because$ angle of incidence = angle of reflection,

$\therefore$ $$\phi = \angle OPT = \angle P'PT' = \angle OTP = \psi$$

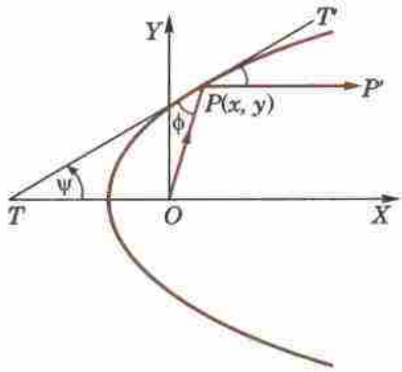


Fig. 12.5

i.e., $$p = \frac{dy}{dx} = \tan \angle XOP = \tan 2\phi$$

$$= \frac{2 \tan \phi}{1 - \tan^2 \phi} = \frac{2p}{1 - p^2}$$

or $$2x = \frac{y}{p} - yp \text{ which is solvable for } x \qquad ...(i)$$

$\therefore$ differentiating (*i*) w.r.t. y, $\dfrac{2}{p} = \dfrac{1}{p} - \dfrac{y}{p^2}\dfrac{dp}{dy} - p - y\dfrac{dp}{dy}$

i.e., $$\left(\frac{1}{p} + p\right) + \left(\frac{1}{p^2} + 1\right) y \frac{dp}{dy} = 0 \quad \text{or} \quad \left(\frac{1}{p} + p\right)\left(1 + \frac{y}{p}\frac{dp}{dy}\right) = 0$$

This gives $\quad dp/p = -\, dy/y$

Integrating, $\quad \log p = \log c - \log y, \quad$ *i.e.,* $p = c/y \qquad ...(ii)$

Thus eliminating p from (*i*) and (*ii*), we have family of curves $y^2 = 2cx + c^2$.

Hence the reflector is a member of the family of paraboloids of revolution $y^2 + z^2 = 2cx + c^2$.

PROBLEMS 12.1

1. Find the equation of the curve which passes through
   (*i*) the point $(3, -4)$ and has the slope $2y/x$ at the point (x, y) on it.
   (*ii*) the origin and has the slope $x + 3y - 1$.
2. At every point on a curve the slope is the sum of the abscissa and the product of the ordinate and the abscissa, and the curve passes through $(0, 1)$. Find the equation of the curve.
3. A curve is such that the length of the perpendicular from origin on the tangent at any point P of the curve is equal to the abscissa of P. Prove that the differential equation of the curve is
   $$y^2 - 2xy\frac{dy}{dx} - x^2 = 0, \text{ and hence find the curve.}$$
4. A plane curve has the property that the tangents from any point on the y-axis to the curve are of constant length a. Find the differential equation of the family to which the curve belongs and hence obtain the curve.
5. Determine the curve whose sub-tangent is twice the abscissa of the point of contact and passes through the point $(1, 2)$. (*Sambalpur, 1998*)
6. Determine the curve in which the length of the sub-normal is proportional to the square of the ordinate.
7. The tangent at any point of a certain curve forms with the coordinate axes a triangle of constant area A. Find the equation to the curve.
8. Find the curve which passes through the origin and is such that the area included between the curve, the ordinate and the x-axis is twice the cube of that ordinate.
9. Find the curve whose (*i*) polar sub-tangent is constant.
   (*ii*) polar sub-normal is proportional to the sine of the vectorial angle.
10. Determine the curve for which the angle between the tangent and the radius vector is twice the vectorial angle. (*Kanpur, 1996*)
11. Find the curve for which the tangent at any point P on it bisects the angle between the ordinate at P and the line joining P to the origin.
12. Find the curve for which the tangent, the radius vector r and the perpendicular from the origin on the tangent form a triangle of area kr^2.

12.3 (1) ORTHOGONAL TRAJECTORIES

Two families of curves such that every member of either family cuts each member of the other family at right angles are called **orthogonal trajectories** *of each other (Fig. 12.6).*

The concept of the orthogonal trajectories is of wide use in applied mathematics especially in field problems. For instance, in an electric field, the paths along which the current flows are the orthogonal trajectories of the equipotential curves and *vice versa*. In fluid flow, the stream lines and the equipotential lines (lines of constant velocity potential) are orthogonal trajectories. Likewise, the lines of heat flow for a body are perpendicular to the isothermal curves. The problem of finding the orthogonal trajectories of a given family of curves depends on the solution of the first order differential equations.

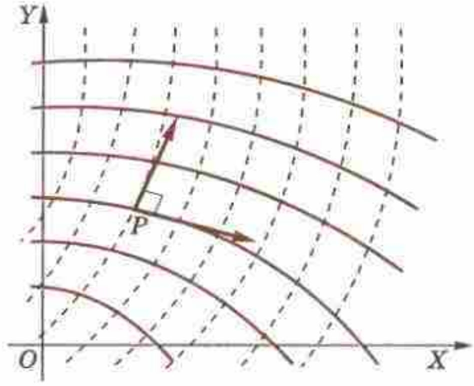


Fig. 12.6

(2) To find the orthogonal trajectories of the family of curves F(x, y, c) = 0.

(*i*) *Form its differential equation in the form $f(x, y, dy/dx) = 0$ by eliminating c.*

(*ii*) *Replace, in this differential equation, dy/dx by $-dx/dy$*, (so that the product of their slopes at each point of intersection is -1).

(*iii*) *Solve the differential equation of the orthogonal trajectories i.e., $f(x, y, -dx/dy) = 0$.*

Example 12.4. *If the stream lines (paths of fluid particles) of a flow around a corner are xy = constant find their orthogonal trajectories (called equipotential lines-§ 20.6)* (*Marathwada, 2008*)

Solution. Taking the axes as the walls, the stream lines of the flow around the corner of the walls is

$$xy = c \qquad ...(i)$$

Defferentiating, we get, $$x\frac{dy}{dx} + y = 0 \qquad ...(ii)$$

as the differential equation of the given family (*i*).

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (*ii*), we obtain $x\left(-\frac{dx}{dy}\right) + y = 0$

or $$xdx - ydy = 0 \qquad ...(iii)$$

as the differential equation of the orthogonal trajectories.

Integrating (*iii*), we get $x^2 - y^2 = c'$ as the required orthogonal trajectories of (*i*) *i.e.*, the *equipotential lines*, shown dotted in Fig. 12.7.

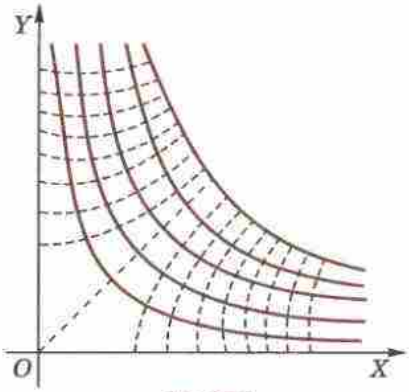


Fig. 12.7

Example 12.5. *Find the orthogonal trajectories of the family of confocal conics $\frac{x^2}{a^2} + \frac{y^2}{a^2+\lambda} = 1$, where λ is the parameter.* (*V.T.U., 2009 S*)

Solution. Differentiating the given equation, we get $\frac{2x}{a^2} + \frac{2y}{a^2+\lambda}\frac{dy}{dx} = 0$

or $$\frac{y}{a^2+\lambda} = -\frac{x}{a^2(dy/dx)} \quad \text{or} \quad \frac{y^2}{a^2+\lambda} = \frac{-xy}{a^2(dy/dx)}$$

Substituting this in the given equation, we get

$$\frac{x^2}{a^2} - \frac{xy}{a^2(dy/dx)} = 1 \quad \text{or} \quad (x^2 - a^2)\frac{dy}{dx} = xy \qquad ...(i)$$

which is the differential equation of the given family.

Changing dy/dx to $-dx/dy$ in (*i*), we get $(a^2 - x^2)\, dx/dy = xy$ as the differential equation of the orthogonal trajectories.

Separating the variables and integrating, we obtain

$$\int y\,dy = \int \frac{a^2 - x^2}{x}\,dx + c \quad \text{or} \quad \frac{1}{2}y^2 = a^2\log x - \frac{1}{2}x^2 + c$$

or $$x^2 + y^2 = 2a^2\log x + c' \qquad [c' = 2c]$$

which is the equation of the required orthogonal trajectories.

Example 12.6. *Find the orthogonal trajectories of a system of confocal and coaxial parabolas.*

Solution. The equation of the family of confocal parabolas having x-axis as their axis, is of the form

$$y^2 = 4a(x + a) \quad ...(i)$$

Differentiating, $$y\frac{dy}{dx} = 2a \quad ...(ii)$$

Substituting the value of a from (ii) in (i), we get $y^2 = 2y\dfrac{dy}{dx}\left(x + \dfrac{1}{2}y\dfrac{dy}{dx}\right)$

i.e., $$y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0 \text{ as the differential equation of the family.} \quad ...(iii)$$

Replacing $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$ in (iii), we obtain $y\left(\dfrac{dx}{dy}\right)^2 - 2x\dfrac{dx}{dy} - y = 0$

or $$y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0 \text{ which is the same as } (iii).$$

Thus we see that a system of confocal and coaxial parabolas is *self-orthogonal, i.e.*, each member of the family (i) cuts every other member of the same family orthogonally.

(3) To find the orthogonal trajectories of the curves F(r, θ, c) = 0.

(i) *Form its differential equation in the form* $f(r, \theta, dr/d\theta) = 0$ *by eliminating* c.

(ii) *Replace in this differential equation,*

$$\frac{dr}{d\theta} \text{ by } -r^2\frac{d\theta}{dr}$$

[$\because$ for the given curve through $P(r, \theta)$ $\tan\phi = rd\theta/dr$ and for the orthogonal trajectory through P

$$\tan\phi' = \tan(90° + \phi) = -\cot\phi = -\frac{1}{r}\frac{dr}{d\theta}$$

Thus for getting the differential equation of the orthogonal trajectory

$$r\frac{d\theta}{dr} \text{ is to be replaced by } -\frac{1}{r}\frac{dr}{d\theta}$$

or $$\frac{dr}{d\theta} \text{ is to be replaced by } -r^2\frac{d\theta}{dr}.]$$

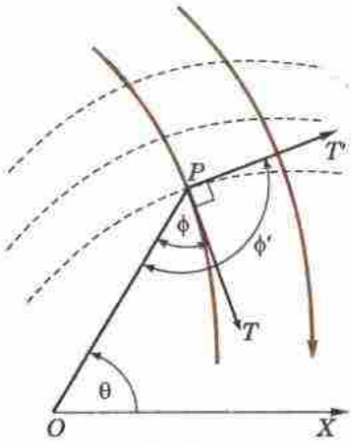


Fig. 12.8

(iii) *Solve the differential equation of the orthogonal trajectories*

i.e., $$f(r, \theta, -r^2 d\theta/dr) = 0.$$

Example 12.7. *Find the orthogonal trajectory of the cardioids* $r = a(1 - \cos\theta)$. *(Kurukshetra, 2005)*

Solution. Differentiating $$r = a(1 - \cos\theta). \quad ...(i)$$

with respect to θ, we get $$\frac{dr}{d\theta} = a\sin\theta \quad ...(ii)$$

Eliminating a from (i) and (ii), we obtain

$$\frac{dr}{d\theta}\cdot\frac{1}{r} = \frac{\sin\theta}{1 - \cos\theta} = \cot\frac{\theta}{2} \text{ which is the differential equation of the given family.}$$

Replacing $dr/d\theta$ by $-r^2\, d\theta/dr$, we obtain

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \cot\frac{\theta}{2} \quad \text{or} \quad \frac{dr}{r} + \tan\frac{\theta}{2}\,d\theta = 0$$

as the differential equation of orthogonal trajectories. It can be rewritten as

$$\frac{dr}{r} = -\frac{(\sin\theta/2)d\theta}{\cos\theta/2}$$

Integrating, $\log r = 2 \log \cos \theta/2 + \log c$

or $$r = c \cos^2 \theta/2 = \frac{1}{2} c(1 + \cos \theta) \quad \text{or} \quad r = a'(1 + \cos \theta)$$

which is the required orthogonal trajectory.

Example 12.8. *Find the orthogonal trajectory of the family of curves $r^n = a \sin n\theta$.* (V.T.U., 2006)

Solution. We have $n \log r = \log a + \log \sin n\theta$.

Differentiating w.r.t. θ, we have

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{n \cos n\theta}{\sin n\theta} \quad \text{or} \quad \frac{1}{r}\frac{dr}{d\theta} = \cot n\theta$$

Replacing $dr/d\theta$ by $-r^2\, d\theta/dr$, we obtain

$$\frac{1}{r}\left(-r^2 \frac{d\theta}{dr}\right) = \cot n\theta \quad \text{or} \quad \tan n\theta.\ d\theta - \frac{dr}{r} = 0$$

Integrating, $$\int \frac{dr}{r} + \int \frac{\sin n\theta}{\cos n\theta}\, d\theta = c,$$

i.e., $$\log r - \frac{1}{n} \log \cos n\theta = c \quad \text{or} \quad \log (r^n/\cos n\,\theta) = nc = \log b. \text{ (say)}$$

or $r^n = b \cos n\theta$, which is the required orthogonal trajectory.

PROBLEMS 12.2

Find the orthogonal trajectories of the family of :

1. Parabolas $y^2 = 4ax$. (Marathwada, 2009)
2. Parabolas $y = ax^2$. (J.N.T.U., 2006)
3. Semi-cubical parabolas $ay^2 = x^3$. (J.N.T.U., 2005)
4. Coaxial circles $x^2 + y^2 + 2\lambda x + c = 2$, λ being the parameter. (J.N.T.U., 2006)
5. Confocal conics $\dfrac{x^2}{a^2+\lambda} + \dfrac{y^2}{b^2+\lambda} = 1$, λ being the parameter. (Kurukshetra, 2006)
6. Cardioids $r = a\,(1 + \cos \theta)$. (J.N.T.U., 2003)
7. $r = 2a\,(\cos \theta + \sin \theta)$ (V.T.U., 2010 S)
8. Confocal and coaxial parabolas $r = 2a/(1 + \cos \theta)$. (Nagpur, 2008)
9. Curves $r^2 = a^2 \cos 2\theta$. (V.T.U., 2009 S)
10. $r^n \cos n\theta = a^n$. (V.T.U. 2011)
11. Show that the family of parabolas $x^2 = 4a\,(y + a)$ is self orthogonal. (Kerala, 2005)
12. Show that the family of curves $r^n = a \sec n\theta$ and $r^n = b \operatorname{cosec} n\theta$ are orthogonal. (Mumbai, 2005)
13. The *electric lines of force* of two opposite charges of the same strength at $(\pm 1, 0)$ are circles (through these points) of the form $x^2 + y^2 - ay = 1$. Find their *equipotential lines* (orthogonal trajectories).

    [**Isogonal trajectories.** *Two families of curves such that every member of either family cuts each member of the other family at a constant angle α (Say), are called* ***isogonal trajectories*** *of each other.* The slopes m, m' of the tangents to the corresponding curves at each point, are connected by the relation $\dfrac{m \square m'}{1 + mm'} = \tan \alpha = \text{const.}$]
14. Find the isogonal trajectories of the family of circles $x^2 + y^2 = a^2$ which intersect at 45°.

12.4 PHYSICAL APPLICATIONS

(1) Let a body of mass m start moving from O along the straight line OX under the action of a force F. After any time t, let it be moving at P where $OP = x$, then

(*i*) its velocity $(v) = \dfrac{dx}{dt}$

(*ii*) its acceleration $(a) = \dfrac{dv}{dt}$ or $\dfrac{vdv}{dx}$ or $\dfrac{d^2x}{dt^2}$

If, however, the body be moving along a curve, then

(i) its velocity $(v) = ds/dt$ and

(ii) its acceleration $(a) = \dfrac{dv}{dt}, v\dfrac{dv}{ds}$ or $\dfrac{d^2s}{dt^2}$.

The quantity mv is called the *momentum*.

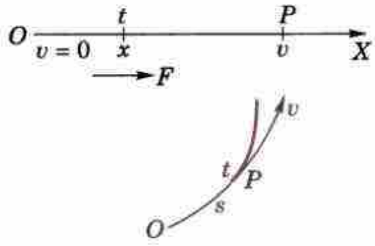


Fig. 12.9

(2) Newton's second law *states that* $F = \dfrac{d}{dt}(mv)$.

If m is constant, then $F = m\dfrac{dv}{dt} = ma$, *i.e., net force = mass × acceleration.*

(3) Hooke's law* *states that tension of an elastic string (or a spring) is proportional to extension of the string (or the spring) beyond its natural length.*

Thus $\mathbf{T = \lambda e/1}$,

where e is the extension beyond the natural length l and λ is the *modulus of elasticity.*

Sometimes for a spring, we write $\mathbf{T = ke}$,

where e is the extension beyond the natural length and k is the *stiffness of the spring.*

(4) Systems of units

I. **F.P.S.** [foot (*ft.*) pound (*lb.*), second (*sec.*)] **system**. If mass m is in *pounds* and acceleration (a) is in *ft/sec*^{2}, then the force $F(= ma)$ is in *poundals*.

II. **C.G.S.** [centimetre (*cm.*), gram (*g*), second (*sec*)] **system**. If mass m is in *grams* and acceleration a is in *cm/sec*^{2} then the force $F(= ma)$ is *dynes*.

III. **M.K.S.** [metre (*m*), kilogram (*kg.*), second (*sec*)] **system**. If mass m is in kilograms and acceleration a in *m/sec*^{2}, then the force $F(= ma)$ is in *newtons* (*nt*).

These are called *absolute units*. If g is the acceleration due to gravity and w is the weight of the body, then w/g is the mass of the body in *gravitational units*.

$$g = 32 \text{ ft/sec}^2 = 980 \text{ cm/sec}^2 = 9.8 \text{ m/sec}^2 \text{ approx.}$$

Example 12.9. Motion of a boat across a stream. *A boat is rowed with a velocity u directly across a stream of width a. If the velocity of the current is directly proportional to the product of the distances from the two banks, find the path of the boat and the distance down stream to the point where it lands.*

Solution. Taking the origin at the point from where the boat starts, let the axes be chosen as in Fig. 12.10.

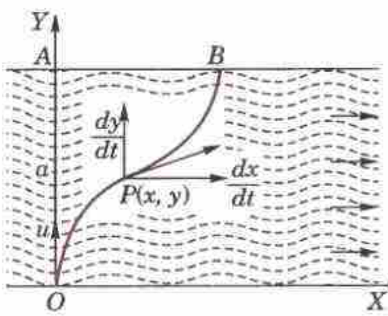


Fig. 12.10

At any time t after its start from O, let the boat be at $P(x, y)$, so that

dx/dt = velocity of the current $= ky(a - y)$

dy/dt = velocity with which the boat is being rowed $= u$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{u}{ky(a-y)} \quad \ldots(i)$$

This gives the direction of the resultant velocity of the boat which is also the direction of the tangent to the path of the boat.

Now (i) is of variables separable form and we can write it as

$$y(a - y)dy = \frac{u}{k}\,dx$$

Integrating, we get $\dfrac{ay^2}{2} - \dfrac{y^3}{3} = \dfrac{u}{k}x + c$

Since $y = 0$ when $x = 0$, $\therefore$ $c = 0$.

Hence the equation to the path of the boat is $x = \dfrac{k}{6u}y^2(3a - 2y)$

Putting $y = a$, we get the distance AB, down stream where the boat lands $= ka^3/6u$.

*Named after an English physicist *Robert Hooke* (1635–1703) who had discovered the law of gravitation earlier than Newton.

Example 12.10. Resisted motion. *A moving body is opposed by a force per unit mass of value cx and resistance per unit of mass of value bv^2 where x and v are the displacement and velocity of the particle at that instant. Find the velocity of the particle in terms of x, if it starts from rest.* (Marathwada, 2008)

Solution. By Newton's second law, the equation of motion of the body is $v\dfrac{dv}{dx} = -cx - bv^2$

or
$$v\frac{dv}{dx} + bv^2 = -cx \qquad ...(i)$$

This is Bernoulli's equation. $\therefore$ Put $v^2 = z$ and $2v\, dv/dx = dz/dx$, so that (i) becomes

$$\frac{dz}{dx} + 2bz = -2cx \qquad ...(ii)$$

This is Leibnitz's linear equation and I.F. $= e^{2bx}$.

$\therefore$ the solution of (ii) is $\quad ze^{2bx} = -\int 2cxe^{2bx}\,dx + c'$ [Integrate by parts]

$$= -2c\left[x\cdot\frac{e^{2bx}}{2b} - \int 1\cdot\frac{e^{2bx}}{2b}\,dx\right] + c' = -\frac{cx}{b}e^{2bx} + \frac{c}{2b^2}e^{2bx} + c'$$

or
$$v^2 = \frac{c}{2b^2} + c'e^{-2bx} - \frac{cx}{b} \qquad ...(iii)$$

Initially $v = 0$ when $x = 0$ $\therefore$ $0 = c/2b^2 + c'$.

Thus, substituting $c' = -c/2b^2$ in (iii), we get $v^2 = \dfrac{c}{2b^2}(1 - e^{-2bx}) - \dfrac{cx}{b}$.

Example 12.11. Resisted vertical motion. *A particle falls under gravity in a resisting medium whose resistance varies with velocity. Find the relation between distance and velocity if initially the particle starts from rest.* (U.P.T.U., 2003)

Solution. After falling a distance s in time t from rest, let v be velocity of the particle. The forces acting on the particle are its weight mg downwards and resistance $m\lambda v$ upwards.

$\therefore$ equating of motion is $\quad m\dfrac{dv}{dt} = mg - m\lambda v$

or
$$\frac{dv}{dt} = g - \lambda v \quad \text{or} \quad \frac{dv}{g - \lambda v} = dt$$

Integrating, $\displaystyle\int \frac{dv}{g - \lambda v} = \int dt + c$ or $-\dfrac{1}{\lambda}\log(g - \lambda v) = t + c$

Since $v = 0$ when $t = 0$, $\therefore$ $c = -\dfrac{1}{\lambda}\log g$

Thus
$$\frac{1}{\lambda}\log\left[\frac{g}{g - \lambda v}\right] = t \quad \text{or} \quad \frac{g - \lambda v}{g} = e^{-\lambda t}$$

or
$$\frac{ds}{dt} = v = \frac{g}{\lambda}(1 - e^{-\lambda t}) \qquad ...(i)$$

Integrating,
$$s = \frac{g}{\lambda}\int (1 - e^{-\lambda t})dt + c' \quad \text{or} \quad s = \frac{g}{\lambda}\left(t + \frac{1}{\lambda}e^{-\lambda t}\right) + c'$$

Since $s = 0$ when $t = 0$, $\therefore$ $c' = -g/\lambda^2$

Thus
$$s = \frac{g}{\lambda}t + \frac{g}{\lambda^2}(e^{-\lambda t} - 1) \qquad ...(ii)$$

Eliminating t from (i) and (ii), we get

$$s = \frac{g}{\lambda^2}\log\left(\frac{g}{g - \lambda v}\right) - \frac{v}{\lambda}$$

which is the desired relation between s and v.

Example 12.12. *A body of mass m, falling from rest is subject to the force of gravity and an air resistance proportional to the square of the velocity (i.e., kv^2). If it falls through a distance x and possesses a velocity v at that instant, prove that*

$$\frac{2kx}{m} = \log \frac{a^2}{a^2 - v^2}, \text{ where } mg = ka^2.$$

Solution. If the body be moving with the velocity v after having fallen through a distance x, then its equation of motion is

$$mv \frac{dv}{dx} = mg - kv^2 \quad \text{or} \quad mv \frac{dv}{dx} = k(a^2 - v^2). \qquad [\because mg = ka^2] \quad ...(i)$$

$\therefore$ separating the variables and integrating, we get $\int \frac{vdv}{a^2 - v^2} = \int \frac{k}{m} dx + c$

or
$$-\frac{1}{2} \log (a^2 - v^2) = \frac{kx}{m} + c \qquad ...(ii)$$

Initially, when $x = 0$, $v = 0$. $\therefore -\frac{1}{2} \log a^2 = c$ ...(iii)

Subtracting (*iii*) from (*ii*), we have $\frac{1}{2} [\log a^2 - \log (a^2 - v^2)] = kx/m$

or
$$\frac{2kx}{m} = \log \left(\frac{a^2}{a^2 - v^2} \right)$$

Obs. *When the resistance becomes equal to the weight, the acceleration becomes zero and particle continues to fall with a constant velocity, called the* **limiting** *or* **terminal** *velocity.* From (*i*), it follows that the acceleration will become zero when $v = a$. *Thus, the limiting velocity, i.e., the maximum velocity which the particle can attain is a.*

Example 12.13. Velocity of escape from the earth. *Find the initial velocity of a particle which is fired in radial direction from the earth's centre and is supposed to escape from the earth. Assume that it is acted upon by the gravitational attraction of the earth only.*

Solution. According to Newton's law of gravitation, the acceleration α of the particle is proportional to $1/r^2$ where r is the variable distance of the particle from the earth's centre. Thus

$$\alpha = v \frac{dv}{dr} = -\frac{\mu}{r^2}$$

where v is the velocity when at a distance r from the earth's centre. The acceleration is negative because v is decreasing. When $r = R$, the earth's radius then $\alpha = -g$, the acceleration of gravity at the surface.

i.e.,
$$-g = -\mu/R^2, \text{ i.e., } \mu = gR^2 \quad \therefore \quad v \frac{dv}{dr} = -\frac{gR^2}{r^2}$$

Separating the variables and integrating, we obtain $\int v dv = -gR^2 \int \frac{dr}{r^2} + c$

i.e.,
$$v^2 = \frac{2gR^2}{r} + 2c \qquad ...(i)$$

On the earth's surface $r = R$ and $v = v_0$ (say), the initial velocity. Then

$$v_0^2 = 2gR + 2c, \quad i.e., \quad 2c = v_0^2 - 2gR$$

Inserting this value of c in (*i*), we get $v^2 = \frac{2gR^2}{r} + v_0^2 - 2gR$

When v vanishes, the particle stops and the velocity will change from positive to negative and the particle will return to the earth. Thus the velocity will remain positive, if and only if $v_0^2 \geq 2gR$ and then the particle projected fron the earth with this velocity will escape from the earth. Hence the minimum such velocity of projection $v_0 = \sqrt{(2gR)}$ *is called the velocity of escape* from the earth [See Problem 9, page 454].

Example 12.14. Rotating cylinder containing liquid. *A cylindrical tank of radius r is filled with water to a depth h. When the tank is rotated with angular velocity ω about its axis, centrifugal force tends to drive the water outwards from the centre of the tank. Under steady conditions of uniform rotation, show that the section of the free surface of the water by a plane through the axis, is the curve*

$$y = \frac{\omega^2}{2g}\left(x^2 - \frac{r^2}{2}\right) + h.$$

Solution. Let the figure represent an axial section of the cylindrical tank. Forces acting on a particle of mass m at $P(x, y)$ on the curve, cut out from the free surface of water, are :

(*i*) the weight mg acting vertically downwards,

(*ii*) the centrifugal force $m\omega^2x$ acting horizontally outwards.

As the motion is steady, P moves just on the surface of the water and, therefore, there is no force along the tangent to the curve. Thus the resultant R of mg and $m\omega^2x$ is along the outward normal to the curve.

$\therefore \quad R \cos \psi = mg$ and $R \sin \psi = m\omega^2x$

whence $$\frac{dy}{dx} = \tan \psi = \frac{m\omega^2 x}{mg} = \frac{\omega^2 x}{g} \qquad ...(i)$$

This is the differential equation of the surface of the rotating liquid.

Integrating (*i*), we get

$$\int dy = \frac{\omega^2}{g}\int x\,dx + c$$

i.e., $$y = \frac{\omega^2 x^2}{2g} + c \qquad ...(ii)$$

To find c, we note that the volume of the liquid remains the same in both cases (Fig. 12.11).

When $x = 0$ in (*ii*), $OA\ (= y) = c$. When $x = r$ in (*ii*), $$h'\ (= y) = \frac{\omega^2 r^2}{2g} + c \qquad ...(iii)$$

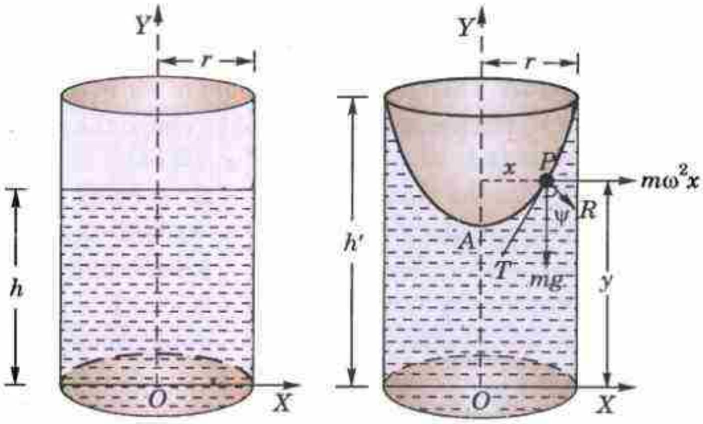


Fig 12.11

Now the volume of the liquid in the non-rotational case $= \pi r^2 h$, and the volume of the liquid in the rotational case

$$= \pi r^2 h' - \int_{OA}^{h'} \pi x^2 dy = \pi r^2 h' - \frac{2\pi g}{\omega^2}\int_c^{h'} (y - c)\,dy \qquad \text{[From (ii)]}$$

$$= \pi r^2 h' - \frac{\pi g}{\omega^2}(h' - c)^2 = \pi r^2\left(\frac{\omega^2 r^2}{4g} + c\right) \qquad \text{[By (iii)]}$$

Thus $$\pi r^2 h = \pi r^2\left(\frac{\omega^2 r^2}{4g} + c\right) \text{ whence } c = h - \frac{\omega^2 r^2}{4g}$$

$\therefore$ (*ii*) becomes, $$y = \frac{\omega^2 x^2}{2g} + h - \frac{\omega^2 x^2}{4g} \quad \text{or} \quad y = \frac{\omega^2}{2g}\left(x^2 - \frac{r^2}{2}\right) + h$$

which is the desired equation of the curve.

Example 12.15. Discharge of water through a small hole. *If the velocity of flow of water through a small hole is $0.6\sqrt{2gy}$ where g is the gravitational acceleration and y is the height of water level above the hole, find the time required to empty a tank having the shape of a right circular cone of base radius a and height h filled completely with water and having a hole of area A_0 in the base.*

Solution. At any time t, let the height of the water level be y and radius of its surface be r (Fig. 12.12) so that

$$\frac{h - y}{r} = \frac{h}{a} \quad \text{or} \quad r = a(h - y)/h$$

$\therefore$ surface area of the liquid $= \pi r^2 = \pi a^2 (1 - y/h)^2$

Volume of water drained through the hole per unit time

$$= 0.6\sqrt{(2gy)}\,A_0 = 4.8\sqrt{y}\,A_0 \qquad [\because\ g = 32$$

$\therefore$ rate of fall of liquid level $= 4.8\,A_0\sqrt{y} + \pi a^2(1 - y/h)^2$

i.e., $$\frac{dy}{dt} = -\frac{4.8\,A_0\sqrt{y}}{\pi a^2(1-y/h)^2}$$ (– ve is taken since the water level decreases)

Hence time to empty the tank (= t)

$$= -\int_h^0 \frac{\pi a^2(1-y/h)^2}{4.8\,A_0\sqrt{y}}\,dy = \frac{\pi a^2}{4.8\,A_0}\int_0^h (y^{-1/2} - 2y^{1/2}/h + y^{3/2}/h^2)\,dy$$

$$= \frac{\pi a^2}{4.8\,A_0}\left|2y^{1/2} - \frac{4}{3h}y^{3/2} + \frac{2}{5h^2}y^{5/2}\right|_0^h = 0.2\,\pi a^2\sqrt{h}/A_0.$$

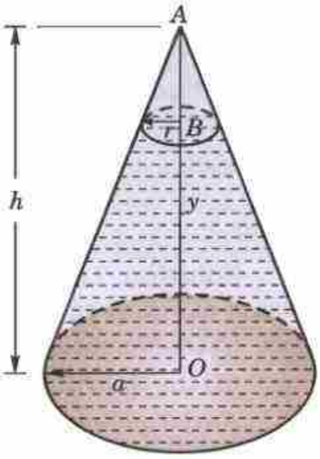


Fig. 12.12

Example 12.16. Atmospheric pressure. *Find the atmospheric pressure p lb. per ft. at a height z ft. above the sea-level, both when the temperature is constant or variable.*

Solution. Take a vertical column of air of unit cross-section.

Let p be the pressure at a height z above the sea-level and $p + \delta p$ at height $z + \delta z$. Let ρ be the density at a height z. (Fig. 12.13)

Now since the thin column δz of air is being pressured upwards with pressure p and downwards with $p + \delta p$, we get by considering its equilibrium;

$$p = p + \delta p + g\rho\delta z.$$

Taking the limit, we get $\qquad dp/dz = -g\rho \qquad$ ...(*i*)

which is the differential equation giving the atmospheric pressure at height z.

(*i*) *When the temperature is constant,* we have by Boyle's law*, $p = k\rho \qquad$ ...(*ii*)

$\therefore$ Substituting the value of ρ from (*ii*) in (*i*), we get

$$\frac{dp}{dz} = -gp/k \quad \text{or} \quad \int\frac{dp}{p} = -\frac{g}{k}\int dz + c \ \text{ or } \log p = -\frac{g}{k}z + c$$

At the sea-level, where $z = 0$, $p = p_0$ (say) then $c = \log p_0$

$$\therefore \qquad \log p - \log p_0 = -\frac{g}{k}z \ \textit{i.e.,}\ \log p/p_0 = -gz/k$$

Hence p is given by $p = p_0 e^{-gz/k}$.

(*ii*) *When the temperature varies,* we have $p = k\rho^n$.

Proceeding as above, we shall find that p is given by $\dfrac{n}{n-1}(p_0^{1-1/n} - p^{1-1/n}) = gk^{-1/n}z$.

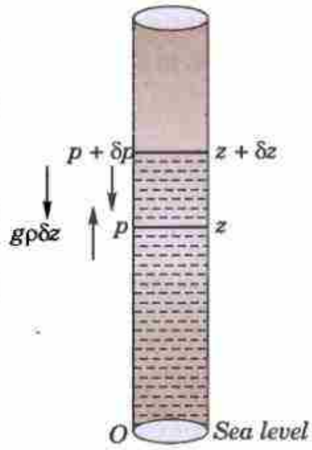


Fig. 12.13

PROBLEMS 12.3

1. A particle of mass m moves under gravity in a medium whose resistance is k times its velocity, where k is a constant. If the particle is projected vertically upwards with a velocity v, show that the time to reach the highest point is $\dfrac{m}{k}\log_e\left(1 + \dfrac{kv}{mg}\right)$.
2. A body of mass m falls from rest under gravity and air resistance proportional to square of velocity. Find velocity as function of time. (*Marathwada, 2008*)
3. A body of mass m falls from rest under gravity in a field whose resistance is mk times the velocity of the body. Find the terminal velocity of the body and also the time taken to acquire one half of its limiting speed.
4. A particle is projected with velocity v along a smooth horizontal plane in the medium whose resistance per unit mass is μ times the cube of the velocity. Show that the distance it has described in time t is $\dfrac{1}{\mu v}(\sqrt{1 + 2\mu v^2 t} - 1)$.

*Named after the English physicist *Robert Boyle* (1627–1691) who was one of the founders of the Royal Society.

5. When a bullet is fired into a sand tank, its retardation is proportional to the square root of its velocity. How long will it take to come to rest if it enters the sand bank with velocity v_0?
6. A particle of mass m is attached to the lower end of a light spring (whose upper end is fixed) and is released. Express the velocity v as a function of the stretch x feet.
7. A chain coiled up near the edge of a smooth table just starts to fall over the edge. The velocity v when a length x has fallen is given by $xv\dfrac{dv}{dx}+v^2=gx$.

   Show that $v = 8\sqrt{(x/3)}$ ft/sec.
8. A toboggan weighing 200 lb., descends from rest on a uniform slope of 5 in 13 which is 15 yards long. If the coefficient of friction is 1/10 and the air resistance varies as the square of the velocity and is 3 lb. weight when the velocity is 10 ft/sec.; prove that its velocity at the bottom is 38.6 ft/sec and show that however long, the slope is the velocity cannot exceed 44 ft per sec.
   [**Hint.** Fig. 12.14. Equation of motion is
   $$\frac{W}{g}\cdot v\frac{dv}{dx} = -\mu R - kv^2 + W\sin\alpha]$$

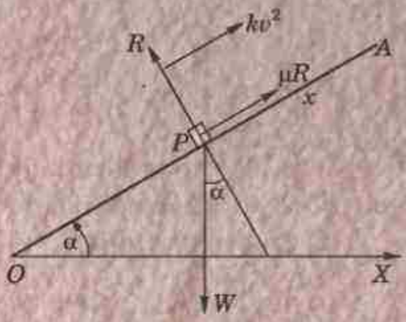


Fig. 12.14

9. Show that a particle projected from the earth's surface with a velocity of 7 miles/sec. will not return to the earth. [Take earth's radius = 3960 miles and $g = 32.17$ ft/sec^2].
10. A cylindrical tank 1.5 m. high stands on its circular base of diameter 1 m. and is initially filled with water. At the bottom of the tank there is a hole of diameter 1 cm., which is opened at some instant, so that the water starts draining under gravity. Find the height of water in the tank at any time t sec. Find the times at which the tank is one-half full, one quarter full, and empty.

    [**Hint.** Take $g = 980$ cm/sec^2 in $v = 0.6\sqrt{(2gy)}$]
11. The rate at which water flows from a small hole at the bottom of a tank is proportional to the square root of the depth of the water. If half the water flows from a cylindrical tank (with vertical axis) in 5 minutes, find the time required to empty the tank.
12. A conical cistern of height h and semi-vertical angle α is filled with water and is held in vertical position with vertex downwards. Water leaks out from the bottom at the rate of kx^2 cubic cms per second, k is a constant and x is the height of water level from the vertex. Prove that the cistern will be empty in $(\pi h \tan^2 \alpha)/k$ seconds.
13. Upto a certain height in the atmosphere, it is found that the pressure p and the density ρ are connected by the relation $p = k\rho^n$ $(n > 1)$. If this relation continued to hold upto any height, show that the density would vanish at a finite height.

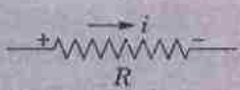
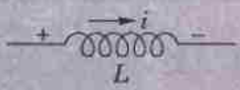
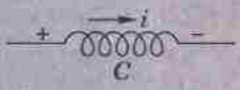
12.5 SIMPLE ELECTRIC CIRCUITS

We shall consider circuits made up of

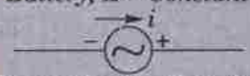
(*i*) three passive elements—resistance, inductance, capacitance and

(*ii*) an active element—voltage source which may be a battery or a generator.

(1) Symbols

Element	*Symbol*	*Unit**
1. *Quantity of electricity*	q	coulomb
2. *Current* (= time rate flow of electricity)	i	ampere (A)
3. *Resistance, R*	 R	ohm (Ω)
4. *Inductance, L*	 L	henry (H)
5. *Capacitance, C*	 C	farad (F)

*These units are respectively named after the French engineer and physicist *Charles Augustin de Coulomb* (1736–1806); French physicist *Andre Marie Ampere* (1775–1836); German physicist *George Simon Ohm* (1789–1854); Italian physicist *Joseph Henry* (1797–1878); American physicist *Michael Faraday* (1791–1867) and the Italian physicist *Alessandro Volta* (1745–1827).

Element	Symbol	Unit
6. *Electromotive force (e.m.f.) or voltage, E*	 *Battery, E = Constant*  *Generator, E = Variable*	volt (V)

7. *Loop* is any closed path formed by passing through two or more elements in series.
8. *Nodes* are the terminals of any of these elements.

(2) Basic relations

(*i*) $i = \dfrac{dq}{dt}$ or $q = \int i dt$ [$\because$ current is the rate of flow of electricity]

(*ii*) Voltage drop across resistance $R = Ri$ [Ohm's Law]

(*iii*) Voltage drop across inductance $L = L\dfrac{di}{dt}$

(*iv*) Voltage drop across capacitance $C = \dfrac{q}{C}$.

(3) Kirchhoff's laws*. The formulation of differential equations for an electrical circuit depends on the following two Kirchhoff's laws which are of cardinal importance :

I. *The algebraic sum of the voltage drops around any closed circuit is equal to the resultant electromotive force in the circuit.*

II. *The algebraic sum of the currents flowing into (or from) any node is zero.*

(4) Differential equations

(*i*) *R, L series circuit.* Consider a circuit containing resistance R and inductance L in series with a voltage source (battery) E. (Fig. 12.15).

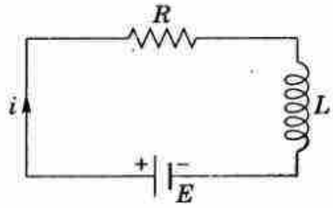


Fig. 12.15

Let i be the current flowing in the circuit at any time t. Then by Kirchhoff's first law, we have sum of voltage drops across R and $L = E$

i.e., $$Ri + L\frac{di}{dt} = E \quad \text{or} \quad \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L} \qquad ...(1)$$

This is a Leibnitz's linear equation.

$$\text{I.F.} = e^{\int \frac{R}{L}dt} = e^{Rt/L} \text{ and therefore, its solution is } i(\text{I.F.}) = \int \frac{E}{L}(\text{I.F.})dt + c$$

or $$i \cdot e^{Rt/L} = \int \frac{E}{L} e^{Rt/L}\, dt + c = \frac{E}{L} \cdot \frac{L}{R} \cdot e^{Rt/L} + c \text{ whence } i = \frac{E}{R} + ce^{-Rt/L} \qquad ...(2)$$

If initially there is no current in the circuit, *i.e.*, $i = 0$, when $t = 0$, we have $c = -E/R$.

Thus (2) becomes $i = \dfrac{E}{R}\left(1 - e^{-Rt/L}\right)$ which shows that i increases with t and attains the maximum value E/R.

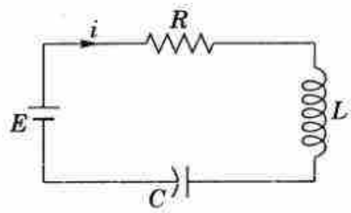


Fig. 12.16

(*ii*) *R, L, C series circuit.* Now consider a circuit containing resistance R, inductance L and capacitance C all in series with a constant e.m.f. E (Fig. 12.16)

If i be the current in the circuit at time t, then the charge q on the condenser $= \int i\, dt$, *i.e.*, $i = \dfrac{dq}{dt}$.

Applying Kirchhoff's law, we have, sum of the voltage drops across R, L and $C = E$.

i.e., $$Ri + L\frac{di}{dt} + \frac{q}{C} = E$$

*Named after the German physicist *Gustav Robert Kirchhoff* (1824–1887).

or $$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E.$$

This is the desired differential equation of the circuit and will be solved in § 14.5.

Example 12.17. *Show that the differential equation for the current i in an electrical circuit containing an inductance L and a resistance R in series and acted on by an electromotive force E sin ωt satisfies the equation L di/dt + Ri = E sin ωt.*

Find the value of the current at any time t, if initially there is no current in the circuit.

(*Kurukshetra, 2005*)

Solution. By Kirchhoff's first law, we have sum of voltage drops across R and $L = E \sin \omega t$

i.e., $$Ri + L\frac{di}{dt} = E \sin \omega t.$$

This is the required differential equation which can be written as $\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L} \sin \omega t$

This is a *Leibnitz's equation*. Its I.F. $= e^{\int \frac{R}{L} dt} = e^{\frac{Rt}{L}}$

$\therefore$ the solution is $i(\text{I.F.}) = \int \frac{E}{L} \sin \omega t \cdot (\text{I.F.})\, dt + c$

or $$ie^{Rt/L} = \frac{E}{L}\int e^{Rt/L} \sin \omega t\, dt + c = \frac{E}{L}\frac{e^{Rt/L}}{\sqrt{[(R/L)^2 + \omega^2]}} \sin\left(\omega t - \tan^{-1}\frac{L\omega}{R}\right) + c$$

or $$i = \frac{E}{\sqrt{(R^2 + \omega^2L^2)}} \sin(\omega t - \phi) + ce^{-Rt/L} \text{ where } \tan \phi = L\omega/R \qquad ...(i)$$

Initially when $t = 0$; $i = 0$. $\therefore$ $0 = \frac{E \sin(-\phi)}{\sqrt{(R^2 + \omega^2L^2)}} + c$, *i.e.*, $c = \frac{E \sin \phi}{\sqrt{(R^2 + \omega^2L^2)}}$

Thus (*i*) takes the form $i = \frac{E \sin(\omega t - \phi)}{\sqrt{(R^2 + \omega^2L^2)}} + \frac{E \sin \phi}{\sqrt{(R^2 + \omega^2L^2)}} \cdot e^{-Rt/L}$

or $$i = \frac{E}{\sqrt{(R^2 + \omega^2L^2)}} [\sin(\omega t - \phi) + \sin \phi \,.\, e^{-Rt/L}]$$ which gives the current at any time t.

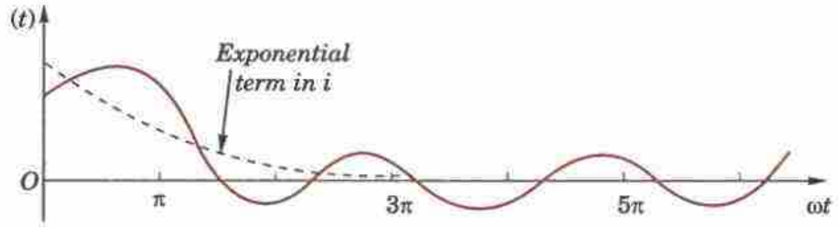


Fig. 12.17

Obs. As t increases indefinitely, the exponential term will approach zero. This implies that after sometime the current $i(t)$ will execute nearly harmonic oscillations only (Fig. 12.17).

PROBLEMS 12.4

1. When a switch is closed in a circuit containing a battery E, a resistance R and an inductance L, the current i builds up at a rate given by $L\, di/dt + Ri = E$.
   Find i as a function of t. How long will it be, before the current has reached one-half its final value if $E = 6$ volts, $R = 100$ ohms and $L = 0.1$ henry ?
2. When a resistance R ohms is connected in series with an inductance L henries with an e.m.f. of E volts, the current i amperes at time t is given by $L\, di/dt + Ri = E$.
   If $E = 10 \sin t$ volts and $i = 0$ when $t = 0$, find i as a function of t.

3. A resistance of 100 Ω, an inductance of 0.5 henry are connected in series with a battery of 20 volts. Find the current in the circuit at $t = 0.5$ sec, if $i = 0$ at $t = 0$. *(Marathwada, 2008)*
4. The equation of electromotive force in terms of current i for an electrical circuit having resistance R and condenser of capacity C in series, is
$$E = Ri + \int \frac{idt}{C}$$
Find the current i at any time t when $E = E_m \sin \omega t$. *(S.V.T.U., 2008, P.T.U., 2006)*
5. A resistance R in series with inductance L is shunted by an equal resistance R with capacity C. An alternating e.m.f. $E \sin pt$ produces currents i_1 and i_2 in two branches. If initially there is no current, determine i_1 and i_2 from the equations
$$L\frac{di_1}{dt} + Ri_1 = E \sin pt \quad \text{and} \quad \frac{i_2}{C} + R\frac{di_2}{dt} = pE \cos pt.$$
Verify that if $R^2C = L$, the total current $i_1 + i_2$ will be $(E \sin pt)/R$.

12.6 NEWTON'S LAW OF COOLING*

According to this law, *the temperature of a body changes at a rate which is proportional to the difference in temperature between that of the surrounding medium and that of the body itself.*

If θ_0 is the temperature of the surroundings and θ that of the body at any time t, then
$$\frac{d\theta}{dt} = -k(\theta - \theta_0), \text{ where } k \text{ is a constant.}$$

Example 12.18. *A body originally at 80°C cools down to 60°C in 20 minutes, the temperature of the air being 40°C. What will be the temperature of the body after 40 minutes from the original ?*

Solution. If θ be the temperature of the body at any time t, then
$$\frac{d\theta}{dt} = -k(\theta - 40), \qquad \text{where } k \text{ is a constant.}$$

Integrating, $$\int \frac{d\theta}{\theta - 40} = -k \int dt + \log c, \qquad \text{where } c \text{ is a constant.}$$

or $$\log(\theta - 40) = -kt + \log c \quad i.e., \quad \theta - 40 = ce^{-kt} \qquad ...(i)$$

When $t = 0$, $\theta = 80°$ and when $t = 20$, $\theta = 60°$. $\therefore$ $40 = c$, and $20 = ce^{-20k}$; $k = \frac{1}{20}\log 2$.

Thus (*i*) becomes $$\theta - 40 = 40e^{-\left(\frac{1}{20}\log 2\right)t}$$

When $t = 40$ min., $$\theta = 40 + 40e^{-2\log 2} = 40 + 40e^{\log(1/4)} = 40 + 40 \times \frac{1}{4} = 50°\text{C}.$$

12.7 HEAT FLOW

The fundamental principles involved in the problems of heat conduction are :

(*i*) *Heat flows from a higher temperature to the lower temperature.*

(*ii*) *The quantity of heat in a body is proportional to its mass and temperature.*

(*iii*) *The rate of heat-flow across an area is proportional to the area and to the rate of change of temperature with respect to its distance normal to the area.*

If q (cal./sec.) be the quantity of heat that flows across a slab of area α (cm^2) and thickness δx in one second, where the difference of temperature at the faces is δT, then by (*iii*) above
$$q = -k\alpha dT/dx \qquad ...(A)$$
where k is a constant depending upon the material of the body and is called the *thermal conductivity*.

*Named after the great English mathematician and physicist *Sir Issac Newton* (1642–1727) whose contributions are of utmost importance. He discovered many physical laws, invented Calculus alongwith *Leibnitz* (see footnote p. 139) and created analytical methods of investigating physical problems. He became professor at Cambridge in1699, but his '*Mathematical Principles of Natural Philosophy*' containing development of classical mechanics had been completed in 1687.

Example 12.19. *A pipe 20 cm in diameter contains steam at 150°C and is protected with a covering 5 cm thick for which $k = 0.0025$. If the temperature of the outer surface of the covering is 40°C, find the temperature half-way through the covering under steady state conditions.*

Solution. Let q cal./sec. be the constant quantity of heat flowing out radially through a surface of the pipe having radius x cm. and length 1 cm (Fig. 12.18). Then the area of the lateral surface (belt) = $2\pi x$.

$\therefore$ the equation (A) above gives

$$q = -k \cdot 2\pi x \cdot \frac{dT}{dx} \quad \text{or} \quad dT = -\frac{q}{2\pi k} \cdot \frac{dx}{x}$$

Integrating, we have

$$T = -\frac{q}{2\pi k} \log_e x + c$$

Since $T = 150$, when $x = 10$. $\therefore$ $150 = -\dfrac{q}{2\pi k} \log_e 10 + c$ ...(*i*)

Again since $T = 40$, when $x = 15$, $40 = -\dfrac{q}{2\pi k} \log_e 15 + c$ ...(*ii*)

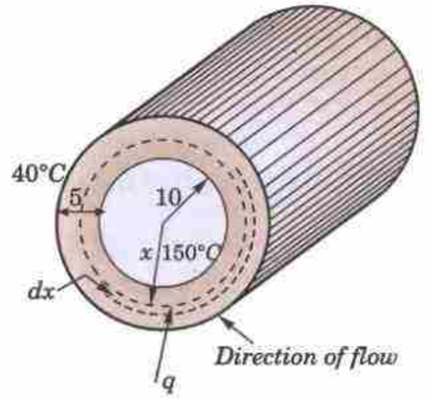


Fig. 12.18

Subtracting (*ii*) from (*i*), $110 = \dfrac{q}{2\pi k} \log_e 1.5$ ...(*iii*)

Let $T = t$, when $x = 12.5$ $\therefore$ $t = -\dfrac{q}{2\pi k} \log_e 12.5 + c$ ...(*iv*)

Subtracting (*i*) from (*iv*), $t - 150 = -\dfrac{q}{2\pi k} \log_e 1.25$ ...(*v*)

Dividing (*v*) by (*iii*), $\dfrac{t - 150}{110} = -\dfrac{\log_e 1.25}{\log_e 1.5}$, whence $t = 89.5$°C.

PROBLEMS 12.5

1. If the temperature of the air is 30°C and the substance cools from 100°C to 70°C in 15 minutes, find when the temperature will 40°C.
2. If the air is maintained at 30°C and the temperature of the body cools from 80°C to 60°C in 12 minutes, find the temperature of the body after 24 minutes.
3. Two friends A and B order coffee and receive cups of equal temperature at the same time. A adds a small amount of cool cream immediately but does not drink his coffee until 10 minutes later, B waits for 10 minutes and adds the same amount of cool cream and begins to drink. Assuming the Newton's law of cooling, decide who drinks the hotter coffee ?
4. A pipe 20 cm. in diameter contains steam at 200°C. It is covered by a layer of insulations 6 cm thick and thermal conductivity 0.0003. If the temperature of the outer surface is 30°C, find the heat loss per hour from two metre length of the pipe.
5. A steam pipe 20 cm. in diameter contains steam at 150°C and is covered with asbestos 5 cm thick. The outside temperature is kept at 60°C. By how much should the thickness of the covering be increased in order that the rate of heat loss should be decreased by 25% ?

12.8 RATE OF DECAY OF RADIO-ACTIVE MATERIALS

This law states that disintegration at any instant is proportional to the amount of material present.

of material at any time t, then $\dfrac{du}{dt} = -ku$, where k is a constant.

Example 12.20. *Uranium disintegrates at a rate proportional to the amount then present at any instant. If M_1 and M_2 grams of uranium are present at times T_1 and T_2 respectively, find the half-life of uranium.*

Solution. Let the mass of uranium at any time t be m grams.

Then the equation of disintegration of uranium is $\frac{dm}{dt} = -\mu m$, where μ is a constant.

Integrating, we get $\int \frac{dm}{dt} = -\mu \int dt + c$ or $\log m = c - \mu t$ ...(i)

Initially, when $t = 0$, $m = M$ (say) so that $c = \log M$ $\therefore$ (i) becomes, $\mu t = \log M - \log m$ ...(ii)

Also when $t = T_1$, $m = M_1$ and when $t = T_2$, $m = M_2$

$\therefore$ From (ii), we get $\mu T_1 = \log M - \log M_1$ ...(iii)

$\mu T_2 = \log M - \log M_2$ ...(iv)

Subtracting (iii) from (iv), we get

$$\mu(T_2 - T_1) = \log M_1 - \log M_2 = \log (M_1/M_2) \text{ whence } \mu = \frac{\log (M_1/M_2)}{T_2 - T_1}$$

Let the mass reduce to half its initial value in time T. *i.e.*, when $t = T$, $m = \frac{1}{2} M$.

$\therefore$ from (ii), we get $\mu T = \log M - \log (M/2) = \log 2$.

Thus $$T = \frac{1}{\mu} \log 2 = \frac{(T_2 - T_1) \log 2}{\log (M_1/M_2)}.$$

12.9 CHEMICAL REACTIONS AND SOLUTIONS

A type of problems which are especially important to chemical engineers are those concerning either chemical reactions or chemical solutions. These can be best explained through the following example :

Example 12.21. *A tank initially contains 50 gallons of fresh water. Brine, containing 2 pounds per gallon of salt, flows into the tank at the rate of 2 gallons per minute and the mixture kept uniform by stirring, runs out at the same rate. How long will it take for the quantity of salt in the tank to increase from 40 to 80 pounds ?* (Andhra, 1997)

Solution. Let the salt content at time t be u lb. so that its rate of change is du/dt

$= 2$ gal. $\times$ 2 lb. $= 4$ lb./min.

If C be the concentration of the brine at time t, the rate at which the salt content decreases due to the out-flow

$= 2$ gal. $\times$ C lb. $= 2C$ lb./min.

$\therefore$ $$\frac{du}{dt} = 4 - 2C \quad ...(i)$$

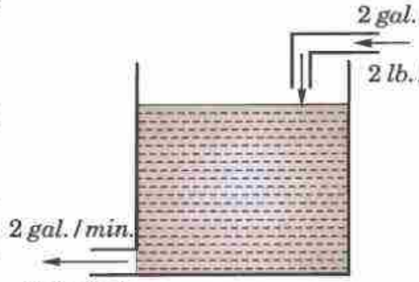


Fig. 12.19

Also since there is no increase in the volume of the liquid, the concentration $C = u/50$.

$\therefore$ (i) becomes $\frac{du}{dt} = 4 - 2\frac{u}{50}$

Separating the variables and integrating, we have

$$\int dt = 25 \int \frac{du}{100 - u} + k \quad \text{or} \quad t = -25 \log_e (100 - u) + k \quad ...(ii)$$

Initially when $t = 0$, $u = 0$ $\therefore$ $0 = -25 \log_e 100 + k$ ...(iii)

Eliminating k from (ii) and (iii), we get $t = 25 \log_e \frac{100}{100 - u}$.

Taking $t = t_1$ when $u = 40$ and $t = t_2$ when $u = 80$, we have

$$t_1 = 25 \log_e \frac{100}{60} \text{ and } t_2 = 25 \log_e \frac{100}{20}$$

$\therefore$ The required time $(t_2 - t_1) = 25 \log_e 5 - 25 \log_e 5/3$

$= 25 \log_e 3 = 25 \times 1.0986 = 27$ min. 28 sec.

PROBLEMS 12.6

1. The number N of bacteria in a culture grew at a rate proportional to N. The value of N was initially 100 and increased to 332 in one hour. What would be the value of N after $1\frac{1}{2}$ hours? *(Nagarjuna, 2008 ; J.N.T.U., 2003)*
2. The rate at which bacteria multiply is proportional to the instantaneous number present. If the original number doubles in 2 hours, in how many hours will it triple? *(Andhra, 2000)*
3. Radium decomposes at a rate proportional to the amount present. If a fraction p of the original amount disappears in 1 year, how much will remain at the end of 21 years?
4. If 30% of radio active substance disappeared in 10 days, low long will it take for 90% of it to disappear? *(Madras, 2000 S)*
5. Under certain conditions cane-sugar in water is converted into dextrose at a rate which is proportional to the amount unconverted at any time. If of 75 gm. at time $t = 0$, 8 gm. are converted during the first 30 minutes, find the amount converted in $1\frac{1}{2}$ hours.
6. In a chemical reaction in which two substances A and B initially of amounts a and b respectively are concerned, the velocity of transformation dx/dt at any time t is known to be equal to the product $(a - x)(b - x)$ of the amounts of the two substances then remaining untransformed. Find t in terms of x if $a = 0.7$, $b = 0.6$ and $x = 0.3$ when $t = 300$ seconds.
7. A tank contains 1000 gallons of brine in which 500 lt. of salt are dissolved. Fresh water runs into the tank at the rate of 10 gallons /minute and the mixture kept uniform by stirring, runs out at the same rate. How long will it be before only 50 lt. of salt is left in the tank?

   [**Hint.** If u be the amount of salt after t minutes, then $du/dt = -10u/1000$.]
8. A tank is initially filled with 100 gallons of salt solution containing 1 lb. of salt per gallon. Fresh brine containing 2 lb. of salt per gallon runs into the tank at the rate of 5 gallons per minute and the mixture assumed to be kept uniform by stirring, runs out at the same rate. Find the amount of salt in the tank at any time, and determine how long it will take for this amount to reach 150 lb.

12.10 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 12.7

Fill up the blanks or choose the correct answer in the following problems :

1. If a coil having a resistance of 15 ohms and an inductance of 10 henries is connected to 90 volts supply then the current after 2 secs is .....
2. A tennis ball dropped from a height of 6 m, rebounds infinitely often. If it rebounds 80% of the distance that it falls, then the total distance for these bounces is .....
3. Radium decomposes at a rate proportional to the amount present. If 5% of the original amount disappears in 50 years then ......% will remain after 100 years.
4. The curve whose polar subtangent is constant is ......
5. The curve in which the length of the subnormal is proportional to the square of the ordinate, is ......
6. The curve in which the portion of the tangent between the axes is bisected at the point of contact, is ......
7. If the stream lines of a flow around a corner are $xy = c$, then the equipotential lines are ......
8. The orthogonal trajectories of a system of confocal and coaxial parabolas is ......
9. When a bullet is fired into a sand tank, its retardation is proportional to $\sqrt{(\text{velocity})}$. If it enters the sand tank with velocity v_o, it will come to rest after ...... seconds.
10. The rate at which bacteria multiply is proportional to the instantaneous number present. If the original number doubles in two hours, then it will triple after ..... hours.
11. Ram and Sunil order coffee and receive cups simultaneously at equal temperature. Ram adds a spoon of cold cream but doesn't drink for 10 minutes, Sunil waits for 10 minutes and adds a spoon of cold cream and begins to drink. Who drinks the hotter coffee?
12. The equation $y - 2x = c$ represents the orthogonal trajectories of the family

    (*i*) $y = ae^{-2x}$ (*ii*) $x^2 + 2y^2 = a$ (*iii*) $xy = a$ (*iv*) $x + 2y = a$.

13. In order to keep a body in air above the earth for 12 seconds, the body should be thrown vertically up with a velocity of

(a) $\sqrt{6}$ g m/sec (b) $\sqrt{12}$ g m/sec (c) 6 g m/sec (d) 12g m/sec.

14. The orthogonal trajectory of the family $x^2 + y^2 = c^2$ is

(a) $x + y = c$ (b) $xy = c$ (c) $x^2 + y^2 = x + y$ (d) $y = cx$. (V.T.U., 2010)

15. If a thermometer is taken outdoors where the temperature is 0°C, from a room having temperature 21°C and the reading drops to 10°C in 1 minute then its reading will be 5°C after ......minutes.

16. The equation of the curve for which the angle between the tangent and the radius vector is twice the vectorial angle is $r^2 = 2a \sin 2\theta$. This satisfies the differential equation

(a) $r\dfrac{dr}{d\theta} = \tan 2\theta$ (b) $r\dfrac{dr}{d\theta} = \cos 2\theta$ (c) $r\dfrac{d\theta}{dr} = \tan 2\theta$ (d) $r\dfrac{d\theta}{dr} = \cos 2\theta$.

17. Two balls of m_1 and m_2 grams are projected vertically upwards such that the velocity of projection of m_1 is double that of m_2. If the maximum height to which m_1 and m_2 rise be h_1 and h_2 respectively then

(a) $h_1 = 2h_2$ (b) $2h_1 = h_2$ (c) $h_1 = 4h_2$ (d) $4h_1 = h_2$.

18. Two balls are projected simultaneously with same velocity from the top of a tower, one vertically upwards and the other vertically downwards. If they reach the ground in times t_1 and t_2, then the height of the tower is

(a) $\dfrac{1}{2} g t_1 t_2$ (b) $\dfrac{1}{2} g(t_1^2 + t_2^2)$ (c) $\dfrac{1}{2} g(t_1^2 - t_2^2)$ (d) $\dfrac{1}{2} g(t_1 + t_2)^2$.

19. A particle projected from the earth's surface with a velocity of 7 miles/sec will return to the earth.

(Taking $g = 32.17$ and earth's radius = 3960 miles) (True/False)

20. If a particle falls under gravity with air resistance k times its velocity, then its velocity cannot exceed g/k. (True/False)

CHAPTER 13

Linear Differential Equations

1. Definitions. 2. Complete solution. 3. Operator *D*. 4. Rules for finding the Complementary function. 5. Inverse operator. 6. Rules for finding the particular integral. 7. Working procedure. 8. Two other methods of finding P.I.— Method of variation of parameters ; Method of undetermined coefficients. 9. Cauchy's and Legendre's linear equations. 10. Linear dependence of solutions. 11. Simultaneous linear equations with constant coefficients. 12. Objective Type of Questions.

13.1 DEFINITIONS

Linear differential equations *are those in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together.* Thus the general linear differential equation of the nth order is of the form

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1}y}{dx^{n-1}} + p_2 \frac{d^{n-2}y}{dx^{n-2}} + \ldots + p_n y = X,$$

where $p_1, p_2, \ldots, p_n$ and X are functions of x only.

Linear differential equations with constant co-efficients are of the form

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1}y}{dx^{n-1}} + k_2 \frac{d^{n-2}y}{dx^{n-2}} + \ldots + k_n y = X$$

where $k_1, k_2, \ldots, k_n$ are constants. Such equations are most important in the study of electro-mechanical vibrations and other engineering problems.

13.2 (1) THEOREM

If y_1, y_2 are only two solutions of the equation

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1}y}{dx^{n-1}} + k_2 \frac{d^{n-2}y}{dx^{n-2}} + \ldots + k_n y = 0 \qquad \ldots(1)$$

then $c_1y_1 + c_2y_2$ $(= u)$ is also its solution.

Since $y = y_1$ and $y = y_2$ are solutions of (1).

$$\therefore \quad \frac{d^n y_1}{dx^n} + k_1 \frac{d^{n-1}y_1}{dx^{n-1}} + k_2 \frac{d^{n-2}y_1}{dx^{n-2}} + \ldots + k_n y_1 = 0 \qquad \ldots(2)$$

and

$$\frac{d^n y_2}{dx^n} + k_1 \frac{d^{n-1}y_2}{dx^{n-1}} + k_2 \frac{d^{n-2}y_2}{dx^{n-2}} + \ldots + k_n y_2 = 0 \qquad \ldots(3)$$

If c_1, c_2 be two arbitrary constants, then

$$\frac{d^n (c_1y_1 + c_2y_2)}{dx^n} + k_1 \frac{d^{n-1}(c_1y_1 + c_2y_2)}{dx^{n-1}} + \ldots + k_n(c_1y_1 + c_2y_2)$$

$$= c_1\left(\frac{d^n y_1}{dx^n} + k_1 \frac{d^{n-1} y_1}{dx^{n-1}} + \ldots + k_n y_1\right) + c_2\left(\frac{d^n y_2}{dx^n} + k_1 \frac{d^{n-1} y_2}{dx^{n-1}} + \ldots + k_n y_2\right)$$

$$= c_1(0) + c_2(0) = 0 \qquad \text{[By (2) and (3)]}$$

i.e.,
$$\frac{d^n u}{dx^n} + k_1 \frac{d^{n-1} u}{dx^{n-1}} + \ldots + k_n y = 0 \qquad \ldots(4)$$

This proves the theorem.

(2) Since the general solution of a differential equation of the nth order contains n arbitrary constants, it follows, from above, that if $y_1, y_2, y_3, \ldots, y_n$, are n independent solutions of (1), then $c_1y_1 + c_2y_2 + \ldots + c_ny_n$ $(= u)$ is its complete solution.

(3) If $y = v$ be any particular solution of

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + k_n y = X \qquad \ldots(5)$$

then
$$\frac{d^n v}{dx^n} + k_1 \frac{d^{n-1} v}{dx^{n-1}} + \ldots + k_n v = X \qquad \ldots(6)$$

Adding (4) and (6), we have $\dfrac{d^n(u+v)}{dx^n} + k_1 \dfrac{d^{n-1}(u+v)}{dx^{n-1}} + \ldots + k_n(u+v) = X$

This shows that $y = u + v$ is the complete solution of (5).

The part u is called the **complementary function (C.F.)** *and the part v is called the* **particular integral (P.I.)** *of (5).*

$\therefore$ the complete solution (C.S.) of (5) is **y = C.F. + P.I.**

Thus in order to solve the question (5), we have to first find the C.F., i.e., the complete solution of (1), and then the P.I., i.e. a particular solution of (5).

13.3 OPERATOR D

Denoting $\dfrac{d}{dx}, \dfrac{d^2}{dx^2}, \dfrac{d^3}{dx^3}$ etc. by D, D^2, D^3 etc., so that

$\dfrac{dy}{dx} = Dy$, $\dfrac{d^2y}{dx^2} = D^2y$, $\dfrac{d^3x}{dx^3} = D^3y$ etc., the equation (5) above can be written in the *symbolic form* $(D^n + k_1D^{n-1} + \ldots + k_n)y = X$, *i.e.,* $f(D)y = X$,
where $f(D) = D^n + k_1D^{n-1} + \ldots + k_n$, *i.e.*, a polynomial in D.

Thus *the symbol D stands for the operation of differentiation and can be treated much the same as an algebraic quantity i.e., f (D) can be factorised by ordinary rules of algebra and the factors may be taken in any order.* For instance

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = (D^2 + 2D - 3)\,y = (D+3)(D-1)y \text{ or } (D-1)(D+3)\,y.$$

13.4 RULES FOR FINDING THE COMPLEMENTARY FUNCTION

To solve the equation $\dfrac{d^n y}{dx^n} + k_1 \dfrac{d^{n-1} y}{dx^{n-1}} + k_2 \dfrac{d^{n-2} y}{dx^{n-2}} + \ldots + k_n y = 0$ ...(1)

where k's are constants.

The equation (1) in *symbolic form is*

$$(D^n + k_1D^{n-1} + k_2D^{n-2} + \ldots + k_n)y = 0 \qquad \ldots(2)$$

Its symbolic co-efficient equated to zero *i.e.*

$$D^n + k_1D^{n-1} + k_2D^{n-2} + \ldots + k_n = 0$$

is called the *auxiliary equation* (A.E.). Let $m_1, m_2, \ldots, m_n$ be its roots.

Case I. *If all the roots be real and different*, then (2) is equivalent to

$$(D - m_1)(D - m_2) \ldots (D - m_n)y = 0 \qquad \ldots(3)$$

Now (3) will be satisfied by the solution of $(D - m_n)y = 0$, *i.e.*, by $\frac{dy}{dx} - m_n y = 0$.

This is a Leibnitz's linear and I.F. $= e^{-m_n x}$

$\therefore$ its solution is $y e^{-m_n x} = c_n$, *i.e.*, $y = c_n e^{m_n x}$

Similarly, since the factors in (3) can be taken in any order, it will be satisfied by the solutions of $(D - m_1)y = 0$, $(D - m_2)y = 0$ etc. *i.e.*, by $y = c_1 e^{m_1 x}$, $y = c_2 e^{m_2 x}$ etc.

Thus the complete solution of (1) is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x}$ ...(4)

Case II. *If two roots are equal (i.e.,* $m_1 = m_2$*),* then (4) becomes

$$y = (c_1 + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

$$y = C e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x} \quad [\because \; c_1 + c_2 = \text{one arbitrary constant } C]$$

It has only $n - 1$ arbitrary constants and is, therefore, not the complete solution of (1). In this case, we proceed as follows :

The part of the complete solution corresponding to the repeated root is the complete solution of $(D - m_1)(D - m_1)y = 0$

Putting $(D - m_1)y = z$, it becomes $(D - m_1) z = 0$ or $\frac{dz}{dx} - m_1 z = 0$

This is a Leibnitz's linear in z and I.F. $= e^{-m_1 x}$. $\therefore$ its solution is $z e^{-m_1 x} = c_1$ or $z = c_1 e^{m_1 x}$

Thus $\quad (D - m_1)y = z = c_1 e^{m_1 x}$ or $\frac{dy}{dx} - m_1 y = c_1 e^{m_1 x}$ ...(5)

Its I.F. being $e^{-m_1 x}$, the solution of (5) is

$$y e^{-m_1 x} = \int c_1 e^{m_1 x} \, dx + c_2 = c_1 x + c_2 \quad \text{or} \quad y = (c_1 x + c_2) e^{m_1 x}$$

Thus the complete solution of (1) is $y = (c_1 x + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$

If, however, the A.E. has three equal roots (*i.e.*, $m_1 = m_2 = m_3$), then the complete solution is

$$y = (c_1 x^2 + c_2 x + c_3) \, e^{m_1 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$$

Case III. *If one pair of roots be imaginary, i.e.,* $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, then the complete solution is

$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

$$= e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x}) + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

$$= e^{\alpha x}[c_1(\cos \beta x + i \sin \beta x) + c_2 (\cos \beta x - i \sin \beta x)] + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

[$\because$ by *Euler's Theorem*, $e^{i\theta} = \cos \theta + i \sin \theta$]

$$= e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x) + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

where $C_1 = c_1 + c_2$ and $C_2 = i(c_1 - c_2)$.

Case IV. *If two points of imaginary roots be equal i.e.,* $m_1 = m_2 = \alpha + i\beta$, $m_3 = m_4 = \alpha - i\beta$, then by case II, the complete solution is

$$y = e^{\alpha x}[(c_1 x + c_2) \cos \beta x + (c_3 x + c_4) \sin \beta x] + \ldots + c_n e^{m_n x}.$$

Example 13.1. *Solve* $\frac{d^2 x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$, *given* $x(0) = 0$, $\frac{dx}{dt}(0) = 15$. *(V.T.U., 2010)*

Solution. Given equation in *symbolic form is* $(D^2 + 5D + 6)\, x = 0$.

Its A.E. is $D^2 + 5D + 6 = 0$, *i.e.*, $(D + 2)(D + 3) = 0$ whence $D = -2, -3$.

$\therefore$ C.S. is $x = c_1 e^{-2t} + c_2 e^{-3t}$ and $\frac{dx}{dt} = -2ae^{-2t} - 3c_2 e^{-3t}$

When $t = 0$, $x = 0$. $\quad \therefore \quad 0 = c_1 + c_2$ (*i*)

When $t = 0$, $dx/dt = 15$ $\quad \therefore \quad 15 = -2c_1 - 3c_2$ ...(*ii*)

Solving (*i*) and (*ii*), $c_1 = 15, c_2 = -15$.
Hence the required solution is $x = 15\,(e^{-2t} - e^{-3t})$.

Example 13.2. *Solve* $\dfrac{d^2x}{dt^2} + 6\dfrac{dx}{dt} + 9x = 0.$

Solution. Given equation in *symbolic form is* $(D^2 + 6D + 9) = 0$

$\therefore$ A.E. is $D^2 + 6D + 9 = 0$, *i.e.*, $(D+3)^2 = 0$ whence $D = -3, -3$.

Hence the C.S. is $x = (c_1 + c_2 t)\, e^{-3t}$.

Example 13.3. *Solve* $(D^3 + D^2 + 4D + 4) = 0.$

Solution. Here the A.E. is $D^3 + D^2 + 4D + 4 = 0$ *i.e.*, $(D^2 + 4)(D + 1) = 0$ $\therefore$ $D = -1, \pm 2i$.

Hence the C.S. is $y = c_1 e^{-x} + e^{0x}(c_2 \cos 2x + c_3 \sin 2x)$

i.e., $y = c_1 e^{-x} + c_2 \cos 2x + c_3 \sin 2x.$

Example 13.4. *Solve (i)* $(D^4 - 4D + 4)\, y = 0$ *(Bhopal, 2008)*
(ii) $(D^2 + 1)^3\, y = 0$ *where* $D \equiv d/dx$.

Solution. (*i*) The A.E. equation is $D^4 - 4D^2 + 4 = 0$ or $(D^2 - 2)^2 = 0$

$\therefore$ $D^2 = 2, 2$ *i.e.*, $D = \pm\sqrt{2}, \pm\sqrt{2}$.

Hence the C.S. is $((c_1 + c_2 x)\, e^{\sqrt{2}x} + (c_3 + c_4 x)\, e^{-\sqrt{2}x})$ [Roots being repeated]

(*ii*) The A.E. equation is $(D^2 + 1)^3 = 0$

$\therefore$ $D = \pm i, \pm i, \pm i.$

Hence the C.S. is $y = e^{ox}\,[(c_1 + c_2 x + c_3 x^2)\cos x + (c_4 + c_5 x + c_6 x^2)\sin x]$

i.e., $y = (c_1 + c_2 + c_3 x^2)\cos x + (c_4 + c_5 x + c_6 x^2)\sin x.$

Example 13.5. *Solve* $\dfrac{d^4x}{dt^4} + 4x = 0.$

Solution. Given equation in *symbolic form is* $(D^4 + 4)\, x = 0$

$\therefore$ A.E. is $D^4 + 4 = 0$ or $(D^4 + 4D^2 + 4) - 4D^2 = 0$ or $(D^2 + 2)^2 - (2D)^2 = 0$

or $(D^2 + 2D + 2)(D^2 - 2D + 2) = 0$

$\therefore$ either $D^2 + 2D + 2 = 0$ or $D^2 - 2D + 2 = 0$

whence $D = \dfrac{-2 \pm \sqrt{(-4)}}{2}$ and $\dfrac{2 \pm \sqrt{(-4)}}{2}$ *i.e.*, $D = -1 \pm i$ and $1 \pm i$.

Hence the required solution is $x = e^{-t}(c_1 \cos t + c_2 \sin t) + e^t (c_3 \cos t + c_4 \sin t)$.

PROBLEMS 13.1

Solve :

1. $\dfrac{d^2x}{dt^2} - 4\dfrac{dx}{dt} + 13x = 0, x(0), \dfrac{dx(0)}{dt} = 2.$ (*V.T.U., 2008*)
2. $y'' - 2y' + 10y = 0, y(0) = 4, y'(0) = 1.$
3. $4y''' + 4y'' + y' = 0.$
4. $\dfrac{d^3y}{dx^3} + y = 0.$ (*V.T.U., 2000 S*)
5. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = 0.$
6. $\dfrac{d^4y}{dx^4} + 8\dfrac{d^2y}{dx^2} + 16y = 0.$ (*J.N.T.U., 2005*)
7. $(4D^4 - 8D^3 - 7D^2 + 11D + 6)y = 0.$ (*V.T.U., 2008*)
8. $(D^2 + 1)^2 (D - 1)y = 0.$
9. If $\dfrac{d^4x}{dt^4} = m^4 x$, show that $x = c_1 \cos mt + c_2 \sin mt + c_3 \cosh mt + c_4 \sinh mt.$

13.5 INVERSE OPERATOR

(1) Definition. $\frac{1}{f(D)}X$ is that function of x, not containing arbitrary constants which when operated upon by $f(D)$ gives X.

i.e.,
$$f(D)\left\{\frac{1}{f(D)}X\right\} = X$$

Thus $\frac{1}{f(D)}X$ *satisfies the equation* $f(D)y = X$ *and is, therefore, its particular integral.*

Obviously, $f(D)$ and $1/f(D)$ are inverse operators.

(2) $$\mathbf{\frac{1}{D}X = \int X dx}$$

Let $$\frac{1}{D}X = y \qquad ...(i)$$

Operating by D, $$D\frac{1}{D}X = Dy \quad i.e., \; X = \frac{dy}{dx}$$

Integrating both sides w.r.t. x, $y = \int X\,dx$, no constant being added as (i) does not contain any constant.

Thus $$\frac{1}{D}X = \int X\,dx.$$

(3) $$\mathbf{\frac{1}{D-a}X = e^{ax}\int Xe^{-ax}dx.}$$

Let $$\frac{1}{D-a}X = y \qquad ...(ii)$$

Operating by $D-a$, $(D-a)\,.\,\frac{1}{D-a}X = (D-a)y.$

or $$X = \frac{dy}{dx} - ay\,,\; i.e.,\; \frac{dy}{dx} - ay = X$$ which is a Leibnitz's linear equation.

$\therefore$ I.F. being e^{-ax}, its solution is

$$ye^{-ax} = \int Xe^{-ax}dx,$$ no constant being added as (ii) doesn't contain any constant.

Thus $$\frac{1}{D-a}X = y = e^{ax}\int Xe^{-ax}dx.$$

13.6 RULES FOR FINDING THE PARTICULAR INTEGRAL

Consider the equation $\frac{d^n y}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + k_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + k_n y = X$

which is *symbolic form of* $(D^n + k_1D^{n-1} + k_2D^{n-2} + \ldots + k_n)y = X$.

$$\therefore \qquad \text{P.I.} = \frac{1}{D^n + k_1D^{n-1} + k_2D^{n-2} + \ldots + k_n}X.$$

Case I. When X = e^{ax}

Since
$$De^{ax} = ae^{ax}$$
$$D^2e^{ax} = a^2e^{ax}$$
....
....
$$D^ne^{ax} = a^ne^{ax}$$

$$\therefore \quad (D^n + k_1D^{n-1} + \ldots + k_n)e^{ax} = (a^n + k_1a^{n-1} + \ldots + k_n)e^{ax}, \quad i.e., \quad f(D)e^{ax} = f(a)e^{ax}$$

Operating on both sides by $\frac{1}{f(D)}$, $\frac{1}{f(D)} f(D) e^{ax} = \frac{1}{f(D)} f(a) e^{ax}$ or $e^{ax} = f(a) \frac{1}{f(D)} e^{ax}$

$\therefore$ dividing by $f(a)$,

$$\mathbf{\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}} \textit{ provided } f(a) \neq 0 \qquad ...(1)$$

If f(a) = 0, *the above rule fails and we proceed further.*

Since a is a root of A.E. $f(D) = D^n + k_1 D^{n-1} + ... + k_n = 0$.

$\therefore$ $D - a$ is a factor of $f(D)$. Suppose $f(D) = (D - a)\, \phi(D)$, where $\phi(a) \neq 0$. Then

$$\frac{1}{f(D)} e^{ax} = \frac{1}{D-a} \cdot \frac{1}{\phi(D)} e^{ax} = \frac{1}{D-a} \cdot \frac{1}{\phi(a)} e^{ax} \qquad \text{[By (1)]}$$

$$= \frac{1}{\phi(a)} \cdot \frac{1}{D-a} e^{ax} = \frac{1}{\phi(a)} \cdot e^{ax} \int e^{ax} \cdot e^{-ax} dx \qquad \text{[By §13.5 (3)]}$$

$$= \frac{1}{\phi(a)} e^{ax} \int dx = x \frac{1}{\phi(a)} e^{ax} \quad i.e., \quad \mathbf{\frac{1}{f(D)} e^{ax} = x \frac{1}{f'(a)} e^{ax}} \qquad ...(2)$$

$$\left[\begin{array}{l} \because f'(D) = (D-a)\phi'(D) + 1 \cdot \phi(D) \\ \therefore f'(a) = 0 \times \phi'(a) + \phi(a) \end{array}\right.$$

If $f'(a) = 0$, then applying (2) again, we get $\mathbf{\frac{1}{f(D)} e^{ax} = x^2 \frac{1}{f''(a)} e^{ax}}$, *provided* $f''(a) \neq 0$ ...(3)

and so on.

Example 13.6. *Find the P.I. of* $(D^2 + 5D + 6)y = e^x$.

Solution. $\text{P.I.} = \frac{1}{D^2 + 5D + 6} e^x$ [Put $D = 1$] $= \frac{1}{1^2 + 5.1 + 6} e^x = \frac{e^x}{12}$.

Example 13.7. *Find the P.I. of* $(D + 2)(D - 1)^2 y = e^{-2x} + 2 \sinh x$.

Solution. $\text{P.I.} = \frac{1}{(D+2)(D-1)^2} [e^{-2x} + 2 \sinh x] = \frac{1}{(D+2)(D-1)^2} [e^{-2x} + e^x - e^{-x}]$

Let us evaluate each of these terms separately.

$$\frac{1}{(D+2)(D-1)^2} e^{-2x} = \frac{1}{D+2} \cdot \left[\frac{1}{(D-1)^2} e^{-2x}\right]$$

$$= \frac{1}{D+2} \cdot \frac{1}{(-2-1)^2} e^{-2x} = \frac{1}{9} \cdot \frac{1}{D+2} e^{-2x}$$

$$= \frac{1}{9} \cdot x \cdot \frac{1}{1} e^{-2x} = \frac{x}{9} e^{-2x} \qquad \left[\because \frac{d}{dD}(D+2) = 1\right]$$

$$\frac{1}{(D+2)(D-1)^2} e^x = \frac{1}{1+2} \cdot \frac{1}{(D-1)^2} e^x = \frac{1}{3} \cdot x^2 \cdot \frac{1}{2} e^x = \frac{x^2}{6} e^x \qquad \left[\because \frac{d^2}{dD^2}(D-1)^2 = 2\right]$$

and
$$\frac{1}{(D+2)(D-1)^2} e^{-x} = \frac{1}{(-1+2)(-1-1)^2} e^{-x} = \frac{e^{-x}}{4}$$

Hence, $\text{P.I.} = \frac{x}{9} e^{-2x} + \frac{x^2}{6} e^x + \frac{1}{4} e^{-x}$.

Case II. When X = sin (ax + b) or cos (ax + b).

Since
$$D \sin(ax + b) = a \cos(ax + b)$$
$$D^2 \sin(ax + b) = -a^2 \sin(ax + b)$$
$$D^3 \sin(ax + b) = -a^3 \cos(ax + b)$$

$$D^4 \sin(ax+b) = a^4 \sin(ax+b)$$

i.e., $$D^2 \sin(ax+b) = (-a^2)\sin(ax+b)$$

$$(D^2)^2 \sin(ax+b) = (-a^2)^2 \sin(ax+b)$$

In general $$(D^2)^r \sin(ax+b) = (-a^2)^r \sin(ax+b)$$

$\therefore$ $$f(D^2)\sin(ax+b) = f(-a^2)\sin(ax+b)$$

Operating on both sides $1/f(D^2)$,

$$\frac{1}{f(D^2)} \cdot f(D^2) \sin(ax+b) = \frac{1}{f(D^2)} f(-a^2) \sin(ax+b)$$

or $$\sin(ax+b) = f(-a^2)\frac{1}{f(D^2)}\sin(ax+b)$$

$\therefore$ Dividing by $f(-a^2)$. $\mathbf{\dfrac{1}{f(D^2)}\sin(ax+b) = \dfrac{1}{f(-a^2)}\sin(ax+b)}$ *provided* $f(-a^2) \neq 0$ ...(4)

If f(– a²) = 0, *the above rule fails and we proceed further.*

Since $\cos(ax+b) + i\sin(ax+b) = e^{i(ax+b)}$ [Euler's theorem]

$\therefore$ $$\frac{1}{f(D^2)}\sin(ax+b) = \text{I.P. of } \frac{1}{f(D^2)}e^{i(ax+b)}$$ [Since $f(-a^2)=0$ $\therefore$ by (2)]

$$= \text{I.P. of } x\frac{1}{f'(D^2)}e^{i(ax+b)}$$ where $D^2 = -a^2$

$\therefore$ $$\mathbf{\frac{1}{f(D^2)}\sin(ax+b) = x\frac{1}{f'(-a^2)}\sin(ax+b)}$$ *provided* $f'(-a^2) \neq 0$ ...(5)

If $f'(-a^2) = 0$, $\mathbf{\dfrac{1}{f(D^2)}.\sin(ax+b) = x^2\dfrac{1}{f''(-a^2)}\sin(ax+b)}$, *provided* $f''(-a^2) \neq 0$, and so on.

Similarly, $\mathbf{\dfrac{1}{f(D^2)}\cos(ax+b) = \dfrac{1}{f(-a^2)}\cos(ax+b)}$, *provided* $f(-a^2) \neq 0$

If $f(-a^2) = 0$, $\mathbf{\dfrac{1}{f(D^2)}\cos(ax+b) = x.\dfrac{1}{f'(-a^2)}\cos(ax+b)}$, *provided* $f'(-a^2) \neq 0$.

If $f'(-a^2) = 0$, $\mathbf{\dfrac{1}{f(D^2)}\cos(ax+b) = x^2\dfrac{1}{f''(-a^2)}\cos(ax+b)}$, *provided* $f''(-a^2) \neq 0$ and so on.

Example 13.8. *Find the P.I. of* $(D^3+1)y = \cos(2x-1)$.

Solution. P.I. $= \dfrac{1}{D^3+1}\cos(2x-1)$ [Put $D^2 = -2^2 = -4$]

$$= \frac{1}{D(-4)+1}\cos(2x-1)$$ [Multiply and divide by $1+4D$]

$$= \frac{(1+4D)}{(1-4D)(1+4D)}\cos(2x-1) = (1+4D).\frac{1}{1-16D^2}\cos(2x-1)$$ [Put $D^2 = -2^2 = -4$]

$$= (1+4D)\frac{1}{1-16(-4)}\cos(2x-1) = \frac{1}{65}[\cos(2x-1) + 4D\cos(2x-1)]$$

$$= \frac{1}{65}[\cos(2x-1) - 8\sin(2x-1)].$$

Example 13.9. *Find the P.I. of* $\dfrac{d^3y}{dx^3} + 4\dfrac{dy}{dx} = \sin 2x$.

Solution. Given equation in *symbolic form* is $(D^3+4D)y = \sin 2x$

$\therefore$ $$\text{P.I.} = \frac{1}{D(D^2+4)}\sin 2x$$ [$\because$ $D^2 + 4 = 0$ for $D^2 = -2^2$, $\therefore$ Apply (5) 477]

$$= x\frac{1}{3D^2+4}\sin 2x$$ $\left[\because \frac{d}{dD}[D^3+4D] = 3D^2+4\right]$

[Put $D^2 = -2^2 = -4$]

$$= x\frac{1}{3(-4)+4}\sin 2x = -\frac{x}{8}\sin 2x.$$

Case III. When X = x^m^.

Here $$\text{P.I.} = \frac{1}{f(D)}x^m = [f(D)]^{-1}x^m.$$

Expand $[f(D)]^{-1}$ in ascending powers of D as far as the term in D^m and operate on x^m term by term. Since the $(m+1)$th and higher derivatives of x^m are zero, we need not consider terms beyond D^m.

Example 13.10. *Find the P.I. of* $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = x^2 + 2x + 4.$

Solution. Given equation in *symbolic form* is $(D^2 + D)y = x^2 + 2x + 4$.

$\therefore$ $$\text{P.I.} = \frac{1}{D(D+1)}(x^2+2x+4) = \frac{1}{D}(1+D)^{-1}(x^2+2x+4)$$

$$= \frac{1}{D}(1 - D + D^2 - \ldots)(x^2+2x+4) = \frac{1}{D}[x^2+2x+4-(2x+2)+2]$$

$$= \int(x^2+4)dx = \frac{x^3}{3} + 4x.$$

Case IV. When X = e^ax^ V, *V being a function of x.*

If u is a function of x, then

$$D(e^{ax}u) = e^{ax}Du + ae^{ax}u + e^{ax}(D+a)u$$

$$D^2(e^{ax}u) = a^{ax}D^2u + 2ae^{ax}Du + a^2e^{ax}u = e^{ax}(D+a)^2u$$

and in general, $$D^n(e^{ax}u) = e^{ax}(D+a)^nu$$

$\therefore$ $$f(D)(e^{ax}u) = e^{ax}f(D+a)u$$

Operating both sides by $1/f(D)$,

$$\frac{1}{f(D)}\cdot f(D)(e^{ax}u) = \frac{1}{f(D)}[e^{ax}f(D+a)u]$$

$$e^{ax}u = \frac{1}{f(D)}[e^{ax}f(D+a)u]$$

Now put $f(D+a)u = V$, *i.e.*, $u = \dfrac{1}{f(D+a)}V$, so that $e^{ax}\dfrac{1}{f(D+a)}V = \dfrac{1}{f(D)}(e^{ax}V)$

i.e., $$\mathbf{\frac{1}{f(D)}(e^{ax}V) = e^{ax}\frac{1}{f(D+a)}V.} \qquad \ldots(6)$$

Example 13.11. *Find P.I. of* $(D^2 - 2D + 4)y = e^x \cos x.$

Solution. $$\text{P.I.} = \frac{1}{D^2-2D+4}e^x\cos x$$ [Replace D by $D + 1$]

$$= e^x\frac{1}{(D+1)^2-2(D+1)+4}\cos x = e^x\frac{1}{D^2+3}\cos x$$ [Put $D^2 = -1^2 = -1$]

$$= e^x\frac{1}{-1+3}\cos x = \frac{1}{2}e^x\cos x.$$

Case V. When X is any other function of x.

Here $\quad \text{P.I.} = \dfrac{1}{f(D)} X.$

If $\quad f(D) = (D - m_1)(D - m_2) \ldots (D - m_n)$, resolving into partial fractions,

$$\frac{1}{f(D)} = \frac{A_1}{D - m_1} + \frac{A_2}{D - m_2} + \ldots + \frac{A_n}{D - m_n}$$

$\therefore$

$$\text{P.I.} = \left[\frac{A_1}{D - m_1} + \frac{A_2}{D - m_2} + \ldots + \frac{A_n}{D - m_n}\right] X$$

$$= A_1 \frac{1}{D - m_1} X + A_2 \frac{1}{D - m_2} X + \ldots + A_n \frac{1}{D - m_n} X$$

$$= A_1 . e^{m_1 x} \int X e^{-m_1 x} dx + A_2 . e^{m_2 x} \int X e^{-m_2 x} dx + \ldots + A_n . e^{m_n x} \int X e^{-m_n x} dx \qquad \text{[By §13.5 ...(3)]}$$

Obs. This method is a general one and can, therefore, be employed to obtain a particular integral in any given case.

13.7 WORKING PROCEDURE TO SOLVE THE EQUATION

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + k_{n-1} \frac{dy}{dx} + k_n y = X$$

of which the *symbolic form* is

$$(D^n + k_1 D^{n-1} + \ldots + k_{n-1} D + k_n) y = X .$$

Step I. To find the complementary function

(*i*) *Write the A.E.*

i.e., $\quad D^n + k_1 D^{n-1} + \ldots + k_{n-1} D + k_n = 0$ and solve it for D.

(*ii*) Write the *C.F.* as follows :

Roots of A.E.	*C.F.*
1. $m_1, m_2, m_3 \ldots$ (real and different roots)	$c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots$
2. $m_1, m_1, m_3 \ldots$ (two real and equal roots)	$(c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \ldots$
3. $m_1, m_1, m_1, m_4 \ldots$ (three real and equal roots)	$(c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + \ldots$
4. $\alpha + i\beta, \alpha - i\beta, m_3 \ldots$ (a pair of imaginary roots)	$e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + \ldots$
5. $\alpha \pm i\beta, \alpha \pm i\beta, m_5 \ldots$ (2 pairs of equal imaginary roots)	$e^{\alpha x}[(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x] + c_5 e^{m_5 x} + \ldots$

Step II. To find the particular integral

From symbolic form P.I. $= \dfrac{1}{D^n + k_1 D^{n-1} + \ldots + k_{n-1} D + k_n} X = \dfrac{1}{f(D)}$ or $\dfrac{1}{\phi(D^2)} X$

(*i*) *When* $X = e^{ax}$

$$\text{P.I.} = \frac{1}{f(D)} e^{ax}, \text{ put } D = a, \qquad [f(a) \neq 0]$$

$$= x \frac{1}{f'(D)} e^{ax}, \text{ put } D = a, \qquad [f(a) = 0, f'(a) \neq 0]$$

$$= x^2 \frac{1}{f''(D)} e^{ax}, \text{ put } D = a, \qquad [f'(a) = 0, f''(a) \neq 0]$$

and so on.

where $\quad f'(D) =$ diff. coeff. of $f(D)$ w.r.t. D

$f''(D) =$ diff. coeff. of $f'(D)$ w.r.t. D, etc.

(*ii*) *When* $X = \sin(ax + b)$ or $\cos(ax + b)$.

$$\text{P.I.} = \frac{1}{\phi(D^2)} \sin(ax + b) \text{ [or } \cos(ax + b)\text{], put } D^2 = -a^2 \qquad [\phi(-a^2) \neq 0]$$

$$= x \frac{1}{\phi'(D^2)} \sin(ax + b) \text{ [or } \cos(ax + b)\text{], put } D^2 = -a^2 \qquad [\phi(-a^2) = 0, \phi'(-a^2) \neq 0]$$

$$= x^2 \frac{1}{\phi''(D^2)} \sin(ax + b) \text{ [or } \cos(ax + b)\text{], put } D^2 = -a^2 \qquad [\phi'(-a^2) \neq 0, \phi''(-a^2) \neq 0]$$

and so on.

where $\phi'(D^2)$ = diff. coeff. of $\phi(D^2)$ w.r.t. D,

$\phi''(D^2)$ = diff. coeff. of $\phi'(D^2)$ w.r.t. D, etc.

(*iii*) *When* $X = x^m$, m being a positive integer.

$$\text{P.I.} = \frac{1}{f(D)} x^m = [f(D)]^{-1} x^m$$

To evaluate it, expand $[f(D)]^{-1}$ in ascending powers of D by Binomial theorem as far as D^m and operate on x^m term by term.

(*iv*) *When* $X = e^{ax}V$, where V is a function of x.

$$\text{P.I.} = \frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D + a)} V$$

and then evaluate $\frac{1}{f(D + a)} V$ as in (*i*), (*ii*), and (*iii*).

(*v*) *When X is any function of x.*

$$\text{P.I.} = \frac{1}{f(D)} X$$

Resolve $\frac{1}{f(D)}$ into partial fractions and operate each partial fraction on X remembering that

$$\frac{1}{D - a} X = e^{ax} \int X e^{-ax} \, dx.$$

Step III. To find the complete solution

Then the C.S. is y = C.F. + P.I.

Example 13.12. *Solve* $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = (1 - e^x)^2$.

Solution. Given equation in *symbolic form* is $(D^2 + D + 1)y = (1 - e^x)^2$

(*i*) *To find C.F.*

Its A.E. is $D^2 + D + 1 = 0$, $\therefore \quad D = \frac{1}{2}(-1 + \sqrt{3}i)$

Thus C.F. $= e^{-x/2} \left(c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x \right)$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2 + D + 1}(1 - 2e^x + e^{2x}) = \frac{1}{D^2 + D + 1}(e^{0x} - 2e^x + e^{2x})$$

$$= \frac{1}{0^2 + 0 + 1} e^{0x} - 2 \cdot \frac{1}{1^2 + 1 + 1} e^x + \frac{1}{2^2 + 2 + 1} e^{2x} = 1 - \frac{2}{3} e^x + \frac{e^{2x}}{7}$$

(*iii*) Hence the C.S. is $y = e^{-x/2} \left(c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x \right) + 1 - \frac{2}{3} e^x + \frac{e^{2x}}{7}$.

Example 13.13. *Solve $y'' + 4y' + 4y = 3 \sin x + 4 \cos x$, $y(0) = 1$ and $y'(0) = 0$.* (J.N.T.U., 2003)

Solution. Given equation in *symbolic form* is $(D^2 + 4D + 4)y = 3 \sin x + 4 \cos x$

(i) *To find C.F.*

Its A.E. is $(D + 2)^2 = 0$ where $D = -2, -2$ $\therefore$ C.F. $= (c_1 + c_2 x)e^{-2x}$.

(ii) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2 + 4D + 4}(3 \sin x + 4 \cos x) = \frac{1}{-1 + 4D + 4}(3 \sin x + 4 \cos x)$$

$$= \frac{4D - 3}{16D^2 - 9}(3 \sin x + 4 \cos x) = \frac{(4D - 3)}{-16 - 9}(3 \sin x + 4 \cos x)$$

$$= \frac{-1}{25}\{3(4 \cos x - 3 \sin x) + 4(-4 \sin x - 3 \cos x)\} = \sin x$$

(iii) C.S. is $y = (c_1 + c_2)\, e^{-2x} + \sin x$

When $x = 0, y = 1,$ $\therefore$ $1 = c_1$

Also $y' = c_2 e^{-2x} + (c_1 + c_2 x)(-2)e^{-2x} + \cos x$.

When $x = 0, y' = 0,$ $\therefore$ $0 = c_2 - 2c_1 + 1$, *i.e.*, $c_2 = 1$.

Hence the required solution is $y = (1 + x)\, e^{-2x} + \sin x$.

Example 13.14. *Solve $(D - 2)^2 = 8(e^{2x} + \sin 2x + x^2)$.*

Solution. (i) *To find C.F.*

Its A.E. is $(D - 2)^2 = 0$, $\therefore$ $D = 2, 2$.

Thus C.F. $= (c_1 + c_2 x)\, e^{2x}$.

(ii) *To find P.I.*

$$\text{P.I.} = 8\left[\frac{1}{(D-2)^2} e^{2x} + \frac{1}{(D-2)^2} \sin 2x + \frac{1}{(D-2)^2} x^2\right]$$

Now $\frac{1}{(D-2)^2} e^{2x} = x^2 \frac{1}{2(1)} e^{2x}$ [$\because$ by putting $D = 2$, $(D - 2)^2 = 0$, $2(D - 2) = 0$]

$$= \frac{x^2 e^{2x}}{2}$$

$$\frac{1}{(D-2)^2} \sin 2x = \frac{1}{D^2 - 4D + 4} \sin 2x = \frac{1}{(-2^2) - 4D + 4} \sin 2x$$

$$= -\frac{1}{4} \int \sin 2x \, dx = -\frac{1}{4}\left(\frac{-\cos 2x}{2}\right) = \frac{1}{8} \cos 2x$$

and

$$\frac{1}{(D-2)^2} x^2 = \frac{1}{4}\left(1 - \frac{D}{2}\right)^{-2} x^2 = \frac{1}{4}\left[1 + (-2)\left(\frac{D}{2}\right) + \frac{(-2)(-3)}{2!}\left(-\frac{D}{2}\right)^2 + \ldots\right] x^2$$

$$= \frac{1}{4}\left(1 + D + \frac{3D^2}{4} + \ldots\right) x^2 = \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right)$$

Thus P.I. $= 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3$.

(iii) Hence the C.S. is $y = (c_1 + c_2 x)\, e^{2x} + 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3$.

Example 13.15. *Find the complete solution of $y'' - 2y' + 2y = x + e^x \cos x$.* (U.P.T.U., 2002)

Solution. Given equation in *symbolic form* is $(D^2 - 2D + 2)y = x + e^x \cos x$

(i) *To find C.F.*

Its A.E. is $D^2 - 2D + 2 = 0$ $\therefore$ $D = \frac{2 \pm \sqrt{(4-8)}}{2} = 1 \pm i$.

Thus C.F. $= e^x (c_1 \cos x + c_2 \sin x)$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2 - 2D + 2}(x) + \frac{1}{D^2 - 2D + 2}(e^x \cos x)$$

$$= \frac{1}{2}\left[1 - \left(D - \frac{D^2}{2}\right)\right]^{-1}(x) + e^x \frac{1}{(D+1)^2 - 2(D+1) + 2}(\cos x)$$

$$= \frac{1}{2}\left(1 + D - \frac{D^2}{2}\right)x + e^x \frac{1}{D^2 + 1}\cos x \qquad \text{[Case of failure]}$$

$$= \frac{1}{2}(x + 1 - 0) + e^x \cdot x \frac{1}{2D}\cos x = \frac{1}{2}(x+1) + \frac{xe^x}{2}\int \cos x\, dx = \frac{1}{2}(x+1) + \frac{xe^x}{2}\sin x$$

(*iii*) Hence the C.S. is $y = e^x(c_1 \cos x + c_2 \sin x) + \frac{1}{2}(x+1) + \frac{xe^x}{2}\sin x$.

Example 13.16. *Solve* $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^{3x} + \sin 2x.$

(*V.T.U., 2008 ; Kottayam, 2005 ; U.P.T.U., 2003*)

Solution. Given equation in *symbolic form* is $(D^2 - 3D + 2)y = xe^{3x} + \sin 2x$

(*i*) *To find C.F.*

Its A.E. is $D^2 - 3D + 2 = 0$ or $(D - 2)(D - 1) = 0$ whence $D = 1, 2$.

Thus C.F. $= c_1e^x + c_2e^{2x}$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2 - 3D + 2}(xe^{3x} + \sin 2x) = \frac{1}{D^2 - 3D + 2}(e^{3x} \cdot x) + \frac{1}{D^2 - 3D + 2}(\sin 2x)$$

$$= e^{3x} \cdot \frac{1}{(D+3)^2 - 3(D+3) + 2}(x) + \frac{1}{-4 - 3D + 2}(\sin 2x)$$

$$= e^{3x} \cdot \frac{1}{D^2 + 3D + 2}(x) - \frac{3D - 2}{9D^2 - 4}(\sin 2x) = \frac{e^{3x}}{2} \cdot \left[1 + \left\{\frac{3D + D^2}{2}\right\}\right]^{-1} x - \frac{(3D - 2)}{9(-4) - 4}(\sin 2x)$$

$$= \frac{e^{3x}}{2}\left(1 - \frac{3D}{2}\ldots\right)x + \frac{1}{40}(6\cos 2x - 2\sin 2x) = \frac{e^{3x}}{2}\left(x - \frac{3}{2}\right) + \frac{1}{20}(3\cos 2x - \sin 2x)$$

(*iii*) Hence the C.S. is $y = c_1e^x + c_2e^{2x} + e^{3x}\left(\frac{x}{2} - \frac{3}{4}\right) + \frac{1}{20}(3\cos 2x - \sin 2x)$.

Example 13.17. *Solve* $\frac{d^2y}{dx^2} - 4y = x \sinh x.$ (*Madras, 2000 S*)

Solution. Given equation in *symbolic form* is $(D^2 - 4)y = x \sinh x$.

(*i*) *To find C.F.*

Its A.E. is $D^2 - 4 = 0$, whence $D = \pm 2$.

Thus C.F. $= c_1e^{2x} + c_2e^{-2x}$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2 - 4}x \sinh x = \frac{1}{D^2 - 4}x\left(\frac{e^x - e^{-x}}{2}\right) = \frac{1}{2}\left[\frac{1}{D^2 - 4}e^x \cdot x - \frac{1}{D^2 - 4}e^{-x} \cdot x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{(D+1)^2 - 4}x - e^{-x}\frac{1}{(D-1)^2 - 4}x\right] = \frac{1}{2}\left[e^x \frac{1}{D^2 + 2D - 3}x - e^{-x}\frac{1}{D^2 - 2D - 3}x\right]$$

$$= \frac{1}{2}\left[\frac{e^x}{-3}\left\{1-\left(\frac{2D}{3}+\frac{D^2}{3}\right)\right\}^{-1}.x-\frac{e^{-x}}{-3}\left\{1+\left(\frac{2D}{3}-\frac{D^2}{3}\right)\right\}^{-1}\cdot x\right]$$

$$= -\frac{1}{6}\left[e^x\left(1+\frac{2D}{3}+...\right)x-e^{-x}\left(1-\frac{2D}{3}+...\right)x\right] = -\frac{1}{6}\left[e^x\left(x+\frac{2}{3}\right)-e^{-x}\left(x-\frac{2}{3}\right)\right]$$

$$= -\frac{x}{3}\left(\frac{e^x-e^{-x}}{2}\right)-\frac{2}{9}\left(\frac{e^x+e^{-x}}{2}\right) = -\frac{x}{3}\sinh x-\frac{2}{9}\cosh x.$$

(*iii*) Hence the C.S. is $y = c_1e^{2x}+c_2e^{-2x}-\frac{x}{3}\sinh x-\frac{2}{9}\cosh x$.

Example 13.18. *Solve* $(D^2-1)y = x\sin 3x+\cos x$.

Solution. (*i*) *To find C.F.*

Its A.E. is $D^2-1=0$, whence $D=\pm 1$. $\therefore$ C.F. $= c_1e^x+c_2e^{-x}$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2-1}(x\sin 3x+\cos x) = \frac{1}{D^2-1}x\,(\text{I.P. of } e^{3ix})+\frac{1}{D^2-1}\cos x$$

$$= \text{I.P. of } \frac{1}{D^2-1}e^{3ix}.x+\frac{1}{(-1)^2-1}\cos x = \text{I.P. of }\left[e^{3ix}\frac{1}{(D+3i)^2-1}x\right]-\frac{\cos x}{2}$$

[Replacing D by $D+3i$]

$$= \text{I.P. of }\left[e^{3ix}\frac{1}{D^2+6iD-10}x\right]-\frac{\cos x}{2}$$

$$= \text{I.P. of }\left[e^{3ix}.\frac{1}{-10}\left(1-\frac{3iD}{5}-\frac{D^2}{10}\right)^{-1}x\right]-\frac{\cos x}{2}$$ [Expand by Binomial theorem]

$$= \text{I.P. of }\left[e^{3ix}.\frac{1}{-10}\left(1+\frac{3iD}{5}+......\right)x\right]-\frac{\cos x}{2} = \text{I.P. of }\left[-\frac{e^{3ix}}{10}\left(x+\frac{3i}{5}\right)\right]-\frac{\cos x}{2}$$

$$= \text{I.P. of }\left[\frac{-1}{10}(\cos 3x+i\sin 3x)\left(x+\frac{3i}{5}\right)\right]-\frac{\cos x}{2}$$

$$= -\frac{1}{10}\,\text{I.P. of }\left[\left(x\cos 3x-\frac{3\sin 3x}{5}\right)+i\left(x\sin 3x+\frac{3}{5}\cos 3x\right)\right]-\frac{\cos x}{2}$$

$$= -\frac{1}{10}\left(x\sin 3x+\frac{3}{5}\cos 3x\right)-\frac{\cos x}{2}.$$

(*iii*) Hence the C.S. is $y = c_1e^x+c_2e^{-x}-\frac{1}{50}(5x\sin 3x+3\cos 3x+25\cos x)$.

Example 13.19. *Solve* $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y = xe^x\sin x$. (*S.V.T.U., 2007 ; J.N.T.U., 2006 ; U.P.T.U., 2005*)

Solution. Given equation in *symbolic form* is $(D^2-2D+1)y = xe^x\sin x$

(*i*) *To find C.F.*

Its A.E. is $D^2-2D+1=0$, *i.e.*, $(D-1)^2=0$

$\therefore$ $D = 1, 1$. Thus C.F. $= (c_1+c_2x)e^x$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{(D-1)^2} e^x \,.\, x \sin x = e^x \,.\, \frac{1}{(D+1-1)^2} x \sin x$$

$$= e^x \frac{1}{D^2} x \sin x = e^x \frac{1}{D} \int x \sin x \, dx \qquad \text{[Integrate by parts]}$$

$$= e^x \frac{1}{D}\left[x(-\cos x) - \int 1 \,.\, (-\cos x)\, dx\right] = e^x \int [-x \cos x + \sin x]\, dx$$

$$= e^x \left[-\left\{x \sin x - \int 1 \,.\, \sin x \, dx\right\} - \cos x\right] = e^x[-x \sin x - \cos x - \cos x]$$

$$= -e^x(x \sin x + 2 \cos x).$$

(*iii*) Hence the C.S. is $y = (c_1 + c_2 x)\, e^x - e^x(x \sin x + 2 \cos x)$.

Example 13.20. *Solve* $(D^4 + 2D^2 + 1)\, y = x^2 \cos x.$ *(Nagarjuna, 2008 ; Rajasthan, 2005)*

Solution. (*i*) *To find C.F.*

Its A.E. is $(D^2 + 1)^2 = 0$ whose roots are $D = \pm i, \pm i$

$\therefore$ C.F. $= (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{(D^2+1)^2} x^2 \cos x = \frac{1}{(D^2+1)^2} x^2 (\text{Re.P. of } e^{ix})$$

$$= \text{Re.P. of} \left\{\frac{1}{(D^2+1)^2} e^{ix} \,.\, x^2\right\} = \text{Re.P. of} \left\{e^{ix} \frac{1}{[(D+i)^2+1]^2} x^2\right\}$$

$$= \text{Re.P. of} \left\{e^{ix} \frac{1}{(D^2+2iD)^2} x^2\right\} = \text{Re.P. of} \left[e^{ix}\left\{-\frac{1}{4D^2}\left(1 - \frac{i}{2}D\right)^{-2} x^2\right\}\right]$$

$$= \text{Re.P. of} \left[-\frac{1}{4} e^{ix} \,.\, \frac{1}{D^2}\left\{1 + 2\frac{iD}{2} + 3\left(\frac{iD}{2}\right)^2 + \ldots\right\} x^2\right]$$

$$= \text{Re.P. of} \left\{-\frac{1}{4} e^{ix} \,.\, \frac{1}{D^2}\left(x^2 + 2ix - \frac{3}{2}\right)\right\} = \text{Re.P. of} \left\{-\frac{1}{4} e^{ix} \,.\, \frac{1}{D}\left(\frac{x^3}{3} + ix^2 - \frac{3}{2}x\right)\right\}$$

$$= -\frac{1}{4} \text{Re.P. of} \left\{e^{ix}\left(\frac{x^4}{12} + \frac{ix^3}{3} - \frac{3}{4}x^2\right)\right\} = \frac{-1}{48} \text{ Re.P. of } \{(\cos x + i \sin x)(x^4 + 4ix^3 - 9x^2)\}$$

$$= -\frac{1}{48} [(x^4 - 9x^2) \cos x - 4x^3 \sin x]$$

(*iii*) Hence the C.S. is $y = (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x + \dfrac{1}{48}\, [4x^3 \sin x - x^2 (x^2 - 9) \cos x]$.

Example 13.21. *Solve* $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x.$ *(J.N.T.U., 2006 ; U.P.T.U., 2004)*

Solution. (*i*) *To find C.F.*

Its A.E. is $D^2 - 4D + 4 = 0$ *i.e.*, $(D-2)^2 = 0$. $\therefore$ $D = 2, 2$

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^{2x}$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{(D-2)^2} (8x^2 e^{2x} \sin 2x) = 8e^{2x} \frac{1}{(D+2-2)^2} (x^2 \sin 2x)$$

$$= 8e^{2x} \frac{1}{D^2} (x^2 \sin 2x) = 8e^{2x} \,.\, \frac{1}{D} \int x^2 \sin 2x \, dx$$

$$= 8e^{2x} \cdot \frac{1}{D}\left\{x^2\left(\frac{-\cos 2x}{2}\right) - \int 2x\left(\frac{-\cos 2x}{2}\right)dx\right\}$$

$$= 8e^{2x}\frac{1}{D}\left\{-\frac{x^2}{2}\cos 2x + x\frac{\sin 2x}{2} - \int 1 . \frac{\sin 2x}{2}\,dx\right\}$$

$$= 8e^{2x}\int\left\{-\frac{x^2}{2}\cos 2x + \frac{x}{2}\sin 2x + \frac{\cos 2x}{4}\right\}dx$$

$$= 8e^{2x}\left[\left\{\frac{-x^2}{2}\frac{\sin 2x}{2} - \int(-x)\frac{\sin 2x}{2}dx\right\} + \left\{\int\frac{x}{2}\sin 2x\,dx\right\} + \frac{\sin 2x}{8}\right]$$

$$= 8e^{2x}\left[\left(\frac{-x^2}{4} + \frac{1}{8}\right)\sin 2x + \int x\sin 2x\,dx\right]$$

$$= 8e^{2x}\left[\left(\frac{1}{8} - \frac{x^2}{4}\right)\sin 2x + x\left(\frac{-\cos 2x}{2}\right) - \int 1 . \left(\frac{-\cos 2x}{2}\right)dx\right]$$

$$= 8e^{2x}\left[\left(\frac{1}{8} - \frac{x^2}{4}\right)\sin 2x - \frac{x\cos 2x}{2} + \frac{\sin 2x}{4}\right]$$

$= e^{2x}[(3 - 2x^2)\sin 2x - 4x\cos 2x]$

(*iii*) Hence the C.S. is $y = e^{2x}[c_1 + c_2x + (3 - 2x^2)\sin 2x - 4x\cos 2x]$.

Example 13.22. *Solve* $\dfrac{d^2y}{dx^2} + a^2y = \sec ax.$

Solution. Given equation in *symbolic form* is $(D^2 + a^2)y = \sec ax$.

(*i*) *To find C.F.*

Its A.E. is $D^2 + a^2 = 0 \quad \therefore \quad D = \pm ia$.

Thus C.F. $= c_1\cos ax + c_2\sin ax$.

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2 + a^2}\sec ax = \frac{1}{(D + ia)(D - ia)}\sec ax \qquad \text{[Resolving into partial fractions]}$$

$$= \frac{1}{2ia}\left[\frac{1}{D - ia} - \frac{1}{D + ia}\right]\sec ax = \frac{1}{2ia}\left[\frac{1}{D - ia}\sec ax - \frac{1}{D + ia}\sec ax\right]$$

Now $\quad \dfrac{1}{D - ia}\sec ax = e^{iax}\displaystyle\int \sec ax . e^{-iax}\,dx \qquad \left[\because \dfrac{1}{D - a}X = e^{ax}\displaystyle\int Xe^{-ax}\,dx\right]$

$$= e^{iax}\int\frac{\cos ax - i\sin ax}{\cos ax}dx = e^{iax}\int(1 - i\tan ax)dx = e^{iax}\left(x + \frac{i}{a}\log\cos ax\right)$$

Changing i to $-i$, we have

$$\frac{1}{D + ia}\sec ax = e^{-iax}\left\{x - \frac{i}{a}\log\cos ax\right\}$$

Thus $\quad \text{P.I.} = \dfrac{1}{2ia}\left[e^{iax}\left\{x + \dfrac{i}{a}\log\cos ax\right\} - e^{-iax}\left\{x - \dfrac{i}{a}\log\cos ax\right\}\right]$

$$= \frac{x}{a}\frac{e^{iax} - e^{-iax}}{2i} + \frac{1}{a^2}\log\cos ax . \frac{e^{iax} + e^{-iax}}{2} = \frac{x}{a}\sin ax + \frac{1}{a^2}\log\cos ax . \cos ax .$$

(*iii*) Hence the C.S. is

$$y = c_1\cos ax + c_2\sin ax + (1/a)x\sin ax + (1/a^2)\cos ax\log\cos ax.$$

PROBLEMS 13.2

Solve :

1. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 6e^{3x} + 7e^{-2x} - \log 2$ (V.T.U., 2005)

2. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = -2\cosh x$. Also find y when $y = 0$, $\frac{dy}{dx} = 1$ at $x = 0$.

3. $\frac{d^2x}{dt^2} + n^2x = k\cos(nt + \alpha)$.

4. $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 3x = \sin t$.

5. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4\cos^2 x$. (Bhopal, 2002 S)

6. $(D^2 - 4D + 3)y = \sin 3x \cos 2x$. (Madras, 2000)

7. $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{-x} + \sin 2x$. (V.T.U., 2004)

8. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{2x} - \cos^2 x$. (Delhi, 2002)

9. $(D^3 - 5D^2 + 7D - 3)y = e^{2x}\cosh x$. (Nagarjuna, 2008)

10. $\frac{d^2y}{dx^2} - y = e^x + x^2e^x$. (Nagpur, 2009)

11. $(D^3 - D)y = 2x + 1 + 4\cos x + 2e^x$. (Mumbai, 2006)

12. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = e^{2x} + \sin x + x$. (V.T.U., 2006)

13. $(D^2 + 1)^2 y = x^4 + 2\sin x \cos 3x$. (Madras, 2006)

14. $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-2x}\sin 2x$. (Bhopal, 2008)

15. $(D^4 + D^2 + 1)y = e^{-x/2}\cos\frac{\sqrt{3}}{2}x$. (Rajasthan, 2006)

16. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = e^x\cos x$. (V.T.U., 2010)

17. $(D^2 + 4D + 3)y = e^{-x}\sin x + xe^{3x}$. (Raipur, 2005 ; Anna, 2002 S)

18. $\frac{d^2y}{dx^2} + 2y = x^2e^{3x} + e^x\cos 2x$.

19. $\frac{d^4y}{dx^4} - y = \cos x \cosh x$.

20. $(D^3 + 2D^2 + D)y = x^2e^{2x} + \sin^2 x$. (P.T.U., 2003)

21. $\frac{d^2y}{dx^2} + 16y = x\sin 3x$. (V.T.U., 2010 S)

22. $(D^2 + 2D + 1)y = x\cos x$. (Rajasthan, 2006)

23. $(D^2 - 1)y = x\sin x + (1 + x^2)e^x$.

24. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$. (S.V.T.U., 2009)

25. $(D^2 + a^2)y = \tan ax$. (V.T.U., 2005)

13.8 TWO OTHER METHODS OF FINDING P.I.

I. Method of variation of parameters. *This method is quite general and applies to equations of the form*

$$y'' + py' + qy = X \quad ...(1)$$

where p, q, and X are functions of x. It gives $P.I. = -y_1\int\frac{y_2X}{W}dx + y_2\int\frac{y_1X}{W}dx \quad ...(2)$

where y_1 *and* y_2 *are the solutions of* $y'' + py' + qy = 0 \quad ...(3)$

and $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ *is called the Wronskian* of* y_1, y_2.

Proof. Let the C.F. of (1) be $y = c_1y_1 + c_2y_2$

Replacing c_1, c_2 (regarded as *parameters*) by unknown functions $u(x)$ and $v(x)$, let the P.I. be

$$y = uy_1 + vy_2 \quad ...(4)$$

Differentiating (4) w.r.t. x, we get $y' = uy'_1 + vy'_2 + u'y_1 + v'y_2$

*Named after the Polish mathematician and philosopher *Hoene Wronsky* (1778–1853).

$$= uy_1' + vy_2' \quad ...(5)$$

on assuming that $u'y_1 + v'y_2 = 0$ ...(6)

Differentiate (4) and substitute in (1). Then noting that y_1 and y_2, satisfy (3), we obtain

$$u'y_1' + v'y_2' = X \quad ...(7)$$

Solving (6) and (7), we get

$$u' = -\frac{y_2 X}{W}, v' = \frac{y_1 X}{W} \qquad \text{where } W = y_1 y_2' - y_2 y_1'$$

Integrating $u = -\int \frac{y_2 X}{W}\, dx$, $v = \int \frac{y_1 X}{W}\, dx$. Substituting these in (4), we get (2).

Example 13.23. *Using the method of variation of parameters, solve*

$$\frac{d^2y}{dx^2} + 4y = \tan 2x. \qquad \text{(V.T.U., 2008 ; Bhopal, 2007 ; S.V.T.U., 2006 S)}$$

Solution. Given equation in *symbolic form* is $(D^2 + 4)y = \tan 2x$.

(i) *To find C.F.*

Its A.E. is $D^2 + 4 = 0$, $\therefore$ $D = \pm 2i$

Thus C.F. is $y = c_1 \cos 2x + c_2 \sin 2x$.

(ii) *To find P.I.*

Here $y_1 = \cos 2x$, $y_2 = \sin 2x$ and $X = \tan 2x$

$$\therefore \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

Thus,

$$\text{P.I.} = -y_1 \int \frac{y_2 X}{W}\, dx + y_2 \int \frac{y_1 X}{W}\, dx$$

$$= -\cos 2x \int \frac{\sin 2x \tan 2x}{2}\, dx + \sin 2x \int \frac{\cos 2x \tan 2x}{2}\, dx$$

$$= -\frac{1}{2}\cos 2x \int (\sec 2x - \cos 2x)dx + \frac{1}{2}\sin 2x \int \sin 2x\, dx$$

$$= -\frac{1}{4}\cos 2x[\log(\sec 2x + \tan 2x) - \sin 2x] - \frac{1}{4}\sin 2x \cos 2x$$

$$= -\frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$$

Hence the C.S. is $y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4}\cos 2x \log(\sec 2x + \tan 2x)$.

Example 13.24. *Solve, by the method of variation of parameters,* $d^2y/dx^2 - y = 2/(1 + e^x)$.

(V.T.U., 2005 ; Hissar, 2005)

Solution. Given equation is $D^2 - 1 = 2/(1 + e^x)$

A.E. is $D^2 - 1 = 0$, $D = \pm 1$, $\therefore$ C.F. $= c_1 e^x + c_2 e^{-x}$

Here $y_1 = e^x$, $y_2 = e^{-x}$ and $X = 2/(1 + e^x)$

$$\therefore \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -e^x e^{-x} - e^x e^{-x} = -2.$$

Thus

$$\text{P.I.} = -y_1 \int \frac{y_2 X}{W}\, dx + y_2 \int \frac{y_1 X}{W}\, dx = -e^x \int \frac{e^{-x}}{-2}\cdot\frac{2}{1+e^x}\, dx + e^{-x}\int \frac{e^x}{-2}\cdot\frac{2}{1+e^x}\, dx$$

$$= e^x \int \left(\frac{1}{e^x} - \frac{1}{1+e^x}\right)dx - e^{-x}\log(1+e^x) = e^x\left[e^{-x} - \int \frac{e^{-x}}{e^{-x}+1}\, dx\right] - e^{-x}\log(1+e^x)$$

$$= e^x[-e^{-x} + \log(e^{-x}+1)] - e^{-x}\log(1+e^x) = -1 + e^x \log(e^{-x}+1) - e^{-x}\log(e^x+1)$$

Hence C.S. is $\quad y = c_1 e^x + c_2 e^{-x} - 1 + e^x \log(e^{-x} + 1) - e^{-x}\log(e^x + 1)$.

Example 13.25. *Solve by the method of variation of parameters* $y'' - 6y' + 9y = e^{3x}/x^2$.
(Nagpur, 2009 ; S.V.T.U., 2009)

Solution. Given equation is $(D^2 - 6D + 9)y = e^{3x}/x^2$

A.E. is $D^2 - 6D + 9 = 0$ *i.e.* $(D-3)^2 = 0$ $\therefore$ C.F. $= (c_1 + c_2 x)e^{3x}$

Here $y_1 = e^{3x}$, $y_2 = xe^{3x}$ and $X = e^{3x}/x^2$

$$\therefore \qquad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x} + 3xe^{3x} \end{vmatrix} = e^{6x}.$$

Thus
$$\text{P.I.} = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx = -e^{3x} \int \frac{xe^{3x}}{e^{6x}} \frac{e^{3x}}{x^2} dx + xe^{3x} \int \frac{e^{3x}}{e^{6x}} \cdot \frac{e^{3x}}{x^2} dx$$
$$= -e^{3x} \int \frac{dx}{x} + xe^{3x} \int x^{-2}\, dx = -e^{3x}(\log x + 1)$$

Hence C.S. is $\quad y = (c_1 + c_2 x)e^{3x} - e^{3x}(\log x + 1)$.

Example 13.26. *Solve, by the method of variation of parameters,* $y'' - 2y' + y = e^x \log x$.
(V.T.U., 2006 ; Kurukshetra, 2005 ; Madras, 2003)

Solution. Given equation in symbolic form is $(D^2 - 2D + 1)\, y = e^x \log x$

(i) To find C.F.

Its A.E. is $(D-1)^2 = 0$, $\therefore$ $D = 1, 1$

Thus C.F. is $y = (c_1 + c_2 x)e^x$

(ii) To find P.I.

Here $y_1 = e^x$, $y_2 = xe^x$ and $X = e^x \log x$

$$\therefore \qquad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & (1+x)\, e^x \end{vmatrix} = e^{2x}$$

Thus
$$\text{P.I.} = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$$
$$= -e^x \int \frac{xe^x \cdot e^x \log x}{e^{2x}} dx + xe^x \int \frac{e^x \cdot e^x \log x}{e^{2x}} dx = -e^x \int x \log x\, dx + xe^x \int \log x\, dx$$
$$= -e^x \left(\frac{x^2}{2} \log x - \int \frac{1}{x} \cdot \frac{x^2}{2} dx \right) + x \cdot e^x \left(x \log x - \int \frac{1}{x} \cdot x\, dx \right)$$
$$= -e^x \left(\frac{x^2}{2} \log x - \frac{x^2}{4} \right) + x\; e^x (x \log x - x) = \frac{1}{4}\, x^2 e^x (2 \log x - 3)$$

Hence C.S. is $y = (c_1 + c_2 x)\, e^x + \dfrac{1}{4}\, x^2 e^x (2 \log x - 3)$.

II. Method of undetermined coefficients

To find the P.I. of $f(D)\, y = X$, we assume a trial solution containing unknown constants which are determined by substitution in the given equation. The trial solution to be assumed in each case, depends on the form of X. Thus when *(i)* $X = 2e^{3x}$, trial solution $= ae^{3x}$.

(ii) $X = 3 \sin 2x$, trial solution $= a_1 \sin 2x + a_2 \cos 2x$

(iii) $X = 2x^3$, trial solution $= a_1 x^3 + a_2 x^2 + a_3 x + a_4$

However when $X = \tan x$ or $\sec x$, this method fails, since the number of terms obtained by differentiating $X = \tan x$ or $\sec x$ is infinite.

The above method holds so long as no term in the trial solution appears in the C.F. If any term of the trial solution appears in the C.F., we multiply this trial solution by the lowest positive integral power of x which is large enough so that none of the terms which are then present, appear in the C.F.

Example 13.27. *Solve* $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 2x^2 + 3e^{-x}$. *(V.T.U., 2008)*

Solution. Here C.F. $= e^{-x}(c_1 \cos\sqrt{3}\,x + c_2 \sin\sqrt{3}\,x)$

Assume P.I. as $y = a_1x^2 + a_2x + a_3 + a_4e^{-x}$

$\therefore$ $Dy = 2a_1x + a_2 - a_4e^{-x}$ and $D^2y = 2a_1 + a_4e^{-x}$

Substituting these in the given equation, we get

$$4a_1x^2 + (4a_1 + 4a_2)x + (2a_1 + 2a_2 + 4a_3) + 3a_4e^{-x} = 2x^2 + 3e^{-x}$$

Equating corresponding coefficients on both sides, we get

$$4a_1 = 2,\ 4a_1 + 4a_2 = 0,\ 2a_1 + 2a_2 + 4a_3 = 0,\ 3a_4 = 3$$

Then $a_1 = \frac{1}{2}, a_2 = -\frac{1}{2}, a_3 = 0, a_4 = 1$. Thus P.I. $= \frac{1}{2}x^2 - \frac{1}{2}x + e^{-x}$

$\therefore$ C.S. is $y = e^{-x}(c_1 \cos\sqrt{3x} + c_2 \sin\sqrt{3x}) + \frac{1}{2}x^2 - \frac{1}{2}x + e^{-x}$.

Example 13.28. *Solve* $(D^2 + 1)y = \sin x$.

Solution. Here C.F. $= c_1 \cos x + c_2 \sin x$

We would normally assume a trial solution as $a_1 \cos x + a_2 \sin x$.

However, since these terms appear in the C.F., we multiply by x and assume the trial P.I. as

$$y = x(a_1 \cos x + a_2 \sin x)$$

$\therefore$ $Dy = (a_1 + a_2x)\cos x + (a_2 - a_1x)\sin x$ and $D^2y = (2a_2 - a_1x)\cos x - (2a_1 + a_2x)\sin x$

Substituting these in the given equation, we get $2a_1 \cos x - 2a_2 \sin x = \sin x$

Equating corresponding coefficients,

$$2a_1 = 0, \quad -2a_2 = 1 \quad \text{so that } a_1 = 0, a_2 = -\frac{1}{2}. \quad \text{Thus P.I.} = -\frac{1}{2}x\sin x$$

$\therefore$ C.S. is $y = c_1 \cos x + c_2 \sin x - \frac{1}{2}x \sin x$.

Example 13.29. *Solve by the method of undetermined coefficients,*

$$\frac{d^2y}{dx^2} - y = e^{3x}\cos 2x - e^{2x}\sin 3x.$$

Solution. Its A.E. is $D^2 - 1 = 0$, $\therefore$ $D = \pm 1$.

Thus C.F. $= c_1e^x + c_2e^{-x}$

Assume P.I. as $y = e^{3x}(c_1 \cos 2x + c_2 \sin 2x) - e^{2x}(c_3 \cos 3x + c_4 \sin 3x)$

$$\therefore \quad \frac{dy}{dx} = e^{3x}\{(3c_1 + 2c_2)\cos 2x + (3c_2 - 2c_1)\sin 2x\} - e^{2x}\{(2c_3 + 3c_4)\cos 3x + (2c_4 - 3c_3)\sin 3x\}$$

and
$$\frac{d^2y}{dx^2} = e^{3x}\{(5c_1 + 12c_2)\cos 2x + (5c_2 - 12c_1)\sin 2x\} - e^{2x}\{(12c_4 - 5c_3)\cos 3x - (5c_4 + 12c_3)\sin 3x\}$$

Substituting these in the given equation, we get

$$e^{3x}\{(4c_1 + 12c_2)\cos 2x + (4c_2 - 12c_1)\sin 2x\} - e^{2x}\{(12c_4 - 6c_3)\cos 3x - (6c_4 + 12c_3)\sin 3x\}$$
$$= e^{3x}\cos 2x - e^{2x}\sin 3x$$

Equating corresponding coefficients,

$$4c_1 + 12c_2 = 1,\ 4c_2 - 12c_1 = 0;\ 12c_4 - 6c_3 = 0,\ 6c_4 + 12c_3 = -1$$

whence $c_1 = 1/40, c_2 = 3/40, c_3 = -1/15, c_4 = -1/30$

Thus P.I. $= \frac{1}{40}e^{3x}(\cos 2x + 3\sin 2x) + \frac{1}{30}e^{2x}(2\cos 3x + \sin 3x)$

Hence C.S. is $y = c_1e^x + c_2e^{-x} + \frac{1}{30}e^{2x}(2\cos 3x + \sin 3x) + \frac{1}{40}e^{3x}(\cos 2x + 3\sin 2x)$.

PROBLEMS 13.3

Solve by the method of variation of parameters :

1. $\dfrac{d^2y}{dx^2} + a^2y = \text{cosec}\, ax.$
2. $\dfrac{d^2y}{dx^2} + y = \sec x.$ *(Bhopal, 2007)*
3. $\dfrac{d^2y}{dx^2} + y = \tan x.$ *(P.T.U., 2005 ; Raipur, 2004)*
4. $\dfrac{d^2y}{dx^2} + y = x \sin x.$ *(S.V.T.U., 2007 ; J.N.T.U., 2005)*
5. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = e^x /]x.$ *(V.T.U., 2006)*
6. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = \dfrac{1}{1+e^{-x}}.$ *(V.T.U., 2010 S ; U.P.T.U., 2005)*
7. $y'' - 2y' + 2y = e^x \tan x.$ *(V.T.U., 2010)*
8. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} = e^x \sin x.$ *(U.P.T.U., 2003)*
9. $\dfrac{d^2y}{dx^2} + y = \dfrac{1}{1+\sin x}.$ *(V.T.U., 2004)*

Solve by the method of undetermined coefficients :

10. $(D^2 - 3D + 2)\, y = x^2 + e^x.$ *(V.T.U., 2003 S)*
11. $\dfrac{d^2y}{dx^2} + y = 2\cos x.$ *(V.T.U., 2000 S)*
12. $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = e^{3x} + \sin x.$ *(V.T.U., 2008)*
13. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = x + \sin x.$ *(V.T.U., 2010)*
14. $(D^2 - 2D + 3)\, y = x^3 + \cos x.$
15. $(D^2 - 2D)\, y = e^x \sin x.$ *(V.T.U., 2006)*

13.9 EQUATIONS REDUCIBLE TO LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

Now we shall study two such forms of linear differential equations with variable coefficients which can be reduced to linear differential equations with constant coefficients by suitable substitutions.

I. Cauchy's homogeneous linear equation*. An equation of the form

$$x^n \frac{d^n y}{dx^n} + k_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \dots + k_{n-1} x \frac{dy}{dx} + k_n y = X \qquad \dots(1)$$

where X is a function of x, is called *Cauchy's homogeneous linear equation.*

Such equations can be reduced to linear differential equations with constant coefficients, by putting

$$x = e^t \quad \text{or} \quad t = \log x. \text{ Then if } D = \frac{d}{dt}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{x}, \quad i.e., \quad x\frac{dy}{dx} = Dy.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2}\frac{dt}{dx} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

i.e., $$x^2\frac{d^2y}{dx^2} = D(D-1)\,y. \text{ Similarly, } x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)\,y \text{ and so on.}$$

After making these substitutions in (1), there results a linear equation with constant coefficients, which can be solved as before.

Example 13.30. *Solve* $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = \log x.$ *(V.T.U., 2010)*

Solution. This is a Cauchy's homogeneous linear.

*See footnote p. 144.

Put $x = e^t$, *i.e.*, $t = \log x$, so that $x\dfrac{dy}{dx} = Dy$, $x^2\dfrac{d^2y}{dx^2} = D(D-1)y$ where $D = \dfrac{d}{dt}$

Then the given equation becomes $[D(D-1) - D + 1]\, y = t$ or $(D-1)^2 y = t$ ...(*i*)

which is a linear equation with constant coefficients.

Its A.E. is $(D-1)^2 = 0$ whence $D = 1, 1$.

$\therefore$ C.F. $= (c_1 + c_2 t)e^t$ and P.I. $= \dfrac{1}{(D-1)^2}\, t = (1-D)^{-2}\, t = (1 + 2D + 3D^2 + \dots)\, t = t + 2$.

Hence the solution of (*i*) is $y = (c_1 + c_2 t)e^t + t + 2$ or, putting $t = \log x$ and $e^t = x$, we get

$y = (c_1 + c_2 \log x)\, x + \log x + 2$ as the required solution of (*i*).

Example 13.31. *Solve* $x^2\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + y = \dfrac{1}{(1-x)^2}$. (*P.T.U., 2003*)

Solution. Put $x = e^t$ *i.e.*, $t = \log x$ so that $x\, dy/dx = Dy$, $x^2\dfrac{d^2y}{dx^2} = D(D-1)y$

Then the given equation becomes

$$[D(D-1) + 3D + 1]\, y = \frac{1}{(1-e^t)^2} \quad \text{or} \quad (D^2 + 2D + 1)\, y = \frac{1}{(1-e^t)^2}$$

Its A.E. is $D^2 + 2D + 1 = 0$ or $(D+1)^2 = 0$ *i.e.*, $D = -1, -1$.

$$\therefore \quad \text{C.F.} = (c_1 + c_2 x)\, e^{-t} = (c_1 + c_2 \log x)\, \frac{1}{x}$$

$$\text{P.I.} = \frac{1}{(D+1)^2}\, \frac{1}{(1-e^t)^2} = \frac{1}{D+1}\, u, \text{ where } u = \frac{1}{D+1} \cdot \frac{1}{(1-e^t)^2} \text{ i.e. } \frac{du}{dt} + u = (1-e^t)^{-2}$$

which is Leibnitz's linear equation having I.F. $= e^t$

$$\therefore \text{ its solution is} \quad ue^t = \int \frac{e^t}{(1-e^t)^2}\, dt = \frac{1}{1-e^t} \quad \text{or} \quad u = \frac{e^{-t}}{1-e^t}$$

$$\therefore \quad \text{P.I.} = \frac{1}{D+1}\left(\frac{e^{-t}}{1-e^t}\right) = e^{-t}\int \frac{1}{1-e^t}\, dt = \frac{1}{x}\int \frac{dx}{x(1-x)}$$

$$= \frac{1}{x}\int \left(\frac{1}{x} + \frac{1}{1-x}\right) dx = \frac{1}{x}\, [\log x - \log(1-x)] = \frac{1}{x} \log \frac{x}{x-1}$$

Hence the solution is
$$y = \left\{c_1 + c_2 \log x + \log \frac{x}{x-1}\right\}\frac{1}{x}.$$

Example 13.32. *Solve* $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = \log x \sin(\log x)$.

(*Kurukshetra, 2006 ; Madras, 2006 ; Kerala, 2005*)

Solution. Putting $x = e^t$ *i.e.* $t = \log x$, the given equation becomes

$[D(D-1) + D + 1]\, y = t \sin t$ *i.e.* $(D^2 + 1)\, y = t \sin t$ ...(*i*)

Its A.E. is $D^2 + 1 = 0$ *i.e.* $D = \pm i$.

$\therefore$ C.F. $= c_1 \cos t + c_2 \sin t$

and
$$P.I. = \frac{1}{D^2+1}\, t \sin t = \frac{1}{D^2+1}\, t \text{ (I.P. of } e^{it})$$

$$= \text{I.P. of } e^{it}\, \frac{1}{(D+i)^2 + 1}\, t = \text{I.P. of } e^{it} \cdot \frac{1}{D^2 + 2iD}\, t$$

$$= \text{I.P. of } e^{it} \frac{1}{2iD\,(1 + D/2i)} t = \text{I.P. of } \frac{1}{2i} e^{it} \frac{1}{D}\left(1 - \frac{iD}{2}\right)^{-1} t$$

$$= \text{I.P. of } \frac{1}{2i} e^{it} \frac{1}{D}\left(1 + \frac{iD}{2} + \dots\right) t = \text{I.P. of } \frac{1}{2i} e^{it} \frac{1}{D}\left(t + \frac{i}{2}\right)$$

$$= \text{I.P. of } \frac{e^{it}}{2i} \int\left(t + \frac{i}{2}\right) dt = \text{I.P. of } \frac{e^{it}}{2i}\left(\frac{t^2}{2} + \frac{it}{2}\right)$$

$$= \text{I.P. of } e^{it}\left(-\frac{i}{4} t^2 + \frac{t}{4}\right) = \text{I.P. of } (\cos t + i \sin t)\left(-\frac{it^2}{4} + \frac{t}{4}\right) = -\frac{t^2}{4}\cos t + \frac{t}{4}\sin t$$

Hence the C.S. of (i) is $y = c_1 \cos t + c_2 \sin t - \dfrac{t^2}{4} \cos t + \dfrac{t}{4} \sin t$

or $\quad y = c_1 \cos(\log x) + c_2 \sin(\log x) - \dfrac{1}{4} (\log x)^2 \cos(\log x) + \dfrac{1}{4} \log(\log x) \sin(\log x)$

which is the required solution.

Example 13.33. *Solve* $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + y = \log x \dfrac{\sin(\log x) + 1}{x}$. *(I.S.M., 2001)*

Solution. Put $x = e^t$, *i.e.*, $t = \log x$ so that $x\dfrac{dy}{dx} = Dy$, $x^2 \dfrac{d^2y}{dx^2} = D(D-1)y$

Then the given equation becomes

$$\{D(D-1) - 3D + 1\}\, y = t\, \frac{\sin t + 1}{e^t} \quad \text{or} \quad (D^2 - 4D + 1)\, y = e^{-t}\, t\, (\sin t + 1)$$

which is a linear equation with constant coefficients.

Its A.E. is $D^2 - 4D + 1 = 0$ whence $D = 2 \pm \sqrt{3}$

$$\therefore \quad \text{C.F.} = c_1 e^{(2+\sqrt{3})t} + c_2 e^{(2-\sqrt{3})t} = e^{2t}(c_1 e^{\sqrt{3t}} + c_2 e^{-\sqrt{3t}})$$

and

$$\text{P.I.} = \frac{1}{D^2 - 4D + 1} e^{-t}\, t\,(\sin t + 1) = e^{-t} \frac{1}{(D-1)^2 - 4(D-1) + 1} t\,(\sin t + 1)$$

$$= e^{-t}\left\{\frac{1}{D^2 - 6D + 6} t + \frac{1}{D^2 - 6D + 6} t \sin t\right\}$$

$$\frac{1}{D^2 - 6D + 6} t = \frac{1}{6}\left(1 - \frac{6D - D^2}{6}\right)^{-1} t = \frac{1}{6}(1 + D)\, t = \frac{1}{6}(t + 1)$$

$$\frac{1}{D^2 - 6D + 6} t \sin t = \text{I.P. of } \frac{1}{D^2 - 6D + 6} e^{it} \cdot t$$

$$= \text{I.P. of } e^{it} \frac{1}{(D+i)^2 - 6(D+i) + 6} t = \text{I.P. of } e^{it} \frac{1}{D^2 + (2i - 6)D + (5 - 6i)} t$$

$$= \text{I.P. of } \frac{e^{it}}{5 - 6i}\left\{1 + \frac{(2i - 6)D + D^2}{5 - 6i}\right\}^{-1} t = \text{I.P. } \frac{e^{it}}{5 - 6i}\left(1 - \frac{2i - 6}{5 - 6i} D\right) t$$

$$= \text{I.P. of } \frac{(5 + 6i)}{61} (\cos t + i \sin t)\left(t - \frac{2i - 6}{5 - 6i}\right)$$

$$= \text{I.P. of } \frac{1}{61} \{(5 \cos t - 6 \sin t) + i(5 \sin t + 6 \cos t)\}\left(t + \frac{42 + 26i}{61}\right)$$

$$= \frac{26}{3721}(5 \cos t - 6 \sin t) + \frac{1}{61}(5 \sin t + \cos t)\left(t + \frac{42}{61}\right)$$

$$= \frac{t}{61}(5 \sin t + \cos t) + \frac{2}{3721}(27 \sin t + 191 \cos t)$$

$\therefore$ $$\text{P.I.} = e^{-t}\left[\frac{1}{6}(t+1) + \frac{1}{61}(5 \sin t + 6 \cos t) + \frac{2}{3721}(27 \sin t + 191 \cos t)\right]$$

Hence $$y = e^{2t}(c_1 e^{\sqrt{3}t} + c_2 e^{-\sqrt{3}t}) + e^{-t}\left[\frac{1}{6}(t+1) + \frac{t}{61}(5 \sin t + 6 \cos t) + \frac{2}{3721}(27 \sin t + 191 \cos t)\right]$$

or $$y = x^2(c_1 x^{\sqrt{3}} + c_2 x^{-\sqrt{3}}) + \frac{1}{x}\left[\frac{1}{6}(\log x + 1) + \frac{\log x}{61}\{5 \sin(\log x) + 6 \cos(\log x)\} + \frac{2}{3721}\{27 \sin(\log x) + 191 \cos(\log x)\}\right].$$

Example 13.34. *Solve* $x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = e^{x^x}$. *(Kurukshetra, 2005 ; U.P.T.U., 2005)*

Solution. Putting $x = e^t$, *i.e.*, $t = \log x$, the given equation becomes

$[D(D-1) + 4D + 2]\, y = e^{e^t}$ *i.e.*, $(D^2 + 3D + 2)\, y = e^{e^t}$

Its A.E. is $D^2 + 3D + 2 = 0$ whence $D = -1, -2$.

$\therefore$ C.F. $= c_1 e^{-t} + c_2 e^{-2t} = c_1 x^{-1} + c_2 x^{-2}$

and $$\text{P.I.} = \frac{1}{(D^2 + 3D + 2)} e^{e^t} = \frac{1}{(D+1)(D+2)} e^{e^t} = \left(\frac{1}{D+1} - \frac{1}{D+2}\right) e^{e^t}.$$

Now $$\frac{1}{D+1} e^{e^t} = \frac{1}{D+1} e^{-t} \cdot e^t e^{e^t} = e^{-t} \frac{1}{(D-1)+1} e^t e^{e^t}$$

$$= e^{-t} \frac{1}{D} e^t e^{e^t} = e^{-t} \int e^{e^t} d(e^t) = x^{-1} \int e^x dx = x^{-1} e^x$$

$$\frac{1}{D+2} e^{e^t} = \frac{1}{D+2} e^{-2t} \cdot e^{2t} e^{e^t} = e^{-2t} \frac{1}{(D-2)+2} e^{2t} e^{e^t}$$

$$= e^{-2t} \frac{1}{D} e^{e^t} e^{2t} = e^{-2t} \int e^{e^t} e^t \, d(e^t)$$

$$= x^{-2} \int e^x \, x \, dx \qquad [\because \; e^t = x]$$

$$= x^{-2}(xe^x - e^x) \qquad \text{[Integrating by parts]}$$

$\therefore$ P.I. $= x^{-1} e^x - x^{-2}(xe^x - e^x) = x^{-2} e^x$

Hence the required solution is $y = c_1 x^{-1} + x^{-2}(c_2 + e^x)$.

II. Legendre's linear equation*. An equation of the form

$$(ax+b)^n \frac{d^n y}{dx^n} + k_1 (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + k_n y = X \qquad \ldots(2)$$

where k's are constants and X is a function of x, is called *Legendre's linear equation*.

Such equations can be reduced to linear equations with constant coefficients by the substitution $ax + b = e^t$, *i.e.*, $t = \log(ax + b)$.

Then, if $$D = \frac{d}{dt}, \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{a}{ax+b} \cdot \frac{dy}{dt} \; \textit{i.e.} \; (ax+b)\frac{dy}{dx} = a\,Dy$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{a}{ax+b}\frac{dy}{dt}\right) = \frac{-a^2}{(ax+b)^2}\frac{dy}{dt} + \frac{a}{ax+b}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} = \frac{a^2}{(ax+b)^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

* A French mathematician *Adrien Marie Legender* (1752 – 1833) who made important contributions to number theory, special functions, calculus of variations and elliptic integrals.

i.e., $(ax+b)^2 \dfrac{d^2y}{dx^2} = a^2 D(D-1)y$. Similarly, $(ax+b)^3 \dfrac{d^3y}{dx^3}$ $a^3D(D-1)(D-2)y$ and so on.

After making these replacements in (2), there results a linear equation with constant coefficients.

Example 13.35. *Solve* $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 2\sin[\log(1+x)]$. ...(*i*)

(*V.T.U., 2009 ; J.N.T.U., 2005 ; Kerala, 2005*)

Solution. This is a Legendre's linear equation.

$\therefore$ put $1+x=e^t$, *i.e.*, $t=\log(1+x)$, so that $(1+x)\dfrac{dy}{dx} = Dy$

and $$(1+x)^2 \frac{d^2y}{dx^2} = D(D-1)y, \text{ where } D = \frac{d}{dt}$$

Then (*i*) becomes $D(D-1)y + Dy + y = 2\sin t$

or $$(D^2+1)y = 2\sin t \qquad ...(ii)$$

This is a linear equation with constant co-efficients

Its A.E. is $D^2+1=0$, whence $D=\pm i$ $\therefore$ C.F. $= c_1\cos t + c_2 \sin t$

and $$\text{P.I.} = 2\frac{1}{D^2+1}\sin t = 2t\cdot\frac{1}{2D}\sin t$$

$$= t\int \sin t\, dt = -t\cos t \qquad [\because \text{ on replacing } D^2 \text{ by } -1^2, D^2+1=0]$$

Hence the solution of (*ii*) is $y = c_1\cos t + c_2\sin t - t\cos t$ and on replacing t by $\log(1+x)$, we get $y = c_1\cos[\log(1+x)] + c_2\sin[\log(1+x)] - \log(1+x)\cos[\log(1+x)]$ as the required solution.

Example 13.36. *Solve* $(2x-1)^2 \dfrac{d^2y}{dx^2} + (2x-1)\dfrac{dy}{dx} - 2y = 8x^2 - 2x + 3$. (*V.T.U., 2006*)

Solution. This is a Legendre's linear equation.

$\therefore$ put $2x-1=e^t$ *i.e.*, $t=\log(2x-1)$ so that $(2x-1)\dfrac{dy}{dx} = 2Dy$

and $$(2x-1)^2\frac{d^2y}{dx^2} = 4D(D-1)y, \text{ where } D = \frac{d}{dt}.$$

Then the given equation becomes

$$4D(D-1)y + 2Dy - 2y = 8\left(\frac{1+e^t}{2}\right)^2 - 2\left(\frac{1+e^t}{2}\right) + 3$$

or $$2D^2y - Dy - y = e^{2t} + \frac{3}{2}e^t + 2 \qquad ...(i)$$

This is a linear equation with constant coefficients.

Its A.E. is $2D^2 - D - 1 = 0$ whence $D = 1, -1/2$.

$\therefore$ $$\text{C.F.} = c_1e^t + c_2e^{-t/2}$$

and $$\text{P.I.} = \frac{1}{2D^2-D-1}\left(e^{2t} + \frac{3}{2}e^t + 2\right) = \frac{1}{2.4-2-1}e^{2t} + \frac{3}{2}\frac{t}{4D-1}e^t + 2\cdot\frac{1}{2.0^2-0-1}e^{0t}$$

$[\because$ on putting $t=1$, $2D^2-D-1=0$

$$= \frac{1}{5}e^{2t} + \frac{3t}{2}\cdot\frac{1}{4-1}e^t - 2 = \frac{1}{5}e^{2t} + \frac{t}{2}e^t - 2$$

Hence the solution of (*i*) is

$$y = c_1e^t + c_2e^{-t/2} + \frac{1}{5}e^{2t} + \frac{1}{2}te^t - 2 \text{ and on replacing } t \text{ by } \log(2x-1),$$

$$y = c_1 (2x - 1) + c_2 (2x - 1)^{-1/2} + \frac{1}{5} (2x - 1)^2 + \frac{1}{2} (2x - 1) \log (2x - 1) - 2.$$

which is the required solution.

PROBLEMS 13.4

Solve :

1. $x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 6y = x^2.$

2. $x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} - 4y = x^4.$

3. $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 4y = (1 + x^2).$ *(S.V.T.U., 2007)*

4. $x \frac{d^2y}{dx^2} - \frac{2y}{x} = x + \frac{1}{x^2}.$ *(V.T.U., 2005 S)*

5. The radial displacement u in a rotating disc at a distance r from the axis is given by $r^2 \frac{d^2u}{dr^2} + r \frac{du}{dr} - u + kr^3 = 0,$ where k is a constant. Solve the equation under the conditions $u = 0$ when $r = 0$, $u = 0$ when $r = a$.

Solve :

6. $x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = \log x.$ *(Bhopal, 2009)*

7. $x^3 \frac{d^3y}{dx^3} + 3x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = x + \log x$ *(Bhopal, 2008)*

8. $x^2y'' + xy' + y = 2\cos^2 (\log x).$ *(V.T.U., 2011)*

9. $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$ *(S.V.T.U., 2006 ; P.T.U., 2003)*

10. $x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + y = \frac{1}{(1 - x)^2}.$ *(P.T.U., 2003)*

11. $x^2 \frac{d^2y}{dx^2} + 5x \frac{dy}{dx} + 4y = x \log x.$ *(U.P.T.U., 2004)*

12. $x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 12y = x^3 \log x.$ *(Bhopal, 2008)*

13. $(2x + 3)^2 \frac{d^2y}{dx^2} - (2x + 3) \frac{dy}{dx} - 12y = 6x.$ *(V.T.U., 2007 ; Kerala, 2005 ; Anna, 2002 S)*

14. $(x - 1)^3 \frac{d^3y}{dx^3} + 2(x - 1)^2 \frac{d^2y}{dx^2} - 4 (x - 1) \frac{dy}{dx} + 4y = 4 \log (x - 1).$ *(Nagpur, 2009)*

15. $(1 + x)^2 \frac{d^2y}{dx^2} + (1 + x) \frac{dy}{dx} + y = \sin [2 \log (1 + x)]$ *(P.T.U., 2006 ; V.T.U., 2004)*

16. $(3x + 2)^2 \frac{d^2y}{dx^2} + 5 (3x + 2) \frac{dy}{dx} - 3y = x^2 + x + 1.$ *(Mumbai, 2006)*

13.10 (1) LINEAR DEPENDENCE OF SOLUTIONS

Consider the initial value problem consisting of the homogeneous linear equation

$$y'' + py' + qy = 0 \quad ...(1)$$

with variable coefficients $p(x)$ and $q(x)$ and two initial conditions $y(x_0) = k_0, y'(x_0) = k_1$ ...(2)

Let its general solution be $y = c_1y_1 + c_2y_2$ ...(3)

which is made up of two linearly dependent solutions y_1 and y_2.*

If $p(x)$ and $q(x)$ are continuous functions on some open interval I and x_0 is any fixed point on I, then the above initial value problem has a **unique solution** *$y(x)$ on the interval I.*

* As in §2.12, y_1, y_2 are said to be *linearly dependent* in an interval I, if and only if there exist numbers λ_1, λ_2 not both zero such that $\lambda_1y_1 + \lambda_2y_2 = 0$ for all x in I.

If no such numbers other than zero exist, then y_1, y_2 are said to be *linearly independent.*

(2) Theorem. *If $p(x)$ and $q(x)$ are continuous on an open interval I, then the solutions y_1 and y_2 of (1) are linearly dependent in I if and only if the Wronskian*† $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 0$ for some x_0 on I. If there is an $x = x_1$ in I at which $W(y_1, y_2) \neq 0$, then y_1, y_2 *are linearly independent on I.*

Proof. If y_1, y_2 are linearly dependent solutions of (1) then there exist two constants c_1, c_2 not both zero, such that

$$c_1y_1 + c_2y_2 = 0 \qquad ...(4)$$

Differentiating w.r.t. x, $c_1y_1' + c_2y_2' = 0$ ...(5)

Eliminating c_1, c_2 from (4) and (5), we get

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 0$$

Conversely, suppose $W(y_1, y_2) = 0$ for some $x = x_0$ on I and show that y_1, y_2 are linearly dependent.

Consider the equation

$$\left.\begin{array}{l} c_1y_1(x_0) + c_2y_2(x_0) = 0 \\ c_1y_1'(x_0) + c_2y_2'(x_0) = 0 \end{array}\right\} \qquad ...(6)$$

which, on eliminating c_1, c_2, give $W(y_1, y_2) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y'_1(x_0) & y'_2(x_0) \end{vmatrix} = 0$

Hence the system has a solution in which c_1, c_2 are not both zero.

Now introduce the function $\bar{y}(x) = c_1y_1(x) + c_2y_2(x)$

Then $y(x)$ is a solution of (1) on I. By (6), this solution satisfies the initial conditions $y(x_0) = 0$ and $y'(x_0) = 0$. Also since $p(x)$ and $q(x)$ are continuous on I, this solution must be unique. But $\mathbf{y = 0}$ is obviously another solution of (1) satisfying the given initial conditions. Hence $\bar{\mathbf{y}} = y$ *i.e.*, $c_1y_1 + c_2y_2 = 0$ in I. Now since c_1, c_2 are not both zero, it implies that y_1 and y_2 are linearly dependent on I.

Example 13.37. *Show that the two functions sin 2x, cos 2x are independent solutions of $y'' + 4y = 0$.*

Solution. Substituting $y_1 = \sin 2x$ (or $y_2 = \cos 2x$) in the given equation we find that y_1, y_2 are its solutions.

Also $$W(y_1, y_2) = \begin{vmatrix} \sin 2x & \cos 2x \\ 2\cos 2x & -2\sin 2x \end{vmatrix} = -2 \neq 0$$

for any value of x. Hence the solutions y_1, y_2 are linearly independent.

PROBLEMS 13.5

Solve :

1. Show that e^{-x}, xe^{-x} are independent solutions of $y'' + 2y' + y = 0$ in any interval.
2. Show that $e^x \cos x$, $e^x \sin x$ are independent solutions of the equation $xy'' - 2y' = 0$.
3. If y_1, y_2 be two solutions of $y'' + p(x)y' + q(x)y = 0$, show that the *Wronskian* can be expressed as $W(y_1, y_2) = ce^{-\int p dx}$

13.11 SIMULTANEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

Quite often we come across linear differential equations in which there are two or more dependent variables and a single independent variable. Such equations are known as *simultaneous linear equations.* Here we shall deal with systems of linear equations with constant coefficients only. Such a system of equations is solved by eliminating all but one of the dependent variables and then solving the resulting equations as before. Each of the dependent variables is obtained in a similar manner.

Example 13.38. *Solve the simultaneous equations :*

$$\frac{dx}{dt} + 5x - 2y = t, \quad \frac{dy}{dt} + 2x + y = 0$$

being given $x = y = 0$ when $t = 0$. (*S.V.T.U., 2009 ; Kurukshetra, 2005*)

† See footnote on p. 486.

Solution. Taking $d/dt = D$, the given equations become

$$(D + 5)\,x - 2y = t \qquad ...(i)$$

$$2x + (D + 1)\,y = 0 \qquad ...(ii)$$

Eliminate x as if D were an ordinary algebraic multiplier. Multiplying (i) by 2 and operating on (ii) by $D + 5$ and then subtracting, we get

$$[-4 - (D + 5)(D + 1)]\,y = 2t \text{ or } (D^2 + 6D + 9)\,y = -2t$$

Its auxiliary equation is $D^2 + 6D + 9 = 0$, *i.e.*, $(D + 3)^2 = 0$

whence $\qquad D = -3, -3 \quad \therefore \quad \text{C.F.} = (c_1 + c_2t)\,e^{-3t}$

and $\qquad \text{P.I.} = \dfrac{1}{(D+3)^2}(-2t) = -\dfrac{2}{9}\left(1+\dfrac{D}{3}\right)^{-2} t = -\dfrac{2}{9}\left(1-\dfrac{2D}{3}+...\right) t = -\dfrac{2t}{9}+\dfrac{4}{27}$

Hence $\qquad y = (c_1 + c_2t)\,e^{-3t} - \dfrac{2t}{9} + \dfrac{4}{27} \qquad ...(iii)$

Now to find x, either eliminate y from (i) and (ii) and solve the resulting equation or substitute the value of y in (ii). Here, it is more convenient to adopt the latter method.

From (iii), $\qquad Dy = c_2e^{-3t} + (c_1 + c_2t)(-3)\,e^{-3t} - \dfrac{2}{9}$

$\therefore$ Substituting for y and Dy in (ii), we get

$$x = -\frac{1}{2}[Dy + y] = \left[\left(c_1 - \frac{1}{2}c_2\right) + c_2t\right]e^{-3t} + \frac{t}{9} + \frac{1}{27} \qquad ...(iv)$$

Hence (iii) and (iv) constitute the solutions of the given equations.

Since $x = y = 0$ when $t = 0$, the equations (iii) and (iv) give

$$0 = c_1 + \frac{4}{27} \text{ and } c_1 - \frac{1}{2}c_2 + \frac{1}{27} = 0 \text{ whence } c_1 = -\frac{4}{27},\ c_2 = -\frac{2}{9}.$$

Hence the desired solutions are

$$x = -\frac{1}{27}(1 + 6t)\,e^{-3t} + \frac{1}{27}(1 + 3t),\ y = -\frac{2}{27}(2 + 3t)\,e^{-3t} + \frac{2}{27}(2 - 3t).$$

Example 13.39. *Solve the simultaneous equations* $\dfrac{dx}{dt} + 2y + \sin t = 0,\ \dfrac{dy}{dt} - 2x - \cos t = 0$ *given that* $x = 0$ *and* $y = 1$ *when* $t = 0$.

Solution. Given equations are

$$Dx + 2y = -\sin t \quad ...(i); \qquad -2x + Dy = \cos t \quad ...(ii)$$

Eliminating x by multiplying (i) by 2 and (ii) by D and then adding, we get

$$4y + D^2y = -2\sin t - \sin t \text{ or } (D^2 + 4)y = -3\sin t$$

Its A.E. is $D = \pm 2i \quad \therefore \quad \text{C.F.} = c_1\cos 2t + c_2\sin 2t$

$$\text{P.I.} = -3\,\frac{1}{D^2+4}\sin t = -3\,\frac{1}{-1+4}\sin t = -\sin t$$

$\therefore \qquad y = c_1\cos 2t + c_2\sin 2t - \sin t \qquad ...(iii)$

and $\qquad dy/dt = -2\sin 2t + 2c_2\cos 2t - \cos t \qquad ...(iv)$

Substituting (iii) in (ii), we get

$$2x = Dy - \cos t = -2c_1\sin 2t + 2c_2\cos 2t - 2\cos t$$

or $\qquad x = -c_1\sin 2t + c_2\cos 2t + -\cos t \qquad ...(v)$

When $t = 0, x = 0, y = 1$, (iii) and (v) give $1 = c_1,\ 0 = c_2 - 1$

Hence $x = \cos 2t - \sin 2t - \cos t,\ y = \cos 2t + \sin 2t - \sin t$.

Example 13.40. *Solve the simultaneous equations*

$$\frac{dx}{dt} + \frac{dy}{dt} - 2y = 2\cos t - 7\sin t,\ \frac{dx}{dt} - \frac{dy}{dt} + 2x = 4\cos t - 3\sin t. \qquad (U.P.T.U., 2001)$$

Solution. Given equations are

$$Dx + (D - 2)y = 2 \cos t - 7 \sin t \quad ...(i)$$

$$(D + 2)x - Dy = 4 \cos t - 3 \sin t \quad ...(ii)$$

Eliminate y by operating on (i) by D and (ii) by $(D - 2)$ and then adding, we get

$$D^2x + (D-2)(D+2)x = -2 \sin t - 7 \cos t + 4(-\sin t - 2\cos t) - 3(\cos t - 2\sin t)$$

or

$$2(D^2 - 2)x = -18 \cos t \text{ or } (D^2 - 2)x = -9 \cos t$$

Its A.E. is $\quad D^2 - 2 = 0$ or $D = \pm \sqrt{2}$, $\quad \therefore$ C.F. $= c_1 e^{\sqrt{st}} + c_2 e^{-\sqrt{2t}}$

$$\text{P.I.} = (-9)\frac{1}{D^2 - 2} \cos t = \frac{-9 \cos t}{-1-2} = 3 \cos t.$$

Hence $x = c_1 e^{\sqrt{2t}} + c_2 e^{-\sqrt{2t}}$ $3 \cos t$.

Now substituting this value of x in (ii), we get

$$Dy = (D+2)(c_1 e^{\sqrt{2t}} + c_2 e^{-\sqrt{2t}} + 3 \cos t) - 4 \cos t + 3 \sin t$$

$$= c_1 \sqrt{2} e^{\sqrt{2t}} + 2c_1 e^{\sqrt{2t}} + c_2(-\sqrt{2} e^{-2t}) + 2c_2 e^{-\sqrt{2t}} - 3 \sin t + 6 \cos t - 4 \cos t + 3 \sin t$$

$$= (2 + \sqrt{2}) c_1 e^{\sqrt{2t}} + (2 - \sqrt{2}) c_2 e^{-\sqrt{2t}} + 2 \cos t$$

Hence $\quad y = (\sqrt{2} + 1) c_1 e^{\sqrt{2t}} - (\sqrt{2} - 1) c_2 e^{-\sqrt{2t}} + 2 \sin t + c_3.$

Example 13.41. *The small oscillations of a certain system with two degrees of freedom are given by the equations*

$$D^2x + 3x - 2y = 0$$

$$D^2x + D^2y - 3x + 5y = 0$$

where $D = d/dt$. If $x = 0, y = 0, Dx = 3, Dy = 2$ when $t = 0$, find x and y when $t = 1/2$.

Solution. Given equations are $(D^2 + 3)x - 2y = 0 \quad ...(i)$

$$(D^2 - 3)x + (D^2 + 5)y = 0 \quad ...(ii)$$

To eliminate x, operate these equations by $D^2 - 3$ and $D^2 + 3$ respectively and subtract (i) from (ii). Then

$$[(D^2 + 3)(D^2 + 5) + 2(D^2 - 3)]\, y = 0 \quad \text{or} \quad (D^4 + 10D^2 + 9)y = 0$$

Its auxiliary equation is $D^4 + 10D^2 + 9 = 0$ whence $D = \pm i, \pm 3i$

Thus $\quad y = c_1 \cos t + c_2 \sin t + c_3 \cos 3t + c_4 \sin 3t \quad ...(iii)$

To find x, we eliminate y from (i) and (ii).

$\therefore$ operating (i) by $D^2 + 5$ and multiplying (ii) by 2 and adding, we get

$$(D^4 + 10D^2 + 9)x = 0. \text{ Thus } x = k_1 \cos t + k_2 \sin t + k_3 \cos 3t + k_4 \sin 3t \quad ...(iv)$$

To find the relations between the constants in (iii) and (iv), substitute these values of x and y either of the given equations, say (i). This gives

$$2(k_1 - c_1)\cos t + 2(k_2 - c_2)\sin t - 2(3k_3 + c_3)\cos 3t - 2(3k_4 + c_4)\sin 3t = 0$$

which must hold for all values of t.

$\therefore$ Equating to zero the coefficients of $\cos t$, $\sin t$, $\cos 3t$ and $\sin 3t$, we get

$$k_1 = c_1, k_2 = c_2, k_3 = -c_3/3, k_4 = -c_4/3$$

Thus $\quad x = c_1 \cos t + c_2 \sin t - \frac{1}{3}(c_3 \cos 3t + c_4 \sin 3t) \quad ...(v)$

Hence (iii) and (iv) constitute the solutions of (i) and (ii).

Since $\quad x = y = 0$, when $t = 0$; $\therefore$ (iii) and (v) give

$$0 = c_1 + c_3 \text{ and } c_1 - \frac{1}{3}c_3 = 0 \text{ i.e. } c_1 = c_3 = 0$$

Thus (iii) and (v) reduce to

and

$$\left.\begin{aligned} y &= c_2 \sin t + c_4 \sin 3t \\ x &= c_2 \sin t - \frac{c_4}{3} \sin 3t \end{aligned}\right\} \quad ...(vi)$$

$\therefore \quad Dx = c_2 \cos t - c_4 \cos 3t \quad$ and $\quad Dy = c_2 \cos t + 3c_4 \cos 3t.$

Since $\quad Dx = 3$ and $Dy = 2$ when $t = 0$

$\therefore \quad 3 = c_2 - c_4$ and $2 = c_2 + 3c_4$, whence $c_2 = 11/4, c_4 = -\dfrac{1}{4}$.

Hence equation (*vi*) becomes $x = \dfrac{1}{4}\ (11 \sin t + \dfrac{1}{3} \sin 3t),\ y = \dfrac{1}{4}\ (11 \sin t - \sin 3t)$ ...(*vii*)

$\therefore$ when $t = 1/2,\ x = \dfrac{1}{4}\left[11 \sin (0.5) + \dfrac{1}{3} \sin (1.5)\right] = \dfrac{1}{4}\left[[11\,(0.4794 + \dfrac{1}{3}(0.9975)\right] = 1.4015$

and $\quad y = \dfrac{1}{4}\,[11 \sin (0.5) - \sin (1.5)] = 1.069.$

Example 13.42. *Solve the simultaneous equations:* $\dfrac{dx}{dt} = 2y, \dfrac{dy}{dt} = 2z, \dfrac{dz}{dt} = 2x$.

(*S.V.T.U., 2006 S ; U.P.T.U., 2004*)

Solution. Differentiating first equation w.r.t. t, $\dfrac{d^2x}{dt^2} = 2\,\dfrac{dy}{dt} = 2\,(2z)$

Again differentiating w.r.t. t, $\dfrac{d^3x}{dt^2} = 4\,\dfrac{dz}{dt} = 4\,(2x)$ ...(*i*)

or $\quad (D^3 - 8)\,x = 0$

Its A.E. is $\quad D^3 - 8 = 0 \quad$ or $\quad (D-2)(D^2 + 2D + 4) = 0$

or $\quad D = 2, -1 \pm i\sqrt{3}$

$\therefore$ the solution of (*i*) is $\quad x = c_1 e^{2t} + e^{-t}\,(c_2 \cos \sqrt{3t} + c_3 \sin \sqrt{3t})$ ...(*ii*)

From the first equation, we have $y = \dfrac{1}{2}\,\dfrac{dx}{dt}$

$$\therefore \quad y = \frac{1}{2}[2c_1e^{2t} + (-1)e^{-t}\,(c_2 \cos \sqrt{3t} + c_3 \sin \sqrt{3t}) + e^t\,(-\sqrt{3}\,c_2 \sin \sqrt{3t} + \sqrt{3}\,c_3 \cos \sqrt{3t})]$$

or $$y = c_1 e^{2t} + \frac{1}{2}\,e^{-t}\left\{(\sqrt{3}c_3 - c_2)\cos\sqrt{3t} - (c_3 + \sqrt{3}c_2)\sin\sqrt{3t}\right\} \quad ...(iii)$$

From the second equation, we have $z = \dfrac{1}{2}\,\dfrac{dy}{dt}$

$$\therefore \quad z = \frac{1}{2}\,2c_1e^{2t} + \frac{1}{4}\Big[(-1)e^{-t}\left\{(\sqrt{3}c_3 - c_2)\cos\sqrt{3t} - (c_3 + \sqrt{3}c_2)\sin\sqrt{3t}\right\}$$
$$+ e^{-t}\left\{\sqrt{3}(c_2 - \sqrt{3}c_3)\sin\sqrt{3t} - \sqrt{3}(c_3 + \sqrt{3}c_2)\cos\sqrt{3t}\right\}\Big]$$
$$= c_1e^{2t} + \frac{1}{4}\,e^{-t}\left\{(-2c_2 - 2\sqrt{3}c_3)\cos\sqrt{3t} - (2\sqrt{3}\,c_2 - 2c_3)\sin\sqrt{3t}\right\}$$

or $$z = c_1e^{2t} - \frac{1}{2}\,e^{-t}\left\{(\sqrt{3}\,c_2 - c_3)\sin\sqrt{3t} + (c_2 + \sqrt{3}c_3)\cos\sqrt{3t}\right\} \quad ...(iv)$$

Hence the equations (*ii*), (*iii*) and (*iv*) taken together give the required solution.

PROBLEMS 13.6

Solve the following simultaneous equations :

1. $\dfrac{dx}{dt} = 5x + y, \dfrac{dy}{dt} = y - 4x.$
2. $\dfrac{dx}{dt} + y = \sin t, \dfrac{dy}{dt} + x = \cos t$; given that $x = 2$ and $y = 0$ when $t = 0$.

(*Bhopal, 2009 ; J.N.T.U., 2006 ; Kerala, 2005*)

3. $\frac{dx}{dt} + 2x + 3y = 0,\ 3x + \frac{dy}{dt} + 2y = 2e^{2t}$. *(Delhi, 2002)*
4. $\frac{dx}{dt} - 7x + y = 0,\ \frac{dy}{dt} - 2x - 5y = 0.$
5. $\frac{dx}{dt} + 2y = e^t,\ \frac{dy}{dt} - 2x = e^{-t}$. *(Bhopal, 2002 S)*
6. $\frac{dx}{dt} + 2x - 3y = t;\ \frac{dy}{dt} - 3x + 2y = e^{2t}$. *(Nagpur, 2009)*
7. $(D - 1)x + Dy = 2t + 1,\ (2D + 1)x + 2Dy = t.$
8. $(D + 1)x + (2D + 1)y = e^t,\ (D - 1)x + (D + 1)y = 1.$
9. $Dx + Dy + 3x = \sin t,\ Dx + y - x = \cos t.$ *(U.P.T.U., 2003)*
10. $t\frac{dx}{dt} + y = 0,\ t\frac{dy}{dt} + x = 0$ given $x(1) = 1,\ y(-1) = 0.$
11. $\frac{dx}{dt} + \frac{dy}{dt} + 3x = \sin t,\ \frac{dx}{dt} + y - x = \cos t.$
12. $\frac{d^2x}{dt^2} - 3x - 4y = 0,\ \frac{d^2y}{dt^2} + x + y = 0.$ *(U.P.T.U., 2005)*
13. $\frac{d^2x}{dt^2} + y = \sin t,\ \frac{d^2y}{dt^2} + x = \cos t.$ *(U.P.T.U., 2004)*
14. A mechanical system with two degrees of freedom satisfies the equations

$$2\frac{d^2x}{dt^2} + 3\frac{dy}{dt} = 4;\ 2\frac{d^2y}{dt^2} - 3\frac{dx}{dt} = 0.$$

Obtain expression for x and y in terms of t, given $x, y, dx/dt, dy/dt$ all vanish at $t = 0$.

13.12 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 13.7

Fill up the blanks or choose the correct answer in the following problems :

1. The complementary function of $(D^4 - a^4)y = 0$ is ....
2. P.I. of the differential equation $(D^2 + D + 1)y = \sin 2x$ is ....
3. P.I. of $y'' - 3y' + 2y = 12$ is ....
4. The Wronskian of x and e^x is ....
5. The C.F. of $y'' - 2y' + y = xe^x \sin x$ is
   (*a*) $C_1e^x + C_2e^{-x}$ (*b*) $(C_1x + C_2)e^x$ (*c*) $(C_1 + C_2x)e^{-x}$ (*d*) None of these. *(V.T.U., 2010)*
6. The general solution of the differential equation $(D^4 - 6D^3 + 12D^2 - 8D)y = 0$ is ....
7. The particular integral of $(D^2 + a^2)y = \sin ax$ is
   (*a*) $-\frac{x}{2a}\cos ax$ (*b*) $\frac{x}{2a}\cos ax$ (*c*) $-\frac{ax}{2}\cos ax$ (*d*) $\frac{ax}{2}\cos ax$.
8. The solution of the differential equation $(D^2 - 2D + 5)^2 y = 0$, is ....
9. The solution of the differential equation $y'' + y = 0$ satisfying the conditions $y(0) = 1$ and $y(\pi/2) = 2$, is ....
10. $e^{-x}(c_1 \cos\sqrt{3x} + c_2 \sin\sqrt{3x}) + c_3e^{2x}$ is the general solution of
    (*a*) $d^3y/dx^3 + 4y = 0$ (*b*) $d^3y/dx^3 - 8y = 0$
    (*c*) $d^3y/dx^3 + 8y = 0$ (*d*) $d^3y/dx^3 - 2d^2y/dx^2 + dy/dx - 2 = 0$.
11. The solution of the differential equation $(D^2 + 1)^2 y = 0$ is ....
12. The particular integral of $d^2y/dx^2 + y = \cos h\ 3x$ is ....
13. The solution of $x^2y'' + xy' = 0$ is ....
14. The general solution of $(D^2 - 2)^2, y = 0$ is.......
15. P.I. of $(D + 1)^2 y = xe^{-x}$ is ....
16. If $f(D) = D^2 - 2,\ \frac{1}{f(D)}e^{2x} =$ ....
17. If $f(D) = D^2 + 5,\ \frac{1}{f(D)}\sin 2x =$ ....
18. The particular integral of $(D + 1)^2 y = e^{-x}$ is ....
19. The general solution of $(4D^3 + 4D^2 + D)y = 0$ is ....

20. P.I. of $(D^2 + 4)y = \cos 2x$ is

(a) $\frac{1}{2}\sin 2x$ (b) $\frac{1}{2}x\sin 2x$ (c) $\frac{1}{4}x\sin 2x$ (d) $\frac{1}{2}x\cos 2x$. *(Bhopal, 2008)*

21. By the method of undetermined coefficients y_p of $y'' + 3y' + 2y = 12x^2$ is of the form

(a) $a + bx + cx^2$ (b) $a + bx$ (c) $ax + bx^2 + cx^3$ (d) None of these. *(V.T.U., 2010)*

22. In the equation $\frac{dx}{dt} + y = \sin t + 1$, $\frac{dy}{dt} + x = \cos t$ if $y = \sin t + 1 + e^{-t}$, then $x =$ ....

23. $(x^2D^2 + xD + 7)y = 2/x$ converted to a linear differential equation with constant coefficients is ....

24. P.I. of $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4$ is

(a) $\frac{x^2}{3} + 4x$ (b) $\frac{x^3}{3} + 4$ (c) $\frac{x^3}{3} + 4x$ (d) $\frac{x^3}{3} + 4x^2$.

25. The solution of the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{3x}$ is given by

(a) $y = C_1e^{x} + C_2e^{2x} + \frac{1}{2}e^{3x}$ (b) $y = C_1e^{-x} + C_2e^{-2x} + \frac{1}{2}e^{3x}$

(c) $y = C_1e^{-x} + C_2e^{2x} + \frac{1}{2}e^{3x}$ (d) $y = C_1e^{-x} + C_2e^{2x} + \frac{1}{2}e^{-3x}$.

26. The particular integral of the differential equation $(D^3 - D)y = e^x + e^{-x}$, $D = \frac{d}{dx}$ is

(a) $\frac{1}{2}(e^x + e^{-x})$ (b) $\frac{1}{2}x(e^x + e^{-x})$ (c) $\frac{1}{2}x^2(e^x + e^{-x})$ (d) $\frac{1}{2}x^2(e^x - e^{-x})$.

27. The complementary function of the differential equation $x^2y'' - xy' + y = \log x$ is ....

28. The homogeneous linear differential equation whose auxiliary equation has roots $1, -1$ is ......

29. The particular integral of $(D^2 - 6D + 9)y = \log 2$ is .... *(V.T.U., 2011)*

30. To transform $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{1}{x}$ into a linear differential equation with constant coefficients, put $x =$ ....

31. The particular integral of $(D^2 - 4)y = \sin 3x$ is

(a) 1/4 (b) $-$ 1/13 (c) 1/5 (d) None of these. *(V.T.U., 2010)*

32. The solution of $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4y = 0$ is ....

33. The differential equation whose auxiliary equation has the roots $0, -1, -1$ is ....

34. Complementary function of $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - y = 2x\log x$ is

(a) $(C_1 + C_2x)e^x$ (b) $(C_1 + C_2\log x)x$ (d) $(C_1 + C_2x)\log x$ (d) $(C_1 + C_2\log x)e^x$. *(Bhopal, 2008)*

35. The general solution of $(D^2 - D - 2)x = 0$ is $x = c_1e^t + c_2e^{-2t}$ (True or False)

36. $\frac{1}{f(D)}(x^2e^{ax}) = \frac{1}{f(D+a)}(e^{ax}x^2)$. (True or False)

CHAPTER 14

Applications of Linear Differential Equations

1. Introduction. 2. Simple harmonic motion. 3. Simple Pendulum, Gain and Loss of Oscillations. 4. Oscillations of a spring. 5. Oscillatory electrical circuits. 6. Electro-mechanical analogy. 7. Deflection of Beams. 8. Whirling of Shafts. 9. Applications of simultaneous linear equations. 10. Objective Type of Questions.

14.1 INTRODUCTION

The linear differential equations with constant coefficients find their most important applications in the study of electrical, mechanical and other linear systems. In fact such equations play a dominant role in unifying the theory of electrical and mechanical oscillatory systems.

We shall begin by explaining the types of oscillations of the mechanical systems and the equivalent electrical circuits. Then we shall study at some length the slightly less striking applications such as deflection of beams and whirling of shafts. At the end, we'll take up some of the applications of simultaneous linear differential equations.

14.2 SIMPLE HARMONIC MOTION

When the acceleration of a particle is proportional to its displacement from a fixed point and is always directed towards it, then the motion is said to be *simple harmonic*.

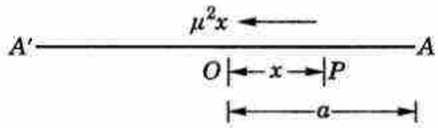


Fig. 14.1

If the displacement of the particle at any time t, from fixed point O is x (Fig. 14.1), then

$$\frac{d^2x}{dt^2} = -\mu^2 x \quad \text{or} \quad (D^2 + \mu^2)x = 0, \qquad ...(1)$$

$\therefore$ its solution is $\quad x = c_1 \cos \mu t + c_2 \sin \mu t \qquad ...(2)$

$\therefore$ its velocity at $\quad P = \dfrac{dx}{dt} = \mu(-c_1 \sin \mu t + c_2 \cos \mu t) \qquad ...(3)$

If the particle starts from rest at A, where $OA = a$,

then from (2), $\quad$ (at $t = 0, x = a$) $\quad a = c_1$

and from (3), $\quad$ (at $t = 0, dx/dt = 0$) $\quad 0 = c_2$.

Thus $\quad x = a \cos \mu t \qquad ...(4)$

and $\quad \dfrac{dx}{dt} = -a\mu \sin \mu t = -\sqrt{(a^2 - x^2)} \qquad ...(5)$

which give the displacement and the velocity of the particle at any time t.

Nature of motion. The particle starts from A towards O under acceleration directed towards O which gradually decreases until it vanishes at O, when the particle has acquired the maximum velocity. On passing through O, retardation begins and the particle comes to an instantaneous rest at A', where $OA' = OA$. It then retraces its path and goes on oscillating between A and A'.

The **amplitude** *or maximum displacement from the centre is a.*

The **periodic time,** *i.e., the time of complete oscillation is* $2\pi/\mu$, for when t is increased by $2\pi/\mu$, the values of x and dx/dt remain unaltered.

The **frequency** *or the number of oscillations per second is*

$$1/\text{periodic time, i.e., } \mu/2\pi.$$

Example 14.1. *In the case of a stretched elastic horizontal string which has one end fixed and a particle of mass m attached to the other, find the equation of motion of the particle given that l is the natural length of the string and e is its elongation due to weight mg. Also find the displacement s of particle when initially* $s = 0$, $v = 0$.

Solution. Let $OA(= l)$ be the elastic horizonta string with the end O fixed and having a particle of mass m attached to the end A. (Fig. 14.2)

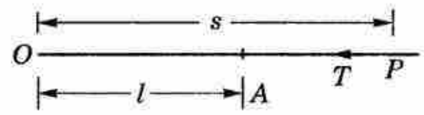


Fig. 14.2

At any time t, let the particle be at P where $OP = s$; so that the elongation $AP = s - l$.

Since for the elongation e, tension $= mg$

$\therefore$ for the elongation $s - l$, tension $T = \dfrac{mg(s-l)}{e}$

Tension being the only horizontal force, the equation of motion is

$$m\frac{d^2s}{dt^2} = -T \quad \text{or} \quad \frac{d^2s}{dt^2} = -\frac{T}{m} = -\frac{g(s-l)}{e} \qquad ...(i)$$

which is the required equation of motion.

Now (i) can be written as $(D^2 + g/e)s = gl/e$, where $D = d/dt$ $\qquad ...(ii)$

$\therefore$ the auxiliary equation is $D^2 + g/e = 0$ or $D = \pm i\sqrt{(g/e)}$

$$\therefore \qquad \text{C.F.} = c_1 \cos\sqrt{(g/e)}\,t + c_2 \sin\sqrt{(g/e)}\,t$$

and
$$\text{P.I.} = \frac{1}{D^2 + g/e}\cdot\frac{gl}{e} = \frac{gl}{e}\cdot\frac{l}{D^2 + g/e}e^{0t} = l$$

Thus the solution of (ii) is

$$s = c_1 \cos\sqrt{(g/e)}\,t + c_2 \sin\sqrt{(g/e)}\,t + l \qquad ...(iii)$$

When $t = 0$, $s = s_0$, $\quad \therefore \quad s_0 = c_1 + 0 + l$ *i.e.*, $c_1 = s_0 - l$

Again from (iii), $\quad \dfrac{ds}{dt} = -c_1\sqrt{(g/e)}\sin\sqrt{(g/e)}\,t + c_2\sqrt{(g/e)}\cos\sqrt{(g/e)}\,t$

When $t = 0$, $ds/dt = 0$; $\therefore \quad 0 = c_2$.

Substituting the values of c_1 and c_2 in (iii), we have

$$s = (s_0 - l)\cos\sqrt{(g/e)}\,t + l \text{ which is the required result.}$$

Example 14.2. *Two particles of masses* m_1 *and* m_2 *are tied to the ends of an elastic string of natural length a and modulus* λ. *They are placed on a smooth table so that the string is just taut and* m_2 *is projected with any velocity directly away from* m_1. *Show that the string will become slack after the lapse of time* $\pi\sqrt{[am_1m_2/\lambda(m_1 + m_2)]}$.

Solution. Taking O as fixed point of reference, let particle m_1 be at O and m_2 at a distance a from m_1 at time $t = 0$ Fig. 14.3. At any time t, let m_1 be of a distance x from O and m_2 be at a distance y from O. Then the equation of motion of m_1 is

Fig. 14.3

$$m_1 d^2x/dt^2 = T \qquad ...(i)$$

and equation of motion of m_2 is $m_2 d^2y/dt^2 = -T$ $\qquad ...(ii)$

where $T = \lambda(y - x)/a$

From (*i*) and (*ii*) $d^2y/dt^2 - d^2x/dt^2 = -\dfrac{T}{m_2} - \dfrac{T}{m_1}$

or $$\frac{d^2(y-x)}{dt^2} = -\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\frac{\lambda(y-x)}{a} \quad \text{or} \quad \frac{d^2u}{dt^2} = -\frac{\lambda(m_1+m_2)u}{m_1 m_2 a} \quad \text{where } u = y - x$$

This is S.H.M. with periodic time $\tau = 2\pi\sqrt{\left\{\dfrac{a m_1 m_2}{\lambda(m_1+m_2)}\right\}}$

The string will acquire its original length (*i.e.* become slack) after time τ_1 of m_2 moving towards m_1 such that

$$\tau_1 = \frac{\tau}{4}+\frac{\tau}{4} = \frac{\tau}{2} = \pi\sqrt{\left\{\frac{a m_1 m_2}{\lambda(m_1+m_2)}\right\}}.$$

Example 14.3. *A particle of mass m executes S.H.M. in the line joining the points A and B, on a smooth table and is connected with these points by elastic strings whose tensions in equilibrium are each T. If l, l′ be the extensions of the strings beyond their natural lengths, find the time of an oscillation.*

Solution. In the equilibrium position, let the particle be at C so that $AC = a + l$ and $BC = a' + l'$, where a, a' are natural lengths of the strings (Fig. 14.4). Then the tensions (at this time) are given by

$$T = \lambda l/a = \lambda' l'/a' \qquad ...(i)$$

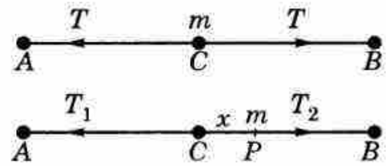


Fig. 14.4

At any time t, let the particle be at P, so that $CP = x$. Then

$$T_1 = \lambda\frac{l+x}{a} \text{ and } T_2 = \lambda\frac{l'-x}{a'}$$

$\therefore$ the equation of motion is $m\dfrac{d^2x}{dt^2} = T_2 - T_1 = \lambda'\dfrac{l'-x}{a'} - \lambda\dfrac{l+x}{a}$

$$= \left(\frac{\lambda' l'}{a'} - \frac{\lambda l}{a}\right) - \left(\frac{\lambda'}{a'}+\frac{\lambda}{a}\right)x = -\left(\frac{T}{l'}+\frac{T}{l}\right)x \qquad \text{[By } (i)]$$

or $$\frac{d^2x}{dt^2} = -\mu x \quad \text{where } \mu = \frac{l+l'}{ll'}\cdot\frac{T}{m}$$

Hence the periodic time $= \dfrac{2\pi}{\sqrt{\mu}} = 2\pi\sqrt{\left\{\dfrac{mll'}{(l+l')T}\right\}}$.

14.3 (1) SIMPLE PENDULUM

A heavy particle attached by a light string to a fixed point and oscillating under gravity constitutes a *simple pendulum*.

Let O be the fixed point, l be the length of the string and A be the position of the bob initially (Fig. 14.5). If P be the position of the bob at any time t, such that arc $AP = s$ and $\angle AOP = \theta$, then $s = l\theta$.

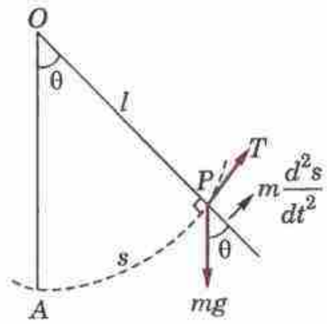


Fig. 14.5

$\therefore$ the equation of motion along PT is $m\dfrac{d^2s}{dt^2} = -mg\sin\theta$

i.e., $$\frac{d^2(l\theta)}{dt^2} = -g\sin\theta$$

or $$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta = -\frac{g}{l}\left(\theta - \frac{\theta^3}{3!}+\ldots\right) = -\frac{g\theta}{l} \text{ to a first approx.}$$

Here the auxiliary equation being $D^2 + g/l = 0$, we have $D = \pm\sqrt{(g/l)}\,i$

$\therefore$ its solution is $\theta = c_1\cos\sqrt{(g/l)}\,t + c_2\sin t$.

Thus the motion of the bob is simple harmonic and the time of an oscillation is $2\pi\sqrt{(l/g)}$.

Obs. The movement of the bob from one end to the other constitutes half an oscillation and is called a *beat* or *a swing*. Thus the time of one beats = $\pi\sqrt{(l/g)}$.

A seconds pendulum beats 86400 times a day for there are 86,400 seconds in 24 hours.

(2) Gain or loss of oscillations. Let a pendulum of length l make n beats in time T, so that

$$T = \text{time of } n \text{ beats} = n\pi\sqrt{(l/g)} \quad \text{or} \quad n = \frac{T}{\pi}(g/l)^{1/2}$$

Taking logs, $\log n = \log (T/\pi) + \frac{1}{2}(\log g - \log l)$.

Taking differentials of both sides, we get $\dfrac{dn}{n} = \dfrac{1}{2}\left(\dfrac{dg}{g} - \dfrac{dl}{l}\right)$.

If only g changes, l remaining constant, $\dfrac{\mathbf{dn}}{\mathbf{n}} = \dfrac{\mathbf{dg}}{\mathbf{2g}}$ ...(1)

If only l changes, g remaining constant, $\dfrac{\mathbf{dn}}{\mathbf{n}} = -\dfrac{\mathbf{dl}}{\mathbf{2l}}$. ...(2)

Example 14.4. *Find how many seconds a clock would lose per day if the length of its pendulum were increased in the ratio 900 : 901.*

Solution. If the original length l of the string be increased to $l + dl$, then

$$\frac{l + dl}{dl} = \frac{901}{900}. \quad \therefore \quad \frac{dl}{l} = \frac{901}{900} - 1 = \frac{1}{900}.$$

$\therefore$ using (2) above, we have $\dfrac{dn}{n} = -\dfrac{dl}{2l} = -\dfrac{1}{1800}$

i.e.,
$$dn = -\frac{n}{1800} = -\frac{86400}{1800} = -48.$$

Since dn is negative, the clock will lose 4 seconds per day.

Example 14.5. *A simple pendulum of length l is oscillating through a small angle θ in a medium in which the resistance is proportional to the velocity. Find the differential equation of its motion. Discuss the motion and find the period of oscillation.*

Solution. Let the position of the bob (of mass m), at any time t be P and O be the point of suspension such that $OP = l$, $\angle AOP = \theta$ and therefore, arc AP $= s = l\theta$. (Fig. 14.6)

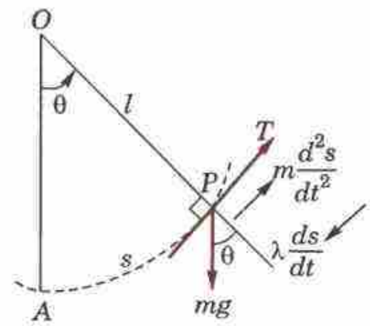


Fig. 14.6

$\therefore$ the equation of motion along the tangent PT is

$$m\frac{d^2s}{dt^2} = -mg \sin\theta - \lambda\frac{ds}{dt} \text{ where } \lambda \text{ is a constant.}$$

or
$$\frac{d^2(l\theta)}{dt^2} + \frac{\lambda}{m}\frac{d(l\theta)}{dt} + g\sin\theta = 0$$

Replacing $\sin\theta$ by θ since it is small and writing $\lambda/m = 2k$, we get

$$\frac{d^2\theta}{dt^2} + 2k\frac{d\theta}{dt} + \frac{g\theta}{l} = 0 \qquad ...(i)$$

which is the required differential equation.

Its auxiliary equation has roots $D = k \pm \sqrt{(k^2 - w^2)}$ where $w = g/l$.

The oscillatory motion of the bob is only possible when $k < w$.

Then the roots of the auxiliary equation are $-k \pm i\sqrt{(w^2 - k^2)}$.

$\therefore$ the solution of (i) is

$$\theta = e^{-kt}$$

which gives a vibratory motion of period $2\pi/\sqrt{(w^2 - k^2)}$.

Example 14.6. *A pendulum of length l has one end of the string fastened to a peg on a smooth plane inclined to the horizon at an angle α. With the string and the weight on the plane, its time of oscillation is t sec. If the pendulum of length l' oscillates in one sec. when suspended vertically, prove that* $\alpha = \sin^{-1}\left(\frac{l}{l't^2}\right)$.

(Kurukshetra, 2006)

Solution. At any time t, let the bob of mass m be at P and O be the point of suspension so that $OP = l$ and $\angle AOP = \theta$ (Fig. 14.7).

The component of weight along the plane being $mg \sin \alpha$, the equation of motion of the bob along the tangent at P is

$$m\frac{d^2s}{dt^2} = -mg \sin \alpha \sin \theta$$

or $$\frac{d^2(l\theta)}{dt^2} = -g \sin \alpha \sin \theta \qquad [\because \quad s = l\theta]$$

or $$\frac{d^2\theta}{dt^2} = -g \sin \alpha\left(\theta - \frac{\theta^3}{3!} + \ldots\right)$$

or $$\frac{d^2\theta}{dt^2} = -\mu\theta \quad \text{where } \mu = \frac{g \sin \alpha}{l}, \text{ to a first approximation.}$$

$\therefore$ the motion being simple harmonic, the time of oscillation t.

$$= \frac{2\pi}{\mu} = 2\pi\sqrt{\frac{l}{g \sin \alpha}} \qquad \ldots(i)$$

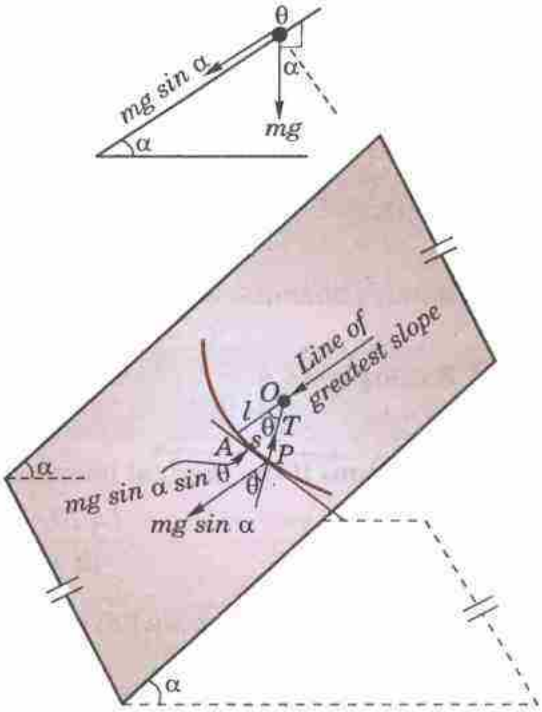


Fig. 14.7

We know that for a pendulum of length l' when suspended vertically, the time of oscillation

$$1 = 2\pi\sqrt{l'/g} \qquad \ldots(ii)$$

$\therefore$ dividing (i) by (ii), we have $t = \sqrt{\left(\frac{l}{l' \sin \alpha}\right)}$

or $$t^2 = l/l' \sin \alpha \quad \text{or} \quad \alpha = \sin^{-1}(l/l't^2).$$

PROBLEMS 14.1

1. A particle is executive simple harmonic motion with amplitude 20 cm and time 4 seconds. Find the time required by the particle in passing between points which are at distances 15 cm and 5 cm from the centre of force and are on the same side of it.
2. At the ends of three successive seconds, the distances of a point moving with S.H.M. from its mean position are x_1, x_2, x_3. Show that the time of a complete oscillation is
$$2\pi/\cos^{-1}\left(\frac{x_1 + x_3}{2x_2}\right).$$
3. An elastic string of natural length $2a$ and modulus λ is stretched between two points A and B distant $4a$ apart on a smooth horizontal table. A particle of mass m is attached to the middle of the string. Show that it can vibrate in line AB with period $2\pi/\omega$, where $\omega^2 = 2\lambda/am$.
4. A particle of mass m moves in a straight line under the action of force $mn^2(OP)$, which is always directed towards fixed point O in the line. If the resistance to the motion is $2\lambda\ mnv$, where v is the speed and $0 < \lambda < 1$, find the displacement x in terms of the time t given that when $t = 0$, $x = 0$ and $dx/dt = u$ where $OP = x$.
5. A point moves in a straight line towards the centre of force μ/(distance^3) starting from rest at a distance a from the centre of force, show that the time of reaching a point b from the centre of force is $a\sqrt{(a^2 - b^2)}/\sqrt{\mu}$ and that its velocity then is $\frac{\sqrt{\mu}}{ab}\sqrt{(a^2 - b^2)}$. *(U.P.T.U., 2001)*

6. A clock loses five seconds a day, find the alteration required in the length of its pendulum in order that it may keep correct time.
7. A clock with a seconds pendulum loses 10 seconds per day at a place where $g = 32$ ft/sec^2. What change in the gravity is necessary to make it accurate ?
8. A seconds pendulum which gains 10 seconds per day at one place loses 10 seconds per day at another; compare the acceleration due to gravity at the two places. *(Kurukshetra, 2005)*
9. Show that the free oscillations of a galvanometer needle, as affected by the viscosity of the surrounding air which varies directly as the angular velocity of the needle, are determined by the equation $\frac{d^2\theta}{dt^2} + K\frac{d\theta}{dt} + \mu\theta = 0$, where k is the co-efficient of viscosity and θ is the angular deflection of the needle at time t. Obtain θ in terms of t and discuss the different cases that can arise.
10. If $I = \frac{d^2\theta}{dt^2} = -mgl \sin\theta$, where I, m, g, l are constant, given that at $t = 0$, $\theta = 0$ and $d\theta/dt = \omega_0 = m\sqrt{(mgl)}/I$, then show that $t = \frac{2}{\omega_0}\log\frac{\pi+\theta}{4}$. *(Nagpur, 2009)*

14.4 OSCILLATIONS OF A SPRING

(*i*) **Free oscillations.** Suppose a mass m is suspended from the end A of a light spring, the other end of which is fixed at O. (Fig. 14.8)

Let e (= AB) be the elongation produced by the mass m hanging in equilibrium. If k be the restoring force per unit stretch of the spring due to elasticity, then for the equilibrium at B,

$$mg = T = ke \qquad \text{...(1)}$$

At any time t, after the motion ensues, let the mass be at P, where $BP = x$. Then the equation of motion of m is

$$m\frac{d^2x}{dt^2} = mg - k(e+x) = -kx \qquad \text{[By (1)]}$$

Or writing $k/m = \mu^2$, it becomes

$$\frac{d^2x}{dt^2} + \mu^2 x = 0 \qquad \text{...(2)}$$

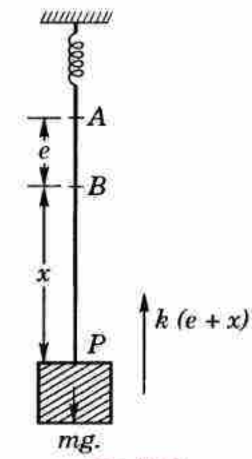


Fig. 14.8

This equation represents the free vibrations of the spring which are of the simple harmonic form having centre of oscillation at B—its equilibrium position and the *period of oscillation*

$$= \frac{2\pi}{\mu} = 2\pi\sqrt{\left(\frac{e}{g}\right)}. \qquad \left[\because \frac{1}{\mu} = \sqrt{\left(\frac{m}{k}\right)} = \sqrt{\left(\frac{e}{g}\right)}, \text{[By (1)]}\right]$$

(*ii*) **Damped oscillations.** If the mass m be subjected to do damping force proportional to velocity (say : $r\,dx/dt$) (Fig. 14.9), then the equation of motion becomes

$$m\frac{d^2x}{dt^2} = mg - k(e+x) - r\frac{dx}{dt}$$

$$= -kx - r\frac{dx}{dt} \qquad \text{[By (1)]}$$

Or writing $r/m = 2\lambda$ and $k/m = \mu^2$, it becomes

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \mu^2 x = 0 \qquad \text{...(3)}$$

$\therefore$ its auxiliary equation is

$$D^2 + 2\lambda D + \mu^2 = 0 \quad \text{whence} \quad D = -\lambda \pm .$$

Case I. *When* $\lambda > \mu$, the roots of the auxiliary equation are real and distinct (say γ_1, γ_2).

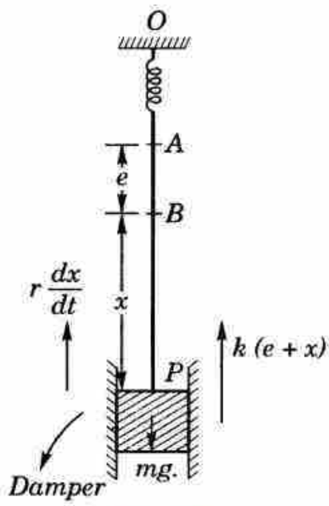


Fig. 14.9

$\therefore$ the solution of (3) is $x = c_1e^{\gamma_1 t} + c_2e^{\gamma_2 t}$ ...(4)

To determine c_1, c_2 let the spring be stretched to a length $x = l$ and then released so that

$x = l$ and $dx/dt = 0$ at $t = 0$.

$\therefore$ from (4), $l = c_1 + c_2$

Also from $\dfrac{dx}{dt} = c_1\gamma_1e^{\gamma_1 t} + c_2\gamma_2e^{\gamma_2 t}$, we get

$0 = c_1\gamma_1 + c_2\gamma_2$

whence $c_1 = \dfrac{-l\gamma_2}{\gamma_1 - \gamma_2}$ and $c_2 = \dfrac{l\gamma_1}{\gamma_1 - \gamma_2}$

Hence the solution of (3) is

$$x = \frac{l}{\gamma_1 - \gamma_2}(\gamma_1 e^{\gamma_2 t} - \gamma_2 e^{\gamma_1 t}) \qquad ...(5)$$

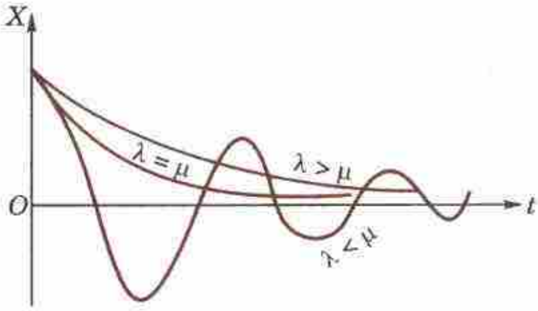


Fig. 14.10

which shows that x is always positive and decreases to zero as $t \to \infty$ (Fig. 14.10).

The restoring force, in this case, is so great that the motion is non-oscillatory and is, therefore, referred to as *over-damped* or *dead-beat* motion.

Case II. *When* $\lambda = \mu$, the roots of the auxiliary equation are real and equal, (each being $= -\lambda$).

$\therefore$ The general solution of (3) becomes $x = (c_1 + c_2t)e^{-\lambda t}$.

As in case I, if $x = l$ and $dx/dt = 0$ at $t = 0$, then $c_1 = l$ and $c_2 = \lambda l$.

Hence the solution of (3) is $x = l\,(1 + \lambda t)e^{-\lambda t}$ which also shows that x is always positive and decreases to zero as $t \to \infty$ (Fig. 14.10).

The nature of motion is similar to that of the previous case and is called the *critically damped motion* for it separates the non-oscillatory motion of case I from the most interesting oscillatory motion of case III.

Case III. *When* $\lambda < \mu$, the roots of the auxiliary equation are imaginary, *i.e.* $D = -\lambda \pm i\alpha$, where $\alpha^2 = \mu^2 - \lambda^2$.

$\therefore$ the solution of (3) is $x = e^{-\lambda t}(c_1 \cos \alpha t + c_2 \sin \alpha t)$

As in case I, $x = l$, $dx/dt = 0$ at $t = 0$, then $c_1 = l$ and $c_2 = \lambda l/\alpha$

Thus the solution of (3) becomes $x = le^{-\lambda t}\left(\cos \alpha t + \dfrac{\lambda}{\alpha}\sin \alpha t\right)$.

which can be put in the form $x = l\sqrt{\left\{1 + \left(\dfrac{\lambda}{\alpha}\right)^2\right\}}e^{-\lambda t}\cos\left\{\alpha - \tan^{-1}\dfrac{\lambda}{\alpha}\right\}$ ...(7)

Here the presence of the trigonometric factor in (7) shows that the *motion is oscillatory*, having

(*a*) the variable amplitude $= l\sqrt{[1 + (\lambda/\alpha)^2]e^{-\lambda t}}$ which decreases with time,

(*b*) the periodic time $T = 2\pi/\alpha$.

But the periodic time of free oscillations is $T' = 2\pi/\mu$.

As $\alpha = \sqrt{(\mu^2 - \lambda^2)} < \mu$

$\therefore$ $\dfrac{2\pi}{\alpha} > \dfrac{2\pi}{\mu}$, *i.e.* $T > T'$.

This shows that the *effect of damping is to increase the period of oscillation and the motion ultimately dies away*. Such a motion is termed as *damped oscillatory motion*.

(*iii*) **Forced oscillations** (*without damping*). If the point of the support of the spring is also vibrating with some external periodic force, then the resulting motion is called the *forced oscillatory motion*.

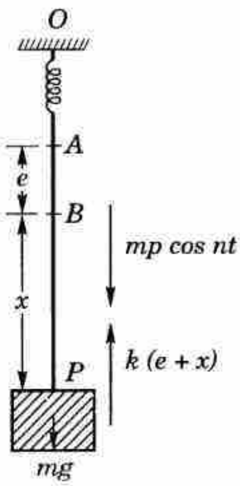


Fig. 14.11

Taking the external periodic force to be $mp \cos nt$, the equation of motion is

$$m\frac{d^2x}{dt^2} = mg - k(e + x) + mp \cos nt$$

$$= -kx + mp \cos nt \qquad [\because \; mg = ke] \qquad ...(8)$$

Or writing $k/m = \mu^2$, (8) takes the form

$$\frac{d^2x}{dt^2} + \mu^2 x = p \cos nt \qquad ...(9)$$

Its C.F. $= c_1 \cos \mu t + c_2 \sin \mu t$ and P.I. $= p \dfrac{1}{D^2 + \mu^2} \cos nt.$

New two cases arise :

Case I. *When* $\mu \neq n$.

$$\text{P.I.} = \frac{p}{\mu^2 - n^2} \cos nt.$$

$\therefore$ the complete solution of (9) is $x = c_1 \cos \mu t + c_2 \sin \mu t + \dfrac{p}{\mu^2 - n^2} \cos nt.$

On writing $c_1 \cos \mu t + c_2 \sin \mu t$ as $r \cos (\mu t + \phi)$, we have

$$x = r \cos (\mu t + \phi) + \frac{p}{\mu^2 - n^2} \cos nt \qquad ...(10)$$

This shows that the motion is compounded of two oscillatory motions : the first (due to the C.F.) *gives free oscillations of period* $2\pi/\mu$, *and the second* (due to the P.I.) *gives forced oscillations of period* $2\pi/n$.

Also we observe that if the frequency of free oscillations is very high (*i.e.*, μ is large), then the amplitude of forced oscillations is small.

Case II. *When* $\mu = n$.

$$\text{P.I.} = pt \cdot \frac{1}{2D} \cos \mu t = \frac{pt}{2} \int \cos \mu t \, dt = \frac{pt}{2\mu} \sin \mu t$$

$\therefore$ the complete solution of (9) is $x = c_1 \cos \mu t + c_2 \sin \mu t + \dfrac{pt}{2\mu} \sin \mu t$

$$= \left(c_2 + \frac{pt}{2\mu}\right) \sin \mu t + c_1 \cos \mu t.$$

Putting $c_2 + pt/2\mu = \rho \cos \psi$ and $c_1 = \rho \sin \psi$, we get

$$x = \rho \sin (\mu t + \psi) \qquad ...(11)$$

This shows that the oscillations are of period $2\pi/\mu$ and amplitude $\rho = \sqrt{[(c_2 + pt/2\mu)^2 + c_1^2]}$, which clearly increases with time (Fig. 14.12).

Thus the amplitude of the oscillations may become abnormally large causing over-strain and consequently breakdown of the system. In practice, however, collapse rarely occurs, though the amplitudes may become dangerously large since there is always some resistance present in the system.

This phenomenon of the impressed frequency becoming equal to the natural frequency of the system, is refered to as **resonance.**

Thus, while designing a machine or a structure, the occurrence of resonance should always be avoided to check the rupture of the system at any stage. That is why, the soldiers break step while marching over a bridge for the fear that their steps may not be in rhym with the natural frequency of the bridge causing its collapse due to 'resonance'.

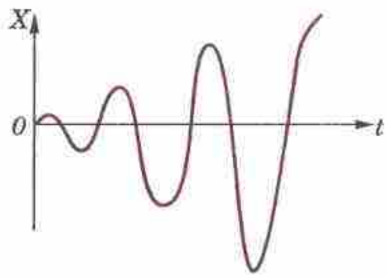


Fig. 14.12

(*iv*) **Forced oscillations** (*with damping*). If, in addition, there is a damping force proportional to velocity (say : $r\, dx/dt$) (Fig. 14.13), then the equation (8) becomes

$$m\frac{d^2x}{dt^2} = mg - k(e + x) + mp \cos nt - r\frac{dx}{dt}$$

$$= -kx + mp \cos nt - r\frac{dx}{dt} \qquad | \because \; mg = ke$$

On writing $r/m = 2\lambda$ and $k/m = \mu^2$, it takes the form

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \mu^2 x = p\cos nt \qquad ...(12)$$

Its auxiliary equation is $D^2 + 2\lambda D + \mu^2 = 0$ whence $D = -\lambda \pm \sqrt{\lambda^2 - \mu^2}$.

$$\therefore \qquad \text{C.F.} = e^{-\lambda t}[c_1 e^{t\sqrt{(\lambda^2-\mu^2)}} + c_2 e^{-t\sqrt{(\lambda^2-\mu^2)}}].$$

It represents the free oscillations of the system which die out as $t \to \infty$.

Also the P.I.

$$= p\frac{1}{D^2 + 2\lambda D + \mu^2}\cos nt = p\frac{1}{-n^2 + 2\lambda D + \mu^2}\cos nt$$

$$= p\frac{(\mu^2 - n^2) - 2\lambda D}{(\mu^2 - n^2)^2 - 4\lambda^2 D^2}\cos nt = p\frac{(\mu^2 - n^2)^2 \cos nt + 2\lambda n \sin nt}{(\mu^2 - n^2)^2 + 4\lambda^2 n^2}$$

Putting $\mu^2 - n^2 = R\cos\theta$ and $2\lambda n = R\sin\theta$, we get

$$\text{P.I.} = \frac{p}{\sqrt{[(\mu^2 - n^2)^2 + 4\lambda^2 n^2]}}\cos(nt - \theta)$$

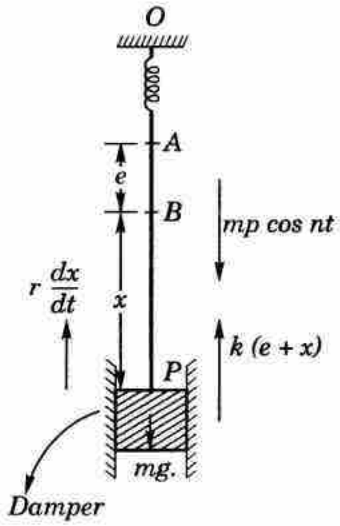


Fig. 14.13

which represents the forced oscillations of the system having

(*a*) a constant amplitude

$$= p/\sqrt{[(\mu^2 - n^2)^2 + 4\lambda^2 n^2]}$$

and (*b*) the period $= 2\pi/n$ which is the same as that of the impressed force.

Thus *with the increase of time, the free oscillations die away while the forced oscillations continue giving the steady state motion.*

Example 14.7. *A body weighing 10 kg is hung from a spring. A pull of 20 kg. wt. will stretch the spring to 10 cm. The body is pulled down to 20 cm below the static equilibrium position and then released. Find the displacement of the body from its equilibrium position at time t sec., the maximum velocity and the period of oscillation.*

Solution. Let O be the fixed end and A, the lower end of the spring (Fig. 14.14).

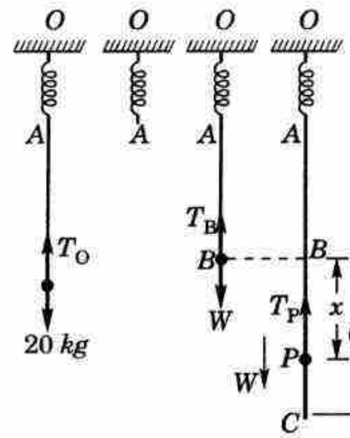


Fig. 14.14

Since a pull of 20 kg wt. at A stretches the spring by 0.1 m.

$$\therefore \qquad 20 = T_0 = k \times 0.1, \text{ i.e. } k = 200 \text{ kg/m}.$$

Let B be the equilibrium position when a body weighing $W = 10$ kg is hung from A ; then

$$10 = T_B = k \times AB$$

i.e.,
$$AB = \frac{10}{200} = 0.05 \text{ m}$$

Now the weight is pulled down to C, where $BC = 0.2$ m. After any time t sec. of its release from C, let the weight be at P where $BP = x$.

Then the tension $T_P = k \times AP = 200(0.05 + x) = 10 + 200x$.

$\therefore$ The equation of motion of the body is

$$\frac{W}{g}\frac{d^2x}{dt^2} = W - T_P, \text{ where } g = 9.8 \text{ m/sec}^2.$$

i.e.,
$$\frac{10}{9.8}\frac{d^2x}{dt^2} = 10 - (10 + 200x) \quad \text{or} \quad \frac{d^2x}{dt^2} = -\mu^2 x, \quad \text{where } \mu = 14.$$

This shows that the motion of the body is simple harmonic about B as centre and the period of oscillation $= 2\pi/\mu = 0.45$ sec.

Also the amplitude of motion being $BC = 0.2$ m., the displacement of the body from B at time t is given by $x = 0.2\cos\mu t = 0.2\cos 14t$ m

and the maximum velocity $= \mu$ (amplitude) $= 14 \times 0.2 = 2.8$ m/sec.

Example 14.8. *A spring fixed at the upper end supports a weight of 980 gm at its lower end. The spring stretches $\frac{1}{2}$ cm under a load of 10 gm and the resistance (in gm wt.) to the motion of the weigth is numerically equal to $\frac{1}{10}$ of the speed of the weight in cm/sec. The weight is pulled down $\frac{1}{4}$ cm. below its equilibrium position and then released. Find the expression for the distance of weight from its equilibrium position at time t during its first upward motion.*

Also find the time it takes the damping factor to drop to $\frac{1}{10}$ of its initial value.

Solution. Let O be the fixed end and A the other end of the spring (Fig. 14.15). Since load of 10 gm attached to A stretches the spring by $\frac{1}{2}$ cm.

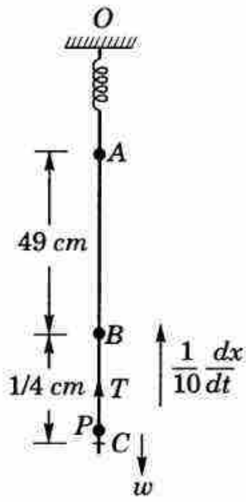


Fig. 14.15

$\therefore$ $$10 = T_0 = k \cdot \frac{1}{2} \text{ i.e., } k = 20 \text{ gm/cm.}$$

Let B be the equilibrium position when 980 gm. weight is attached to A, then

$$980 = T_B = k \times AB, \quad i.e., \quad AB = \frac{980}{20} = 49 \text{ cm.}$$

Now the 980 gm weight is pulled down to C, where $BC = \frac{1}{4}$ cm.

After any time t of its release from C, let the weight be at P, where $BP = x$. Then the tension

$$T = k \times AP = 20\,(49 + x) = 980 + 20x \text{ and the resistance to motion } = \frac{1}{10}\frac{dx}{dt}.$$

$\therefore$ the equation of motion is

$$\frac{980}{g}\frac{d^2x}{dt^2} = w - T - \frac{1}{10}\frac{dx}{dt} \qquad [\because g = 980 \text{ cm/sec}^2 \text{ (p. 449)}$$

$$= 980 - (980 + 20x) - \frac{1}{10}\frac{dx}{dt} \quad i.e. \quad 10\frac{d^2x}{dt^2} + \frac{dx}{dt} + 200x = 0 \qquad ...(i)$$

Its auxiliary equation is $10D^2 + D + 200 = 0$,

whence $$D = \frac{-1 + \sqrt{[1 - 4 \times 10 \times 200]}}{20} = \frac{-1 + i(89.4)}{20} = -0.05 \pm i\,(4.5)$$

$\therefore$ the solution of (i) is $x = e^{-0.05t}[c_1 \cos (4.5)t + c_2 \sin (4.5)t]$ ...(ii)

Also $$\frac{dx}{dt} = e^{-0.05t}(-0.05)\,[c_1 \cos (4.5)t + c_2 \sin (4.5)t] + e^{-0.05t}\,[-c_1 \sin (4.5)t + c_2 \cos (4.5)t](4.5) \qquad ...(iii)$$

Initially when the mass is at C, $t = 0$, $x = \frac{1}{4}$ cm. and $dx/dt = 0$.

From (ii), $c_1 = \frac{1}{4}$, and from (iii) $0 = (-0.05)c_1 + c_2(4.5)$, *i.e.*, $c_2 = -0.003$.

Thus, substituting these value in (ii), we get

$$x = e^{-0.05t}[0.25 \cos (4.5)t + 0.003 \sin (4.5)t]$$

which gives the displacement of the weight from the equilibrium position at any time t.

Here damping factor $= re^{-0.05t}$, where r is a constant of proportionality.

Its initial value $= re^0 = r$.

Suppose after time t, the damping factor $= r/10$. $\therefore$ $r/10 = re^{-0.05t}$ or $e^{t/20} = 10$.

Thus $$t = 20 \log_e 10 = 20 \times 2.3 = 46 \text{ sec.}$$

Example 14.9. *A spring which stretches by an amount e under a force $m\lambda^2 e$ is suspended from a support O and has a mass m at the lower end. Initially the mass is at rest in its equilibrium position at a point A below O. A vertical oscillation is now given to the support O such that at any time ($t > 0$) its displacement below its initial position is a sin nt. Show that the displacement x of the mass below A is given by*

$$d^2x/dt^2 + \lambda^2 x = \lambda^2 a \sin nt.$$

Hence show that if $n \neq \lambda$, the displacement is given by $x = \lambda a\,(\lambda \sin nt - n \sin \lambda t)/(\lambda^2 - n^2)$. What happens when $n = \lambda$?

Solution. If k be the stiffness of the spring then $m\lambda^2 e = ke$ *i.e.,* $k = m\lambda^2$.

Also in equilibrium $mg = ke$ ...(*i*)

Initially the mass is in equilibrium at A (Fig. 14.7). At time t, the support P is given a downward displacement $a \sin nt$. If the mass is displaced through a further distance x from A, then the equation of motion of the mass is given by

$$m\frac{d^2x}{dt^2} = mg - k(x+e) + ka \sin nt$$
$$= -kx + ka \sin nt \qquad \text{[By (i)]}$$

or $$\frac{d^2x}{dt^2} + \lambda^2 x = \lambda^2 a \sin nt \qquad [\because \; k = m\lambda^2]$$

or $$(D^2 + \lambda^2)x = \lambda^2 a \sin nt \qquad ...(ii)$$

Its A.E. $= c_1 \cos \lambda t + c_2 \sin \lambda t$

$$\text{P.I.} = \frac{1}{D^2+\lambda^2}\lambda^2 a \sin nt.$$

Now two cases arise :

Case I. *When $n \neq \lambda$*

$$\text{P.I.} = \lambda^2 a \frac{1}{n^2+\lambda^2} \sin nt$$

$\therefore$ the complete solution of (*ii*) is $x = c_1 \cos \lambda t + c_2 \sin \lambda t + \dfrac{\lambda^2 a}{\lambda^2 - n^2}\sin nt$ ...(*iii*)

$$\therefore \quad \frac{dx}{dt} = -c_1\lambda \sin \lambda t + c_2 \lambda \cos \lambda t + \frac{\lambda^2 an}{\lambda^2 - n^2}\cos nt$$

Initially when $t = 0$, $x = 0$ and $dx/dt = 0$.

$\therefore$ $c_1 = 0$ and $0 = c_2\lambda + \lambda^2 an/(\lambda^2 - n^2)$ *i.e.,* $c_2 = \lambda an/(\lambda^2 - n^2)$

Thus, substituting the values of c_1 and c_2 in (*iii*), we have

$$x = -\frac{\lambda an}{\lambda^2 - n^2}\sin \lambda t + \frac{\lambda^2 a}{\lambda^2 - n^2}\sin nt = \frac{\lambda a}{\lambda^2 - n^2}(\lambda \sin nt - n \sin \lambda t)$$

Case II. *When $n = \lambda$*

$$\text{P.I.} = \lambda^2 a \frac{1}{D^2+\lambda^2}\sin nt = \lambda^2 at \,.\, \frac{1}{2D}\sin \lambda t = \frac{\lambda^2 at}{2}\int \sin \lambda t\, dt = -\frac{\lambda at}{2}\cos \lambda t$$

$\therefore$ the complete solution is

$$x = c_1 \cos \lambda t + c_2 \sin \lambda t - \frac{\lambda at}{2}\cos \lambda t \qquad ...(iv)$$

$$\therefore \quad \frac{dx}{dt} = -c_1\lambda \sin \lambda t + c_2\lambda \cos \lambda t + \frac{\lambda^2 at}{2}\sin \lambda t - \frac{\lambda a}{2}\cos \lambda t$$

When $t = 0$, $x = 0$ and $dx/dt = 0$

$\therefore$ $0 = c_1$ and $0 = c_2\lambda - \lambda a/2$ *i.e.,* $c_2 = a/2$.

Thus, substituting the values of c_1 and c_2 in (*iv*), we get

$$x = \frac{a}{2}\sin \lambda t - \frac{\lambda at}{2}\cos \lambda t$$
$$= \frac{a}{2}(\sin \lambda t - \lambda t \cos \lambda t) \qquad [\text{Put } 1 = r\cos\phi \text{ and } \lambda t = r \sin \phi]$$
$$= \frac{ar}{2}\sin(\lambda t - \phi)$$

Its amplitude $\left(\frac{ar}{2}\right) = \frac{a}{2}\sqrt{(1 + \lambda^2 t^2)}$, which increases with time. Hence the phenomenon of *resonance* occurs.

Example 14.10. *A spring of negligible weight which stretches 1 inch under tension of 2 lb is fixed at one end and is attached to a weight of w lb at the other. It is found that resonance occurs when an axial periodic force 2 cos 2t lb acts on the weight. Show that when the free vibrations have died out, the forced vibrations are given by x = ct sin 2t, and find the values of w and c.*

Solution. As a weight of 2 lb attached to the lower end A of the spring stretched it by $\frac{1}{12}$ ft.

$$\therefore \qquad 2 = T = k \cdot \frac{1}{12}, \quad i.e., \ k = 24 \text{ lb/ft.}$$

Let B be the equilibrium position of the weight w attached to A (Fig. 14.16), then

$$w = T_B = k \times AB = 24 \times AB$$

$$\therefore \qquad AB = w/24 \text{ ft.}$$

At any time t, let the weight be at P, where $BP = x$.

Then the tension T at $P = k \times AP = 24\left(\frac{w}{24} + x\right) = w + 24x$

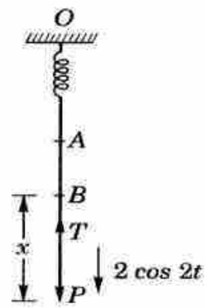


Fig. 14.16

$\therefore$ its equation of motion is

$$\frac{w}{g}\frac{d^2x}{dt^2} = -T + w + 2\cos 2t = -w - 24x + w + 2\cos 2t$$

or
$$w\frac{d^2x}{dt^2} + 24gx = 2g\cos 2t \qquad \ldots(i)$$

The phenomenon of **resonance** *occurs when the period of free oscillations is equal to the period of forced oscillations.*

Writing (i) as $\frac{d^2x}{dt^2} + \mu^2 x = \frac{2g}{w}\cos 2t$, where $\mu^2 = 24g/w$, the period of free oscillations is found to be $2\pi/\mu$ and the period of the force $(2g/w)\cos 2t$ is π.

$$\therefore \qquad 2\pi/\mu = \pi \quad \text{or} \quad 24g/w = \mu^2 = 4. \quad \text{Thus the weight, } w = 6g.$$

Taking this value of w, (i) takes the form

$$\frac{d^2x}{dt^2} + 4x = \frac{1}{3}\cos 2t \qquad \ldots(ii)$$

We know that the free oscillations are given by the C.F. and the forced oscillations by the P.I.

Thus, when the free oscillations have died out, the forced oscillations are given by the P.I. of (ii).

Now P.I. of $(ii) = \frac{1}{3}\cdot\frac{1}{D^2+4}\cos 2t = \frac{1}{3}t\cdot\frac{1}{2D}\cos 2t = \frac{1}{12}t\sin 2t.$

Hence $\qquad c = \frac{1}{12}.$

PROBLEMS 14.2

1. An elastic string of natural length a is fixed at one end and a particle of mass m hangs freely from the other end. The modulus of elasticity is mg. The particle is pulled down a further distance l below its equilibrium position and released from rest. Show that the motion of the particle is simple harmonic and find the pariodicity.
2. A mass of 4 lb suspended from a light elastic string of natural length 3 feet extends it to a distance 2 feet. One end of the string is fixed and a mass of 2 lb is attached to other. The mass is held so that the string is just unstretched and is then let go. Find the amplitude, the period and the maximum velocity of the ensuing simple harmonic motion.

3. A light elastic string of natural length l has one extremity fixed at a point A and the other end attached to a stone, the weight of which in equilibrium would extend the string to a depth l_1. Show that if the stone be dropped from rest at A, it will come to instantaneous rest at a depth $\sqrt{(l_1^2 - l^2)}$ below the equilibrium position.
4. A 4 lb weight on a string stretches it 6 in. Assuming that a damping force in lb wt. equal to λ times the instantaneous velocity in ft/sec. acts on the weight, show that the motion is over damped, critically damped or oscillatory according as $\lambda >=< 2$. Find the period of oscillation when $\lambda = 1.5$.
5. A mass of 200 gm is tied at the end of a spring which extends to 4 cm under a force 196,000 dynes. The spring is pulled 5 cm and released. Find the displacement t seconds after release if there be a damping force of 2000 dynes per cm per second.
6. A body weighing 16 lb is suspended by a spring in a fluid whose resistance in lb wt. is twice the speed of the body in ft/sec. A pull of 25 lb wt. would stretch the spring 3 inches. The body is drawn 3 inches below the equilibrium position in the fluid and then released. Find the period of oscillations and the time required for the damping factor to be reduced to one-tenth of its initial value. *(Sambhalpur, 1998)*
7. A mass M suspended from the end of a helical spring is subjected to a periodic force $f = F \sin \omega t$ in the direction of its length. The force f is measured positive vertically downwards and at zero time M is at rest. If the spring stiffness is S, prove that the displacement of M at time t from the commencement of motion is given by

$$x = \frac{F}{M(p^2 - \omega^2)}\left[\sin \omega t - \frac{\omega}{p}\sin pt\right]$$

where $p^2 = S/M$ and damping effects are neglected. *(U.P.T.U., 2002)*
8. A vertical spring having 4.5 lb/ft. has 16 lb wt. suspended from it. An external force of 24 sin $9t$ $(t \geq 0)$ lb wt. is applied. A damping force given numerically in lb. wt. by four times its velocity in ft/sec, is assumed to act. Initially the weight is at rest at its equilibrium position. Determine the position of the weight at any time. Also find the amplitude, period and the frequency of the steady-state solution.
9. A body weighing 4 lb hangs at rest on a spring producing in the spring an extension of 1ft. The upper end of the spring in now made to execute a vertical simple harmonic oscillation $x = \sin 4t$, x being measured vertically downwards in feet. If the body is subject to a frictional resistance whose magnitude in lb wt. is one-quarter of its velocity in feet per second, obtain the differential equation for the motion of the body and find the expression for its displacement at time t, when t is large.
10. A body executes damped forced vibrations given by the equation

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + b^2x = e^{-kt} \sin nt.$$

Solve the equation for both the cases, when $n^2 \neq b^2 - k^2$ and $n^2 = b^2 - k^2$. *(U.P.T.U., 2004)*

14.5 OSCILLATORY ELECTRICAL CIRCUIT

(i) L-C circuit

Consider an electrical circuit containing an inductance L and capacitance C (Fig. 14.17).

Let i be the current and q the charge in the condenser plate at any time t, so that the voltage drop across

$$L = L\frac{di}{dt} = L\frac{d^2q}{dt^2}$$

and the voltage drop across $C = q/C$.

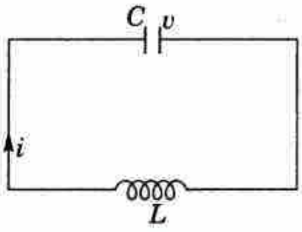


Fig. 14.17

As there is no applied e.m.f. in the circuit, therefore, by Kirchhoff's first law, we have

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = 0.$$

Or dividing by L and writing $1/LC = \mu^2$, we get $\dfrac{d^2q}{dt^2} + \mu^2 q = 0$ ...(1)

This equation is precisely same as (2) on page 507 and, therefore, *it represents free electrical oscillations of the current having period* $2\pi/\mu = 2\pi\sqrt{LC}$.

Thus the discharging of a condenser through an inductance L is same as the motion of the mass m at the end of a spring.

(*ii*) **L–C–R circuit**

Now consider the discharge of a condenser C through an inductanc L and the resistance R (Fig. 14.18). Since the voltage drop across L, C and R are respectively

$$L\frac{d^2q}{dt^2}, \frac{q}{C} \text{ and } R\frac{dq}{dt}.$$

$\therefore$ by Kirchhoff's law, we have $L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{q}{C} = 0$ ...(2)

Or writing $R/L = 2\lambda$ and $1/LC = \mu^2$, we have $\dfrac{d^2q}{dt^2} + 2\lambda\dfrac{dq}{dt} + \mu^2 q = 0$

This equation is same as (3) on page 507 and, therefore has the same solution as for the mass m on a spring with a damper.

Thus *the charging or discharging of a condenser through the resistance R and an inductance L is an electrical analogue of the damped oscillations of mass m on a spring.*

(*iii*) **L–C circuit with e.m.f.** = *p cos nt*.

The equation (1) for an L–C circuit (Fig. 14.19), now becomes $L\dfrac{d^2q}{dt^2} + \dfrac{q}{C} = p \cos nt.$

Or writing $1/LC = \mu^2$, we have $\dfrac{d^2q}{dt^2} + \mu^2 q = \dfrac{p}{L}\cos nt$ ...(3)

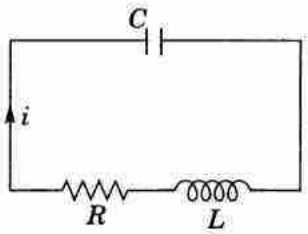


Fig. 14.18

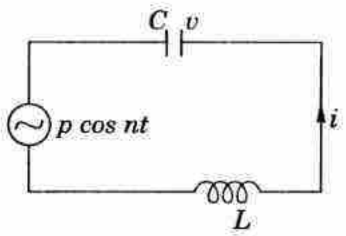


Fig. 14.19

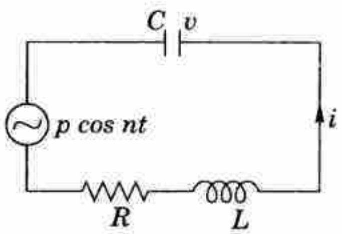


Fig. 14.20

This equation is of the same form as (9) on page 509 and, therefore, has the solution as for the motion of a mass m on a spring with external periodic force $p \cos nt$ acting on it.

Thus the condenser placed in series with source of e.m.f. (= p cos nt) *and discharging through a coil containing inductance L is an electrical analogue of the forced oscillations of the mass m on a spring.*

An electrical instance of *resonance* phenomena occurs while tuning a radio-station, for the natural frequency of the tuning of L–C circuit is made equal to the frequency of the desired radio-station, giving the maximum output of the receiver at the said receiving station.

(*iv*) **L–C–R circuit with e.m.f.** = *p cos nt*.

The equation of (2) above, now becomes $L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{q}{C} = p \cos nt.$ (Fig. 14.20)

Or writing $R/L = 2\lambda$ and $1/LC = \mu^2$ as before, we have

$$\frac{d^2q}{dt^2} + 2\lambda\frac{dq}{dt} + \mu^2 q = \frac{p}{L}\cos nt \quad ...(4)$$

This equation is exactly same as (12) on page 510 and, therefore, its C.F. represents the free oscillations of the circuit whereas the P.I. represents the forced oscillations.

Here also as t increases, the free oscillations die out while the forced oscillations persist giving steady motion.

Thus the L–C–R circuit with a source of alternating e.m.f. is an electrical equivalent of the mechanical phenomena of forced oscillations with resistance.

14.6 ELECTRO-MECHANICAL ANALOGY

We have just seen, how merely by renaming the variables, the differential equation representing the oscillation of a weight on a spring represents an analogous electrical circuit. As electrical circuits are easy to assemble and the currents and voltages are accurately measured with ease, this affords a practical method of studying the oscillations of complicated mechanical systems which are expensive to make and unwieldy to handle by considering an equivalent electrical circuit. While making an electrical equivalent of a mechanical system, the following correspondences between the elements should be kept in mind, noting that the circuit may be in series or in parallels :

Mech. System	*Series circuit*	*Parallel circuit*
Displacement	Current i	Voltage E
Force or couple	Voltage E	Current i
Mass m or $M.I.$	Inductance L	Capacitance C
Damping force	Resistance R	Conductance $1/R$
Spring modulus	Elastance $1/C$	Susceptance $1/L$

Example 14.11. *An uncharged condenser of capacity C is charged by applying an e.m.f.* $E \sin t/\sqrt{(LC)}$, *through leads of self-inductance L and negligible resistance. Prove that at any time t, the charge on one of the plates is* $\frac{EC}{2}\left\{\sin\frac{t}{\sqrt{(LC)}} - \frac{t}{\sqrt{(LC)}}\cos\frac{t}{\sqrt{(LC)}}\right\}$ (U.P.T.U., 2003)

Solution. If q be the change on the condenser, the differential equation of the circuit is

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = E\sin\frac{t}{\sqrt{(LC)}} \qquad ...(i)$$

Its A.E. is $LD^2 + 1/C = 0$ or $D = \pm 1/\sqrt{(LC)}$

$\therefore$ C.F. $= c_1 \cos t/\sqrt{LC} + c_2 \sin t/\sqrt{(LC)}$

and P.I. $= \dfrac{1}{LD^2 + \frac{1}{C}} E \sin\dfrac{t}{\sqrt{(LC)}}$ $\qquad \left[\text{Putting } D^2 = -\frac{1}{LC}, \text{denom.} = 0\right]$

$$= Et\frac{1}{2LD}\sin\frac{t}{\sqrt{(LC)}} = \frac{Et}{2L}\int \sin\frac{t}{\sqrt{(LC)}}dt = -\frac{Et}{2L}\sqrt{(LC)}\cos\frac{t}{\sqrt{(LC)}} = -\frac{Et}{2}\sqrt{\frac{C}{L}}\cos\frac{t}{\sqrt{LC}}$$

Thus the C.S. of (i) is $q = c_1\cos\dfrac{t}{\sqrt{LC}} + c_2\sin\dfrac{t}{\sqrt{LC}} - \dfrac{Et}{2}\sqrt{\dfrac{C}{L}}\cos\dfrac{t}{\sqrt{LC}}$

When $t = 0, q = 0, c_1 = 0$

$$\therefore \quad q = c_2\sin\frac{t}{\sqrt{LC}} - \frac{Et}{2}\sqrt{\frac{C}{L}}\cos\frac{t}{\sqrt{LC}} \qquad ...(ii)$$

Differentiating (ii) w.r.t. t, we get

$$\frac{dq}{dt} = \frac{c_2}{\sqrt{LC}}\cos\frac{t}{\sqrt{LC}} - \frac{E}{2}\sqrt{\frac{C}{L}}\left\{\cos\frac{t}{\sqrt{LC}} - \frac{t}{\sqrt{LC}}\sin\frac{t}{\sqrt{LC}}\right\}$$

Also when $t = 0$, $dq/dt = i = 0$,

$$\therefore \quad \frac{c_2}{\sqrt{LC}} - \frac{E}{2}\sqrt{\frac{C}{L}} = 0 \quad \text{or} \quad c_2 = \frac{EC}{2}.$$

Substituting the value of c_2 in (ii), q at any time t is given by

$$q = \frac{EC}{2}\left\{\sin\frac{t}{\sqrt{LC}} - \frac{t}{\sqrt{LC}}\cos\frac{t}{\sqrt{LC}}\right\}.$$

Example 14.12. *In an L—C—R circuit, the charge q on a plate of a condenser is given by*

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E\sin pt.$$

The circuit is tuned to resonance so that $p^2 = 1/LC$. *If initially the current i and the charge q be zero, show that, for small values of R/L, the current in the circuit at time t is given by*

$(Et/2L)\sin pt.$ (U.P.T.U., 2004)

Solution. Given differential equation is $(LD^2 + RD + 1/C)q = E \sin pt$ ...(i)

Its auxiliary equation is $LD^2 + RD + 1/C = 0$,

which gives $$D = \frac{1}{2L}\left[-R \pm \sqrt{\left(R^2 - \frac{4L}{C}\right)}\right] = -\frac{R}{2L} + \sqrt{\left(\frac{R^2}{4L^2} - \frac{1}{LC}\right)}$$

As R/L is small, therefore, to the first order in R/L,

$$D = -\frac{R}{2L} \pm i\frac{1}{\sqrt{(LC)}} = -\frac{R}{2L} \pm ip \qquad \left[\because\ p^2 = \frac{1}{LC}\right]$$

$\therefore$ $$\text{C.F.} = e^{-(Rt/2L)}(c_1 \cos pt + c_2 \sin pt)$$
$$= (1 - Rt/2L)(c_1 \cos pt + c_2 \sin pt) \text{ rejecting terms in } (R/L)^2 \text{ etc.}$$

and $$\text{P.I.} = \frac{1}{LD^2 + RD + 1/C} E \sin pt = E\frac{1}{-Lp^2 + RD + 1/C}\sin pt$$
$$= \frac{E}{R}\int \sin pt\, dt = -\frac{E}{Rp}\cos pt \qquad \left[\because p^2 = \frac{1}{LC}\right]$$

Thus the complete solution of (i) is $q = \left(1 - \frac{Rt}{2L}\right)(c_1 \cos pt + c_2 \sin pt) - \frac{E}{Rp}\cos pt$ ...(ii)

$\therefore$ $$i = \frac{dq}{dt} = \left(1 - \frac{Rt}{2L}\right)(-c_1 \sin pt + c_2 \cos pt)\,p - \frac{R}{2L}(c_1 \cos pt + c_2 \sin pt) + \frac{E}{R}\sin pt \quad ...(iii)$$

Initially, when $t = 0, q = 0, i = 0$ $\therefore$ from (ii), $0 = c_1 - E/Rp$ $\therefore$ $c_1 = E/Rp$ and from (iii),

$$0 = c_2 p - Rc_1/2L \therefore c_2 = Rc_1/2Lp = E/2Lp^2$$

Thus, substituting these values of c_1 and c_2 in (iii), we get

$$i = \left(1 - \frac{Rt}{2L}\right)\left(-\frac{E}{Rp}\sin pt + \frac{E}{2Lp^2}\cos pt\right)p - \frac{R}{2L}\left(\frac{E}{Rp}\cos pt + \frac{E}{2Lp^2}\sin pt\right) + \frac{E}{R}\sin pt$$

$$= \frac{Et}{2L}\sin pt. \qquad [\because\ R/L \text{ is small}]$$

PROBLEMS 14.3

1. Show that the frequency of free vibrations in a closed electrical circuit with inductance L and capacity C in series is $\frac{30}{\pi\sqrt{(LC)}}$ per minute.
2. The differential equation for a circuit in which self-inductance and capacitance neutralize each other is $L\frac{d^2i}{dt^2} + \frac{i}{C} = 0$. Find the current i as a function of t given that I is the maximum current, and $i = 0$ when $t = 0$.
3. A constant e.m.f. E at $t = 0$ is applied to a circuit consisting of inductance L, resistance R and capacitance C in series. The initial values of the current and the charge being zero, find the current at any time t, if $CR^2 < 4L$. Show that the amplitudes of the successive vibrations are in geometrical progression.
4. The damped LCR circuit is governed by the equation $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$ where, L, R, C are positive constants. Find the conditions under which the circuit is over damped, under damped and critically damped. Find also the critical resistance. *(U.P.T.U., 2005)*
5. A condenser of capacity C discharged through an inductance L and resistance R in series and the charge q at time t satisfies the equation $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$. Given that $L = 0.25$ henries, $R = 250$ ohms, $C = 2 \times 10^{-6}$ farads, and that when $t = 0$, charge q is 0.002 coulombs and the current $dq/dt = 0$, obtain the value of q in terms of t.
6. An e.m.f. $E \sin pt$ is applied at $t = 0$ to a circuit containing a capacitance C and inductance L. The current i satisfies the equation $L\frac{di}{dt} + \frac{1}{C}\int i\, dt = E \sin pt$. If $p^2 = 1/LC$ and initially the current i and the charge q are zero, show that the current at time t is $(Et/2L) \sin pt$, where $i = dq/dt$.

7. For an L—R—C circuit, the charge q on a plate of the condenser is given by $L\dfrac{d^2q}{dt^2}+R\dfrac{dq}{dt}+\dfrac{q}{C}=E\sin\omega t$, where $i=\dfrac{dq}{dt}$. The circuit is tuned to resonance so that $\omega^2 = 1/LC$.

   If $CR^2 < 4L$ and initially $q = 0$, $i = 0$, show that $q=\dfrac{E}{R\omega}\left[e^{-Rt/2C}\left(\cos pt+\dfrac{R}{2Lp}\sin pt\right)-\cos\omega t\right]$

   where $$p^2=\frac{1}{LC}-\frac{R^2}{4L^2}.$$ (*U.P.T.U., 2003*)

8. An alternating E.M.F. $E\sin pt$ is applied to a circuit at $t = 0$. Given the equation for the current i as $L\dfrac{d^2i}{dt^2}+R\dfrac{di}{dt}+\dfrac{i}{C}=pE\cos pt$, find the current i when (*i*) $CR^2 > 4L$, (*ii*) $CR^2 < 4L$.

14.7 DEFLECTION OF BEAMS

Consider a uniform beam as made up of fibres running lengthwise. We have to find its deflection under given loadings.

In the bent form, the fibres of the lower half are stretched and those of upper half are compressed. In between these two, there is a layer of unstrained fibres called the *neutral surface*. The fibre which was initially along the x-axis (the central horizontal axis of the beam) now lies in the neutral surface, in the form of a curve called the *deflection curve* or the *elastic curve*. We shall encounter differential equations while finding the equation of this curve.

Consider a cross-section of the beam cutting the elastic curve in P and the neutral surface in the line AA'—called the neutral axis of this section (Fig. 14.21).

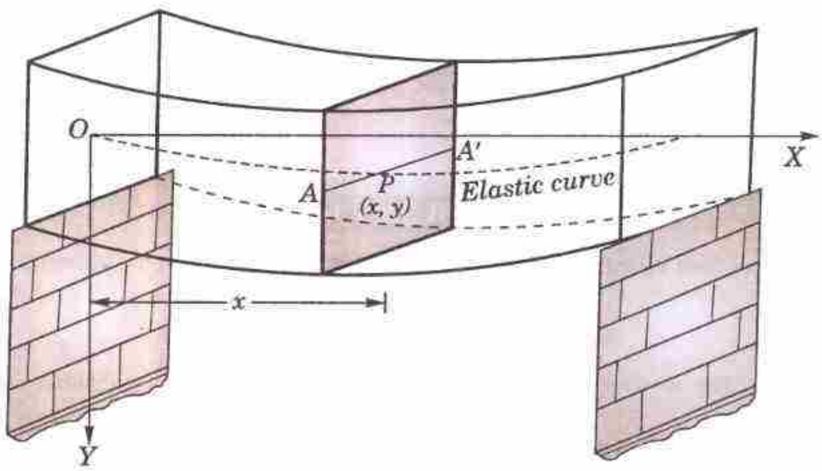


Fig. 14.21

It is well-known from mechanics that the bending moment M about AA', of all forces acting on either side of the two portions of the beam separated by this cross-section, is given by the *Bernoulli-Euler law*

$$M = EI/R$$

where E = modulus of elasticity of the beam,

I = moment of inertia of the cross-section about AA',

and R = radius of curvature of the elastic curve at $P(x, y)$.

If the deflection of the beam is small, the slope of the elastic curve is also small so that we may neglect $(dy/dx)^2$ in the formula,

$$R=\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}\Bigg/\frac{d^2y}{dx^2}.$$ Thus for small deflections, $R = 1/(d^2y/dx^2)$.

Hence (1) *Bending moment* $\mathbf{M} = \mathbf{EI}\,\dfrac{\mathbf{d^2y}}{\mathbf{dx^2}}$

(2) *Shear force* $\left(=\dfrac{dM}{dx}\right) = EI\,\dfrac{d^3y}{dx^3}$;

(3) *Intensity of loading* $\left(=\dfrac{d^2M}{dx^2}\right) = EI\,\dfrac{d^4y}{dx^4}$

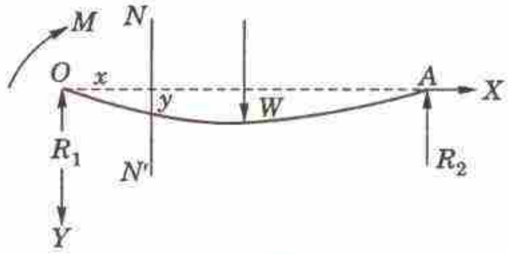


Fig. 14.22

(4) *Convention of signs.* The sum of the moments about a section NN' due to external forces on the left of the section, if anti-clockwise is taken as positive and if clockwise (as in Fig. 14.22) is taken as negative.

The deflection y downwards and length x to the right are taken as positive. The slope dy/dx will be positive if downwards in the direction of x-positive.

(5) *End conditions.* The arbitrary constants appearing in the solution of the differential equation (1) for a given problem are found from the following end conditions :

(*i*) *At a freely supported end* (Fig. 14.23), there being no deflection and no bending moment, we have $y = 0$ and $d^2y/dx^2 = 0$.

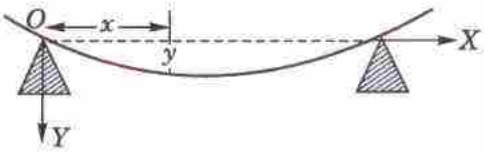


Fig. 14.23

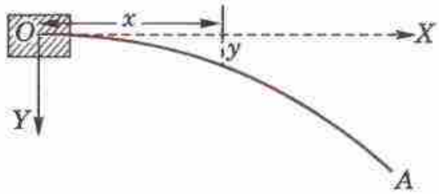


Fig. 14.24

(*ii*) *At a (horizontal) fixed end* (Fig. 14.24), the deflection and the slope of the beam being both zero, we have

$$y = 0 \text{ and } dy/dx = 0.$$

(*ii*) *At a perfectly free end* (*A* in Fig. 14.24), there being no bending moment or shear force, we have

$$\frac{d^2y}{dx^2} = 0 \quad \text{and} \quad \frac{d^3y}{dx^3} = 0$$

(6) *A member of a structure or a machine when subjected to end thrusts only is called a* **strut** *and a vertical strut is called a* **column**.

There are four possible ways of the end fixation of a strut:

(*i*) Both ends fixed, called a *built-in* or *encastre* strut.

(*ii*) One end fixed and the other freely supported, hinged or pin-jointed.

(*iii*) One end fixed and the other end free, called a *cantilever.*

(*iv*) Both ends freely supported or pin-jointed.

Example 14.13. *The deflection of a strut of length l with one end (x = 0) built-in and the other supported and subjected to end thrust P, satisfies the equation*

$$\frac{d^2y}{dx^2} + a^2y = \frac{a^2R}{P}(l - x).$$

Prove that the deflection curve is $y = \dfrac{R}{P}\left(\dfrac{\sin ax}{a} - l\cos ax + l - x\right)$, *where* $al = \tan al$.

(*U.P.T.U., 2001*)

Solution. Given differential equation is $(D^2 + a^2)y = \dfrac{a^2R}{P}(l - x)$ ...(*i*)

Its auxiliary equation is $D^2 + a^2 = 0$, whence $D = \pm\, ai$.

$$\therefore \qquad C.F. = \frac{1}{D^2 + a^2}\,\frac{a^2R}{P}(l - x) = \frac{R}{P}\left(1 + \frac{D^2}{a^2}\right)^{-1}(l - x)$$

$$= \frac{R}{P}\left(1 - \frac{D^2}{a^2} + \ldots\right)(l - x) = \frac{R}{P}(l - x)$$

Thus the complete solution of (*i*) is $y = c_1 \cos ax + c_2 \sin ax + \dfrac{R}{P}(l - x)$ ...(*ii*)

Also $$\frac{dy}{dx} = -c_1 a \sin ax + c_2 a \cos ax - \frac{R}{P}$$ ...(*iii*)

Now as the end O is built in (Fig. 14.25). $\therefore\ y = dy/dx = 0$ at $x = 0$.

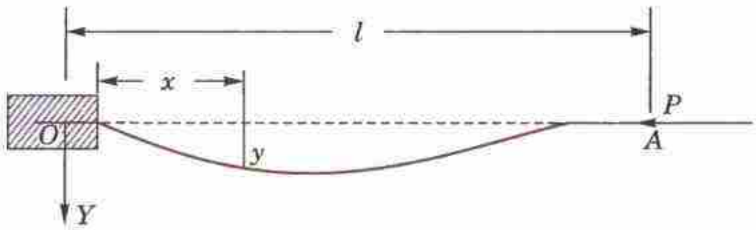


Fig. 14.25

$\therefore$ from (*ii*) and (*iii*), we have

$$0 = c_1 + Rl/P \text{ and } 0 = c_2 a - R/P$$

whence $$c_1 = -Rl/P \quad \text{and} \quad c_2 = R/aP$$

Thus (*ii*) becomes $$y = \frac{R}{P}\left(\frac{\sin ax}{a} - l\cos ax + l - x\right)$$ ...(*iv*)

which is the desired equation of the deflection curve.

The end A being freely supported $y = 0$ when $x = l$ (We don't need the other condition $d^2y/dx^2 = 0$].

$\therefore$ (*iv*) gives $$0 = \frac{R}{P}\left(\frac{\sin al}{a} - l\cos al\right) \text{ whence } al = \tan al.$$

Example 14.14. *A horizontal tie-rod is freely pinned at each end. It carries a uniform load w lb per unit length and has a horizontal pull P. Find the central deflection and the maximum bending moment, taking the origin at one of its ends.*

Solution. Let OA be the given beam of length l (Fig.14.26).

At each end there is a vertical reaction $R = wl/2$.

The external forces acting to the left of the section NN' are :

(*i*) the horizontal pull P, (*ii*) the reaction $R = wl/2$ and (*iii*) the weight of the portion $ON = wx$ acting mid-way.

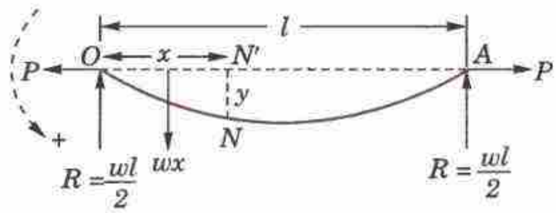


Fig. 14.26

Taking moments about, N, we have

$$EI\frac{d^2y}{dx^2} = Py - \frac{wl}{2}x + wx \cdot \frac{x}{2}$$

or $$EI\frac{d^2y}{dx^2} - Py = \frac{w}{2}(x^2 - lx) \quad \text{or} \quad \frac{d^2y}{dx^2} - a^2 y = \frac{w}{2EI}(x^2 - lx), \text{ where } a^2 = \frac{P}{EI}$$ ...(*i*)

This is the differential equation of the elastic curve. Its auxiliary equation is $D^2 - a^2 = 0$, whence $D = \pm a$.

$\therefore$ $$\text{C.F.} = c_1 \cosh ax + c_2 \sinh ax$$

and $$\text{P.I.} = \frac{1}{D^2 - a^2}\frac{w}{2EI}(x^2 - lx) = \frac{-w}{2EIa^2}\left(1 - \frac{D^2}{a^2}\right)^{-1}(x^2 - lx)$$

$$= -\frac{w}{2P}\left(1 + \frac{D^2}{a^2}\ldots\right)(x^2 - lx) = -\frac{w}{2P}\left(x^2 - lx + \frac{2}{a^2}\right).$$

Thus the complete solution of (*i*) is $y = c_1 \cosh ax + c_2 \sinh ax - \dfrac{W}{2P}\left(x^2 - lx + \dfrac{2}{a^2}\right)$ ...(*ii*)

At the end O, $y = 0$ when $x = 0$, [We don't need the other condition $d^2y/dx^2 = 0$]

$\therefore$ (*ii*) gives $0 = c_1 - w/Pa^2$, or $c_1 = w/Pa^2$ ...(*iii*)

At the end A, $y = 0$ when $x = l$, [We don't need the other condition $d^2y/dx^2 = 0$]

$\therefore$ (*ii*) gives $0 = c_1 \cosh al + c_2 \sinh al - w/Pa^2$ or $c_2 \sinh al = \dfrac{W}{Pa^2}(1 - \cosh al)$

whence $$c_2 = -\frac{w}{Pa^2} \tanh \frac{al}{2} \qquad ...(iv)$$

Substituting these values of c_1 and c_2 in (*ii*), we get

$$y = \frac{w}{Pa^2}\left(\cosh ax - \tanh \frac{al}{2} \sinh ax\right) - \frac{w}{2P}\left(x^2 - lx + \frac{2}{a^2}\right)$$

which gives the deflection of the beam at N.

Thus the central deflection $= y$ (at $x = l/2$)

$$= \frac{w}{Pa^2}\left(\cosh \frac{al}{2} - \tanh \frac{al}{2} \sinh \frac{al}{2} - 1\right) + \frac{wl^2}{8P} = \frac{w}{Pa^2}\left(\text{sech}\, \frac{al}{2} - 1\right) + \frac{wl^2}{8P}$$

Also the bending moment is maximum at the point of maximum deflection ($x = l/2$).

$\therefore$ *The maximum bending moment*

$$= EI\frac{d^2y}{dx^2}(\text{at } x = l/2) = Py + \frac{w}{2}(x^2 - lx)\ (\text{at } x = l/2) = \frac{w}{a}\left(\text{sech}\, \frac{al}{2} - 1\right)$$

Example 14.15. *A cantilever beam of length l and weighing w lb/unit is subjected to a horizontal compressive force P applied at the free end. Taking the origin at the free end and y-axis upwards, establish the differential equation of the beam and hence find the maximum deflection.*

Solution. Let $N(x, y)$ be any point of the beam referred to axes through the free end as shown (Fig. 14.27).

The external forces acting to the left of the section NN', are

(*i*) the compressive force P,

(*ii*) the weight of the portion $ON = wx$ acting midway.

$\therefore$ Taking moments about N, we get $EI\,\dfrac{d^2y}{dx^2} = -Py - wx \cdot \dfrac{x}{2}$

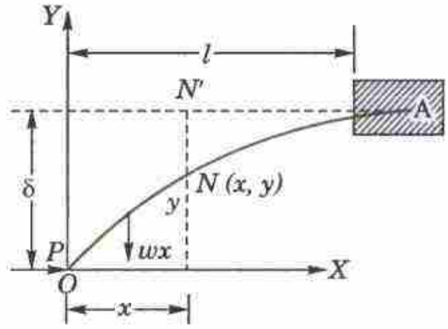


Fig. 14.27

or $$EI\,\frac{d^2y}{dx^2} + Py = -\frac{wx^2}{2} \qquad ...(i)$$

which is the desired differential equation.

Dividing by EI and taking $P/EI = n^2$, we get

$$\frac{d^2y}{dx^2} + n^2y = -\frac{wn^2}{2P} \cdot x^2$$

Its auxiliary equation is $D^2 + n^2 = 0$, whence $D = \pm ni$.

C.F. $= c_1 \cos nx + c_2 \sin nx$

$$\therefore \quad \text{P.I.} = \frac{1}{D^2 + n^2}\left(-\frac{wn^2}{2P}x^2\right) = -\frac{w}{2P}\left(1 + \frac{D^2}{n^2}\right)^{-1} x^2 = -\frac{w}{2P}\left(1 - \frac{D^2}{n^2} + \ldots\right)x^2 = \frac{w}{2P}\left(\frac{2}{n^2} - x^2\right)$$

Thus the complete solution of (*i*) is $y = c_1 \cos nx + c_2 \sin nx + \dfrac{w}{2P}\left(\dfrac{2}{n^2} - x^2\right)$ ...(*ii*)

The boundary conditions at the fixed end are

$x = l$, $y = \delta$, the maximum deflection and $dy/dx = 0$.

Using the first condition (*i.e.* $y = \delta$, when $x = l$), (*ii*) gives

$$\delta = c_1 \cos nl + c_2 \sin nl + \frac{w}{2P}\left(\frac{2}{n^2} - l^2\right) \qquad ...(iii)$$

Differentiating (*ii*), we get $\frac{dy}{dx} = n(-c_1 \sin nx + c_2 \cos nx) - \frac{wx}{P}$.

Applying the second condition, it gives $0 = n(-c_1 \sin nl + c_2 \cos nl) - wl/P$ ...(*iv*)

Also imposing the boundary condition for the free end (*i.e.* $x = 0$, $d^2y/dx^2 = 0$) on

$$\frac{d^2y}{dx^2} = -n^2(c_1 \cos nx + c_2 \sin nx) - \frac{w}{P},$$

we get $$0 = -n^2 c_1 - w/P, \text{ i.e., } c_1 = -w/Pn^2.$$

Substituting this value of c_1 in (*iv*), we get $c_2 = \frac{wl}{Pn} \sec nl - \frac{w}{Pn^2} \tan nl$

Thus, substituting the values of c_1 and c_2 in (*iii*), we get

the *maximum deflection* $\delta = \frac{w}{Pn^2}\left(1 - \frac{l^2n^2}{2} - \sec nl + nl \tan nl\right)$.

14.8 WHIRLING OF SHAFTS

(1) Critical or whirling speeds. A shaft seldom rotates about its geometrical axis for there is always some non-symmetrical crookedness in the shaft. In fact, the dead weight of the shaft causes some deflection which tends to become large at certain speeds. Such speeds at which the deflection of the shaft reaches a stage, where the shaft will fracture unless the speed is lowered are called the *critical or whirling speeds* of the shaft.

(2) Differential equation of the rotating shaft. Consider a shaft of weight W per unit length which is rotating with angular velocity ω.

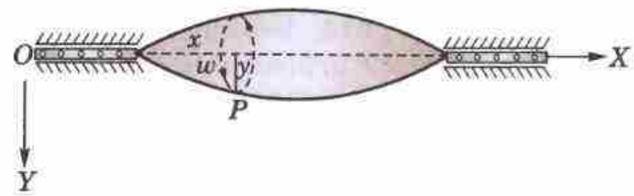


Fig. 14.28

Take its original horizontal position and the vertical downwards through the end O as the axes of x and y (Fig. 14.28). We know that for a uniformly loaded beam, the intensity of loading at $P(x, y) = EI\, d^4y/dx^4$.

$\therefore$ the restoring force (*i.e.* the internal action to oppose bending at $P(x, y) = EI\, d^4y/dx^4$.

Also the centrifugal force per unit length at $P = mr\omega^2$, *i.e.* $\frac{Wy}{g}\ \omega^2$.

As the restoring force arising out of the rigidity or stiffness of the shaft balances the centrifugal force which causes further deflection.

$$\therefore \qquad EI\frac{d^4y}{dx^4} = \frac{W}{g} y\omega^2 \quad \text{or} \quad \frac{d^4y}{dx^4} - a^4y = 0, \text{ where } a^4 = \frac{W\omega^2}{gEI}$$

which is the desired differential equation.

Its auxiliary equation being $D^4 - a^4 = 0$, we have

$$D = \pm a, \pm ai.$$

Hence its solution is $y = c_1e^{ax} + c_2e^{-ax} + c_3 \cos ax + c_4 \sin ax$ which may be put in the form

$$y = A \cosh ax + B \sinh ax + C \cos ax + D \sin ax.$$

(3) End conditions. To determine the arbitrary constants A, B, C, D we use the following end conditions :

(*i*) *At an end in a short or flexible bearings* (Fig. 14.29), there being no deflection and also no bending moment, we have

$$y = 0 \text{ and } \frac{d^2y}{dx^2} = 0.$$

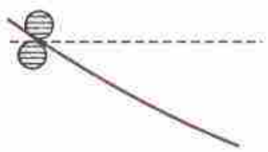


Fig. 14.29

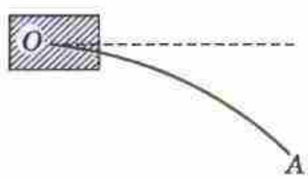


Fig. 14.30

(*ii*) *At an end in long or fixed bearings* (Fig 14.30), the deflection and the slope of the shaft being both zero, we have

$$y = 0 \text{ and } \frac{dy}{dx} = 0.$$

(*ii*) *At a perfectly free end* (such as *A* in Fig. 14.30), there being no bending moment and no shear force, we have

$$\frac{d^2y}{dx^2} = 0 \text{ and } \frac{d^3y}{dx^3} = 0.$$

Example 14.16. *The differential equation for the displacement y of a whirling shaft when the weight of the shaft is taken into account is*

$$EI\,\frac{d^4y}{dx^4} - \frac{W\omega^2}{g}y = W.$$

Taking the shaft of length 2l with the origin at the centre and short bearings at both ends, show that the maximum deflection of the shaft is

$$\frac{g}{2\omega^2}(\text{sech } al + \sec al - 2).$$

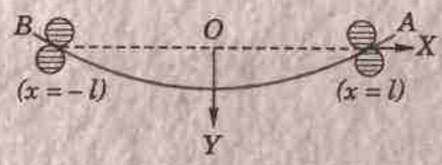


Fig. 14.31

Solution. Given differential equation can be written as

$$\frac{d^4y}{dx^4} - a^4y = \frac{W}{EI}, \text{ where } a^4 = \frac{W\omega^2}{EIg} \qquad ...(i)$$

Its C.F. $= A\cosh ax + B\sinh ax + C\cos ax + D\sin ax$

and

$$\text{P.I.} = \frac{1}{D^4 - a^4}\cdot\frac{W}{EI} = \frac{W}{EI}\cdot\frac{1}{D^4 - a^4}e^{0.x} = -\frac{W}{EIa^4} = -\frac{g}{\omega^2}$$

Thus the complete solution of (*i*) is

$$y = A\cosh ax + B\sinh ax + C\cos ax + D\sin ax - \frac{g}{\omega^2} \qquad ...(ii)$$

Differentiating it twice, we get

$$\frac{1}{a}\frac{dy}{dx} = A\sinh ax + B\cosh ax - C\sin ax + D\cos ax$$

$$\frac{1}{a^2}\frac{d^2y}{dx^2} = A\cosh ax + B\sinh ax - C\cos ax - D\sin ax \qquad ...(iii)$$

As the end A of the shaft is in short bearings (Fig. 14.31)

∴ when $x = l;\ y = 0,\ d^2y/dx^2 = 0$

∴ from (*ii*) and (*iii*), we have

$$0 = A\cosh al + B\sinh al + C\cos al + D\sin al - \frac{g}{\omega^2} \qquad ...(iv)$$

$$0 = A\cosh al + B\sinh al - C\cos al - D\sin al \qquad ...(v)$$

Similarly *at the end B*, $x = -l,\ y = 0,\ d^2y/dx^2 = 0$.

∴ from (*ii*) and (*iii*), we get

$$0 = A\cosh al - B\sinh al + C\cos al - D\sin al - \frac{g}{\omega^2} \qquad ...(vi)$$

$$0 = A\cosh al - B\sinh al - C\cos al + D\sin al \qquad ...(vii)$$

Adding (*iv*) and (*vi*), and (*v*) and (*vii*), we get

$$A\cosh al + C\cos al = \frac{g}{\omega^2} \quad \text{and} \quad A\cosh al - C\cos al = 0.$$

whence

$$A = \frac{g}{2\omega^2\cosh al} \text{ and } C = \frac{g}{2\omega^2\cos al}$$

Again subtracting (*vi*) from (*iv*) and (*vii*) from (*v*), we get

$$D \sinh al + D \sin al = 0 \text{ and } B \sinh al - D \sin al = 0, \text{ whence } B = 0 \text{ and } D = 0.$$

Substituting the values of A, B, C and D in (*ii*), we get

$$y = \frac{g}{2\omega^2}\left[\frac{\cosh ax}{\cosh al} + \frac{\cos ax}{\cos al} - 2\right]$$

Thus the maximum deflection = value of y at the centre (x = 0)

$$= \frac{g}{2\omega^2}(\text{sech } al + \sec al - 2).$$

Example 14.17. *The whirling speed of a shaft of length l is given by*

$$\frac{d^4y}{dx^4} - m^4y = 0 \text{ where } m^4 = \frac{W\omega^2}{gEI},$$

and y is the displacement at distance x from one end. If the ends of the shaft are constrained in long bearings, show that the shaft will whirl when $\cos ml \cosh ml = 1$.

Solution. The solution of the given differential equation is

$$y = A \cosh mx + B \sinh mx + C \cos mx + D \sin mx \quad ...(i)$$

which on differentiation gives,

$$\frac{1}{m}\frac{dy}{dx} = A \sinh mx + B \cosh mx - C \sin mx + D \cos mx \quad ...(ii)$$

Fig. 14.32

As the end O of the shaft is fixed in long bearings (Fig. 14.32).

∴ when $x = 0$, $y = 0$, $dy/dx = 0$,

∴ from (*i*) and (*ii*), we have

$$0 = A + C \quad \text{or} \quad C = -A \quad ...(iii)$$

and $$0 = B + D \quad \text{or} \quad D = -B \quad ...(iv)$$

Similarly, at the end A, $x = l$, $y = 0$, $dy/dx = 0$.

∴ From (*i*) and (*ii*), we have

$$0 = A \cosh ml + B \sinh ml + C \cos ml + D \sin ml \quad ...(v)$$

$$0 = A \sinh ml + B \cosh ml - C \sin ml + D \cos ml \quad ...(vi)$$

Substituting the values of C and D in (*v*) and (*vi*), we get

$$A(\cosh ml - \cos ml) + B(\sinh ml - \sin ml) = 0$$

and $$A(\sinh ml + \sin ml) + B(\cosh ml - \cos ml) = 0$$

Eliminating A and B from these equations, we get

$$\frac{\cosh ml - \cos ml}{\sinh ml - \sin ml} = -\frac{B}{A} = \frac{\sinh ml + \sin ml}{\cosh ml - \cos ml}$$

or $$\cosh^2 ml - 2 \cosh ml \cos ml + \cos^2 ml = \sinh^2 ml - \sin^2 ml$$

or $$-2 \cosh ml \cos ml + 2 = 0 \text{ or } \cos ml \cosh ml = 1$$

which must be satisfied when the shaft whirls.

The solution of this equation gives $ml = 4.73 = 3\pi/2$ radians approximately.

$$\therefore \quad \omega\sqrt{\left(\frac{W}{gEI}\right)}\, l^2 = m^2l^2 = \frac{9\pi^2}{4}$$

Thus the *whirling speed of a shaft with ends in long bearings.*

$$= \omega = \frac{9\pi^2}{4l^2}\sqrt{\left(\frac{gEI}{W}\right)} \text{ approximately.}$$

Obs. 1. *When the shaft has one long bearing and the other short bearing*, the condition to be satisfied is $\tan ml = \tanh ml$, of which the solution is $ml = 3.927$

or $$\omega\sqrt{\left(\frac{W}{gEI}\right)}\,.\,l^2 = m^2l^2 = (3.927)^2 = 15.4 \text{ nearly.}$$

Thus *the whirling speed* $= \omega = \dfrac{15.4}{l^2}\sqrt{\left(\dfrac{gEI}{W}\right)}$

Obs. 2. *When the shaft has both short bearings,* the condition to be satisfied is $\sin ml = 0$ *i.e.* $ml = \pi$ (least non-zero value).

$$\therefore \quad \omega\sqrt{\left(\frac{W}{gEI}\right)}\,.\,l^2 = m^2 l^2 = \pi^2. \text{ Thus } \textit{the whirling speed} = \omega = \frac{\pi^2}{l^2}\sqrt{\left(\frac{gEI}{W}\right)}.$$

Obs. 3. *When the shaft has one long bearing,* the condition to the satisfied is $\cos ml \cosh ml = -1$.

Its solution gives $ml = 1.865$ [See Example 1.25]

$$\therefore \quad \omega\sqrt{\left(\frac{W}{gEI}\right)}\,.\,l^2 = m^2l^2 = (1.865)^2 = 3.5 \text{ nearly. Thus } \textit{the whirling speed } \omega = \frac{3.5}{l^2}\sqrt{\left(\frac{gEI}{W}\right)}.$$

PROBLEMS 14.4

1. A horizontal tie-rod of length $2l$ with concentrated load W at the centre and ends freely hinged, satisfies the differential equation $EI\,\dfrac{d^2y}{dx^2} = Py - \dfrac{W}{2}\,x$. With conditions $x = 0, y = 0$ and $x = l$, $dy/dx = 0$, prove that the deflection δ and the bending moment M at the centre $(x = l)$ are given by $\delta = \dfrac{W}{2Pn}\,(nl - \tanh nl)$ and $M = -\dfrac{W}{2n}\tanh nl$, where $n^2EI = P$.
2. A light horizontal strut AB is freely pinned at A and B. It is under the action of equal and opposite compressive forces P at its ends and it carries a load W at its centre. Then for $0 < x < l/2$, $EI\,\dfrac{d^2y}{dx^2} + Py + \dfrac{1}{2}\,Wx = 0$. Also $y = 0$ at $x = 0$ and $dy/dx = 0$ at $x = l/2$.

   Prove that $y = \dfrac{W}{2P}\left(\dfrac{\sin nx}{n\cos nl/2} - x\right)$ where $n^2 = \dfrac{P}{EI}$.
3. A uniform horizontal strut of length l freely supported at both ends, carries a uniformly distributed load W per unit length. If the thrust at each end is, P, prove that the maximum deflection is $\dfrac{W}{Pa^2}\left(\sec\dfrac{al}{2} - 1\right) - \dfrac{Wl^2}{8P}$, where $\dfrac{P}{EI} = a^2$.

   Prove also that the maximum bending moment is of the magnitude $\dfrac{W}{a^2}\left(\sec\dfrac{al}{2} - 1\right)$.
4. The shape of a strut of length l subjected to an end thrust P and lateral load w per unit length, when the ends are built in, is given by $EI\,\dfrac{d^2y}{dx^2} + Py = \dfrac{wx^2}{2} - \dfrac{wlx}{2} + M$, where M is the moment at a fixed end. Find y in terms of x, given that $y = 0$, $dy/dx = 0$ at $x = 0$ and $dy/dx = 0$ at $x = l/2$.
5. A light horizontal strut of length l is clamped at one end carries a vertical load W at the free end. If the horizontally thrust at the free end is P, show that the strut satisfies the differential equation

   $El\,\dfrac{d^2y}{dx^2} = (\delta - y)\,P + W(l - x)$, where y is the displacement of a point at a distance x from the fixed end and δ, the deflection at the free end.

   Prove that the deflection at the free end is given by $\dfrac{W}{nP}(\tan nl - nl)$, where $n^2EI = P$.
6. A long column fixed at one end $(x = 0)$ and hinged at the other $(x = l)$ is under the action of axial load P. If a force F is applied laterally at the hinge to prevent lateral movement, show that it satisfies the equation $\dfrac{d^2y}{dx^2} + n^2y = \dfrac{En^2}{P}$ $(l - x)$, where $EIn^2 = P$. Hence determine the equation of the deflection curve.
7. A long column of length l is fixed at one end and is completely free at the other end. If y is the lateral deflection at a point distance x from the fixed end, when load P is axially applied, find the differential equation satisfied by x and y. Show that the deflection curve is given by $y = a\,\{1 - \cos\,\sqrt{P/EI)}\,x\}$ and find the least value of the critical load (a is the lateral deflection of the free end).

8. The differential equation for the displacement y of a heavy whirling shaft is $\frac{d^4y}{dx^4} = a^4\left(y + \frac{g}{\omega^2}\right)$, where $a^4 = \frac{W\omega^2}{gEI}$. If both ends are in short bearings, the ends being $x = 0$ and $x = l$, find the bending moment of the centre of the shaft.

14.9 APPLICATIONS OF SIMULTANEOUS LINEAR EQUATIONS

So far we have considered engineering systems having only one degree of freedom. The analysis of a system having more than one degree of freedom depends on the solution of simultaneous linear equations. In fact such equations form the basis of the theory of projectiles and the coupled circuits having self and mutual inductance. The details of such applications are best explained through the following examples :

Example 14.18. Projectile with resistance. *Find the path of a particle projected with a velocity v at an angle α to the horizon in a medium whose resistance, apart from gravity, varies as velocity. Also find the greatest height attained.*

Solution. Let the axes of x and y be respectively horizontal and vertical with origin at the point of projection (Fig. 14.33).

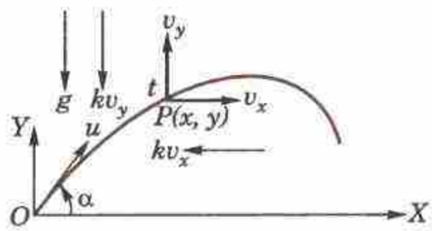


Fig. 14.33

Let $P(x, y)$ be the position of the projectile at the time t, where the velocity components parallel to the axes are

$$v_x = \frac{dx}{dt}, v_y = \frac{dy}{dt}$$

∴ the equations of motion are:

Parallel to x-axis	*Parallel to y-axis*
$m\frac{dv_x}{dt} = -mkv_x$	$m\frac{dv_y}{dt} = -mg - mkv_y$
or $\frac{dv_x}{dt} = -kv_x$	$\frac{dv_y}{dt} = -(g + kv_y)$

Separating the variables and integrating, we have

$\int \frac{dv_x}{v_x} = -k\int dt + c_1$	$\frac{dv_y}{g + kv_y} = -\int dt + c_2$
or $\log v_x = -kt + c_1$	$\frac{1}{k}\log(g + kv_y) = -t + c_2.$

Initially when $t = 0$, $v_x = u\cos\alpha$, $v_y = u\sin\alpha$.

$\log u\cos\alpha = c_1$	$\frac{1}{k}\log(g + ku\sin\alpha) = c_2$

Subtracting,

$\log\left(\frac{v_x}{u\cos\alpha}\right) = -kt$	$\frac{1}{k}\log\left(\frac{g + kv_y}{g + ku\sin\alpha}\right) = -t$
or $\frac{dx}{dt} = v_x = u\cos\alpha e^{-kt}$ ...(*i*)	$\frac{dy}{dt} = v_y = \frac{1}{k}[(g + ku\sin\alpha)e^{-kt} - g]$ ...(*ii*)

Again integrating, we get

$$x = \frac{u\cos\alpha}{-k}e^{-kt} + c_3,\ y = -\frac{1}{k}\left(\frac{g}{k} + u\sin\alpha\right)e^{-kt} - \frac{g}{k}t + c_4$$

Initially when $t = 0, x = 0, y = 0$,

$$\therefore \quad 0 = \frac{u\cos\alpha}{k} + c_3,\ 0 = -\frac{1}{k}\left(\frac{g}{k} + u\sin\alpha\right) + c_4$$

Subtracting, we get $\quad x = \frac{u\cos\alpha}{k}(1 - e^{-kt})$ ...(*iii*)

$$y = \frac{1}{k}\left(\frac{g}{k} + u \sin \alpha\right)\left(1 - e^{-kt}\right) - \frac{gt}{k} \qquad ...(iv)$$

Eliminating t from (*iii*) and (*iv*), we obtain $y = \left(\frac{g}{k} + u \sin \alpha\right)\frac{x}{u \cos \alpha} + \frac{g}{k^2} \log\left(1 - \frac{kx}{u \cos \alpha}\right)$

which is the required equation of the trajectory.

The projectile will attain the greatest height when $dy/dt = 0$.

i.e., when $e^{-kt} = g/(g + ku \sin \alpha)$, *i.e.*, at time $t = \frac{1}{k} \log\left(1 + \frac{ku \sin \alpha}{g}\right)$. [From (*ii*)]

Subtituting the value of t in (*iv*), we get the greatest height attained

$$(= y) = \frac{u \sin \alpha}{k} - \frac{g}{k^2} \log\left(1 + \frac{ku \sin \alpha}{g}\right).$$

Example 14.19. *Two particles each of mass m gm are suspended from two springs of same stiffness k as in Fig. 14.34. After the system comes to rest, the lower mass is pulled l cm downwards and released. Discuss their motion.*

Solution. Let x and y denote the displacement of the upper and lower masses at time t from their respective positions of equilibrium.

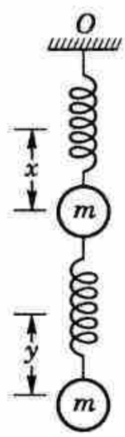


Fig. 14.34

Then the stretch of the upper spring is x and that of the lower spring is $y - x$.

$\therefore$ the restoring force acting on the upper mass

$$= -kx + k(y - x) = k(y - 2x)$$

and that on the lower mass $= -k(y - x)$.

Thus their equations of motion are

$$m\frac{d^2x}{dt^2} = k(y - 2x) \text{ and } m\frac{d^2y}{dt^2} = -k(y - x)$$

or $\qquad (mD^2 + 2k)x - ky = 0 \qquad ...(i)$

and $\qquad (mD^2 + k)y - kx = 0 \qquad ...(ii)$

Operating (*i*) by $(mD^2 + k)$ and adding to k times (*ii*), we get

$[(mD^2 + k)(mD^2 + 2k) - k^2]\, x = 0$ or $(D^4 + 3\lambda D^2 + \lambda^2)x = 0$, where $\lambda^2 = k/m$.

Its auxiliary equation is $D^4 + 3\lambda D^2 + \lambda^2 = 0$

which gives $\qquad D^2 = \dfrac{-3\lambda \pm \sqrt{(9\lambda^2 - 4\lambda^2)}}{2} = -2.62\lambda \text{ or } -0.38\lambda = -\alpha^2, -\beta^2$ (say)

so that $D = \pm i\alpha, \pm i\beta$.

Thus $\quad x = c_1 \cos \alpha t + c_2 \sin \alpha t + c_3 \cos \beta t + c_4 \sin \beta t \qquad ...(iii)$

Also from (*i*), $y = \left(\frac{D^2}{\lambda} + 2\right)x = (2 - \alpha^2/\lambda)(c_1 \cos \alpha t + c_2 \sin \alpha t) + (2 - \beta^2/\lambda)(c_3 \cos \beta t + c_4 \sin \beta t) \qquad ...(iv)$

Initially when $\quad t = 0, x = y = l, dx/dt = dy/dt = 0$.

$\therefore$ from (*iii*), $\quad l = c_1 + c_3;\ 0 = \alpha c_2 + \beta c_4$

and from (*iv*) $\quad l = (2 - \alpha^2/\lambda)c_1 + (2 - \beta^2/\lambda)c_3$ and $0 = (2 - \alpha^2/\lambda)\alpha c_2 + (2 - \beta^2/\lambda)\beta c_4$

whence $\qquad c_1 = \dfrac{l(\lambda - \beta^2)}{\alpha^2 - \beta^2},\ c_3 = \dfrac{l(\lambda - \alpha^2)}{\beta^2 - \alpha^2},\ c_3 = c_4 = 0.$

Substituting these values of constants in (*iii*) and (*iv*), we get x and y which show that the motion of the spring is a combination of two simple harmonic motions of periods $2\pi/\alpha$ and $2\pi/\beta$.

Example 14.20. *Two coils of a transformer are identical with resistance R, inductance L, mutual inductance M and a voltage E is impressed on the primary. Determine the currents in the coils at any instant, assuming that there is no current in either initially.*

Solution. Let i_1, i_2 ampere be the currents flowing through the primary and secondary coils at time t sec (Fig. 14.35). Then by Kirchhoff's law, we know that sum of the voltage drops across R, L and M = applied voltage.

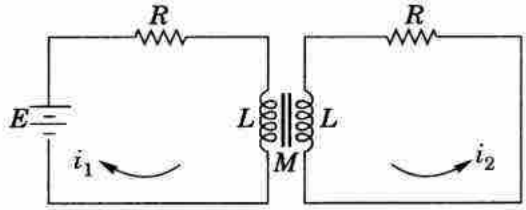


Fig. 14.35

$\therefore$ for the primary circuit,

$$Ri_1 + L\frac{di_1}{dt} + M\frac{di_2}{dt} = E$$

and for the secondary circuit, $Ri_2 + L\dfrac{di_2}{dt} + M\dfrac{di_1}{dt} = 0.$

Replacing d/dt by D and rearranging the terms,

$$(LD + R)i_1 + MDi_2 = E \quad \text{...(i)}$$

$$MDi_1 + (LD + R)i_2 = 0 \quad \text{...(ii)}$$

Eliminating i_2, we get $[(LD + R)^2 - M^2D^2]i_1 = (LD + R)E$

i.e., $$[(L^2 - M^2)D^2 + 2LRD + R^2]i_1 = RE \quad \text{...(iii)}$$

Its auxiliary equation is $(L^2 - M^2)D^2 + 2LRD + R^2 = 0$ whence $D = \dfrac{-R}{L+M}, \dfrac{-R}{L-M}$.

As L is usually $> M$, therefore, both values of D are negative and real.

$$\therefore \quad \text{C.F.} = c_1 e^{-\frac{Rt}{L+M}} + c_2 e^{-\frac{Rt}{L-M}} \text{ and P.I.} = RE \,.\, \frac{1}{(L^2 - M^2)D^2 + 2LRD + R^2} e^{0.t} = E/R.$$

Thus the complete solution of (*iii*) is $i_1 = c_1 e^{-Rt/(L+M)} + c_2 e^{-Rt(L-M)} + E/R$ ...(*iv*)

and from (*ii*), we have $i_2 = -\dfrac{MD}{LD+R} i_1$

$$= -\frac{MD}{LD+R}\left(c_1 e^{-Rt/(L+M)} + c_2 e^{-Rt/(L-M)}\right) - \frac{MD}{LD+R}\left(\frac{E}{R}\right)$$

$$= -\frac{Mc_1}{L\left(\dfrac{-R}{L+M}\right)+R} \,.\, De^{-Rt/(L+M)} - \frac{Mc_2}{L\left(\dfrac{-R}{L-M}\right)+R} \,.\, De^{-Rt/(L-M)}$$

$$= c_1 e^{-Rt/(L+M)} - c_2 e^{-Rt/(L-M)}$$

Initially, when $t = 0, i_1 = i_2 = 0$.

$\therefore \quad c_1 + c_2 = -E/R, c_1 - c_2 = 0 \qquad \therefore c_1 = c_2 = -E/2R.$

Substituting the values of c_1, c_2 in (*iv*) and (*v*), we get

$$i_1 = \frac{E}{2R}\left[2 - e^{-Rt/(L+M)} - e^{-Rt/(L-M)}\right] \quad \text{...(vi)}$$

and $$i_2 = \frac{E}{2R}\left[e^{-Rt/(L-M)} - e^{-Rt/(L+M)}\right] \quad \text{...(vii)}$$

Thus (*vi*) and (*vii*) give the currents at any instant.

PROBLEMS 14.5

1. A particle is projected with velocity u, at an elevation α. Neglecting air resistance, show that the equation to its path is the parabola $y = x \tan \alpha - \dfrac{gx^2}{2u^2 \cos^2 \alpha}$. Also find the time of flight and range on the horizontal plane.
2. An inclined plane makes angle α with the horizontal. A projectile is launched from the bottom of the inclined plane with speed V in a direction making angle β with the horizontal. Set up the differential equations and find (*i*) the range on the incline, (*ii*) the maximum range up the incline.
3. A particle of unit mass is projected with velocity u at an inclination α above the horizon in a medium whose resistance is k times the velocity. Show that its direction will again make an angle α with the horizon after a time $\dfrac{1}{k}\log\left\{1 + \dfrac{2ku}{g}\sin\alpha\right\}$.

4. A particle moving in a plane is subjected to a force directed towards a fixed point O and proportional to the distance of the particle from O. Show that the differential equations of motion are of the form $\frac{d^2x}{dt^2} = -k^2x$, $\frac{d^2y}{dt^2} = -k^2y$. Find the cartesian equation of the path of the particle if $x = 1, y = 0, \frac{dx}{dt} = 0$ and $dy/dt = 2$, when $t = 0$.
5. The currents i_1 and i_2 in mesh are given by the differential equations

$$\frac{di_1}{dt} - \omega i_2 = a \cos pt, \frac{di_2}{dt} + \omega i_1 = a \sin pt.$$ Find the currents i_1 and i_2 if $i_1 = i_2 = 0$ at $t = 0$.
6. The currents i_1 and i_2 in two coupled circuits are given by

$$L\frac{di_1}{dt} + Ri_1 + R(i_1 - i_2) = E; L\frac{di_2}{dt} + Ri_2 - R(i_1 - i_2) = 0,$$

where L, R, E are constants. Find i_1 and i_2 in terms of t given that $i_1 = i_2 = 0$ at $t = 0$.
7. The motion of a particle is governed by the equations

$$\frac{d^2x}{dt^2} - n\frac{dy}{dt} = 0, \frac{d^2y}{dt^2} + n\frac{dx}{dt} = n^2a,$$ when $x = y = \frac{dx}{dt} = \frac{dy}{dt} = 0$ at $t = 0$. Find x and y in terms of t.
8. Under certain conditions, the motion of an electron is given by the equations $m\frac{d^2x}{dt^2} + eH\frac{dy}{dt} = eE$ and $m\frac{d^2y}{dt^2} - eH\frac{dx}{dt} = 0$. Find the path of the electron, if it started from rest at the origin.
9. The voltage V and the current i at a distance x from the source satisfy the equations $-dV/dt = Ri, -di/dx = GV$, where R, G are constants. If $V = V_0$ at $x = 0$ and $V = 0$ at the receiving end $x = l$, show that $V = V_0 \sinh n(l-x)/\sinh nl$, $i = V_0/(G/R). \cosh n(l-x) / \sinh nl$, where $n^2 = RG$.

14.10 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 14.6

Fill up the blanks or choose the correct answer in the following problems:

1. A particle executing simple harmonic motion of amplitude 5 cm has a speed of 8 cm/sec when at a distance of 3 cm from the centre of the path. The period of the motion of the particle will be
(*a*) $\pi/2$ sec (*b*) π sec (*c*) 2π sec (*c*) 4π sec.
2. A ball of mass m is suspended from a fixed point O by a light string of natural length l and modulus of elasticity λ. If the ball is displaced vertically, its motion will be S.H.M. of period
(*a*) $2\pi\sqrt{(m/\lambda l)}$ (*b*) $2\pi\sqrt{(ml/\lambda)}$ (*c*) $2\pi\sqrt{(l/m\lambda)}$ (*d*) $2\pi\sqrt{(\lambda m/l)}$.
3. The periodic time of the motion described by the differential equation $\frac{d^2x}{dt^2} + 4x = 0$ is
(*a*) $\pi/2$ (*b*) π (*c*) 2π.
4. A particle is projected with a velocity u at an angle of 60° to the horizontal. The time of flight of the projectile is equal to
(*a*) $\sqrt{3u/2g}$ (*b*) $\sqrt{3u/g}$ (*c*) u/g (*d*) $u/2g$.
5. A body of 6.5 kg is suspended by two strings of lengths 5 and 12 metres attached to two points in the same horizontal line whose distance apart is 13 meters. The tension of the strings are
(*a*) 2 kg & 6.5 kg (*b*) 2.5 kg & 6 kg (*c*) 2.25 kg & 6.25 kg (*d*) 3 kg & 5.5 kg.
6. A particle is projected at an angle of 30° to the horizontal with a velocity of 1962 cm/sec then the time of flight is
(*a*) 1 sec (*b*) 2 sec (*c*) 2.5 sec (*d*) 3 sec.
7. A point moves with S.H.M. whose period is 4 seconds. If it starts from rest at a distance of 4 meters from the centre of its path, then the time it takes, before it has described 2 metres is
(*a*) $\frac{1}{3}$ second (*b*) $\frac{2}{3}$ second (*c*) $\frac{3}{4}$ second (*d*) $\frac{4}{5}$ second.

8. If the length of the pendulum of a clock be increased in the ratio 720 : 721, it would loose .... seconds per day.
9. The frequency of free vibrations in a closed circuit with inductance L and capacity C in series is .... per minute.
10. If a clock with a seconds pendulum loses 10 seconds per day at a place having $g = 32$ ft/sec^2, g should be increased by .... ft/sec^2, to keep correct time.
11. The soldiers break step while marching over a bridge for the fear that their steps may not be in rhym with the natural frequency of the bridge causing its collapse due to ....
12. A horizontal tie-rod is freely pinned at each end. If it carries a uniform load w lb per unit length and has a horizontal pull P, then the differential equation of the elastic curve is ....
13. The conditions for an end of a whirling shaft to be in fixed bearings are .... and ....

CHAPTER 15

Differential Equations of Other Types

1. Introduction. 2. Equations of the form $d^2y/dx^2 = f(x)$. 3. Equations of the form $d^2y/dx^2 = f(y)$. 4. Equations which do not contain y. 5. Equations which do not contain x. 6. Equations whose one solution is known. 7. Equations which can be solved by changing the independent variable. 8. Total differential equation : $Pdx + Qdy + Rdz = 0$. 9. Simultaneous total differential equations. 10. Equations of the form $dx/P = dy/Q = dz/R$.

15.1 INTRODUCTION

In this chapter, we propose to study some other important types of ordinary differential equations which require special methods for their solution and have varied applications as illustrated side by side.

15.2 EQUATIONS OF THE FORM $d^2y/dx^2 = f(x)$

Integrating with respect to x, we have $\dfrac{dy}{dx} = \int f(x)dx + c = F(x)$. (say)

Again integrating, we get $y = \int F(x)dx + c'$ as the required solution.

In general, the solution of the equations of the form $\dfrac{d^n y}{dx^n} = f(x)$ is obtained by integrating it n times successively.

Example 15.1. *Solve* $\dfrac{d^2y}{dx^2} = xe^x$.

Solution. Integrating, we get $\dfrac{dy}{dx} = xe^x - \int e^x\, dx + c_1 = (x-1)e^x + c_1$

Again integrating, we get

$$y = (x-1)e^x - \int e^x dx + c_1x + c_2 = (x-2)e^x + c_1x + c_2.$$

PROBLEMS 15.1

Solve :

1. $\dfrac{d^2y}{dx^2} = x^2 \sin x$.
2. $\dfrac{d^3y}{dx^3} = x + \log x$.

3. A beam of length $2l$ with uniform load w per unit length is freely supported at both ends. Prove that the maximum deflection of the beam is $\dfrac{5wl^4}{24EI}$.

[**Hint.** Taking the origin at the left end, we have $EI\, d^4y/dx^4 = w$. At each end, $y = 0$ and $d^2y/dx^2 = 0$.]

4. For a cantilever beam of length l with a uniform load of w per unit length, show that the maximum deflection at the free end is wl^4/EI, where the symbols have the usual meaning.

15.3 EQUATIONS OF THE FORM $d^2y/dx^2 = f(y)$

Multiplying both sides by $2dy/dx$, we have $2\dfrac{dy}{dx}\cdot\dfrac{d^2y}{dx^2} = 2f(y)\dfrac{dy}{dx}$

Integrating with respect to x, $\left(\dfrac{dy}{dx}\right)^2 = 2\int f(y)\ dy + c = F(y)$ (say)

or
$$\frac{dy}{dx} = \sqrt{[F(y)]}$$

Separating the variables and integrating, we get $\displaystyle\int \frac{dy}{\sqrt{[F(y)]}} = x + c$, whence follows the desired solution.

Such equations occur quite frequently in Dynamics.

Example 15.2. *Solve $d^2y/dx^2 = 2(y^3 + y)$ under the conditions $y = 0$, $dy/dx = 1$, when $x = 0$.* *(U.P.T.U., 2003)*

Solution. Multiplying by $2\, dy/dx$, the given equation becomes

$$2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} = 4(y^3 + y)\ \frac{dy}{dx}$$

Integrating w.r.t. x, $\left(\dfrac{dy}{dx}\right)^2 = 4\left(\dfrac{y^4}{4} + \dfrac{y^2}{2}\right) + c = y^4 + 2y^2 + c$ ...(*i*)

As $dy/dx = 1$ for $y = 0$, $\therefore\ c = 1$

$\therefore$ (*i*) takes the form $(dy/dx)^2 = y^4 + 2y^2 + 1 = (y^2 + 1)^2$ or $dy/dx = y^2 + 1$

Separating the variables and integrating, we have $\displaystyle\int \frac{dy}{1 + y^2} = \int dx + c'$

or
$$\tan^{-1} y = x + c' \quad ...(ii)$$

Thus (*ii*) becomes $\tan^{-1} y = x$ or $y = \tan x$ which is the required solution.

Example 15.3. *A point moves in a straight line towards a centre of force $\mu/(\text{distance})^3$, starting from rest at a distance 'a' from the centre of force, show that the time of reaching a point distant 'b' from the centre of force is $\dfrac{a}{\sqrt{\mu}}\sqrt{(a^2 - b^2)}$ and that its velocity is $\dfrac{\sqrt{\mu}}{ab}\sqrt{(a^2 - b^2)}$.* *(U.P.T.U., 2001)*

Solution. Let O be the centre of force and A the point of start so that $OA = a$. At any time t, let the point be at P where $OP = x$ so that

$$\frac{d^2x}{dt^2} = \frac{-\mu}{x^3} \quad ...(i)$$

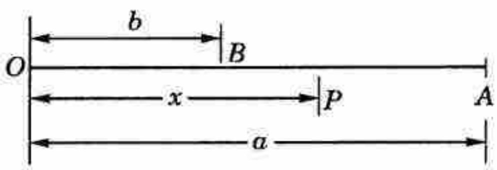


Fig. 15.1

Multiplying both sides by $2\, dx/dt$, we get

$$\frac{2dx}{dt}\cdot\frac{d^2x}{dt^2} = -\frac{\mu}{x^3}\cdot\frac{2dx}{dt}$$

Integrating both sides, we obtain

$$\left(\frac{dx}{dt}\right)^2 = -\mu \int \frac{2}{x^3}\frac{dx}{dt}\,.\,dt + c = +\frac{\mu}{x^2} + c$$

When $x = a$, velocity $dx/dt = 0$. $\therefore\ c = -\mu/a^2$.

$$\therefore \qquad \left(\frac{dx}{dt}\right)^2 = \mu\left(\frac{1}{x^2} - \frac{1}{a^2}\right) = \frac{\mu(a^2 - x^2)}{a^2x^2} \qquad ...(ii)$$

At B $(x = b)$, velocity towards $O = \dfrac{\sqrt{\mu(a^2 - b^2)}}{ab}$

Again (ii) can be rewritten as $\dfrac{-ax\,dx}{\sqrt{(a^2 - x^2)}} = \sqrt{\mu}\,dt$ [– ve is taken since point is moving towards O]

Integrating both sides, we get

$$\sqrt{\mu}\int dt = -\int \frac{ax\,dx}{\sqrt{(a^2 - x^2)}} + c' \qquad \text{or} \qquad \sqrt{\mu}t = a\sqrt{(a^2 - x^2)} + c' \qquad ...(iii)$$

Since $t = 0$ at $x = a$, $\therefore\quad c' = 0$

Thus (iii) gives $t = \dfrac{a}{\sqrt{\mu}}\sqrt{(a^2 - x^2)}$

Hence at B $(x = b)$ $t = \dfrac{a}{\sqrt{\mu}}\sqrt{(a^2 - b^2)}$.

PROBLEMS 15.2

Solve :

1. $d^2y/dx^2 = 3\sqrt{y}$ given that $y = 1$, $dy/dx = 2$ when $x = 0$.
2. $\dfrac{d^2y}{dx^2} = \dfrac{36}{y^2}$, given that when $x = 0$, $\dfrac{dy}{dx} = 0$, $y = 8$.
3. If $d^2r/dt^2 = \omega^2 r$, find the value of r in terms of t, if $r = a$ and $dr/dt = v$, when $t = 0$.
4. The motion of a particle let fall from a point outside the earth is given by $d^2x/dt^2 = -ga^2/x^2$. Given that $x = h$ and $dx/dt = 0$, when $t = 0$, find t in terms of x.
5. A particle is acted upon by a force $\mu\,(x + a^4/x^3)$ per unit mass towards the origin, where x is the distance from the origin at time t. If it starts from rest at a distance a, show that it will arrive at the origin in time $\pi/(4\sqrt{\mu})$.

15.4 EQUATIONS WHICH DO NOT CONTAIN y

A second order equation of this form is

$$f(d^2y/dx^2, dy/dx, x) = 0$$

On putting $dy/dx = p$ and $d^2y/dx^2 = dp/dx$, it becomes

$$f(dp/dx, p, x) = 0.$$

This is an equation of the first order in x and p and can, therefore, be solved easily.

If its solution is $(p =)\ dy/dx = \phi(x)$, then $y = \int \phi(x)dx + c$ is the required solution.

Obs. This method may be used to reduce any such equation of the nth order to one of the $(n-1)$th order. If, however, the lowest derivative in such an equation is d^ry/dx^r

(i) put $d^ry/dx^r = p$; (ii) find p and therefrom find y, (*See* Ex. 15.5).

Example 15.4. *Solve* $x\dfrac{d^2y}{dx^2} = \sqrt{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]}$.

Solution. Putting $dy/dx = p$ and $d^2y/dx^2 = dp/dx$, the given equation becomes

$$xdp/dx = \sqrt{(1+p^2)}.$$

Separating the variables and integrating, we get

$$\int \frac{dp}{\sqrt{(1+p^2)}} = \int \frac{dx}{x} + \text{constant}$$

or $$\log\left[p + \sqrt{(1+p^2)}\right] = \log x + \log c = \log cx.$$

$\therefore$ $$p + \sqrt{(1+p^2)} = cx \quad \text{or} \quad 1 + p^2 = (cx - p)^2$$

or $$(p =)\ \frac{dy}{dx} = \frac{1}{2}\left(cx - \frac{1}{cx}\right).$$

$\therefore$ integrating again, we have $y = \dfrac{1}{2}\left(c\dfrac{x^2}{2} - \dfrac{1}{c}\log x\right) + c'$ as the required solution.

Example 15.5. *Solve* $\dfrac{d^4y}{dx^4} \cdot \dfrac{d^3y}{dx^3} = 1.$

Solution. Putting $d^3y/dx^3 = p$ and $d^4y/dx^4 = dp/dx$, the given equation becomes $\dfrac{dp}{dx}\ p = 1.$

Integrating w.r.t. x, $\int pdp = x + c_1$, *i.e.* $p^2/2 = x + c_1$ or $(p =)\ d^3y/dx^3 = \sqrt{2}(x + c_1)^{1/2}.$

Integrating thrice successively, we get

$$\frac{d^2y}{dx^2} = \sqrt{2}\frac{(x+c_1)^{3/2}}{3/2} + c_2,\ \frac{dy}{dx} = \frac{2\sqrt{2}}{3} \cdot \frac{(x+c_1)^{5/2}}{5/2} + c_2x + c_3$$

$$y = \frac{4\sqrt{2}}{15}\frac{(x+c_1)^{7/2}}{7/2} + c_2\frac{x^2}{2} + c_3x + c_4$$

Hence $y = \dfrac{8\sqrt{2}}{105}(x + c_1)^{7/2} + \dfrac{1}{2}c_2x^2 + c_3x + c_4$ is the desired solution.

PROBLEMS 15.3

Solve the following equations :

1. $x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 6x = 0.$

2. $(1 + x^2)\dfrac{d^2y}{dx^2} + 1 + \left(\dfrac{dy}{dx}\right)^2 = 0.$

3. $2x\dfrac{d^3y}{dx^3} \cdot \dfrac{d^2y}{dx^2} = \left(\dfrac{d^2y}{dx^2}\right)^2 - a^2.$

4. $\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2} = a\dfrac{d^2y}{dx^2}.$

5. A particle of mass m grammes is constrained to move in a horizontal circular path of radius a cm and is subjected to a resistance proportional to the square of the speed at any instant. Show that the differential equation of motion is of the form $m\dfrac{d^2\theta}{dt^2} + \mu a\left(\dfrac{d\theta}{dt}\right)^2 = 0.$ If the particle starts with an angular velocity ω, find its angular displacement θ at time t sec.

6. When the inner of two concentric spheres of radii r_1 and $r_2 (r_1 < r_2)$ carries an electric charge, the differential equation for the potential v at any point between two spheres at a distance r from their common centre is $\dfrac{d^2v}{dt^2} + \dfrac{2}{r}\dfrac{dv}{dr} = 0.$ Solve for v given $v = v_1$ when $r = r_1$ and $v = v_2$ when $r = r_2$.

15.5 EQUATIONS WHICH DO NOT CONTAIN x

A second order equation of this form is

$$f(d^2y/dx^2, dy/dx, y) = 0.$$

On putting $\frac{dy}{dx} = p$ and $\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\cdot\frac{dy}{dx} = p\frac{dp}{dy}$, it becomes

$$f(p\, dp/dy, p, y) = 0.$$

This is an equation of the first order in y and p and can, therefore, be solved easily.

Example 15.6. *Solve* $y\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(\frac{dy}{dx} - 2y\right) = 0.$

Solution. On putting $dy/dx = p$ and $d^2y/dx^2 = p\, dp/dy$, the given equation becomes

$$yp\frac{dp}{dy} + p(p - 2y) = 0.$$

This gives *either* $p = 0$, of which the solution is $y = c$;

or $$\left(y\frac{dp}{dy} + p\right) - 2y = 0 \quad i.e., \quad (ydp + pdy) = 2ydy \; i.e., d(py) = 2ydy.$$

Integrating, $\quad py = 2\int ydy + c_1 = y^2 + c_1.$

Separating the variables and integrating, we get

$$\int\frac{ydy}{y^2 + c_1} = \int dx + c_2 \quad \text{or} \quad \frac{1}{2}\log(y^2 + c_1) = x + c_2 \text{ whence } y^2 + c_1 = c_3e^{2x}$$

Hence the required solutions are $y = c$ and $y^2 + c_1 = c_3 e^{2x}$.

Example 15.7. *Find the curve in which the radius of curvature is twice the normal and in the opposite direction.*

Solution. At any point $P(x, y)$ of a curve, the radius of curvature

$$\rho = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \Big/ \frac{d^2y}{dx^2}$$

and the length of the normal (PN)

$$= y\sqrt{[1 + (dy/dx)^2]}.$$

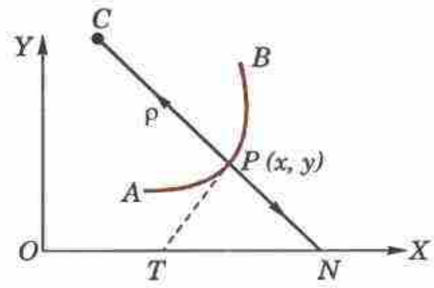


Fig. 15.2

Also we know that ρ is measured inwards and the normal is measured outwards, *i.e.*, both of them are positive when measured in opposite directions. So the sign will be positive (or negative) according as ρ and the normal run in the opposite (or same) directions.

Thus for the given curve $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \div \frac{d^2y}{dx^2} = 2y\sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}$

or $$1 + \left(\frac{dy}{dx}\right)^2 = 2y\frac{d^2y}{dx^2}.$$

On putting $dy/dx = p$ and $d^2y/dx^2 = p\, dp/dy$, the given equation becomes

$$1 + p^2 = 2y\,.\,p\, dp/dy.$$

$\therefore$ separating variables and integrating, we have

$$\int\frac{2pdp}{1 + p^2} = \int\frac{dy}{y} + \text{constant}$$

or $$\log(1 + p^2) = \log y + \log a = \log ay$$

$\therefore$ $$1 + p^2 = ay \text{ or } (p =)\, dy/dx = \sqrt{(ay - 1)}$$

$\therefore$ separating the variables and integrating, we get

$$\int dx + b = \int (ay-1)^{-1/2} dy$$

or
$$x + b = \frac{2}{a}(ay-1)^{1/2} \quad \text{or} \quad a^2(x+b)^2 = 4(ay-1)$$

which is required equation of the curve and represents a system of parabolas having axes parallel to y-axis.

PROBLEMS 15.4

Solve the following equations :

1. $2\dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dx}\right)^2 + 4 = 0.$

2. $y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 1.$

3. $y\dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dx}\right)^2 = y^2 \log y.$

4. $y(1-\log y)\dfrac{d^2y}{dx^2} + (1+\log y)\left(\dfrac{dy}{dx}\right)^2 = 0.$

5. Find the curve in which the radius of curvature is equal to the normal and is in the same direction.

15.6 EQUATIONS WHOSE ONE SOLUTION IS KNOWN

Consider the equation $d^2y/dx^2 + P\,dy/dx + Q = R$, where P, Q and R are functions of x only. If $y = u(x)$ is a known solution of this equation, then put $y = uv$ in it. It *reduces the differential equation to one of first order in dv/dx which can be completely solved.*

One integral belonging to the C.F. can be found by inspection as follows ;

(*i*) If $1 + P + Q = 0$, then $y = e^x$ is a solution,

(*ii*) If $I - P + Q = 0$, then $y = e^{-x}$ is a solution,

(*iii*) If $P + Qx = 0$, then $y = x$ is a solution.

Example 15.8. *Solve* $x\dfrac{d^2y}{dx^2} - (2x-1)\dfrac{dy}{dx} + (x-1)y = 0.$ *(Bhopal, 2008 S)*

Solution. The given equation is $\dfrac{d^2y}{dx^2} - \left(2 - \dfrac{1}{x}\right)\dfrac{dy}{dx} + \left(1 - \dfrac{1}{x}\right)y = 0$ ...(*i*)

Here $1 + P + Q = 1 - (2 - 1/x) + (1 - 1/x) = 0$

$\therefore$ $y = e^x$ is a part of C.F. of (*i*)

Now let $y = e^x v$ ...(*ii*)

so that $\dfrac{dy}{dx} = e^x v + e^x\dfrac{dv}{dx}$ ...(*iii*) and $\dfrac{d^2y}{dx^2} = e^x v + 2e^x\dfrac{dv}{dx} + e^x\dfrac{d^2v}{dx^2}$ ...(*iv*)

Substituting (*iv*), (*iii*) and (*ii*) in (*i*), we get

$$x\left(e^x v + 2e^x\frac{dv}{dx} + e^x\frac{d^2v}{dx^2}\right) - (2x-1)\left(e^x v + e^x\frac{dv}{dx}\right) + (x-1)e^x v = 0$$

or cancelling e^x, it becomes $x\dfrac{d^2v}{dx^2} + \dfrac{dv}{dx} = 0$ or $x\dfrac{dp}{dx} + p = 0$, where $p = \dfrac{dv}{dx}$.

Integrating, we get $\displaystyle\int \frac{dp}{p} = -\int \frac{dx}{x} + c$ or $\log p = -\log x + \log c_1$

i.e.,
$$p = \frac{c_1}{x} \quad \text{or} \quad \frac{dv}{dx} = \frac{c_1}{x}.$$

Again integrating, we obtain $v = c_1 \log x + c_2$

Hence the complete solution of (*i*) is $y = e^x(c_1 \log x + c_2)$.

Example 15.9. *Solve $(1-x^2)\,y'' - 2xy' + 2y = 0$ given that $y = x$ is a solution.* (B.P.T.U., 2005 S)

Solution. Let $y = xv$ so that $y' = v + x\dfrac{dv}{dx}$

and $$y'' = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}$$

Substituting these in the given equation, we get

$$(1-x^2)\left(x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}\right) - 2x\left(v + x\frac{dv}{dx}\right) + 2xv = 0$$

or $$(x-x^3)\frac{d^2v}{dx^2} + (2-4x^2)\frac{dv}{dx} = 0$$

or $$(x-x^3)\frac{dp}{dx} + (2-4x^2)\,p = 0 \text{ where } p = \frac{dv}{dx}$$

Integrating, we get $\displaystyle\int \frac{dp}{p} + \int \frac{2-4x^2}{x-x^3}\,dx = c$

or $$\log p + \int \frac{2}{x}dx - \int \frac{dx}{1-x} - \int \frac{dx}{1+x} = c$$

or $$\log p + 2\log x + \log(1-x) - \log(1+x) = \log c_1$$

$$px^2(1-x)/(1+x) = c_1 \text{ or } \frac{dv}{dx} = \frac{c_1(1+x)}{x^2(1-x)}$$

Again integrating, $v = c_1 \displaystyle\int\left(\frac{2}{x} + \frac{1}{x^2} + \frac{2}{1-x}\right)dx + c_2$

or $$v = c_1\{2\log(x/1-x) - 1/x\} + c_2$$

Hence the required complete solution is $y = x\,[c_1\{\log(x/1-x)^2 - 1/x\} + c_2]$

Obs. Here $P + Qx = 0$. That is why $y = x$ is a solution of the given equation.

PROBLEMS 15.5

1. If $y = e^{x^2}$ is a solution of $y'' - 4xy' + (4x^2-2)\,y = 0$, find a second independent solution. (U.P.T.U., 2004)
2. Solve $x^2y'' - (x^2+2x)\,y' + (x+2)y = x^3e^x$.
3. Solve $x\dfrac{d^2y}{dx^2} - (2x-1)\dfrac{dy}{dx} + (x-1)\,y = e^x$ given that $y = e^x$ is one integral. (Bhopal, 2007 S)
4. Solve $\sin^2 x\dfrac{d^2y}{dx^2} = 2y$, given that $y = \cot x$ is a solution. (Bhopal, 2007)
5. Solve $\dfrac{d^2y}{dx^2} - \cot x\dfrac{dy}{dx} - (1-\cot x)\,y = e^x \sin x$.

15.7 EQUATIONS WHICH CAN BE SOLVED BY CHANGING THE INDEPENDENT VARIABLE

Consider the equation $$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad ...(1)$$

To change the independent variable x to z, let $z = f(x)$

Then $$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} \qquad ...(2)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dz}\cdot\frac{dz}{dx}\right) = \frac{dy}{dz}\cdot\frac{d^2z}{dx^2} + \left(\frac{dz}{dx}\right)^2\frac{d^2y}{dz^2} \qquad ...(3)$$

Substituting (2) and (3) in (1), we get $\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$ ...(4)

where $$P_1 = \left(\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right)\Bigg/\left(\frac{dz}{dx}\right)^2, Q_1 = Q\Bigg/\left(\frac{dz}{dx}\right)^2, R_1 = R\Bigg/\left(\frac{dz}{dx}\right)^2$$

Now equation (4) can be solved by taking Q_1 = a constant.

Example 15.10. *Solve, by changing the independent variable,* $x\frac{d^2y}{dx^2} - \frac{dy}{dx} + 4x^3y = x^5$ *(U.P.T.U., 2003)*

Solution. Given equation is $\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + 4x^2y = x^4$ ...(*i*)

Here $P = -1/x$, $Q = 4x^2$ and $R = x^4$.

Choose z so that $Q/(dz/dx)^2$ = const. or $(dz/dx)^2 = 4x^2$ (say)

or $$\frac{dz}{dx} = 2x \quad \text{or} \quad z = x^2$$

Changing the independent variable x to z by $z = x^2$, we get

$$\frac{d^2y}{dz^2} + P \cdot \frac{dy}{dz} + Q_1 y = R_1 \qquad ...(ii)$$

where $P_1 = \left(\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right)\Bigg/\left(\frac{dz}{dx}\right)^2 = [2 + (-x^{-1})\, 2x]/4x^2 = 0$

$$Q_1 = \frac{Q}{(dz/dx)^2} = \frac{4x^2}{4x^2} = 1, R_1 = \frac{R}{(dz/dx)^2} = \frac{x^4}{4x^2} = \frac{x^2}{4} = \frac{z}{4}$$

$\therefore$ (*ii*) takes the form $\frac{d^2y}{dz^2} + y = \frac{z}{4}$ or $(D^2 + 1)\,y = \frac{z}{4}$

Its A.E. is $D^2 + 1 = 0$, *i.e.*, $D = \pm i$

$$\text{C.F.} = c_1 \cos z + c_2 \sin z$$

$$\text{P.I.} = \frac{1}{D^2+1}\frac{z}{4} = \frac{1}{4}(1 + D^2)^{-1} z = \frac{1}{4}(1 - D^2 \,.........)\, z = \frac{z}{4}.$$

Hence the complete solution of (*i*) is

$$y = c_1 \cos z + c_2 \sin z + \frac{z}{4} \quad \text{or} \quad y = c_1 \cos x^2 + c_2 \sin x^2 + \frac{x^2}{4}.$$

Example 15.11. *Solve* $\frac{d^2y}{dx^2} + \cot x \frac{dy}{dx} + 4y \operatorname{cosec}^2 x = 0.$ ...(*i*)

Solution. Here $P = \cot x$, $Q = 4 \operatorname{cosec}^2 x$

Choosing z so that $Q\Bigg/\left(\frac{dz}{dx}\right)^2$ = const. or $\left(\frac{dz}{dx}\right)^2 = \operatorname{cosec}^2 x$ (say)

$$dz/dx = \operatorname{cosec} x \text{ or } z = \int \operatorname{cosec} x\, dx = \log \tan x/2$$

Changing the independent variable x to z, we get

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1 \qquad ...(ii)$$

where $P_1 = \left(\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right)\Bigg/\left(\frac{dz}{dx}\right)^2 = (-\operatorname{cosec} x \cot x + \cot x \operatorname{cosec} x)/\operatorname{cosec}^2 x = 0$

$$Q_1 = Q\Bigg/\left(\frac{dz}{dx}\right)^2 = \frac{4\operatorname{cosec}^2 x}{\operatorname{cosec}^2 x} = 4, R_1 = 0$$

$\therefore$ (*ii*) takes the form $\dfrac{d^2y}{dz^2} + 4y = 0$

Its solution is $y = c_1 \cos(2z) + c_2 \sin(2z)$

i.e., $$y = c_1 \cos(2 \log \tan x/2) + c_2 \sin(2 \log \tan x/2)$$

This is the required complete solution of (*i*).

PROBLEMS 15.6

Solve the following equations (by changing the independent variable) :

1. $\dfrac{d^2y}{dx^2} + \tan x \dfrac{dy}{dx} + y \cos^2 x = 0.$ *(Bhopal, 2005)*
2. $\dfrac{d^2y}{dx^2} + \dfrac{2}{x}\dfrac{dy}{dx} + \dfrac{y}{x^4} = 0.$
3. $\dfrac{d^2y}{dx^2} - \cot x \dfrac{dy}{dx} - \sin^2 xy = 0.$
4. $x\dfrac{d^2y}{dx^2} + (4x^2 - 1)\dfrac{dy}{dx} + 4x^3y = 2x^3.$ *(U.P.T.U., 2006)*
5. $\cos x \dfrac{d^2y}{dx^2} + \sin x \dfrac{dy}{dx} - 2y \cos^3 x = 2 \cos^5 x.$ *(Bhopal, 2006 S)*

15.8 TOTAL DIFFERENTIAL EQUATIONS

(1) An ordinary differential equation of the first order and first degree involving three variables is of the form

$$P + Q\frac{dy}{dx} + R\frac{dz}{dx} = 0 \qquad ...(1)$$

where P, Q, R are functions of x, y, z and x is the independent variable.

In terms of differentials, (1) can be written as

$$Pdx + Qdy + Rdz = 0 \qquad ...(2)$$

which is integrable only if

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \qquad ...(3)$$

(2) Rule *to solve* $Pdx + Qdy + Rdz = 0$

If the condition of integrability is satisfied, consider one of the variables say : z, as constant so that dz = 0. Then integrate the equation Pdx + Qdy = 0. Replace the arbitrary constant appearing in its integral by φ(z). Now differentiate the integral just obtained with respect to x, y, z. Finally, compare this result with the given differential equation to determine φ(z).

Example 15.12. *Solve* $(y^2 + yz)dx + (z^2 + zx)dy + (y^2 - xy)dz = 0.$

Solution. Here $P = y^2 + yz,\ Q = z^2 + zx,\ R = y^2 - xy.$

$$\therefore \quad P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$= (y^2 + yz)[2z + x - (2y - x)] + (z^2 + zx)[-y - y] + (y^2 - xy)[(2y + z) - z] = 0$$

Hence the condition of integrability is satisfied.

Considering z as constant, the given equation becomes

$$(y^2 + yz)dx + (z^2 + zx)dy = 0, \quad \text{or} \quad \frac{dx}{z(z+x)} + \frac{dy}{y(y+z)} = 0$$

Integrating and noting that z is a constant, we get

$$\frac{1}{z}\int \frac{dx}{z+x} + \frac{1}{z}\int \left(\frac{1}{y} - \frac{1}{y+z}\right) dy = \text{constant}$$

i.e., $$\log(z + x) + \log y - \log(y + z) = \text{constant.}$$

i.e., $$\frac{y(z+x)}{y+z} = \text{constant} = \phi(z), \text{ say} \qquad ...(i)$$

or $$y(z + x) - (y + z)\,\phi(z) = 0$$

Differentiating w.r.t. x, y, z, we obtain

$$y(dz + dx) + (z + x)dy - [(y + z)\phi'(z)dz + (dy + dz)\phi(z)] = 0$$

or $$ydx + [z + x - \phi(z)]dy + [y - (y + z)\,\phi'(z) - \phi(z)]\,dz = 0 \qquad ...(ii)$$

Comparing (ii) with the given differential equation, we get

$$\frac{y^2 + yz}{y} = \frac{z^2 + zx}{z + x - \phi(z)} = \frac{y^2 - xy}{y - (y + z)\phi'(z) - \phi(z)}.$$

The relation $\dfrac{y^2 + yz}{y} = \dfrac{z^2 + zx}{z + x - \phi(z)}$ reduces to (i). $\therefore$ it gives no information about $\phi(z)$.

Taking $\dfrac{y^2 + yz}{y} = \dfrac{y^2 - xy}{y - (y + z)\phi'(z) - \phi(z)}$, we get

$$y^2 - xy = (y + z)\,[y - (y + z)\phi'(z) - \phi(z)] = y^2 + yz - (y + z)^2\,\phi'(z) - (y + z)\,\phi(z)$$

$$= y^2 + yz - (y + z)^2\,\phi'(z) - y(z + x) \qquad \text{[From } (i)]$$

$$= y^2 - xy - (y + z)^2\,\phi'(z)$$

i.e., $(y + z)^2\,\phi'(z) = 0$, *i.e.,* $\phi'(z) = 0$ so that $\phi(z) = c$

Hence the required solution is $y(z + x) = (y + z)\,c$. [From (i)]

Obs. *Sometimes the integral is readily obtained by simply regrouping the terms in the given equation* as is illustrated below.

Example 15.13. *Solve* $xdx + zdy + (y + 2z)\,dz = 0$.

Solution. Regrouping the terms, we can write the given equation as

$$xdx + (ydz + zdy) + 2z\,dz = 0$$

of which the integral is $\dfrac{x^2}{2} + yz + z^2 = c$.

PROBLEMS 15.7

Solve :

1. $(mz - ny)dx + (nx - lz)dy + (ly - mx)dz = 0$.
2. $(y^2 + z^2 - x^2)dx - 2xydy - 2xzdz = 0$.
3. $yzdx - 2zxdy - 3xydz = 0$.
4. $(2xz - yz)dx + (2yz - zx)dy - (x^2 - xy + y^2)dz = 0$.
5. $(x + z)^2dy + y^2(dx + dz) = 0$.
6. $(yz + xyz)dx + (zx + xyz)\,dy + (xy + xyz)\,dz = 0$.

15.9 SIMULTANEOUS TOTAL DIFFERENTIAL EQUATIONS

These equations in three variables are given by

$$\left.\begin{aligned} Pdx + Qdy + Rdz = 0 \\ P'dx + Q'dy + R'dz = 0 \end{aligned}\right\} \qquad ...(1)$$

where P, Q, R and P', Q', R' are any functions of x, y, z.

(a) If each of these equations is integrable and have solutions $f(x, y, z) = c$ and $Y(x, y, z) = c¢$ respectively, then these taken together constitute the solution of the simultaneous equations (1).

(b) If one or both the equations (1) is not integrable, then we write these as follows :

$$\frac{dx}{QR' - Q'R} = \frac{dy}{RP' - R'P} = \frac{dz}{PQ' - P'Q}$$

and solve these by the methods explained below.

15.10 EQUATIONS OF THE FORM dx/P = dy/Q = dz/R

(1) Method of grouping

See if it is possible to take two fractions $dx/P = dz/R$ from which y can be cancelled or is absent, leaving equations in x and z only.

If so, integrate it by giving $\phi(x, z) = c$. ...(1)

Again see if one variable say : x is absent or can be removed may be with the help of (1), from the equation $dy/Q = dz/R$.

Then integrate it by giving $\psi(y, z) = c'$ ...(2)

These two independent solutions (1) and (2) taken together constitute the complete solution required.

Example 15.14. *Solve* $\dfrac{dx}{z^2 y} = \dfrac{dy}{z^2 x} = \dfrac{dz}{y^2 x}$.

Solution. Taking the first two fractions and cancelling z^2, we get

$$\frac{dx}{y} = \frac{dy}{x} \quad \text{or} \quad xdx - ydy = 0$$

which on integration gives $x^2 - y^2 = c$. ...(*i*)

Again taking the second and third fractions and cancelling x, we have

$$\frac{dy}{z^2} = \frac{dz}{y^2}, \text{ i.e., } y^2 dy - z^2 dz = 0.$$

Its integral is $y^3 - z^3 = c'$. ...(*ii*)

Thus (*i*) and (*ii*) taken together constitute the required solution of the given equations.

(2) Method of multipliers

By a proper choice of the multipliers l, m, n which are not necessarily constants, we write

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{ldx + mdy + ndz}{lP + mQ + nR} \text{ such that } lP + mQ + nR = 0.$$

Then $ldx + mdy + ndz = 0$ can be solved giving the integral $\phi(x, y, z) = c$ ...(1)

Again search for another set of multipliers λ, μ, γ

so that $\lambda P + \mu Q + \gamma R = 0$

giving $\lambda dx + \mu dy + \gamma dz = 0$,

which on integration gives the solution $\psi(x, y, z) = c'$ ...(2)

These two solutions (1) and (2) taken together constitute the required solution.

Example 15.15. *Solve* $\dfrac{dx}{x(y^2 - z^2)} = \dfrac{dy}{-y(z^2 + x^2)} = \dfrac{dz}{z(x^2 + y^2)}$.

Solution. Using the multipliers x, y, z

$$\text{each fraction} = \frac{xdx + ydy + zdz}{x^2(y^2 - z^2) - y^2(z^2 + x^2) + z^2(x^2 + y^2)} = \frac{xdx + ydy + zdz}{0}$$

$\therefore$ $xdx + ydy + zdz = 0$, which on integration gives the solution $x^2 + y^2 + z^2 = c$ ...(*i*)

Again using the multipliers $1/x, -1/y, -1/z$

$$\text{each fraction} = \frac{\frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz}{(y^2 - z^2) + (z^2 + x^2) - (x^2 + y^2)} = \frac{\frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz}{0} \text{ so that } \frac{dx}{x} - \frac{dy}{y} - \frac{dz}{z} = 0$$

which on integration gives $\log x - \log y - \log z = \text{constant}$ or $yz = c'x$. ...(*ii*)

Hence the solution of the given equation is $x^2 + y^2 + z^2 = c$; $yz = c'x$.

PROBLEMS 15.8

Solve :

1. $\dfrac{xdx}{y^2 z} = \dfrac{dy}{xz} = \dfrac{dz}{y^2}$.
2. $\dfrac{dx}{mz - ny} = \dfrac{dy}{nx - lz} = \dfrac{dx}{ly - mx}$.
3. $\dfrac{dx}{x^2 - yz} = \dfrac{dy}{y^2 - zx} = \dfrac{dz}{z^2 - xy}$.
4. $\dfrac{dx}{y - zx} = \dfrac{dy}{yz + x} = \dfrac{dz}{x^2 + y^2}$.
5. $\dfrac{dx}{x(y^2 - z^2)} = \dfrac{dy}{y(z^2 - x^2)} = \dfrac{dz}{z(x^2 - y^2)}$.
6. $\dfrac{dx}{x^2 - y^2 - z^2} = \dfrac{dy}{2xy} = \dfrac{dz}{2xz}$.

CHAPTER 16

Series Solution of Differential Equations and Special Functions

16.1 INTRODUCTION

Many differential equations arising from physical problems are linear but have variable coefficients and do not permit a general solution in terms of known functions. Such equations can be solved by numerical methods (Chapter 28), but in many cases it is easier to find a solution in the form of an infinite convergent series.

The series solution of certain differential equations give rise to special functions such as Bessel's function, Legendre's polynomial, Lagurre's polynomial, Hermite's polynomial, Chebyshev polynomials. Strum-Lioville problem based on the orthogonality of functions is also included which shows that Bessel's, Legendre's and other equations can be considered from a common point of view. These special functions have many applications in engineering.

16.2 VALIDITY OF SERIES SOLUTION OF THE EQUATION

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \qquad ...(i)$$

can be determined with the help of the following theorems :

Def. 1. If $P_0(a) \neq 0$, then $x = a$ is called and **ordinary point** of (i), otherwise a **singular point.**

2. *A singular point $x = a$ of (1) is called* **regular** *if, when (i) is put in the form*

$$\frac{d^2y}{dx^2} + \frac{Q_1(x)}{x-a}\frac{dy}{dx} + \frac{Q_2(x)}{(x-a)^2}y = 0,$$

$Q_1(x)$ and $Q_2(x)$ possess derivatives of all orders in the neighbourhood of a.

3. *A singular point which is not regular is called an* **irregular singular point.**

Theorem I. *When $x = a$ is an ordinary point of (i), its every solution can be expressed in the form*

$$y = a_0 + a_1(x-a) + a_2(x-a)^2 + ... \qquad ...(ii)$$

Theorem II. *When $x = a$ is a regular singularity of (i), at least one of the solutions can be expressed as*

$$y = (x-a)^m [a_0 + a_1(x-a) + a_2(x-a)^2 + ...] \qquad ...(iii)$$

Theorem III. *The series (ii) and (iii) are convergent at every point within the circle of convergence at a.* A solution in series will be valid only if the series is convergent.

16.3 SERIES SOLUTION WHEN X = 0 IS AN ORDINARY POINT OF THE EQUATION

$$P_0 \frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = 0 \quad ...(1)$$

where P's are polynomials in x and $P_0 \neq 0$ at $x = 0$.

(*i*) Assume its solution to be of the form $y = a_0 + a_1x + a_2x^2 + ... + a_nx^n + ...$ ...(2)

(*ii*) Calculate dy/dx, d^2y/dx^2 from (2) and substitute the values of y, dy/dx, d^2y/dx^2 in (1).

(*iii*) Equate to zero the coefficients of the various powers of x and determine a_2, a_3, a_4 ... in terms of a_0, a_1. (The result obtained by equating to zero is the coefficient of x^n that is called the *recurrence relation*).

(*iv*) Substituting the values of a_2, a_3, a_4 ... in (2), we get the desired series solution having a_0, a_1 as its arbitrary constants.

Example 16.1. *Solve in series the equation* $\frac{d^2y}{dx^2} + xy = 0.$ *(V.T.U., 2010)*

Solution. Here $x = 0$ is an ordinary point since coefficient of $y'' \neq 0$ at $x = 0$.

Assume its solution is $y = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n + ...$ ...(*i*)

Then $$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + ... + na_nx^{n-1} + ...$$

and $$\frac{d^2y}{dx^2} = 2 . 1a_2 + 3 . 2a_3x + 4 . 3a_4x^2 + ... + n(n-1)a_nx^{n-2} + ...$$

Substituting in the given differential equation

$$1 . 1a_2 + 3 . 2a_3x + 4 . 3a_4x^2 + ... + n(n-1)a_nx^{n-2} + ... + x(a_0 + a_1x + a_2x^2 + ... + a_nx^n + ...) = 0$$

or $$2 . 1a_2 + (3 . 2a_3 + a_0)x + (4 . 3a_4 + a_1)x^2 + (5 . 4a_5 + a_2)x^3 + ... + [(n+2)(n+1)a_{n+2} + a_{n-1}]x^n + ... = 0.$$

Equating to zero the co-efficients of the various powers of x,

$a_2 = 0,$ [Coeff. of $x^0 = 0$]

$3 . 2a_3 + a_0 = 0$, *i.e.*, $a_3 = -\frac{a_0}{3!}$ [Coeff. of $x = 0$]

$4 . 3a_4 + a_1 = 0$, *i.e.*, $a_4 = -\frac{2a_1}{4!}$ [Coeff. of $x^2 = 0$]

$5 . 4a_5 + a_2 = 0$, *i.e.*, $a_5 = -\frac{a_2}{5 . 4} = 0$ and so on. [Coeff. of $x^3 = 0$]

In general, $(n+2)(n+1)a_{n+2} + a_{n-1} = 0$ [Coeff. of $x^n = 0$]

i.e., $$a_{n+2} = \frac{-a_{n-1}}{(n+2)(n+1)} \quad ..(ii)$$

which is the *recurrence relation.*

Putting $n = 4, 5, 6, ...$ in (*ii*) successively, $a_6 = -\frac{a_3}{6 . 5} = \frac{4a_0}{6!}$; $a_7 = -\frac{a_4}{7 . 6} = \frac{5 . 2a_1}{7!}$

$$a_8 = -\frac{a_5}{8 . 7} = 0; a_9 = -\frac{a_6}{9 . 8} = -\frac{7 . 4a_0}{9!} \text{ and so on.}$$

Substituting these values in (*i*), we get

$$y = a_0\left(1 - \frac{x^3}{3!} + \frac{1 . 4x^6}{6!} - \frac{1 . 4 . 7x^9}{9!} + ...\right) + a_1\left(x - \frac{2x^4}{4!} + \frac{2 . 5x^7}{7!} - ...\right)$$

which is the required solution.

Example 16.2. *Solve in series* $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0.$ *(Bhopal, 2008 ; U.P.T.U., 2006)*

Solution. Here $x = 0$ is an ordinary point since coefficient of $y'' \neq 0$ at $x = 0$.

Assume the solution of the given equation to be

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots \qquad \ldots(i)$$

Then $$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + \ldots + na_nx^{n-1} + \ldots$$

and $$\frac{d^2y}{dx^2} = 2a_2 + 3.2a_3x + 4.3a_4x^2 + \ldots + n(n-1)a_nx^{n-2} + \ldots$$

Substituting in the given equation, we get

$$(1-x^2)[2a_2 + 3.2a_3x + 4.3a_4x^2 + \ldots + n(n-1)a_nx^{n-2} + \ldots]$$
$$- x[a_1 + 2a_2x + 3a_3x^2 + \ldots + na_nx^{n-1} + \ldots] + 4[a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots] = 0$$

Equating to zero the coefficients of the various powers of x,

$$2a_2 + 4a_0 = 0 \quad i.e., \quad a_2 = -2a_0 \qquad [\text{coeff. of } x^0 = 0]$$

$$3.2a_3 - a_1 + 4a_1 = 0 \quad i.e., \quad a_3 = -\frac{1}{2}a_1 \qquad [\text{coeff. of } x^1 = 0]$$

$$4.3a_4 - 2a_2 - 2a_2 + 4a_2 = 0 \quad i.e., \quad a_4 = 0 \qquad [\text{coeff. of } x^2 = 0]$$

$$5.4a_5 - 3.2a_3 - 3a_3 + 4a_3 = 0 \qquad [\text{coeff. of } x^3 = 0]$$

i.e., $$20a_5 - 5a_3 = 0 \quad i.e., \quad a_5 = -\frac{a_1}{8} \text{ and so on.}$$

In general, $(n+2)(n+1)a_{n+2} - n(n-1)a_n - na_n + 4a_n = 0$

or $$a_{n+2} = \frac{n-2}{n+1}a_n \qquad \ldots(ii)$$

which is the *recurrence relation*

Putting $n = 4, 5, 6, 7\ldots$ in (ii) successively,

$$a_6 = 0; \quad a_7 = \frac{3}{6}a_5 = -\frac{3}{6}\frac{a_1}{8}; a_8 = 0; a_9 = -\frac{5.3}{8.6}\cdot\frac{a_1}{8}\ldots$$

Substituting these values in (i), we get

$$y = a_0(1-2x^2) + a_1x\left(1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{3}{6}\cdot\frac{x^6}{8} - \frac{5.3}{8.6}\cdot\frac{x^8}{8} - \ldots\right).$$

PROBLEMS 16.1

Solve the following equations in series :

1. $\frac{d^2y}{dx^2} + y = 0$, given $y(0) = 0$. *(B.P.T.U., 2005 S)*
2. $\frac{d^2y}{dx^2} + x^2y = 0$.
3. $y'' + xy' + y = 0$. *(V.T.U., 2008)*
4. $(1-x^2)y'' + 2y = 0$, given $y(0) = 4$, $y'(0) = 5$. *(P.T.U., 2006)*
5. $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$. *(S.V.T.U., 2008)*
6. $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 2y = 0$. *(U.P.T.U., 2004)*

16.4 FROBENIUS* METHOD : Series solution when x = 0 is a regular sigularity of the equation

$$\mathbf{P_0\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2y = 0} \qquad \ldots(1)$$

*A German mathematician *F.G. Frobenius* (1849–1917) who is known for his contributions to the theory of matrices and groups.

(*i*) Assume the solution to be $y = x^m(a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n + ...)$ ...(2)

(*ii*) Substitute from (2) for y, dy/dx, d^2y/dx^2 in (1) as before.

(*iii*) Equate to zero the coefficient of the lowest degree term in x. It gives a quadratic equation known as the *indicial equation*.

(*iv*) Equating to zero the coefficients of the other powers of x, find the values of $a_1, a_2, a_3,...$ in terms of a_0.

The complete solution depends on the nature of roots of the indicial equation.

Case I. When roots of the indicial equation are distinct and do not differ by an integer, *the complete solution is*

$$y = c_1(y)_{m_1} + c_2(y)_{m_2}$$

where m_1, m_2 are the roots.

Example 16.3. *Solve in series the equation* $9x(1-x)\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 4y = 0.$

(*Madras, 2006 ; Roorkee, 2000*)

Solution. Here $x = 0$ is a singular point since coefficient of $y'' = 0$ at $x = 0$.

Substituting $y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + ...$

$$\therefore \quad \frac{dy}{dx} = ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + ...$$

and
$$\frac{d^2y}{dx^2} = m(m-1)a_0x^{m-2} + (m+1)ma_1x^{m-1} + (m+2)(m+1)a_2x^m + ...$$

in the given equation, we obtain

$$9x(1-x)\,[m(m-1)a_0x^{m-2} + (m+1)ma_1x^{m-1} + (m+2)(m+1)a_2x^m + ...]$$
$$- 12[ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + ...] + 4[a_0x^m + a_1x^{m+1} + a_2x^{m+2} + ...] = 0.$$

The lowest power of x is x^{m-1}. Its coefficient equated to zero gives

$$a_0(9m(m-1) - 12m] = 0, \text{ i.e., } m(3m-7) = 0 \quad \text{as} \quad a_0 \neq 0.$$

Thus the roots of the *indicial equation* are $m = 0, 7/3$. *i.e., Roots are distinct and do not differ by an integer.*

The coefficient of x^m equated to zero gives $a_1\{9(m+1)m - 12(m+1)\} + a_0\{4 - 9m(m-1)\} = 0$

i.e., $$3a_1(3m-4)(m+1) - a_0(3m-4)(3m+1) = 0$$

i.e., $$3a_1(m+1) = a_0(3m+1).$$

Similarly $3a_2(m+2) = a_1(3m+4)$, $3a_3(m+3) = a_2(5m+7)$ and so on.

$$\therefore \quad a_1 = \frac{3m+1}{3(m+1)}a_0,\ a_2 = \frac{(3m+4)a_1}{3(m+2)} = \frac{(3m+4)(3m+1)}{3^2(m+2)(m+1)}a_0,\ a_3 = \frac{(3m+7)(3m+4)(3m+1)}{3^3(m+3)(m+2)(m+1)}a_0 \text{ etc.}$$

When $m = 0$, $a_1 = \frac{1}{3}a_0$, $a_2 = \frac{1.4}{3.6}a_0$, $a_3 = \frac{1.4.7}{3.6.9}a_0$ etc. giving the particular solution

$$y_1 = a_0\left[1 + \frac{1}{3}x + \frac{1.4}{3.6}x^2 + \frac{1.4.7}{3.6.9}x^3 + ...\right]$$

When $m = 7/3$, the particular solution is

$$y_2 = a_0x^{7/3}\left[1 + \frac{8}{10}x + \frac{8.11}{10.13}x^2 + \frac{8.11.14}{10.13.16}x^3 + ...\right]$$

Thus the complete solution is $y = c_1y_1 + c_2y_2$

i.e., $$y = C_1\left[1 + \frac{1}{3}x + \frac{1.4}{3.6}x^2 + \frac{1.4.7}{3.6.9}x^3 + ...\right]$$
$$+ C_2x^{7/3}\left[1 + \frac{8}{10}x + \frac{8.11}{10.13}x^2 + \frac{8.11.14}{10.13.16}x^3 + ...\right],$$ where $C_1 = c_1a_0$, $C_2 = c_2a_0$.

Case II. When roots of the indicial equation are equal *the complete solution is*

$$y = c_1(y)_{m_1} + c_2\left(\frac{\partial y}{\partial m}\right)_{m_1}$$

where m_1, m_1 are the roots.

Example 16.4. *Solve in series the equation* $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + xy = 0.$ *(V.T.U., 2010 ; S.V.T.U., 2007)*

Solution. Substituting $y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots$ ...(*i*)

$$\frac{dy}{dx} = ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \ldots$$

and
$$\frac{d^2y}{dx^2} = m(m-1)a_0x^{m-2} + (m+1)ma_1x^{m-1} + (m+2)(m+1)a_2x^m + \ldots$$

in the given equation, we obtain

$$x[m(m-1)a_0x^{m-2} + (m+1)ma_1x^{m-1} + (m+2)(m+1)a_2x^m + \ldots]$$
$$+ [ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \ldots]$$
$$+ x[a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots] = 0.$$

The lowest power of x is x^{m-1}. Its coefficient equated to zero gives $a_0[m(m-1) + m] = 0$.

i.e., $m^2 = 0$ as $a_0 \neq 0$. $\therefore$ $m = 0, 0$.

The coefficients of $x^m, x^{m+1}, \ldots$ equated to zero give

$$a_1[(m+1)m + m + 1] = 0, \quad i.e., \quad a_1 = 0$$
$$a_2(m+2)^2 + a_0 = 0, a_3(m+3)^2 + a_1 = 0, a_4(m+4)^2 + a_2 = 0 \text{ and so on.}$$

Clearly $a_3 = a_5 = a_7 \ldots\ldots = 0$.

Also
$$a_2 = -\frac{a_0}{(m+2)^2}, a_4 = -\frac{a_2}{(m+4)^2} = \frac{a_0}{(m+2)^2(m+4)^2} \text{ etc.}$$

$$\therefore \quad y = a_0x^m\left[1 - \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+2)^2(m+4)^2} - \frac{x^6}{(m+2)^2(m+4)^2(m+6)^2} + \ldots\ldots\right] \quad \ldots(ii)$$

Putting $m = 0$, the first solution is

$$y_1 = a_0\left[1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots\ldots\right] \quad \ldots(iii)$$

This gives only one solution instead of two. To get the second solution, differentiate (*ii*) partially w.r.t. m.

$$\frac{\partial y}{\partial m} = y \log x + a_0x^m\left\{\frac{x^2}{(m+2)^2}\frac{2}{m+2} - \frac{x^4}{(m+2)^2(m+4)^2}\left[\frac{2}{m+2} + \frac{2}{m+4}\right] + \ldots\right\}$$

$\therefore$ the second solution is $y_2 = \left(\dfrac{\partial y}{\partial m}\right)_{m=0}$

$$= y_1 \log x + a_0\left\{\frac{1}{2^2}x^2 - \frac{1}{2^2 \cdot 4^2}\left(1 + \frac{1}{2}\right)x^4 + \frac{1}{2^2 \cdot 4^2 \cdot 6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right)x^6 - \ldots\right\} \quad \ldots(iv)$$

Hence the complete solution is $y = c_1y_1 + c_2y_2$. [From (*iii*) & (*iv*)]

i.e.,
$$y = (C_1 + C_2 \log x)\left[1 - \frac{1}{2^2}x^2 + \frac{1}{2^2 \cdot 4^2}x^4 - \frac{1}{2^2 \cdot 4^2 \cdot 6^2}x^6 + \ldots\right]$$
$$+ C_2\left\{\frac{1}{2^2}x^2 - \frac{1}{2^2 \cdot 4^2}\left(1 + \frac{1}{2}\right)x^4 + \frac{1}{2^2 \cdot 4^2 \cdot 6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right)x^6 - \ldots\right\}$$

where $C_1 = a_0c_1, C_2 = a_0c_2$.

Obs. The above differential equation is called *Bessel's equation of order zero*, y_1 is called *Bessel function of the first kind of order zero* and is denoted by $J_0(x)$. It is absolutely convergent for all values of x whether real or complex.

y_2 is called the *Bessel function of the second kind of order zero or the Neumann function* and is denoted by $Y_0(x)$.

Thus the complete solution of the *Bessel's equation of order zero is* $y = AJ_0(x) + BY_0(x)$.

Case III. When roots of indicial equation are distinct and differ by an integer, making a coefficient of y infinite.

Let m_1 and m_2 be the roots such that $m_1 < m_2$. If some of the coefficients of y series become infinite when $m = m_1$, we modify the form of y by replacing a_0 by $b_0(m - m_1)$. Then the complete solution is

$$y = C_1(y)_{m_2} + C_2\left(\frac{\partial y}{\partial m}\right)_{m_1}$$

Obs. 1. Two independent solution can also be obtained by putting $m = m_1$ (lesser of the two roots) in the modified form of y and $\partial y/\partial m$.

Obs. 2. *If one of the coefficients* (say : a_1) *becomes* **indeterminate** *when* $m = m_2$, *the complete solution is given by putting* $m = m_2$ *in* y *which contains two arbitrary constants.*

Example 16.5. *Obtain the series solution of the equation*

$$x(1-x)\frac{d^2y}{dx^2} - (1+3x)\frac{dy}{dx} - y = 0.$$

Solution. Here $x = 0$ is a singular point, since coefficient of y'' is zero at $x = 0$.

$\therefore$ substituting $\quad y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots \quad$ ...(*i*)

$$\frac{dy}{dx} = ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \ldots$$

and
$$\frac{d^2y}{dx^2} = m(m-1)a_0x^{m-2} + (m+1)ma_1x^{m-1} + (m+2)(m+1)a_2x^m + \ldots$$

in the given equation, we obtain

$$x(1-x)[m(m-1)a_0x^{m-2} + (m+1)ma_1x^{m-1} + (m+2)(m+1)a_2x^m + \ldots]$$
$$-(1+3x)[ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \ldots] - [a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots] = 0$$

Equating to zero the coefficients of the lowest power of x, we get $a_0[m(m-1) - m] = 0$, $(a_0 \neq 0)$,

i.e., $\quad m(m-2) = 0$, *i.e.* $m = 0, 2$ *i.e.*, the two roots are distinct and differ by an integer.

Equating to zero the coefficients of successive powers of x, we get

$$(m-1)a_1 = (m+1)a_0,\ ma_2 = (m+2)a_1,\ (m+1)a_3 = (m+3)a_2 \text{ and so on.}$$

i.e.,
$$a_1 = \frac{m+1}{m-1}a_0,\ a_2 = \frac{(m+1)(m+2)}{(m-1)m}a_0,\ a_3 = \frac{(m+1)(m+2)(m+3)}{(m-1)m(m+1)}a_0 \text{ etc.}$$

Thus (*i*) becomes

$$y = a_0x^m\left[1 + \frac{m+1}{m-1}x + \frac{(m+1)(m+2)}{(m-1)m}x^2 + \frac{(m+1)(m+2)(m+3)}{(m-1)m(m+1)}x^3 + \ldots\right] \quad \ldots(ii)$$

Putting $m = 2$ (greater of the two roots) in (*ii*), the first solution is

$$y_1 = a_0x^2\left[1 + 3x + \frac{3.4}{2}x^2 + \frac{4.5}{2}x^3 + \ldots\right]$$

If we put $m = 0$ in (*ii*), the coefficients become infinite.

To obviate this difficulty, put $a_0 = b_0(m - 0)$ so that

$$y = b_0x^m\left[m + \frac{m(m+1)}{m-1}x + \frac{(m+1)(m+2)}{m-1}x^2 + \frac{(m+1)(m+2)(m+3)}{(m-1)(m+1)}x^3 + \ldots\right]$$

$$\therefore \quad \frac{\partial y}{\partial m} = b_0x^m\log x\left[m + \frac{m(m+1)x}{m-1} + \frac{(m+1)(m+2)}{m-1}x^2 + \frac{(m+1)(m+2)(m+3)}{(m-1)(m+1)}x^3 + \ldots\right]$$
$$+ b_0x^m\left[1 + \frac{m^2-2m-1}{(m-1)^2}x + \frac{m^2-m-5}{(m-1)^2}x^2 + \frac{m^2-2m-11}{(m-1)^2}x^3 + \ldots\right]$$

$\therefore$ the second solution is $y_2 = \left(\dfrac{\partial y}{\partial m}\right)_{m=0}$

$$= b_0 \log x\,[-1.2x^2 - 2.3x^3 - 3.4x^4 - \ldots] + b_0[1 - x - 5x^2 - 11x^3 - \ldots]$$

Hence the complete solution is $y = c_1y_1 + c_2y_2$

i.e.,
$$y = \frac{1}{2}\,c_1a_0[1.2x^2 + 2.3x^3 + 3.4x^4 + \ldots] - b_0c_2 \log x\,[1.2x^2 + 2.3x^3 + 3.4x^4 + \ldots]$$
$$- b_0c_2\,[-1 + x + 5x^2 + 11x^3 + \ldots]$$

i.e.,
$$y = (C_1 + C_2 \log x)\,(1.2x^2 + 2.3x^3 + 3.4x^4 + \ldots) + C_2\,(-1 + x + 5x^2 + 11x^3 + \ldots)$$

where $C_1 = \dfrac{1}{2}\,c_1a_0,\ C_2 = -b_0c_2$

Example 16.6. *Solve in series* $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (x^2 - 4)\,y = 0.$ *(Bhopal, 2008 S; Rajasthan, 2003)*

Solution. $x = 0$ is a singular point, since coeff. of y'' is zero at $x = 0$.

Substituting
$$y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots \qquad \ldots(i)$$
$$\frac{dy}{dx} = ma_0x^{m-1} + (m+1)\,a_1x^m + (m+2)\,a_2x^{m+1} + \ldots$$

and
$$\frac{d^2y}{dx^2} = m\,(m-1)\,a_0x^{m-2} + (m+1)\,ma_1x^{m-1} + (m+2)\,(m+1)\,a_2x^m + \ldots$$

in the given equation, we get
$$x^2\,[m\,(m-1)\,a_0x^{m-2} + (m+1)\,ma_1x^{m-1} + (m+2)\,(m+1)\,a_2x^m + \ldots]$$
$$+ x\,[ma_0x^{m-1} + (m+1)\,a_1x^m + (m+2)\,a_2x^{m+1} + \ldots]$$
$$+ (x^2 - 4)\,[a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots] = 0$$

Equating to zero the coefficients of the lowest power of x.
$$a_0\,[m\,(m-1) + m - 4] = 0 \text{ so that } m = \pm 2.$$

i.e., the two roots are distinct and differ by an integer.

Now equating to zero the coefficiens of successive powers of x, we get
$$m(m+4)\,a_2 = -a_0,\ i.e.,\ a_2 = \frac{-1}{m\,(m+4)}\,a_0,\ a_3 = 0$$
$$a_4 = \frac{1}{(m+2)\,(m+6)}\cdot\frac{1}{m\,(m+4)}\,a_0,\ a_5 = a_7 = \ldots = 0.$$
$$a_6 = \frac{-a_0}{m\,(m+2)\,(m+4)^2\,(m+6)\,(m+8)} \text{ etc.}$$

Substituting these values in (i), we get
$$y = a_0x^m\left[1 - \frac{x^2}{m\,(m+4)} + \frac{x^4}{m\,(m+2)\,(m+4)\,(m+6)} - \frac{x^6}{m\,(m+2)\,(m+4)^2\,(m+6)\,(m+8)} + \ldots\right] \qquad \ldots(ii)$$

Putting $m = 2$ (greater of the two roots) in (ii), the first solution is
$$y_1 = a_0x^2\left\{1 - \frac{x^2}{2\cdot 6} + \frac{x^4}{2\cdot 4\cdot 6\cdot 8} - \frac{x^6}{2\cdot 4\cdot 6^2\cdot 8\cdot 10} + \ldots\right\}$$

If we put $m = -2$ in (ii), the coefficients become infinite. To obviate this difficulty, let $a_0 = b_0\,(m+2)$, so that
$$y = b_0x^m\left[(m+2)\left\{1 - \frac{x^2}{m\,(m+4)}\right\} + \frac{x^4}{m\,(m+4)\,(m+6)} - \frac{x^6}{m\,(m+4)^2\,(m+6)\,(m+8)} + \ldots\right]$$

$$\therefore \quad \frac{\partial y}{\partial m} = b_0 x^m \log x \left[(m+2)\left\{1 - \frac{x^2}{m(m+4)}\right\} + \frac{x^4}{m(m+4)(m+6)} - \ldots \right]$$
$$+ b_0 x^m \left[1 - \frac{(m+2)}{m(m+4)} \left\{ \frac{1}{m+2} - \frac{1}{m} - \frac{1}{m+4} \right\} x^2 \right.$$
$$\left. + \frac{1}{m(m+4)(m+6)} \left\{ -\frac{1}{m} - \frac{1}{m+4} - \frac{1}{m+6} \right\} x^4 + \ldots \right]$$

The second solution is $y_2 = \left(\frac{\partial y}{\partial m}\right)_{m=-2}$

$$= b_0 x^{-2} \log x \left[-\frac{x^4}{2^2 \cdot 4} + \frac{x^6}{2^3 . 4 \cdot 6} \cdots \right] + b_0 x^{-2} \left[1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \ldots \right]$$

Hence the complete solution $y = c_1 y_1 + c_2 y_2$

i.e.,
$$y = C_1 x^2 \left\{ 1 - \frac{x^2}{2 \cdot 6} + \frac{x^4}{2 \cdot 4 \cdot 6 \cdot 8} - \frac{x^6}{2 \cdot 4 \cdot 6^2 \cdot 8 \cdot 10} + \ldots \right\}$$
$$+ C_2 \left[x^2 \log x \left\{ -\frac{1}{2^2 \cdot 4} + \frac{x^4}{2^3 \cdot 4 \cdot 6} \cdots \right\} + x^{-2} \left\{ 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \cdots \right\} \right]$$

where $C_1 = c_1 a_0$, $C_2 = c_2 b_0$.

Example 16.7. *Solve in series $xy'' + 2y' + xy = 0$.* (*U.P.T.U., 2003*)

Solution. Here $x = 0$ is a singular point since coefficient of $y'' = 0$ at $x = 0$.

$\therefore$ Substituting $y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + \ldots$ ...(*i*)

$$\frac{dy}{dx} = m a_0 x^{m-1} + (m+1) a_1 x^m + (m+2) a_2 x^{m+1} + (m+3) a_3 x^{m+2} + \ldots$$

and
$$\frac{d^2y}{dx^2} = m(m-1) a_0 x^{m-2} + (m+1) m a_1 x^{m-1} + (m+2)(m+1) a_2 x^m + \ldots.$$

in the given equation, we get

$$x\,[m(m-1) a_0 x^{m-2} + (m+1) m a_1 x^{m-1} + (m+2)(m+1) a_2 x^m + \ldots]$$
$$+ 2\,[m a_0 x^{m-1} + (m+1) a_1 x^m + (m+2) a_2 x^{m+1} + (m+3) a_3 x^{m+2} + \ldots]$$
$$+ x\,[a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + \ldots]$$

Equating to zero, the coefficients of the lowest power of x,

$$m(m-1) a_0 + 2m a_0 = 0 \text{ so that } m = 0, -1.$$

i.e., the roots are distinct of and differ by an integer.

Equating to zero, the coefficient of x^m, we get

$$(m+1) m a_1 + 2(m+1) a_1 = 0 \;\; i.e. \;\; (m+1)(m+2) a_1 = 0$$

or $\quad (m+1) a_1 = 0 \qquad [\because \; m + 2 \neq 0]$

When $m = -1$, $a_1 = 0/0$ *i.e., indeterminate.*

Hence the complete solution will be given by putting $m = -1$ in y itself (containing two arbitrary constants a_0 and a_1).

Now equating to zero, the coefficients of successive powers of x, we get

$(m+2)(m+3) a_2 + a_0 = 0$ [Coeff. of $x^{m+1} = 0$]

$(m+3)(m+4) a_3 + a_1 = 0$ [Coeff. of $x^{m+2} = 0$]

$(m+4)(m+5) a_4 + a_2 = 0$ [Coeff. of $x^{m+3} = 0$]

$(m+5)(m+6) a_5 + a_3 = 0$ etc. [Coeff. of $x^{m+4} = 0$]

i.e.,
$$a_2 = -\frac{a_0}{(m+2)(m+3)}, a_3 = \frac{-a_1}{(m+3)(m+4)}, a_4 = \frac{a_0}{(m+2)(m+3)(m+4)(m+5)},$$

$$a_5 = \frac{a_1}{(m+3)(m+4)(m+5)(m+6)} \text{ and so on.}$$

Substituting the values in (*i*), we get

$$y = x^m\left[a_0 + a_1 x - \frac{a_0}{(m+2)(m+3)}x^2 - \frac{a_1}{(m+3)(m+4)}x^3\right.$$
$$\left. + \frac{a_0}{(m+2)(m+3)(m+4)(m+5)}x^4 + \frac{a_1}{(m+3)(m+4)(m+5)(m+6)}x^5 - \cdots\right]$$

Putting $m = -1$, the complete solution is

$$y = x^{-1}\left\{a_0\left(1 - \frac{x^2}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} - \cdots\right) + a_1\left(x - \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdots\right)\right\}$$
$$= x^{-1}(a_0 \cos x + a_1 \sin x).$$

PROBLEMS 16.2

Solve the following equations in power series :

1. $4x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0.$ (*P.T.U., 2005*)
2. $y'' + xy' + (x^2 + 2)y = 0.$ (*P.T.U., 2007*)
3. $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - y = 0.$
4. $3x\dfrac{d^2y}{dx^2} + (1-x)\dfrac{dy}{dx} - y = 0.$ (*S.V.T.U., 2008*)
5. $x\dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + 2y = 0.$ (*J.N.T.U., 2006*)
6. $2x^2y'' + xy' - (x+1)y = 0.$ (*U.P.T.U., 2005*)
7. $8x^2\dfrac{d^2y}{dx^2} + 10x\dfrac{dy}{dx} - (1+x)y = 0.$ (*P.T.U., 2009*)
8. $2x(1-x)\dfrac{d^2y}{dx^2} + (1-x)\dfrac{dy}{dx} + 3y = 0.$ (*U.P.T.U., 2004*)
9. $x(1-x)\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 0.$
10. $(2x + x^3)\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 6xy = 0.$ (*Bhopal, 2008*)

16.5 BESSEL'S EQUATION*

One of the most important differential equations in applied mathematics is

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \qquad \text{...(1)}$$

which is known as *Bessel's equation of order n*. Its particular solutions are called *Bessel functions of order n*. Many physical problems involving vibrations or heat conduction in cylindrical regions give rise to this equation.

Substituting $y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots$

(1) takes the form

$$a_0(m^2 - n^2)x^m + a_1\{(m+1)^2 - n^2\}x^{m+1} + \{a_2[(m+2)^2 - n^2] + a_0\}x^{m+2} + \ldots = 0.$$

Equating to zero the coefficient of x^m, we obtain the indicial equation $m^2 - n^2 = 0$ (as $a_0 \neq 0$) where $m = n$ or $-n$.

$$a_1 = a_3 = a_5 = a_7 = \ldots = 0$$

and $$a_2 = -\frac{a_0}{(m+2)^2 - n^2},\ a_4 = -\frac{a_2}{(m+4)^2 - n^2} \text{ etc.}$$

These give $$y = a_0x^m\left(1 - \frac{1}{(m+2)^2 - n^2}x^2 + \frac{1}{[(m+2)^2 - n^2][(m+4)^2 - n^2]}x^4 - \cdots\right)$$

* Named after the German mathematician and astronomer *Friederich Wilhelm Bessel* (1784 – 1846) whose paper on Bessel functions appeared in 1826. He studied Astronomy of his own and became director of Königsberg observatory.

For $m = n$, we get

$$y_1 = a_0 x^n \left\{1 - \frac{1}{4(n+1)} x^2 + \frac{1}{4^2 \cdot 2!\,(n+1)(n+2)} x^4 - \frac{1}{4^3 \cdot 3!\,(n+1)(n+2)(n+3)} x^6 + \cdots\right\} \qquad ...(2)$$

and for $m = -n$, we have

$$y_2 = a_0 x^{-n} \left\{1 - \frac{1}{4(-n+1)} x^2 + \frac{1}{4^2 \cdot 2!\,(-n+1)(-n+2)} x^4 - \frac{1}{4^3 \cdot 3!\,(-n+1)(-n+2)(-n+3)} x^6 + \cdots\right\} \qquad ...(3)$$

Case I. *When n is not integral or zero,* the complete solution of (1) is $y = c_1 y_1 + c_2 y_2$.

If we take $a_0 = 1/2^n\,\Gamma(n+1)$, then the solution given by (2) is called the *Bessel function of the first kind of order n* and is denoted by $J_n(x)$. Thus

$$J_n(x) = \left(\frac{x}{2}\right)^n \left\{\frac{1}{\Gamma(n+1)} - \frac{1}{1!\,\Gamma(n+2)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma(n+3)}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\,\Gamma(n+4)}\left(\frac{x}{2}\right)^6 + \cdots\right\} \quad (n > 0)$$

i.e.

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{r!\,\Gamma(n+r+1)} \qquad ...(4)$$

and corresponding to (3), we have $J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{-n+2r} \frac{1}{r!\,\Gamma(-n+r+1)}$ ...(5)

which is called the *Bessel function of the first kind of order* $-n$.

Hence complete solution of the Bessel's equation (1) may be expressed in the form.

$$y = AJ_n(x) + BJ_{-n}(x). \qquad ...(6)$$

Case II. *When n is zero,* $y_1 = y_2$ and the complete solution of (1), which reduces to the *Bessel's equation of order zero,* is obtained as in Example 16.4.

Case III. *When n is integral,* y_2 fails to give a solution for positive values of n and y_1 fails to give a solution for negative values. Thus another independent integral of the Bessel's equation (1) is needed to form its general solution. We now proceed to find an independent solution of (1), when n is an integer.

Let $y = u(x)J_n(x)$ be a solution of (1). Substituting the values of y, $y' = u'J_n + uJ_n'$ and $y'' = u''J_n + 2u'J_n' + uJ_n''$ in (1), we obtain

$$x^2(u''J_n + 2u'J_n' + uJ_n'') + x(u'J_n + uJ_n') + (x^2 - n^2)uJ_n = 0$$

or

$$u\{x^2 J_n'' + xJ_n' + (x^2 - n^2)J_n\} + x^2u''J_n + 2x^2u'J_n' + xu'J_n = 0. \qquad ...(7)$$

Now since J_n is a solution of (1), therefore, $x^2J_n'' + xJ_n' + (x^2 - n^2)J_n = 0$

$\therefore$ (7) reduces to $x^2u''J_n + 2x^2u'J_n' + xu'J_n = 0$.

Dividing throughout by $x^2u'J_n$, it becomes $\dfrac{u''}{u'} + 2\dfrac{J_n'}{J_n} + \dfrac{1}{x} = 0$

i.e.,

$$\frac{d}{dx}(\log u') + 2\frac{d}{dx}(\log J_n) + \frac{d}{dx}(\log x) = 0 \text{ or } \frac{d}{dx}\{\log(u'J_n^2x)\} = 0.$$

Integrating, $\log(u'J_n^2x) = \log B$, whence $xu'J_n^2 = B$.

$$\therefore \qquad u' = \frac{B}{xJ_n^2} \text{ or } u = B\int \frac{dx}{xJ_n^2} + A.$$

Thus

$$y = AJ_n(x) + BJ_n(x)\int \frac{dx}{x[J_n(x)]^2}.$$

Hence the complete solution of the Bessel's equation (1) is

$$y = AJ_n(x) + BY_n(x) \qquad ...(8) \quad (\textit{V.T.U., 2006})$$

where

$$Y_n(x) = J_n(x)\int \frac{dx}{x[J_n(x)]^2} \qquad ...(9)$$

$Y_n(x)$ is called the *Bessel function of the second kind of order n or Neumann function**.

* Named after the German mathematician and physicist *Carl Neumann* (1832–1925) whose work on potential theory gave impetus for development of integral equations by *Volterra* of Rome, *Fredholm* of Stockholm and *Hilbert* of Gottingen.

Obs. Putting $k = -n + r$, *i.e.* $r = k + n$, and noting that $\Gamma(k+1) = k\,!$ where k is an integer, (5) may be written as

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+n}\,(x/2)^{2k+n}}{(k+n)!\,k!} = (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k\,(x/2)^{n+2k}}{k\,!\,\Gamma(k+n+1)}$$

Hence $J_{-n}(x) = (-1)^n J_n(x)$. ...(10) (*Bhopal, 2008 ; S.V.T.U., 2008 ; V.T.U., 2006*)

16.6 RECURRENCE FORMULAE FOR $J_n(x)$

The following recurrence formulae can easily be derived from the series expression for $J_n(x)$:

(1) $\dfrac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$.

(2) $\dfrac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$.

(3) $J_n(x) = \dfrac{x}{2n}[J_{n-1}(x) + J_{n+1}(x)]$.

(4) $J_n'(x) = \dfrac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$.

(5) $J_n'(x) = \dfrac{n}{x} J_n(x) - J_{n+1}(x)$.

(6) $J_{n+1}(x) = \dfrac{2n}{x} J_n(x) - J_{n-1}(x)$.

These formulae are very useful in the solution of boundary value problems and in establishing the various properties of Bessel functions.

Proofs. **(1)** Multiplying (4) of page 551 by x^n, we have

$$x^n J_n(x) = x^n \sum_{r=0}^{\infty} \frac{(-1)^r\,(x/2)^{n+2r}}{r\,!\,\Gamma(n+r+1)} = \sum_{r=0}^{\infty} \frac{(-1)^r\,x^{2(n+r)}}{2^{n+2r}\,r\,!\,\Gamma(n+r+1)}$$

$$\therefore \quad \frac{d}{dx}[x^n J_n(x)] = \sum_{r=0}^{\infty} \frac{(-1)^r\,2(n+r)x^{2(n+r)-1}}{2^{n+2r}\,r\,!\,\Gamma(n+r+1)} = x^n \sum_{r=0}^{\infty} \frac{(-1)^r\,(x/2)^{n-1+2r}}{r\,!\,\Gamma(n-1+r+1)} = x^n J_{n-1}(x).$$

(*Bhopal, 2008 ; V.T.U., 2005 ; U.P.T.U., 2005*)

(2) Multiplying (4) of page 551 by x^{-n}, we have

$$x^{-n} J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r\,x^{2r}}{2^{n+2r}\,r!\,\Gamma(n+r+1)}$$

$$\therefore \quad \frac{d}{dx}[x^{-n} J_n(x)] = \sum_{r=0}^{\infty} \frac{(-1)^r\,2r\,x^{2r-1}}{2^{n+2r}\,r\,!\,\Gamma(n+r+1)} = -x^{-n} \sum_{r=1}^{\infty} \frac{(-1)^{r-1}\,x^{n+1+2(r-1)}}{2^{n+1+2(r-1)}(r-1)\,!\,\Gamma(n+r+1)}$$

$$= -x^{-n} \sum_{k=0}^{\infty} \frac{(-1)^k\,(x/2)^{n+1+2k}}{k\,!\,\Gamma(n+1+k+1)} = -x^{-n} J_{n+1}(x), \text{ where } k = r-1.$$

(*P.T.U., 2006 ; B.P.T.U., 2005*)

(3) From (1), we have $x^n J_n'(x) + nx^{n-1} J_n(x) = x^n J_{n-1}(x)$

or dividing by x^n, $\quad J_n'(x) + (n/x) J_n(x) = J_{n-1}(x)$ ...(*i*)

Similarly from (2), we get $x^{-n} J_n'(x) - nx^{-n-1} J_n(x) = -x^{-n} J_{n+1}(x)$

or $\quad -J_n'(x) + \dfrac{n}{x} J_n(x) = J_{n+1}(x)$ ...(*ii*)

Adding (*i*) and (*ii*), we obtain $\dfrac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$

i.e., $\quad J_n(x) = \dfrac{x}{2n}[J_{n-1}(x) + J_{n+1}(x)]$ (*S.V.T.U., 2008 ; Anna, 2005 S*)

(4) Subtracting (*ii*) from (*i*), we get $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

i.e., $\quad J_n'(x) = \dfrac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$. (*S.V.T.U., 2007 ; P.T.U., 2005*)

(5) is another way of writing (*ii*). (*J.N.T.U., 2006 ; Anna, 2005*)

(6) is another way of writing (3). (*Madras, 2006 ; V.T.U., 2005*)

16.7 (1) EXPANSIONS FOR J_0 AND J_1

We have from (4) of page 551,

$$J_0(x) = 1 - \frac{1}{(1!)}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \cdots \quad ...(1)$$

and
$$J_1(x) = \frac{x}{2}\left[1 - \frac{1}{1!\,2!}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,3!}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\,4!}\left(\frac{x}{2}\right)^6 + \cdots\right] \quad \textit{(B.P.T.U., 2005)} \; ...(2)$$

Because of their special importance, the values of $J_0(x)$ and $J_1(x)$ are given in Appendix 2 : Table II to four decimal places at intervals of 0.1. With the help of these values, the graphs of $J_0(x)$ and $J_1(x)$ can be drawn as shown in Fig. 16.1, for $x > 0$. Their close resemblance to graphs of $\cos x$ and $\sin x$ is interesting.

Obs. The roots of the equation $J_0(x) = 0$ are useful in some physical problems. This equation has no complex roots but an infinite number of real roots. Its first four roots are $x = 2.4, 5.52, 8.65, 11.79$ approximately.

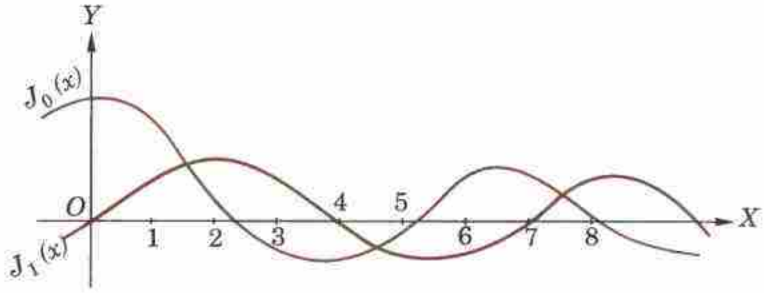


Fig. 16.1

16.8 VALUE OF $J_{1/2}$

We may think that $J_0(x)$ is the simplest of the J's but actually $J_{1/2}(x)$ is simpler, for it can be expressed in a finite form. Taking $n = \frac{1}{2}$ in (4) of page 551, we have

$$J_{1/2}(x) = \left(\frac{x}{2}\right)^{1/2}\left\{\frac{1}{\Gamma\left(\frac{3}{2}\right)} - \frac{1}{1!\,\Gamma\left(\frac{5}{2}\right)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma\left(\frac{7}{2}\right)}\left(\frac{x}{2}\right)^4 - \ldots\right\}$$

$$= \left(\frac{x}{2}\right)^{1/2}\left\{\frac{1}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} - \frac{1}{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^2 + \frac{1}{2\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^4 - \ldots\right\}$$

$$= \frac{\sqrt{x}}{\sqrt{2\Gamma\left(\frac{1}{2}\right)}}\left\{\frac{2}{1!} - \frac{2x^2}{3!} + \frac{2x^4}{5!} - \ldots\right\}$$

Now multiplying the series by $x/2$ and outside by $2/x$, we get

$$J_{1/2}(x) = \frac{\sqrt{2}}{\sqrt{x}\sqrt{\pi}}\left\{\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right\} = \sqrt{\left(\frac{2}{\pi x}\right)}\sin x. \quad ...(3) \quad \textit{(V.T.U., 2009 ; J.N.T.U., 2003)}$$

Similarly taking $n = \frac{1}{2}$ in (5) of page 551, it can be shown that

$$J_{-\frac{1}{2}}(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\cos x. \quad ...(4) \quad \textit{(Anna, 2005 ; W.B.T.U., 2005 ; V.T.U., 2003)}$$

Example 16.8. *Express $J_5(x)$ in terms of $J_0(x)$ and $J_1(x)$.* *(Bhopal, 2008 S ; V.T.U., 2001)*

Solution. We know that

$$J_n(x) = \frac{x}{2n}[J_{n-1}(x) + J_{n+1}(x)] \text{ i.e. } J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x)$$

Putting $n = 1, 2, 3, 4$ successively, $J_2(x) = \frac{2}{x}J_1(x) - J_0(x) \quad ...(i) \qquad J_3(x) = \frac{4}{x}J_2(x) - J_1(x) \quad ...(ii)$

$$J_4(x) = \frac{6}{x}J_3(x) - J_2(x) \quad ...(iii) \qquad J_5(x) = \frac{8}{x}J_4(x) - J_3(x) \quad ...(iv)$$

Substituting the value of $J_2(x)$ in (ii), we have

$$J_3(x) = \frac{4}{x}\left\{\frac{2}{x}J_1(x) - J_0(x)\right\} - J_1(x) = \left(\frac{8}{x^2} - 1\right)J_1(x) - \frac{4}{x}J_0(x) \qquad ...(v)$$

(W.B.T.U., 2005 ; Madras, 2003)

Now substituting the values of $J_3(x)$ from (v) and $J_2(x)$ from (i) in (iii), we get

$$J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right)J_1(x) + \left(1 - \frac{24}{x^2}\right)J_0(x) \qquad ...(vi) \quad (V.T.U., 2003\ S)$$

Finally putting the values of $J_4(x)$ from (vi) and $J_3(x)$ from (v) in (iv), we obtain

$$J_5(x) = \left(\frac{384}{x^4} - \frac{72}{x^2} - 1\right)J_1(x) + \left(\frac{12}{x} - \frac{192}{x^3}\right)J_0(x).$$

Example 16.9. *Prove that* $J_{5/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\left\{\frac{3-x^2}{x^2}\sin x - \frac{3}{x}\cos x\right\}$. *(J.N.T.U., 2006)*

Solution. We know that $J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x)$ ...(i)

Putting $n = \frac{1}{2}$, we get $J_{3/2}(x) = \frac{1}{x}J_{1/2}(x) - J_{-1/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\left(\frac{\sin x}{x} - \cos x\right)$

(Bhopal, 2007 ; V.T.U., 2006)

Again putting $n = \frac{3}{2}$ in (i), we get $J_{5/2}(x) = \frac{3}{x}J_{3/2}(x) - J_{1/2}(x)$

$$= \frac{3}{x}\left[\sqrt{\left(\frac{2}{\pi x}\right)}\left(\frac{\sin x}{x} - \cos x\right)\right] - \sqrt{\left(\frac{2}{\pi x}\right)}\sin x = \sqrt{\left(\frac{2}{\pi x}\right)}\left[\frac{3-x^2}{x^2}\sin x - \frac{3}{x}\cos x\right]$$

which is the required result.

Example 16.10. *Prove that*

(a) $J_n''(x) = \frac{1}{4}[J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)]$. *(J.N.T.U., 2006)*

(b) $\frac{d}{dx}[xJ_n(x)J_{n+1}(x)] = x[J_n^2(x) - J_{n+1}^2(x)]$. *(V.T.U., 2006)*

Solution. *(a)* We know that $J_n'(x) = \frac{1}{2}\{J_{n-1}(x) - J_{n+1}(x)\}$ ...(i)

Differentiating both sides, we get $J_n''(x) = \frac{1}{2}\{J_{n-1}'(x) - J_{n+1}'(x)\}$ ...(ii)

Changing n to $n-1$ in (i), we obtain $J_{n-1}'(x) = \frac{1}{2}\{J_{n-2}(x) - J_n(x)\}$ ...(iii)

Changing n to $n+1$ in (i), we have $J_{n+1}'(x) = \frac{1}{2}\{J_n(x) - J_{n+2}(x)\}$ ...(iv)

Substituting the values of $J_{n-1}'(x)$ and $J_{n+1}'(x)$ from (iii) and (iv) in (ii), we get

$$J_n'' = \frac{1}{4}[J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)]$$

(b) $\frac{d}{dx}[xJ_n(x)J_{n+1}(x)] = J_n(x)J_{n+1}(x) + x[J_n(x)J'_{n+1}(x) + J_n'(x)J_{n+1}(x)]$ ...(i)

From (5) of § 16.6, we have $J_n'(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)$ ...(ii)

Changing n to $n+1$ in (i) of page 499, we get $J'_{n+1}(x) = J_n(x) - \frac{n+1}{x}J_{n+1}(x)$ ...(iii)

Now substituting from (*iii*) and (*ii*) in (*i*), we get

$$\frac{d}{dx}[xJ_n(x)\,J_{n+1}(x)] = J_n(x)\,J_{n+1}(x) + x\left[J_n(x)\left\{J_n(x) - \frac{n+1}{x}J_{n+1}(x)\right\} + \left\{\frac{n}{x}J_n(x) - J_{n+1}(x)\right\}J_{n+1}(x)\right]$$

$$= x\{J_n^2(x) - J_{n+1}^2(x)\}.$$

Example 16.11. *Prove that :*

(*a*) $\int J_3(x)dx = c - J_2(x) - \frac{2}{x}J_1(x).$

(*b*) $\int xJ_0^2(x)dx = \frac{1}{2}x^2\{J_0^2(x) + J_1^2(x)\}.$ (*U.P.T.U., 2004 ; Osmania, 2002*)

Solution. (*a*) We know that $\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$ [§ 16.6 (2)] ...(*i*)

or $$\int x^{-n}J_{n+1}(x)dx = -x^{-n}J_n(x) \quad ...(ii)$$

$\therefore$ $$\int J_3(x)\,dx = \int x^2\cdot x^{-2}J_3(x)dx + c \quad \text{[Integrate by parts]}$$

$$= x^2.\int x^{-2}J_3(x)dx - \int 2x\left[\int x^{-2}J_3(x)dx\right]dx + c$$

$$= x^2[-x^{-2}J_2(x)] - \int 2x[-x^{-2}J_2(x)]\,dx + c \quad \text{[By (ii) when } n = 2]$$

$$= c - J_2(x) + \int \frac{2}{x}J_2(x)\,dx = c - J_2(x) - \frac{2}{x}J_1(x) \quad \text{[By (ii) when } n = 1]$$

(*b*) $$\int xJ_0^2(x)dx = \int J_0^2(x)\cdot x dx \quad \text{[Integrate by parts]}$$

$$= J_0^2(x)\cdot\frac{1}{2}x^2 - \int 2J_0(x)J_0'(x).\frac{1}{2}x^2\,dx$$

$$= \frac{1}{2}x^2J_0^2(x) + \int x^2J_0(x)J_1(x)\,dx \quad \text{[By (i) when } n = 0]$$

$$= \frac{1}{2}x^2J_0^2(x) + \int xJ_1(x)\cdot\frac{d}{dx}[xJ_1(x)]\,dx \quad \left[\because \frac{d}{dx}[xJ_1(x)] = xJ_0(x) \text{ by § 16.6 (1)}\right]$$

$$= \frac{1}{2}x^2J_0^2(x) + \frac{1}{2}[xJ_1(x)]^2 = \frac{1}{2}x^2[J_0^2(x) + J_1^2(x)].$$

16.9 GENERATING FUNCTION FOR $J_n(x)$

To prove that $$e^{\frac{1}{2}x(t-1/t)} = \sum_{n=-\infty}^{\infty} t^nJ_n(x).$$

We have $$e^{\frac{1}{2}x(t-t^{-1})} = e^{xt/2}\times e^{-x/2t}$$

$$= \left[1 + \left(\frac{xt}{2}\right) + \frac{1}{2!}\left(\frac{xt}{2}\right)^2 + \frac{1}{3!}\left(\frac{xt}{2}\right)^3 + \cdots\right]\times\left[1 - \left(\frac{x}{2t}\right) + \frac{1}{2!}\left(\frac{x}{2t}\right)^2 - \frac{1}{3!}\left(\frac{x}{2t}\right)^3 + \cdots\right]$$

The coefficient of t^n in this product

$$= \frac{1}{n!}\left(\frac{x}{2}\right)^n - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{2!(n+2)!}\left(\frac{x}{2}\right)^{n+4} - \ldots = J_n(x).$$

As all the integral powers of t, both positive and negative occur, we have

$$e^{\frac{1}{2}x(t-t^{-1})} = J_0(x) + tJ_1(x) + t^2J_2(x) + t^3J_3(x) + \ldots + t^{-1}J_{-1}(x) + t^{-2}J_{-2}(x) + t^{-3}J_{-3}(x) + \ldots$$

$$= \sum_{n=-\infty}^{\infty} t^nJ_n(x)$$ (*V.T.U., 2007*)

This shows that Bessel functions of various orders can be derived as coefficients of different powers of t in the expansion of $e^{\frac{1}{2}x(t-1/t)}$. For this reason, it is known as the *generating function of Bessel functions*.

Example 16.12. *Show that*

(a) $J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\,d\theta$, *n being an integer.* *(V.T.U., 2006)*

(b) $J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\cos\phi)\,d\phi$. *(Madras, 2006)*

(c) $J_0^2 + 2J_1^2 + 2J_2^2 + 2J_3^2 + \ldots = 1$. *(Kerala M. Tech, 2005 ; U.P.T.U., 2003; V.T.U., 2003 S)*

Solution. (*a*) We know that

$$e^{\frac{1}{2}x(t-1/t)} = J_0(x) + tJ_1(x) + t^2J_2(x) + t^3J_3(x) + \ldots + t^{-1}J_{-1}(x) + t^{-2}J_{-2}(x) + t^{-3}J_{-3}(x) + \ldots$$

Since $J_{-n}(x) = (-1)^n J_n(x)$

$$\therefore \quad e^{\frac{1}{2}x(t-1/t)} = J_0 + J_1(t - 1/t) + J_2(t^2 + 1/t^2) + J_3(t^3 - 1/t^3) + \ldots \quad \ldots(i)$$

Now put $t = \cos\theta + i\sin\theta$

so that $t^p = \cos p\theta + i\sin p\theta$ and $1/t^p = \cos p\theta - i\sin p\theta$

giving $t^p + 1/t^p = 2\cos p\theta$ and $t^p - 1/t^p = 2i\sin p\theta$.

Substituting these in (*i*), we get

$$e^{ix\sin\theta} = J_0 + 2[J_2\cos 2\theta + J_4\cos 4\theta + \ldots] + 2i\,[J_1\sin\theta + J_3\sin 3\theta + \ldots] \quad \ldots(ii)$$

Since $e^{ix\sin\theta} = \cos(x\sin\theta) + i\sin(x\sin\theta)$.

$\therefore$ equating the real and imaginary parts in (*ii*), we get

$$\cos(x\sin\theta) = J_0 + 2\,[J_2\cos 2\theta + J_4\cos 4\theta + \ldots] \quad \ldots(iii)$$

$$\sin(x\sin\theta) = 2[J_1\sin\theta + J_3\sin 3\theta + \ldots] \quad \ldots(iv)$$

which are known as *Jacobi series**. *(V.T.U., 2006)*

Now multiplying both sides of (*iii*) by $\cos n\theta$ and both sides of (*iv*) by $\sin n\theta$ and integrating each of the resulting expressions between 0 and π, we obtain

$$\frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta)\cos n\theta\,d\theta = \begin{cases} J_n(x), & n \text{ even or zero} \\ 0, & n \text{ odd} \end{cases}$$

and

$$\frac{1}{\pi}\int_0^{\pi}\sin(x\sin\theta)\sin n\theta\,d\theta = \begin{cases} 0 & n \text{ even} \\ J_n(x), & n \text{ odd} \end{cases}$$

Hence generally, if n is a positive integer,

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi}[\cos(x\sin\theta)\cos n\theta + \sin(x\sin\theta)\sin n\theta]\,d\theta = \frac{1}{\pi}\int_0^{\pi}\cos(n\theta - x\sin\theta)\,d\theta.$$

[This is Bessel's original definition of $J_n(x)$ given in 1824 while investigating Planetary motion.]

(*b*) Changing θ to $\frac{1}{2}\pi - \phi$ in (*iii*), we get

$$\cos(x\cos\phi) = J_0 + 2J_2\cos(\pi - 2\phi) + 2J_4\cos(2\pi - 4\phi) + \ldots$$
$$= J_0 - 2J_2\cos 2\phi + 2J_4\cos 4\phi - \ldots$$

Integrating both sides w.r.t. ϕ from 0 to π, we get

$$\int_0^{\pi}\cos(x\cos\phi)\,d\phi = \int_0^{\pi}\{J_0(x) - 2J_2(x)\cos 2\phi + 2J_4(x)\cos 4\phi - \cdots\}\,d\phi$$

$$= \left| J_0(x)\cdot\phi - 2J_2(x)\cdot\frac{1}{2}\sin 2\phi + 2J_4(x)\cdot\frac{1}{4}\sin 4\phi - \cdots \right|_0^{\pi} = J_0(x)\,.\,\pi \text{ whence follows the result.}$$

* See footnot p. 215.

(c) Squaring (iii) and (iv) and integrating w.r.t. ϕ from 0 to π and noting that (m, n being integers),

$$\int_0^\pi \cos m\theta \cos n\theta \, d\theta = \int_0^\pi \sin m\theta \sin n\theta \, d\theta = 0, \qquad (m \neq n)$$

and $$\int_0^\pi \cos^2 n\theta \, d\theta = \int_0^\pi \sin^2 n\theta \, d\theta = \pi/2, \text{ we obtain}$$

$$[J_0(x)]^2 \pi + 4\,[J_2(x)]^2 \frac{\pi}{2} + 4\,[J_4(x)]^2 \frac{\pi}{2} + \ldots = \int_0^\pi \cos^2(x \sin\theta)\, d\theta$$

$$4\,[J_1(x)]^2 \frac{\pi}{2} + 4\,[J_3(x)]^2 \frac{\pi}{2} + \ldots = \int_0^\pi \sin^2(x \sin\theta)\, d\theta$$

Adding, $$\pi\,[J_0^2 + 2J_1^2 + 2J_2^2 + 2J_3^2 + \cdots] = \int_0^\pi d\theta = \pi$$

Hence $$J_0^2 + 2J_1^2 + 2J_2^2 + 2J_3^2 + \cdots = 1.$$

PROBLEMS 16.3

1. Compute $J_0(2)$, $J_1(1)$ correct to three decimal places.

2. Show that (i) $J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^4}\right) J_0(x)$. (ii) $J_1(x) + J_3(x) = \frac{4}{x} J_2(x)$ (P.T.U., 2003)

3. Show that

   (i) $J_{-1/2}(x) = J_{1/2}(x) \cot x$. (S.V.T.U., 2008) (ii) $J'_{1/2}(x) J_{-1/2}(x) - J'_{-1/2}(x) J_{1/2}(x) = 2/\pi x$ (Delhi, 2002)

   (iii) $J_{-3/2}(x) = -\sqrt{\left(\frac{2}{\pi x}\right)}\left(\sin x + \frac{\cos x}{x}\right)$ (iv) $J_{-5/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\left(\frac{3}{x}\sin x + \frac{3 - x^2}{x^2}\cos x\right)$

   (V.T.U., 2000)

4. Prove that (i) $\frac{d}{dx} J_0(x) = -J_1(x)$. (ii) $\frac{d}{dx}[xJ_1(x)] = xJ_0(x)$.

   (iii) $\frac{d}{dx}[x^n J_n(ax)] = ax^n J_{n-1}(ax)$. (Madras, 2000 S) (iv) $J'_n(x) = -\frac{n}{2} J_n(x) + J_{n-1}(x)$ (P.T.U., 2009 S)

5. Show by the use of recurrence formula, that

   (i) $J_0''(x) = \frac{1}{2}[J_2(x) - J_0(x)]$ (ii) $J_1''(x) = J_1(x) - \frac{1}{x} J_2(x)$.

   (iii) $4J_0'''(x) + 3J_0'(x) + J_3(x) = 0$. (Osmania, 2003)

6. Prove that

   (i) $\frac{d}{dx}\{J_n^2(x)\} = \frac{x}{2n}[J_{n-1}^2(x) - J_{n+1}^2(x)]$ (S.V.T.U., 2008 ; Kerala M.E., 2005)

   (ii) $\frac{d}{dx}\{J_n^2(x) + J_{n+1}^2(x)\} = 2\left\{\frac{n}{2} J_n^2(x) - \frac{n+1}{x} J_{n+1}^2(x)\right\}$. (U.P.T.U., 2005 ; V.T.U., 2000 S)

7. Prove that (i) $\int_0^{\pi/2} \sqrt{(\pi x)}\, J_{1/2}(2x)\, dx = 1$. (P.T.U., 2005) (ii) $\int_0^r xJ_0(ax)dx = \frac{r}{a} J_1(ar)$.

   (iii) $\int x^2 J_1(x) dx = x^2 J_2(x)$. (P.T.U., 2007)

8. Prove that (i) $\int J_0(x) J_1(x)\, dx = -\frac{1}{2}[J_0(x)]^2$. (ii) $\int_0^\infty e^{-ax} J_0(bx) dx = \frac{1}{\sqrt{a^2 + b^2}}$.

9. Starting with the series of § 16.9, prove that

   $$2nJ_n(x) = x\{J_{n-1}(x) + J_{n+1}(x)\} \text{ and } xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x).$$

10. Establish the *Jacobi series*

    $$\cos(x\cos\theta) = J_0 - 2J_2 \cos 2\theta + 2J_4 \cos 4\theta - \ldots$$

    $$\sin(x\cos\theta) = 2[J_1 \cos\theta - J_3 \cos 3\theta + J_5 \cos 5\theta - \ldots]$$ (Madras, 2003 S)

11. Prove that (i) $\sin x = 2[J_1 - J_3 + J_5 - \ldots]$ (Anna, 2005 S)

    (ii) $\cos x = J_0 - 2J_2 + 2J_4 - 2J_6 + \ldots$ (Kerala M. Tech., 2005)

    (iii) $1 = J_0 + 2J_2 + 2J_4 + 2J_6 + \ldots$

16.10 EQUATIONS REDUCIBLE TO BESSEL'S EQUATION

In many problems, we come across such differential equations which can easily be reduced to Bessel's equation and, therefore, can be solved by means of Bessel functions.

(**1**) *To reduce the differential equation* $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (k^2x^2 - n^2)\, y = 0$ *to Bessel form.*

Put $t = kx$, so that $\dfrac{dy}{dx} = k\,\dfrac{dy}{dt}$ and $\dfrac{d^2y}{dx^2} = k^2\,\dfrac{d^2y}{dt^2}$.

Then (**1**) becomes $t^2\,\dfrac{d^2y}{dt^2} + t\,\dfrac{dy}{dt} + (t^2 - n^2)y = 0$

$\therefore$ its solution is $y = c_1J_n(t) + c_2J_{-n}(t)$, n is non-integral,

or $\qquad y = c_1J_n(t) + c_2Y_n(t)$, n is integral.

Hence the solution of (1) is

$$y = c_1J_n(kx) + c_2J_{-n}(kx),\ n \text{ is non-integral}$$

or
$$y = c_1J_n(kx) + c_2Y_n(kx),\ n \text{ is integral.}$$

(**2**) *To reduce the differential equation* $x\dfrac{d^2y}{dx^2} + a\,\dfrac{dy}{dx} + k^2xy = 0$ *to Bessel's equation,* *(Madras, 2006)*

put $\qquad y = x^nz,$

so that
$$\frac{dy}{dx} = x^n\,\frac{dz}{dx} + nx^{n-1}z \text{ and } \frac{d^2y}{dx^2} = x^n\,\frac{d^2z}{dx^2} + 2nx^{n-1}\,\frac{dz}{dx} + n(n-1)x^{n-2}z$$

Then (**2**) takes the form $x^{n+1}\,\dfrac{d^2z}{dx^2} + (2n + a)x^n\,\dfrac{dz}{dx} + [k^2x^2 + n^2 + (a-1)n]x^{n-1}z = 0.$

Dividing throughout by x^{n-1} and putting $2n + a = 1$,

$$x^2\,\frac{d^2z}{dx^2} + x\,\frac{dz}{dx} + (k^2x^2 - n^2)z = 0.$$

Its solution by (1) is $z = c_1J_n(kx) + c_2J_{-n}(kx)$, n is non-integral

or $\qquad z = c_1J_n(kx) + c_2Y_n(kx)$, n is integral

Hence the solution of (2) is $y = x^n\,[c_1J_n(kx) + c_2J_{-n}(kx)]$, n is non-integral

or $\qquad y = x^n\,[c_1J_n(kx) + c_2Y_n(kx)]$, n is integral, where $n = (1 - a)/2$.

(**3**) *To reduce the differential equation* $x\,\dfrac{d^2y}{dx^2} + c\,\dfrac{dy}{dx} + k^2x^ry = 0$ *to Bessel form,* put $x = t^m$, *i.e.* $t = x^{1/m}$,

so that
$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{1}{m}t^{1-m}\,\frac{dy}{dt}$$

and
$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{1}{m}t^{1-m}\,\frac{dy}{dt}\right)\cdot\frac{1}{m}\,t^{1-m}\ \frac{1}{m^2}\,t^{2-2m}\,\frac{d^2y}{dt^2} + \frac{1-m}{m^2}\,t^{1-2m}\,\frac{dy}{dt}$$

Then (**3**) takes the form $\dfrac{1}{m^2}\,t^{2-m}\,\dfrac{d^2y}{dt^2} + \dfrac{1-m+cm}{m^2}\,t^{1-m}\,\dfrac{dy}{dt} + k^2t^{mr}\,y = 0$

or multiplying throughout by m^2/t^{1-m}, $t\,\dfrac{d^2y}{dt^2} + (1 - m + cm)\,\dfrac{dy}{dt} + (km)^2t^{mr+m-1}\,y = 0.$

In order to reduce it to (2), we set $mr + m - 1 = 1$, *i.e.* $m = 2/(r + 1)$

and $\qquad a = 1 - m + cm = (r + 2c - 1)/(r + 1).$

Thus it reduces to $t\,\dfrac{d^2y}{dt^2} + a\,\dfrac{dy}{dt} + (km)^2ty = 0$ which is similar to (2).

Hence the solution of (3) is $y = x^{n/m}\{c_1J_n\,(k_mx^{1/m}) + c_2J_{-n}(kmx^{1/m})\}$, n is a fraction

or $\qquad y = x^{n/m}\,\{c_1J_n(kmx^{1/m}) + c_2Y_n(kmx^{1/m})\}$, n is an integer

where
$$n = \frac{1-a}{2} = \frac{1-c}{1+r} \text{ and } m = \frac{2}{1+r}.$$

Example 16.13. *Solve the differential equations :*

(i) $y'' + \dfrac{y'}{x} + \left(8 - \dfrac{1}{x^2}\right) y = 0.$ *(ii)* $4y'' + 9xy = 0.$ *(iii)* $xy'' + y' + \dfrac{1}{4}y = 0.$ *(Anna, 2005)*

Solution. (*i*) Rewriting the given equation as $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (8x^2 - 1)y = 0,$

and comparing with (**1**) above, we see that $n = 1$ and $k = 2\sqrt{2}$.

$\therefore$ The solution of the given equation is $y = c_1J_n(kx) + c_2Y_n(kx)$

i.e.,
$$y = c_1J_1(2\sqrt{2x}) + c_2Y_1(2\sqrt{2x}).$$

(*ii*) Rewriting the given equation as $x\dfrac{d^2y}{dx^2} + \dfrac{9}{4}x^2y = 0$ ...(α)

and comparing with (**3**) above, we find that $c = 0$, $k = 3/2$ and $r = 2$.

$$\therefore \quad n = \frac{1-c}{1+r} = \frac{1}{3}, \quad m = \frac{2}{1+r} = \frac{2}{3} \quad \text{and} \quad \frac{n}{m} = \frac{1}{2}.$$

Hence the solution of (α) is $y = x^{n/m}\{c_1J_n(kmx^{1/m}) + c_2Y_{-n}(kmx^{1/m})\}$

$$y = \sqrt{x}\,[c_1J_{1/3}(x^{3/2}) + c_2J_{-1/3}(x^{3/2})].$$

(*iii*) Multiplying by x, the given equation becomes

$$x^2y'' + xy' + \frac{1}{4}xy = 0 \qquad \text{...(α)}$$

Comparing with (**3**) above, we get $c = 1$, $k = 1/2$ & $r = 0$. $\therefore$ $m = \dfrac{2}{1+r} = 2$, $n = \dfrac{1-c}{1+r} = 0$ & $\dfrac{n}{m} = 0$

Hence the solution of (α)

$$y = x^{n/m}\{c_1J_n(kmx^{1/m}) + c_2Y_n(kmx^{1/m})\} = x^0\left\{c_1J_0\left(\frac{1}{2}2x^{1/2}\right) + c_2Y_0\left(\frac{1}{2}.2x^{1/2}\right)\right\}$$

i.e.,
$$y = c_1J_0(\sqrt{x}) + c_2Y_0(\sqrt{x})$$

16.11 (1) ORTHOGONALITY OF BESSEL FUNCTIONS

We shall prove that

$$\int_0^1 xJ_n(\alpha x)J_n(\beta x)\,dx = \begin{cases} 0, & \alpha \neq \beta \\ \frac{1}{2}[J_{n+1}(\alpha)]^2, & \alpha = \beta \end{cases}$$, *where* α, β *are the roots of* $J_n(x) = 0$.

We know that the solution of the equation

$$x^2u'' + xu' + (\alpha^2x^2 - n^2)u = 0 \qquad \text{...(1)}$$

and
$$x^2v'' + xv' + (\beta^2x^2 - n^2)v = 0 \qquad \text{...(2)}$$

are $u = J_n(\alpha x)$ and $v = J_n(\beta x)$ respectively.

Multiplying (1) by v/x and (2) by u/x and subtracting, we get

$$x(u''v - uv'') + (u'v - uv') + (\alpha^2 - \beta^2)xuv = 0$$

or
$$\frac{d}{dx}\{x(u'v - uv')\} = (\beta^2 - \alpha^2)\,xuv.$$

Now integrating both sides from 0 to 1,

$$(\beta^2 - \alpha^2)\int_0^1 xuv\,dx = [x(u'v - uv')]_0^1 = (u'v - uv')_{x=1} \qquad \text{...(3)}$$

Since $u = J_n(\alpha x)$,

$$\therefore \quad u' = \frac{d}{dx}[J_n(\alpha x)] = \frac{d}{d(\alpha x)}[J_n(\alpha x)].\frac{d(\alpha x)}{dx} = \alpha J_n'(\alpha x)$$

Similarly, $v = J_n(\beta x)$ and $v' = \beta J_n'(\beta x)$. Substituting these values in (3), we get

$$\int_0^1 xJ_n(\alpha x)J_n(\beta x)\,dx = \frac{\alpha J_n'(\alpha)J_n(\beta) - \beta J_n(\alpha)J_n'(\beta)}{\beta^2 - \alpha^2} \quad \text{...(4)}$$

If α and β are distinct roots of $J_n(x) = 0$, then $J_n(\alpha) = J_n(\beta) = 0$, and (4) reduces to

$$\int_0^1 xJ_n(\alpha x)J_n(\beta x)\,dx = 0 \quad \text{...(5)}$$

This is known as the *orthogonality relation of Bessel functions.*

When $\beta = \alpha$, the right side of (4) is of 0/0 form. Its value can be found by considering α as a root of $J_n(x) = 0$ and β as a variable approaching α. Then (4) gives

$$\underset{\beta\to\alpha}{\text{Lt}} \int_0^1 xJ_n(\alpha x)J_n(\beta x)\,dx = \underset{\beta\to\alpha}{\text{Lt}} \frac{\alpha J_n'(\alpha)J_n(\beta)}{\beta^2 - \alpha^2}$$

or by L'Hospital's rule, $$\int_0^1 xJ_n^2(\alpha x)\,dx = \underset{\beta\to\alpha}{\text{Lt}} \frac{\alpha J_n'(\alpha)J_n'(\beta)}{2\beta} = \frac{1}{2}[J_n'(\alpha)]^2$$

$$= \frac{1}{2}[J_{n+1}(\alpha)]^2 \quad \text{...(6)} \quad \text{[By (5) of p. 552]}$$

Obs. If however, the interval be from 0 to 1, it can be shown that

$$\int_0^1 xJ_n^2(\alpha x)\,dx = \frac{1}{2}[J_n'(\alpha)]^2 \quad \text{where } \alpha \text{ is the root of } J_n(x) = 0. \quad \text{...(7)} \quad (\textit{V.T.U., 2006})$$

(2) Fourier-Bessel expansion. If $f(x)$ is a continuous function having finite number of oscillations in the interval $(0, a)$, then we can write

$$f(x) = c_1J_n(\alpha_1 x) + c_2J_n(\alpha_2 x) + \ldots + c_nJ_n(\alpha_n x) + \ldots \quad \text{...(8)}$$

where $\alpha_1, \alpha_2 \ldots$ are the positive roots of $J_n(x) = 0$.

To determine the coefficients c_n, multiply both sides of (8) by $xJ_n(\alpha_n x)$ and integrate from 0 to a. Then all integrals on the right of (1) vanish by (5), except the term in c_n. This gives

$$\int_0^a xf(x)J_n(\alpha_n x)\,dx = c_n \int_0^a xJ_n^2(\alpha_n x)\,dx = c_n \frac{a^2}{2} J_{n+1}^2(a\alpha_n) \quad \text{[By (7)]}$$

$$\therefore \quad c_n = \frac{2}{a^2 J_{n+1}^2(a\alpha_n)} \int_0^a xf(x)J_n(\alpha_n x)\,dx$$

Equation (8) is known as the *Fourier-Bessel expansion of f (x).*

Example 16.14. *If $\alpha_1, \alpha_2, \ldots \alpha_n$ are the positive roots of $J_0(x) = 0$, show that*

$$\frac{1}{2} = \sum_{n=1}^{\infty} [J_0(\alpha_n x)/\alpha_n J_1(\alpha_n)].$$

Solution. If $$f(x) = c_1J_n(\alpha_1 x) + c_2J_n(\alpha_2 x) + \ldots + c_rJ_n(\alpha_n x) + \ldots \quad \text{...}(i)$$

then $$c_r = \frac{2}{a^2 J_{n+1}^2(a\alpha_r)} \int_0^a x\,f(x)J_n(\alpha_n x)\,dx$$

Taking $f(x) = 1$, $a = 1$ and $n = 0$, we get

$$c_r = \frac{2}{J_1^2(\alpha_r)} \int_0^1 xJ_0(\alpha_r x)\,dx = \frac{2}{J_1^2(\alpha_r)} \left| \frac{xJ_1(\alpha_r x)}{\alpha_r} \right|_0^1 = \frac{2}{\alpha_r J_1(\alpha_r)}$$

From (*i*), $$1 = \sum_{r=1}^{\infty} \frac{2}{\alpha_r J_1(\alpha_r)} J_0(\alpha_r x) \quad \text{or} \quad \frac{1}{2} = \sum_{n=1}^{\infty} \frac{J_0(\alpha_n x)}{\alpha_n J_1(\alpha_n)}.$$

Example 16.15. *Expand $f(x) = x^2$ in the interval $0 < x < 2$ in terms of $J_2(\alpha_n x)$, where α_n are determined by $J_2(2\alpha_n) = 0$.*

Solution. Let the Fourier-Bessel expansion of $f(x)$ be $x^2 = \sum_{n=1}^{\infty} c_n J_2(\alpha_n x)$.

Multiplying both sides by $xJ_2(\alpha_n x)$ and integrating w.r.t. x from 0 to 2, we get

$$\int_0^2 x^3 J_2(\alpha_n x)dx = c_n \int_0^2 xJ_2^2(\alpha_n x)dx = c_n \frac{(2)^2}{2} J_3^2(2\alpha_n) \qquad \text{[By (7)]}$$

or
$$\left| \frac{x^3 J_3(\alpha_n x)}{\alpha_n} \right|_0^2 = 2c_n J_3^2(2\alpha_n)$$

$\therefore$
$$c_n = \frac{4}{\alpha_n J_3(2\alpha_n)}$$

Hence
$$x^2 = 4\sum_{n=1}^{\infty} \frac{J_2(\alpha_n x)}{\alpha_n J_3(2\alpha_n)}.$$

16.12 BER AND BEI FUNCTIONS

Consider the differential equation

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} - ixy = 0 \qquad ...(1)$$

which occurs in certain problems of electrical engineering. This is equation (1) of §16.10 with $n = 0$ and $k^2 = -i$, so that its particular solution is

$$y = J_0(kx) = J_0[(-i)^{1/2} x] = J_0(i^{3/2} x)$$

Replacing $i^{3/2} x$ in the series for $J_0(x)$ [§16.8], we get

$$y = 1 - \frac{i^3x^2}{2^2} + \frac{i^6x^4}{(2\,!)^2\, 2^4} - \frac{i^9x^6}{(3\,!)^2 2^6} + \frac{i^{12}x^8}{(4!)^2\, 2^8} - ...$$

$$= \left[1 - \frac{x^4}{2^2 \cdot 4^2} + \frac{x^8}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} - ...\right] + i\left[\frac{x^2}{2^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \frac{x^{10}}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2 \cdot 10^2} - ...\right] \qquad ...(2)$$

which is complex for x real. The series in the above brackets are taken to define *Bessel-real* (*or ber*) and *Bessel-imaginary* (*or bei*) *functions*.

Thus
$$ber\, x = 1 + \sum_{m=1}^{\infty} (-1)^m \frac{x^{4m}}{2^2 \cdot 4^2 \cdot 6^2 ...(4m)^2} \qquad ...(3)$$

and
$$bei\, x = -\sum_{m=1}^{\infty} (-1)^m \frac{x^{4m-2}}{2^2 \cdot 4^2 \cdot 6^2 ...(4m-2)^2} \qquad ...(4)$$

so that $y = ber\, x + i\, bei\, x$ is a solution of (1).

Tables giving numerical values of *ber* x and *bei* x are also available.

Example 16.16. *Prove that* (*i*) $\frac{d}{dx}(x\, ber'\, x) = -x\, bei\, x$ (*ii*) $\frac{d}{dx}(x\, bei'\, x) = x\, ber\, x.$

Solution. We have $x\, ber'x = x\sum_{m=1}^{\infty} (-1)^m \frac{4mx^{4m-1}}{2^2 \cdot 4^2 \cdot 6^2 ...(4m)^2}$

$$= \sum_{m=1}^{\infty} (-1)^m \cdot \frac{x^{4m}}{2^2 \cdot 4^2 \cdot 6^2 ...(4m-2)^2\, 4m} = -\int_0^{x} x\, bei\, x\, dx$$

or
$$\frac{d}{dx}(x\, ber'\, x) = -x\, bei\, x$$

Again
$$\int_0^x x\, ber\, x\, dx = \frac{x^2}{2} + \sum_{m=1}^{\infty} (-1)^m \frac{x^{4m+2}}{2^2 \cdot 4^2 \cdot 6^2 ...(4m)^2 (4m+2)}$$

$$= -\sum_{m=1}^{\infty} (-1)^m \frac{x^{4m-2}}{2^2 \cdot 4^2 \cdot 6^2 ...(4m-4)^2 (4m-2)} = x\, bei'\, x \quad \text{or} \quad \frac{d}{dx}(x\, bei'\, x) = x\, ber\, x.$$

PROBLEMS 16.4

Obtain the solutions of the following differential equations in terms of Bessel functions :

1. $y'' + \dfrac{y'}{x} + \left(1 - \dfrac{1}{9x^2}\right) y = 0.$

2. $y'' + \dfrac{y'}{2} + \left(1 - \dfrac{1}{6.25x^2}\right) y = 0.$

3. $xy'' + \alpha y' + k^2 xy = 0.$ *(V.T.U., 2010)*

4. $x^2y'' - xy' + 4x^2y = 0.$

5. $xy'' + y = 0.$

6. Show that (*i*) $x^n J_n(x)$ is a solution of the equation $xy'' + (1 - 2n) y' + xy = 0.$ *(V.T.U., 2001)*
(*ii*) $x^{-n} J_n(x)$ is a solution of the equation $xy'' + (1 + 2n)y' + xy = 0.$

7. Show that under the transformation $y = u/\sqrt{x}$, Bessel equation becomes

$$u'' + \left(1 + \frac{1 - 4n^2}{4x^2}\right) u = 0.$$ Hence find the solution of this equation.

8. By the use of substitution $y = u/\sqrt{x}$, show that the solution of the equation $x^2 \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} + \left(x^2 - \dfrac{1}{4}\right) y = 0$ can be written in the form $y = c_1 \dfrac{\sin x}{\sqrt{x}} + c_2 \dfrac{\cos x}{\sqrt{x}}.$

9. Show that $\int_0^p x(ber^2 x + bei^2 x)\, dx = p(ber\, p\, bei'\, p - bei\, p\, ber'p).$

10. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the positive roots of $J_0(x) = 0$, prove that

$$x^2 = 2 \sum_{n=1}^{\infty} \frac{\alpha_n^2 - 4}{\alpha_n^3 J_1(\alpha_n)} J_0(\alpha_n x).$$

11. Expand $f(x) = x^3$ in the interval $0 < x < 3$ in terms of functions $J_1(\alpha_n x)$ where α_n are determined by $J_1(3\alpha) = 0.$

16.13 LEGENDRE'S EQUATION*

Another differential equation of importance in Applied Mathematics, particularly in boundary value problems for spheres, is *Legendre's equation,*

$$(1 - x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + n(n + 1)y = 0 \qquad \ldots(1)$$

Here n is a real number. But in most applications only integral values of n are required.

Substituting $\qquad y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \ldots\ (a_0 \neq 0),$

(1) takes the form

$$a_0(m)(m - 1)x^{m-2} + a_1(m + 1)mx^{m-1} + \ldots + [a_{r+2}(m + r + 2)(m + r + 1)$$
$$-\{(m + r)(m + r + 1) - n(n + 1)\}a_r]x^{m+r} + \ldots = 0$$

Equating to zero the coefficient of the lowest power of x, *i.e.*, of x^{m-2}, we get

$$a_0 m(m - 1) = 0, m = 0, 1 \qquad [\because\ a_0 \neq 0]$$

Equating to zero the coefficients of x^{m-1} and x^{m+r}, we get $a_1(m + 1)m = 0 \qquad \ldots(2)$

$$a_{r+2}(m + r + 2)(m + r + 1) - \{(m + r)(m + r + 1) - n(n + 1)\}a_r = 0 \qquad \ldots(3)$$

When $m = 0$, (2) is satisfied and therefore, $a_1 \neq 0$. Then (3) gives, taking $r = 0, 1, 2, \ldots$ in turn,

$$a_2 = -\frac{n(n+1)}{2!}a_0, \quad a_3 = -\frac{(n-1)(n+2)}{3!}a_1$$

$$a_4 = \frac{-(n-2)(n+3)}{4.3}a_2 = \frac{n(n-2)(n+1)(n+3)}{4!}a_0$$

$$a_5 = -\frac{(n-3)(n+4)}{5.4}a_3 = \frac{(n-1)(n-3)(n+2)(n+4)}{5!}a_1, \text{ etc.}$$

Hence for $m = 0$, there are two independent solutions of (1) :

$$y_1 = a_0 \left\{1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \ldots\ldots\right\} \qquad \ldots(4)$$

*See footnote p. 493.

$$y_2 = a_1 \left\{ x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} x^5 - \ldots\ldots \right\} \qquad \ldots(5)$$

When $m = 1$, (2) shows that $a_1 = 0$. Therefore, (3) gives

$$a_3 = a_5 = a_7 = \ldots = 0$$

and
$$a_2 = -\frac{(n-1)(n+2)}{3!} a_0$$

$$a_4 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!} a_0, \text{ etc.}$$

Thus for $m = 1$, we get the solution (5) again. Hence $y = y_1 + y_2$ is the general solution of (1).

If n is a positive even integer, the series (4) terminates at the term in x^n and y_1 becomes a polynomial. Similarly if n is an odd integer, (5) becomes a polynomial of degree n. Thus, whenever n is a positive integer, the general solution of (1) consists of a polynomial solution and an infinite series solution.

These polynomial solutions, with a_0 or a_1 so chosen that the value of the polynomial is 1 for $x = 1$, are called *Legendre polynomials* of order n and are denoted by $P_n(x)$. The infinite series solution with (a_0 or a_1 properly chosen) is called *Legendre function of the second kind* and is denoted by $Q_n(x)$. *(V.T.U., 2006)*

16.14 (1) RODRIGUE'S FORMULA*

We shall prove that $P_n(x) = \dfrac{1}{n!\,2^n} \dfrac{d^n}{dx^n}(x^2 - 1)^n \qquad \ldots(1)$

Let $\qquad v = (x^2 - 1)^n$. Then $v_1 = \dfrac{dv}{dx} = 2nx(x^2 - 1)^{n-1}$

i.e., $\qquad (1 - x^2)v_1 + 2nxv = 0 \qquad \ldots(2)$

Differentiating (2), $(n + 1)$ times by Leibnitz's theorem

$$(1 - x^2)v_{n+2} + (n+1)(-2x)v_{n+1} + \frac{1}{2!}(n+1)n(-2)v_n + 2n[xv_{n+1} + (n+1)v_n = 0$$

or
$$(1 - x^2)\frac{d^2(v_n)}{dx^2} - 2x\frac{d(v_n)}{dx} + n(n+1)v_n = 0$$

which is Legendre's equation and cv_n is its solution. Also its finite series solution is $P_n(x)$.

$$\therefore \qquad P_n(x) = cv_n = c\frac{d^n}{dx^n}(x^2 - 1)^n \qquad \ldots(3)$$

To determine the constant c, put $x = 1$ in (3). Then

$$1 = c\left[\frac{d^n}{dx^n}\{(x-1)^n(x+1)^n\}\right]_{x=1}$$

$$= c[n!\,(x+1)^n]$$

$+$ terms containing $(x - 1)$ and its powers$]_{x=1}$

$$= c\,.\,n!\,2^n,\ i.e.\ c = 1/n!\,2^n.$$

Substituting this value of c in (3), we get (1), which is known as the *Rodrigue's formula*.

(V.T.U., 2008 ; Bhopal, 2007 ; U.P.T.U., 2004)

Obs. All roots of $P_n(x) = 0$ are real and lie between -1 and $+1$. *(Madras, 2003 S)*

(2) Legendre polynomials. Using (1), we get

$P_0(x) = 1, \qquad P_1(x) = x,$

$P_2(x) = \dfrac{1}{2}(3x^2 - 1), \qquad P_3(x) = \dfrac{1}{2}(5x^3 - 3x),$

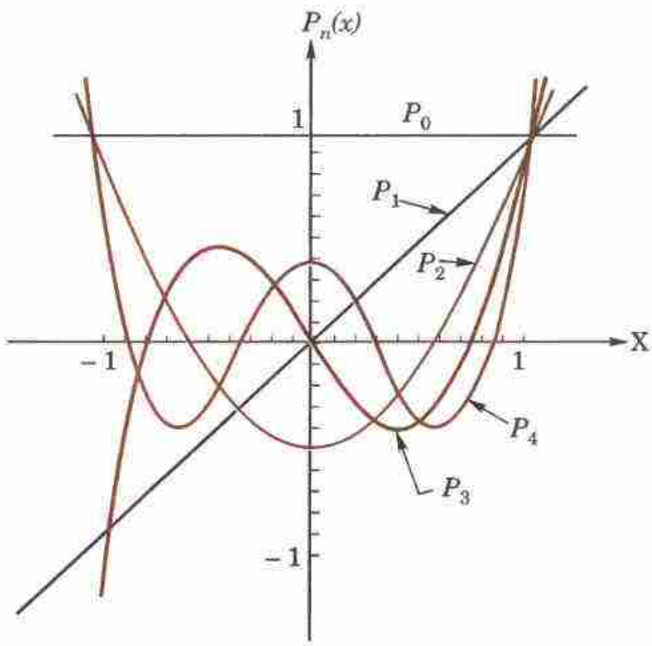


Fig. 16.2. Legendre polynomials.

* Named after the French mathematician and economist *Olinde Rodrigue* (1794–1851).

$$P_4(x) = \frac{1}{8}\ (35x^4 - 30x^2 + 3),\ P_5(x) = \frac{1}{8}\ (63x^5 - 70x^3 + 15x), \text{ etc.}$$ (*V.T.U., 2009*)

In general, we have $P_n(x) = \sum_{r=0}^{N} \frac{(-1)^r\ (2n-2r)!}{2^n r!\,(n-r)!\,(n-2r)!}\, x^{n-2r}$ ...(4)

where $N = \frac{1}{2} n$ or $\frac{1}{2}(n-1)$ according as n is even or odd.

Let us derive (4) from (1).

By Binomial theorem, $(x^2-1)^n = \sum_{r=0}^{n} {}^nC_r (x^2)^{n-r}\,(-1)^r = \sum_{r=0}^{n} (-1)^r \frac{n!}{r!\,(n-r)!} x^{2n-2r}$

$\therefore$ by (1), $P_n = \frac{1}{n!\,2^n} \sum_{n=0}^{n} \frac{(-1)^r\ n!}{r!\,(n-r)!} \frac{d^n\ (x^{2n-2r})}{dx^n} = \sum_{r=0}^{N} \frac{(-1)^r\ (2n-2r)!}{2^n r!\,(n-r)!\,(n-2r)!} x^{n-2r}$

This is same as (4), and the last term ($r = N$) is such that the power of x (*i.e.*, $n - 2r$) for this term is either 0 or 1.

Example 16.17. *Express $f(x) = x^4 + 3x^3 - x^2 + 5x - 2$ in terms of Legendre polynomials.* (*V.T.U., 2010 ; S.V.T.U., 2007*)

Solution. Since $P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$. $\therefore x^4 = \frac{8}{35} P_4(x) + \frac{6}{7}x^2 - \frac{3}{35}$

$$\therefore \quad f(x) = \left[\frac{8}{35} P_4(x) + \frac{6}{7}x^2 - \frac{3}{35}\right] + 3x^3 - x^2 + 5x - 2$$

$$= \frac{8}{35} P_4(x) + 3x^3 - \frac{1}{7}x^2 + 5x - \frac{73}{35} \qquad \left[\because\ x^3 = \frac{2}{5} P_3(x) + \frac{3}{5}x\ ;\ x^2 = \frac{2}{3} P_2(x) + \frac{1}{3}\right]$$

$$= \frac{8}{35} P_4(x) + 3\left[\frac{2}{5} P_3(x) + \frac{3}{5}x\right] - \frac{1}{7}\left[\frac{2}{3} P_2(x) + \frac{1}{3}\right] + 5x - \frac{73}{35}$$

$$= \frac{8}{35} P_4(x) + \frac{6}{5} P_3(x) - \frac{2}{21} P_2(x) + \frac{34}{5}x - \frac{224}{105} \qquad [\because\ x = P_1(x),\ 1 = P_0(x)]$$

$$= \frac{8}{35} P_4(x) + \frac{6}{5} P_3(x) - \frac{2}{21} P_2(x) + \frac{34}{5} P_1 x - \frac{224}{105} P_0(x).$$

Example 16.18. *Show that for any function $f(x)$, for which the nth derivative is continuous,*

$$\int_{-1}^{1} f(x)\,P_n(x) dx = \frac{1}{2^n\ n!} \int_{-1}^{1} (1-x^2)^n\ f^n(x)\,dx.$$

Solution. Using Rodrigue's formula : $P_n(x) = \frac{1}{2^n\ n!} \frac{d^n}{dx^n}(x^2-1)^n$,

$$\int_{-1}^{1} f(x)\,P_n(x)\,dx = \frac{1}{2^n\ n!} \int_{-1}^{1} f(x) \frac{d^n (x^2-1)^n}{dx^n}\,dx \qquad \text{[Integrate by parts]}$$

$$= \frac{1}{2^n\ n!}\left[\left| f(x) \cdot \frac{d^{n-1}(x^2-1)^n}{dx^{n-1}}\right|_{-1}^{1} - \int_{-1}^{1} f'(x) \cdot \frac{d^{n-1}(x^2-1)^n}{dx^{n-1}} dx\right]$$

$$= \frac{(-1)}{2^n\ n!} \int_{-1}^{1} f'(x) \frac{d^{n-1}}{dx^{n-1}}\ (x^2-1)^n\,dx$$

$$= \frac{(-1)^2}{2^n\ n!} \int_{-1}^{1} f''(x) \frac{d^{n-2}}{dx^{n-2}}\ (x^2-1)^n\,dx \qquad \text{[Again integrating by parts]}$$

$$= \frac{(-1)^n}{2^n\, n!} \int_{-1}^{1} f^n(x)\, (x^2-1)^n\, dx \qquad \text{[Integrating by parts } (n-2) \text{ times]}$$

$$= \frac{(-1)^{2n}}{2^n\, n!} \int_{-1}^{1} f^n(x)\,(1-x^2)^n\, dx = \frac{1}{2^n n!} \int_{-1}^{1} f^n\,(x)\,(1-x^2)^n\, dx$$

16.15 GENERATING FUNCTION FOR $P_n(x)$

To show that $(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x)$.

Since $$(1-z)^{-\frac{1}{2}} = 1 + \frac{1}{2} z + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!} z^2 + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!} z^3 + \ldots$$

$$= 1 + \frac{2!}{(1!)^2\, 2^2} z + \frac{4!}{(2!)^2\, 2^4} z^2 + \frac{6!}{(3!)^2\, 2^6} z^3 + \ldots$$

$\therefore$ $$[1-t(2x-t)]^{-\frac{1}{2}} = 1 + \frac{2!}{(1!)^2\, 2^2}\, t(2x-t) + \frac{4!}{(2!)^2\, 2^4}\, t^2(2x-t)^2 + \ldots$$

$$+ \frac{(2n-2r)!}{[(n-r)!]^2\, 2^{2n-2r}}\, t^{n-r}(2x-t)^{n-r} + \ldots + \frac{(2n)!}{(n!)^2\, 2^{2n}}\, t^n\,(2x-t)^n + \ldots \qquad \ldots(1)$$

The term in t^n from the term containing $t^{n-r}\,(2x-t)^{n-r}$

$$= \frac{(2n-2r)!}{[(n-r)!]^2\, 2^{2n-2r}}\, t^{n-r}\,.\,{}^{n-r}C_r\,(-t)^r\,(2x)^{n-2r}$$

$$= \frac{(2n-2r)!}{[(n-r)!]^2\, 2^{2n-2r}} \times \frac{(n-r)!}{r!\,(n-2r)!}\,(-1)^r t^n\,.\,(2x)^{n-2r} = \frac{(-1)^r\,(2n-2r)!}{2^n\, r!\,(n-r)!\,(n-2r)!}\, x^{n-2r}\,.\,t^n.$$

Collecting all terms in t^n which will occur in the term containing $t^n\,(2x-t)^n$ and the preceding terms, we see that terms in t^n.

$$= \sum_{r=0}^{N} \frac{(-1)^r\,(2n-2r)!}{2^n\, r!\,(n-r)!\,(n-2r)!}\, x^{n-2r}\,.\,t^n = P_n(x) t^n$$

where $N = \frac{1}{2}\, n$ or $\frac{1}{2}\,(n-1)$ according as n is even or odd.

Hence (1) may be written as $$[1-t(2x-t)]^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n \qquad \ldots(2)$$

(Kerala M.E., 2005 ; U.P.T.U., 2005)

This shows that $P_n(x)$ is the coefficient of t^n in the expansion of $(1-2xt+t^2)^{-1/2}$. That is why, it is known as the *generating function of Legendre polynomials.*

Cor. 1. $P_n(1) = 1$. *(V.T.U., 2003 S ; Delhi, 2002)*

Taking $x = 1$ in (2), we have $(1-2t+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(1)\, t^n$

i.e., $$\sum_{n=0}^{\infty} P_n(1)\, t^n = (1-t)^{-1} = 1 + t + t^2 + \ldots + t^n + \ldots$$

Equating coefficients of t^n, we get $P_n(1) = 1$.

Cor. 2. $P_n(-1) = (-1)^n$. *(B.P.T.U., 2005 S ; V.T.U., 2003)*

Taking $x = -1$ in (2), we have

$$\sum_{n=0}^{\infty} P_n(-1)\, t^n = (1+t)^{-1} = 1 - t + t^2 - \ldots + (-1)^n\, t^n + \ldots$$

Equating coefficients of t^n, we get the desired result.

Cor. 3. $P_n(0) = \begin{cases} (-1)^{n/2} \dfrac{1 \times 3 \times 5 \ldots (n-1)}{2 \times 4 \times 6 \times \ldots n}, & \text{when } n \text{ is even} \\ 0, & \text{when } n \text{ is odd} \end{cases}$ (V.T.U., 2005)

Putting $x = 0$ in (2), we get $\sum_{n=0}^{\infty} P_n(0)\ t^n = (1 + t^2)^{-1/2}$

$$= 1 - \frac{1}{2}t^2 + \frac{1.3}{2.4}\ t^4 \ldots + (-1)^r \frac{1 \cdot 3 \cdot 5 \cdots (2r+1)}{2 \cdot 4 \cdot 6 \cdots 2r} t^{2r} + \ldots$$

Equating coefficient of t^{2m}, we get $P_{2m}(0) = (-1)^m \dfrac{1 \cdot 3 \cdot 5 \cdots (2m-1)}{2 \cdot 4 \cdot 6 \cdots 2m}$

Similarly equating coefficients of t^{2m+1}, we have $P_{2m+1}(0) = 0$.

Cor. 4. $P'_n(1) = \dfrac{1}{2} n(n+1)$ (U.P.T.U. 2003)

Since $P_n(x)$ is a solution of Legendre's eduation, $(1 - x^2) P''_n(x) - 2xP'_n(x) + n(n+1) P_n(x) = 0$

Putting $x = 1$, $-2P'_n(1) + n(n+1) P_n(1) = 0$ or $P'_n(1) = \dfrac{1}{2}\ n(n+1)$ $[\because\ P_n(1) = 1]$

16.16 RECURRENCE FORMULAE FOR $P_n(x)$

The following recurrence formulae can be easily derived from the generating function for $P_n(x)$:

(1) $(n+1) P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$ **(2)** $nP_n(x) = xP'_n(x) - P'_{n-1}(x)$

(3) $(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$ **(4)** $P'_n(x) = xP'_{n-1}(x) + nP_{n-1}(x)$.

(5) $(1 - x^2)P'_n(x) = n\,[P_{n-1}(x) - xP_n(x)]$.

Proofs. **(1)** We know that $(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n$ ...(*i*)

Differentiating partially w.r.t. t, we get

$$-\frac{1}{2}\ (1 - 2xt + t^2)^{-3/2}(-2x + 2t) = \Sigma nP_n(x)\ t^{n-1}$$

or $$(x - t)(1 - 2xt + t^2)^{-1/2} = (1 - 2xt + t^2)\ \Sigma nP_n(x)\ t^{n-1}$$

or $$(x - t)\ \Sigma P_n(x)\ t^n = (1 - 2xt + t^2)\ \Sigma nP_n(x)\ t^{n-1}$$

Equating coefficients of t^n from both sides, we get

$$xP_n(x) - P_{n-1}(x) = (n+1) P_{n+1}(x) - 2nxP_n(x) + (n-1) P_{n-1}(x)$$

whence follows the required result. (S.V.T.U., 2007 ; V.T.U., 2003)

(2) Differentiating (*i*) partially w.r.t. x,

$$-\frac{1}{2}\ (1 - 2xt + t^2)^{-3/2} . (-2t) = \Sigma P_n'(x)t^n$$

i.e., $$t(1 - 2tx + t^2)^{-3/2} = \Sigma P_n'(x)\ t^n \qquad \ldots(ii)$$

Again differentiating (*i*) partially w.r.t. t, we have

$$(x - t)(1 - 2tx + t^2)^{-3/2} = \Sigma nP_n(x)t^{n-1} \qquad \ldots(iii)$$

Dividing (*iii*) by (*ii*), we get $\dfrac{x-t}{t} = \dfrac{\Sigma nP_n(x)t^{n-1}}{\Sigma P_n'(x)t^n}$

i.e., $$\Sigma nP_n(x)\ t^n = (x - t)\ \Sigma P_n'(x)\ t^n$$

Equating coefficients of t^n from both sides, we get (2). (J.N.T.U., 2006 ; U.P.T.U., 2006)

(3) Differentiating **(1)** w.r.t. x, we get

$$(n+1)P'_{n+1}(x) = (2n+1)P_n(x) + (2n+1)xP'_n(x) - nP'_{n-1}(x) \qquad \ldots(iv)$$

Substituting for $xP'_n(x)$ from (2) in (*iv*), we obtain

$$(n+1) P'_{n+1}(x) = (2n+1) P_n(x) + (2n+1) [nP_n(x) + P'_{n-1}(x)] - nP'_{n-1}(x)$$

or $$(2n+1) P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$$ (Madras, 2006)

(4) Rewriting (*iv*) as

$$(n+1)P'_{n+1}(x) = (2n+1)P_n(x) + (n+1)xP_n'(x) + n[xP_n'(x) - P'_{n-1}(x)]$$
$$= (2n+1)P_n(x) + (n+1)xP_n'(x) + n^2P_n(x) \quad \text{[by (2)]}$$
$$= (n+1)xP'_n(x) + (n^2+2n+1)P_n(x)$$

or
$$P'_{n+1}(x) = xP'_n(x) + (n+1)P_n(x)$$

Replacing n by $(n-1)$, we get (4).

(5) Rewriting (2) and (4) as

$$xP_n'(x) - P'_{n-1}(x) = nP_n(x) \qquad ...(v)$$

and
$$P_n'(x) - xP'_{n-1}(x) = nP_{n-1}(x) \qquad ...(vi)$$

Multiplying (*v*) by x and substracting from (*vi*), we get

$$(1-x^2)P_n'(x) = n[P_{n-1}(x) - xP_n(x)].$$

Example 16.19. *Prove that* $(2n+1)(1-x^2)P_n'(x) = n(n+1)[P_{n-1}(x) - P_{n+1}(x)]$.

Solution. We have the recurrence formula

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

or
$$(\overline{n+1}+n)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$$

or
$$(n+1)[xP_n(x) - P_{n+1}(x)] = n[P_{n-1}(x) - xP_n(x)]$$
$$= (1-x^2)P_n'(x) \quad [\because (1-x^2)P_n'(x) = n[P_{n-1}(x) - xP_n(x)] \qquad ...(i)$$

or
$$xP_n(x) = P_{n+1}(x) + \frac{(1-x^2)P_n'(x)}{n+1} \qquad ...(ii)$$

Also from (*i*)
$$xP_n(x) = P_{n-1}(x) - \frac{(1-x^2)P_n'(x)}{n} \qquad ...(iii)$$

From (*ii*) and (*iii*),
$$P_{n-1}(x) - \frac{(1-x^2)P_n'(x)}{n} = P_{n+1}(x) + \frac{(1-x^2)P_n'(x)}{n+1}$$

or
$$n(n+1)P_{n-1}(x) - (n+1)(1-x^2)P'_n(x) = n(n+1)P_{n+1}(x) + n(1-x^2)P'_n(x)$$

or
$$(2n+1)(1-x^2)P'_n(x) = n(n+1)[P_{n-1}(x) - P_{n+1}(x)]$$

16.17 (1) ORTHOGONALITY OF LEGENDRE POLYNOMIALS

We shall prove that,
$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \begin{cases} 0, & m \neq n \\ \dfrac{2}{2n+1}, & m = n \end{cases}$$

We know that the solutions of

$$(1-x)^2u'' - 2xu' + m(m+1)u = 0 \qquad ...(1)$$

and
$$(1-x^2)v'' - 2xv' + n(n+1)v = 0 \qquad ...(2)$$

are $P_m(x)$ and $P_n(x)$ respectively.

Multiplying (1) by v and (2) by u and subtracting, we get

$$(1-x^2)(u''v - uv'') - 2x(u'v - uv') + [m(m+1) - n(n+1)]uv = 0$$

or
$$\frac{d}{dx}\{(1-x^2)(u'v - uv')\} + (m-n)(m+n+1)uv = 0.$$

Now integrating from -1 to 1, we get

$$(m-n)(m+n+1)\int_{-1}^{1} uv\,dx = \Big|(1-x^2)(uv' - u'v)\Big|_{-1}^{1} = 0.$$

Hence
$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0. \qquad (m \neq n) \qquad ...(3)$$

This is known as the *orthogonality property of Legendre polynomials.*

(S.V.T.U., 2008 ; Madras, 2006 ; V.T.U., 2006)

When $m = n$, we have from Rodrigue's formula,

$$(n\,!\,2^n)^2 \int_{-1}^{1} P_n^2(x)\,dx = \int_{-1}^{1} D^n(x^2-1)^n \cdot D^n(x^2-1)^n\,dx \qquad \text{[Integrate by parts]}$$

$$= \left| D^n(x^2-1)^n \cdot D^{n-1}(x^2-1)^n \right|_{-1}^{1} - \int_{-1}^{1} D^{n+1}(x^2-1)^n \,.\, D^{n-1}(x^2-1)^n\,dx$$

Since $D^{n-1}(x^2-1)^n$ has x^2-1 as a factor, the first term on the right vanishes for $x = \pm 1$. Thus

$$(n\,!\,2^n)^2 \int_{-1}^{1} P_n^2(x)\,dx = -\int_{-1}^{1} D^{n+1}(x^2-1)^n \,.\, D^{n-1}(x^2-1)^n\,dx$$

[Integrate by parts $(n-1)$ times]

$$= (-1)^n \int_{-1}^{1} D^{2n}(x^2-1)^n \,.\, (x^2-1)^n\,dx = (-1)^n \int_{-1}^{1} (2n)\,!\,(x^2-1)^n\,dx$$

$$= 2(2n)\,! \int_{0}^{1} (1-x^2)^n\,dx \qquad \text{[Put } x = \sin\theta\text{]}$$

$$= 2(2n)\,! \int_{0}^{\pi/2} \cos^{2n+1}\theta\,d\theta = 2(2n)\,! \frac{2n(2n-2)\cdots 4\cdot 2}{(2n+1)(2n-1)\cdots 2\,.\,1}$$

$$= 2(2n)\,!\,[2n(2n-2)\ldots 4\,.\,2]^2/(2n+1)\,! = \frac{2}{2n+1}\,(2^n n\,!)^2$$

Hence $\displaystyle\int_{-1}^{1} P_n^2(x)dx = 2/(2n+1).$ ...(4) (*Bhopal, 2008 ; V.T.U., 2007 ; J.N.T.U., 2006*)

(2) Fourier-Legendre expansion of f(x). If $f(x)$ be a function defined from $x = -1$ to $x = 1$, we can write

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x) \qquad \text{...(5)}$$

To determine the coefficient c_n, multiply both sides by $P_n(x)$ and integrate from -1 to 1. Then (3) and (4) give

$$\int_{-1}^{1} f(x)P_n(x)dx = c_n \int_{-1}^{1} P_n^2(x)\,dx = \frac{2c_n}{2n+1} \quad \text{or} \quad c_n = \left(n+\frac{1}{2}\right)\int_{-1}^{1} f(x)P_n(x)\,dx$$

Equation (5) is known as *Fourier-Legendre expansion of* $f(x)$.

Example 16.20. *Show that* $\displaystyle\int_{-1}^{1} xP_n(x)\,P_{n-1}(x)\,dx = \frac{2n}{4n^2-1}$.

Solution. The recurrence formula (1) can be written as

$$(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$$

or $$(2n-1)\,xP_{n-1} = nP_n + (n-1)P_{n-2} \qquad \text{[Changing } n \text{ to } n-1\text{]}$$

Multiplying by P_n, we get $\displaystyle xP_nP_{n-1} = \frac{1}{2n-1}[nP_n^2 + (n-1)P_nP_{n-2}]$

Integrating both sides w.r.t. x from $x = -1$ to $x = 1$, we get

$$\int_{-1}^{1} xP_nP_{n-1}dx = \frac{n}{2n-1}\int_{-1}^{1} P_n^2 dx + \frac{n-1}{2n-1}\int_{-1}^{1} P_nP_{n-2}\,dx$$

$$= \frac{n}{2n-1}\left(\frac{2}{2n+1}\right) + \frac{n-1}{2n-1}(0), \text{ by Orthogonal property}$$

Hence $\displaystyle\int_{-1}^{1} xP_n(x)P_{n-1}(x)\,dx = \frac{2n}{4n^2-1}.$

Example 16.21. *Show that* $\displaystyle\int_{-1}^{1} (1-x^2)\,P_m'(x)\,dx, = \begin{cases} 0, & \text{when } m \neq n \\ \dfrac{2n(n+1)}{2n+1}, & \text{when } m = n \end{cases}$

(*S.V.T.U., 2008 ; U.P.T.U., 2006*)

Solution. Integrating by parts,

$$\int_{-1}^{1}(1-x^2)\ P'_m(x)\,P'_n(x)dx = \Big|(1-x^2)P'_m(x)\cdot P_n(x)\Big|_{-1}^{1} - \int_{-1}^{1}\cdot\frac{d}{dx}\left\{(1-x^2)\,P'_m(x)\right\}P_n\,dx$$

$$= -\int_{-1}^{1} P_n\left\{(1-x^2)\,P''_m(x) - 2x\,P'_m(x)\right\}dx \qquad \text{...(i)}$$

Now $P_m(x)$ being a solution of Legendre's equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + m(m+1)\,y = 0, \text{ we have}$$

$$(1-x^2)\,P''_m(x) - 2x\,P'_m(x) = -m(m+1)\,P_m(x)$$

Substituting this in (*i*), we get

$$\int_{-1}^{1}(1-x^2)\ P'_m(x)\,P'_n(x)\,dx = -\int_{-1}^{1}P_n\ \{-m(m+1)\,P_m(x)\}\,dx$$

$$= m(m+1)\int_{-1}^{1}P_m(x)\,P_n(x)dx \qquad \text{...(ii)}$$

When $m \neq n$, $\int_{-1}^{1}P_m(x)\,P_n(x)\,dx = 0$, by orthogonality property.

$$\therefore \quad \int_{-1}^{1}(1-x^2)\ P'_m(x)\,P'_n(x)\,dx = m(m+1)\,.\,0 = 0 \qquad \text{[from (ii)]}$$

When $m = n$, $\int_{-1}^{1}P_m(x)\,P_n(x)\,dx = \dfrac{2}{2n+1}$, by orthogonality property.

$$\therefore \quad \int_{-1}^{1}(1-x^2)\ P'_m(x)\,P'_n(x)\,dx = n(n+1)\,.\,\frac{2}{2n+1} = \frac{2n(n+1)}{(2n+1)}.$$

Example 16.22. *Show that* $\int_{-1}^{1}x^2\ P_{n-1}P_{n+1}\,dx = \dfrac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$.

(J.N.T.U., 2006 ; Kerala M. Tech., 2005)

Solution. We have from the recurrence relation (1),

$$(2n+1)\,x\,P_n = (n+1)\,P_{n+1} + n\,P_{n-1}$$

$$\therefore \quad xP_{n-1} = \frac{1}{2n-1}\ \{nP_n + (n-1)\,P_{n-2}\}$$

and

$$xP_{n+1} = \frac{1}{2n+3}\ \{(n+2)\,P_{n+2} + (n+1)\,P_n\}P$$

$$\therefore \quad x^2P_{n-1}P_{n+1} = \frac{1}{(2n-1)(2n+3)}\{n(n+2)\,P_nP_{n+2} + n(n+1)P_n^{\,2}$$

$$+ (n-1)(n+2)P_{n-2}P_{n+2} + (n^2-1)P_nP_{n-2}\}$$

Integrating both sides from -1 to 1 and using orthogonality of Legendre polynomials, we get

$$\int_{-1}^{1}x^2P_{n-1}P_{n+1}\,dx = \frac{n(n+1)}{(2n-1)(2n+3)}\int_{-1}^{1}P_n^{\,2}\,dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}.$$

Example 16.23. *If* $f(x) = 0, \quad -1 < x \le 0$

$\qquad = x, \quad 0 < x < 1,$

show that $f(x) = \frac{1}{4}P_0(x) + \frac{1}{2}P_1(x) + \frac{5}{16}\,P_2(x) - \frac{3}{32}P_4(x) + \ldots\ldots$ *(U.P.T.U., 2003)*

Solution. Let $\qquad f(x) = \sum_{n=0}^{\infty} c_nP_n(x)$

Then c_n is given by $c_n = \left(n+\frac{1}{2}\right)\int_{-1}^{1} f(x)\, P_n(x)\, dx$

$$= \left(n+\frac{1}{2}\right)\left[\int_{-1}^{0} 0\,.\, P_n(x)\, dx + \int_{0}^{1} xP_n(x)\, dx\right] = \left(n+\frac{1}{2}\right)\int_{0}^{1} x\, P_n(x)\, dx$$

$\therefore$
$$c_0 = \frac{1}{2}\int_0^1 xP_0(x)\, dx = \frac{1}{2}\int_0^1 x\, dx = \frac{1}{4}$$

$$c_1 = \frac{3}{2}\int_0^1 xP_1(x)\, dx = \frac{3}{2}\int_0^1 x^2\, dx = \frac{1}{2}$$

$$c_2 = \frac{5}{2}\int_0^1 xP_2(x)\, dx = \frac{5}{2}\int_0^1 x\,.\,\frac{3x^2-1}{2}\, dx = \frac{5}{4}\left|\frac{3x^4}{4}-\frac{x^2}{2}\right|_0^1 = \frac{5}{16}$$

$$c_3 = \frac{7}{2}\int_0^1 xP_3(x)\, dx = \frac{7}{2}\int_0^1 x\,.\,\frac{5x^3-3x}{2}\, dx = \frac{7}{4}\left|5\frac{x^5}{5}-3\frac{x^3}{3}\right|_0^1 = 0$$

$$c_4 = \frac{9}{2}\int_0^1 xP_4(x)\, dx = \frac{9}{2}\int_0^1 x\,.\,\frac{35x^4-30x^2+3}{8}\, dx$$

$$= \frac{9}{16}\left|35\frac{x^6}{6}-35\frac{x^4}{4}+3\frac{x^2}{2}\right|_0^1 = -\frac{3}{32} \text{ and so on.}$$

Hence
$$f(x) = \frac{1}{4}P_0(x) + \frac{1}{2}P_1(x) + \frac{5}{16}P_2(x) - \frac{3}{32}P_4(x) + \dots .$$

PROBLEMS 16.5

1. Show that $P_n(-x) = (-1)^n P_n(x)$. (*Bhopal, 2008 ; V.T.U., 2003 S*)

2. Prove that (*i*) $P_{2n}'(0) = 0$ (*ii*) $P_{2n+1}'(0) = \frac{(-1)^n (2n+1)!}{2^{2n}(n!)^2}$. (*iii*) $P_n'(-1) = (-1)^n \frac{n(n+1)}{2}$ (*S.V.T.U., 2008*)

3. Express the following in terms of Legendre polynomials : (*i*) $5x^3 + x$
   (*ii*) $x^3 + 2x^2 - x - 3$, (*Osmania, 2003*) (*iii*) $4x^3 + 6x^2 + 7x + 2$. (*S.V.T.U., 2008*)
   (*iv*) $x^4 + 3x^3 - x^2 + 5x - 2$ (*Bhopal, 2008 ; Madras, 2006*)

4. Prove that (*i*) $(1-x^2)\, P_n'(x) = (n+1)\,[xP_n(x) - P_{n+1}(x)]$,
   (*ii*) $P_n(x) = P'_{n+1}(x) - 2xP_n'(x) + P'_{n-1}(x)$. (*iii*) $P_n(x)\, P_{n+1/2}(x) = \frac{\sqrt{\pi}}{2^{2n+1}}\, P_{2m}(x)$ (*Anna, 2005 S*)

5. Prove that (*i*) $\int_{-1}^{1} [P_2(x)]^2\, dx = \frac{2}{5}$. (*P.T.U., 2002*) (*ii*) $\int_0^1 P_{2n}(x)\, dx = 0$.

6. Prove that $\int_{-1}^{1} P_n(x)(1-2hx+h^2)^{-1/2}\, dx = \frac{2h^n}{2n+1}$.

7. Show that $\int_{-1}^{1} (1-x^2)[P_n'(x)]^2\, dx = \frac{2n(n+1)}{2n+1}$. (*U.P.T.U., 2006 ; Kerala M.E., 2005*)

8. Using Rodrigue's formula, show that $P_n(x)$ satisfies the differential equation
$$\frac{d}{dx}\left[(1+x^2)\frac{d}{dx}[P_n(x)]\right] + n\,(n+1)\, P_n(x) = 0.$$

9. Expand the following functions in terms of Legendre polynomials in the interval $-1 < x < 1$:
   (*i*) $f(x) = x^3 + 2x^2 - x - 3$ (*V.T.U., 2008*) (*ii*) $f(x) = x^4 + x^3 + 2x^2 - x - 3$.

10. If $f(x) = 0, \quad -1 < x < 0$
    $= 1, \quad 0 < x < 1$, show that $f(x) = \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \dots\dots$

16.18 OTHER SPECIAL FUNCTIONS

The following special functions occur in numerous engineering problems. We state below their important properties which can be verified by similar methods :

(1) Laguerre's polynomials*. These are the solutions of *Laguerre's differential equation*

$$xy'' + (1-x)y' + ny = 0 \qquad ...(1)$$

These polynomials $L_n(x)$, are given by the corresponding Rodrigue's formula

$$L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x}) \qquad ...(2)$$

In particular, $L_0(x) = 1$; $L_1(x) = 1 - x$, $L_2(x) = 2 - 4x + x^2$; $L_3(x) = 6 - 18x + 9x^2 - x^3$. *(Madras, 2006)*

Their *generating function* is given by

$$\frac{e^{-xt/(1-t)}}{1-t} = \sum_{n=0}^{\infty} \frac{L_n(x)}{n!} t^n \qquad ...(3)$$

The *orthogonal property* for these polynomials is

$$\int_0^{\infty} e^{-x} L_m(x) L_n(x)\, dx = \begin{cases} 0, & m \neq n \\ (n!)^2, & m = n \end{cases} \qquad ...(4)$$

(2) Hermite's polynomials†. These are the solutions of Hermite's differential equation

$$y'' - 2xy' + 2ny = 0 \qquad ...(5)$$

These polynomials $H_n(x)$, are given by the Rodrigue's formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^{2n}}{dx^n}(e^{-x^2}) \qquad ...(6)$$

In particular, $H_0(x) = 1$; $H_1(x) = 2x$; $H_2(x) = 4x^2 - 2$; $H_3(x) = 8x^3 - 12x$. *(Madras, 2006)*

Their *generating function is* given by

$$e^{2tx - t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n \qquad ...(7)$$ *(Madras, 2002 S)*

The *orthogonal property* of these polynomials is

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)\, dx = \begin{cases} 0, & m \neq n \\ 2^n\, n!\sqrt{\pi}, & m = n \end{cases} \qquad ...(8)$$

(3) Chebyshev polynomials.** These polynomials denoted by $T_n(x)$, are the solutions of the differential equation

$$(1 - x^2)y'' - xy' + n^2 y = 0 \qquad ...(9)$$

Their *generating function* is

$$\frac{1 - xt}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} T_n(x)\, t^n \qquad ...(10)$$

and $$T_n(x) = \frac{n}{2} \sum_{r=0}^{N} (-1)^r \frac{(n-r-1)!}{r!(n-2r)!} (2x)^{n-2r} \qquad ...(11)$$

(J.N.T.U., 2006)

where $N = \frac{n}{2}$, if n is even and $N = \frac{1}{2}(n-1)$, if n is odd.

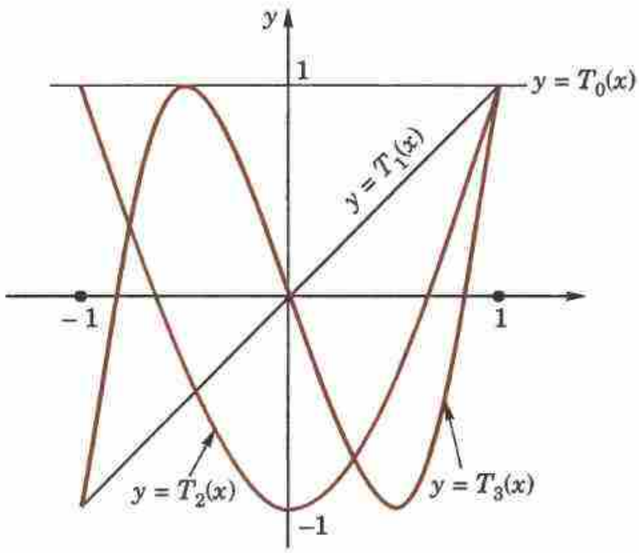


Fig. 16.3. Graphs of $T_0(x)$, $T_1(x)$, $T_2(x)$, $T_3(x)$.

* Named after the French mathematician *Edmond Laguerre* (1834–86) who is known for his work in infinite series and geometry.

† See footnote p. 68.

** Named after the Russian mathematician *Pafnuti Chebyshev* (1821–1894) who is known for his work in the theory of numbers and approximation theory.

In particular, $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$. Also, we have the recurrence relation

$$T_{n+1}(x) = 2x\, T_n(x) - T_{n-1}(x) \qquad \text{...(12)} \quad (Bhopal,\ 2002)$$

which defines T_{n+1} in terms of T_n and T_{n-1}.

Their *orthogonal property* is

$$\int_{-1}^{1} (1-x^2)^{-1/2}\, T_m(x) T_n(x)\, dx = \begin{cases} 0, & m \neq n \\ \dfrac{\pi}{2}, & m = n \\ \pi, & m = n = 0 \end{cases} \qquad \text{...(13)}$$

Example 16.24. *Prove that* $\int_0^{\infty} e^{-x} L_m(x)\, L_n(x)\, dx = 0,\ m \neq n.$ *(Anna, 2006)*

Solution. Since $L_m(x)$ and $L_n(x)$ are the solutions of the Laguerre's differential equation (1).

$$\therefore \qquad xL_m'' + (1-x)L_m' + mL_m = 0 \qquad \text{...}(i)$$

$$xL_n'' + (1-x)L_n' + nL_n = 0 \qquad \text{...}(ii)$$

Multiplying (*i*) by L_n and (*ii*) by L_m and subtracting, we get

$$x(L_nL_m'' - L_mL_n'') + (1-x)(L_nL_m' - L_mL_n') = (n-m)\, L_mL_n$$

or

$$\frac{d}{dx}(L_nL_m' - L_mL_n') + \frac{1-x}{x}(L_nL_m' - L_mL_n') = \frac{(n-m)\, L_mL_n}{x}$$

This is Leibnitz's linear equation and its

$$\text{I.F.} = e^{\int \left(\frac{1}{x} - 1\right) dx} = e^{\log x - x} = xe^{-x}.$$

$$\therefore \quad \text{Its solution is } \left| (L_nL_m' - L_mL_n')\, xe^{-x} \right|_0^{\infty} = \int_0^{\infty} \frac{(n-m)L_mL_n}{x}\, xe^{-x}\, dx$$

or

$$\int_0^{\infty} e^{-x} L_mL_n\, dx = \left| \frac{(L_nL_m' - L_mL_n')xe^{-x}}{n-m} \right|_0^{\infty} = 0 \text{ which proves the result.}$$

Example 16.25. *Prove that* $H_n(x) = (-1)^n\, e^{x^2} \dfrac{d^{2n}}{dx^n}(e^{-x^2})$.

Solution. The generating function for $H_n(x)$ is $e^{2tx - t^2} = e^{x^2} \cdot e^{-(t-x)^2} = \sum\limits_{n=0}^{\infty} H_n(x) \cdot \dfrac{t^n}{n!}$

Then

$$\left[\frac{\partial^n}{\partial t^n}(e^{2tx - t^2})\right]_{t=0} = H_n(x) \qquad \text{...}(i)$$

Also

$$\left[\frac{\partial^n}{\partial t^n}(e^{2tx - x^2})\right]_{t=0} = e^{x^2}\left[\frac{\partial^n}{\partial t^n}\{e^{-(t-x)^2}\}\right]_{t=0}$$

$$= e^{x^2}\left[\frac{\partial^n}{\partial(-x)^n}\{e^{-(t-x)^2}\}\right]_{t=0} = (-1)^n \frac{d^n}{dx^n}(e^{-x^2}) \qquad \text{...}(ii)$$

Equating (*i*) and (*ii*), we get the desired result.

PROBLEMS 16.6

1. Using the generating function (3) page 571, obtain the recurrence formula
$$L_{n+1}(x) = (2n + 1 - x)\, L_n(x) - n^2 L_{n-1}(x).$$
2. Show that (*i*) $nL_{n-1}(x) = nL'_{n-1}(x) - L'_n(x)$. (*ii*) $L'_n(x) = L'_{n-1}(x) - L_{n-1}(x)$. (Anna, 2005)
3. Show that (*i*) $H_{2n}(0) = (-1)^n \dfrac{2n!}{n!}$ (*ii*) $H_{2n+1}(0) = 0$ (Anna, 2005)

4. Prove that (i) $H_n'(x) = 2n\, H_{n-1}(x)$ (ii) $\frac{d^m}{dx^m}[H_n(x)] = \frac{2^m \cdot n!}{(n-m)!} H_{n-m}(x)$, $m < n$.

5. Using the generating function (7) page 515, obtain the recurrence formula $2xH_n(x) = 2nH_{n-1}(x) + H_{n+1}(x)$.

6. Prove that (i) $\int_{-\infty}^{\infty} e^{-x^2} H_2(x) H_3(x) = 0$, (ii) $\int_{-\infty}^{\infty} e^{-x^2} \{H_2(x)\}^2 = 8\sqrt{\pi}$. *(Madras, 2003)*

7. Express x^3 in terms of Chebyshev polynomials T_1 and T_3. *(U.P.T.U., 2009)*

8. Show that (i) $T_5 = 16x^5 - 20x^3 + 5x$. *(Bhopal, 2002)*
   (ii) $(1 - x^2)T_n' = nT_{n-1}(x) - nxT_n(x)$. *(Osmania, 2003)*

9. Prove that $\frac{1 - t^2}{1 - 2xt + t^2} = T_0(x) + 2\sum_{n=1}^{\infty} T_n(x)\, t^n$. *(J.N.T.U., 2006)*

16.19 (1) STRUM*-LIOUVILLE† PROBLEM

Legendre's equation $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$ ...(i)
can be written as, $[(1 - x^2)y']' + \lambda y = 0$ $[\lambda = n(n + 1)]$

Bessel's equation $X^2 \frac{d^2y}{dx^2} + X\frac{dy}{dx} + (X^2 - n^2)y = 0$ can be transformed by putting $X = kx$ (so that $\frac{dy}{dx} = \frac{dy}{dx}\cdot\frac{dx}{dX} = \frac{y'}{k}, \frac{d^2y}{dX^2} = \frac{y''}{k^2}$ to the form

$$x^2y'' + xy' + (k^2x^2 - n^2)\,y = 0$$

or $$(xy'' + y') + (\lambda x - n^2/x)\,y = 0 \qquad [\lambda = k^2]$$

or $$(xy')' + (\lambda x - n^2/x)\,y = 0 \qquad ...(ii)$$

Both the equations (i) and (ii) are of the form

$$[r(x)\,y']' + [\lambda p(x) + q(x)]y = 0 \qquad ...(1)$$

which is known as the *Strum-Liouville equation*. Similarly Laguerre's, Hermite's equations etc. can also be reduced to (1). Thus all the above equations of engineering utility can be considered with a common approach by means of Strum-Liouville's equation.

Eqution (1) considered on some interval $a \le x \le b$, satisfying the conditions

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0,\ \beta_1 y(b) + \beta_2 y'(b) = 0 \qquad ...(2)$$

with the real constants : α_1, α_2 not both zero and β_1, β_2 not both zero. The conditions (2) at the end points are called *boundary condtions.*

A differential equation together with the boundary conditions, is called a **boundary value problem.** *Equation (1) together with boundary conditions (2) is called a* **Strum-Liouville problem.**

Obviously $y = 0$ is a solution of the problem for any value of the parameter λ which is a trivial solution and as such is of no practical utility. Any other solution of (1) satisfying (2) is called an *eigen function* of the problem and the corresponding value of λ is called an eign value of the problem.

A special case. Taking $r = p = 1$ and $q = 0$ in (1), we get

$$y'' + \lambda y = 0 \qquad ...(3)$$

Also if $\alpha_1 = \beta_1 = 1$ and $\alpha_2 = \beta_2 = 0$, then the boundary conditions (2) become

$$y(a) = 0, \quad y(b) = 0 \qquad ...(4)$$

Thus (3) and (4) constitute the *simplest form of Strum-Liouville problem.*

(2) Orthogonality. Of the various properties of eigen functions of Strum-Liouville problem the orthogonality is of special importance.

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* Named after the Swiss mathematician *J.C.F. Strum* (1803–1855) who later became Poisson's successor at Sorbonne university, Paris.

† Named after the French professor *Joseph Liouville* (1809–1882) who is known for his important contributions to complex analysis, special functions, number theory and differential geometry.

Def. *Two functions $y_m(x)$ and $y_n(x)$ defined on some interval $a \le x \le b$, are said to be orthogonal on this interval w.r.t. the weight function $p(x) > 0$, if*

$$\int_a^b p(x)\, y_m(x) y_n(x)\, dx = 0 \text{ for } m \ne n.$$

The **norm** *of y_m, denoted by $\| y_m \|$, is defined to be the non-negative square root of $\int_a^b p(x)\,[y_m(x)]^2\, dx$. Thus*

$$\| y_m \| = \sqrt{\left\{\int_a^b p(x)\,[y_m(x)]^2\, dx\right\}}$$

The functions which are orthogonal on $a \le x \le b$ and have norm equal to 1, are called **orthonormal** on this interval.

(3) Orthogonality of eigen functions.

Theorem. *If (i) the functions p, q, r and r′ in the Strum-Liouville equation (1) be continuous in $a \le x \le b$;*

(ii) $y_m(x), y_n(x)$ be two eigen functions of the Strum-Liouville problem corresponding to eigen values λ_m and λ_n respectively ;

then $y_m(x)$ and $y_n(x)$ $(m \ne n)$ are orthogonal on that interval w.r.t. the weight function $p(x)$.

Proof. Since y_m and y_n satisfy (1) above

$$(ry'_m)' + (\lambda_m p + q)\, y_m = 0$$
$$(ry'_n)' + (\lambda_n p + q)\, y_n = 0$$

Multiplying the first equation by y_n and the second by $-y_m$ and adding, we get

$$(\lambda_m - \lambda_n)\, p y_m y_n = y_m(ry'_n) - y_n(ry'_m)$$
$$= \frac{d}{dx}\,[(ry'_n)\, y_m - (ry'_m) y_n], \text{ after differentiation.}$$

Now integrating both sides w.r.t. x from a to b, we obtain

$$(\lambda_m - \lambda_n) \int_a^b p y_m y_n\, dx = \left[(ry'_n) y_m - (ry'_m) y_n\right]_a^b$$
$$= r(b)\,[y'_n(b)\, y_m(b) - y'_m(b)\, y_n(b)] - r(a)\,[y'_n(a)\, y_m(a) - y'_m(a)\, y_n(a)] \qquad \text{...(A)}$$

The R.H.S. will vanish if the boundary conditions are of one of the following forms :

I. $y(a) = y(b) = 0$; II. $y'(a) = y'(b) = 0$; III. $\alpha_1 y(a) + \alpha_2 y'(a) = 0$, $\beta_1 y(b) + \beta_2 y'(b) = 0$ where either α_1 and α_2 is not zero and either β_1 or β_2 is not zero.

Thus in each case (A) reduces to $\int_a^b p y_m y_n\, dx = 0 \quad (m \ne n)$

which shows that the eigen functions y_m and y_n are orthogonal on $a \le x \le b$ w.r.t. the weight function $p(x) = 0$.

Obs. The third form of the boundary conditions in fact contains the first two forms as special cases.

Cor. 1. Orthogonality of Legendre polynomials has already been established directly in § 16.17. But if follows at once from the above theorem.

We have already seen in para (1) that Legendre's equation is Strum-Liouville equation

$$[(1 - x^2)\, y']' + \lambda y = 0 \qquad [\lambda = n(n+1)]$$

with $r(x) = 1 - x^2$, $p(x) = 1$ and $q(x) = 0$.

Since $y(-1) = y(1) = 0$ and for $n = 0, 1, 2, \ldots, \lambda = 0, 1.2, 2.3, \ldots$, the Legendre polynomials are the solutions of the problem *i.e.*, these are the eigen functions. Thus it follows by the above theorem, that they are orthogonal on $-1 \le x \le 1$.

Cor. 2. Orthogonality of Bessel functions has also been established directly in § 16.11. But it can easily be seen to follow from the above theorem.

In para (1), we transformed the Bessel's equation

$$X^2 \frac{d^2 J_n}{dx^2} + X \frac{dJ_n}{dx} + (X^2 - n^2)\, J_n(x) = 0$$

into $[xJ'_n(kx)]' + (k^2 x - n^2/x)\, J_n(kx) = 0$ which is Strum-Liouville equation with $r(x) = x$, $p(x) = x$, $q(x) = -n^2/x$ and $\lambda = k^2$. Since $r(0) = 0$, it follows from the above theorem that those solutions of $J_n(kx)$ which are zero at $x = 0$ form an orthogonal set on $0 \le x \le R$ with weight function $p(x) = x$.

Example 16.26. *For the Strum-Liouville problem $y'' + \lambda y = 0$, $y(0) = 0$, $y(l) = 0$, find the eigen functions and show that they are orthogonal.*

Solution. For $\lambda = -\gamma^2$, the general solution of the equation is $y(x) = c_1 e^{\gamma x} + c_2 e^{-\gamma x}$

The above boundary conditions give $c_1 = c_2 = 0$ and $y = 0$ which is not an eigen function.

For $\lambda = \gamma^2$, the general solution is $y(x) = A \cos \gamma x + B \sin \gamma x$

The first boundary condition gives $y(0) = A = 0$ and the second boundary condition gives $y(l) = B \sin \gamma l = 0$, $\gamma = 0, \pm \pi/l, \pm 2\pi/l, \ldots$ Thus the eigen values are $\lambda = 0, \pi^2/l^2, 4\pi^2/l^2, \ldots$ and taking $B = 1$, the corresponding eigen functions are

$$y_n(x) = \sin (n\pi x/l) \qquad n = 0, 1, 2, \ldots$$

From the above theorem, it follows that the said eigen functions are orthogonal on the interval $0 \le x \le l$.

Obs. This problem concerns an elastic string stretched between fixed points $x = 0$ and $x = l$ and allowed to vibrate. Here $y(x)$ is the space function of the deflection $u(x, t)$ of the string where t is the time. (See § 18.4).

PROBLEMS 16.7

Find the eigen functions of each of the following *Strum-Liouville problems* and verify their orthogonality :

1. $y'' + \lambda y = 0, y(0) = 0, y(\pi) = 0$.
2. $y'' + \lambda y = 0, y(0) = 0, y'(l) = 0$.
3. $y'' + \lambda y = 0, y'(0) = 0, y'(\pi) = 0$.
4. $y'' + \lambda y = 0, y(\pi) = y(-\pi), y'(\pi) = y'(-\pi)$.
5. $(xy')' + \lambda x^{-1} y = 0, y(1) = 0, y'(e) = 0$.

Transform each of the following equations to the *Strum-Liouville equations* indicating the weight function :

6. *Laguerre's equation* : $xy'' + (1 - x)y' + ny = 0$.
7. Hermite's equation : $y'' - 2xy' + 2ny = 0$.

16.20 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 16.8

Fill up the blanks or choose the correct answer in the following problems :

1. In terms of Legendre polynomials $2 - 3x + 4x^2$ is ......
2. $J_{-1/2} =$ ....
3. $\int_{-1}^{1} P_n^2(x)\, dx =$ ....
4. $P_{2n+1}(0) =$ ....
5. $\int_{-1}^{1} x^m P_n(x)\, dx =$ ......(m being an integer $< n$)
6. The recurrence relation connecting $J_n(x)$ to $J_{n-1}(x)$ and $J_{n+1}(x)$ is ....
7. Orthogonality relation for Bessel functions is ....
8. Bessel's equation of order zero is ....
9. $J_{1/2} =$ ....
10. $\frac{d}{dx}[x^n J_n(x)] =$ ....
11. Value of $P_2(x)$ is ....
12. $\int_{-1}^{1} P_3(x) P_4(x)\, dx =$ ....
13. $P_n(-1) = (-1)^n$ (True or False)
14. Rodrigue's formula for $P_n(x)$ is ....
15. $\int_0^1 x J_n(\alpha x) J_n(\beta x)\, dx = 0$, if ....
16. Expansion of $5x^3 + x$ interms of Legendre polynomials is ....
17. Generating function of $P_n(x)$ is ....
18. $\frac{d}{dx}[J_0(x)] =$ ....
19. Bessel equation of order 4 is $x^2y'' + xy' + (x^2 - 4)y = 0$. (True or False)
20. $\frac{d}{dx}[x^2 J_2(x)] = x^2 J_1(x)$. (True or False)
21. Legendre's polynomial of first degree $= x$. (True or False)

22. If α is a root of $P_n(x) = 0$, then $P_{n+1}(\alpha)$ and $P_{n-1}(\alpha)$ are of opposite signs. (True or False)
23. $x = 0$ is a regular singular point of $2x^2y'' + 3xy' + (x^2 - 4)y = 0$. (True or False)
24. $\cos x = 2J_1 - 2J_3 + 2J_5 - \ldots\ldots$ (True or False)
25. If J_0 and J_1 are Bessel functions, then $J_1'(x)$ is given by

(a) $-J_0$ (b) $J_0(x) - 1/x\, J_1(x)$ (c) $J_0(x) + \dfrac{1}{x} J_1(x)$.

26. If $J_n(x)$ is the Bessel function of first kind, then $\int_0^{\pi} [J_{-2}(x) - J_2(x)]\, dx =$

(a) 2 (b) -2 (c) 0 (d) 1.

27. If $J_{n+1}(x) = \dfrac{2}{x} J_n(x) - J_0(x)$, then n is

(a) 0 (b) 2 (c) -1 (d) none of these.

28. The series $x - \dfrac{x^3}{2^2(1!)^2} + \dfrac{x^5}{2^4(2!)^2} - \dfrac{x^7}{2^6(3!)^2} + \ldots\ldots \infty$ equals

(a) $J_{1/2}(x)$ (b) $J_0(x)$ (c) $xJ_0(x)$ (d) $xJ_{1/2}(x)$.

29. If $\int_{-1}^{1} P_n(x)\, dx = 2$, then n is

(a) 0 (b) 1 (c) -1 (d) none of these.

30. The value of $\int_{-1}^{1} (2x + 1)P_3(x)\, dx$ where $P_3(x)$ is the third degree Legendre polynomial, is

(a) 1 (b) -1 (c) 2 (d) 0.

31. The value of the integral $\int_{-1}^{1} x^3 P_3(x)\, dx$, where $P_3(x)$ is a Legendre polynomial of degree 3, is

(a) 0 (b) $\dfrac{2}{35}$ (c) $\dfrac{4}{35}$ (d) $\dfrac{11}{35}$.

32. The polynomial $2x^2 + x + 3$ in terms of Legendre polynomials is

(a) $\dfrac{1}{3}(4P_2 - 3P_1 + 11P_0)$ (b) $\dfrac{1}{3}(4P_2 + 3P_1 - 11P_0)$

(c) $\dfrac{1}{3}(4P_2 + 3P_1 + 11P_0)$ (d) $\dfrac{1}{3}(4P_2 - 3P_1 - 11P_0)$.

33. If $P_n(x)$ be the Legendre polynomial, then $P_n'(-x)$ is equal to

(a) $(-1)^n P_n(x)$ (b) $(-1)^n P_n'(x)$ (c) $(-1)^{n+1} P_n'(x)$ (d) $P_n''(x)$.

34. Legendre polynomial $P_5(x) = \lambda(63x^5 - 70x^3 + 15x)$ where λ is equal to

(a) 1/2 (b) 1/5 (c) 1/8 (d) 1/10.

35. $\int_{-1}^{1} (1 + x) P_n(x)\, dx$, $(n > 1)$, is equal to

(a) $\dfrac{1}{2n+1}$ (b) $\dfrac{2}{2n+1}$ (c) $\dfrac{n}{2n+1}$ (d) 0.

36. The singular points of the differential equation $x^3(x - 1)y'' + 2(x - 1)y' + y - 0$ are .... (P.T.U., 2009)

CHAPTER 17

Partial Differential Equations

17.1 INTRODUCTION

The reader has, already been introduced to the notion of partial differential equations. Here, we shall begin by studying the ways in which partial differential equations are formed. Then we shall investigate the solutions of special types of partial differential equations of the first and higher orders.

In what follows x and y will, usually be taken as the independent variables and z, the dependent variable so that $z = f(x, y)$ and we shall employ the following notation :

$$\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial^2 z}{\partial x^2} = r, \frac{\partial^2 z}{\partial x \partial y} = s, \frac{\partial^2 z}{\partial y^2} = t.$$

17.2 FORMATION OF PARTIAL DIFFERENTIAL EQUATIONS

Unlike the case of ordinary differential equations which arise from the elimination of arbitrary constants; *the partial differential equations can be formed either by the elimination of arbitrary constants or by the elimination of arbitrary functions from a relation involving three or more variables.* The method is best illustrated by the following examples :

Example 17.1. *Derive a partial differential equation (by eliminating the constants) from the equation*

$$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}. \qquad ...(i)$$

Solution. Differentiating (i) partially with respect to x and y, we get

$$2\frac{\partial z}{\partial x} = \frac{2x}{a^2} \quad \text{or} \quad \frac{1}{a^2} = \frac{1}{x}\frac{\partial z}{\partial x} = \frac{p}{x}$$

and

$$\frac{2\partial z}{\partial y} = \frac{2y}{b^2} \quad \text{or} \quad \frac{1}{b^2} = \frac{1}{y}\frac{\partial z}{\partial y} = \frac{q}{y}$$

Substituting these values of $1/a^2$ and $1/b^2$ in (i), we get

$$2z = xp + yq$$

as the desired partial differential equation of the first order.

Example 17.2. *Form the partial differential equations (by eliminating the arbitrary functions) from*
(*a*) $z = (x + y)\,\phi\,(x^2 - y^2)$ (*P.T.U., 2009*)
(*b*) $z = f(x + at) + g(x - at)$ (*V.T.U., 2009*) (*c*) $f(x^2 + y^2, z - xy) = 0$ (*S.V.T.U., 2007*)

Solution. (*a*) We have $z = (x + y)\,\phi\,(x^2 - y^2)$

Differentiating z partially with respect to x and y,

$$p = \frac{\partial z}{\partial x} = (x + y)\,\phi'\,(x^2 - y^2)\,.\,2x + \phi\,(x^2 - y^2), \quad ...(i)$$

$$q = \frac{\partial z}{\partial y} = (x + y)\,\phi'\,(x^2 - y^2)\,.\,(-\,2y) + \phi\,(x^2 - y^2) \quad ...(ii)$$

From (*i*), $$p - \frac{z}{x + y} = 2x\,(x + y)\,\phi'\,(x^2 - y^2)$$

From (*ii*), $$q - \frac{z}{x + y} = -\,2y\,(x + y)\,\phi'\,(x^2 - y^2)$$

Division gives $$\frac{p - z/(x + y)}{q - z/(x + y)} = -\,\frac{x}{y}$$

i.e., $[p(x + y) - z]y + [q\,(x + y) - z]x$

i.e., $(x + y)\,(py + qx) - z(x + y) = 0$

Hence $py + qz = z$ is required equation.

(*b*) We have $z = f\,(x + at) + g(x - at)$ ...(*i*)

Differentiating z partially with respect to x and t,

$$\frac{\partial z}{\partial x} = f'\,(x + at) + g'(x - at),\ \frac{\partial^2 z}{\partial x^2} = f''\,(x + at) + g''(x - at) \quad ...(ii)$$

$$\frac{\partial z}{\partial t} = af'\,(x + at) - ag'(x - at),\ \frac{\partial^2 z}{\partial t^2} = a^2 f''\,(x + at) + a^2 g''(x - at) = a^2\frac{\partial^2 z}{\partial x^2} \quad \text{[By (ii)]}$$

Thus the desired partial differential equation is $\dfrac{\partial^2 z}{\partial t^2} = a^2\,\dfrac{\partial^2 z}{\partial x^2}$

which is an equation of the second order and (*i*) is its solution.

(*c*) Let $x^2 + y^2 = u$ and $z - xy = v$ so that $f(u, v) = 0$.

Differentiating partially w.r.t. x and y, we have

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right) = 0$$

or $$\frac{\partial f}{\partial u}(2x) + \frac{\partial f}{\partial v}(-\,y + p) = 0 \quad ...(i)$$

and $$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right) = 0 \quad \text{or} \quad \frac{\partial f}{\partial u}(2y) + \frac{\partial f}{\partial v}(-\,x + q) = 0 \quad ...(ii)$$

Eliminating $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$ from (*i*) and (*ii*), we get

$$\begin{vmatrix} 2x & -\,y + p \\ 2y & -\,x + q \end{vmatrix} = 0 \quad \text{or} \quad xq - yp = x^2 - y^2.$$

Example 17.3. *Find the differential equation of all planes which are at a constant distance a from the origin.* (*V.T.U., 2009 S ; Kurukshetra, 2006*)

Solution. The equation of the plane in 'normal form' is

$$lx + my + nz = a \quad ...(i)$$

where l, m, n are the d.c.s of the normal from the origin to the plane.

Then $l^2 + m^2 + n^2 = 1$ or $n = \sqrt{(1 - l^2 - m^2)}$

$\therefore$ (*i*) becomes $lx + my + \sqrt{(1 - l^2 - m^2)}\ z = a$ ...(*ii*)

Differentiating partially w.r.t. x, we get

$$l + \sqrt{(1 - l^2 - m^2)}\ . p = 0 \quad ...(iii)$$

Differentiating partially w.r.t. y, we get

$$m + \sqrt{(1 - l^2 - m^2)}\ . q = 0 \quad ...(iv)$$

Now we have to eliminate l, m from (*ii*), (*iii*) and (*iv*).

From (*iii*), $l = -\sqrt{(1 - l^2 - m^2)}\ . p$ and $m = -\sqrt{(1 - l^2 - m^2)}\ . q$

Squaring and adding, $l^2 + m^2 = (1 - l^2 - m^2)(p^2 + q^2)$

or $$(l^2 + m^2)(1 + p^2 + q^2) = p^2 + q^2 \text{ or } 1 - l^2 - m^2 = 1 - \frac{p^2 + q^2}{1 + p^2 + q^2} = \frac{1}{1 + p^2 + q^2}$$

Also $$l = -\frac{p}{\sqrt{(1 + p^2 + q^2)}} \text{ and } m = -\frac{q}{\sqrt{(1 + p^2 + q^2)}}$$

Substituting the values of l, m and $1 - l^2 - m^2$ in (*ii*), we obtain

$$\frac{-px}{\sqrt{(1 + p^2 + q^2)}} - \frac{qy}{\sqrt{(1 + p^2 + q^2)}} + \frac{1}{\sqrt{(1 + p^2 + q^2)}}\ z = a$$

or $z = px + qy + a\sqrt{(1 + p^2 + q^2)}$ which is the required partial differential equation.

PROBLEMS 17.1

From the partial differential equation (by eliminating the arbitrary constants from :

1. $z = ax + by + a^2 + b^2$.
2. $(x - a)^2 + (y - b)^2 + z^2 = c^2$. (*Kottayam, 2005*)
3. $(x - a)^2 + (y - b)^2 = z^2 \cot^2 \alpha$ (*Anna, 2009*)
4. $z = a \log \left\{\frac{b(y-1)}{1-x}\right\}$ (*J.N.T.U., 2002 S*)
5. Find the differential equation of all spheres of fixed radius having their centres in the xy-plane. (*Madras 2000 S*)
6. Find the differential equation of all spheres whose centres lie on the z-axis. (*Kerala, 2005*)

Form the partial differential equations (by eliminating the arbitrary functions) from :

7. $z = f(x^2 - y^2)$ (*S.V.T.U., 2008*)
8. $z = f(x^2 + y^2) + x + y$ (*Anna, 2009*)
9. $z = yf(x) + xg(y)$. (*V.T.U., 2004*)
10. $z = x^2 f(y) + y^2 g(x)$. (*Anna, 2003*)
11. $z = f(x) + e^y g(x)$.
12. $xyz = \phi(x + y + z)$.
13. $z = f_1(x) f_2(y)$.
14. $z = e^{my} \phi(x - y)$. (*P.T.U., 2002*)
15. $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$. (*V.T.U., 2010 ; J.N.T.U., 2010 ; Madras, 2000*)
16. $z = f_1(y + 2x) + f_2(y - 3x)$. (*Kurukshetra, 2005*)
17. $v = \frac{1}{r}[f(r - at) + F(r + at)]$.
18. $z = xf_1(x + t) + f_2(x + t)$.
19. $F(xy + z^2, x + y + z) = 0$. (*V.T.U., 2006*)
20. $F(x + y + z, x^2 + y^2 + z^2) = 0$. (*S.V.T.U., 2007*)
21. If $u = f(x^2 + 2yz, y^2 + 2zx)$, prove that $(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0$.

17.3 SOLUTIONS OF A PARTIAL DIFFERENTIAL EQUATION

It is clear from the above examples that a partial differential equation can result both from elimination of arbitrary constants and from the elimination of arbitrary functions.

The solution $f(x, y, z, a, b) = 0$ ...(1)

of a first order partial differential equation which contains two arbitrary constants is called a *complete integral*.

A solution obtained from the complete integral by assigning particular values to the arbitrary constants is called a particular integral.

If we put $b = \phi(a)$ in (1) and find the envelope of the family of surfaces $f[x, y, z, \phi(a)] = 0$, then we get a solution containing an arbitrary function ϕ, which is called the *general integral*.

The envelope of the family of surfaces (1), with parameters a and b, if it exists, is called a *singular integral*. The singular integral differs from the particular integral in that it is not obtained from the complete integral by giving particular values to the constants.

17.4 EQUATIONS SOLVABLE BY DIRECT INTEGRATION

We now consider such partial differential equations which can be solved by direct integration. In place of the usual constants of integration, we must, however, use arbitrary functions of the variable held fixed.

Example 17.4. *Solve* $\dfrac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x - y) = 0.$ *(V.T.U., 2010)*

Solution. Integrating twice with respect to x (keeping y fixed),

$$\frac{\partial^2 z}{\partial x \partial y} + 9x^2y^2 - \frac{1}{2}\cos(2x - y) = f(y)$$

$$\frac{\partial z}{\partial y} + 3x^3y^2 - \frac{1}{4}\sin(2x - y) = xf(y) + g(y).$$

Now integrating with respect to y (keeping x fixed)

$$z + x^3y^3 - \frac{1}{4}\cos(2x - y) = x\int f(y)dy + \int g(y)dy + w(x)$$

The result may be simplified by writing

$$\int f(y)dy = u(y) \text{ and } \int g(y)dy = v(y).$$

Thus $z = \dfrac{1}{4}\cos(2x - y) - x^3y^3 + xu(y) + v(y) + w(x)$ where u, v, w are arbitrary functions.

Example 17.5. *Solve* $\dfrac{\partial^2 z}{\partial x^2} + z = 0$, *given that when* $x = 0$, $z = e^y$ *and* $\dfrac{\partial z}{\partial x} = 1$.

Solution. If z were function of x alone, the solution would have been $z = A\sin x + B\cos x$, where A and B are constants. Since z is a function of x and y, A and B can be arbitrary functions of y. Hence the solution of the given equation is $z = f(y)\sin x + \phi(y)\cos x$

$$\therefore \qquad \frac{\partial z}{\partial x} = f(y)\cos x - \phi(y)\sin x$$

When $x = 0$; $z = e^y$, $\quad \therefore\ e^y = \phi(y)$. $\quad$ When $x = 0$, $\dfrac{\partial z}{\partial x} = 1$, $\quad \therefore\ 1 = f(y)$.

Hence the desired solution is $z = \sin x + e^y\cos x$.

Example 17.6. *Solve* $\dfrac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$, *for which* $\dfrac{\partial z}{\partial y} = -2\sin y$ *when* $x = 0$ *and* $z = 0$ *when* y *is an odd multiple of* $\pi/2$. *(V.T.U., 2010 S)*

Solution. Given equation is $\dfrac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$

Integrating w.r.t. x, keeping y constant, we get

$$\frac{\partial z}{\partial y} = -\cos x \sin y + f(y) \qquad \ldots(i)$$

When $x = 0$, $\dfrac{\partial z}{\partial y} = -2 \sin y$, $\quad \therefore \quad -2 \sin y = -\sin y + f(y)$ or $f(y) = -\sin y$

$\therefore$ (*i*) becomes $\dfrac{\partial z}{\partial y} = -\cos x \sin y - \sin y$

Now integrating w.r.t. y, keeping x constant, we get

$$z = \cos x \cos y + \cos y + g(x) \qquad ...(ii)$$

When y is an odd multiple of $\pi/2$, $z = 0$.

$\therefore \qquad 0 = 0 + 0 + g(x)$ or $g(x) = 0 \qquad [\because \cos(2n+1)\pi/2 = 0]$

Hence from (*ii*), the complete solution is $z = (1 + \cos x) \cos y$.

PROBLEMS 17.2

Solve the following equations :

1. $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{x}{y} + a.$
2. $\dfrac{\partial^2 z}{\partial x^2} = xy.$
3. $\dfrac{\partial^2 u}{\partial x dt} = e^{-t} \cos x.$
4. $\dfrac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y).$
5. $\dfrac{\partial^2 z}{\partial y^2} = z$, gives that when $y = 0$, $z = e^x$ and $\dfrac{\partial z}{\partial y} = e^{-x}$
6. $\dfrac{\partial^2 z}{\partial x^2} = a^2 z$ given that when $x = 0$, $\dfrac{\partial z}{\partial x} = a \sin y$ and $\dfrac{\partial z}{\partial y} = 0.$

17.5 LINEAR EQUATIONS OF THE FIRST ORDER

A linear partial differential equation of the first order, commonly known as *Lagrange's Linear equation**, is of the form

$$Pp + Qq = R \qquad ...(1)$$

where P, Q and R are functions of x, y, z. This equation is called a *quasi-linear equation*. When P, Q and R are independent of z it is known as *linear equation*.

Such an equation is obtained by eliminating an arbitrary function ϕ from $\phi(u, v) = 0 \qquad ...(2)$

where u, v are some functions of x, y, z.

Differentiating (2) partially with respect to x and y.

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p\right) = 0 \text{ and } \frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} q\right) = 0.$$

Eliminating $\dfrac{\partial \phi}{\partial u}$ and $\dfrac{\partial \phi}{\partial v}$, we get $\begin{vmatrix} \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial z} p & \dfrac{\partial v}{\partial x} + \dfrac{\partial v}{\partial z} p \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} q & \dfrac{\partial v}{\partial y} + \dfrac{\partial v}{\partial z} q \end{vmatrix} = 0$

which simplifies to $\left(\dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial z} - \dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial y}\right) p + \left(\dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial z}\right) q = \left(\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x}\right) \qquad ...(3)$

This is of the same form as (1).

Now suppose $u = a$ and $v = b$, where a, b are constants, so that

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = du = 0$$

$$\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz = dv = 0.$$

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*See footnote p. 142.

By cross-multiplication, we have

$$\frac{dx}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}}=\frac{dy}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}}=\frac{dz}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}.$$

or $$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \qquad \text{...(4) [By virtue of (1) and (3)]}$$

The solutions of these equations are $u = a$ and $v = b$.

$\therefore$ $\phi(u, v) = 0$ is the required solution of (1).

Thus to solve the equation $Pp + Qq = R$.

(*i*) *form the subsidiary equations* $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$.

(*ii*) *solve these simultaneous equations by the method of § 16.10 giving $u = a$ and $v = b$ as its solutions.*

(*iii*) *write the complete solution as $\phi(u, v) = 0$* or $u = f(v)$.

Example 17.7. *Solve* $\frac{y^2z}{x}p + xzq = y^2$. *(Kottayam, 2005)*

Solution. Rewriting the given equation as

$$y^2zp + x^2zq = y^2x,$$

The subsidiary equations are $\frac{dx}{y^2z}=\frac{dy}{x^2z}=\frac{dz}{y^2x}$

The first two fractions give $x^2dx = y^2dy$.

Integrating, we get $x^3 - y^3 = a$ ...(*i*)

Again the first and third fractions give $xdx = zdz$

Integrating, we get $x^2 - z^2 = b$ ...(*ii*)

Hence from (*i*) and (*ii*), the complete solution is

$$x^3 - y^3 = f(x^2 - z^2).$$

Example 17.8. *Solve* $(mz - ny)\frac{\partial z}{\partial x} + (nx - lz)\frac{\partial z}{\partial y} = ly - mx$. *(V.T.U., 2010 ; S.V.T.U., 2009)*

Solution. Here the subsidiary equations are $\frac{dx}{mz-ny}=\frac{dy}{mx-lz}=\frac{dz}{ly-mx}$

Using multipliers x, y, and z, we get each fraction $= \frac{xdx + ydy + zdz}{0}$

$\therefore$ $xdx + ydy + zdz = 0$ which on integration gives $x^2 + y^2 + z^2 = a$ ...(*i*)

Again using multipliers l, m and n, we get each fraction $= \frac{ldx + mdy + ndz}{0}$

$\therefore$ $ldx + mdy + ndx = 0$ which on integration gives $lx + my + nz = b$ ...(*ii*)

Hence from (*i*) and (*ii*), the required solution is $x^2 + y^2 + z^2 = f(lx + my + nz)$.

Example 17.9. *Solve* $(x^2 - y^2 - z^2)p + 2xyq = 2xz$. *(V.T.U., 2010 ; Anna, 2009 ; S.V.T.U., 2008)*

Solution. Here the subsidiary equations are $\frac{dx}{x^2-y^2-z^2}=\frac{dy}{2xy}=\frac{dz}{2xz}$

From the last two fractions, we have $\frac{dy}{y}=\frac{dz}{z}$

which on integration gives $\log y = \log z + \log a$ or $y/z = a$ ...(*i*)

Using multipliers x, y and z, we have

$$\text{each fraction} = \frac{xdx + ydy + zdz}{x(x^2+y^2+z^2)} \quad \therefore \quad \frac{2xdx + 2ydy + 2zdz}{x^2+y^2+z^2}=\frac{dz}{z}$$

which on integration gives $\log(x^2+y^2+z^2)=\log z+\log b$

or $$\frac{x^2+y^2+z^2}{z}=b \qquad ...(ii)$$

Hence from (*i*) and (*ii*), the required solution is $x^2+y^2+z^2=zf(y/z)$.

Example 17.10. *Solve* $x^2(y-z)p+y^2(z-x)q=z^2(x-y)$. *(P.T.U., 2009 ; Bhopal, 2008 ; S.V.T.U. 2007)*

Solution. Here the subsidiary equations are

$$\frac{dx}{x^2(y-z)}=\frac{dy}{y^2(z-x)}=\frac{dz}{z^2(x-y)}$$

Using the multipliers $1/x$, $1/y$ and $1/z$, we have

$$\text{each fraction}=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$$

$\therefore$ $$\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0 \text{ which on integration gives}$$

$$\log x+\log y+\log z=\log a \quad \text{or} \quad xyz=a \qquad ...(i)$$

Using the multipliers $\frac{1}{x^2},\frac{1}{y^2}$ and $\frac{1}{z^2}$, we get

$$\text{each fraction}=\frac{\frac{1}{x^2}dx+\frac{1}{y^2}dy+\frac{1}{z^2}dz}{0}$$

$\therefore$ $$\frac{dx}{x^2}+\frac{dy}{y^2}+\frac{dz}{z^2}=0, \text{ which on integrating gives}$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0 \qquad ...(ii)$$

Hence from (*i*) and (*ii*), the complete solution is

$$xyz=f\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right).$$

Example 17.11. *Solve* $(x^2-yz)p+(y^2-zx)q=z^2-xy$. *(Bhopal, 2008 ; V.T.U., 2006 ; Madras, 2000)*

Solution. Here the subsidiary equations are

$$\frac{dx}{x^2-yz}=\frac{dy}{y^2-zx}=\frac{dz}{z^2-xy} \qquad ...(i)$$

$$\text{Each of these equations}=\frac{dx-dy}{x^2-y^2-(y-x)z}=\frac{dy-dz}{y^2-z^2-x(z-y)}$$

i.e., $$\frac{d(x-y)}{(x-y)(x+y+z)}=\frac{d(y-z)}{(y-z)(x+y+z)} \quad \text{or} \quad \frac{d(x-y)}{x-y}=\frac{d(y-z)}{y-z}$$

Integrating, $$\log(x-y)=\log(y-z)+\log c \quad \text{or} \quad \frac{x-y}{y-z}=c \qquad ...(ii)$$

$$\text{Each of the subsidiary equations } (i)=\frac{xdx+ydy+zdz}{x^3+y^3+z^3-3xyz}$$

$$=\frac{xdx+ydy+zdz}{(x+y+z)(x^2+y^2+z^2-yz-zx-xy)} \qquad ...(iii)$$

$$\text{Also each of the subsidiary equations}=\frac{dx+dy+dz}{x^2+y^2+z^2-yz-zx-xy} \qquad ...(iv)$$

Equating (*iii*) and (*iv*) and cancelling the common factor, we get

$$\frac{xdx + ydy + zdz}{x + y + z} = dx + dy + dz$$

or $$\int (xdx + ydy + zdz) = \int (x + y + z) d\,(x + y + z) + c'$$

or $$x^2 + y^2 + z^2 = (x + y + z)^2 + 2c' \quad \text{or} \quad xy + yz + zx + c' = 0 \qquad ...(v)$$

Combining (*ii*) and (*v*), the general solution is

$$\frac{x - y}{y - z} = f\,(xy + yz + zx).$$

PROBLEMS 17.3

Solve the following equations :

1. $xp + yq = 3z$.
2. $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$.
3. $(z - y)p + (x - z)q = y - x$.
4. $p \cos (x + y) + q \sin (x + y) = z$.
5. $pyz + qzx = xy$.
6. $p \tan x + q \tan y = \tan z$.
7. $p - q = \log (x + y)$.
8. $xp - yq = y^2 - x^2$ (*J.N.T.U., 2002 S*)
9. $(y + z)p - (z + x)\, q = x - y$.
10. $x(y - z)p + y(z - x)q = z(x - y)$. (*Bhopal, 2007*)
11. $x(y^2 - z^2)p + y(z^2 - x^2)q - z(x^2 - y^2) = 0$. (*V.T.U., 2010 ; Anna, 2008*)
12. $y^2p - xyq = x(z - 2y)$. (*S.V.T.U., 2008*)
13. $(y^2 + z^2)p - xyq + zx = 0$. (*P.T.U., 2009 ; V.T.U., 2009*)
14. $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$. (*Kerala, 2005*)
15. $px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)$.

17.6 NON-LINEAR EQUATIONS OF THE FIRST ORDER

Those equations in which p and q occur other than in the first degree are called *non-linear partial differential equations of the first order*. The *complete solution* of such an equation contains only two arbitrary constants (*i.e.*, equal to the number of independent variables involved) and the particular integral is obtained by giving particular values to the constants.]

Here we shall discuss four standard forms of these equations.

Form I. f(p, q) = 0, *i.e., equations containing p and q only.*

Its complete solution is $z = ax + by + c$ ...(1)

where a and b are connected by the relation $f\,(a, b) = 0$ ...(2)

[Since from (1), $p = \dfrac{\partial z}{\partial x} = a$ and $q = \dfrac{\partial z}{\partial y} = b$, which when substituted in (2) give $f\,(p, q) = 0$].

Expressing (2) as $b = \phi(a)$ and substituting this value of b in (1), we get the required solution as $z = ax + \phi(a)y + c$ in which a and c are arbitrary constants.

Example 17.12. *Solve* $p - q = 1$. (*Anna, 2009*)

Solution. The complete solution is $z = ax + by + c$ where $a - b = 1$

Hence $z = ax + a - 1y + c$ is the desired solution.

Example 17.13. *Solve* $x^2p^2 + y^2q^2 = z^2$. (*Anna, 2008 ; Bhopal, 2008 ; Kerala, 2005 ; Kurukshetra, 2005*)

Solution. Given equation can be reduced to the above form by writing it as

$$\left(\frac{x}{z}\cdot\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z}\cdot\frac{\partial z}{\partial y}\right)^2 = 1 \qquad ...(i)$$

and setting $$\frac{dx}{x} = du, \frac{dy}{y} = dv, \frac{dz}{z} = dw \text{ so that } u = \log x, v = \log y, w = \log z.$$

Then (*i*) becomes $$\left(\frac{\partial w}{\partial u}\right)^2 + \left(\frac{\partial w}{\partial v}\right)^2 = 1$$

i.e., $$P^2 + Q^2 = 1 \quad \text{where} \quad P = \frac{\partial w}{\partial u} \quad \text{and} \quad Q = \frac{\partial w}{\partial v}.$$

Its complete solution is $w = au + bv + c$ ...(*ii*)

where $a^2 + b^2 = 1$ or $b = \sqrt{(1-a^2)}$.

$\therefore$ (*ii*) becomes $w = au + \sqrt{(1-a^2)}\, v + c$

or $$\log z = a \log x + \sqrt{(1-a^2)} \log y + c$$ which is the required solution.

Form II. f(z, p, q) = 0, *i.e., equations not containing x and y.*

As a trial solution, assume that z is a function of $u = x + ay$, where a is an arbitrary constant.

$\therefore$ $$p = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{dz}{du} \qquad q = \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = a\frac{dz}{du}$$

Substituting the values of p and q in $f(z, p, q) = 0$, we get

$$f\left(z, \frac{\partial z}{\partial u}, a\frac{dz}{du}\right) = 0$$ which is an ordinary differential equation of the first order.

Rewiting it as $\frac{dz}{du} = \phi(z, a)$ it can be easily integrated giving

$F(z, a) = u + b$, or $x + ay + b = F(z, a)$ which is the desired complete solution.

Thus to solve $f(z, p, q) = 0$,

(*i*) *assume* $u = x + ay$ *and substitute* $p = dz/du$, $q = a\, dz/du$ *in the given equation;*

(*ii*) *solve the resulting ordinary differential equation in z and u;*

(*iii*) *replace u by* $x + ay$.

Example 17.14. *Solve* $p(1+q) = qz$. (*Madras, 2000 S*)

Solution. Let $u = x + ay$, so that $p = dz/du$ and $q = a\, dz/du$.

Substituting these values of p and q in the given equation, we have

$$\frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = az\frac{dz}{du} \text{ or } a\frac{dz}{du} = az - 1 \quad \text{or} \quad \int \frac{a\,dz}{az-1} = \int du + b$$

or $$\log(az-1) = u + b \text{ or } \log(az-1) = x + ay + b$$

which is the required complete solution.

Example 17.15. *Solve* $q^2 = z^2p^2(1-p^2)$. (*J.N.T.U., 2005 ; Kerala, 2005*)

Solution. Setting $u = y + ax$ and $z = f(u)$, we get

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = a\frac{dz}{du} \text{ and } q = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = \frac{dz}{du}$$

$\therefore$ The given equation becomes $\left(\frac{dz}{du}\right)^2 = a^2z^2\left(\frac{dz}{du}\right)^2\left\{1 - a^2\left(\frac{dz}{du}\right)^2\right\}$ ...(*i*)

or $$a^4z^2\left(\frac{dz}{du}\right)^2 = a^2z^2 - 1 \quad \text{or} \quad \frac{dz}{du} = \frac{\sqrt{(a^2z^2-1)}}{a^2z}$$

Integrating, $$\int \frac{a^2z}{\sqrt{(a^2z^2-1)}}dz = \int du + c \quad \text{or} \quad (a^2z^2-1)^{1/2} = u + c$$

i.e., $$a^2z^2 = (y + ax + c)^2 + 1 \qquad [\because u = y + ax]$$

The second factor in (*i*) is $dz/du = 0$. Its solution is $z = c'$.

Example 17.16. *Solve* $z^2(p^2x^2 + q^2) = 1$. (*Bhopal, 2008 S*)

Solution. Given equation can be reduced to the above form by writing it as

$$z^2\left[\left(x\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]=1 \qquad ...(i)$$

Putting $X = \log x$, so that $x\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial X}$, (i) takes the standard form

$$z^2\left[\left(\frac{\partial z}{\partial X}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]=1 \qquad ...(ii)$$

Let $u = X + ay$ and put $\dfrac{\partial z}{\partial X}=\dfrac{dz}{du}$ and $\dfrac{\partial z}{\partial y}=a\dfrac{dz}{du}$ in (ii), so that

$$z^2\left[\left(\frac{dz}{du}\right)^2+a^2\left(\frac{dz}{du}\right)^2\right]=1 \quad \text{or} \quad \sqrt{(1+a^2)}\,zdz=\pm\, du$$

Integrating, $\sqrt{(1+a^2)}\,z^2 = \pm\, 2u + b = \pm\, 2(X+ay)+b$

or $z^2\sqrt{(1+a^2)} = \pm\, 2\,(\log x + ay) + b$

which is the complete solution required.

Form III. f(x, p) = F(y, q), *i.e., equations in which z is absent and the terms containing x and p can be separated from those containing y and q.*

As a trial solution assume that $f(x, p) = F(y, q) = a$, say

Then solving for p, we get $p = \phi(x)$

and solving for q, we get $q = \psi(y)$

Since $dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy = pdx + qdy$

$\therefore$ $dz = \phi(x)dx + \psi(y)dy$

Integrating, $z = \int \phi(x)dx + \int \psi(y)dy + b$

which is the desired complete solution containing two constants a and b.

Example 17.17. *Solve $p^2 + q^2 = x + y$.* *(Bhopal, 2006 ; Madras, 2003)*

Solution. Given equation is $p^2 - x = y - q^2 = a$, say

$\therefore$ $p^2 - x = a$ gives $p = \sqrt{(a+x)}$

and $y - q^2 = a$ gives $q = \sqrt{(y-a)}$

Substituting these values of p and q in $dz = pdx + qdy$, we get

$$dz = \sqrt{(a+x)}\,dx + \sqrt{(y-a)}\,dy$$

$\therefore$ integrating gives, $z = \dfrac{2}{3}(a+x)^{3/2} + \dfrac{2}{3}(y-a)^{3/2} + b$

which is the required complete solution.

Example 17.18. *Solve $z^2(p^2 + q^2) = x^2 + y^2$.* *(Bhopal, 2008)*

Solution. The equation can be reduced to the above form by writing it as

$$\left(z\frac{\partial z}{\partial x}\right)^2+\left(z\frac{\partial z}{\partial y}\right)^2 = x^2+y^2 \qquad ...(i)$$

and putting $zdz = dZ$, *i.e.*, $Z = \dfrac{1}{2}z^2$

$\therefore$ $\dfrac{\partial Z}{\partial x}=\dfrac{\partial Z}{\partial z}\cdot\dfrac{\partial z}{\partial x}=z\dfrac{\partial z}{\partial x}=P$

and $\dfrac{\partial Z}{\partial y}=\dfrac{\partial Z}{\partial z}\cdot\dfrac{\partial z}{\partial y}=z\dfrac{\partial z}{\partial y}=Q$

$\therefore$ (*i*) becomes $\quad P^2 + Q^2 = x^2 + y^2$

or $\quad P^2 - x^2 = y^2 - Q^2 = a$, say.

$$\therefore \quad P = \sqrt{(x^2 + a)} \text{ and } Q = \sqrt{(y^2 - a)}.$$

$\therefore$ $\quad dZ = Pdx + Qdy$ gives

$$dZ = \sqrt{(x^2 + a)}\, dx + \sqrt{(y^2 - a)}\, dy$$

Integrating, we have

$$Z = \frac{1}{2} x \sqrt{(x^2 + a)} + \frac{1}{2} a \log [x + \sqrt{(x^2 + a)}] + \frac{1}{2} y \sqrt{(y^2 - a)} - \frac{1}{2} a \log [y + \sqrt{(y^2 - a)}] + b$$

or

$$z^2 = x \sqrt{(x^2 + a)} + y \sqrt{(y^2 - a} + a \log \frac{x + \sqrt{(x^2 + a)}}{y + \sqrt{(y^2 - a)}} + 2b$$

which is the required complete solution.

Example 17.19. *Solve* $(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1.$

(Bhopal, 2006 ; Rajasthan, 2006 ; V.T.U., 2003)

Solution. This equation can be reduced to the form $f(x, q) = F(y, q)$ by putting $u = x + y$, $v = x - y$ and taking $z = z(u, v)$.

Then $\quad p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u} \cdot \dfrac{\partial u}{\partial x} + \dfrac{\partial z}{\partial v} \cdot \dfrac{\partial v}{\partial x} = P + Q$

and $\quad q = \dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u} \cdot \dfrac{\partial u}{\partial y} + \dfrac{\partial z}{\partial v} \cdot \dfrac{\partial v}{\partial y} = P - Q$, where $P = \dfrac{\partial z}{\partial u}, Q = \dfrac{\partial z}{\partial v}$

Substituting these, the given equation reduces to

$$u(2P)^2 + v(2Q)^2 = 1 \quad \text{or} \quad 4P^2 u = 1 - 4Q^2 v = a \text{ (say)}$$

$$P = \pm \frac{1}{2}\sqrt{\frac{a}{u}}, Q = \pm \frac{1}{2}\sqrt{\frac{1-a}{v}}$$

$$\therefore \quad dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv = Pdu + Qdv$$

$$= \pm \frac{\sqrt{a}}{2} \frac{du}{\sqrt{u}} \pm \frac{\sqrt{1-a}}{2} \frac{dv}{\sqrt{v}}$$

Integrating, we have $\quad z = \pm \sqrt{a}\sqrt{u} \pm \sqrt{1-a}\sqrt{v} + b$

or $\quad z = \pm \sqrt{a(x + y)} \pm \sqrt{(1-a)(x - y)} + b$

which is the required complete solution.

Form IV. z = px + qy + f(p, q): *an equation analogous to the Clairaut's equation (§ 11.14).*

Its complete solution is $z = ax + by + f(a, b)$ which is obtained by writing a for p and b for q in the given equation.

Example 17.20. *Solve* $z = px + qy + \sqrt{(1 + p^2 + q^2)}.$ *(Anna, 2009)*

Solution. Given equation is of the form $z = px + qy + f(p, q)$ where $f(p, q) = \sqrt{(1 + p^2 + q^2)}$

$\therefore$ Its complete solution is $z = ax + by + \sqrt{(1 + a^2 + b^2)}$.

PROBLEMS 17.4

Obtain the complete solution of the following equations :

1. $pq + p + q = 0$.
2. $p^2 + q^2 = 1$. *(Osmania, 2000)*
3. $z = p^2 + q^2$. *(Anna, 2005 S ; J.N.T.U., 2002 S)*
4. $p(1 - q^2) = q(1 - z)$ *(Anna, 2006)*
5. $yp + xq + pq = 0$.
6. $p + q = \sin x + \sin y$.

7. $p^2 - q^2 = x - y$.
8. $\sqrt{p} + \sqrt{q} = x + y$.
9. $p^2 + q^2 = x^2 + y^2$. (*Osmania, 2003*)
10. $z = px + qy + \sin(x + y)$.
11. $\sqrt{p} + \sqrt{q} = 2x$. (*J.N.T.U., 2006*)
12. $z = px + qy - 2\sqrt{(pq)}$.
13. $(x - y)(px - qy) = (p - q)^2$. [**Hint.** Use $x + y = u, xy = v$]

17.7 CHARPIT'S METHOD*

We now explain a general method for finding the complete integral of a non-linear partial differential equation which is due to Charpit.

Consider the equation

$$f(x, y\, z, p, q) = 0 \quad ...(1)$$

Since z depends on x and y, we have

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy \quad ...(2)$$

Now if we can find another relation involving x, y, z, p, q such as $\phi(x, y, z, p, q) = 0$ ...(3)
then we can solve (1) and (3) for p and q and substitute in (2). This will give the solution provided (2) is integrable.

To determine ϕ, we differentiate (1) and (3) with respect to x and y giving

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}p + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = 0 \quad ...(4)$$

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial z}p + \frac{\partial \phi}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial \phi}{\partial q}\frac{\partial q}{\partial x} = 0 \quad ...(5)$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}q + \frac{\partial f}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial y} = 0 \quad ...(6)$$

$$\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z}q + \frac{\partial \phi}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial \phi}{\partial q}\frac{\partial q}{\partial y} = 0 \quad ...(7)$$

Eliminating $\frac{\partial p}{\partial x}$ between the equations (4) and (5), we get

$$\left(\frac{\partial f}{\partial x}\frac{\partial \phi}{\partial p} - \frac{\partial \phi}{\partial x}\frac{\partial f}{\partial p}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial \phi}{\partial p} - \frac{\partial \phi}{\partial z}\frac{\partial f}{\partial p}\right)p + \left(\frac{\partial f}{\partial q}\frac{\partial \phi}{\partial p} - \frac{\partial \phi}{\partial q}\frac{\partial f}{\partial p}\right)\frac{\partial q}{\partial x} = 0 \quad ...(8)$$

Also eliminating $\frac{\partial q}{\partial y}$ between the equations (6) and (7), we obtain

$$\left(\frac{\partial f}{\partial y}\frac{\partial \phi}{\partial q} - \frac{\partial \phi}{\partial y}\frac{\partial f}{\partial q}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial \phi}{\partial q} - \frac{\partial \phi}{\partial z}\frac{\partial f}{\partial q}\right)q + \left(\frac{\partial f}{\partial p}\frac{\partial \phi}{\partial q} - \frac{\partial \phi}{\partial p}\frac{\partial f}{\partial q}\right)\frac{\partial p}{\partial y} = 0 \quad ...(9)$$

Adding (8) and (9) and using $\frac{\partial q}{\partial x} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial p}{\partial y}$,

we find that the last terms in both cancel and the other terms, on rearrangement, give

$$\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial p} + \left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial q} + \left(-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial z} + \left(-\frac{\partial f}{\partial p}\right)\frac{\partial \phi}{\partial x} + \left(-\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial y} = 0 \quad ...(10)$$

i.e.,

$$\left(-\frac{\partial f}{\partial p}\right)\frac{\partial \phi}{\partial x} + \left(-\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial y} + \left(-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial z} + \left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial p} + \left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial q} = 0 \quad ...(11)$$

This is Lagrange's linear equation (§ 17.5) with x, y, z, p, q as independent variables and ϕ as the dependent variable. Its solution will depend on the solution of the subsidiary equations

*Charpit's memoir containing this method was presented to the Paris Academy of Sciences in 1784.

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{\partial \phi}{0}$$

An integral of these equations involving p or q or both, can be taken as the required relation (3), which alongwith (1) will give the values of p and q to make (2) integrable. Of course, we should take the simplest of the integrals so that it may be easier to solve for p and q.

Example 17.21. *Solve* $(p^2 + q^2)y = qz$. (*V.T.U., 2007 ; Hissar, 2005*)

Solution. Let $f(x, y, z, p, q) = (p^2 + q^2)y - qz = 0$ ...(*i*)

Charpit's subsidiary equations are

$$\frac{dx}{-2py} = \frac{dy}{z - 2qy} = \frac{dz}{-qz} = \frac{dp}{-pq} = \frac{dq}{p^2}$$

The last two of these give $pdp + qdq = 0$

Integrating, $\quad p^2 + q^2 = c^2$ ...(*ii*)

Now to solve (*i*) and (*ii*), put $p^2 + q^2 = c^2$ in (*i*), so that $q = c^2y/z$

Substituting this value of q in (*ii*), we get $p = c\sqrt{(z^2 - c^2y^2)}/z$

Hence $$dz = pdx + qdy = \frac{c}{z}\sqrt{(z^2 - c^2y^2)}\,dx + \frac{c^2y}{z}dy$$

or $$zdz - c^2y\,dy = c\sqrt{(z^2 - c^2y^2)}\,dx \quad \text{or} \quad \frac{\frac{1}{2}d(z^2 - c^2y^2)}{\sqrt{(z^2 - c^2y^2)}} = c\,dx$$

Integrating, we get $\sqrt{(z^2 - c^2y^2)} = cx + a$ or $z^2 = (a + cx)^2 + c^2y^2$ which is the required complete integral.

Example 17.22. *Solve* $2xz - px^2 - 2qxy + pq = 0$. (*Rajasthan, 2006*)

Solution. Let $f(x, y, z, p, q) = 2xz - px^2 - 2qxy + pq = 0$ ...(*i*)

Charpit's subsidiary equations are

$$\frac{dx}{x^2 - q} = \frac{dy}{2xy - p} = \frac{dz}{px^2 - 2pq + 2qxy} = \frac{dp}{2z - 2qy} = \frac{dq}{0}$$

$\therefore \quad dq = 0 \quad \text{or} \quad q = a.$

Putting $q = a$ in (*i*), we get $p = \dfrac{2x(z - ay)}{x^2 - a}$

$$\therefore \quad dz = pdx + qdy = \frac{2x(z - ay)}{x^2 - a}dx + ady \quad \text{or} \quad \frac{dz - ady}{z - ay} = \frac{2x}{x^2 - a}dx$$

Integrating, $\quad \log(z - ay) = \log(x^2 - a) + \log b$

or $$z - ay = b(x^2 - a) \quad \text{or} \quad z = ay + b(x^2 - a)$$

which is the required complete solution.

Example 17.23. *Solve* $2z + p^2 + qy + 2y^2 = 0$. (*J.N.T.U., 2005 ; Kurukshetra, 2005*)

Solution. Let $f(x, y, z, p, q) = 2z + p^2 + qy + 2y^2$

Charpit's subsidiary equations are

$$\frac{dx}{-2p} = \frac{dy}{-y} = \frac{dz}{-(2p^2 + qy)} = \frac{dp}{2p} = \frac{dq}{4y + 3q}$$

From first and fourth ratios,

$$dp = -dx \quad \text{or} \quad p = -x + a$$

Substituting $p = a - x$ in the given equation, we get

$$q = \frac{1}{y}[-2z - 2y^2 - (a - x)^2]$$

$\therefore \qquad dz = pdx + qdy = (a-x)dx - \dfrac{1}{y}[2z + 2y^2 + (a-x)^2]dy$

Multiplying both sides by $2y^2$,

$$2y^2dz + 4yz\,dy = 2y^2(a-x)dx - 4y^3dy - 2y(a-x)^2dy$$

Integrating $\qquad 2zy^2 = -[y^2(a-x)^2 + y^4] + b$

or $\qquad y^2[(x-a)^2 + 2z + y^2] = b$, which is the desired solution.

PROBLEMS 17.5

Solve the following equations :

1. $z = p^2x + q^2x$.
2. $z^2 = pq\,xy$. *(Anna, 2009 ; V.T.U., 2004)*
3. $1 + p^2 = qz$.
4. $pxy + pq + qy = yz$. *(J.N.T.U., 2006 ; Kurukshetra, 2006)*
5. $p(p^2+1) + (b-z)q = 0$.
6. $q + xp = p^2$. *(Osmania, 2003)*

17.8 HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

An equation of the form

$$\frac{\partial^n z}{\partial x^n} + k_1\frac{\partial^n z}{\partial x^{n-1}\partial y} + \dots + k_n\frac{\partial^n z}{\partial y^n} = F(x,y) \qquad \dots(1)$$

in which k's are constants, is called a *homogeneous linear partial differential equation of the nth order with constant coefficients.* It is called homogeneous because all terms contain derivatives of the same order.

On writing, $\dfrac{\partial^r}{\partial x^r} = D^r$ and $\dfrac{\partial^r}{\partial y^r} = D'^r$. (1) becomes $(D^n + k_1D^{n-1}D'^r + D' + \dots + k_nD'^n)\,z = F(x,y)$

or briefly $\qquad f(D, D')z = F(x,y) \qquad \dots(2)$

As in the case of ordinary linear equations with constant coefficients the complete solution of (1) consists of two parts, namely : the *complementary function* and *the particular integral.*

The complementary function is the complete solution of the equation $f(D, D')z = 0$, which must contain n arbitrary functions. The particular integral is the particular solution of equation (2).

17.9 RULES FOR FINDING THE COMPLEMENTARY FUNCTION

Consider the equation $\dfrac{\partial^2 z}{\partial x^2} + k_1\dfrac{\partial^2 z}{\partial x\partial y} + k_2\dfrac{\partial^2 z}{\partial y^2} = 0 \qquad \dots(1)$

which in *symbolic form is* $(D^2 + k_1DD' + k_2D'^2)z = 0 \qquad \dots(2)$

Its symbolic operator equated to zero, *i.e.*, $D^2 + k_1DD' + k_2D'^2 = 0$ is called the *auxiliary equation* (A.E.)

Let its root be $\qquad D/D' = m_1, m_2$.

Case I. *If the roots be real and distinct* then (2) is equivalent to

$$(D - m_1D')(D - m_2D')z = 0 \qquad \dots(3)$$

It will be satisfied by the solution of

$$(D - m_2D')z = 0, \text{ i.e., } p - m_2q = 0.$$

This is a Lagrange's linear and the subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m_2} = \frac{dz}{0}, \text{ whence } y + m_2x = a \text{ and } z = b.$$

$\therefore$ its solution is $z = \phi(y + m_2x)$.

Similarly (3) will also be satisfied by the solution of

$$(D - m_1D')z = 0, \quad i.e., \quad \text{by } z = f(y + m_1x)$$

Hence the complete solution of (1) is $z = f(y + m_1x) + \phi(y + m_2x)$.

Case II. *If the roots be equal* (*i.e.*, $m_1 = m_2$), then (2) is equivalent to

$$(D - m_1D')^2z = 0 \quad ...(4)$$

Putting $(D - m_1D')z = u$, it becomes $(D - m_1D')u = 0$ which gives

$$u = \phi(y + m_1x)$$

$\therefore$ (4) takes the form $(D - m_1'D)z = \phi(y + m_1x)$ or $p - m_1q) = \phi(y + m_1x)$

This is again Lagrange's linear and the subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m_1} = \frac{dz}{\phi(y + m_1x)}$$

giving $y + m_1x = a$ and $dz = \phi(a)\, dx$, *i.e.*, $z = \phi(a)x + b$

Thus the complete solution of (1) is

$$z - x\phi(y + m_1x) = f(y + m_1x). \quad i.e., \quad z = f(y + m_1x) + x\phi(y + m_1x).$$

Example 17.24. *Solve* $2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0.$

Solution. Given equation in *symbolic form* is $(2D^2 + 5DD' + 2D'^2)z = 0$.

Its *auxiliary equation* is $2m^2 + 5m + 2 = 0$, where $m = D/D'$.

which gives $m = -2, -1/2.$

Here the complete solution is $z = f_1(y - 2x) + f_2(y - \frac{1}{2}x)$

which may be written as $z = f_1(y - 2x) + f_2(2y - x)$.

Example 17.25. *Solve* $4r + 12s + 9t = 0.$ *(P.T.U., 2010)*

Solution. Given equation in *symbolic form* is $(4D^2 + 12DD' + 9D'^2)z = 0$

for $r = \frac{\partial^2 z}{\partial x^2} = D^2z,\ s = \frac{\partial^2 z}{\partial x \partial y} = DD'z$ and $t = \frac{\partial^2 z}{\partial y^2} = D'^2z.$

$\therefore$ Its *auxiliary equation* is $4m^2 + 12m + 9 = 0$, whence $m = -3/2, -3/2$

Hence the complete solution is $z = f_1(y - 1.5x) + xf_2(y - 1.5x)$.

17.10 RULES FOR FINDING THE PARTICULAR INTEGRAL

Consider the equation $(D^2 + k_1DD' + k_2D'^2)z = F(x, y)$ *i.e.*, $f(D, D')z = F(x, y)$.

$$\therefore \quad \text{P.I.} = \frac{1}{f(D, D')}F(x, y)$$

Case I. *When* $F(x, y) = e^{ax + by}$

Since $De^{ax+by} = ae^{ax+by}$; $D'e^{ax+by} = be^{ax+by}$

$\therefore$ $D^2e^{ax+by} = a^2e^{ax+by}$; $DD'e^{ax+by} = abe^{ax+by}$

and $D'^2e^{ax+by} = b^2e^{ax+by}$

$\therefore$ $(D^2 + k_1DD' + k_2D'^2)e^{ax+by} = (a^2 + k_1ab + k_2b^2)\, e^{ax+by}$

i.e., $f(D, D')e^{ax+by} = f(a, b)\, e^{ax+by}$

Operating both sides by $1/f(D, D')$, we get

$$\text{P.I.} = \frac{1}{f(D, D')}e^{ax+by} = \frac{1}{f(a, b)}e^{ax+by}$$

Case II. *When* $F(x, y) = \sin(mx + ny)$ or $\cos(mx + ny)$

Since $D^2 \sin(mx + ny) = -m^2 \sin(mx + ny)$

$DD' \sin(mx + ny) = -mn \sin(mx + ny)$

and $D'^2 \sin(mx + ny) = -n^2 \sin(mx + ny).$

$\therefore$ $f(D^2, DD', D'^2) \sin(mx + ny) = f(-m^2, -mn, -n^2) \sin(mx + ny)$

Operating both sides by $1/f(D^2, DD', D'^2)$, we get

$$\text{P.I.} = \frac{1}{f(D^2, DD', D'^2)} \sin(mx+ny) = \frac{1}{f(-m^2 - mn, -n^2)} \sin(mx+ny)$$

Similarly about the P.I. for cos $(mx + ny)$.

Case III. *When* $F(x, y) = x^m y^n$, m and n being constants.

$$\therefore \qquad \text{P.I.} = \frac{1}{f(D, D')} x^m y^n = [f(D, D')]^{-1} x^m y^n.$$

To evaluate it, we expand $[f(D, D')]^{-1}$ in ascending powers of D or D' by Binomial theorem and then operate on $x^m y^n$ term by term.

Case IV. *When $F(x, y)$ is any function of x and y.*

$$\therefore \qquad \text{P.I.} = \frac{1}{f(D, D')} F(x, y)$$

To evaluate it, we resolve $1/f(D, D')$ into partial fractions treating $f(D, D')$ as a function of D alone and operate each partial fraction on $F(x, y)$ remembering that

$$\frac{1}{D - mD'} F(x, y) = \int F(x, c - mx)\, dx$$

where c is replaced by $y + mx$ after integration.

17.11 WORKING PROCEDURE TO SOLVE THE EQUATION

$$\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \ldots + k^n \frac{\partial^n z}{\partial y^n} = F(x, y).$$

Its *symbolic form* is $\quad (D^n + k_1 D^{n-1} D' + \ldots + k_n D'^n) z = F(x, y)$

or briefly $\quad f(D, D') z = F(x, y)$

Step I. To find the C.F.

(*i*) *Write the A.E.*

i.e., $\quad m^n + k_1 m^{n-1} + \ldots + k_n = 0$ *and solve it for m.*

(*ii*) *Write the C.F. as follows*

Roots of A.E.	*C.F.*
1. $m_1, m_2, m_3 \ldots$ (distinct roots)	$f_1(y + m_1 x) + f_2(y + m_2 x) + f_3(y + m_3 x) + \ldots$
2. $m_1, m_1, m_3 \ldots$ (two equal roots)	$f_1(y + m_1 x) + x f_2(y + m_1 x) + f_3(y + m_3 x) + \ldots$
3. $m_1, m_1, m_1 \ldots$ (three equal roots)	$f_1(y + m_1 x) + x f_2(y + m_1 x) + x^2 f_3(y + m_1 x) + \ldots$

Step II. To find the P.I.

From the *symbolic form*, P.I. $= \dfrac{1}{f(D, D')} F(x, y)$.

(*i*) *When* $F(x, y) = e^{ax + by}$ P.I. $= \dfrac{1}{f(D, D')} e^{ax+by}$ [Put $D = a$ and $D' = b$]

(*ii*) *When* $F(x, y) = \sin(mx + ny)$ or $\cos(mx + ny)$

$$\text{P.I.} = \frac{1}{f(D^2, DD', D'^2)} \sin \text{ or } \cos (mx + ny) \qquad [\text{Put } D^2 = -m^2, DD' = -mn, D'^2 = -n^2]$$

(*iii*) *When* $F(x, y) = x^m y^n$, P.I. $= \dfrac{1}{f(D, D')} x^m y^n = [f(D, D')]^{-1} x^m y^n$.

Expand $[f(D, D')]^{-1}$ in ascending powers of D or D′ and operate on $x^m y^n$ term by term.

(*iv*) *When $F(x, y)$ is any function of x and y* P.I. $= \dfrac{1}{f(D, D')} F(x, y)$.

Resolve $1/f(D, D')$ into partial fractions considering $f(D, D')$ as a function of D alone and operate each partial fraction on $F(x, y)$ remembering that

$$\frac{1}{D - mD'} F(x, y) = \int F(x, c - mx)\, dx \text{ where } c \text{ is replaced by } y + mx \text{ after integration.}$$

Example 17.26. *Solve $(D^2 + 4DD' - 5D'^2)z = \sin(2x + 3y)$.* *(Madras, 2006)*

Solution. A.E. of the given equation is $m^2 + 4m - 5 = 0$ *i.e.*, $m = 1, -5$

$\therefore$ C.F. $= f_1(y + x) + f_2(y - 5x)$

$$\text{P.I.} = \frac{1}{D^2 + 4DD' - 5D'^2} \sin(2x + 3y) \quad [\text{Put } D^2 = -2^2, DD' = -2 \times 3, D'^2 = -3^2$$

$$= \frac{1}{-4 + 4(-6) - 5(-9)} \sin(2x + 3y) = \frac{1}{17} \sin(2x + 3y).$$

Hence the C.S. is $z = f_1(y + x) + f_2(y - 5x) + \frac{1}{17} \sin(2x + 3y)$.

Example 17.27. *Solve* $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \cos x \cos 2y$. *(Bhopal, 2008 S)*

Solution. Given equation in *symbolic form* is $(D^2 - DD')z = \cos x \cos 2y$.

Its A.E. is $m^2 - m = 0$, whence $m = 0, 1$.

$\therefore$ C.F. $= f_1(y) + f_2(y + x)$

$$\text{P.I.} = \frac{1}{D^2 - DD'} \cos x \cos 2y = \frac{1}{2} \frac{1}{D^2 - DD'} [\cos(x + 2y) + \cos(x - 2y)]$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - DD'} \cos(x + 2y)\right. \qquad [\text{Put } D^2 = -1, DD' = -2]$$

$$\left. + \frac{1}{D^2 - DD'} \cos(x - 2y)\right] \qquad [\text{Put } D^2 = -1, DD' = 2]$$

$$= \frac{1}{2}\left[\frac{1}{-1 + 2} \cos(x + 2y) + \frac{1}{-1 - 2} \cos(x - 2y)\right] = \frac{1}{2} \cos(x + 2y) - \frac{1}{6} \cos(x - 2y)$$

Hence the C.S. is $z = f_1(y) + f_2(y + x) + \frac{1}{2} \cos(x + 2y) - \frac{1}{6} \cos(x - 2y)$.

Example 17.28. *Solve* $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} = 2e^{2x} + 3x^2 y$. *(S.V.T.U., 2007)*

Solution. Given equation in *symbolic form* is

$$(D^3 - 2D^2 D')z = 2e^{2x} + 3x^2 y$$

Its A.E. is $m^3 - 2m^2 = 0$, whence $m = 0, 0, 2$.

$\therefore$ C.F. $= f_1(y) + x f_2(y) + f_3(y + 2x)$

$$\text{P.I.} = \frac{1}{D^3 - 2D^2 D'} (2e^{2x} + 3x^2 y) = 2\frac{1}{D^3 - 2D^2 D'} e^{2x} + 3\frac{1}{D^3(1 - 2D'/D)} x^2 y$$

$$= 2\frac{1}{2^3 - 2 \cdot 2^2(0)} e^{2x} + \frac{3}{D^3}(1 - 2D'/D)^{-1} x^2 y = \frac{1}{4} e^{2x} + \frac{3}{D^3}\left(1 + \frac{2D'}{D} + \frac{4D'^2}{D^2} + \ldots\right) x^2 y$$

$$= \frac{1}{4} e^{2x} + \frac{3}{D^3}\left(x^2 y + \frac{2}{D} x^2 \cdot 1\right) = \frac{1}{4} e^{2x} + \frac{3}{D^3}\left(x^2 y + \frac{2}{3} x^3\right) \qquad \left[\because \frac{1}{D} f(x) = \int f(x)\, dx\right]$$

$$= \frac{1}{4} e^{2x} + 3y \frac{x^5}{3 \cdot 4 \cdot 5} + 2 \cdot \frac{x^6}{4 \cdot 5 \cdot 6} \qquad \left[\because \frac{1}{D^3} f(x) = \int\left[\int\left(\int f(x)\, dx\right) dx\right] dx\right]$$

$$= \frac{e^{2x}}{4} + \frac{x^5 y}{20} + \frac{x^6}{60}$$

Hence the C.S. is $z = f_1(y) + xf_2(y) + f_3(y + 2x) + \frac{1}{60}(15e^{2x} + 3x^5y + x^6)$.

Example 17.29. *Solve* $r - 4s + 4t = e^{2x+y}$.

Solution. Given equation is $\dfrac{\partial^2 z}{\partial x^2} - 4\dfrac{\partial^2 z}{\partial x \partial y} + 4\dfrac{\partial^2 z}{\partial y^2} = e^{2x+y}$.

i.e., in *symbolic form* $(D^2 - 4DD' + 4D'^2)z = e^{2x+y}$.

Its A.E. is $(m - 2)^2 = 0$, whence $m = 2, 2$.

$\therefore$ C.F. $= f_1(y + 2x) + xf_2(y + 2x)$

$$\text{P.I.} = \frac{1}{(D - 2D')^2} e^{2x+y}$$

The usual rule fails because $(D - 2D')^2 = 0$ for $D = 2$ and $D' = 1$.

$\therefore$ to obtain the P.I., we find from $(D - 2D')u = e^{2x+y}$, the solution

$$u = \int F(x, c - mx)\,dx = \int e^{2x+(c-2x)}\,dx = xe^c = xe^{2x+y} \qquad [\because\ y = c - mx = c - 2x]$$

and from $(D - 2D')z = u = xe^{2x+y}$, the solution

$$z = \int xe^{2x+(c-2x)}dy = \frac{1}{2}x^2e^c = \frac{1}{2}x^2e^{2x+y} \qquad [\because\ y = c - mx = c - 2x]$$

Hence the C.S. is $z = f_1(y + 2x) + xf_2(y + 2x) + \dfrac{1}{2}x^2e^{2x+y}$.

Example 17.30. *Solve* $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = \cos(2x + y)$. *(P.T.U., 2010 ; S.V.T.U., 2009)*

Solution. Given equation in *symbolic form* is $(D^2 + DD' - 6D'^2)z = \cos(2x + y)$

Its A.E. is $m^2 + m - 6 = 0$ whence $m = -3, 2$.

$\therefore$ C.F. $= f_1(y - 3x) + f_2(y + 2x)$.

Since $D^2 + DD' - 6D'^2 = -2^2 - (2)(1) - 6(-1)^2 = 0$

$\therefore$ It is a case of failure and we have to apply the general method.

$$\text{P.I.} = \frac{1}{D^2 + DD' - 6D'^2}\cos(2x + y) = \frac{1}{(D + 3D')(D - 2D')}\cos(2x + y)$$

$$= \frac{1}{D + 3D'}\left[\int \cos(2x + \overline{c - 2x})\,dx\right]_{c \to y+2x} = \frac{1}{D + 3D'}\left[\int \cos c\,dx\right]_{c \to y+2x}$$

$[\because\ y = c - mx = c - 2x$

$$= \frac{1}{D + 3D'}x\cos(y + 2x) = \left[\int x\cos(\overline{c + 3x} + 2x)\,dx\right]_{c \to y-3x} = \left[\int x\cos(5x + c)\,dx\right]_{c \to y-3x}$$

$$= \left[\frac{x\sin(5x + c)}{5} + \frac{\cos(5x + c)}{25}\right]_{c \to y-3x} \qquad \text{[Integrating by parts]}$$

$$= \frac{x}{5}\sin(5x + \overline{y - 3x}) + \frac{1}{25}\cos(5x + \overline{y - 3x}) = \frac{x}{5}\sin(2x + y) + \frac{1}{25}\cos(2x + y)$$

Hence the C.S. is

$$z = f_1(y - 3x) + f_2(y + 2x) + \frac{x}{5}\sin(2x + y) + \frac{1}{25}\cos(2x + y)$$

$$z = f_1(y - 3x) + f_2(y + 2x) + \frac{x}{5}\sin(2x + y) + \frac{1}{25}\cos(2x + y).$$

Example 17.31. *Solve* $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y \cos x.$ (*Anna, 2005 S ; U.P.T.U., 2003*)

or $r + s - 6t = y \cos x.$ (*Bhopal, 2008 ; S.V.T.U., 2008*)

Solution. Its *symbolic form* is $(D^2 + DD' - 6D'^2)z = y \cos x$
and the A.E. is $m^2 + m - 6 = 0$, whence $m = -3, 2$.

$$\therefore \quad \text{C.F.} = f_1(y - 3x) + f_2(y + 2x)$$

$$\text{P.I.} = \frac{1}{(D - 2D')(D + 3D')} y \cos x = \frac{1}{D - 2D'}\left[\int (c + 3x) \cos x \, dx\right]_{c \to y - 3x}$$

$[\because \; y = c - mx = c + 3x]$

$$= \frac{1}{D - 2D'}[(c + 3x) \sin x + 3 \cos x]_{c \to y - 3x} \qquad \text{[Integrating by parts]}$$

$$= \frac{1}{D - 2D'}(y \sin x + 3 \cos x) = \left[\int \{(c - 2x) \sin x \; 3 \cos x\} dx\right]_{c \to y - 2x}$$

$$= [(c - 2x)(-\cos x) - (-2)(-\sin x) + 3 \sin x]_{c \to y + 2x}$$

$$= -y \cos x + \sin x$$

Hence the C.S. is $z = f_1(y - 3x) + f_2(y + 2x) + \sin x - y \cos x$.

Example 17.32. *Solve* $4\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 16 \log (x + 2y).$

Solution. Its *symbolic form* is $4D^2 - 4DD' + D'^2 = 16 \log (x + 2y)$
and the A.E. is $4m^2 - 4m + 1 = 0$, $m = 1/2, 1/2$.

$$\therefore \quad \text{C.F.} = f_1\left(y + \frac{1}{2}x\right) + x f_2\left(y + \frac{1}{2}x\right)$$

$$\text{P.I.} = \frac{1}{(2D - D')^2} 16 \log (x + 2y) = 4\frac{1}{\left(D - \frac{1}{2}D'\right)}\left\{\frac{1}{D - \frac{1}{2}D'}\log (x + 2y)\right\}$$

$$= 4\frac{1}{D - \frac{1}{2}D'}\left[\int \log\left\{x + 2\left(c - \frac{x}{2}\right)\right\} dx\right]_{c \to y + x/2} \qquad [\because \; y = c - mx = c - x/2]$$

$$= 4\frac{1}{D - \frac{1}{2}D'}\left[\int \log (2c) dx\right]_{c \to y + x/2} = 4\frac{1}{D - \frac{1}{2}D'}[x \log (x + 2y)]$$

$$= 4\left[\int \left\{x \log\left[x + 2\left(c - \frac{x}{2}\right)\right]\right\} dx\right]_{c \to y + x/2} = 4\left[\log 2c \int x \, dx\right]_{c \to y + x/2} = 2x^2 \log (x + 2y)$$

Hence the C.S. is $z = f_1\left(y + \frac{x}{2}\right) + x f_2\left(y + \frac{x}{2}\right) + 2x^2 \log (x + 2y)$.

PROBLEMS 17.6

Solve the following equations :

1. $\frac{\partial^3 z}{\partial x^3} - 4\frac{\partial^3 x}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial x \partial y^2} = 0.$

2. $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial y^3} = e^{x + 2y}.$ (*Burdwan, 2003*)

3. $(D^2 - 2DD' + D'^2)z = e^{x + y}.$ (*Bhopal, 2007*)

4. $\frac{\partial^3 z}{\partial x^3} - 4\frac{\partial^3 z}{\partial x^2 \partial y} + 5\frac{\partial^3 z}{\partial x \partial y^2} - 2\frac{\partial^3 z}{\partial y^3} = e^{2x + y}.$ (*Bhopal, 2008*)

5. $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin x.$ (P.T.U., 2009 S)

6. $\frac{\partial^2 y}{\partial t^2} - a^2 \frac{\partial^2 y}{\partial x^2} = E \sin pt.$

7. $\frac{\partial^3 z}{\partial x^3} - \frac{4\partial^3 z}{\partial z^2 \partial y} + 4\frac{\partial^3 z}{\partial x \partial y^2} = 2\sin(3x + 2y).$ (S.V.T.U., 2007)

8. $(D^3 - 7DD'^2 - 6D'^3)z = \cos(x + 2y) + 4.$ (Anna, 2008)

9. $\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = e^{2x-y} + e^{x+y} + \cos(x + 2y).$ (U.P.T.U., 2006)

10. $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y.$ (U.P.T.U., 2003)

11. $(D^2 - DD')z = \cos 2y (\sin x + \cos x).$

12. $(D^2 - D'^2)z = e^{x-y} \sin(x + 2y).$ (Anna, 2009)

13. $(D^2 + 3DD' + 2D'^2)z = 24xy.$

14. $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = x^2 + xy + y^2.$

15. $(D^2 - DD' - 2D'^2)z = (y - 1)e^x.$ (Bhopal, 2006)

16. $(D^3 + D^2D' - DD'^2 - D'^3)z = e^x \cos 2y.$

17. $(D^2 + 2DD' + D'^2)z = 2\cos y - x \sin y.$ (P.T.U., 2005)

17.12 NON-HOMOGENEOUS LINEAR EQUATIONS

If in the equation $\qquad f(D, D')z = F(x, y)$ ...(1)

the polynomial expression $f(D, D')$ is not homogeneous, then (1) is a non-homogeneous linear partial differential equation. As in the case of homogeneous linear partial differential equations, its complete solution = C.F. + P.I.

The methods to find P.I. are the same as those for homogeneous linear equations.

To find the C.F., we factorize $f(D, D')$ into factors of the form $D - mD' - c$. To find the solution of $(D - mD' - c)z = 0$, we write it as $p - mq = cz$ ...(2)

The subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{cz}$$

Its integrals are $y + mx = a$ and $z = be^{cx}$.

Taking $b = \phi(a)$, we get $z = e^{cx}\,\phi(y + mx)$

as the solution of (2). The solution corresponding to various factors added up, give the C.F. of (1).

Example 17.32. *Solve* $(D^2 + 2DD' + D'^2 - 2D - 2D')z = \sin(x + 2y).$ (U.P.T.U., 2004)

Solution. Here $\qquad f(D, D') = (D + D')(D + D' - 2)$

Since the solution corresponding to the factor $D - mD' - c$ is known to be

$$z = e^{cx}\,\phi(y + mx)$$

$\therefore$
$$\text{C.F.} = \phi_1(y - x) + e^{2x} f_2(y - x)$$

$\therefore$
$$\text{P.I.} = \frac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'} \sin(x + 2y)$$

$$= \frac{1}{-1 + 2(-2) + (-4) - 2D - 2D'} \sin(x + 2y)$$

$$= -\frac{1}{2(D + D') + 9} \sin(x + 2y) = -\frac{2(D + D') - 9}{4(D^2 + 2DD' + D'^2) - 81} \sin(x + 2y)$$

$$= \frac{1}{39}[2\cos(x + 2y) - 3\sin(x + 2y)]$$

Hence the complete solution is

$$z = \phi_1(y - x) + e^{2x}\phi_2(y - x) + \frac{1}{39}[2\cos(x + 2y) - 3\sin(x + 2y)].$$

PROBLEMS 17.7

Solve the following equations :

1. $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial x\partial y}+\dfrac{\partial z}{\partial y}-z=e^{-x}$.
2. $(D-D'-1)(D-D'-2)z=e^{2x-y}$.
3. $(D+D'-1)(D+2D'-3)z=4+3x+6y$.
4. $\dfrac{\partial^2 z}{\partial x^2}-\dfrac{\partial^2 z}{\partial x\partial y}+\dfrac{\partial z}{\partial y}=x^2+y^2$. *(Madras, 2000 S)*
5. $(D^2+DD'+D'-1)z=\sin(x+2y)$. *(S.V.T.U., 2009)*
6. $(2DD'+D'^2-3D')z=3\cos(3x-2y)$.

17.13 NON-LINEAR EQUATIONS OF THE SECOND ORDER

We now give a method due to *Monge**, for integrating the equation $Rr+Ss+Tt=V$ ...(1)
in which R, S, T, V are functions of x, y, z, p and q.

Since $dp=\dfrac{\partial p}{\partial x}dx+\dfrac{\partial p}{\partial y}dy=rdx+tdy$, and $dq=sdx+tdy$,

we have $r=(dp-tdy)/dx$ and $t=(dq-sdx)/dy$.

Substituting these values of r and t in (1), and rearranging the terms, we get

$$(Rdpdy+Tdqdx-Vdxdy)-s(Rdy^2-Sdydx+Tdx^2)=0 \quad ...(2)$$

Let us consider the equations

$$Rdy^2-Sdydx+Tdx^2=0 \quad ...(3)$$

$$Rdpdy+Tdqdx-Vdxdy=0 \quad ...(4)$$

which are known as *Monge's equations.*

Since (3) can be factorised, we obtain its integral first. In case the factors are different, we may get two distinct integrals of (3). Either of these together with (4) will give an integral of (4). If need be, we may also use the relation $dz=pdx+qdy$ while solving (3) and (4).

Let $u(x,y,z,p,q)=a$ and $v(x,y,z,p,q)=b$ be the integrals of (3) and (4) respectively. Then $u=a$, $v=b$ evidently constitute a solution of (2) and therefore, of (1) also. Taking $b=\phi(a)$, we find a general solution of (1) to be $v=\phi(u)$, which should be further integrated by methods of first order equations.

Example 17.34. *Solve $(x-y)(xr-xs-ys+yt)=(x+y)(p-q)$.* *(S.V.T.U., 2007)*

Solution. Monge's equations are

$$xdy^2+(x+y)dy\,dx+ydx^2=0 \quad ...(i)$$

$$xdpdy+ydqdx-\frac{x+y}{x-y}(p-q)\,dydx=0 \quad ...(ii)$$

(*i*) may be factorised as $(xdy+ydx)(dx+dy)=0$ whose integrals are $xy=c$ and $x+y=c$.

Taking $xy=c$ and dividing each term of (*ii*) by xdy or its equivalent $-ydx$, we get

$$dp-dq-\frac{dx-dy}{x-y}(p-q)=0 \quad \text{or} \quad \frac{d(p-q)}{p-q}-\frac{d(x-y)}{x-y}=0$$

This gives on integration $(p-q)/(x-y)=c$.

Hence a first integral of the given equation is $p-q=(x-y)\,\phi(xy)$ which is a Lagrange's linear equation. Its subsidiary equations are

$$\frac{dx}{1}=\frac{dy}{-1}=\frac{dz}{(x-y)\phi(xy)}$$

From the first two equations, we have $x+y=a$

Using this, we have

$$dz=-\phi(ax-x^2)\,.\,(a-2x)\,dx \quad \text{which gives } z=\phi_1(ax-x^2)+b$$

Writing $b=\phi_2(a)$ and $a=x+y$, we get

$$z=\phi_1(xy)+\phi_2(x+y).$$

* Named after *Gaspard Monge* (1746–1818), Professor at Paris.

Obs. Had we started with the integral $x + y = c$ and divided each term of (*ii*) by dx or $-dy$, we would have arrived at the same solution.

Example 17.35. *Solve* $y^2r - 2ys + t = p + 6y$. (*Osmania, 2002*)

Solution. Monge's equations are $y^2dy^2 + 2ydydx + dx^2 = 0$ ...(*i*)

and $$y^2dpdy + dqdx - (p + 6y)dydx = 0 \quad ...(ii)$$

(*i*) gives $$(ydy + dx)^2 = 0 \text{ i.e. } y^2 + 2x = c \quad ...(iii)$$

Putting $ydy = -dx$ in (*ii*), we get

$$ydp - dq + (p + 6y)dy = 0 \quad \text{or} \quad (ydp + pdy) - dq + 6ydy = 0$$

whose integral is $$py - q + 3y^2 = a$$

Combining this with (*iii*), we get the integral $py - q + 3y^2 = \phi(y^2 + 2x)$

The subsidiary equations for this Lagrange's linear equation are

$$\frac{dx}{y} = \frac{dy}{-1} = \frac{dz}{\phi(y^2+2x)-3y^2}$$

From the first two equations, we have $y^2 + 2x = c$

Using this, we have $dz + [\phi(c) - 3y^2]\, dy = 0$

whose solution is $$z + y\phi(c) - y^3 = b.$$

Hence the required solution is $z = y^3 - y\phi(y^2 + 2x) + \psi(y^2 + 2x)$.

PROBLEMS 17.8

Solve :

1. $(q + 1)s = (p + 1)t$.
2. $r - t\cos^2 x + p\tan x = 0$.
3. $2x^2r - 5xys + 2y^2t + 2(px + qy) = 0$. (*J.N.T.U., 2006*)
4. $xy(t - r) + (x^2 - y^2)(s - 2) = py - qx$.
5. $q^2r - 2pqs + p^2t = pq^2$.
6. $(1 + q)^2r - 2(1 + p + q + pq)s + (1 + p)^2t = 0$.

17.14 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 17.9

Fill up the blanks or choose the correct answer in each of the following problems :

1. The equation $\dfrac{\partial^2 z}{\partial x^2} + 2xy\left(\dfrac{\partial z}{\partial x}\right)^2 + \dfrac{\partial z}{\partial y} = 5$ is of order .... and degree ....
2. The complementary function of $(D^2 - 4DD' + 4D'^2)z = x + y$ is ......
3. The solution of $\dfrac{\partial^2 z}{\partial y^2} = \sin(xy)$ is ......
4. A solution of $(y - z)p + (z - x)q = x - y$ is ....
5. The particular integral of $(D^2 + DD')z = \sin(x + y)$ is ....
6. The partial differential equation obtained from $z = ax + by + ab$ by eliminating a and b is ......
7. Solution of $\sqrt{p} + \sqrt{q} = 1$ is ......
8. Solution of $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$ is ......
9. Solution of $p - q = \log(x + y)$.
10. The order of the partial differential equation obtained by eliminating f from $z = f(x^2 + y^2)$, is ......
11. The solution of $x\,\partial z/\partial x = 2x + y$ is
12. By eliminating a and b from $z = a(x + y) + b$, the p.d.e. formed is ......
13. The solution of $[D^3 - 3D^2D' + 2DD'^2]z = 0$ is ......
14. By eliminating the arbitrary constants from $z = a^2x + ay^2 + b$, the partial differential equation formed is ...... (*Anna, 2008*)
15. A solution of $u_{xy} = 0$ is of the form ....
16. If $u = x^2 + t^2$ is a solution of $c^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$, then $c =$ ......

17. The general solution of $u_{xx} = xy$ is ....

18. The complementary function of $r - 7s + 6t = e^{x+y}$ is ....

19. The solution of $xp + yq = z$ is

(*i*) $f(x^2, y^2) = 0$ (*ii*) $f(xy, yz)$ (*iii*) $f(x, y) = 0$ (*iv*) $f\left(\frac{x}{y}, \frac{y}{z}\right) = 0$.

20. The solution of $(y - z)p + (z - x)q = x - y$, is

(*i*) $f(x^2 + y^2 + z^2) = xyz$ (*ii*) $f(x + y + z) = xyz$

(*iii*) $f(x + y + z) = x^2 + y^2 + z^2$ (*iv*) $f(x^2 + y^2 + z^2, xyz) = 0$.

21. The partial differential equation from $z = (c + x)^2 + y$ is

(*i*) $z = \left(\frac{\partial z}{\partial x}\right)^2 + y$ (*ii*) $z = \left(\frac{\partial z}{\partial y}\right)^2 + y$ (*iii*) $z = \frac{1}{4}\left(\frac{\partial z}{\partial x}\right)^2 + y$ (*iv*) $z = \frac{1}{4}\left(\frac{\partial z}{\partial y}\right)^2 + y$.

22. The solution of $p + q = z$ is

(*i*) $f(xy, y \log z) = 0$ (*ii*) $f(x + y, y + \log z) = 0$

(*iii*) $f(x - y, y - \log z) = 0$ (*iv*) None of these.

23. Particular integral of $(2D^2 - 3DD' + D'^2)z = e^{x+2y}$ is

(*i*) $\frac{1}{2}e^{x+2y}$ (*ii*) $-\frac{x}{2}e^{x+2y}$ (*iii*) xe^{x+2y} (*iv*) x^2e^{x+2y}.

24. The solution of $\frac{\partial^3 z}{\partial x^3} = 0$ is

(*i*) $z = (1 + x + x^2) f(y)$ (*ii*) $z = (1 + y + y^2) f(x)$

(*iii*) $z = f_1(x) + y f_2(x) + y^2 f_3(x)$ (*iv*) $z = f_1(y) + x f_2(y) + x^2 f_3(y)$.

25. Particular integral of $(D^2 - D'^2)z = \cos(x + y)$ is

(*i*) $x \cos(x + y)$ (*ii*) $\frac{x}{2}\cos(x + y)$ (*iii*) $x \sin(x + y)$ (*iv*) $\frac{x}{2}\sin(x + y)$

26. The solution of $\partial^2 z/\partial x^2 = \partial^2 z/\partial y^2$ is

(*i*) $z = f_1(y + x) + f_2(y - x)$ (*ii*) $z = f_1(y + x) + f_1(y - x)$

(*iii*) $z = f(x^2 - y^2)$ (*iv*) $z = f(x^2 + y^2)$.

27. $xu_x + yu_y = u^2$ is a non-linear partial differential equation. (True or False)

28. $xu_x + u_{xx} = 0$ is a non-linear partial differential equation. (True or False)

29. $u = x^2 - y^2$ is a solution of $u_{xx} + u_{yy} = 0$. (True or False)

30. $u = e^{-t} \sin x$ is a solution of $\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0$. (True or False)

31. $x\frac{\partial u}{\partial x} + t\frac{\partial u}{\partial t} = 2u$ is an ordinary differential equation. (True or False)

CHAPTER 18

Applications of Partial Differential Equations

18.1 INTRODUCTION

In physical problems, we always seek a solution of the differential equation which satisfies some specified conditions known as the boundary conditions. The differential equation together with these boundary conditions, constitute a *boundary value problem*.

In problems involving ordinary differential equations, we may first find the general solution and then determine the arbitrary constants from the initial values. But the same process is not applicable to problems involving partial differential equations for the general solution of a partial differential equation contains arbitrary functions which are difficult to adjust so as to satisfy the given boundary conditions. Most of the boundary value problems involving linear partial differential equations can be solved by the following method.

18.2 METHOD OF SEPARATION OF VARIABLES

It involves a solution which breaks up into a product of functions each of which contains only one of the variables. The following example explains this method :

Example 18.1. *Solve (by the method of separation of variables) :*

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0. \qquad \textit{(P.T.U., 2009 S ; Bhopal 2008 ; U.P.T.U., 2005)}$$

Solution. Assume the trial solution $z = X(x)Y(y)$ ...(*i*)

where X is a function of x alone and Y that of y alone.

Substituting this value of z in the given equation, we have

$$X''Y - 2X'Y + XY' = 0 \qquad \text{where } X' = \frac{dX}{dx},\ Y' = \frac{dY}{dy} \text{ etc.}$$

Separating the variables, we get $\dfrac{X'' - 2X'}{X} = -\dfrac{Y'}{Y}$ ...(*ii*)

Since x and y are independent variables, therefore, (*ii*) can only be true if each side is equal to the same constant, a (say).

$\therefore \qquad \dfrac{X'' - 2X'}{X} = a$, *i.e.* $X'' - 2X' - aX = 0$ ...(*iii*)

and $\qquad -Y'/Y = a$, *i.e.*, $Y' + aY = 0$ ...(*iv*)

To solve the ordinary linear equation (*iii*), the auxiliary equation is

$$m^2 - 2m - a = 0, \quad \text{whence } m = 1 \sqrt{(1 + a)}.$$

$\therefore$ the solution of (*iii*) is $X = c_1 e^{[1 + \sqrt{(1+a)}]x} + c_2 e^{[1 - \sqrt{(1+a)}]x}$

and the solution of (*iv*) is $\qquad Y = c_3 e^{-ay}$.

Substituting these values of X and Y in (*i*), we get

$$z = \{c_1 e^{[1 + \sqrt{(1+a)}]x} + c_2 e^{[1 - \sqrt{(1+a)}]x}\} \cdot c_3 e^{-ay}$$

i.e., $$z = \{k_1 e^{[1 + \sqrt{(1+a)}]x} + k_2 e^{[1 - \sqrt{(1+a)}]x}\}\, e^{-ay}$$

which is the required complete solution.

Obs. In practical problems, the unknown constants a, k_1, k_2 are determined from the given boundary conditions.

Example 18.2. *Using the method of separation of variables, solve* $\dfrac{\partial u}{\partial x} = 2\dfrac{\partial u}{\partial t} + u$ *where* $u(x, 0) = 6e^{-3x}$.

(V.T.U., 2009 ; Kurukshetra, 2006 ; Kerala, 2005)

Solution. Assume the solution $u(x, t) = X(x)T(t)$

Substituting in the given equation, we have

$$X'T = 2XT' + XT \quad \text{or} \quad (X' - X)\,T = 2XT'$$

or $$\frac{X' - X}{2X} = \frac{T'}{T} = k \text{ (say)}$$

$\therefore \qquad X' - X - 2kX = 0$ or $\dfrac{X'}{X} = 1 + 2k$ ...(*i*) and $\dfrac{T'}{T} = k$ ...(*ii*)

Solving (*i*), $\qquad \log X = (1 + 2k)x + \log c$ or $X = ce^{(1+2k)x}$

From (*ii*), $\qquad \log T = kt + \log c'$ or $T = c'e^{kt}$

Thus $\qquad u(x, t) = XT = cc'\, e^{(1+2k)x}\, e^{kt}$ ...(*iii*)

Now $\qquad 6e^{-3x} = u(x, 0) = cc'\, e^{(1+2k)x}$

$\therefore \qquad cc' = 6$ and $1 + 2k = -3$ or $k = -2$

Substituting these values in (*iii*), we get

$u = 6e^{-3x}\, e^{-2t}$ *i.e.*, $u = 6e^{-(3x+2t)}$ which is the required solution.

PROBLEMS 18.1

Solve the following equations by the method of *separation of variables* :

1. $py^3 + qx^2 = 0$. (*V.T.U., 2011 ; S.V.T.U., 2008*)
2. $x^2 \dfrac{\partial u}{\partial x} + y^2 \dfrac{\partial u}{\partial y} = 0$. (*V.T.U., 2008*)
3. $\dfrac{\partial u}{\partial x} = 4\dfrac{\partial u}{\partial y}$, given that $u(0, y) = 8e^{-3y}$. (*J.N.T.U., 2006*)
4. $4\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 3u$, given $u = 3e^{-y} - e^{-5y}$ when $x = 0$. (*S.V.T.U., 2008*)
5. $3\dfrac{\partial u}{\partial x} + 2\dfrac{\partial u}{\partial y} = 0$, $u(x, 0) = 4e^{-x}$. (*V.T.U., 2008 S*)
6. Find a solution of the equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial y} + 2u$ in the form $u = f(x)\, g(y)$. Solve the equation subject to the conditions $u = 0$ and $\partial u/\partial x = 1 + e^{-3y}$, when $x = 0$ for all values of y. (*Andhra, 2000*)

18.3 PARTIAL DIFFERENTIAL EQUATIONS OF ENGINEERING

A number of problems in engineering give rise to the following well-known partial differential equations :

(*i*) *Wave equation* : $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$.

(*ii*) *One dimensional heat flow equation* : $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$.

(*iii*) *Two dimensional heat flow equation* which in steady state becomes the two dimensional *Laplace's equation* : $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

(*iv*) *Transmission line equations.*

(*v*) *Vibrating membrane.* Two dimensional wave equation.

(*vi*) *Laplace's equation* in three dimensions.

Besides these, the partial differential equations frequently occur in the theory of Elasticity and Hydraulics.

Starting with the method of separation of variables, we find their solutions subject to specific boundary conditions and the combination of such solution gives the desired solution. Quite often a certain condition is not applicable. In such cases, the most general solution is written as the sum of the particular solutions already found and the constants are determined using Fourier series so as to satisfy the remaining conditions.

18.4 VIBRATIONS OF A STRETCHED STRING—WAVE EQUATION

Consider a tightly stretched elastic string of length l and fixed ends A and B and subjected to constant tension T (Fig. 18.1). The tension T will be considered to be large as compared to the weight of the string so that the effects of gravity are negligible.

Let the string be released from rest and allowed to vibrate. We shall study the subsequent motion of the string, with no external forces acting on it, assuming that each point of the string makes small vibrations at right angles to the equilibrium position AB, of the string entirely in one plane.

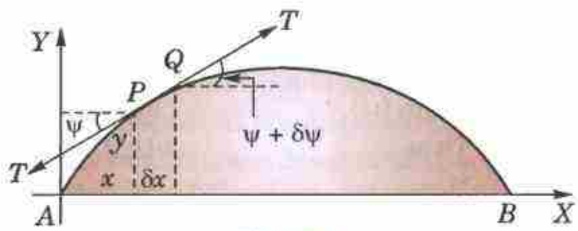

Fig. 18.1

Taking the end A as the origin, AB as the x-axis and AY perpendicular to it as the y-axis ; so that the motion takes place entirely in the xy-plane. Figure 18.1 shows the string in the position APB at time t. Consider the motion of the element PQ of the string between its points $P(x, y)$ and $Q(x + \delta x, y + \delta y)$, where the tangents make angles ψ and $\psi + \delta\psi$ with the x-axis. Clearly the element is moving upwards with the acceleration $\partial^2 y/\partial t^2$. Also the vertical component of the force acting on this element.

$$= T \sin(\psi + \delta\psi) - T \sin\psi = T[\sin(\psi + \delta\psi) - \sin\psi]$$

$$= T[\tan(\psi + \delta\psi) - \tan\psi], \text{ since } \psi \text{ is small} = T\left[\left\{\frac{\partial y}{\partial x}\right\}_{x+\delta x} - \left\{\frac{\partial y}{\partial x}\right\}_x\right]$$

If m be the mass per unit length of the string, then by Newton's second law of motion, we have

$$m\delta x \,.\, \frac{\partial^2 y}{\partial t^2} = T\left[\left\{\frac{\partial y}{\partial x}\right\}_{x+\delta x} - \left\{\frac{\partial y}{\partial x}\right\}_x\right] \quad i.e., \quad \frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\left[\frac{\left\{\frac{\partial y}{\partial x}\right\}_{x+\delta x} - \left\{\frac{\partial y}{\partial x}\right\}_x}{\delta x}\right]$$

Taking limits as $Q \to P$ *i.e.*, $dx \to 0$, we have $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$, where $c^2 = \dfrac{T}{m}$ ...(1)

This is the partial differential equation giving the transverse vibrations of the string. It is also called the one dimensional *wave equation*.

(2) Solution of the wave equation. Assume that a solution of (1) is of the form

$z = X(x)T(t)$ where X is a function of x and T is a function of t only.

Then $$\frac{\partial^2 y}{\partial t^2} = X \,.\, T'' \text{ and } \frac{\partial^2 y}{\partial x^2} = X'' \,.\, T$$

Substituting these in (1), we get $XT'' = c^2 X''T$ *i.e.*, $\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}$ ...(2)

Clearly the left side of (2) is a function of x only and the right side is a function of t only. Since x and t are independent variables, (2) can hold good if each side is equal to a constant k (say). Then (2) leads to the ordinary differential equations :

$$\frac{d^2X}{dx^2} - kX = 0 \quad ...(3) \qquad \text{and} \qquad \frac{d^2T}{dt^2} - kc^2T = 0 \quad ...(4)$$

Solving (3) and (4), we get

(*i*) *When k is positive* and $= p^2$, say $X = c_1e^{px} + c_2e^{-px}$; $T = c_3e^{cpt} + c_4e^{-cpt}$.

(*ii*) *When k is negative* and $= -p^2$ say $X = c_5 \cos px + c_6 \sin px$; $T = c_7 \cos cpt + c_8 \sin cpt$.

(*iii*) *When k is zero*. $X = c_9x + c_{10}$; $T = c_{11}t + c_{12}$.

Thus the various possible solutions of wave-equation (1) are

$$y = (c_1e^{px} + c_2e^{-px})(c_3e^{cpt} + c_4e^{-cpt}) \quad ...(5)$$

$$y = (c_5 \cos px + c_6 \sin px)(c_7 \cos cpt + c_8 \sin cpt) \quad ...(6)$$

$$y = (c_9x + c_{10})(c_{11}t + c_{12}) \quad ...(7)$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. As we will be dealing with problems on vibrations, y must be a periodic function of x and t. Hence their solution must involve trigonometric terms. Accordingly the solution given by (6), *i.e.*, of the form

$$y = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt) \quad ...(8)$$

is the only suitable solution of the *wave equation*. (*Bhopal, 2008*)

Example 18.3. *A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $y = a \sin (\pi x/l)$ from which it is released at time $t = 0$. Show that the displacement of any point at a distance x from one end at time t is given by*

$y(x, t) = a \sin (\pi x/l) \cos (\pi ct/l)$. (*V.T.U., 2010 ; S.V.T.U., 2008 ; Kerala, 2005 ; U.P.T.U., 2004*)

Solution. The vibration of the string is given by $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ ...(*i*)

As the end points of the string are fixed, for all time,

$$y(0, t) = 0 \quad ...(ii) \qquad \text{and} \qquad y(l, t) = 0 \quad ...(iii)$$

Since the initial transverse velocity of any point of the string is zero,

therefore, $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad ...(iv)$$

Also $$y(x, 0) = a \sin (\pi x/l) \quad ...(v)$$

Now we have to solve (*i*) subject to the *boundary conditions* (*ii*) and (*iii*) and *initial conditions* (*iv*) and (*v*). Since the vibration of the string is periodic, therefore, the solution of (*i*) is of the form

$$y(x, t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt) \quad ...(vi)$$

By (*ii*), $y(0, t) = C_1(C_3 \cos cpt + C_4 \sin cpt) = 0$

For this to be true for all time, $C_1 = 0$.

Hence $$y(x, t) = C_2 \sin px(C_3 \cos cpt + C_4 \sin cpt) \quad ...(vii)$$

and $$\frac{\partial y}{\partial t} = C_2 \sin px \{C_3(-cp \,.\, \sin cpt) + C_4(cp \,.\, \cos cpt)\}$$

$\therefore$ By (*iv*), $\left(\frac{\partial y}{\partial t}\right)_{t=0} = C_2 \sin px \,.\, (C_4\, cp) = 0$, whence $C_2C_4cp = 0$.

If $C_2 = 0$, (*vii*) will lead to the trivial solution $y(x, t) = 0$,

$\therefore$ the only possibility is that $C_4 = 0$.

Thus (*vii*) becomes $y(x, t) = C_2C_3 \sin px \cos cpt$ ...(*viii*)

$\therefore$ By (*iii*), $\quad y(l, t) = C_2C_3 \sin pl \cos cpt = 0$ for all t.

Since C_2 and $C_3 \neq 0$, we have $\sin pl = 0$. $\therefore$ $pl = n\pi$, *i.e.*, $p = n\pi/l$, where n is an integer.

Hence (*i*) reduces to $\quad y(x, t) = C_2C_3 \sin \dfrac{n\pi x}{l} \cos \dfrac{n\pi ct}{l}$.

[These are the solutions of (*i*) satisfying the boundary conditions. These functions are called the **eigen functions** corresponding to the **eigen values** $\lambda_n = cn\pi/l$ of the vibrating string. The set of values $\lambda_1, \lambda_2, \lambda_3, \ldots$ is called its **spectrum.**]

Finally, imposing the last condition (*v*), we have $y(x, 0) = C_2C_3 \sin \dfrac{n\pi x}{l} = a \sin \dfrac{\pi x}{l}$

which will be satisfied by taking $C_2C_3 = a$ and $n = 1$.

Hence the required solution is $y(x, t) = a \sin \dfrac{\pi x}{l} \cos \dfrac{\pi ct}{l}$ ...(*ix*)

Obs. We have from (*ix*) $\dfrac{\partial^2 y}{\partial t^2} = -a\left(\dfrac{\pi c}{l}\right)^2 \sin \dfrac{\pi x}{l} \cos \dfrac{\pi ct}{l} = -\left(\dfrac{\pi c}{l}\right)^2 y$.

This shows that the motion of each point $y(x, t)$ of the string is simple harmonic with period $= 2\pi/(\pi c/l)$, *i.e.*, $2l/c$.

Thus we can look upon (*ix*) as a sine wave $y = y_0 \sin (\pi x/l)$ of wave length l, wave-velocity c and amplitude $y_0 = a \cos (\pi ct/l)$ which varies harmonically with time t. Whatever t may be, $y = 0$ when $x = 0, l, 2l, 3l$ etc. and these points called *nodes*, remain undisturbed during wave motion. Thus (*ix*) represents a *stationary sine wave* of varying amplitudes whose frequency is $c/2l$. Such waves often occur in electrical and mechanical vibratory systems.

Example 18.4. *A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y = y_0 \sin^3 (\pi x/l)$. If it is released from rest from this position, find the displacement $y(x, t)$.*

(*Rajasthan, 2006 ; V.T.U., 2003 ; J.N.T.U., 2002*)

Solution. The equation of the vibrating string is $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ ...(*i*)

The boundary conditions are $\quad y(0, t) = 0, y(l, t) = 0$ ...(*ii*)

Also the initial conditions are $y(x, 0) = y_0 \sin^3 \left(\dfrac{\pi x}{l}\right)$ ...(*iii*)

and $\quad \left(\dfrac{\partial y}{\partial t}\right)_{t=0} = 0$ ...(*iv*)

Since the vibration of the string is periodic, therefore, the solution of (*i*) is of the form

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

By (*ii*), $\quad y(0, t) = c_1(c_3 \cos cpt + c_4 \sin cpt) = 0$

For this to be true for all time, $c_1 = 0$.

$\therefore \quad y(x, t) = c_2 \sin px\,(c_3 \cos cpt + c_4 \sin cpt)$

Also by (*ii*), $\quad y(l, t) = c_2 \sin pl\,(c_3 \cos cpt + c_4 \sin cpt) = 0$ for all t.

This gives $pl = n\pi$ or $p = n\pi/l$, n being an integer.

Thus $\quad y(x, t) = c_2 \sin \dfrac{n\pi x}{l}\left(c_3 \cos \dfrac{cn\pi t}{l} + c_4 \sin \dfrac{cn\pi t}{l}\right)$ ...(*v*)

$$\frac{\partial y}{\partial t} = \left(c_2 \sin \frac{n\pi x}{l}\right)\frac{cn\pi}{l}\left(-c_3 \sin \frac{cn\pi t}{l} + c_4 \cos \frac{cn\pi t}{l}\right)$$

By (*iv*), $\quad \left(\dfrac{\partial y}{\partial t}\right)_{t=0} = \left(c_2 \sin \dfrac{n\pi x}{l}\right)\dfrac{cn\pi}{l} . c_4 = 0$, *i.e.* $c_4 = 0$.

Thus (*v*) becomes $y(x, t) = c_2c_3 \sin \dfrac{n\pi x}{l} \cos \dfrac{n\pi ct}{l} = b_n \sin \dfrac{n\pi x}{l} \cos \dfrac{n\pi ct}{l}$

Adding all such solutions the general solution of (*i*) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \qquad \ldots(vi)$$

$\therefore$ from (iii), $$y_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

or $$y_0 \left\{ \frac{3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}}{4} \right\} = b_1 \sin \frac{\pi x}{l} + b_2 \sin \frac{2\pi x}{l} + b_3 \sin \frac{3\pi x}{l} + \ldots$$

Comparing both sides, we have

$$b_1 = 3\, y_0/4,\ b_2 = 0,\ b_3 = -\, y_0/4,\ b_4 = b_5 = \ldots = 0.$$

Hence from (vi), the desired solution is

$$y(x, t) = \frac{3y_0}{4} \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l} \cos \frac{3\pi ct}{l}.$$

Example 18.5. *A tightly stretched flexible string has its ends fixed at $x = 0$ and $x = l$. At time $t = 0$, the string is given a shape defined by $F(x) = \mu x(l - x)$, where μ is a constant, and then released. Find the displacement of any point x of the string at any time $t > 0$.*

(Bhopal, 2008 ; Madras, 2006 ; J.N.T.U., 2005 ; P.T.U., 2005)

Solution. The equation of the string is $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ ...(i)

The boundary conditions are $y(0, t) = 0,\ y(l, t) = 0$ ...(ii)

Also the initial conditions are $y(x, 0) = \mu x(l - x)$ ...(iii)

and $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad \text{...(iv)}$$

The solution of (i) is of the form

$$y(x, t) = (c_1 \cos px + c_2 \sin px)\,(c_3 \cos cpt + c_4 \sin cpt)$$

By (ii), $$y(0, t) = c_1(c_3 \cos cpt + c_4 \sin cpt) = 0$$

For this to be true for all time, $c_1 = 0$.

$\therefore$ $$y(x, t) = c_2 \sin px\,(c_3 \cos cpt + c_4 \sin cpt)$$

Also by (ii) $$y(l, t) = c_2 \sin pl(c_3 \cos cpt + c_4 \sin cpt) = 0 \text{ for all } t.$$

This gives $pl = n\pi$ or $p = n\pi/l$, n being an integer.

Thus $$y(x, t) = c_2 \sin \frac{n\pi x}{l} \left(c_3 \cos \frac{n\pi ct}{l} + c_4 \sin \frac{n\pi ct}{l} \right) \quad \text{...(v)}$$

$$\frac{\partial y}{\partial t} = \left(c_2 \sin \frac{n\pi x}{l} \right) \frac{n\pi c}{l} \left(-\, c_3 \sin \frac{n\pi ct}{l} + c_4 \cos \frac{n\pi ct}{l} \right)$$

$\therefore$ by (iv) $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \left(c_2 \sin \frac{n\pi x}{l} \right) \frac{n\pi c}{l} \,.\, c_4 = 0$$

Thus (v) becomes $$y(x, t) = c_2 c_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} = b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$$

Adding all such solutions, the general solution of (i) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \quad \text{...(vi)}$$

From (iii), $$\mu(lx - x^2) = y(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where $$b_n = \frac{2}{l} \int_0^l \mu(lx - x^2) \sin \frac{n\pi x}{l}\, dx, \text{ by Fourier half-range sine series}$$

$$= \frac{2\mu}{l} \left\{ \left| (lx - x^2)\left(-\frac{\cos n\pi x/l}{n\pi/l} \right) \right|_0^l - \int_0^l (l - 2x)\left(-\frac{\cos n\pi x/l}{n\pi/l} \right) dx \right\}$$

$$= \frac{2\mu}{l} \cdot \frac{1}{n\pi} \left\{ \int_0^l (l - 2x) \frac{\cos n\pi x}{l} dx \right\} = \frac{2\mu}{n\pi} \left\{ \left| (l - 2x) \frac{\sin n\pi x/l}{n\pi/l} \right|_0^l - \int_0^l (-2) \frac{\sin n\pi x/l}{n\pi/l} dx \right\}$$

$$= \frac{2\mu}{n\pi} \cdot \frac{2l}{n\pi} \int_0^l \sin \frac{n\pi x}{l} dx = \frac{4\mu l}{n^2\pi^2} \left| \frac{-\cos n\pi x/l}{n\pi/l} \right|_0^l = \frac{4\mu l^2}{n^3\pi^3} \{1 - (-1)^n\}$$

Hence from (vi), the desired solution is

$$y(x, t) = \frac{4\mu l^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^3} \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$$

$$= \frac{8\mu l^2}{\pi^3} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^3} \sin \frac{(2m-1)\pi}{l} x \cos \frac{(2m-1)\pi ct}{l}.$$

Example 18.6. *A tightly stretched string of length l with fixed ends is initially in equilibrium position. It is set vibrating by giving each point a velocity $v_0 \sin^3 \pi x/l$. Find the displacement y(x, t).*

(S.V.T.U., 2008 ; V.T.U., 2008 ; U.P.T.U., 2006)

Solution. The equation of the vibrating string is $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ ...(i)

The boundary conditions are $y(0, t) = 0, y(l, t) = 0$ ...(ii)

Also the initial conditions are $y(x, 0) = 0$ ...(iii)

and $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin^3 \frac{\pi x}{l} \qquad ...(iv)$$

Since the vibration of the string is periodic, therefore, the solution of (i) is of the form

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

By (ii), $y(0, t) = c_1(c_3 \cos cpt + c_4 \sin cpt) = 0$

For this to be true for all time $c_1 = 0$.

$\therefore$ $y(x, t) = c_2 \sin px\,(c_3 \cos cpt + c_4 \sin cpt)$

Also $y(l, t) = c_2 \sin pl\,(c_3 \cos cpt + c_4 \sin cpt) = 0$ for all t.

This gives $pl = n\pi$ or $p = \dfrac{n\pi}{l}$, n being an integer.

Thus $$y(x, t) = c_2 \sin \frac{n\pi x}{l} \left(c_3 \cos \frac{cn\pi}{l} t + c_4 \sin \frac{cn\pi}{l} t \right)$$

By (iii), $0 = c_2 c_3 \sin \dfrac{n\pi x}{l}$ for all x *i.e.*, $c_2 c_3 = 0$

$\therefore$ $y(x, t) = b_n \sin \dfrac{n\pi x}{l} \sin \dfrac{cn\pi t}{l}$ where $b_n = c_2 c_4$

Adding all such solutions, the general solution of (i) is

$$y(x, t) = \sum b_n \sin \frac{n\pi x}{l} \sin \frac{cn\pi t}{l} \qquad ...(v)$$

Now $$\frac{\partial y}{\partial t} = \sum b_n \sin \frac{n\pi x}{l} \cdot \frac{cn\pi}{l} \cos \frac{cn\pi t}{l}$$

By (iv), $$v_0 \sin^3 \frac{\pi x}{l} = \left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum \frac{cn\pi}{l} b_n \sin \frac{n\pi x}{l}$$

or $$\frac{v_0}{4}\left(3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}\right) = \sum \frac{cn\pi}{l} b_n \sin \frac{n\pi x}{l} \qquad [\because \sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta]$$

$$= \frac{c\pi}{l} b_1 \sin \frac{\pi x}{l} + \frac{2c\pi}{l} b_2 \sin \frac{2\pi x}{l} + \frac{3c\pi}{l} b_3 \sin \frac{3\pi x}{l} + \ldots$$

Equating coefficients from both sides, we get

$$\frac{3v_0}{4} = \frac{c\pi}{l} b_1, \quad 0 = \frac{2c\pi}{l} b_2, \quad -\frac{v_0}{4} = \frac{3c\pi}{l} b_3, \ldots$$

$$\therefore \qquad b_1 = \frac{3lv_0}{4c\pi}, \quad b_3 = -\frac{lv_0}{12c\pi}, \quad b_2 = b_4 = b_3 = \ldots = 0$$

Substituting in (v), the desired solution is

$$y = \frac{lv_0}{12c\pi}\left(9 \sin\frac{\pi x}{l} \sin\frac{c\pi t}{l} - \sin\frac{3\pi x}{l} \sin\frac{3c\pi t}{l}\right).$$

Example 18.7. *A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is vibrating by giving to each of its points a velocity $\lambda x(l - x)$, find the displacement of the string at any distance x from one end at any time t.* (Anna, 2009 ; U.P.T.U., 2002)

Solution. The equation of the vibrating string is $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ ...(*i*)

The boundary conditions are $y(0, t) = 0, y(l, t) = 0$ ...(*ii*)

Also the initial conditions are $y(x, 0) = 0$ ...(*iii*)

and $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \lambda x(l - x) \qquad \ldots(iv)$$

As in example 18.6, the general solution of (*i*) satisfying the conditions (*ii*) and (*iii*) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} . \sin\frac{n\pi ct}{l} \qquad \ldots(v)$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} . \cos\frac{n\pi ct}{l} . \left(\frac{n\pi c}{l}\right)$$

By (*iv*), $$\lambda x(l - x) = \left(\frac{\partial y}{\partial t}\right)_{t=0} = \frac{\pi c}{l} \sum_{n=1}^{\infty} nb_n \sin\frac{n\pi x}{l}$$

$$\therefore \qquad \frac{\pi c}{l} nb_n = \frac{2}{l}\int_0^l \lambda x(l - x) \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{2\lambda}{l}\left| (lx - x^2)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - (l - 2x)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right) + (-2)\left(\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right)\right|_0^l$$

$$= \frac{4\lambda l^2}{n^3\pi^3}(1 - \cos n\pi) = \frac{4\lambda l^2}{n^3\pi^3}\,[1 - (-1)^n]$$

or $$b_n = \frac{4\lambda l^3}{c\pi^4 n^4}[1 - (-1)^n] = \frac{8\lambda l^3}{c\pi^4(2m - 1)^4} \text{ taking } n = 2m - 1.$$

Hence, from (v), the desired solution is

$$y = \frac{8\lambda l^3}{c\pi^4} \sum_{m=1}^{\infty} \frac{1}{(2m - 1)^4} \sin\frac{(2m - 1)\pi x}{l} \sin\frac{(2m - 1)\pi ct}{l}.$$

Example 18.8. *The points of trisection of a string are pulled aside through the same distance on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string at subsequent time and show that the mid-point of the string always remains at rest.* (Kerala, 2005)

Solution. Let B and C be the points of the trisection of the string $OA(= l)$ (Fig. 18.2). Initially the string is held in the form $OB'C'A$, where $BB' = CC' = a$(say).

The displacement $y(x, t)$ of any point of the string is given by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad ...(i)$$

and the boundary conditions are

$$y(0, t) = 0 \qquad ...(ii)$$

$$y(l, t) = 0 \qquad ...(iii)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \qquad ...(iv)$$

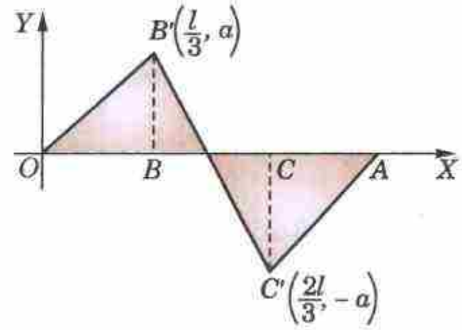

Fig. 18.2

The remaining condition is that at $t = 0$, the string rests in the form of the broken line $OB'C'A$. The equation of OB' is $y = (3a/l)\, x$;

the equation of $B'C'$ is $\quad y - a = \dfrac{-2a}{(l/3)}\left(x - \dfrac{l}{3}\right)$, *i.e.*, $y = \dfrac{3a}{l}(l - 2x)$

and the equation of $C'A$ is $\quad y = \dfrac{3a}{l}(x - l)$

Hence the fourth boundary condition is

$$\left.\begin{aligned} y(x, 0) &= \frac{3a}{l}\, x,\ 0 \le x \le \frac{l}{3} \\ &= \frac{3a}{l}(l - 2x),\ \frac{l}{3} \le x \le \frac{2l}{3} \\ &= \frac{3a}{l}(x - l),\ \frac{2l}{3} \le x \le l \end{aligned}\right\} \qquad ...(v)$$

As in example 18.6, the solution of (*i*) satisfying the boundary conditions (*ii*), (*iii*) and (*iv*), is

$$y(x, t) = b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \qquad [\text{Where } b_n = C_2C_3]$$

Adding all such solutions, the most general solution of (*i*) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \qquad ...(vi)$$

Putting $t = 0$, we have $y(x, 0) = \displaystyle\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$ $\qquad ...(vii)$

In order that the condition (*v*) may be satisfied, (*v*) and (*vii*) must be same. This requires the expansion of $y(x, 0)$ into a Fourier half-range sine series in the interval $(0, l)$.

$\therefore$ by (1) of § 10.7,

$$b_n = \frac{2}{l}\left[\int_0^{l/3} \frac{3ax}{l} \sin \frac{n\pi x}{l}\, dx + \int_{l/3}^{2l/3} \frac{3a}{l}(l - 2x) \sin \frac{n\pi x}{l}\, dx + \int_{2l/3}^{l} \frac{3a}{l}(x - l) \sin \frac{n\pi x}{l}\, dx\right]$$

$$= \frac{6a}{l^2}\left[\left| x\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - 1\left\{-\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right|_0^{l/3}\right.$$

$$+ \left|(l - 2x)\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - (-2)\left\{\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right|_{l/3}^{2l/3}$$

$$\left. + \left|(x - l)\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - (1)\,.\left\{-\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right|_{2l/3}^{l}\right]$$

$$= \frac{6a}{l^2}\left[\left(-\frac{l^2}{3n\pi}\cos\frac{n\pi}{3} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{3}\right) + \frac{l^2}{3n\pi}\cos\frac{2n\pi}{3} - \frac{2l^2}{n^2\pi^2}\sin\frac{2n\pi}{3} + \frac{l^2}{3n\pi}\cos\frac{n\pi}{3}\right.$$

$$\left. + \frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{3} - \left(\frac{l^2}{3n\pi}\cos\frac{2n\pi}{3} + \frac{l^2}{n^2\pi^2}\sin\frac{2n\pi}{3}\right)\right]$$

$$= \frac{6a}{l^2} \cdot \frac{3l^2}{n^2\pi^2}\left(\sin\frac{n\pi}{3} - \sin\frac{2n\pi}{3}\right)$$

$$= \frac{18a}{n^2\pi^2} \sin\frac{n\pi}{3}\,[1 + (-1)^n] \qquad \left[\because \sin\frac{2n\pi}{3} = \sin\left(n\pi - \frac{n\pi}{3}\right) = -(-1)^n \sin\frac{n\pi}{3}\right]$$

Thus $b_n = 0$, when n is odd.

$$= \frac{36a}{n^2\pi^2} \sin\frac{n\pi}{3}, \text{ when } n \text{ is even.}$$

Hence (vi) gives

$$y(x, t) = \sum_{n=2,4,\ldots}^{\infty} \frac{36a}{n^2\pi^2} \sin\frac{n\pi}{3} \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l} \qquad [\text{Take } n = 2m]$$

$$= \frac{9a}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \sin\frac{2m\pi}{3} \sin\frac{2m\pi x}{l} \cos\frac{2m\pi ct}{l} \qquad \ldots(vii)$$

Putting $x = l/2$ in (vii), we find that the displacement of the mid-point of the string, *i.e.* $y(l/2, t) = 0$, because $\sin m\pi = 0$ for all integral values of m.

This shows that the mid-point of the string is always at rest.

(3) D'Alembert's solution of the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \ldots(1)$$

Let us introduce the new independent variables $u = x + ct$, $v = x - ct$ so that y becomes a function of u and v.

Then $\dfrac{\partial y}{\partial x} = \dfrac{\partial y}{\partial u} + \dfrac{\partial y}{\partial v}$

and
$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) = \frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}$$

Similarly,
$$\frac{\partial^2 y}{\partial t^2} = c^2\left(\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right)$$

Substituting in (1), we get $\dfrac{\partial^2 y}{\partial u \partial v} = 0$ ...(2)

Integrating (2) w.r.t. v, we get $\dfrac{\partial y}{\partial u} = f(u)$ ...(3)

where $f(u)$ is an arbitrary function of u. Now integrating (3) w.r.t. u, we obtain

$$y = \int f(u)\,du + \psi(v)$$

where $\psi(v)$ is an arbitrary function of v. Since the integral is a function of u alone, we may denote it by $\phi(u)$. Thus

$$y = \phi(u) + \psi(v)$$

i.e.
$$y(x, t) = \phi(x + ct) + \psi(x - ct) \qquad \ldots(4)$$

This is the *general solution of the wave equation* (1).

Now to determine ϕ and ψ, suppose initially $u(x, 0) = f(x)$ and $\partial y(x, 0)/\partial t = 0$.

Differentiating (4) w.r.t. t, we get $\dfrac{\partial y}{\partial t} = c\phi'(x + ct) - c\psi'(x - ct)$

At $t = 0$, $\qquad \phi'(x) = \psi'(x)$ ...(5)

and $\qquad y(x, 0) = \phi(x) + \psi(x) = f(x)$ ...(6)

(5) gives, $\qquad \phi(x) = \psi(x) + k$

$\therefore$ (6) becomes $2\psi(x) + k = f(x)$

or
$$\psi(x) = \frac{1}{2}[f(x) - k] \text{ and } \phi(x) = \frac{1}{2}[f(x) + k]$$

Hence the solution of (4) takes the form

$$y(x, t) = \frac{1}{2}[f(x + ct) + k] + \frac{1}{2}[f(x - ct) - k] = f(x + ct) + f(x - ct) \quad ...(7)$$

which is the *d'Alembert's solution* of the wave equation* (1) *(V.T.U., 2011 S)*

Obs. The above solution gives a very useful method of solving partial differential equations by change of variables.

Example 18.9. *Find the deflection of a vibrating string of unit length having fixed ends with initial velocity zero and initial deflection $f(x) = k(\sin x - \sin 2x)$.* *(V.T.U., 2011)*

Solution. By d'Alembert's method, the solution is

$$y(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)]$$

$$= \frac{1}{2}[k\{\sin(x + ct) - \sin 2(x + ct)\} + k\{\sin(x - ct) - \sin 2(x - ct)\}]$$

$$= k[\sin x \cos ct - \sin 2x \cos 2ct]$$

Also $\quad y(x, 0) = k(\sin x - \sin 2x) = f(x)$

and $\quad \partial y(x, 0)/\partial t = k(-c \sin x \sin ct + 2c \sin 2x \sin 2ct)_{t=0} = 0$

i.e., the given boundary conditions are satisfied.

PROBLEMS 18.2

1. Solve completely the equation $\partial^2 y/\partial t^2 = c^2 \partial^2 y/\partial x^2$, representing the vibrations of a string of length l, fixed at both ends, given that $y(0, t) = 0$; $y(l, t) = 0$; $y(x, 0) = f(x)$ and $\partial y(x, 0)/\partial t = 0$, $0 < x < l$. *(Bhopal, 2007 S ; U.P.T.U., 2005)*
2. Solve the wave equation $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ under the conditions $u(0, t) = 0$, $u(l, t) = 0$ for all t ; $u(x, 0) = f(x)$ and $\left(\dfrac{\partial u}{\partial t}\right)_{t=0} = g(x)$, $0 < x < l$.
3. Find the solution of the wave equation $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$, corresponding to the triangular initial deflection
$$f(x) = \frac{2k}{l} x \text{ when } 0 < x < \frac{l}{2} . = \frac{2k}{l}(l - x) \text{ when } \frac{l}{2} < x < l,$$
and initial velocity zero. *(Bhopal, 2006 ; Kerala, M.E., 2005)*
4. A tightly stretched string of length l has its ends fastened at $x = 0$, $x = l$. The mid-point of the string is then taken to height h and then released from rest in that position. Find the lateral displacement of a point of the string at time t from the instant of release. *(Anna, 2005)*
5. A tightly stretched string with fixed end points at $x = 0$ and $x = 1$, is initially in a position given by
$$f(x) = \begin{cases} x, 0 \le x \le \frac{1}{2} \\ 1 - x, \frac{1}{2} \le x \le 1 \end{cases}$$
If it is released from this position with velocity a, perpendicular to the x-axis, show that the displacement $u(x, t)$ at any point x of the string at any time $t > 0$, is given by
$$u(x, t) = \frac{4\sqrt{2}}{\pi^2}\left[\sum_{n=1}^{\infty}\left\{\frac{\sin[(4pi - 3)\pi x]\cos[(4pi - 3)\pi at - \pi/4]}{(4n - 3)^2} - \frac{\sin[(4pi - 1)\pi x]\cos[(4pi - 1)\pi at - \pi/4]}{(4n - 1)^2}\right\}\right]$$
6. If a string of length l is initially at rest in equilibrium position and each of its points is given a velocity v such that $v = cx$ for $< x < l/2$
$c(l \,.\, x)$ for $l/2 < x < l$, determine the displacement $y(x, t)$ at anytime t. *(Anna, 2008)*
7. Using d'Alembert's method, find the deflection of a vibrating string of unit length having fixed ends, with initial velocity zero and initial deflection :
(*i*) $f(x) = a(x - x^2)$ *(Kerala, M. Tech., 2005)* (*ii*) $f(x) = a \sin^2 \pi x$.

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*See footnote of p. 373.

18.5 (1) ONE-DIMENSIONAL HEAT FLOW

Consider a homogeneous bar of uniform cross-section $\alpha(\text{cm}^2)$. Suppose that the sides are covered with a material impervious to heat so that the stream lines of heat-flow are all parallel and perpendicular to the area α. Take one end of the bar as the origin and the direction of flow as the positive x-axis (Fig. 18.3). Let ρ be the density (gr/cm^3), s the specific heat (cal./gr. deg.) and k the thermal conductivity (cal./cm. deg. sec.).

Let $u(x, -t)$ be the temperature at a distance x from O. If δu be the temperature change in a slab of thickness δx of the bar, then by § 12.7 (*ii*) p. 466, the quantity of heat in this slab $= s\rho\alpha\, \delta x\delta u$. Hence the rate of increase of heat in this slab, *i.e.*, $s\rho\alpha\delta x \dfrac{\partial u}{\partial t} = R_1 - R_2$, where R_1 and R_2 are respectively the rate (cal./sec.) of inflow and outflow of heat.

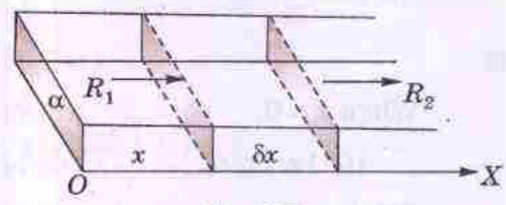


Fig. 18.3

Now by (*A*) of p. 466, $R_1 = -k\alpha\left(\dfrac{\partial u}{\partial x}\right)_x$ and $R_2 = -k\alpha\left(\dfrac{\partial u}{\partial x}\right)_{x+\delta x}$

the negative sign appearing as a result of (*i*) on p. 466.

Hence $s\rho\alpha\delta x \dfrac{\partial u}{\partial t} = -k\alpha\left(\dfrac{\partial u}{\partial x}\right)_x + k\alpha\left(\dfrac{\partial u}{\partial x}\right)_{x+\delta x}$ *i.e.*, $\dfrac{\partial u}{\partial t} = \dfrac{k}{s\rho}\left\{\dfrac{(\partial u/\partial x)_{x+\delta x} - (\partial u/\partial x)_x}{\delta x}\right\}$

Writing $k/s\rho = c^2$, called the *diffusivity* of the substance (cm^2/sec.), and taking the limit as $\delta x \to 0$, we get

$$\frac{\partial \mathbf{u}}{\partial \mathbf{t}} = \mathbf{c}^2 \frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} \qquad \text{...(1)}$$

This is the *one-dimensional heat-flow equation.* (*V.T.U., 2011*)

(2) Solution of the heat equation. Assume that a solution of (1) is of the form

$$u(x, t) = X(x) \,.\, T(t)$$

where X is a function of x alone and T is a function of t only.

Substituting this in (1), we get

$$XT' = c^2X''T, \text{ i.e., } X''/X = T'/c^2T \qquad \text{...(2)}$$

Clearly the left side of (2) is a function of x only and the right side is a function of t alone. Since x and t are independent variables, (2) can hold good if each side is equal to a constant k (say). Then (2) leads to the ordinary differential equations

$$\frac{d^2X}{dx^2} - kX = 0 \quad \text{...(3)} \qquad \text{and} \qquad \frac{dT}{dt} - kc^2T = 0 \qquad \text{...(4)}$$

Solving (3) and (4), we get

(*i*) *When k is positive* and $= p^2$, say :

$$X = c_1e^{px} + c_2e^{-px},\ T = c_3\, e^{c^2p^2t}\ ;$$

(*ii*) *When k is negative* and $= -p^2$, say :

$$X = c_4 \cos px + c_5 \sin px,\ T = c_6\, e^{-c^2p^2t}\ ;$$

(*iii*) *When k is zero :*

$$X = c_7x + c_8,\ T = c_9.$$

Thus the various possible solutions of the heat-equation (1) are

$$u = (c_1e^{px} + c_2e^{-px})\,c_3e^{c^2p^2t} \qquad \text{...(5)}$$

$$u = (c_4 \cos px + c_5 \sin px)\,c_6e^{-c^2p^2t} \qquad \text{...(6)}$$

$$u = (c_7x + c_8)c_9 \qquad \text{...(7)}$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. As we are dealing with problems on heat conduction, it must be a transient solution, *i.e.*, u is to decrease with the increase of time t. Accordingly, the solution given by (6), *i.e.*, of the form

$$u = (C_1 \cos px + C_2 \sin px)\,e^{-c^2p^2t} \qquad \text{...(8)}$$

is the only suitable solution of the heat equation.

Example 18.10. *Solve the equation* $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ *with boundary conditions* $u(x, 0) = 3 \sin n\pi x$, $u(0, t) = 0$ *and* $u(1, t) = 0$, *where* $0 < x < 1$, $t > 0$.

Solution. The solution of the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ ...(*i*)

is $$u(x, t) = (c_1 \cos px + c_2 \sin px)\, e^{-p^2 t} \qquad ...(ii)$$

When $x = 0$, $u(0, t) = c_1 e^{-p^2 t} = 0$ *i.e.*, $c_1 = 0$.

$\therefore$ (*ii*) becomes $u(x, t) = c_2 \sin px e^{-p^2 t}$ ...(*iii*)

When $x = 1$, $u(1, t) = c_2 \sin p \,.\, e^{-p^2 t} = 0$ or $\sin p = 0$

i.e., $p = n\pi$.

$\therefore$ (*iii*) reduces to $u(x, t) = b_n e^{-(n\pi)^2 t} \sin n\pi x$ where $b_n = c_2$

Thus the general solution of (*i*) is $u(x, t) = \sum b_n e^{-n^2\pi^2 t} \sin n\pi x$ ...(*iv*)

When $t = 0$, $3 \sin n\pi x = u(0, t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin n\pi x$

Comparing both sides, $b_n = 3$

Hence from (*iv*), the desired solution is

$$u(x, t) = 3 \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \sin n\pi x.$$

Example 18.11. *Solve the differential equation* $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ *for the conduction of heat along a rod without radiation, subject to the following conditions :*

(*i*) *u is not infinite for* $t \to \infty$, (*ii*) $\frac{\partial u}{\partial x} = 0$ *for* $x = 0$ *and* $x = l$,

(*ii*) $u = lx - x^2$ *for* $t = 0$, *between* $x = 0$ *and* $x = l$. (*P.T.U., 2007*)

Solution. Substituting $u = X(x)T(t)$ in the given equation, we get

$$XT' = \alpha^2 X''T \quad i.e., \quad X''/X = \frac{T'}{\alpha^2 T} = -k^2 \text{ (say)}$$

$$\therefore \quad \frac{d^2X}{dx^2} + k^2 X = 0 \text{ and } \frac{dT}{dt} + k^2\alpha^2 T = 0 \qquad ...(1)$$

Their solutions are $X = c_1 \cos kx + c_2 \sin kx$, $T = c_3 e^{-k^2\alpha^2 t}$ ...(2)

If k^2 is changed to $-k^2$, the solutions are

$$X = c_4 e^{kx} + c_5 e^{-kx},\ T = c_6 e^{k^2\alpha^2 t} \qquad ...(3)$$

If $k^2 = 0$, the solutions are $X = c_7 x + c_8$, $T = c_9$ ...(4)

In (3), $T \to \infty$ for $t \to \infty$ therefore, u also $\to \infty$ *i.e.*, the given condition (*i*) is not satisfied. So we reject the solutions (3) while (2) and (4), satisfy this condition.

Applying the condition (*ii*) to (4), we get $c_7 = 0$.

$$\therefore \quad u = XT = c_8 c_9 = a_0 \quad \text{(say)} \qquad ...(5)$$

From (2), $$\frac{\partial u}{\partial x} = (-c_1 \sin kx + c_2 \cos kx)\, kc_3 e^{-k^2\alpha^2 t}$$

Applying the condition (*ii*), we get $c_2 = 0$ and $-c_1 \sin kl + c_2 \cos kl = 0$

i.e., $c_2 = 0$ and $kl = n\pi$ (n an integer)

$$\therefore \quad u = c_1 \cos kx \,.\, c_3 e^{-k^2\alpha^2 t} = a_n \cos\left(\frac{n\pi x}{l}\right) \frac{e^{-n^2\pi^2\alpha^2 t}}{l^2} \qquad ...(6)$$

Thus the general solution being the sum of (5) and (6), is

$$u = a_0 + \Sigma a_n \cos(n\pi x/l)\, e^{-n^2\pi^2\alpha^2 t/l^2} \qquad ...(7)$$

Now using the condition (*iii*), we get

$$lx - x^2 = a_0 + \Sigma a_n \cos(n\pi x/l)$$

This being the expansion of $lx - x^2$ as a half-range cosine series in (0, l), we get

$$a_0 = \frac{1}{l}\int_0^l (lx - x^2)\, dx = \frac{1}{l}\left|\frac{lx^2}{2} - \frac{x^3}{3}\right|_0^l = \frac{l^2}{6}$$

and

$$a_n = \frac{2}{l}\int_0^l (lx - x^2)\cos\frac{n\pi x}{l}\, dx = \frac{2}{l}\left|(lx - x^2)\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right)\right.$$

$$\left. - (l - 2x)\left(-\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right) + (-2)\left(-\frac{l^3}{n^3\pi^3}\sin\frac{n\pi x}{l}\right)\right|_0^l$$

$$= \frac{2}{l}\left\{0 - \frac{l^3}{n^2\pi^2}(\cos n\pi + 1) + 0\right\} = -\frac{4l^2}{n^2\pi^2} \text{ when } n \text{ is even, otherwise } 0.$$

Hence taking $n = 2m$, the required solution is

$$u = \frac{l^2}{6} - \frac{l^2}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\cos\left(\frac{2m\pi x}{l}\right)e^{-4m^2\pi^2\alpha^2 t/l^2}.$$

Example 18.12. (*a*) *An insulated rod of length l has its ends A and B maintained at 0°C and 100°C respectively until steady state conditions prevail. If B is suddenly reduced to 0°C and maintained at 0°C, find the temperature at a distance x from A at time t.* (*U.P.T.U., 2005*)

(*b*) *Solve the above problem if the change consists of raising the temperature of A to 20°C and reducing that of B to 80°C.* (*Madras, 2000 S*)

Solution. (*a*) Let the equation for the conduction of heat be

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \qquad ...(i)$$

Prior to the temperature change at the end B, when $t = 0$, the heat flow was independent of time (*steady state condition*). When u depends only on x, (*i*) reduces to $\partial^2 u/\partial x^2 = 0$.

Its general solution is $u = ax + b$ ...(*ii*)

Since $u = 0$ for $x = 0$ and $u = 100$ for $x = l$, therefore, (*ii*) gives $b = 0$ and $a = 100/l$.

Thus the *initial condition* is expressed by $u(x, 0) = \dfrac{100}{l}x$ ...(*iii*)

Also the *boundary conditions* for the subsequent flow are

$$u(0, t) = 0 \text{ for all values of } t \qquad ...(iv)$$

and

$$u(l, t) = 0 \text{ for all values of } t \qquad ...(v)$$

Thus we have to find a temperature function $u(x, t)$ satisfying the differential equation (*i*) subject to the initial condition (*iii*) and the boundary conditions (*iv*) and (*v*).

Now the solution of (*i*) is of the form

$$u(x, t) = (C_1\cos px + C_2\sin px)\, e^{-c^2p^2t} \qquad ...(vi)$$

By (*iv*), $u(0, t) = C_1 e^{-c^2p^2t} = 0$, for all values of t.

Hence $C_1 = 0$ and (*vi*) reduces to $u(x, t) = C_2 \sin px \,.\, e^{-c^2p^2t}$ ...(*vii*)

Applying (*v*), (*vii*) gives $u(l, t) = C_2\sin pl \,.\, e^{-c^2p^2t} = 0$, for all values of t.

This requires $\sin pl = 0$ *i.e.*, $pl = n\pi$ as $C_2 \neq 0$. $\therefore\; p = n\pi/l$, where n is any integer.

Hence (*vii*) reduces to $u(x, t) = b_n \sin\dfrac{n\pi x}{l}\cdot e^{-c^2n^2\pi^2t/l^2}$, where $b_n = C_2$.

[These are the solutions of (i) satisfying the boundary conditions (iv) and (v). These are the **eigen functions** corresponding to the **eigen values** $\lambda_n = cn\pi/l$, of the problem.]

Adding all such solutions, the most general solution of (i) satisfying the boundary conditions (iv) and (v) is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot e^{-c^2n^2\pi^2t/l^2} \quad ...(viii)$$

Putting $t = 0$, $$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad ...(ix)$$

In order that the condition (iii) may be satisfied, (iii) and (ix) must be same. This requires the expansion of $100x/l$ as a half-range Fourier sine series in $(0, l)$. Thus

$$\frac{100x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \text{ where } b_n = \frac{2}{l}\int_0^l \frac{100x}{l} \cdot \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{200}{l^2}\left[x\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - (1)\left\{-\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right]_0^l = \frac{200}{l^2}\left(-\frac{l^2}{n\pi}\cos n\pi\right) = \frac{200}{n\pi}(-1)^{n+1}$$

Hence $(viii)$ gives $u(x, t) = \dfrac{200}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l}\cdot e^{-(cn\pi/l)^2 t}$

(b) Here the initial condition remains the same as (iii) above, and the boundary conditions are

$$u(0, t) = 20 \text{ for all values of } t \quad ...(x)$$

$$u(l, t) = 80 \text{ for all values of } t \quad ...(xi)$$

In part (a), the boundary values (*i.e.*, the temperature at the ends) being zero, we were able to find the desired solution easily. Now the boundary values being non-zero, we have to modify the procedure.

We split up the temperature function $u(x, t)$ into two parts as

$$u(x, t) = u_s(x) + u_t(x, t) \quad ...(xii)$$

where $u_s(x)$ is a solution of (i) involving x only and satisfying the boundary conditions (x) and (xi); $u_t(x, t)$ is then a function defined by (xii). Thus $u_s(x)$ is a steady state solution of the form (ii) and $u_t(x, t)$ may be regarded as a transient part of the solution which decreases with increase of t.

Since $u_s(0) = 20$ and $u_s(l) = 80$, therefore, using (ii) we get

$$u_s(x) = 20 + (60/l)x \quad ...(xiii)$$

Putting $x = 0$ in (xii), we have by (x),

$$u_t(0, t) = u(0, t) - u_s(0) = 20 - 20 = 0 \quad ...(xiv)$$

Putting $x = l$ in (xii), we have by (xi),

$$u_t(l, t) = u(l, t) - u_s(l) = 80 - 80 = 0 \quad ...(xv)$$

Also $$u_t(x, 0) = u(x, 0) - u_s(x) = \frac{100x}{l} - \left(\frac{60x}{l} + 20\right) \quad \text{[by } (iii) \text{ and } (xiii)]$$

$$= \frac{40x}{l} - 20 \quad ...(xvi)$$

Hence (xiv) and (xv) give the boundary conditions and (xvi) gives the initial condition relative to the transient solution. Since the boundary values given by (xiv) and (xv) are both zero, therefore, as in part (a), we have $u_t(x, t) = (C_1 \cos px + C_2 \sin px)\, e^{-c^2p^2t}$

By (xiv), $u_t(0, t) = C_1 e^{-c^2p^2t} = 0$, for all values of t.

Hence $C_1 = 0$ and $$u_t(x, t) = C_2 \sin px \,.\, e^{-c^2p^2t} \quad ...(xvii)$$

Applying (xv), it gives $u_t(l, t) = C_2 \sin pl e^{-c^2p^2t} = 0$ for all values of t.

This requires $\sin pl = 0$, *i.e.* $pl = n\pi$ as $C_2 \neq 0$. $p = n\pi/l$, when n is any integer.

Hence $(xvii)$ reduces to $u_t(x, t) = b_n \sin \dfrac{n\pi x}{l} e^{-c^2n^2\pi^2t/l^2}$ where $b_n = C_2$.

Adding all such solutions, the most general solution of (*xvii*) satisfying the boundary conditions (*xiv*) and (*xv*) is

$$u_t(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2} \quad ...(xviii)$$

Putting $t = 0$, we have $u_t(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$ ...(*xix*)

In order that the condition (*xvi*) may be satisfied, (*xvi*) and (*xix*) must be same. This requires the expansion of $(40/l)\,x - 20$ as a half-range Fourier sine series in $(0, l)$. Thus

$$\frac{40x}{l} - 20 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \qquad \text{where} \quad b_n = \frac{2}{l}\int_0^l \left(\frac{40x}{l} - 20\right) \sin \frac{n\pi x}{l}\, dx = -\frac{40}{nx}\,(1 + \cos n\pi)$$

i.e., $b_n = 0$, when n is odd ; $= -80/n\pi$, when n is even

Hence (*xviii*) becomes $u_t(x, t) = \sum_{n=2,4...}^{\infty} \left(\frac{-80}{n\pi}\right) \sin \frac{n\pi x}{l} \cdot e^{-c^2 n^2 \pi^2 t/l^2}$ [Take $n = 2m$]

$$= -\frac{40}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \sin \frac{2m\pi x}{l} \cdot e^{-4c^2 m^2 \pi^2 t/l^2} \quad ...(xx)$$

Finally combining (*xiii*) and (*xx*), the required solution is

$$u(x, t) = \frac{40x}{l} + 20 - \frac{40}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \sin \frac{2m\pi x}{l} \cdot e^{-4c^2 m^2 \pi^2 t/l^2}.$$

Example 18.13. *The ends A and B of a rod 20 cm long have the temperature at 30°C and 80°C until steady-state prevails. The temperature of the ends are changed to 40°C and 60°C respectively. Find the temperature distribution in the rod at time t.*

Solution. Let the heat equation be $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ ...(*i*)

In steady state condition, u is independent of time and depends on x only, (*i*) reduces to

$$\partial^2 u / \partial x^2 = 0. \quad ...(ii)$$

Its solution is $u = a + bx$

Since $u = 30$ for $x = 0$ and $u = 80$ for $x = 20$, therefore $a = 30$, $b = (80 - 30)/20 = 5/2$

Thus the initial conditions are expressed by

$$u\,(x, 0) = 30 + \frac{5}{2}x \quad ...(iii)$$

The boundary conditions are $u\,(0, t) = 40$, $u\,(20, t) = 60$

Using (*ii*), the steady state temperature is

$$u\,(x, 0) = 40 + \frac{60 - 40}{20}\, x = 40 + x \quad ...(iv)$$

To find the temperature u in the intermediate period,

$$u\,(x, t) = u_s\,(x) + u_t\,(x, t)$$

where $u_s\,(x)$ is the steady state temperature distribution of the form (*iv*) and $u_t\,(x, t)$ is the transient temperature distribution which decreases to zero as t increases.

Since $u_t\,(x, t)$ satisfies one dimensional heat equation

$$\therefore \quad u(x, t) = 40 + x + \sum_{n=1}^{\infty} (a_n \cos px + b_n \sin px)\, e^{-p^2 t} \quad ...(v)$$

$$u(0, t) = 40 = 40 + \sum_{n=1}^{\infty} a_n e^{-p^2 t} \quad \text{whence } a_n = 0.$$

$\therefore$ (v) reduces to $$u(x, t) = 40 + x + \sum_{n=1}^{\infty} b_n \sin px e^{-p^2 t} \quad ...(vi)$$

Also $$u(20, t) = 60 = 40 + 20 + \sum_{n=1}^{\infty} b_n \sin 20\, p e^{-p^2 t}$$

or $$\sum_{n=1}^{\infty} b_n \sin 20\, p e^{-p^2 t} = 0 \text{ i.e., } \sin 20p = 0 \text{ i.e., } p = n\pi/20$$

Thus (vi) becomes $$u(x, t) = 40 + x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20} e^{-n\pi t/20} \quad ...(vii)$$

Using (iii), $$30 + \frac{5}{2}x = u(0, t) = 40 + x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}$$

or $$\frac{3x}{2} - 10 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}$$

where $b_n = \frac{2}{20} \int_0^{20} \left(\frac{3x}{2} - 10\right) \sin \frac{n\pi x}{20} dx = -\frac{20}{n\pi}(1 + 2\cos n\pi)$

Hence from (vii), the desired solution is

$$u = 40 + x - \frac{20}{\pi} \sum_{n=1}^{\infty} \frac{1 + 2\cos n\pi}{n} \sin \frac{n\pi x}{20} e^{-(n\pi/20)^2 t}.$$

Example 18.14. Bar with insulated ends. *A bar 100 cm long, with insulated sides, has its ends kept at 0°C and 100°C until steady state conditions prevail. The two ends are then suddenly insulated and kept so. Find the temperature distribution.*

Solution. The temperature $u(x, t)$ along the bar satisfies the equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad ...(i)$$

By law of heat conduction, the rate of heat flow is proportional to the gradient of the temperature. Thus, if the ends $x = 0$ and $x = l$ (= 100 cm) of the bar are insulated (Fig. 18.4) so that no heat can flow through the ends, the boundary conditions are

$$\frac{\partial u(0, t)}{\partial x} = 0, \frac{\partial u(l, t)}{\partial x} = 0 \text{ for all } t \quad ...(ii)$$

Initially, under steady state conditions, $\frac{\partial^2 u}{\partial x^2} = 0$. Its solution is $u = ax + b$.

Since $u = 0$ for $x = 0$ and $u = 100$ for $x = l$ $\therefore$ $b = 0$ and $a = 1$.

Thus the initial condition is $u(x, 0) = x \quad 0 < x < l$. ...(iii)

Now the solution of (i) is of the form $u(x, t) = (c_1 \cos px + c_2 \sin px) e^{-c^2 p^2 t}$ ...(iv)

Differentiating partially w.r.t. x, we get

$$\frac{\partial u}{\partial x} = (-c_1 p \sin px + c_2 p \cos px) e^{-c^2 p^2 t} \quad ...(v)$$

Putting $x = 0$, $\left(\frac{\partial u}{\partial x}\right)_0 = c_2 p e^{-c^2 p^2 t} = 0$ for all t. [By (ii)]

$\therefore$ $c_2 = 0$

Putting $x = l$ in (v), $\left(\frac{\partial u}{\partial x}\right)_l = -c_1 p \sin pl e^{-c^2 p^2 t}$ for all t. [By (ii)]

$\therefore$ $c_1 p \sin pl = 0$ *i.e.*, p being $\neq 0$, either $c_1 = 0$ or $\sin pl = 0$.

When $c_1 = 0$, (iv) gives $u(x, t) = 0$ which is a trivial solution, therefore $\sin pl = 0$.

or $\quad pl = n\pi \quad$ or $\quad p = n\pi/l, \quad n = 0, 1, 2, ...$

Hence (*iv*) becomes $u(x, t) = c_1 \cos \dfrac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t / l^2}$.

$\therefore$ the most general solution of (*i*) satisfying the boundary conditions (*ii*) is

$$u(x, t) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2} = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2} \qquad \text{(where } A_n = c_1) \quad \ldots(vi)$$

Putting $t = 0$, $u(x, 0) = A_0 + \sum_{n=1}^{\infty} A_n \cos \dfrac{n\pi x}{l} = x$ [by (*iii*)]

This requires the expansion of x into a half range cosine series in $(0, l)$.

Thus $$x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x/l \qquad \text{where } a_0 = \frac{2}{l}\int_0^l x\,dx = l$$

and $$a_n = \frac{2}{l}\int_0^l x \cos \frac{n\pi x}{l}\, dx = \frac{2l}{n^2\pi^2}(\cos n\pi - 1)$$
$= 0$, where n is even ; $= -4l/n^2\pi^2$, when n is odd.

$\therefore$ $$A_0 = \frac{a_0}{2} = l/2, \text{ and } A_n = a_n = 0 \text{ for } n \text{ even} ; = -4l/n^2\pi^2 \text{ for } n \text{ odd.}$$

Hence (*vi*) takes the form

$$u(x, t) = \frac{l}{2} + \sum_{n=1,3,\ldots}^{\infty} \frac{4l}{n^2\pi^2} \cos \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2}$$

$$= \frac{l}{2} - \frac{4l}{\pi^2}\sum_{1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{l} e^{-c^2(2n-1)^2\pi^2 t/l^2} \quad \ldots(vii)$$

This is the required temperature at a point P_1 distant x from end A at any time t.

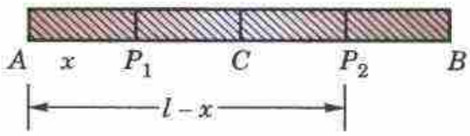


Fig. 18.4

Obs. *The sum of the temperatures at any two points equidistant from the centre is always 100°C, a constant.*

Let P_1, P_2 be two points equidistant from the centre C of the bar so that $CP_1 = CP_2$ (Fig. 18.4).

If $AP_1 = BP_2 = x$ (say), then $AP_2 = l - x$.

$\therefore$ Replacing x by $l - x$ in (*vii*), we get the temperature at P_2 as

$$u(l - x, t) = \frac{l}{2} - \frac{4l}{\pi^2}\sum_{1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi(l-x)}{l} e^{\frac{-c^2(2n-1)^2\pi^2 t}{l^2}}$$

$$= \frac{l}{2} + \frac{4l}{\pi^2}\sum_{1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{l} e^{\frac{-c^2(2n-1)^2\pi^2 t}{l^2}} \quad \ldots(viii)$$

$$\left\{\because \cos \frac{(2n-1)\pi(l-x)}{l} = \cos\left[2n\pi - \pi - \frac{(2n-1)\pi x}{l}\right] = -\cos\frac{(2n-1)\pi x}{l}\right.$$

Adding (*vii*) and (*viii*), we get $u(x, t) + u(l - x, t) = l = 100°C$.

PROBLEMS 18.3

1. A homogeneous rod of conducting material of length 100 cm has its ends kept at zero temperature and the temperature initially is

$$u(x, 0) = x, \qquad 0 \le x \le 50$$
$$= 100 - x, \qquad 50 \le x \le 100.$$

Find the temperature $u(x, t)$ at any time. *(Bhopal, 2007 ; S.V.T.U., 2007 ; Kurukshetra, 2006)*

2. Find the temperature $u(x, t)$ in a homogeneous bar of heat conducting material of length l, whose ends are kept at temperature 0°C and whose initial temperature in (°C) is given by $ax(l - x)/l^2$. (*P.T.U., 2009*)
3. A rod 30 cm. long, has its ends A and B kept at 20° and 80°C respectively until steady state conditions prevail. The temperature at each end is then suddenly reduced to 0°C and kept so. Find the resulting temperature function $u(x, t)$ taking $x = 0$ at A. (*Anna, 2008*)
4. A bar of 10 cm long, with insulated sides has its ends A and B maintained at temperatures 50°C and 100°C respectively, until steady-state conditions prevail. The temperature A is suddenly raised to 90°C and at the same time that at B is lowered to 60°C. Find the temperature distribution in the bar at time t. (*P.T.U., 2010*)

   Show that the temperature at the middle point of the bar remains unaltered for all time, regardless of the material of the bar.
5. Solve the following boundary value problem :
$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \qquad \frac{\partial u(0, t)}{\partial x} = 0, \frac{\partial u(l, t)}{\partial x} = 0, \; u(x, 0) = x. \qquad (S.V.T.U., 2008)$$
6. The temperatures at one end of a bar, 50 cm long with insulated sides, is kept at 0°C and that the other end is kept at 100°C until steady-state conditions prevail. The two ends are then suddenly insulated, so that the temperature gradient is zero at each end thereafter. Find the temperature distribution.
7. Find the solution of $\frac{\partial \theta}{\partial t} = k \frac{\partial^2 \theta}{\partial x^2}$, such that

   (*i*) θ is not infinite when $t \to +\infty$; (*ii*) $\left.\begin{array}{ll} \frac{\partial \theta}{\partial x} = 0 & \text{when } x = 0 \\ \theta = 0, & \text{when } x = l \end{array}\right\}$ for all values of t ;

   (*iii*) $\theta = \theta_0$, when $t = 0$, for all values of x between 0 and l. (*S.V.T.U., 2008*)
8. Find the solution of $\frac{\partial V}{\partial t} = k \frac{\partial^2 V}{\partial x^2}$ having given that $V = V_0 \sin nt$ when $x = 0$ for all values of t and $V = 0$ when x is very large.

18.6 TWO-DIMENSIONAL HEAT FLOW

Consider the flow of heat in a metal plate of uniform thickness α (cm), density ρ (gr/cm^3), specific heat s (cal/gr deg) and thermal conductivity k (cal/cm sec deg). Let XOY plane be taken in one face of the plate (Fig. 18.5). If the temperature at any point is independent of the z-coordinate and depends only on x, y and time t, then the flow is said to be two-dimensional. In this case, the heat flow is in the XY-plane only and is zero along the normal to the XY-plane.

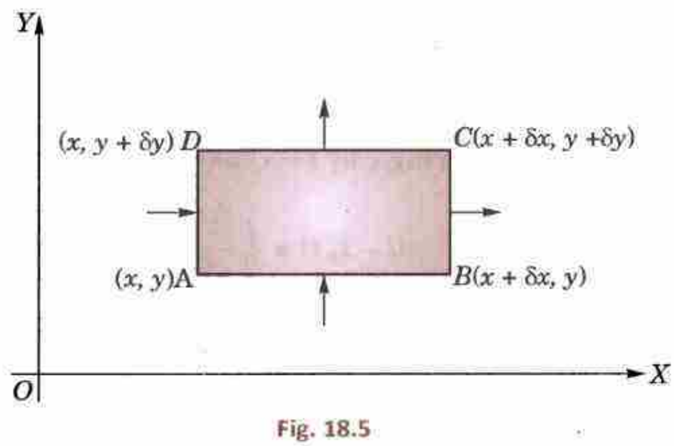


Fig. 18.5

Consider a rectangular element $ABCD$ of the plane with sides δx and δy. By (*A*) on p. 466, the amount of heat entering the element in 1 sec. from the side AB

$$= -k\alpha\delta x \left(\frac{\partial u}{\partial y}\right)_y$$

and the amount of heat entering the element in 1 second from the side $AD = -k\alpha\delta y \left(\frac{\partial u}{\partial x}\right)_x$

The quantity of heat flowing out through the side CD per sec. $= -k\alpha\delta x \left(\frac{\partial u}{\partial y}\right)_{y+\delta y}$

and the quantity of heat flowing out through the side BC per second $= -k\alpha\delta y \left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$

Hence the total gain of heat by the rectangular element $ABCD$ per second

$$= -k\alpha\delta x \left(\frac{\partial u}{\partial y}\right)_y - k\alpha\delta y \left(\frac{\partial u}{\partial x}\right)_x + k\alpha\delta x \left(\frac{\partial u}{\partial y}\right)_{y+\delta y} + k\alpha\delta y \left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$$

$$= k\alpha\delta x\left[\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y\right] + k\alpha\delta y\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

$$= k\alpha\delta x\delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\partial x} + \frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\partial y}\right] \quad ...(1)$$

Also the rate of gain of heat by the element

$$= \rho\delta x\delta y\alpha s\,\frac{\partial u}{\partial t} \quad ...(2)$$

Thus equating (1) and (2),

$$k\alpha\delta x\delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x} + \frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\delta y}\right] = \rho\delta x\delta y\alpha s\,\frac{\partial u}{\partial t}.$$

Dividing both sides by $\alpha\delta x\delta y$ and taking limits as $\delta x \to 0$, $\delta y \to 0$, we get

$$k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \rho s\frac{\partial u}{\partial t}$$

i.e., $$\frac{\partial \mathbf{u}}{\partial \mathbf{t}} = \mathbf{c}^2\left(\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2}\right) \text{ where } c^2 = k/\rho s \text{ is the } \textit{diffusivity}. \quad ...(3)$$

Hence the equation (3) gives the temperature distribution of the plane in the *transient state.*

Cor. In the *steady state*, u is independent of t, so that $\partial u/\partial t = 0$ and the above equation reduces to,

$$\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2} = \mathbf{0}$$

which is the well known **Laplace's equation** *in two dimensions.*

Obs. When the stream lines are curves in space, *i.e.*, the heat flow is three dimensional, we shall similarly arrive at the equation

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

In a *steady state*, it reduces to $\dfrac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \dfrac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2} + \dfrac{\partial^2 \mathbf{u}}{\partial \mathbf{z}^2} = \mathbf{0}$

which is the *three dimensional* **Laplace's equation.**

18.7 SOLUTION OF LAPLACE'S EQUATION

$$\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2} = \mathbf{0} \quad ...(1)$$

Let $u = X(x)Y(y)$ be a solution of (1).

Substituting it in (1), we get $\dfrac{d^2X}{dx^2}Y + X\dfrac{\partial^2 Y}{\partial y^2} = 0$

or separating the variables, $$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} \quad ...(2)$$

Since x and y are independent variables, (2) can hold good only if each side of (2) is equal to a constant k (say). Then (2) leads to the ordinary differential equations

$$\frac{d^2X}{dx^2} - kX = 0 \text{ and } \frac{d^2Y}{dy^2} + kY = 0.$$

Solving these equations, we get

(*i*) *When k is positive* and is equal to p^2, say

$$X = c_1 e^{px} + c_2 e^{-px},\ Y = c_3 \cos py + c_4 \sin py$$

(*ii*) *When k is negative*, and is equal to $-p^2$, say

$$X = c_5 \cos px + c_6 \sin px,\ Y = c_7 e^{py} + c_8 e^{-py}$$

(*iii*) *When k is zero* ; $X = c_9 x + c_{10}$, $Y = c_{11} y + c_{12}$.

Thus the various possible solutions of (1) are

$$u = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \quad ...(3)$$

$$u = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py}) \quad ...(4)$$

$$u = (c_9 x + c_{10})(c_{11} y + c_{12}) \quad ...(5)$$

Of these we take that solution which is consistent with the given boundary conditions.

(V.T.U., 2011 S ; Kerala, 2005)

Temperature distribution in long plates

Example 18.15. *An infinitely long plane uniform plate is bounded by two parallel edges and an end at right angles to them. The breadth is π ; this end is maintained at a temperature u_0 at all points and other edges are at zero temperature. Determine the temperature at any point of the plate in the steady-state.*

(P.T.U., 2005 ; J.N.T.U., 2002 S)

Solution. In the steady state (Fig. 18.6), the temperature $u(x, y)$ at any point $P(x, y)$ satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ...(i)$$

The boundary conditions are $u(0, y) = 0$ for all values of y ...(*ii*)

$u(\pi, y) = 0$ for all values of y ...(*iii*)

$u(x, \infty) = 0$ in $0 < x < \pi$ ...(*iv*)

$u(x, 0) = u_0$ in $0 < x < \pi$ ...(*v*)

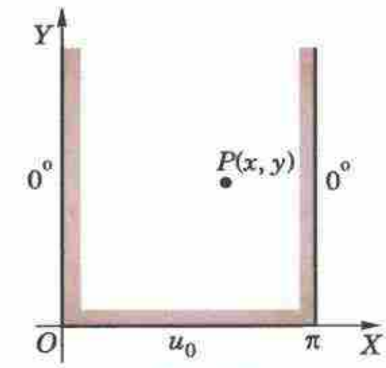


Fig. 18.6

Now the three possible solutions of (*i*) are

$$u = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \quad ...(vi)$$

$$u = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py}) \quad ...(vii)$$

$$u = (c_9 x + c_{10})(c_{11} y + c_{12}) \quad ...(viii)$$

Of these, we have to choose that solution which is consistent with the physical nature of the problem. The solution (*vi*) cannot satisfy the condition (*ii*) for $u \neq 0$ for $x = 0$, for all values of y. The solution (*viii*) cannot satisfy the condition (*iv*). Thus the only possible solution is (*vii*), *i.e.* of the form

$$u(x, y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py}) \quad ...(ix)$$

By (*ii*), $u(0, y) = C_1(C_3 e^{py} + C_4 e^{-py}) = 0$ for all y.

Hence $C_1 = 0$ and (*ix*) reduces to

$$u(x, y) = C_2 \sin px\, (C_3 e^{py} + C_4 e^{-py}) \quad ...(x)$$

By (*iii*), $u(\pi, y) = C_2 \sin p\pi (C_3 e^{py} + C_4 e^{-py}) = 0$, for all y.

This requires $\sin p\pi = 0$, *i.e.* $p\pi = n\pi$ as $C_2 \neq 0$. $\therefore$ $p = n$, an integer.

Also to satisfy the condition (*iv*), *i.e.*, $u = 0$ as $y \to \infty$, $C_3 = 0$.

Hence (*x*) takes the form $u(x, y) = b_n \sin nx \,.\, e^{-ny}$, where $b_n = C_2 C_4$.

$\therefore$ the most general solution satisfying (*ii*), (*iii*) and (*iv*) is of the form

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin nx \,.\, e^{-ny} \quad ...(xi)$$

Putting $y = 0$, $$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin nx \quad ...(xii)$$

In order that the condition (*v*) may be satisfied, (*v*) and (*xii*) must be same. This requires the expansion of u as a half-range Fourier sine series in $(0, \pi)$. Thus

$$u = \sum_{n=1}^{\infty} b_n \sin nx \text{ where } b_n = \frac{2}{\pi}\int_0^{\pi} u_0 \sin nx\, dx = \frac{2u_0}{n\pi}\,[1-(-1)^n]$$

i.e., $b_n = 0$, if n is even ; $= 4u_0/n\pi$, if n is odd.

Hence (*xi*) becomes $u(x, y) = \dfrac{4u_0}{\pi}\left[e^{-y}\sin x + \dfrac{1}{3}e^{-3y}\sin 3x + \dfrac{1}{5}e^{-5y}\sin 5x + ...\right]$.

Temperature distribution in finite plates

Example 18.16. *Solve the Laplace equation* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ *subject to the conditions* $u(0, y) = u(l, y) = u(x, 0) = 0$ *and* $u(x, a) = \sin n\pi x/l$. *(V.T.U., 2011 ; J.N.T.U., 2006 ; Kerala M. Tech., 2005, U.P.T.U., 2004)*

Solution. The three possible solutions of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad ...(i)$$

are
$$u = (c_1e^{px} + c_2e^{-px})(c_3\cos py + c_4\sin py) \qquad ...(ii)$$
$$u = (c_5\cos px + c_6\sin px)(c_7e^{py} + c_8e^{-py}) \qquad ...(iii)$$
$$u = (c_9x + c_{10})(c_{11}y + c_{12}) \qquad ...(iv)$$

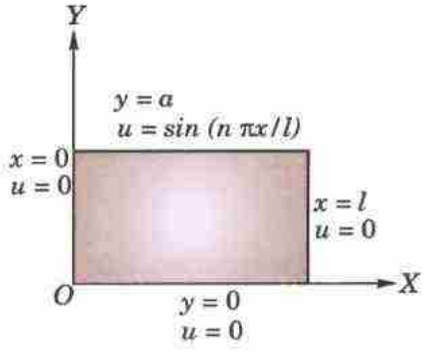


Fig. 18.7

We have to solve (*i*) satisfying the following boundary conditions

$$u(0, y) = 0 \quad ...(v) \qquad u(l, y) = 0 \quad ...(vi)$$
$$u(x, 0) = 0 \quad ...(vii) \qquad u(x, a) = \sin n\pi x/l \quad ...(viii)$$

Using (*v*) and (*vi*) in (*ii*), we get

$$c_1 + c_2 = 0, \quad \text{and} \quad c_1e^{pl} + c_2e^{-pl} = 0$$

Solving these equations, we get $c_1 = c_2 = 0$ which lead to trivial solution. Similarly, we get a trivial solution by using (*v*) and (*vi*) in (*iv*). Hence the suitable solution for the present problem is solution (*iii*). Using (*v*) in (*iii*), we have $c_5(c_7e^{py} + c_8e^{-py}) = 0$ *i.e.*, $c_5 = 0$

$\therefore$ (*iii*) becomes $\qquad u = c_6 \sin px(c_7e^{py} + c_8e^{-py}) \qquad ...(ix)$

Using (*vi*), we have $c_6 \sin pl\,(c_7e^{py} + c_8e^{-py}) = 0$

$\therefore$ either $c_6 = 0$ or $\sin pl = 0$

If we take $c_6 = 0$, we get a trivial solution.

Thus $\sin pl = 0$ whence $pl = n\pi$ or $p = n\pi/l$ where $n = 0, 1, 2, ...$

$\therefore$ (*ix*) becomes $u = c_6 \sin(n\pi x/l)(c_1e^{n\pi y/l} + c_8e^{-n\pi y/l}) \qquad ...(x)$

Using (*vii*), we have $0 = c_6 \sin n\pi x/l\,.\,(c_7 + c_8)$ *i.e.*, $c_8 = -c_7$.

Thus the solution suitable for this problem is

$$u(x, y) = b_n \sin\frac{n\pi x}{l}(e^{n\pi y/l} - e^{-n\pi y/l}) \quad \text{where} \quad b_n = c_6c_7$$

Now using the condition (*viii*), we have

$$u(x, a) = \sin\frac{n\pi x}{l} = b_n \sin\frac{n\pi x}{l}\,(e^{n\pi a/l} - e^{-n\pi a/l}),$$

we get
$$b_n = \frac{1}{(e^{n\pi a/l} - e^{-n\pi a/l})}$$

Hence the required solution is

$$u(x, y) = \frac{e^{n\pi y/l} - e^{-n\pi y/l}}{e^{n\pi a/l} - e^{-n\pi a/l}}\sin\frac{n\pi x}{l} = \frac{\sinh(n\pi y/l)}{\sinh(n\pi a/l)}\sin\frac{n\pi x}{l}.$$

Example 18.17. *The function* $v(x, y)$ *satisfies the Laplace's equation in rectangular coordinates* (x, y) *and for points within the rectangle* $x = 0, x = a, y = 0, y = b$, *it satisfies the conditions* $v(0, y) = v(a, y) = v(x, b) = 0$ *and* $v(x, 0) = x(a - x)$, $0 < x < a$. *Show that* $v(x, y)$ *is given by*

$$v(x, y) = \frac{8a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi x/a}{(2n+1)^3}\frac{\sinh(2n+1)\pi(b-y)/a}{\sinh(2n+1)\pi b/a}$$ *(Madras, 2003)*

Solution. The only possible solution of

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad ...(i)$$

is of the form $v(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$ ...(ii)

The boundary conditions are

$$v(0, y) = 0\ ; \quad v(a, y) = 0 \quad ...(iii)$$

$$v(x, b) = 0 \quad ...(iv)$$

$$v(x, 0) = x(a - x),\ 0 < x < a. \quad ...(v)$$

Using (*iii*) $\quad v(0, y) = c_1(c_3 e^{py} + c_4 e^{-py}) = 0 \quad i.e., \quad c_1 = 0.$

$\therefore$ (*ii*) becomes $\quad v(x, y) = c_2 \sin px\,(c_3 e^{py} + c_4 e^{-py})$ ...(vi)

Again using (*iii*), $\quad v(a, y) = c_2 \sin pa\,(c_3 e^{py} + c_4 e^{-py}) = 0.$

i.e., $\quad \sin pa = 0$, *i.e.* $pa = n\pi$ or $p = n\pi/a$

$\therefore$ (*vi*) becomes $\quad v(x, y) = c_2 \sin \dfrac{n\pi x}{a}\left(c_3 e^{\frac{n\pi y}{a}} + c_4 e^{-\frac{n\pi y}{a}}\right)$

or $\quad v(x, y) = \sin \dfrac{n\pi x}{a}\,(A e^{n\pi y/a} + B e^{-n\pi y/a})$ where $A = c_2 c_3, B = c_2 c_4$ ...(vii)

Now using (*iv*), $\quad v(x, b) = \sin \dfrac{n\pi x}{a}\left(A e^{\frac{n\pi b}{a}} + B e^{-\frac{n\pi b}{a}}\right) = 0$

i.e., $\quad A e^{n\pi b/a} + B e^{-n\pi b/a} = 0$ or $A e^{n\pi b/a} - B e^{-n\pi b/a} = -\dfrac{1}{2} b_n$ (say)

Thus (*vii*) becomes $\quad v(x, y) = \sin \dfrac{n\pi x}{a} \cdot \dfrac{1}{2} b_n \left\{e^{n\pi(b-y)/a} - e^{-n\pi(b-y)/a}\right\}$

$$= b_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi(b-y)}{a}$$

$\therefore$ the most general solution of (*i*) is

$$v(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi(b-y)}{a} \quad ...(viii)$$

Using the condition (*v*), we have

$$x(a - x) = v(x, 0) = \sum_{n=1}^{\infty} b_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi x}{a}$$

where $\quad b_n \sinh \dfrac{n\pi b}{a} = \dfrac{2}{a}\displaystyle\int_0^a x(a - x) \sin \dfrac{n\pi x}{a}\, dx$

$$= \frac{2}{a}\left| (ax - x^2)\left(\frac{-\cos n\pi x/a}{n\pi/a}\right) - (a - 2x)\left(-\frac{\sin n\pi x/a}{(n\pi/a)^2}\right) + (-2)\left\{\frac{\cos n\pi x/a}{(n\pi/a)^3}\right\}\right|_0^a$$

$$= 0 - 0 + \frac{4a^2}{n^3\pi^3}(1 - \cos n\pi)$$

$$= \frac{8a^2}{n^3\pi^3} \text{ when } n \text{ is odd, otherwise zero when } n \text{ is even.}$$

Hence from (*viii*), the required solution is

$$v(x, y) = \frac{8a^2}{\pi^3} \sum_{n=1,3,5,\ldots}^{\infty} \frac{\sinh n\pi(b-y)/a}{n^3 \sinh n\pi b/a} \sin \frac{n\pi x}{a}$$

or

$$v(x, y) = \frac{8a^2}{\pi^3} \sum_{n=0}^{\infty} \frac{\sinh (2n+1)\pi(b-y)/a}{(2n+1)^3 \sinh (2n+1)\pi b/a} \sin \frac{(2n+1)\pi x}{a}.$$

PROBLEMS 18.4

1. A long rectangular plate of width a cm. with insulated surface has its temperature v equal to zero on both the long sides and one of the short sides so that $v(0, y) = 0$, $v(a, y) = 0$, $v(x, \infty) = 0$, $v(x, 0) = kx$. Show that the steady-state temperature within the plate is
$$v(x, y) = \frac{2ak}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot e^{-n\pi y/a} \sin \frac{n\pi x}{a}.$$ (*J.N.T.U., 2005*)

2. A rectangular plate with insulated surface is 8 cm. wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along one short edge $y = 0$ is given by
$$u(x, 0) = 100 \sin (\pi x/8), \quad 0 < x < 8;$$
while the two long edges $x = 0$ and $x = 8$ as well as the other short edge are kept at 0°C, show that the steady-state temperature at any point of the plane is given by
$$u(x, y) = 100e^{-\pi y/8} \sin (\pi x/8).$$

3. A rectangular plate with insulated surface is 10 cm. wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature of the short edge $y = 0$ is given by
$$u = 20\,x \qquad \text{for } 0 \le x \le 5$$
and
$$u = 20(10 - x) \qquad \text{for } 5 \le x \le 10$$
and the two long edges $x = 0$, $x = 10$ as well as the other short edge are kept at 0°C, prove that the temperature u at any point (x, y) is given by
$$u = \frac{40}{\pi^2} \sum_{1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{10} \cdot e^{-(2n-1)\pi y/10}$$ (*Anna, 2009*)

4. Solve $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ for $0 < x < \pi$, $0 < y < \pi$, with conditions given : $u(0, y) = u(\pi, y) = u(x, \pi) = 0$, $u(x, 0) = \sin^2 x$.

5. A square plate is bounded by the lines $x = 0$, $y = 0$, $x = 20$ and $y = 20$. Its faces are insulated. The temperature along the upper horizontal edge is given by
$$u(x, 20) = x(20 - x), \text{ when } 0 < x < 20,$$
while other three edges are kept at 0°C. Find the steady state temperature in the plate. (*Madras, 2003*)

6. The temperature u is maintained at 0° along three edges of a square plate of length 100 cm. and the fourth edge is maintained at 100° until steady-state conditions prevail. Find an expression for the temperature u at any point (x, y). Hence show that the temperature at the centre of the plate
$$= \frac{200}{\pi}\left[\frac{1}{\cosh \pi/2} - \frac{1}{3 \cosh 3\pi/2} + \frac{1}{5 \cosh 5\pi/2} - \dots\right].$$

7. A square thin metal plate of side a is bounded by the lines $x = 0$, $x = a$, $y = 0$, $y = a$. The edges $x = 0$, $y = a$ are kept at zero temperature, the edge $y = 0$ is insulated and the edge $x = a$ is kept at constant temperature T_0. Show that in the steady state conditions, the temperature $u(x, y)$ at the point (x, y) is given by
$$u(x, y) = \frac{4T_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sinh \dfrac{(2n-1)\pi x}{2a} \cos \dfrac{(2n-1)\pi y}{2a}}{(2n-1) \sinh \dfrac{(2n-1)\pi}{2}}.$$

8. A rectangular plate has sides a and b. Taking the side of length a as OX and that of length b as OY and other sides to be $x = a$ and $y = b$, the sides $x = 0$, $x = a$, $y = b$ are insulated and the edge $y = 0$ is kept at temperature $u_0 \cos \dfrac{\pi x}{a}$. Find the temperature $u(x, y)$ in the steady-state.

18.8 (1) LAPLACE'S EQUATION IN POLAR COORDINATES

In the study of steady-state temperature distribution in a rectangular plate, it is usually convenient to employ Cartesian coordinates as hitherto done. Sometimes Polar coordinates (r, θ) are found to be more useful and the Cartesian form of Laplace's equation is replaced by its polar form :
$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0$$ (See Ex. 5.24, p. 213-214)

(2) Solution of Laplace's equation

$$r^2\frac{\partial^2 u}{\partial r^2}+r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial \theta^2}=0 \quad ...(1)$$

Assume that a solution of (1) is of the form $u = R(r) \,.\, \phi(\theta)$ where R is a function of r alone and ϕ is a function of θ only.

Substituting it in (1), we get $r^2R''\phi + rR'\phi + R\phi'' = 0$ or $\phi(r^2R'' + rR') + R\phi'' = 0$.

Separating the variables $\dfrac{r^2R''+rR'}{R}=-\dfrac{\phi''}{\phi}$ ...(2)

Clearly the left side of (2) is a function of r only and the right side is a function of θ alone. Since r and θ are independent variables, (2) can hold good only if each side is equal to a constant k (say). Then (2) leads to the ordinary differential equations

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-kR=0 \quad ...(3) \qquad \text{and} \qquad \frac{d^2\phi}{d\theta^2}+k\phi=0 \quad ...(4)$$

Putting $r = e^z$, (3) reduces to $\dfrac{d^2R}{dz^2}-kR=0$ ...(5)

Solving (5) and (4), we get

(*i*) *When k is positive* and $= p^2$, say :

$$R = c_1e^{pz} + c_2e^{-pz} = c_1r^p + c_2r^{-p},\ \phi = c_3 \cos p\theta + c_4 \sin p\theta$$

(*ii*) *When k is negative* and $= -p^2$, say

$$R = c_5 \cos pz + c_6 \sin pz = c_5 \cos (p \log r) + c_6 \sin (p \log r),\ \phi = c_7e^{p\theta} + c_8e^{-p\theta}$$

(*iii*) *When k is zero* :

$$R = c_9z + c_{10} = c_9 \log r + c_{10},\ \phi = c_{11}\theta + c_{12}$$

Thus the three possible solutions of (1) are

$$u = (c_1r^p + c_2r^{-p})\,(c_3 \cos p\theta + c_4 \sin p\theta) \quad ...(6)$$

$$u = [c_5 \cos (p \log r) + c_6 \sin (p \log r)]\,(c_7e^{p\theta} + c_8e^{-p\theta}) \quad ...(7)$$

$$u = (c_9 \log r + c_{10})\,(c_{11}\theta + c_{12}) \quad ...(8)$$

Of these solutions, we have to take that solution which is consistent with the physical nature of the problem. The general solution will consist of a sum of terms of type (6), (7) or (8). (*S.V.T.U., 2008*)

Example 18.18. *The diameter of a semi-circular plate of radius a is kept at 0°C and the temperature at the semi-circular boundary is T°C. Show that the steady state temperature in the plate is given by*

$$u(r,\theta)=\frac{4T}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\left(\frac{r}{a}\right)^{2n-1}\sin(2n-1)\theta. \qquad \textit{(Kerala M. Tech., 2005)}$$

Solution. Take the centre of the circle as the pole and bounding diameter as the initial line as in Fig. 18.8. Let the steady state temperature at any point $P(r, \theta)$ be $u(r, \theta)$, so that u satisfies the equation

$$r^2\frac{\partial^2 u}{\partial r^2}+r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial \theta^2}=0 \quad ...(i)$$

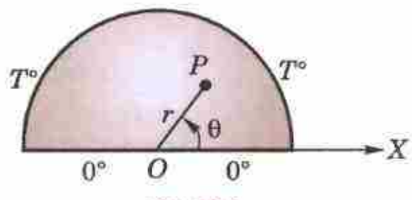


Fig. 18.8

The boundary conditions are :

$$u(r, 0) = 0 \quad \text{in } 0 \le r \le a \quad ...(ii)$$

$$u(r, \pi) = 0 \quad \text{in } 0 \le r \le a \quad ...(iii)$$

and

$$u(a, \theta) = T \quad ...(iv)$$

The three possible solutions of (*i*) are

$$u = (c_1r^p + c_2r^{-p})\,(c_3 \cos p\theta + c_4 \sin p\theta) \quad ...(v)$$

$$u = [c_5 \cos (p \log r) + c_6 \sin (p \log r)]\,(c_7e^{p\theta} + c_8e^{-p\theta}) \quad ...(vi)$$

$$u = (c_9 \log r + c_{10})\,(c_{11}\theta + c_{12}) \quad ...(vii)$$

From (*ii*) and (*iii*), $u = 0$ when $r = 0$ *i.e.*, u must be finite at the origin. Thus the solutions (*vi*) and (*vii*) are to be rejected. Hence the only suitable solution is (*v*).

By (ii), $u(r, \theta) = (c_1 r^p + c_2 r^{-p})\, c_3 = 0$

Hence $c_3 = 0$ and (v) becomes $u(r, \theta) = (c_1 r^p + c_2 r^{-p})\, c_4 \sin p\theta$ ...$(viii)$

By (iii), $u(r, \pi) = (c_1 r^p + c_2 r^{-p})\, c_4 \sin p\pi = 0.$

As $c_4 \neq 0$, $\sin p\pi = 0$, *i.e.*, $p = n$, where n is any integer.

Hence $(viii)$ reduces to $u(r, \theta) = (c_1 r^n + c_2 r^{-n})\, c_4 \sin n\theta$ ...(ix)

Since $u = 0$, when $r = 0$, $\therefore$ $c_2 = 0$ and (ix) becomes

$u(r, \theta) = b_n r^n \sin n\theta$, where $b_n = c_1 c_4$.

$\therefore$ the most general solution of (i) is of the form

$$u(r, \theta) = \sum_{n=1}^{\infty} b_n r^n \sin n\theta \qquad ...(x)$$

Putting $r = a$,

$$u(a, \theta) = \sum_{n=1}^{\infty} b_n a^n \sin n\theta. \qquad ...(xi)$$

In order that (iv) may be satisfied, (iv) and (xi) must be same. This requires the expansion of T as a half-range Fourier sine series in $(0, \pi)$. Thus

$$T = \sum_{n=1}^{\infty} B_n \sin n\theta \quad \text{where} \quad B_n = \frac{2}{\pi}\int_0^{\pi} T \sin n\theta\, d\theta = \frac{2T}{n\pi}(1 - \cos n\pi) \quad \text{and } B_n = b_n a^n$$

$$\therefore \qquad b_n = \frac{B_n}{a^n} = \frac{2T}{n\pi a^n}(1 - \cos n\pi)$$

i.e., $b_n = 0$, if n is even

$$= \frac{4T}{n\pi a^n}, \text{ if } n \text{ is odd.}$$

Hence (x) gives $u(r, \theta) = \dfrac{4T}{\pi}\left\{\dfrac{(r/a)}{1}\sin\theta + \dfrac{(r/a)^3}{3}\sin 3\theta + \dfrac{(r/a)^5}{5}\sin 5\theta + \dots\right\}$

Example 18.19. *The bounding diameter of a semi-circular plate of radius a cm is kept at 0°C and the temperature along the semi-circular boundary is given by*

$$u(a, \theta) = \begin{cases} 50\theta, & \text{when } 0 < \theta \leq \pi/2 \\ 50(\pi - \theta) & \text{when } \pi/2 < \theta < \pi \end{cases}$$

Find the steady-state temperature function u (r, θ). (Madras, 2003)

Solution. We know that $u(r, \theta)$ satisfies the equation

$$r^2 \frac{\partial^2 u}{\partial\theta^2} + r\frac{\partial u}{\partial\theta} + \frac{\partial^2 u}{\partial\theta^2} = 0 \qquad ...(i)$$

The boundary conditions are $u(r, \theta) = 0$, $u(r, \pi) = 0$ ...(ii)

and $u(a, \theta) = 50\theta$ for $0 \leq \theta \leq \pi/2$; $u(a, \theta) = 50(\pi - \theta)$ for $\pi/2 \leq \theta \leq \pi$ ...(iii)

As in example 18.18, the most general solution of (i) satisfying the boundary conditions (ii) is of the form

$$u(r, \theta) = \sum_{n=1}^{\infty} b_n r^n \sin n\theta \qquad ...(iv)$$

Putting $r = a$,

$$r(a, \theta) = \sum_{n=1}^{\infty} b_n a^n \sin n\theta$$

In order that the boundary condition (iii) is satisfied, we have $r(a, \theta) = \sum_{n=1}^{\infty} B_n \sin n\theta$

where $b_n a^n = B_n = \dfrac{2}{\pi}\left\{\displaystyle\int_0^{\pi/2} 50\theta \sin n\theta\, d\theta + \int_{\pi/2}^{\pi} 50(\pi - \theta)\sin n\theta\, d\theta\right\}$ ...(v)

$$= \frac{100}{\pi}\left\{\left|\theta\left(\frac{-\cos n\theta}{\theta}\right)-(1)\left(\frac{-\sin n\theta}{n^2}\right)\right|_0^{\pi/2}+\left|(\pi-\theta)\left(\frac{-\cos n\theta}{n}\right)-(-1)\left(\frac{-\sin n\theta}{n^2}\right)\right|_{\pi/2}^{\pi}\right\}$$

$$= \frac{100}{\pi}\left\{-\frac{\pi}{2n}\cos\frac{n\pi}{2}+\frac{\sin n\pi/2}{n^2}+\frac{\pi}{2n}\cos\frac{n\pi}{2}+\frac{\sin n\pi/2}{n^2}\right\} = \frac{200}{\pi n^2}\sin n\pi/2.$$

When n is even $B_n = 0$, so taking $n = 1, 3, 5$ etc, (*iv*) gives

$$u(r,\theta) = \sum_{n=1,3,5,\ldots}^{\infty}\left(\frac{200}{\pi n^2}\sin\frac{n\pi}{2}\right)\frac{1}{a^n}\cdot r^n \sin n\theta$$

$$= \frac{200}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{(2m-1)^2}\left(\frac{r}{a}\right)^{2m-1}\sin(2m-1)\theta.$$

[Taking $n = 2m - 1$, $n = 1, 3, 5, \ldots$; gives $m = 1, 2, 3$, $\sin n\pi/2 = \sin(2m-1)\pi/2 = (-1)^{m-1}$. This gives the required temperature function.

PROBLEMS 18.5

1. A semi-circular plate of radius a has its circumference kept at temperature $u(a, \theta)$, $= k\theta(\pi - \theta)$ while the boundary diameter is kept at zero temperature. Find the steady state temperature distribution $u(r, \theta)$ of the plate assuming the lateral surfaces of the plate to be insulated.
2. A semi-circular plate of radius 10 cm has insulated faces and heat flows in plane curves. The bounding diameter is kept at 0°C and on the circumference the temperature distribution maintained is $u(10, \theta) = (400/\pi)(\pi\theta - \theta^2)$, $0 \le \theta \le \pi$. Determine the temperature distribution $u(r, \theta)$ at any point on the plate.
3. A plate in the shape of truncated quadrant of a circle, is bounded by $r = a$, $r = b$ and $\theta = 0$, $\theta = \pi/2$. It has its faces insulated and heat flows in plane curves. It is kept at temperature 0°C along three of the edges while along the edge $r = a$, it is kept at temperature $\theta(\pi/2 - \theta)$. Determine the temperature distribution.
4. Determine the steady state temperature at the points on the sector $0 \le \theta \le \pi/4$, $0 \le r \le a$ of a circular plate, if the temperature is maintained at 0°C along the side edges and at a constant temperature k°C along the curved edges.
5. Find the steady-state temperature in a circular plate of radius a which has one-half of its circumference at 0°C and the other half at 60°C.
6. If the radii of the inner and outer boundaries of a circular annulus area 10 cm and 20 cm and

$$u(10, \theta) = 15\cos\theta,\ u(20, \theta) = 30\sin\theta,$$

find the value of $u(r, \theta)$ in the annulus. [$u(r, \theta)$ satisfies Laplace equation in the interior of the annulus.]
7. A plate in the form of a ring is bounded by the lines $r = 2$ and $r = 4$. Its surfaces are insulated and the temperature along the boundaries are

$$u(2, \theta) = 10\sin\theta + 6\cos\theta,\ u(4, \theta) = 17\sin\theta + 15\cos\theta$$

Find the steady-state temperature $u(r, \theta)$ in the ring.

18.9 (1) VIBRATING MEMBRANE—TWO DIMENSIONAL WAVE EQUATION

We shall now derive the equation for the vibrations of a tightly stretched membrane, such as the membrane of a drum. We shall assume that the membrane is uniform and the tension T in it per unit length is the same in all directions at every point.

Consider the forces acting on an element $\delta x \delta y$ of the membrane (Fig. 18.9). Forces $T\delta x$ and $T\delta y$ act on the edges along the tangent to the membrane. Let u be its small displacement perpendicular to the xy-plane, so that the forces $T\delta y$ on its opposite edges of length δy make angles α and β to the horizontal. So their vertical component

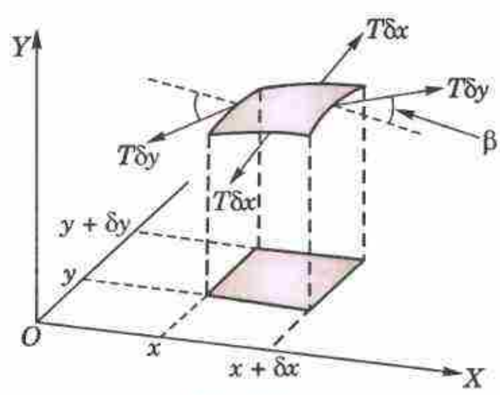


Fig. 18.9

$= T\delta y \sin\beta - T\delta y \sin\alpha$

$= T\delta y(\tan\beta - \tan\alpha)$ approximately, since α and β are small

$$= T\delta y\left\{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right\} = T\delta y\,\frac{\partial^2 u}{\partial x^2}\,\delta x, \text{ up to a first order of approximation.}$$

Similarly, the vertical component of the force $T\delta x$ acting on the edges of length δx

$$= T\delta x\,\frac{\partial^2 u}{\partial y^2}\,\delta y$$

If m be the mass per unit area of the membrane, then the equation of motion of the element is

$$m\delta x\delta y\,\frac{\partial^2 u}{\partial t^2} = T\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]\delta x\delta y \quad \text{or} \quad \frac{\partial^2 \mathbf{u}}{\partial \mathbf{t}^2} = \mathbf{c}^2\left[\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2}\right] \quad \text{where} \quad c^2 = T/m \qquad ...(1)$$

This is the wave equation in two dimensions.

(2) Solution of the two-dimensional wave equation - Rectangular membrane. Assume that a solution of (1) is of the form $u = X(x)Y(y)T(t)$

Substituting this in (1) and dividing by XYT, we get

$$\frac{1}{c^2T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2}$$

This can hold good if each member is a constant. Choosing the constants suitably, we have

$$\frac{d^2X}{dx^2} + k^2X = 0,\ \frac{d^2Y}{dy^2} + l^2Y = 0 \quad \text{and} \quad \frac{d^2T}{dt^2} + (k^2 + l^2)\,c^2T = 0$$

Hence a solution of (1) is

$$u = (c_1 \cos kx + c_2 \sin kx)\,(c_3 \cos ly + c_4 \sin ly) \times [c_5 \cos \sqrt{(k^2 + l^2)}ct + c_6 \sin \sqrt{(k^2 + l^2)}ct\,] \qquad ...(2)$$

Now suppose the membrane is rectangular and is stretched between the lines $x = 0, x = a, y = 0, y = b$. Then the condition $u = 0$ when $x = 0$ gives

$$0 = c_1(c_3 \cos ly + c_4 \sin ly)[c_5 \cos \sqrt{(k^2 + l^2)ct} + c_6 \sin \sqrt{(k^2 + l^2)ct}\,] \quad i.e., \quad c_1 = 0.$$

Then putting $c_1 = 0$ in (2) and applying the condition $u = 0$ when $x = a$, we get $\sin ka = 0$ or $k = m\pi/a$. (m being an integer)

Similarly, applying the conditions $u = 0$, when $y = 0$ and $y = b$, we obtain

$$c_3 = 0 \quad \text{and} \quad l = n\pi/b \qquad (n \text{ being an integer})$$

Thus the solution (2) becomes

$$u(x, y, t) = c_2c_4 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\,(c_5 \cos p_{mn}t + c_6 \sin p_{mn}t)$$

where $p_{mn} = \pi c\sqrt{[(m/a)^2 + (n/b)^2]}$ ...(3)

[These are the solutions of the wave equation (1) which are zero on the boundary of the rectangular membrane. These functions are called **eigen functions** and the numbers p_{mn} are the **eigen values** of the vibrating membrane.]

Choosing the constants c_2 and c_4 so that $c_2c_4 = 1$, we can write the general solution of the equation (1) as

$$u = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\,(A_{mn} \cos pt + B_{mn} \sin pt) \qquad ...(4)$$

If the membrane starts from rest from the initial position $u = f(x, y)$, *i.e.*, $\frac{\partial u}{\partial t} = 0$ when $t = 0$, then (3) gives $B_{mn} = 0$.

Also using the condition $u = f(x, y)$ when $t = 0$, we get

$$f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

This is *double Fourier series.* Multiplying both sides by $\sin(m\pi x/a)\sin(n\pi y/b)$ and integrating from $x = 0$ to $x = a$ and $y = 0$ to $y = b$, every term on the right except one, becomes zero. Hence we obtain

$$\int_0^a \int_0^b f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dydx = \frac{ab}{4} A_{mn} \qquad ...(5)$$

which gives the coefficients in the solution and is called the **generalised Euler's formula.**

Rectangular Membranes

Example 18.20. *Find the deflection $u(x, y, t)$ of the square membrane with $a = b = 1$ and $c = 1$, if the initial velocity is zero and the initial deflection is $f(x, y) = A \sin \pi x \sin 2\pi y$.*

Solution. Taking $a = b = 1$ and $f(x, y) = A \sin \pi x \sin 2\pi y$, in (5), we get

$$A_{mn} = 4 \int_0^1 \int_0^1 A \sin \pi x \sin 2\pi y \sin m\pi x \sin n\pi y\, dydx$$

$$= 4A \int_0^1 \sin \pi x \sin m\pi x\, dx \left(\int_0^1 \sin 2\pi y \sin n\pi y\, dy \right) = 0, \quad \text{for } m \neq 1$$

$$= 4A \left(\frac{1}{2}\right) \int_0^1 \sin 2\pi y \sin n\pi y\, dy, \text{ for } m = 1 \qquad \left[\because \int_0^1 \sin \pi x \sin \pi x\, dx = \tfrac{1}{2}\right]$$

i.e.,
$$A_{mn} = 2A \int_0^1 \sin 2\pi y \sin n\pi y\, dy = 0, \text{ for } n \neq 2$$

$$= 2A\left(\frac{1}{2}\right), \text{ for } n = 2.$$

$\therefore$ $A_{12} = A$. Also from (3), $p_{mn} = \pi\sqrt{(m^2 + n^2)}$ $\qquad [\because\ a = b = 1 = c]$

$\therefore$ $p_{12} = \pi\sqrt{(1^2 + 2^2)} = \sqrt{5}\,\pi$.

Hence from (4), the required solution is $u(x, y, t) = A \sin \pi x \sin 2\pi y \cos (\sqrt{5}\,\pi t)$.

Example 18.21. *Find the vibration $u(x, y, t)$ of a rectangular membrane $(0 < x < a,\ 0 < y < b)$ whose boundary is fixed given that it starts from rest and $u(x, y, 0) = hxy(a - x)(b - y)$.*

Solution. Proceeding as in § 18.9 (**2**), we have from (4),

$$u = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (A_{mn} \cos pt + B_{mn} \sin pt) \text{ where } p = \pi c\sqrt{[(m/a)^2 + (n/b)^2}$$

Since the membrane starts from rest $\partial u/\partial t = 0$ when $t = 0$,

$$\therefore \quad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (- A_{mn}\, p \sin pt + pB_{mn} \cos pt) = 0 \text{ when } t = 0$$

This gives $B_{mn} = 0$

$$\therefore \quad u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos pt \qquad ...(i)$$

Then
$$hxy(a - x)(b - y) = u(x, y, 0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

where
$$A_{mn} = \frac{2}{a} \cdot \frac{2}{b} \int_0^a \int_0^b hxy(a - x)(b - y) \,.\, \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dydx$$

$$= \frac{4h}{ab} \left\{ \int_0^a x(a - x) \sin \frac{m\pi x}{a} dx \right\} \left\{ \int_0^b y(b - y) \sin \frac{n\pi y}{b} dy \right\}$$

$$= \frac{4h}{ab} \left| (ax - x^2)\left(\frac{-\cos m\pi x/a}{m\pi/a}\right) - (a - 2x)\left\{\frac{-\sin \frac{m\pi x}{a}}{(m\pi/a)^2}\right\} + (-2)\frac{\cos m\pi x/a}{(m\pi/a)^3} \right|_0^a$$

$$\times \left| (by - y^2)\left(\frac{-\cos n\pi y/b}{n\pi/b}\right) - (b - 2y)\left\{\frac{-\sin n\pi/b}{(n\pi/b)^2}\right\} + (-2)\frac{\cos n\pi y/b}{(n\pi/b)^3} \right|_0^b$$

$$= \frac{4h}{ab} \frac{2a^3}{m^3\pi^3} \cdot \frac{2b^3}{n^3\pi^3} (1 - \cos m\pi)(1 - \cos n\pi)$$

Hence from (i), we get

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos pt$$

where $A_{mn} = \dfrac{16ha^2b^2}{m^3n^3\pi^6}(1 - \cos m\pi)(1 - \cos n\pi)$ and $p = \pi c\sqrt{[(m/a)^2 + (n/b)^2]}$

Circular Membranes*

Example 18.22. *A circular membrane of unit radius fixed along its boundary starts vibrating from rest and has initial deflection $u(r, 0) = f(r)$. Show that the deflection $u(r, t)$ of the membrane at any instant is given by*

$$u(r, t) = \sum_{m=1}^{\infty} A_m \cos (c\alpha_m t) \,.\, J_o(\alpha_m r) \text{ where } A_m = \frac{2}{J_1^2(\alpha_m)} \int_0^1 r f(r) J_0(\alpha_m r)\, dr,$$

and $\alpha_m (m = 1, 2, ...)$ are the positive roots of the Bessel function $J_0(k) = 0$.

Solution. The vibrations of a plane circular membrane are governed by 2-dimensional wave equation in polar coordinates *i.e.*,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right)$$

For a radially symmetric membrane (in which u does not depend on θ) the above equation reduces to

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) \qquad ...(i)$$

For the given membrane fixed along its boundary, the boundary condition is

$$u(1, t) = 0 \quad \text{for all } t \geq 0 \qquad ...(ii)$$

For solutions not depending on θ,

initial deflection $u(r, 0) = f(r)$ ...(iii)

and initial velocity $\left(\dfrac{\partial u}{\partial t}\right)_{t=0} = 0$ ...(iv)

which are the initial conditions. We find the solutions $u(r, t) = R(r)T(t)$ ...(v)

satisfying the boundary condition (ii).

Differentiating and substituting (v) in (i), we get

$$\frac{\partial^2 T/\partial t^2}{c^2 T} = \frac{1}{R}\left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r} \frac{\partial R}{\partial r} \right) = -k^2 \text{ (say)}$$

This leads to $\dfrac{\partial^2 T}{\partial t^2} + p^2 T = 0$ where $p = ck$ ...(vi)

and
$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r} \frac{\partial R}{\partial r} + k^2 R = 0 \qquad ...(vii)$$

Now putting $s = kr$, (vii) transforms to $\dfrac{d^2 R}{ds^2} + \dfrac{1}{s} \dfrac{dR}{ds} + R = 0$ which is Bessel's equation. Its general solution $R = aJ_0(s) + bY_0(s)$ where J_0 and Y_0 are Bessel's functions of the first and second kind of order zero.

Since the deflection of the membrane is always finite, we must have $b = 0$. Then taking $a = 1$, we get

$$R(r) = J_0(s) = J_0(kr)$$

On the boundary of the circular membrane, we must have $J_0(k) = 0$, which is satisfied for

$$k = \alpha_m,\ m = 1, 2, ...$$

*Drums, telephones and microphones provide examples of circular membrane and as such are quite useful in engineering.

Thus the solutions of (*vii*) are $R(r) = J_0(\alpha_m r)$, $m = 1, 2, \ldots$ and the corresponding solutions of (*vi*) are $T(t) = A_m \cos p_m t + B_m \sin p_m t$, where $p_m = ck_m = c\alpha_m$.

Hence the general solution of (*i*) satisfying (*ii*) are

$$u(r, t) = (A_m \cos p_m t + B_m \sin p_m t)\, J_0(\alpha_m r)$$

which are the *eigen functions* of the problem and the corresponding *eigen values* are p_m.

To find that solution which also satisfies the initial conditions (*iii*) and (*iv*), consider the series

$$u(r, t) = \sum_{m=1}^{\infty} (A_m \cos p_m t + B_m \sin p_m t)\, J_0(\alpha_m r)$$

Putting $t = 0$ and using (*iii*), we get $u(r, 0) = \sum_{m=1}^{\infty} A_m J_0(\alpha_m r) = f(r)$

Here, the constants A_m must be the coefficients of Fourier-Bessel series [(8) page 560] with $m = 0$, *i.e.*,

$$A_m = \frac{2}{J_1^2(\alpha_m)} \int_0^1 r f(r)\, J_0(\alpha_m r)\, dr$$

Using (*iv*), we get $B_m = 0$. Hence the result.

PROBLEMS 18.6

1. A tightly stretched unit square membrane starts vibrating from rest and its initial displacement is $k \sin 2\pi x \sin \pi y$. Show that the deflection at any instant is $k \sin 2\pi x \sin \pi y \cos(\sqrt{5}\,\pi ct)$.
2. Find the deflection $u(r, t)$ of the circular membrane of unit radius if $c = 1$, the initial velocity is zero and the initial deflection is $0.25\,(1 - r^2)$.

18.10 TRANSMISSION LINE

Consider a cable l km in length, carrying an electric current with resistance R ohms/km, inductance L henries/km ; capacitance C farads/km and leakance G mhos/km (Fig. 18.10).

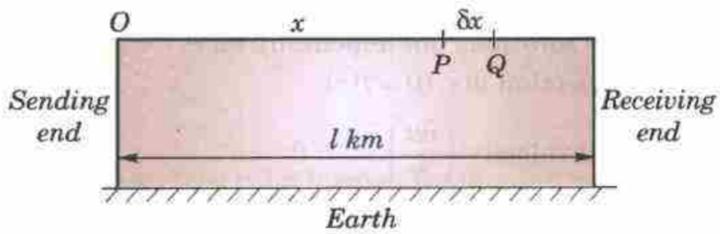


Fig. 18.10

Let the instantaneous voltage and current at any point P, distant x km from the sending end O, and at time t sec be $v(x, t)$ and $i(x, t)$ respectively. Consider a small length $PQ(= \delta x)$ of the cable.

Now since the voltage drop across the segment δx

= voltage drop due to resistance + voltage drop due to inductance

$$\therefore \qquad -\delta v = iR\delta x + L\delta x \cdot \frac{\partial i}{\partial t}$$

and dividing by δx and taking limits as $\delta x \to 0$, we get

$$-\frac{\partial v}{\partial x} = Ri + L\frac{\partial i}{\partial t} \qquad \ldots(1)$$

Similarly the current loss between P and Q

= current lost due to capacitance and leakance

$$\therefore \qquad -\delta i = C\frac{\partial v}{\partial t}\delta x + Gv\delta x$$ from which as before, we get

$$-\frac{\partial i}{\partial x} = C\frac{\partial v}{\partial t} + Gv \qquad \ldots(2)$$

Rewriting the simultaneous partial differential equations (1) and (2) as

$$\left(R + L\frac{\partial}{\partial t}\right) i + \frac{\partial v}{\partial x} = 0 \qquad \ldots(3)$$

and $$\frac{\partial i}{\partial x} + \left(C\frac{\partial}{\partial t} + G\right)v = 0, \quad ...(4)$$

we shall eliminate i and v in turn.

$\therefore$ operating (3) by $\frac{\partial}{\partial x}$ and (4) by $\left(R + L\frac{\partial}{\partial t}\right)$ and subtracting

$$\frac{\partial^2 v}{\partial x^2} - \left(R + L\frac{\partial}{\partial t}\right)\left(C\frac{\partial}{\partial t} + G\right)v = 0$$

or $$\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2} + (LG + RC)\frac{\partial v}{\partial t} + RGv \quad ...(5)$$

Again operating (3) by $\left(C\frac{\partial}{\partial t} + G\right)$ and (4) by $\frac{\partial}{\partial x}$ and subtracting

$$\left(C\frac{\partial}{\partial t} + G\right)\left(R + L\frac{\partial}{\partial t}\right)i - \frac{\partial^2 i}{\partial x^2} = 0$$

or $$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} + (LG + RC)\frac{\partial i}{\partial t} + RGi \quad ...(6)$$

which is (5) with v replaced by i. The equations (5) and (6) are called the *telephone equations*.

Cor. 1. *If* $L = G = 0$, the equations (5) and (6) become

$$\frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t} \quad ...(7) \qquad \frac{\partial^2 i}{\partial x^2} = RC\frac{\partial i}{\partial t} \quad ...(8)$$

which are known as the *telegraph equations*.

Rewriting (7) as $\frac{\partial v}{\partial t} = \frac{1}{RC}\frac{\partial^2 v}{\partial x^2}$, we observe that it is similar to the heat equation [(1) p. 611].

Cor. 2. *If* $R = G = 0$, the equations (5) and (6) become

$$\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2} \quad ...(9) \qquad \frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} \quad ...(10)$$

which are called the *radio equations*.

Rewriting (9) as $\frac{\partial^2 v}{\partial t^2} = k^2\frac{\partial^2 v}{\partial x^2}$ where $k^2 = \frac{1}{LC}$,

its general solution is $v(x, t) = f_1(x + kt) + f_2(x - kt)$. [See (4) p. 609]

Similarly from (10), $i(x, t) = F_1(x + kt) + F_2(x - kt)$.

Thus the voltage $v(x, t)$ for the current $i(x, t)$ at any point along the lossless transmission line can be obtained by the superposition of a progressive wave and a receding wave travelling with equal velocities (k). This is the case of oscillations of $v(x, t)$ and $i(x, t)$ at high frequencies.

Cor. 3. *If* $L = C = 0$, *e.g.*, in the case of a submarine cable, then (5) gives

$$\frac{\partial^2 v}{\partial x^2} = GRv, \text{ i.e. } (D^2 - GR)v = 0$$

$$\therefore \quad v(x) = A\cosh(\sqrt{GR}\,.\,x) + B\sinh(\sqrt{GR}\,.\,x) \quad ...(11)$$

Since by (1), $$Ri = -\frac{\partial v}{\partial x} = -\sqrt{GR}\,[A\sinh(\sqrt{GR}\,.\,x) + B\cosh(\sqrt{GR}\,.\,x)]$$

$$\therefore \quad i(x) = -\sqrt{G/R}\,[A\sinh(\sqrt{GR}\,.\,x) + B\cosh(\sqrt{GR}\,.\,x)] \quad ...(12)$$

If $v(0) = v_0$ and $i(0) = i_0$, then $v_0 = A$ and $i_0 = -\sqrt{(G/R)}B$.

Hence writing $\sqrt{(GR)} = \gamma$ and $\sqrt{(R/G)} = z_0$, (11) and (12) give

$$v(x) = v_0\cosh\gamma x - i_0 z_0\sinh\gamma x \quad ...(13)$$

and $$i(x) = i_0\cosh\gamma x - \frac{v_0}{z_0}\sinh\gamma x. \quad ...(14)$$

Obs. Steady-state solutions. We have so far considered the transient state solutions only. The steady-state solutions of transmission lines are however, obtained by assuming $v = Ve^{i\omega t}$ and $i = Ie^{i\omega t}$, where V and I are complex functions of x only. Substituting these in (5) and (6), we get two ordinary linear differential equations of the second order which can be solved at once.

Example 18.23. *Neglecting R and G, find the e.m.f. v(x, t) in a line of length l, t seconds after the ends were suddenly grounded, given that* $i(x, 0) = i_0$ *and* $v(x, 0) = e_1 \sin \frac{\pi x}{l} + e_5 \sin \frac{5\pi x}{l}$. (S.V.T.U., 2008)

Solution. Since R and G are negligible, we use the *Radio equation* $\frac{\partial^2 v}{\partial x^2} = LC \frac{\partial^2 v}{\partial t^2}$ ...(*i*)

Since the ends are suddenly grounded, we have the boundary conditions

$$v(0, t) = 0, v(l, t) = 0 \quad ...(ii)$$

Also the initial conditions are $i(x, 0) = i_0$

and $$v(x, 0) = e_1 \sin \pi x/l + e_5 \sin 5\pi x/l \quad ...(iii)$$

$\therefore$ $$\frac{\partial i}{\partial x} = -c\frac{\partial v}{\partial t} \quad \text{gives} \quad \frac{\partial v}{\partial t}(x, 0) = 0 \quad ...(iv)$$

Let $v = X(x)T(t)$ be the solution of (*i*).

$\therefore$ (*i*) gives $$X''T = LCXT''$$

$$\frac{X''}{X} = LC\frac{T''}{T} = -k^2 \text{ (say)}$$

$\therefore$ $$X'' + k^2X = 0 \quad \text{and} \quad T'' + (k^2/LC)T = 0$$

Solving these equations, we get

$$v = (c_1 \cos kx + c_2 \sin kx)\left(c_3 \cos \frac{k}{\sqrt{LC}}.t + c_4 \sin \frac{k}{\sqrt{LC}}t\right)$$

Using the boundary conditions (*ii*), we get

$$c_1 = 0 \quad \text{and} \quad k = n\pi/l.$$

$\therefore$ $$v = \sin \frac{n\pi x}{l}\left(a_n \cos \frac{n\pi}{l\sqrt{LC}}t + b_n \sin \frac{n\pi}{l\sqrt{LC}}t\right)$$

Using the initial condition (*iv*), we get $b_n = 0$

$\therefore$ $$v = a_n \sin \frac{n\pi x}{l} \cos \frac{n\pi}{l\sqrt{LC}}t$$

Thus the most general solution of (*i*) is

$$v = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \cos \frac{n\pi t}{l\sqrt{LC}}$$

Finally by the initial condition (*iii*), we have

$$e_1 \sin \frac{\pi x}{l} + e_5 \sin \frac{5\pi x}{l} = \sum a_n \sin \frac{n\pi x}{l}$$

$\therefore$ $$a_1 = e_1 \quad \text{and} \quad a_5 = e_5$$ while all other a's are zero.

Hence $$v = e_1 \sin \frac{\pi x}{l} \cos \frac{\pi t}{l\sqrt{LC}} + e_5 \sin \frac{5\pi x}{l} \cos \frac{5\pi t}{l\sqrt{LC}}$$

which is the required solution.

Example 18.24. *A telephone line 3000 km. long has a resistance of 4 ohms/km. and a capacitance of 5×10^{-7} farad/km. Initially both the ends are grounded so that the line is uncharged. At time t = 0, a constant e.m.f. E is applied to one end, while the other end is left grounded. Assuming the inductance and leakance to be negligible, show that the steady state current of the grounded end at the end of 1 sec. is 5.3%.*

Solution. Since $L = 0$, $G = 0$, we use the telegraph equation

$$\frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t}$$

Let $v = X(x)T(t)$ be its solution so that

$$TX'' = RCXT' \quad \text{or} \quad \frac{X''}{X} = RC\frac{T'}{T} = -k^2 \text{ (say)}$$

$\therefore$ $$X'' + k^2X = 0 \text{ and } T' + (k^2/RC)T = 0$$

Solving these equations, we get

$$X = c_1 \cos kx + c_2 \sin kx,\ T = c_3 e^{-k^2 t/RC}$$

giving $$v = (c_1 \cos kx + c_2 \sin kx)\, c_3 e^{-k^2 t/RC} \qquad ...(i)$$

When $t = 0$; $v = 0$ at $x = 0$ and $v = 0$ at $x = l$

$\therefore$ $$0 = c_1 c_3 \,;\ 0 = (c_1 \cos kl + c_2 \sin kl)c_3$$

i.e., $$c_1 c_3 = 0 \text{ and } kl = n\pi \ (n \text{ an integer})$$

Putting these in (i) and adding a linear term, we have

$$v = a_0 x + b_0 + \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\, e^{-n^2\pi^2 t/RCl^2} \qquad ...(ii)$$

The end conditions of the problem are

$$v = 0 \text{ at } x = 0 \text{ and } v = E \text{ at } x = l$$

Using these, (ii) gives $b_0 = 0$ and $a_0 = E/l$

Then (ii) becomes $$v = \frac{E}{l}x + \sum b_n \sin\frac{n\pi x}{l}\, e^{-n^2\pi^2 t/RCl^2}$$

Also $v = 0$ when $t = 0$, we get $-Ex/l = \Sigma b_n \sin n\pi x/l$

This requires the expansion of $(-Ex/l)$ as a half-range sine series in $(0, l)$.

$\therefore$ $$b_n = \frac{2}{l}\int_0^l \left(\frac{-Ex}{l}\right)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[\left(\frac{-Ex}{l}\right)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - \left(\frac{-E}{l}\right)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_0^l = \frac{2}{l}\left(\frac{El}{n\pi}\cos n\pi\right) = \frac{2E}{n\pi}(-1)^n.$$

Thus $$v = \frac{Ex}{l} + \frac{2E}{\pi}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\, e^{-n^2\pi^2 t/RCl^2} \qquad ...(iii)$$

Also when $L = 0$, $\dfrac{-\partial v}{\partial x} = Ri$

i.e., $$i = -\frac{1}{R}\frac{\partial v}{\partial x} = -\frac{E}{lR} - \frac{2E}{lR}\sum_{n=1}^{\infty} (-1)^n \cos\frac{n\pi x}{l}\, e^{-n^2\pi^2 t/RCl^2}$$

At the grounded end $(x = 0)$, the current is

$$i = -\frac{E}{lR} - \frac{2E}{lR}\sum_{n=1}^{\infty} (-1)^n e^{-n^2\pi^2 t/RCl^2}$$

When $t = 1$ sec, $$i = -\frac{E}{lR}\left(1 - 2e^{\pi^2/RCl^2} + 2e^{-4\pi^2/RCl^2} - ...\right) \qquad ...(iv)$$

Since $$\frac{\pi^2}{RCl^2} = \frac{(3.14)^2}{4(5\times 10^{-7})(3000)^2} = 0.548$$

$\therefore$ $$e^{-\pi^2/RCl^2} = e^{-0.548} = 0.578$$

When $t \to \infty$, $i \to -E/lR$

Hence from (*iv*), we have

$$i = -\frac{E}{lR}\{1-2(0.578)+2(0.578)^4-2(0.578)^9+\ldots\}$$
$$= -\frac{E}{lR}\{1-1.156+0.223-0.014+\ldots\}$$
$$= i_\infty(0.053) = 5.3\% \text{ of } i_\infty.$$

Example 18.25. *A transmission line 1000 kilometers long is initially under steady-state conditions with potential 1300 volts at the sending end (x = 0) and 1200 volts at the receiving end (x = 1000). The terminal end of the line is suddenly grounded, but the potential at the source is kept at 1300 volts. Assuming the inductance and leakance to be negligible, find the potential v(x, t).* (Andhra, 2000)

Solution. The equation of the telegraph line is

$$\frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t} \quad \text{or} \quad \frac{\partial v}{\partial t} = \frac{1}{RC}\frac{\partial^2 v}{\partial x^2} \qquad \ldots(i)$$

$$v_s = \text{initial steady voltage satisfying } \frac{\partial^2 v}{\partial x^2} = 0 = 1300 - x/10 = v(x, 0) \qquad \ldots(ii)$$

v_s' = steady voltage (after grounding the terminal end) when steady conditions are ultimately reached = $1300 - 1.3x$

$\therefore$ $v(x, t) = v_s' + v_t(x, t)$ where $v_t(x, t)$ is the transient part

$$= 1300 - 1.3x + \sum_{n=1}^{\infty} b_n e^{-(n^2\pi^2 t)/(l^2 RC)} \sin\frac{n\pi x}{l} \qquad \text{[By (viii), p. 614]} \qquad \ldots(iii)$$

where l = 1000 kilometers.

Putting $t = 0$, we have from (*ii*) and (*iii*)

$$1300 - 0.1x = v(x, 0) = 1300 - 1.3x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

i.e.
$$1.2x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \text{ where } b_n = \frac{2}{l}\int_0^1 1.2\sin\frac{n\pi x}{l}\,dx = \frac{2400}{\pi}\cdot\frac{(-1)^{n+1}}{n}$$

Hence
$$v(x, t) = 1300 - 1.3x + \frac{2400}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} e^{-(n^2\pi^2 t)/(l^2 RC)} \sin\frac{n\pi x}{1000}.$$

PROBLEMS 18.7

1. Find the current i and voltage e in a line of length l, t seconds after the ends are suddenly grounded, given that $i(x, 0) = i_0$, $e(x, 0) = e_0 \sin(\pi x/l)$.
   Also R and G are negligible.
2. Show that a transmission line with negligible resistance and leakage propagates waves of current and potential with a velocity equal to $l/\sqrt{(LC)}$, where L is the self-inductance and C is the capacitance.
3. A steady voltage distribution of 20 volts at the sending end and 12 volts at the receiving end is maintained in a telephone wire of length l. At time $t = 0$, the receiving end is grounded. Find the voltage and current t sec later. Neglect leakance and inductance.
4. Obtain the solution of the *radio equation*

$$\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2}$$

appropriate to the case when a periodic e.m.f. V_0 cost pt is applied at the end $x = 0$ of the line.

18.11 LAPLACE'S EQUATION IN THREE DIMENSIONS

We have seen that the three dimensional heat flow equation in steady state reduces to

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad ...(1)$$

which is the *Laplace's equation in three dimensions.*

Laplace's equation also arises in the study of gravitational potential at (x, y, z) of a particle of mass m situated at (ξ, η, ζ) given by

$$\frac{Gm}{r} \text{ where } r = \sqrt{[(x-\zeta)^2 + (y-\eta)^2 + (z-\zeta)^2]}$$

This function is called the *potential of the gravitational field* and satisfies the Laplace's equation.

If a mass of density ρ at (ξ, η, ζ) is distributed throughout a region R, then the gravitational potential u at an external point (x, y, z) is given by

$$u(x, y, z) = G \iiint_R \frac{\rho}{r}\, d\xi\, d\eta\, d\zeta \qquad ...(2)$$

Since $\nabla^2(1/r) = 0$ and ρ is independent of x, y and z, we get

$$\nabla^2 u = \iiint_R \rho\, \nabla^2 (1/r)\, d\xi\, d\eta\, d\zeta = 0.$$

This shows that the gravitational potential defined by (2) also obeys Laplace's equation.

Thus Laplace's equation (1) is one of the most important equations arising in connection with numerous problems of physics and engineering. *The theory of its solutions is called the* **potential theory** *and its solutions are called the* **harmonic functions.**

In most of the problems leading to Laplace's equation, it is required to solve the equation subject to certain boundary conditions. A proper choice of coordinate system makes the solution of the problem simpler. Now we proceed to take up the solutions of (1) in its other forms.

18.12 SOLUTIONS OF THREE DIMENSIONAL LAPLACE'S EQUATION

(1) **Cartesian form of $\nabla^2 u = 0$** is $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ ...(1)

Let $\qquad u = X(x)Y(y)Z(z)$ ...(2)

be a solution of (1). Substituting (2) in (1) and dividing by XYZ, we obtain

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\cdot\frac{d^2Z}{dz^2} = 0 \qquad ...(3)$$

which is of the form $F_1(x) + F_2(y) + F_3(z) = 0$.

As x, y, z are independent, this will hold good only if F_1, F_2, F_3 are constants. Assuming these constants to be $k^2, l^2, -(k^2 + l^2)$ respectively, (3) leads to the equations

$$\frac{d^2X}{dx^2} - k^2X = 0,\ \frac{d^2Y}{dy^2} - l^2Y = 0,\ \frac{d^2Z}{dz^2} + (k^2 + l^2)Z = 0$$

Their solutions are $\qquad X = c_1e^{kx} + c_2e^{-kx},\ Y = c_3e^{ly} + c_4e^{-ly}$

$$Z = c_5 \cos \sqrt{(k^2 + l^2)}z + c_6 \sin \sqrt{(k^2 + l^2)}z$$

Thus a possible solution of (1) is

$$u = (c_1e^{kx} + c_2e^{-kx})(c_3e^{ly} + c_4e^{-ly})\{c_5 \cos \sqrt{(k^2 + l^2)}z + c_6 \sin \sqrt{(k^2 + l^2)}z\}.$$

Since the three constants could have been taken as $-k^2, -l^2$ and $k^2 + l^2$, an alternative solution of (1) will be

$$u = (c_1 \cos kx + c_2 \sin kx)(c_3 \cos ly + c_4 \sin ly)\{c_5e^{\sqrt{(k^2+l^2)}z} + c_6e^{-\sqrt{(k^2+l^2)}z}\}.$$

(2) **Cylindrical form of $\nabla^2 u = 0$** is $\dfrac{\partial^2 u}{\partial \rho^2} + \dfrac{1}{\rho}\dfrac{\partial u}{\partial \rho} + \dfrac{1}{\rho^2}\dfrac{\partial^2 u}{\partial \phi^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ ...(1)

Let $u = R(\rho)\, H(\phi)\, Z(z)$ [(*iv*) p. 359]
be a solution of (1). Substituting it in (1), and dividing by RHZ, we get

$$\frac{1}{R}\left(\frac{d^2R}{d\rho^2}+\frac{1}{\rho}\frac{dR}{d\rho}\right)+\frac{1}{\rho^2H}\frac{d^2H}{d\phi^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=0 \qquad ...(2)$$

Assuming that $\dfrac{d^2H}{d\phi^2} = -n^2H$ and $\dfrac{d^2Z}{dz^2} = k^2Z$, ..(3)

(2) reduces to $$\frac{1}{R}\left(\frac{d^2R}{d\rho^2}+\frac{1}{\rho}\frac{dR}{d\rho}\right)-\frac{n^2}{\rho^2}+k^2=0$$

or $$\rho^2\frac{d^2R}{d\rho^2}+\rho\frac{dR}{d\rho}+(k^2\rho^2-n^2)R=0.$$

This is Bessel's equation [§ 16.10 (1)] and its solution is $R = c_1J_n(k\rho) + c_2Y_n(k\rho)$.

Also the solutions of equations (3) are

$$H = c_3 \cos n\phi + c_4 \sin n\phi,\ Z = c_5e^{kz} + c_6e^{-kz}$$

Thus a solution of (1) is

$$u = [c_1J_n(k\rho) + c_2Y_n(k\rho)]\,[c_3 \cos n\phi + c_4 \sin n\phi]\,[c_5e^{kz} + c_6e^{-kz}]$$

which is known as a *cylindrical harmonic*. (*Assam, 1999*)

(3) Spherical form of $\nabla^2 u = 0$ is

$$\frac{\partial^2u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2u}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2u}{\partial\phi^2}=0 \qquad ...(1)\quad [(iv)\text{ p. }361]$$

Let $u = R(r)\, G(\theta)\, H(\phi)$ be a solution of (1).

Then $$\frac{1}{R}\left(r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}\right)+\frac{1}{G}\left(\frac{d^2G}{d\theta^2}+\cot\theta\frac{dG}{d\theta}\right)+\frac{1}{H\sin^2\theta}\frac{d^2H}{d\phi^2}=0$$

Putting $$\frac{1}{R}\left(r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}\right)=n(n+1) \qquad ...(2) \qquad \text{and} \quad \frac{1}{H}\frac{d^2H}{d\phi^2}=-m^2, \qquad ...(3)$$

the above equation takes the form

$$\frac{d^2G}{d\theta^2}+\cot\theta\frac{dG}{d\theta}+[n(n+1)-m^2\,\text{cosec}^2\,\theta]G=0 \qquad ...(4)$$

Now differentiating the *Legendre's equation* (§ 16.13)

$$(1-x^2)y''-2xy'+n(n+1)y=0,$$

m times with respect to x and writing $u = d^my/dx^m$, we get

$$(1-x^2)u''-2(m+1)xu'+(n-m)(n+m+1)u=0 \qquad ...(5)$$

Now putting $G = (1-x^2)^{m/2}\,u$ in (5), we get

$$(1-x^2)\frac{d^2G}{dx^2}-2x\frac{dG}{dx}+\left[n(n+1)-\frac{m^2}{1-x^2}\right]G=0 \qquad ...(6)$$

Now putting $x = \cos\theta$ in (6), it reduces to (4) and its solution is

$$G = c_1P_n^m(\cos\theta) + c_2Q_n^m(\cos\theta)$$

The solution of (3) is $H = c_3 \cos m\phi + c_4 \sin m\phi$

To solve (2), write $R = r^k$, so that $k(k-1) + 2k = n(n+1)$ which gives $k = n$ or $-(n+1)$

Thus $$R = c_5r^n + c_6r^{-n-1}$$

Hence the general solution of (1) is

$$u = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\{c_1P_n^m(\cos\theta)+c_2Q_n^m(\cos\theta)\}(c_3\cos m\phi+c_4\sin m\phi)\times(c_5r^n+c_6r^{-n-1})$$

Any solution of (1) is known as a *spherical harmonic*.

Example 18.26. *Find the potential in the interior of a sphere of unit radius when the potential on the surface is* $f(\theta) = \cos^2 \theta$.

Solution. Take the origin at the centre of the given sphere S. Since the potential is independent of ϕ on S, so also is the potential at any point. Therefore, the Laplace's equation in spherical co-ordinates reduces to

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0 \qquad ...(i)$$

Putting $u(r, \theta) = R(r)\, G(\theta)$ in (i) and proceeding as in § 18.12 (3), we obtain the equations

$$\frac{\partial^2 G}{d\theta^2} + \cot\theta \frac{dG}{d\theta} + n(n+1)G = 0 \qquad ...(ii)$$

and

$$\frac{1}{R}\left(r^2 \frac{d^2 R}{dr^2} + 2r\frac{dR}{dr}\right) = n(n+1) \qquad ...(iii)$$

Putting $\cot\theta = v$, (ii) takes the form

$$(1 - v^2)\frac{d^2 G}{dv^2} - 2v\frac{dG}{dv} + n(n+1)G = 0$$

which is Legendre's equation. Its solutions are

$$G = P_n(v) = P_n(\cos\theta) \text{ for } n = 0, 1, 2, ...$$

The solutions of (iii) are $R_n(r) = r^n$, $\overline{R_n}\,(r) = 1/r^{n+1}$.

Hence the equation (i) has the following two sets of solutions

$$u_n(r, \theta) = c_n r^n P_n(\cos\theta) \text{ and } \bar{u}_n(r, \theta) = c_n P_n(\cos\theta)/r^{n+1}, \text{ where } n = 0, 1, 2, ...$$

For points inside S, we have the general equation $u(r, \theta) = \sum_{n=0}^{\infty} c_n r^n P_n(\cos\theta)$ ...(iv)

On the boundary of S, $u(1, \theta) = f(\theta)$ $\therefore$ $f(\theta) = \sum_{n=0}^{\infty} c_n P_n(\cos\theta)$

which is Fourier-Legendre expansion of $f(\theta)$. Hence by (5) p. 560, we have

$$c_n = \left(n + \frac{1}{2}\right)\int_{-1}^{1} f(\theta)\, P_n(x)\, dx \text{ where } x = \cos\theta.$$

$$= \left(n + \frac{1}{2}\right)\int_{-1}^{1} x^2 P_n(x) dx \qquad [\because f(\theta) = \cos^2\theta]$$

$$= \left(n + \frac{1}{2}\right)\int_{-1}^{1}\left[\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right]P_n(x)\, dx \qquad [\because P_2(x) = \tfrac{1}{2}(3x^2 - 1)]$$

Using the orthogonality of Legendre polynomials, we get

$c_n = 0$, except for $n = 0, 2$. Hence

$$c_0 = \frac{1}{2}\cdot\frac{1}{3}\int_{-1}^{1} P_0^2(x)\, dx = \frac{1}{3},\ c_2 = \frac{5}{2}\cdot\frac{2}{3}\int_{-1}^{0} P_2^2(x)\, dx = \frac{2}{3}.$$

Substituting in (iv), we get $u(r, \theta) = \frac{1}{3} + \frac{2}{3}r^2 P_2(\cos\theta)$ or $u(r, \theta) = \frac{1}{3} + r^2(\cos^2\theta - \frac{1}{3})$.

PROBLEMS 18.8

1. Show that a solution of Laplace's equation in cylindrical co-ordinates, which remains finite at $r = 0$, may be expressed in the form

$$u = \sum_{n=0}^{\infty} J_n(kr)\,\{e^{kz}(A_n \cos n\theta + B_n \sin n\theta) + e^{-kz}(C_n \cos n\theta + D_n \sin n\theta)\}.$$

2. The potential on the surface of a unit sphere is $f(\theta) = \cos 2\theta$. Show that the potential at all points of space is given by

$$u(r, \theta) = 2r^2(\cos^2\theta - 1/3) - \frac{1}{3} \text{ for } r < 1,$$

and
$$u(r, \theta) = 2r^{-3}(\cos^2\theta - 1/3) - r^{-1}/3 \text{ for } r > 1.$$

3. Show that in spherical polar coordinates (r, θ, ϕ), Laplace's equation possesses solutions of the form

$$(Ar^n + B/r^{n+1})P_n(\mu)e^{\pm im\phi},$$

where $\mu = \cos\theta$, A, B, m, n are constants and $P_n(\mu)$ satisfies Legendre's equation

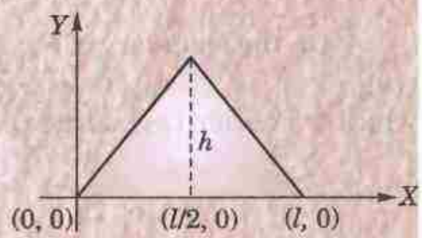
$$(1-\mu^2)\frac{d^2P_n}{d\mu^2} - 2\mu\frac{dP_n}{d\mu} + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\}P_n = 0.$$

18.13 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 18.9

Fill up the blanks in each of the following questions :

1. The radio equations for the potential and current are ....
2. The partial differential equation representing variable heat flow in three dimensions, is ....
3. Temperature gradient is defined as ....
4. The differential equation $f_{xx} + 2f_{xy} + 4f_{yy} = 0$ is classified as ....
5. The partial differential equation of the transverse vibrations of a string is ....
6. The steady state temperature of a rod of length l whose ends are kept at 30° and 40° is ....
7. The equation $u_t = c^2u_{xx}$ is classified as ....
8. The two dimensional steady state heat flow equation in polar coordinates is ....
9. The mathematical function of the initial form of the string given by the following graph is ....



10. When a vibrating string fastened to two points l apart, has an initial velocity u_0, its initial conditions are ....
11. In two dimensional heat flow, the temperature along the normal to the xy-plane is ....
12. If a square plate has its faces and the edge $y = 0$ insulated, its edges $x = 0$ and $x = a$ are kept at zero temperature and the fourth edge is kept at temperature u, then the boundary conditions for this problem are ....
13. If the ends $x = 0$ and $x = l$ are insulated in one dimensional heat flow problems, then the boundary conditions are ....
14. D'Alembert's solution of the wave equation is ....
15. The partial differential equation of 2-dimensional heat flow in ....
16. A rod 50 cm long with insulated sides has its end A and B kept at 20° and 70°C respectively. The steady state temperature distribution of the rod is .... *(Anna, 2008)*
17. The three possible solutions of Laplace equation in polar coordinates are ....
18. Solution of $\dfrac{\partial u}{\partial x} = 4\dfrac{\partial u}{\partial y}$, given $u(0, y) = 8e^{-3y}$, is ....
19. Solution of $\dfrac{\partial z}{\partial x} + 4z = \dfrac{\partial z}{\partial t}$, given $z(x, 0) = 4e^{-3x}$, is ....
20. In the equation $\dfrac{\partial u}{\partial t} = \alpha^2\dfrac{\partial^2 u}{\partial x^2}, \alpha^2$ represents ....
21. The telegraph equations for potential and current are ......
22. The general solution of one-dimensional heat flow equation when both ends of the bar are kept at zero temperature, is of the form ....
23. The three possible solutions of Laplace equation $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$ are ....

CHAPTER 19

Complex Numbers and Functions

19.1 COMPLEX NUMBERS

Definition. *A number of the form $x + iy$, where x and y are real numbers and $i = \sqrt{(-1)}$, is called a* **complex number.**

x is called the *real part* of $x + iy$ and is written as $R(x + iy)$ and y is called the *imaginary part* and is written as $I(x + iy)$.

A pair of complex numbers $x + iy$ and $x - iy$ are said to be conjugate of each other.

Properties : (1) *If $x_1 + iy_1 = x_2 + iy_2$, then $x_1 - iy_1 = x_2 - iy_2$.*

(**2**) *Two complex numbers $x_1 + iy_1$ = and $x_2 + iy_2$ are said to be equal when*

$$R(x_1 + iy_1) = R(x_2 + iy_2), \text{ i.e., } x_1 = x_2$$

and

$$I(x_1 + iy_1) = I(x_2 + iy_2), \text{ i.e., } y_1 = y_2.$$

(**3**) *Sum, difference, product and quotient of any two complex numbers is itself a complex number.*

If $x_1 + iy_1$ and $x_2 + iy_2$ be two given complex numbers, then

(i) their sum $= (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$

(ii) their difference $= (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)$

(i) their product $= (x_1 + iy_1) + (x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$

and (iv) their quotient $= \dfrac{x_1 + iy_1}{x_2 + iy_2} = \dfrac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \dfrac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\,\dfrac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}$

(**4**) *Every complex number $x + iy$ can always be expressed in the form r (cos θ + i sin θ).*

Put $\quad R(x + iy)$, *i.e.*, $x = r \cos\theta$ ...(i)

and $\quad I(x + iy)$, *i.e.*, $y = r \sin\theta$ ...(ii)

Squaring and adding, we get $x^2 + y^2 = r^2$ *i.e.* $r = \sqrt{(x^2 + y^2)}$ (taking positive square root only)

Dividing (ii) by (i), we get $y/x = \tan\theta$ *i.e.* $\theta = \tan^{-1}(y/x)$.

Thus $x + iy = r(\cos\theta + i\sin\theta)$ where $r = \sqrt{(x^2 + y^2)}$ and $\theta = \tan^{-1}(y/x)$.

Definitions. *The number* $r = +\sqrt{(x^2 + y^2)}$ *is called the* **modulus** *of* $x + iy$ *and is written as mod* $(x + iy)$ *or* $|x + iy|$.

The angle θ *is called the amplitude or argument of* $x + iy$ *and is written as amp* $(x + iy)$ *or arg* $(x + iy)$.

Evidently the amplitude θ has an infinite number of values. The value of θ which lies between $-\pi$ and π is called the *principal value of the amplitude.* Unless otherwise specified, we shall take amp (z) to mean the principal value.

Note. *$\cos\theta + i\sin\theta$ is briefly written as cis θ (pronounced as 'sis θ')*

(**5**) *If the conjugate of* $z = x + iy$ be $\bar{z}$, *then*

(*i*) $R(z) = \dfrac{1}{2}(z + \bar{z}),\ I(z) = \dfrac{1}{2i}(z - \bar{z})$ (*ii*) $|z| = \sqrt{R^2(z) + I^2(z)} = |\bar{z}|$

(*iii*) $z\bar{z} = |z|^2$ (*iv*) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(*v*) $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$ (*vi*) $\overline{(z_1/z_2)} = \bar{z}_1/\bar{z}_2$, where $\bar{z}_2 \neq 0$.

Example 19.1. *Reduce $1 - \cos\alpha + i\sin\alpha$ to the modulus amplitude form.*

Solution. Put $1 - \cos\alpha = r\cos\theta$ and $\sin\alpha = r\sin\theta$

$\therefore$ $$r = (1 - \cos\alpha)^2 + \sin^2\alpha = 2 - 2\cos\alpha = 4\sin^2\alpha/2$$

i.e., $$r = 2\sin\alpha/2$$

and $$\tan\theta = \frac{\sin\alpha}{1 - \cos\alpha} = \frac{2\sin\alpha/2\cos\alpha/2}{2\sin^2\alpha/2} = \cot\alpha/2$$

$$= \tan\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) \quad \therefore \theta = \frac{\pi - \alpha}{2}.$$

Thus $1 - \cos\alpha + i\sin\alpha = 2\sin\dfrac{\alpha}{2}\left[\cos\dfrac{\pi - \alpha}{2} + i\sin\dfrac{\pi - \alpha}{2}\right]$.

Example 19.2. *Find the complex number z if arg $(z + 1) = \pi/6$ and arg $(z - 1) = 2\pi/3$.* (*Mumbai, 2009*)

Solution. Let $z = x + iy$ so that $z + 1 = (x + 1) + iy$ and $(z - 1) = (x - 1) + iy$

Since $arg\,(z + 1) = \pi/6$, $\therefore \tan^{-1}\left(\dfrac{y}{x+1}\right) = 30°$

i.e., $$\frac{y}{x+1} = \tan 30° = 1/\sqrt{3}, \text{ or } \sqrt{3}y = x + 1 \qquad ...(i)$$

Also since $\arg(z - 1) = 2\pi/3$, $\therefore \tan^{-1}\left(\dfrac{y}{x+1}\right) = 120°$

i.e., $$\frac{y}{x-1} = \tan 120° = -\sqrt{3}, \text{ or } y = -\sqrt{3}x + \sqrt{3} \text{ or } \sqrt{3}y = -3x + 3 \qquad ...(ii)$$

Subtracting (*ii*) from (*i*), we get $4x - 2 = 0$ *i.e.,* $x = 1/2$

From (*i*), $\sqrt{3}\,y = 1/2 + 1$, *i.e.,* $y = \sqrt{3}/2$

Hence $z = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$.

Example 19.3. *Find the real values of x, y so that $-3 + ix^2y$ and $x^2 + y + 4i$ may represent complex conjugate numbers.*

Solution. If $z = -3 + ix^2y$, then $\bar{z} = x^2 + y + 4i$

so that $$z = (x^2 + y) - 4i$$

$\therefore$ $$-3 + ix^2y = x^2 + y - 4i$$

Equating real and imaginary parts from both sides, we get

$$-3 = x^2 + y, x^2y = -4$$

Eliminating x, $(y + 3)y = -4$

or $y^2 + 3y - 4 = 0$ *i.e.*, $y = 1$ or -4

When $y = 1$, $x^2 = -3 - 1$ or $x = +2i$ which is not feasible

When $y = -4$, $x^2 = 1$ or $x = \pm 1$

Hence $x = 1$, $y - 4$ or $x = -1, y = -4$.

19.2 (1) GEOMETRIC REPRESENTATION OF IMAGINARY NUMBERS

Let all the real numbers be represented along $X'OX$, the positive real numbers being along OX and negative ones along OX'. Let OA be equal to one unit of measurement (Fig. 19.1).

Take a point L on OX such that $OL = x\ (OA)$.

Then L on OX represents the positive real number x and $i.ix = i^2x = -x$ is represented by a point L' on OX' distant OL from O.

From this we infer that the multiplication of the real number x by i twice amounts to the rotation of OL through two right angles to the position OL'.

Thus it naturally follows that the multiplication of a real number by i is equivalent to the rotation of OL through one right angle to the position OL''.

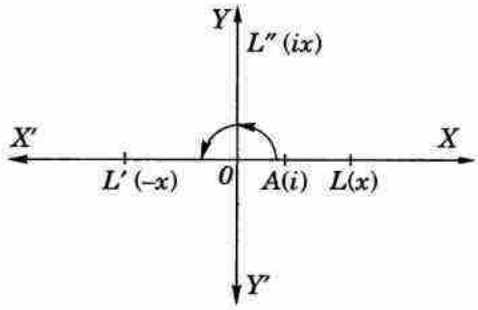


Fig. 19.1

Hence, if $Y'OY$ be a line perpendicular to the real axis $X'OX$, then all imaginary numbers are represented by points on $Y'OY$, called the **imaginary axis**, *the positive ones along OY and negative ones along OY'.**

Obs. Geometric interpretation of i*. From the above, it is clear that i is an operation which when multiplied to any real number makes it imaginary and rotates its direction through a right angle on the complex plane.

(2) Geometric representation of complex numbers†

Consider two lines $X'OX$, $Y'OY$ at right angles to each other.

Let all the real numbers be represented by points on the line $X'OX$ (called the *real axis*), positive real numbers being along OX and negative ones along OX'. Let the point L on OX represent the real number x (Fig. 19.2).

Since the multiplication of a real number by i is equivalent to the rotation of its direction through a right angle. Therefore, let all the imaginary numbers be represented by points on the line $Y'OY$ (called the *imaginary axis*), the positive ones along OY and negative ones along OY'. Let the point M on OY represent the imaginary number iy.

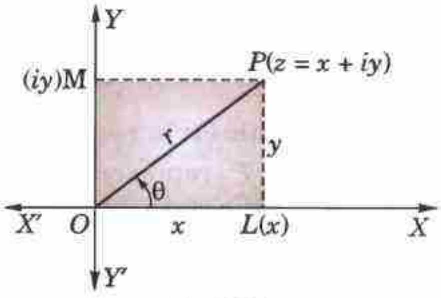


Fig. 19.2

Complete the rectangle $OLPM$. *Then the point whose cartesian coordinates are (x, y) uniquely represents the complex number $z = x + iy$ on the complex plane z.* The diagram in which this representation is carried out is called the **Argand's diagram**.

If (r, θ) be the polar coordinates of P, then r is the modulus of z and θ is its amplitude.

Obs. Since a complex number has magnitude and direction, therefore, it can be represented like a vector. *Hereafter we shall often refer to the complex number $z = x + iy$ as*

(i) the point z whose co-ordinates are (x, y) or (ii) the vector z from O to $P(x, y)$.

Example 19.4. *The centre of a regular hexagon is at the origin and one vertex is given by $\sqrt{3} + i$ on the Argand diagram. Determine the other vertices.*

---

* The first mathematician to propose a geometric representation of imaginary number i was *Kuhn of Denzig* (1750–51).

† The geometric representation of complex numbers came into mathematics through the memoire of *Jean Robert Argand*, Paris 1806.

Solution. Let $\vec{OA} = \sqrt{3} + i$ so that

$OA = 2$ and $\angle XOA = \tan^{-1} 1/\sqrt{3} = 30°$. (Fig. 19.3)

Being a regular hexagon, $OB = OC = 2$

$$\angle XOB = 30° + 60° = 90°$$

and $$\angle XOC = 30° + 120° = 150°$$

$\therefore$ $$\vec{OB} = 2(\cos 90° + i \sin 90°) = 2i$$

$$\vec{OC} = 2(\cos 150° + i \sin 150°) = -\sqrt{3} + i$$

Since $\vec{AD}, \vec{BE}, \vec{CF}$ are bisected at O,

$\therefore$ $$\vec{OD} = -\vec{OA} = -\sqrt{3} - i$$

$$\vec{OE} = -\vec{OB} = -2i \text{ and } \vec{OF} = -\vec{OC} = \sqrt{3} - i.$$

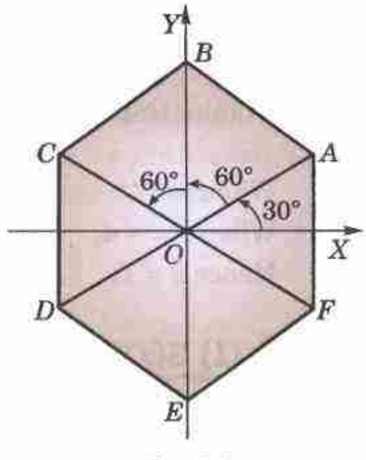


Fig. 19.3

19.3 (1) GEOMETRIC REPRESENTATION OF $z_1 + z_2$

Let P_1, P_2 represent the complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. (Fig. 19.4)

Complete the parallelogram OP_1PP_2. Draw P_1L, P_2M and $PN \perp$ s to OX. Also draw $P_1K \perp PN$.

Since $$ON = OL + LN = OL + OM = x_1 + x_2 \quad [\because LN = P_1K = OM]$$

and $$NP = NK + KP = LP_1 + MP_2 = y_1 + y_2.$$

The coordinates of P are $(x_1 + x_2, y_1 + y_2)$ and it represents the complex number

$$z = x_1 + x_2 + i(y_1 + y_2) = (x_1 + iy_1) + (x_2 + iy_2) = z_1 + z_2.$$

Thus the point P which is the extremity of the diagonal of the parallelogram having OP_1 and OP_2 as adjacent sides, represents the sum of the complex numbers $P_1(z_1)$ and $P_2(z_2)$ such that

$$|z_1 + z_2| = OP \text{ and amp } (z_1 + z_2) = \angle XOP.$$

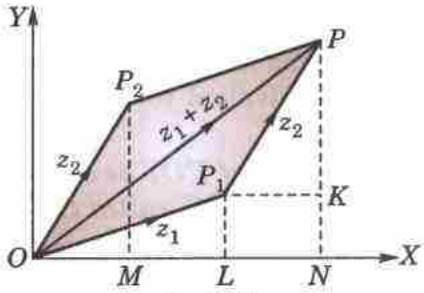


Fig. 19.4

Obs. Vectorially, we have $\vec{OP_1} + \vec{P_1P} = \vec{OP}$.

(2) Geometric representation of $z_1 - z_2$

Let P_1, P_2 represent the complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ (Fig. 19.5). Then the subtraction of z_2 from z_1 may be taken as addition of z_1 to $-z_2$.

Produce P_2O backwards to R such that $OR = OP_2$. Then the coordinates of R are evidently $(-x_2, -y_2)$ and so it corresponds to the complex number $-x_2 - iy_2 = -z_2$.

Complete the parallelogram $ORQP_1$, then the sum of z_1 and $(-z_2)$ is represented by OQ *i.e.*, $z_1 - z_2 = \vec{OQ} = \vec{P_2P_1}$.

Hence the complex number $z_1 - z_2$ is represented by the vector P_2P_1.

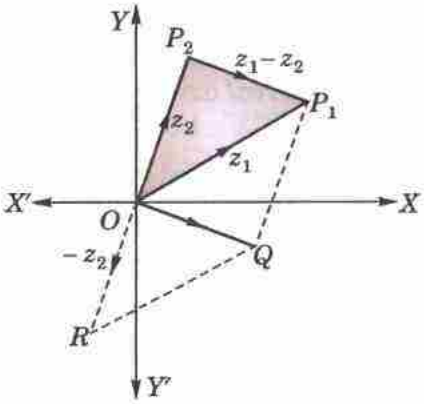


Fig. 19.5

Obs. By means of the relation $\vec{P_2P_1} = \vec{OP_1} - \vec{OP_2}$, any vector $\vec{P_2P_1}$ may be referred to the origin.

Example 19.5. *Find the locus of $P(z)$ when*
(*i*) $|z - a| = k$;
(*ii*) *amp* $(z - a) = \alpha$, where k and α are constants. (*Gorakhpur, 1999*)

Solution. Let a, z be represented by A and P in the complex plane, O being the origin (Fig. 19.6).

Then $z - a = \vec{OP} - \vec{OA} = \vec{AP}$

(*i*) $|z - a| = k$ means that $AP = k$.

Thus the locus of P(z) is a circle whose centre is A(a) and radius k.

(*ii*) amp $(z - a)$, *i.e.*, amp $(\overrightarrow{AP}) = \alpha$, means that AP always makes a constant angle with the X-axis.

Thus the locus of P(z) is a straight line through A(a) making an ∠α with OX.

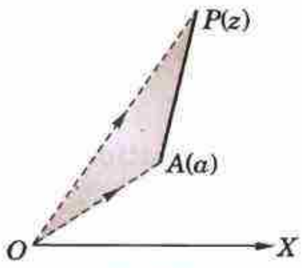


Fig. 19.6

Example 19.6. *Determine the region in the z-plane represented by*
(*i*) $1 < |z + 2i| \le 3$ (*ii*) $R(z) > 3$ (*iii*) $\pi/6 \le amp\ (z) \le \pi/3$.

Solution. (*i*) $|z + 2i| = 1$ is a circle with centre $(-2i)$ and radius 1 and $|z + 2i| = 3$ is a circle with the same centre and radius 3.

Hence $1 < |z + 2i| \le 3$ represents the region outside the circle $|z + 2i| = 1$ and inside (including circumference of) the circle $|z + 2i| = 3$ [Fig. 19.7].

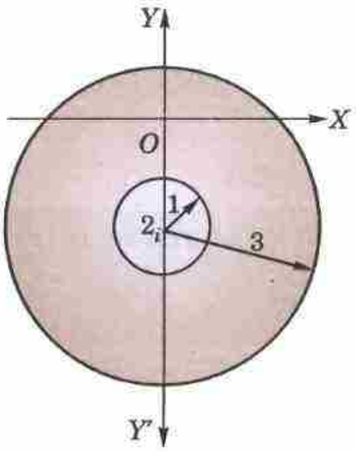


Fig. 19.7

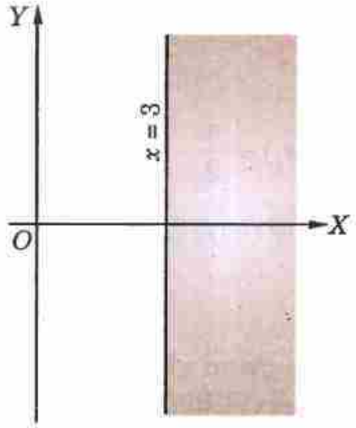


Fig. 19.8

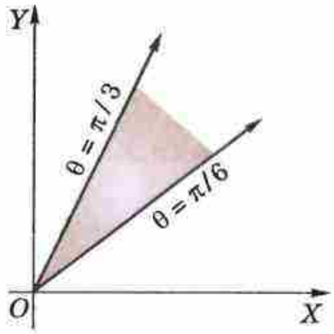


Fig. 19.9

(*ii*) $R(z) > 3$, defines all points (z) whose real part is greater than 3. Hence it represents the region of the complex plane to the right of the line $x = 3$ [Fig. 19.8].

(*iii*) If $z = r(\cos\theta + i\sin\theta)$, then amp $(z) = \theta$.

$\therefore$ $\pi/6 \le$ amp $(z) \le \pi/3$ defines the region bounded by and including the lines $\theta = \pi/6$ and $\theta = \pi/3$. [Fig. 19.9].

Example 19.7. *If z_1, z_2 be any two complex numbers, prove that*
(*i*) $|z_1 + z_2| \le |z_1| + |z_2|$ [*i.e., the modulus of the sum of two complex numbers is less than or at the most equal to the sum of their moduli*].
(*ii*) $|z_1 - z_2| \ge |z_1| - |z_2|$ [*i.e., the modulus of the difference of two complex numbers is greater than or at the most equal to the difference of their moduli*].

Solution. Let P_1, P_2 represent the complex numbers z_1, z_2 (Fig. 19.10). Complete the parallelogram OP_1PP_2, so that

$$|z_1| = OP_1,\ |z_2| = OP_2 = P_1P,$$

and $$|z_1 + z_2| = OP.$$

Now from ΔOP_1P, $OP \le OP_1 + P_1P$, the sign of equality corresponding to the case when O, P_1, P are collinear.

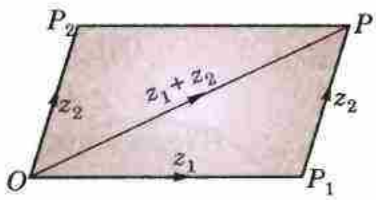


Fig. 19.10

Hence $$|z_1 + z_2| \le |z_1| + |z_2| \quad ...(i)$$

Again $$|z_1| = |(z_1 - z_2) + z_2| \le |z_1 - z_2| + |z_2| \quad \text{[By (i)]}$$

Thus $$|z_1 - z_2| \ge |z_1| - |z_2| \quad ...(ii)$$

Obs. $|z_1 + z_2 + z_3| \le |z_1| + |z_2| + |z_3|$.
In general, $|z_1 + z_2 + ... + z_n| \le |z_1| + |z_2| + ... + |z_n|$.

Example 19.8. *If $|z_1 + z_2| = |z_1 - z_2|$, prove that the difference of amplitudes of z_1 and z_2 is $\pi/2$.* (*Mumbai, 2007*)

Solution. Let $z_1 + z_2 = r(\cos\theta + i\sin\theta)$ and $z_1 - z_2 = r(\cos\phi + i\sin\phi)$

Then
$$2z_1 = r[(\cos\theta + \cos\phi) + i(\sin\theta + \sin\phi)]$$
$$= r\left\{2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2} + 2i\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}\right\}$$

or
$$z_1 = r\cos\frac{\theta-\phi}{2}\left(\cos\frac{\theta+\phi}{2} + i\sin\frac{\theta+\phi}{2}\right) \text{ i.e., } \operatorname{amp}(z_1) = \frac{\theta+\phi}{2} \quad \text{...(i)}$$

Also
$$2z_2 = r(\cos\theta - \cos\phi) + i(\sin\theta - \sin\phi)$$
$$= 2r\sin\frac{\theta-\phi}{2}\left(-\sin\frac{\theta+\phi}{2} + i\cos\frac{\theta+\phi}{2}\right)$$

or
$$z_2 = r\sin\frac{\theta-\phi}{2}\left\{\cos\left(\frac{\pi}{2} + \frac{\theta+\phi}{2}\right) + i\sin\left(\frac{\pi}{2} + \frac{\theta+\phi}{2}\right)\right\}$$

i.e.,
$$\operatorname{amp}(z_2) = \frac{\pi}{2} + \frac{\theta+\phi}{2} \quad \text{...(ii)}$$

Hence [(*ii*) – (*i*)], gives $\operatorname{amp}(z_2) - \operatorname{amp}(z_1) = \dfrac{\pi}{2}$.

Example 19.9. *Show that the equation of the ellipse having foci at z_1, z_2 and major axis $2a$, is $|z - z_1| + |z - z_2| = 2a$.*

Also find its eccentricity.

Solution. Let $P(z)$ be any point on the given ellipse (Fig. 19.11) having foci at $S(z_1)$ and $S'(z_2)$ so that $SP = |z - z_1|$ and $S'P = |z - z_2|$.

We know that $SP + S'P = AA'$ $(= 2a)$

i.e., $\quad |z - z_1| + |z - z_2| = 2a$

which is the desired equation of the ellipse.

Also we know that $SS' = 2ae$, e being the eccentricity.

or $\quad \left|\vec{OS'} - \vec{OS}\right| = 2ae \quad$ or $\quad |z_2 - z_1| = 2ae$

or $\quad |z_1 - z_2| = 2ae$ whence $e = |z_1 - z_2|/2a$.

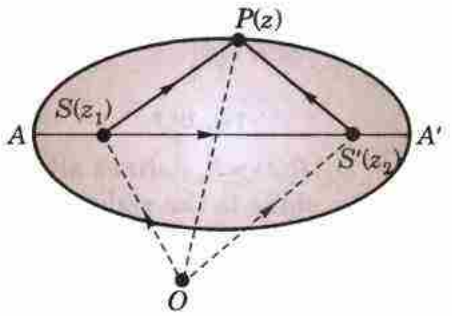


Fig. 19.11

(3) Geometric Representation of $\mathbf{z_1 z_2}$. Let P_1, P_2 represent the complex numbers

$$z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$

and
$$z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

Measure off $OA = 1$ along OX (Fig. 19.12). Construct ΔOP_2P on OP_2 directly similar to ΔOAP_1,

so that $\quad OP/OP_1 = OP_2/OA$ *i.e.,* $OP = OP_1 \,.\, OP_2 = r_1r_2$

and $\quad \angle AOP = \angle AOP_2 + \angle P_2OP = \angle AOP_2 + \angle AOP_1 = \theta_2 + \theta_1$

$\therefore$ P represents the number

$$r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].$$

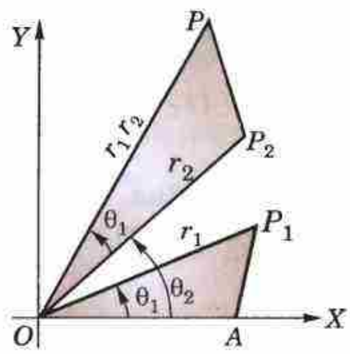


Fig. 19.12

Hence *the product of two complex numbers z_1, z_2 is represented by the point P, such that* (*i*) $|\mathbf{z_1z_2}| = |\mathbf{z_1}| \,.\, |\mathbf{z_2}|$.

(*ii*) $\mathbf{amp\,(z_1z_2) = amp\,(z_1) + amp\,(z_2)}$.

Cor. The effect of multiplication of any complex number z by $\cos\theta + i\sin\theta$ is to rotate its direction through an angle θ, for the modulus of $\cos\theta + i\sin\theta$ is unity.

(4) Geometric representation of $\mathbf{z_1/z_2}$.

Let P_1, P_2 represent the complex numbers

$$z_1 = x_1 + iy_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$$

and $$z_2 = x_2 + iy_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$$

Measure off $OA = 1$, construct triangle OAP on OA directly similar to the triangle OP_2P_1 (Fig. 19.13), so that

$$\frac{OP}{OA} = \frac{OP_1}{OP_2} \quad i.e., \quad OP = \frac{OP_1}{OP_2} = \frac{r_1}{r_2}$$

and $$\angle XOP = \angle P_2OP_1 = \angle AOP_1 - \angle AOP_2 = \theta_1 - \theta_2.$$

$\therefore$ P represents the number

$$(r_1/r_2)\,[\cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2)].$$

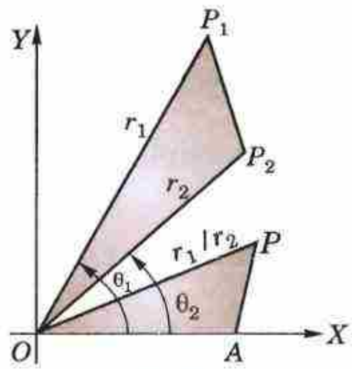


Fig. 19.13

Hence the complex number z_1/z_2 is represented by the point P, such that

(*i*) $|z_1/z_2| = |z_1|/|z_2|$

(*ii*) **amp (z_1/z_2) = amp (z_1) – amp (z_2).**

Note. *If $P_1(z_1)$, $P_2(z_2)$ and $P_3(z_3)$ be any three points, then*

$$\text{amp}\left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \angle P_1P_2P_3.$$

Join O, the origin, to P_1, P_2, and P_3. Then from the figure 19.14, we have

$$\vec{P_2P_1} = z_1 - z_2 \quad \text{and} \quad \vec{P_2P_3} = z_3 - z_2$$

$$\therefore \quad \text{amp}\left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \text{amp}\left[\frac{\vec{P_2P_3}}{\vec{P_2P_1}}\right]$$

$$= \text{amp}\,(\vec{P_2P_3}) - \text{amp}\,(\vec{P_2P_1}) = \beta - \alpha = \angle P_1P_2P_3.$$

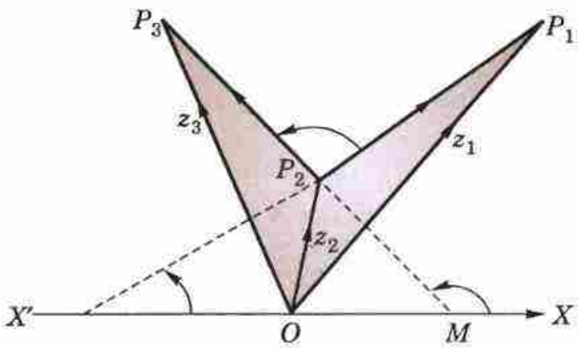


Fig. 19.14

Example 19.10. *Find the locus of the point z, when*

(*i*) $\left|\dfrac{z-a}{z-b}\right| = k$ (*ii*) *amp* $\left(\dfrac{z-a}{z-b}\right) = \alpha$ *where k and α are constants.*

Solution. Let $A(a)$ and $B(b)$ be any two fixed points on the complex plane and let $P(z)$ be any variable point (Fig. 19.15).

(*i*) Since $|z - a| = AP$ and $|z - b| = BP$.

$\therefore$ The point P moves so that $\left|\dfrac{z-a}{z-b}\right| = \dfrac{|z-a|}{|z-b|} = \dfrac{AP}{BP} = k$

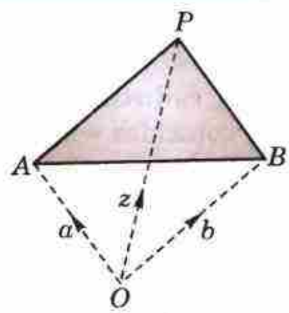


Fig. 19.15

i.e., P moves so that its distances from two fixed points are in a constant ratio, which is obviously the *Appollonius* circle.

When $k = 1$, $BP = AP$ *i.e.*, P moves so that its distance from two fixed points are always equal and thus the locus of P is the right bisector of AB.

Hence the locus of P(z) is a circle (unless k = 1, when the locus is the right bisector of AB).

Obs. For different values of k, the equation represents family of non-intersecting coaxial circles having A and B as its limiting points.

(*ii*) From the figure 19.16, we have amp $\left(\dfrac{z-a}{z-b}\right) = \angle APB = \alpha.$

Hence the locus of P(z) is the arc APB of the circle which passes through the fixed points A and B.

If, however, $P'(z')$ be a point on the lower arc AB of this circle, then

amp $\left(\dfrac{z'-a}{z'-b}\right) = \angle BP'A = \alpha - \pi$, which shows that the locus of P' is the arc $AP'B$ of the same circle.

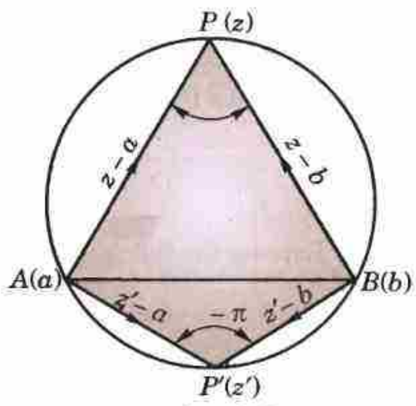


Fig. 19.16

Obs. For different values of α from $-\pi$ to π, the equation represents a family of intersecting coaxial circles having AB as their common radical axis.

Example 19.11. *If z_1, z_2 be two complex numbers, show that*

$$(z_1 + z_2)^2 + (z_1 - z_2)^2 = 2[|z_1|^2 + |z_2|^2].$$

Solution. Let $z_1 = r_1 \text{ cis } \theta_1$ and $z_2 = r_2 \text{ cis } \theta_2$ so that

$$|z_1 + z_2|^2 = (r_1 \cos\theta_1 + r_2 \cos\theta_2)^2 + (r_1 \sin\theta_1 + r_2 \sin\theta_2)^2$$
$$= r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_2 - \theta_1)$$

and
$$|z_1 - z_2|^2 = (r_1 \cos\theta_1 - r_2 \cos\theta_2)^2 + (r_1 \sin\theta_1 - r_2 \sin\theta_2)^2$$
$$= r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)$$

$$\therefore \quad |z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(r_1^2 + r_2^2) = 2\left\{|z_1|^2 + |z_2|^2\right\}.$$

Example 19.12. *If z_1, z_2, z_3 be the vertices of an isosceles triangle, right angled at z_2, prove that*

$$z_1^2 + z_3^2 + 2z_2^2 = 2z_2(z_1 + z_3).$$

Solution. Let $A(z_1), B(z_2), C(z_3)$ be the vertices of ΔABC such that

$$AB = BC \text{ and } \angle ABC = \pi/2. \text{ (Fig. 19.17)}$$

Then $\quad |z_1 - z_2| = |z_3 - z_2| = r$ (say).

If amp $(z_1 - z_2) = \theta$ then amp $(z_3 - z_2) = \pi/2 + \theta$

$$\therefore \quad z_1 - z_2 = r(\cos\theta + i\sin\theta),$$

and
$$z_3 - z_2 = r\left[\cos\left(\frac{\pi}{2} + \theta\right) + i\sin\left(\frac{\pi}{2} + \theta\right)\right] = r(-\sin\theta + i\cos\theta)$$

i.e.,
$$z_3 - z_2 = ir(\cos\theta + i\sin\theta) = i(z_1 - z_2)$$

or
$$(z_3 - z_2)^2 = -(z_1 - z_2)^2 \text{ or } z_1^2 + z_3^2 + 2z_2^2 = 2z_3(z_1 + z_2).$$

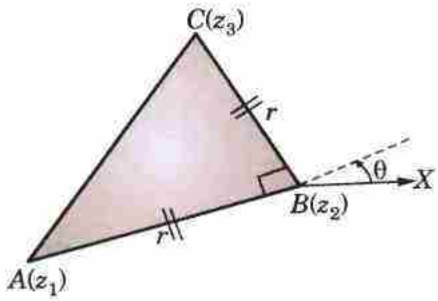


Fig. 19.17

Example 19.13. *If z_1, z_2, z_3 be the vertices of an equilateral triangle, prove that*

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$

Solution. Since ΔABC is equilateral, therefore, BC when rotated through $60°$ coincides with BA (Fig. 19.18). But to turn the direction of a complex number through an $\angle\theta$, we multiply it by $\cos\theta + i\sin\theta$.

$$\therefore \quad \vec{BC}\,(\cos\pi/3 + i\sin\pi/3) = \vec{BA}$$

i.e.,
$$(z_3 - z_2)\left(\frac{1 + i\sqrt{3}}{2}\right) = z_1 - z_2$$

or
$$i\sqrt{3}\,(z_3 - z_2) = 2z_1 - z_2 - z_3$$

Squaring,
$$-3(z_3 - z_2)^2 = (2z_1 - z_2 - z_3)^2$$

or
$$4(z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1) = 0$$

whence follows the required condition.

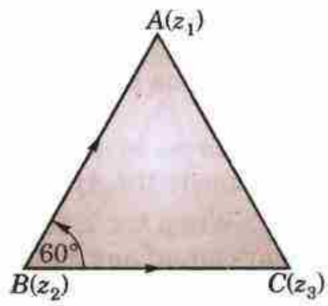


Fig. 19.18

PROBLEMS 19.1

1. Express the following in the modulus-amplitude form:

   (*i*) $1 + \sin\alpha + i\cos\alpha$ (*ii*) $\dfrac{1}{(2+1)^2} - \dfrac{1}{(2-1)^2}$. (*V.T.U., 2011 S*)

2. If $\dfrac{1}{x+iy} + \dfrac{1}{u+iv} = 1$; x, y, u, v being real quantities, express v in terms of x and y.

3. If x and y are real, solve the equation $\frac{iy}{ix+1} - \frac{3y+4i}{3x+y} = 0$.
4. If $\alpha - i\beta = \frac{1}{a - ib}$, prove that $(\alpha^2 + \beta^2)(a^2 + b^2) = 1$. (*Mumbai, 2008 S*)
5. Find what curve $z\bar{z} + (1+i)z + (1-i)\bar{z} = 0$ represents?
6. In an Argand diagram, show that $9 + i$, $4 + 13i$, $-8 + 8i$ and $-3 - 4i$ form a square.
7. If $|z_1| = |z_2|$ and amp (z_1) + amp $(z_2) = 0$, then show that z_1 and z_2 are conjugate complex numbers.
8. A rectangle is constructed in the complex plane and its sides parallel to the axes and its centre is situated at the origin. If one of the vertices of the rectangle is $1 + i\sqrt{3}$, find the complex numbers representing the other three vertices of the rectangle. Find also the area of the rectangle.
9. An equilateral triangle constructed in the complex plane has its one vertex at the point $1 + i\sqrt{3}$. Find the complex numbers representing the other two vertices, O the origin being its circumcentre.
10. The centre of a regular hexagon is at the origin and one vertex is given by $1 + i$ on the Argand diagram. Find the remaining vertices.
11. What domain of the z-plane is represented by
    (*i*) $2 \le |z + 3| < 4$ (*ii*) $I(z) > 2$
    (*iii*) $\pi/3 <$ amp $(z) < \pi/2$ (*iv*) $|z + 2| + |z - 2| < 4$.
12. If $|z^2 - 1| = |z|^2 + 1$, prove that z lies on the imaginary axis. (*Mumbai, 2007*)
13. What are the loci given by (*i*) $|z - 1| + |z + 1| = 3$ (*ii*) $|z - 3| = k\,|z + 1|$ for $k = 1$ and 2.
14. Find the locus of z given by :
    (*i*) $|z| = |z - 2|$. (*ii*) $|3z - 1| = |z - 3|$.
15. Find the locus of z :
    (*i*) when $\frac{z+i}{z+2}$ is real, (*ii*) when $\frac{z-i}{z-2}$ is purely imaginary. (*Osmania, 2003 S*)

19.4 DE MOIVRE'S THEOREM*

Statement : *If n be* (*i*) *an integer, positive or negative* $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$;
(*ii*) *a fraction, positive or negative, one of the values of* $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$.

Proof. Case I. *When n is a positive integer.*

By actual multiplication

$$\text{cis}\,\theta_1\,\text{cis}\,\theta_2 = (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$$
$$= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2), \text{ i.e., } \text{cis}\,(\theta_1 + \theta_2)$$

Similarly $\text{cis}\,\theta_1\,\text{cis}\,\theta_2\,\text{cis}\,\theta_3 = \text{cis}\,(\theta_1 + \theta_2)\,\text{cis}\,\theta_3 = \text{cis}\,(\theta_1 + \theta_2 + \theta_3)$

Proceeding in this way,

$$\text{cis}\,\theta_1\,\text{cis}\,\theta_2\,\text{cis}\,\theta_3 \ldots \text{cis}\,\theta_n = \text{cis}\,(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n)$$

Now putting $\theta_1 = \theta_2 = \theta_3 = \ldots = \theta_n = \theta$, we obtain $(\text{cis}\,\theta)^n = \text{cis}\,n\theta$.

Case II. *When n is a negative integer.*

Let $n = -m$, where m is a + ve integer.

$$\therefore \quad (\text{cis}\,\theta)^n = (\text{cis}\,\theta)^{-m} = \frac{1}{(\text{cis}\,\theta)^m} = \frac{1}{\text{cis}\,m\theta} \qquad \text{(By case I)}$$

$$= \frac{\cos m\theta - i\sin m\theta}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}$$

[Multiplying the num. and denom. by $(\cos m\theta - i\sin m\theta)$]

*One of the remarkable theorems in mathematics; called after the name of its discoverer *Abraham De Moivre* (1667–1754), a French Mathematician.

$$= \frac{\cos m\theta - i \sin m\theta}{\cos^2 m\theta + \sin^2 m\theta} = \cos m\theta - i \sin m\theta$$

$$= \cos(-m\theta) + i \sin(-m\theta) = \text{cis}(-m\theta) = \text{cis } n\theta \qquad [\because \; -m = n]$$

Case III. *When n is a fraction, positive or negative.*

Let $n = p/q$, where q is a + ve integer and p is any integer + ve or – ve

Now $(\text{cis } \theta/q)^q = \text{cis }(q \cdot \theta/q) = \text{cis } \theta$

$\therefore$ Taking qth root of both sides cis (θ/q) is one of the q values of $(\text{cis } \theta)^{1/q}$,

i.e., one of the values of $(\text{cis } \theta)^{1/q} = \text{cis } \theta/p$

Raise both sides to power p, then one of the values of $(\text{cis } \theta)^{p/q} = (\text{cis } \theta/q)^p = \text{cis }(p/q)\,\theta$ *i.e.*, one of the values of $(\text{cis } \theta)^n = \text{cis } n\theta$. (By case I and II)

Thus *the theorem is completely established for all rational values of n.*

Cor. **1.** $\text{cis } \theta_1 \cdot \text{cis } \theta_2 \ldots \text{cis } \theta_n = \text{cis }(\theta_1 + \theta_2 + \ldots + \theta_n)$

2. $(\cos \theta - i \sin \theta)^n = \cos n\theta - i \sin n\theta = (\cos \theta + i \sin \theta)^{-n}$

3. $(\text{cis } m\theta)^n = \text{cis } mn\theta = (\text{cis } n\theta)^m$.

Example 19.14. *Simplify* $\dfrac{(\cos 3\theta + i \sin 3\theta)^4 (\cos 4\theta - i \sin 4\theta)^5}{(\cos 4\theta + i \sin 4\theta)^3 (\cos 5\theta + i \sin 5\theta)^{-4}}$.

Solution. We have, $(\cos 3\theta + i \sin 3\theta)^4 = \cos 12\theta + i \sin 12\theta = (\cos \theta + i \sin \theta)^{12}$

$(\cos 4\theta - i \sin 4\theta)^5 = \cos 20\theta - i \sin 20\theta = (\cos \theta + i \sin \theta)^{-20}$

$(\cos 4\theta + i \sin 4\theta)^3 = \cos 12\theta + i \sin 12\theta = (\cos \theta + i \sin \theta)^{12}$

$(\cos 5\theta + i \sin 5\theta)^{-4} = \cos 20\theta - i \sin 20\theta = (\cos \theta + i \sin \theta)^{-20}$

$$\therefore \quad \text{The given expression} = \frac{(\cos \theta + i \sin \theta)^{12} (\cos \theta + i \sin \theta)^{-20}}{(\cos \theta + i \sin \theta)^{12} (\cos \theta + i \sin \theta)^{-20}} = 1.$$

Example 19.15. *Prove that*

$$(1 + \cos \theta + i \sin \theta)^n + (1 + \cos \theta - i \sin \theta)^n = 2^{n+1} \cos^n (\theta/2) . (\cos n\theta/2).$$

Solution. Put $1 + \cos \theta = r \cos \alpha$, $\sin \theta = r \sin \alpha$.

$$\therefore \qquad r^2 = (1 + \cos \theta)^2 + \sin^2 \theta = 2 + 2 \cos \theta = 4 \cos^2 \theta/2 \quad i.e., \quad r = 2 \cos \theta/2$$

and
$$\tan \alpha = \frac{\sin \theta}{1 + \cos \theta} = \frac{2 \sin \theta/2 \cdot \cos \theta/2}{2 \cos^2 \theta/2} = \tan \theta/2 \quad i.e., \quad \alpha = \theta/2.$$

$$\therefore \qquad \text{L.H.S.} = [r(\cos \alpha + i \sin \alpha)]^n + [r(\cos \alpha - i \sin \alpha)]^n$$

$$= r^n[(\cos \alpha + i \sin \alpha)^n + (\cos \alpha - i \sin \alpha)^n] = r^n (\cos n\alpha + i \sin n\alpha + \cos n\alpha - i \sin n\alpha)$$

$$= r^n \cdot 2 \cos n\alpha \qquad \text{[Substituting the values of } r \text{ and } \alpha]$$

$$= 2^{n+1} \cos^n (\theta/2) \cos (n\theta/2).$$

Example 19.16. *If* $2 \cos \theta = x + \dfrac{1}{x}$, *prove that*

(*i*) $2 \cos r\theta = x^r + 1/x^r$, (*ii*) $\dfrac{x^{2n} + 1}{x^{2n-1} + x} = \dfrac{\cos n\theta}{\cos (n-1)\theta}$ (*Madras, 2000 S*)

Solution. Since $x + 1/x = 2 \cos \theta$ $\quad \therefore \quad x^2 - 2x \cos \theta + 1 = 0$

whence
$$x = \frac{2 \cos \theta \pm \sqrt{(4 \cos^2 \theta - 4)}}{2} = \cos \theta \pm i \sin \theta.$$

(*i*) Taking the + ve sign, $x^r = (\cos \theta + i \sin \theta)^r = \cos r\theta + i \sin r\theta$ (*S.V.T.U., 2009*)

and $\quad x^{-r} = (\cos \theta + i \sin \theta)^{-r} = \cos r\theta - i \sin r\theta$

Adding $x^r + 1/x^r = 2 \cos r\theta$. Similarly with the – ve sign, the same result follows.

(*ii*)
$$\frac{x^{2n}+1}{x^{2n-1}+x} = \frac{(\cos\theta + i\sin\theta)^{2n}+1}{(\cos\theta+i\sin\theta)^{2n-1}+\cos\theta+i\sin\theta}$$
$$= \frac{\cos 2n\theta + i\sin 2n\theta + 1}{\cos(2n-1)\theta + i\sin(2n-1)\theta + \cos\theta + i\sin\theta}$$
$$= \frac{(1+\cos 2n\theta) + i\sin 2n\theta}{(\cos\overline{2n-1}\,\theta + \cos\theta) + i(\sin\overline{2n-1}\,\theta + \sin\theta)}$$
$$= \frac{2\cos^2 n\theta + 2i\sin n\theta\cos\theta}{2\cos n\theta\cos\overline{n-1}\,\theta + 2i\sin n\theta\cos\overline{n-1}\,\theta}$$
$$= \frac{\cos n\theta\,(2\cos n\theta + 2i\sin n\theta)}{\cos\overline{n-1}\,\theta\,(2\cos n\theta + 2i\sin n\theta)} = \frac{\cos n\theta}{\cos\overline{n-1}\,\theta}.$$

Example 19.17. *If* $\sin\alpha + \sin\beta + \sin\gamma = \cos\alpha + \cos\beta + \cos\gamma = 0$,
prove that (*i*) $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$
(*ii*) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$
(*iii*) $\sin 4\alpha + \sin 4\beta + \sin 4\gamma = 2\Sigma \sin 2(\alpha+\beta)$
(*iv*) $\sin(\alpha+\beta) + \sin(\beta+\gamma) + \sin(\gamma+\alpha) = 0$. (*Mumbai, 2009*)

Solution. Let $a = \text{cis}\,\alpha,\ b = \text{cis}\,\beta$ and $c = \text{cis}\,\gamma$.

Then $a + b + c = (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma) = 0$ ...(1)

(*i*) $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c} = (\cos\alpha + i\sin\alpha)^{-1} + (\cos\beta + i\sin\beta)^{-1} + (\cos\gamma + i\sin\gamma)^{-1}$

$$= \sum \frac{\cos\alpha - i\sin\alpha}{\cos\alpha - i\sin\alpha}\cdot\frac{1}{\cos\alpha + i\sin\alpha} = \sum(\cos\alpha - i\sin\alpha)$$

$= (\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma) = 0$ (Given)

or $bc + ca + ab = 0$ ...(2)

$\therefore\ a^2 + b^2 + c^2 = (a+b+c)^2 - 2(bc + ca + ab) = 0$ [By (1) & (2) ...(3)]

or $(cis\,\alpha)^2 + (cis\,\beta)^2 + (cis\,\gamma)^2 = cis\,2\alpha + cis\,2\beta + cis\,2\gamma = 0$

Equating imaginary parts from both sides, we get

$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$$

(*ii*) Since $a + b + c = 0$, $\therefore\ a^3 + b^3 + c^3 = 3abc$

or $(cis\,\alpha)^3 + (cis\,\beta)^3 + (cis\,\gamma)^3 = 3\,cis\,\alpha\,cis\,\beta\,cis\,\gamma$

or $cis\,3\alpha + cis\,3\beta + cis\,3\gamma = 3\,cis\,(\alpha+\beta+\gamma)$

Equating imaginary parts from both sides, we get

$$\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$$

(*iii*) From (1), $a + b = -c$ or $(a+b)^2 = c^2$ or $a^2 + b^2 - c^2 = -2ab$

Again squaring, $a^4 + b^4 + c^4 + 2a^2b^2 - 2b^2c^2 - 2c^2a^2 = 4a^2b^2$

i.e., $a^4 + b^4 + c^4 = 2(a^2b^2 + b^2c^2 + c^2a^2)$

or $(cis\,\alpha)^4 + (cis\,\beta)^4 + (cis\,\gamma)^4 = 2\sum (cos\,\alpha)^2\ (cis\,\beta)^2$

or $cis\,4\alpha + cis\,4\beta + cis\,4\gamma = 2\sum cis\,2\alpha\,cis\,2\beta = 2\sum cis\,2(\alpha+\beta)$

Equating imaginary parts from both sides, we get

$$\sin 4\alpha + \sin 4\beta + \sin 4\gamma = 2\sum \sin 2(\alpha+\beta)$$

(*iv*) From (2), $ab + bc + ca = 0$

or $cis\,\alpha\,cis\,\beta + cis\,\beta\,cis\,\gamma + cis\,\gamma\,cis\,\alpha = 0$

or $cis\,(\alpha+\beta) + cis\,(\beta+\gamma) + cis\,(\gamma+\alpha) = 0$

Equating imaginary parts from both sides, we get

$$\sin(\alpha+\beta) + \sin(\beta+\gamma) + \sin(\gamma+\alpha) = 0$$

PROBLEMS 19.2

1. Prove that (*i*) $\dfrac{(\cos 5\theta - i \sin 5\theta)^2 (\cos 7\theta + i \sin 7\theta)^{-3}}{(\cos 4\theta - i \sin 4\theta)^9 (\cos \theta + i \sin \theta)^5} = 1$

(*ii*) $\dfrac{(\cos \alpha + i \sin \alpha)^4}{(\sin \beta + i \cos \beta)^5} = \sin (4\alpha + 5\beta) - i \cos (4\alpha + 5\beta).$ (*iii*) $\left(\dfrac{\cos \theta + i \sin \theta)^4}{\sin \theta + i \cos \theta}\right) = \cos 8\theta + i \sin 8\theta.$

2. If $p = \text{cis}\ \theta$ and $q = \text{cis}\ \phi$, show that

(*i*) $\dfrac{p-q}{p+q} = i \tan \dfrac{\theta - \phi}{2}$ (*Mumbai, 2008*) (*ii*) $\dfrac{(p+q)(pq-1)}{(p-q)(pq+1)} = \dfrac{\sin \theta + \sin \phi}{\sin \theta - \sin \phi}.$ (*Kurukshetra, 2005*)

3. If $a = \text{cis}\ 2\alpha$, $b = \text{cis}\ 2\beta$, $c = \text{cis}\ 2\gamma$ and $d = \text{cis}\ 2\delta$, prove that

(*i*) $\sqrt{\dfrac{ab}{c}} + \sqrt{\dfrac{c}{ab}} = 2 \cos (\alpha + \beta - \gamma)$ (*ii*) $\sqrt{\dfrac{ab}{cd}} + \sqrt{\dfrac{cd}{ab}} = 2 \cos (\alpha + \beta - \gamma - \delta).$

4. If $x_r = \text{cis}\ (\pi/2^r)$, show that $\underset{n \to \infty}{\text{Lt}}\ x_1 x_2 x_3 \ldots x_n = -1.$ (*S.V.T.U., 2009 ; Mumbai, 2007*)

5. Find the general value of θ which satisfies the equation

$$(\cos \theta + i \sin \theta)(\cos 2\theta + i \sin 2\theta) \ldots (\cos n\theta + i \sin n\theta) = 1.$$

6. Prove that (*i*) $(a + ib)^{m/n} + (a - ib)^{m/n} = 2(a^2 + b^2)^{m/2n} \cos \left(\dfrac{m}{n} \tan^{-1} \dfrac{b}{a}\right).$

(*ii*) $(1 + i)^n + (1 - i)^n = 2^{n/2 + 1} \cos n\pi/4.$

7. Simplify $[\cos \alpha - \cos \beta + i (\sin \alpha - \sin \beta)]^n + [\cos \alpha - \cos \beta - i (\sin \alpha - \sin \beta)]^n$

8. Prove that (*i*) $(1 + \sin \theta + i \cos \theta)^n + (1 + \sin \theta - i \cos \theta)^n = 2^{n+1} \cos^n \left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right) \cos \left(\dfrac{n\pi}{4} - \dfrac{n\theta}{2}\right).$

(*ii*) $\left[\dfrac{1 + \sin \alpha + i \cos \alpha}{1 + \sin \alpha - i \cos \alpha}\right]^n = \cos \left(\dfrac{n\pi}{2} - n\alpha\right) + i \sin \left(\dfrac{n\pi}{2} - n\alpha\right).$ (*S.V.T.U., 2006*)

9. If $2 \cos \theta = x + 1/x$ and $2 \cos \phi = y + 1/y$, show that one of the values of

(*i*) $x^m y^n + \dfrac{1}{x^m y^n}$ is $2 \cos (m\theta + n\phi).$ (*S.V.T.U., 2007*)

(*ii*) $\dfrac{x^m}{y^n} + \dfrac{y^n}{x^m}$ is $2 \cos (m\theta - n\phi).$ (*Nagpur, 2009*)

10. If α, β be the roots of $x^2 - 2x + 4 = 0$, prove that $\alpha^n + \beta^n = 2^{n+1} \cos n\pi/3.$ (*Delhi, 2002*)

11. If α, β are the roots of the equation $z^2 \sin^2 \theta - z \sin \theta + 1 = 0$, then prove that

(*i*) $\alpha^n + \beta^n = 2 \cos n\theta \operatorname{cosec}^n \theta$ (*ii*) $\alpha^n \beta^n = \operatorname{cosec}^{2n} \theta.$ (*Mumbai, 2009*)

12. If $x^2 - 2x \cos \theta + 1 = 0$, show that $x^{2n} - 2x^n \cos n\theta + 1 = 0.$

13. If $x = \cos \alpha + i \sin \alpha$, $y = \cos \beta + i \sin \beta$, $z = \cos \gamma + i \sin \gamma$ and $x + y + z = 0$, then prove that

$$x^{-1} + y^{-1} + z^{-1} = 0.$$

14. If $\sin \theta + \sin \phi + \sin \psi = 0 = \cos \theta + \cos \phi + \cos \psi$, prove that

(*i*) $\cos 2\theta + \cos 2\phi + \cos 2\psi = 0$ (*Mumbai, 2009*)

(*ii*) $\cos 3\theta + \cos 3\phi + \cos 3\psi = 3 \cos (\theta + \phi + \psi)$

(*iii*) $\cos 4\theta + \cos 4\phi + \cos 4\psi = 2 \sum \cos 2(\phi + \psi).$

15. If $\cos \alpha + \cos \beta + \cos \gamma = 0$ and $\sin \alpha + \sin \beta + \sin \gamma = 0$, prove that

(*i*) $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 3/2$

(*ii*) $\cos (\alpha + \beta) + \cos (\beta + \gamma) + \cos (\gamma + \alpha) = 0$ (*Mumbai, 2009 ; S.V.T.U., 2008*)

16. If $\sin \alpha + 2 \sin \beta + 3 \sin \gamma = 0$, $\cos \alpha + 2 \cos \beta + 3 \cos \gamma = 0$, prove that $\sin 3\alpha + 8 \sin 3\beta + 27 \sin 3\gamma = 18 \sin (\alpha + \beta + \gamma)$ and $\cos 3\alpha + 8 \cos 3\beta + 27 \sin 3\gamma = 18 \cos (\alpha + \beta + \gamma).$

19.5 ROOTS OF A COMPLEX NUMBER

There are q and only q distinct values of $(\cos \theta + i \sin \theta)^{1/q}$, *q being an integer.*

Since $\cos \theta = \cos (2n\pi + \theta)$ and $\sin \theta = \sin (2n\pi + \theta)$, where n is any integer.

$$\therefore \qquad \text{cis}\ \theta = \text{cis}\ (2n\pi + \theta).$$

By De Moivre's theorem one of the values of

$$(\text{cis } \theta)^{1/q} = [\text{cis } (2n\pi + \theta)]^{1/q} = \text{cis } (2n\pi + \theta)/q \quad ...(1)$$

Giving n the values 0, 1, 2, 3, ..., $(q - 1)$ successively, we get the following q values of $(\text{cis } \theta)^{1/q}$;

$$\left.\begin{array}{ll} \text{cis } \theta/q & (\text{for } n = 0) \\ \text{cis } (2\pi + \theta)/q & (\text{for } n = 1) \\ \text{cis } (4\pi + \theta)/q & (\text{for } n = 2) \\ \cdots\cdots & \cdots\cdots \\ \text{cis } [2(q-1)\pi + \theta]/q & (\text{for } n = q - 1) \end{array}\right\} \quad ...(2)$$

Putting $n = q$ in (1), we get a value of $(\text{cis } \theta)^{1/q} = \text{cis } (2\pi + \theta/q) = \text{cis } \theta/q$, which is the same as the value of $n = 0$.

Similarly for $n = q + 1$, we get a value of $(\text{cis } \theta)^{1/q}$ to be cis $(2\pi + \theta)/q$, which is the same as the value for $n = 1$ and so on.

Thus, the values of $(\text{cis } \theta)^{1/q}$ for $n = q, q + 1, q + 2$ etc. are the mere repetition of the q values obtained in (2).

Moreover, the q values given by (2) are clearly distinct from each other, for no two of the angles involved therein are equal or differ by a multiple of 2π.

Hence $(\text{cis } \theta)^{1/q}$ has q and only q distinct values given by (2).

Obs. $(\text{cis } \theta)^{p/q}$ *where p/q is a rational fraction in its lowest terms, has also q and only q distinct values*; which are obtained by putting $n = 0, 1, 2, ..., q - 1$ successively in cis $p\,(2n\pi + \theta)/q$.

Note that $(\text{cis } \theta)^{6/15}$ has only 5 distinct values and not 15; because 6/15 in its lowest terms = 2/5

$\therefore$ *In order to find the distinct values of* $(\text{cis } \theta)^{p/q}$ *always see that* p/q *is in its lowest terms.*

Note. The above discussion can usefully be employed for extracting any assigned root of a given quantity. We have only to express it in the form $r(\cos\theta + i \sin\theta)$ and proceed as above.

Example 19.18. *Find the cube roots of unity and show that they form an equilateral triangle in the Argand diagram.*

Solution. If x be a cube root of unity, then

$$x = (1)^{1/3} = (\cos 0 + i \sin 0)^{1/3} = (\text{cis } 0)^{1/3} = (\text{cis } 2n\pi)^{1/3} = \text{cis } 2n\pi/3$$

where $n = 0, 1, 2$.

$\therefore$ the three vlaues of x are cis $0 = 1$,

$$\text{cis } 2\pi/3 = \cos 120° + i \sin 120° = -\frac{1}{2} + i\frac{\sqrt{3}}{2},$$

and

$$\text{cis } 4\pi/3 = \cos 240° + i \sin 240° = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

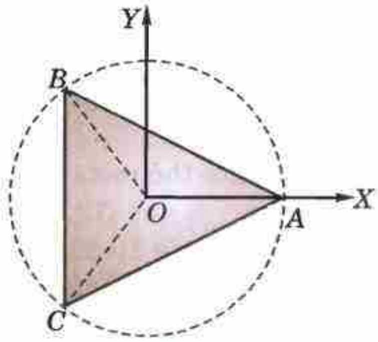


Fig. 19.19

These three cube roots are represented by the points A, B, C on the Argand diagram such that $OA = OB = OC$ and $\angle AOB = 120°$, $\angle AOC = 240°$ (Fig. 19.19).

$\therefore$ these points lie on a circle with centre O and unit radius such that $\angle AOB = \angle BOC = \angle COA = 120°$ *i.e.*, $AB = BC = CA$.

Hence A, B, C form an equilateral triangle.

Example. 19.19. *Find all the values of* $\left(\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)^{3/4}$.

Also show that the continued product of these values is 1. (*Nagpur, 2009*)

Solution. Put $1/2 = r\cos\theta$ and $\sqrt{3}/2 = r \sin\theta$ so that $r = 1$ and $\theta = \pi/3$

$$\therefore \quad (1/2 + \sqrt{3}i/2)^{3/4} = [(\cos \pi/3 + i \sin \pi/3)^3]^{1/4} = (\text{cis } \pi)^{1/4}$$

$$= [\text{cis } (2n + 1)\pi]^{1/4} = \text{cis } (2n + 1)\,\pi/4 \text{ where } n = 0, 1, 2, 3.$$

Hence the required values are cis $\pi/4$, cis $3\pi/4$, cis $5\pi/4$ and cis $7\pi/4$.

$$\therefore \quad \text{their continued product} = \text{cis}\left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}\right) = \text{cis } 4\pi = 1.$$

Example 19.20. *Use De Moivre's theorem to solve the equation.* *(P.T.U., 2005)*

$$x^4 - x^3 + x^2 - x + 1 = 0.$$

Solution. '$x^4 - x^3 + x^2 - x + 1$' is a G.P. with common ratio $(-x)$, therefore

$$\frac{1-(-x)^5}{1-(-x)} = 0, \quad x \neq -1 \quad \text{or} \quad x^5 + 1 = 0$$

i.e., $$x^5 = -1 = \text{cis } \pi = \text{cis } (2n+1)\pi$$

$\therefore$ $$x = [\text{cis } (2n+1)\pi]^{1/5} = \text{cis } (2n+1)\pi/5, \text{ where } n = 0, 1, 2, 3, 4$$

Hence the values are cis $\pi/5$, cis $3\pi/5$, cis π, cis $7\pi/5$, cis $9\pi/5$

or $$\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}, \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}, -1, \cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5}, \cos\frac{\pi}{5} - i\sin\frac{\pi}{5}$$

Rejecting the value -1 which corresponds to the factor $x + 1$, the required roots are :

$$\cos \pi/5 \pm i \sin \pi/5, \cos 3\pi/5 \pm i \sin 3\pi/5.$$

Example 19.21. *Show that the roots of the equation $(x-1)^n = x^n$, n being a positive integer are $\frac{1}{2}(1 + i \cot r\pi/n)$, where r has the values 1, 2, 3, ..., $n-1$.*

Solution. Given equation is $\left(\dfrac{x-1}{x}\right)^n = 1$ or $1 - \dfrac{1}{x} = (1)^{1/n}$

or $$\frac{1}{x} = 1 - (1)^{1/n} = 1 - \text{cis } \frac{2r\pi}{n}, r = 0, 1, 2, \dots (n-1). \qquad [\because \ 1 = \text{cis } 2\pi r]$$

or $$= \left(1 - \cos\frac{2r\pi}{n}\right) - i\sin\frac{2r\pi}{n} = 2\sin^2\frac{r\pi}{n} - 2i\sin\frac{r\pi}{n}\cos\frac{r\pi}{n}$$

$\therefore$ $$x = \frac{1}{2\sin\frac{r\pi}{n}} \cdot \frac{1}{\left(\sin\frac{r\pi}{n} - i\cos\frac{r\pi}{n}\right)} = \frac{\sin\frac{r\pi}{n} + i\cos\frac{r\pi}{n}}{2\sin\frac{r\pi}{n}}$$

$$= \frac{1}{2}\left(1 + i\cot\frac{r\pi}{n}\right), r = 1, 2, \dots (n-1). \qquad [\because \ \cot 0 \to \infty]$$

Hence the roots of the given equation are $\frac{1}{2}(1 + i \cot r\pi/n)$ where $r = 1, 2, 3, \dots (n-1)$.

Example 19.22. *Find the 7th roots of unity and prove that the sum of their nth powers always vanishes unless n be a multiple number of 7, n being an integer, and then the sum is 7.*

(Mumbai, 2008 ; Kurukshetra, 2005)

Solution. We have $$(1)^{1/7} = (\cos 2r\pi + i\sin 2r\pi)^{1/7} = \text{cis } \frac{2r\pi}{7} = \left(\text{cis } \frac{2\pi}{7}\right)^r$$

Putting $r = 0, 1, 2, 3, 4, 5, 6$, we find that 7th roots of unity are $1, \rho, \rho^2, \rho^3, \rho^4, \rho^5, \rho^6$ where $\rho = \cos 2\pi/7$.

$\therefore$ sum S of the nth powers of these roots $= 1 + \rho^n + \rho^{2n} + \dots + \rho^{6n}$ ...(*i*)

$$= \frac{1-\rho^{7n}}{1-\rho^n}, \text{ being a G.P. with common ratio } \rho$$

When n is not a multiple of 7, $\rho^{7n} = (\rho^7)^n = (\text{cis } 2\pi)^n = 1$.

i.e., $$1 - \rho^{7n} = 0 \text{ and } 1 - \rho^n \neq 0, \text{ as } n \text{ is not a multiple of 7.}$$

Thus $S = 0$.

When n is a multiple of 7 = 7p (say)

From (*i*), $$S = 1 + (\rho^7)^p + (\rho^7)^{2p} + \dots + (\rho^7)^{6p} = 1 + 1 + 1 + 1 + 1 + 1 + 1 = 7.$$

Example 19.23. *Find the equation whose roots are $2\cos\pi/7$, $2\cos 3\pi/7$, $2\cos 5\pi/7$.*

Solution. Let $y = \cos\theta + i\sin\theta$, where $\theta = \pi/7, 3\pi/7, \dots, 13\pi/7$.

Then $y^7 = (\cos\theta + i\sin\theta)^7 = \cos 7\theta + i\sin 7\theta = -1$ or $y^7 + 1 = 0$

or $(y+1)(y^6 - y^5 + y^4 - y^3 + y^2 - y + 1) = 0$

Leaving the factor $y + 1$ which corresponds to $\theta = \pi$,

We get $y^6 - y^5 + y^4 - y^3 + y^2 - y + 1 = 0$ ...(*i*)

Its roots are $y = \text{cis}\,\theta$ where $\theta = \pi/7, 3\pi/7, 5\pi/7, 9\pi/7, 11\pi/7, 13\pi/7$.

Dividing (*i*) by y^3, $(y^3 + 1/y^3) - (y^2 + 1/y^2) + (y + 1/y) - 1 = 0$

or $\{(y + 1/y)^3 - 3(y + 1/y)\} - \{(y + 1/y)^2 - 2\} - (y + 1/y) - 1 = 0$

or $x^3 - x^2 - 2x + 1 = 0$ ...(*ii*)

where $x = y + 1/y = 2\cos\theta$.

Now since $\cos 13\pi/7 = \cos \pi/7$, $\cos 11\pi/7 = \cos 3\pi/7$, $\cos 9\pi/7 = \cos 5\pi/7$

Hence the roots of (*ii*) are $2\cos\frac{\pi}{7}, 2\cos\frac{3\pi}{7}, 2\cos\frac{5\pi}{7}$.

PROBLEMS 19.3

1. Find all the values of
   (*i*) $(1+i)^{1/4}$ (*ii*) $(-1+i)^{2/5}$
   (*iii*) $(-1+i\sqrt{3})^{3/2}$ (*iv*) $(1+i\sqrt{3})^{1/3} + (1-i\sqrt{3})^{1/3}$.
2. If w is a complex cube root of unity, prove that $1 + w + w^2 = 0$.
3. Find all the values of $(-1)^{1/6}$.
4. Mark by points on the Argand diagram, all the values of $(1+i\sqrt{3})^{1/5}$ and verify that they form a pentagon.
5. Use De Moivre's theorem to solve the following equations :
   (*i*) $x^5 + 1 = 0$ (*ii*) $x^7 + x^4 + x^3 + 1 = 0$
   (*iii*) $x^9 + x^5 - x^4 - 1 = 0$ (*Madras, 2000*) (*iv*) $(x-1)^5 + x^5 = 0$.
6. Find the roots common to the equations $x^4 + 1 = 0$ and $x^6 - i = 0$.
7. Solve the equation $x^{12} - 1 = 0$ and find which of its roots satisfy the equation $x^4 + x^2 + 1 = 0$.
8. Show that the roots of $(x+1)^7 = (x-1)^7$ are given by $\pm i \cot r\pi/7$, $r = 1, 2, 3$. (*Mumbai, 2008*)
9. Prove that the nth roots of unity form a geometric progression. (*Mumbai, 2007*)
   Also show that the sum of these n roots is zero and their product is $(-1)^{n-1}$.
10. Find the equation whose roots are $2\cos 2\pi/7$, $2\cos 4\pi/7$, $2\cos 6\pi/7$.

19.6 (1) TO EXPAND sin nθ, cos nθ AND tan nθ IN POWERS OF sin θ, cos θ AND tan θ RESPECTIVELY (n BEING A POSITIVE INTEGER)

We have $\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$ (By De Moivre's theorem)

$= \cos^n\theta + {}^nC_1\cos^{n-1}\theta\,(i\sin\theta) + {}^nC_2\cos^{n-2}\theta\,(i\sin\theta)^2 + {}^nC_3\cos^{n-3}\theta\,(i\sin\theta)^3 + \dots$

(By Binomial theorem)

$= (\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + \dots) + i\,({}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + \dots)$

Equating real and imaginary parts from both sides, we get

$$\cos n\theta = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - \dots \quad ...(1)$$

$$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta - \dots \quad ...(2)$$

Replacing every $\sin^2\theta$ by $1 - \cos^2\theta$ in (1) and every $\cos^2\theta$ by $1 - \sin^2\theta$ in (2), we get the desired expansions of $\cos n\theta$ and $\sin n\theta$.

Dividing (2) by (1),

$$\tan n\theta = \frac{{}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta - \dots}{\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - \dots}$$

and dividing numerator and denominator by $\cos^n\theta$, we get

$$\tan n\theta = \frac{{}^nC_1\tan\theta - {}^nC_3\tan^3\theta + {}^nC_5\tan^5\theta - \dots}{1 - {}^nC_2\tan^2\theta + {}^nC_4\tan^4\theta - \dots}.$$

Example 19.24. *Express cos 6θ in terms of cos θ.* *(Madras, 2002)*

Solution. We know that $\cos n\theta = \cos^n \theta - {}^nC_2 \cos^{n-2}\theta \sin^2\theta + {}^nC_4 \cos^{n-4}\theta \sin^4\theta - \ldots$

Put $n = 6$, then $\cos 6\theta = \cos^6\theta - {}^6C_2 \cos^4\theta \sin^2\theta + {}^6C_4 \cos^2\theta \sin^4\theta - {}^6C_6 \sin^6\theta$

$$= \cos^6\theta - 15\cos^4\theta(1-\cos^2\theta) + 15\cos^2\theta(1-\cos^2\theta)^2 - (1-\cos^2\theta)^3$$

$$= 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1.$$

(2) Addition formulae for any number of angles

We have, $\cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)$

$$= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots(\cos\theta_n + i\sin\theta_n)$$

Now $\cos\theta_1 + i\sin\theta_1 = \cos\theta_1(1 + i\tan\theta_1)$, $\cos\theta_2 + i\sin\theta_2 = \cos\theta_2(1 + i\tan\theta_2)$ and so on.

$\therefore$ $\cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)$

$$= \cos\theta_1 \cos\theta_2 \ldots \cos\theta_n (1 + i\tan\theta_1)(1 + i\tan\theta_2)\ldots(1 + i\tan\theta_n)$$

$$= \cos\theta_1\cos\theta_2\ldots\cos\theta_n [1 + i(\tan\theta_1 + \tan\theta_2 + \ldots + \tan\theta_n)$$
$$+ i^2(\tan\theta_1\tan\theta_2 + \tan\theta_2\tan\theta_3 + \ldots) + i^3(\tan\theta_1\tan\theta_2\tan\theta_3 + \ldots) + \ldots + \ldots]$$

$$= \cos\theta_1\cos\theta_2\ldots\cos\theta_n(1 + is_1 - s_2 - is_3 + s_4 + \ldots)$$

where $s_1 = \tan\theta_1 + \tan\theta_2 + \ldots + \tan\theta_n$, $s_2 = \Sigma\tan\theta_1\tan\theta_2$, $s_3 = \Sigma\tan\theta_1\tan\theta_2\tan\theta_3$ etc.

Equating real and imaginary parts, we have

$$\cos(\theta_1 + \theta_2 + \ldots + \theta_n) = \cos\theta_1\cos\theta_2\ldots\cos\theta_n(1 - s_2 + s_4 - \ldots)$$
$$\sin(\theta_1 + \theta_2 + \ldots + \theta_n) = \cos\theta_1\cos\theta_2\ldots\cos\theta_n(s_1 - s_3 + s_5 - \ldots)$$

and by division, we get $\mathbf{\tan(\theta_1 + \theta_2 + \ldots + \theta_n) = \dfrac{s_1 - s_3 + s_5 - \ldots}{1 - s_2 + s_4 - s_6 + \ldots}}$.

Example 19.25. *If* $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi/2$, *show that* $xy + yz + zx = 1$. *(P.T.U., 2003)*

Solution. Let $\tan^{-1}x = \alpha$, $\tan^{-1}y = \beta$, $\tan^{-1}z = \gamma$ so that $x = \tan\alpha$, $y \tan\beta$, $z = \tan\gamma$

We know that $$\tan(\alpha + \beta + \gamma) = \frac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1 - \tan\alpha\tan\beta - \tan\beta\tan\gamma - \tan\gamma\tan\alpha}$$

$\therefore$ $$\tan\pi/2 = \frac{x + y + z - xyz}{1 - xy - yz - zx} \quad \text{or} \quad 1 - xy - yz - zx = 0$$

Hence $xy + yz + zx = 1$.

Example 19.26. *If* $\theta_1, \theta_2, \theta_3$ *be three values of* θ *which satisfy the equation* $\tan 2\theta = \lambda\tan(\theta + \alpha)$ *and such that no two of them differ by a multiple of* π, *show that* $\theta_1 + \theta_2 + \theta_3 + \alpha$ *is a multiple of* π.

Solution. Given equation can be written as $\dfrac{2t}{1 - t^2} = \lambda\dfrac{t + \tan\alpha}{1 - t\,.\tan\alpha}$ where $t = \tan\theta$

or $$\lambda t^3 + (\lambda - 2)\tan\alpha\,.\,t^2 + (2 - \lambda)t - \lambda\tan\alpha = 0$$

$\therefore$ $\tan\theta_1$, $\tan\theta_2$, $\tan\theta_3$, being its roots, we have

$$s_1 = \Sigma\tan\theta_1 = -\frac{\lambda - 2}{\lambda}\tan\alpha \qquad \text{[By § 1.3]}$$

$$s_2 = \Sigma\tan\theta_1\tan\theta_2 = \frac{2 - \lambda}{\lambda} \quad \text{and} \quad s_3 = \tan\alpha$$

$\therefore$ $$\tan(\theta_1 + \theta_2 + \theta_3) = \frac{s_1 - s_3}{1 - s_2} = \frac{(-1 + 2/\lambda)\tan\alpha - \tan\alpha}{1 - (2/\lambda - 1)}$$

$$= -\tan\alpha = \tan(n\pi - \alpha)$$

Thus $\theta_1 + \theta_2 + \theta_3 = n\pi - \alpha$, whence follows the result.

(3) To expand $\sin^m\theta$, $\cos^n\theta$ or $\sin^m\theta\cos^n\theta$ in a series of sines or cosines of multiples of θ

If $z = \cos\theta + i\sin\theta$ then $1/z = \cos\theta - i\sin\theta$.

By De Moivre's theorem, $z^p = \cos p\theta + i\sin p\theta$ and $1/z^p = \cos p\theta - i\sin p\theta$

$\therefore$ $z + 1/z = 2\cos\theta$, $z - 1/z = 2i\sin\theta$; $z^p + 1/z^p = 2\cos p\theta$, $z^p - 1/z^p = 2i\sin p\theta$

These results are used to expand the powers of $\sin\theta$ or $\cos\theta$ or their products in a series of sines or cosines of multiples of θ.

Example 19.27. *Expand $\cos^8\theta$ in a series of cosines of multiples of θ.*

Solution. Let $z = \cos\theta + i\sin\theta$, so that $z + 1/z = 2\cos\theta$ and $z^p + 1/z^p = 2\cos p\theta$.

$\therefore$ $(2\cos\theta)^8 = (z + 1/z)^8$

$$= z^8 + {}^8C_1 z^7 \cdot \frac{1}{z} + {}^8C_2 z^6 \cdot \frac{1}{z^2} + {}^8C_3 z^5 \cdot \frac{1}{z^3} + {}^8C_4 z^4 \cdot \frac{1}{z^4} + {}^8C_5 z^3 \cdot \frac{1}{z^5} + {}^8C_6 z^2 \cdot \frac{1}{z^6} + {}^8C_7 z \cdot \frac{1}{z^7} + \frac{1}{z^8}$$

$$= (z^8 + 1/z^8) + {}^8C_1(z^6 + 1/z^6) + {}^8C_2(z^4 + 1/z^4) + {}^8C_3(z^2 + 1/z^2) + {}^8C_4$$

$$= (2\cos 8\theta) + 8(2\cos 6\theta) + 28(2\cos 4\theta) + 56(2\cos 2\theta) + 70.$$

Hence $\cos^8\theta = \dfrac{1}{128}[\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35]$.

Example 19.28. *Expand $\sin^7\theta\cos^3\theta$ in a series of sines of multiples of θ.*

Solution. Let $z = \cos\theta + i\sin\theta$

so that $z + 1/z = 2\cos\theta$, $z - 1/z = 2i\sin\theta$ and $z^p - 1/z^p = 2i\sin p\theta$.

$\therefore$ $(2i\sin\theta)^7 (2\cos\theta)^3 = (z - 1/z)^7(z + 1/z)^3$

$$= (z - 1/z)^4 [(z - 1/z)(z + 1/z)]^3 = (z - 1/z)^4 (z^2 - 1/z^2)^3$$

$$= \left(z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4}\right)\left(z^6 - 3z^2 + \frac{3}{z^2} - \frac{1}{z^6}\right)$$

$$= \left(z^{10} - \frac{1}{z^{10}}\right) - 4\left(z^8 - \frac{1}{z^8}\right) + 3\left(z^6 - \frac{1}{z^6}\right) + 8\left(z^4 - \frac{1}{z^4}\right) - 14\left(z^2 - \frac{1}{z^2}\right)$$

$$= 2i\sin 10\theta - 4(2i\sin 8\theta) + 3(2i\sin 6\theta) + 8(2i\sin 4\theta) - 14(2i\sin 2\theta)$$

Since $i^7 = -i$,

$\therefore$ $\sin^7\theta\cos^3\theta = -\dfrac{1}{2^9}[\sin 10\theta - 4\sin 8\theta + 3\sin 6\theta + 8\sin 4\theta - 14\sin 2\theta]$.

Obs. The expansion of $\sin^m\theta\cos^n\theta$ is a series of sines or cosines of multiples of θ according as m is odd or even.

PROBLEMS 19.4

1. Express $\sin 6\theta/\sin\theta$ as a polynomial in $\cos\theta$?

Prove that (2–5) :

2. $\sin 7\theta/\sin\theta = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta$.
3. $\dfrac{1+\cos 7\theta}{1+\cos\theta} = (x^3 - x^2 - 2x + 1)^2$, where $x = 2\cos\theta$. (*Madras, 2002*)
4. $2(1 + \cos 8\theta) = (x^4 - 4x^2 + 2)^2$ where $x = 2\cos\theta$.
5. $\tan 5\theta = \dfrac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$ where $t = \tan\theta$.
6. If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, show that $x + y + z = xyz$.
7. If α, β, γ be the roots of the equation $x^3 + px^2 + qx + p = 0$, prove that $\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma = n\pi$ radians except in one particular case.

Prove that (8-12) :

8. $\cos^7\theta = \dfrac{1}{16}(\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta)$. (*Madras, 2003 S*)
9. $\cos^6\theta - \sin^6\theta = \dfrac{1}{16}(\cos 6\theta + 15\cos 2\theta)$. (*Mumbai, 2007*)
10. $\sin^8\theta = 2^{-7}(\cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35)$.
11. $32\sin^4\theta\cos^2\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$.
12. $\sin^5\theta\cos^2\theta = \dfrac{1}{64}(\sin 7\theta - 3\sin 5\theta + \sin 3\theta + 5\sin\theta)$. (*Madras, 2003*)

13. Expand $\cos^5\theta \sin^7\theta$ in a series of sines of multiples of θ ? (*Madras, 2002*)
14. If $\cos^5\theta = A\cos\theta + B\cos 3\theta + C\cos 5\theta$, find $\sin^5\theta$ in terms of A, B, C.
15. If $\sin^4\theta\cos^3\theta = A_1\cos\theta + A_3\cos 3\theta + A_5\cos 5\theta + A_7\cos 7\theta$, prove that
$$A_1 + 9A_3 + 25A_5 + 49A_7 = 0.$$

19.7 COMPLEX FUNCTION

Definition. *If for each value of the complex variable z (= $x + iy$) in a given region R, we have one or more values of w (= $u + iv$), then w is said to be a* **complex function** *of z* and we write $w = u(x, y) + iv(x, y) = f(z)$ where u, v are real functions of x and y.

If to each value of z, there corresponds one and only one value of w, then w is said to be a *single-valued function of z* otherwise a *multi-valued function*. For example, $w = 1/z$ is a single-valued function and $w = \sqrt{z}$ is a multi-valued function of z. The former is defined at all points of the z-plane except at $z = 0$ and the latter assumes two values for each value of z except at $z = 0$.

19.8 EXPONENTIAL FUNCTION OF A COMPLEX VARIABLE

(1) Definition. When x is real, we are already familiar with the exponential function

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots \infty.$$

Similarly, *we define the exponential function of the complex variable $z = x + iy$, as*

$$\mathbf{e^z \text{ or } exp(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots + \frac{z^n}{n!} + \dots \infty} \qquad \dots(i)$$

(2) Properties :

I. Exponential form of $z = re^{i\theta}$

Putting $x = 0$ in (i), we get

$$e^{iy} = 1 + \frac{iy}{1!} + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \dots \infty$$

$$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots\right) = \cos y + i\sin y$$

Thus $\quad e^z = e^x \cdot e^{iy} = e^x(\cos y + i\sin y)$

Also $\quad x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$. Thus, $z = re^{i\theta}$

II. e^z is periodic function having imaginary period $2\pi i$, [$\because \quad e^{z+2n\pi i} = e^z \cdot e^{2n\pi i} = e^z$].

III. e^z is not zero for any value of z.

Since $\quad e^z = e^{x+iy} = re^{i\theta}$ or $e^x \cdot e^{iy} = re^{i\theta}$

$\therefore \quad r = e^x > 0, y = \theta, |e^{iy}| = 1,$

Thus $\quad |e^z| = |e^x| \cdot |e^{iy}| = e^x \neq 0.$

IV. $\quad e^{\bar{z}} = \overline{e^z}$

Since $\quad e^{\bar{z}} = e^{x-iy} = e^x \cdot e^{-iy} = e^x(\cos y - i\sin y)$

$$= \overline{e^x(\cos y + i\sin y)} = \overline{e^z}$$

19.9 CIRCULAR FUNCTIONS OF A COMPLEX VARIABLE

(1) Definitions:

Since $\quad e^{iy} = \cos y + i\sin y$ and $e^{-iy} = \cos y - i\sin y$.

$\therefore$ the circular functions of real angles can be written as

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}, \cos y = \frac{e^{iy} + e^{-iy}}{2} \text{ and so on.}$$

It is, therefore, natural to *define the circular functions of the complex variable z by the equations :*

$$\mathbf{\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \cos z = \frac{e^{iz} + e^{-iz}}{2}, \tan z = \frac{\sin z}{\cos z}}$$

with cosec z, sec z and cot z as their respective reciprocals.

(2) Properties :

I. Circular functions are periodic : $\sin z$, $\cos z$ are periodic functions having real period 2π while $\tan z$, $\cot z$ have period π. [*a* $\sin (z + 2n\pi) = \sin z$, $\tan (z + n\pi) = \tan z$ etc.]

II. Even and odd functions : cos z, sec z are even functions while sin z, cosec z are odd functions. [$\because$ $\cos z = \dfrac{e^{-iz} + e^{iz}}{2} = \cos z$, and $\sin (-z) = \dfrac{e^{-iz} - e^{iz}}{2i} = \dfrac{e^{iz} - e^{-iz}}{2i} = -\sin z$]

III. Zeros of sin z are given by $z = \pm 2n\pi$ *and zeros of cos z are given by* $z = \pm \frac{1}{2} (2n + 1)\pi$, $n = 0, 1, 2, ...$

IV. All the formulae for real circular functions are valid for complex circular functions

e.g., $\sin^2 z + \cos^2 z = 1$, $\sin (z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$.

(3) Euler's theorem $e^{iz} = \cos z + i \sin z$.

By definition $\cos z + i \sin z = \dfrac{e^{iz} + e^{-iz}}{2} + i \dfrac{e^{iz} - e^{-iz}}{2i} = e^{iz}$ where $z = x + iy$.

Also we have shown that $e^{iy} = \cos y + i \sin y$, where y is real.

Thus $e^{i\theta} = \mathbf{\cos \theta + i \sin \theta}$, *where θ is real or complex. This is called the Euler's theorem.**

Cor. *De Moivre's theorem for complex numbers*

Whether θ is real or complex, we have

$$(\cos \theta + i \sin \theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i \sin n\theta$$

Thus De Moivre's theorem is true for all θ (real or complex).

Example 19.29. *Prove that (i)* $[\sin (\alpha + \theta) - e^{i\alpha} \sin \theta]^n = \sin^n \alpha \, e^{-in\theta}$

(ii) $\sin (\alpha - n\theta) + e^{-i\alpha} \sin n\theta = e^{-in\theta} \sin \alpha$.

Solution. (*i*) L.H.S. $= [\sin \alpha \cos \theta + \cos \alpha \sin \theta - (\cos \alpha + i \sin \alpha) \sin \theta]^n$

$$= (\sin \alpha \cos \theta - i \sin \alpha \sin \theta)^n$$
$$= \sin^n \alpha (\cos \theta - i \sin \theta)^n = \sin^n \alpha (e^{-i\theta})^n = \sin^n \alpha e^{-in\theta}$$

(*ii*) L.H.S. $= \sin \alpha \cos n\theta - \cos \alpha \sin n\theta + (\cos \alpha - i \sin \alpha) \sin n\theta$

$$= \sin \alpha \cos n\theta - i \sin \alpha \sin n\theta$$
$$= \sin \alpha (\cos n\theta - i \sin n\theta) = \sin \alpha \, . \, e^{-in\theta}.$$

Example 19.30. *Given* $\dfrac{1}{\rho} = \dfrac{1}{L\rho i} + C\rho i + \dfrac{1}{R}$, *where L, ρ, R are real, express ρ in the form* $Ae^{i\theta}$ *giving the values of A and θ.*

Solution.
$$\frac{1}{\rho} = \frac{R + L\rho^2 CR(-1) + L\rho i}{L\rho Ri} = \frac{(R - L\rho^2 CR) + iLR}{L\rho Ri}$$

or
$$\rho = L\rho \frac{Ri}{(R - L\rho^2 CR) + iL\rho} \times \frac{(R - L\rho^2 CR) - iL\rho}{(R - L\rho^2 CR) - iL\rho}$$
$$= \frac{L^2\rho^2 R + iL\rho R\,(R - L\rho^2 CR)}{(R - L\rho^2 CR)^2 + (L\rho)^2} = A(\cos \theta + i \sin \theta), \text{ say}$$

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*See footnote p. 205.

Equating real and imaginary parts, we have

$$A\cos\theta = \frac{L^2\rho^2 R}{(R - L\rho^2 CR)^2 + (L\rho)^2} \quad ...(i)$$

$$A\sin\theta = \frac{L\rho R(R - L\rho^2 CR)}{(R - L\rho^2 CR)^2 + (L\rho)^2} \quad ...(ii)$$

Squaring and adding (*i*) and (*ii*),

$$A^2 = \frac{(L^2\rho^2 R)^2 + (L\rho R)^2 (R - L\rho^2 CR)^2}{[(R - L\rho^2 CR)^2 + (L\rho)^2]^2} \quad \text{or} \quad A = \frac{L\rho R}{\sqrt{[(R - L\rho^2 CR)^2 + (L\rho)]^2}} \quad ...(iii)$$

Dividing (*ii*) by (*i*),

$$\tan\theta = \frac{R - L\rho^2 CR}{L\rho} \quad \text{or} \quad \theta = \tan^{-1}\left\{\frac{R(1 - LC\rho^2)}{L\rho}\right\} \quad ...(iv)$$

Hence $P = A(\cos\theta + i\sin\theta) = Ae^{i\theta}$

where A and θ are given by (*iii*) and (*iv*).

19.10 HYPERBOLIC FUNCTIONS

(1) Definitions: *If x be real or complex,*

(*i*) $\dfrac{e^x - e^{-x}}{2}$ *is defined as* **hyperbolic sine of x** and is written as **sinh x.**

(*ii*) $\dfrac{e^x + e^{-x}}{2}$ *is defined as* **hyperbolic cosine of x** *and is written as* **cosh x.**

Thus $\sinh x = \dfrac{e^x - e^{-x}}{2}$ and $\cosh x = \dfrac{e^x + e^{-x}}{2}$

Also we define,

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}; \ \coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}; \ \operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

(2) Properties

I. Periodic functions : sinh z and cosh z are periodic functions having imaginary period $2\pi i$.

[$\because$ $\sinh(z + 2\pi i) = \sinh z$; $\cosh(z + 2\pi i) = \cosh z$]

II. Even and odd functions : cosh z is an even function while sinh z is an odd function

III. $\sinh 0 = 0$, $\cosh 0 = 1$, $\tanh 0 = 0$.

IV. **Relations between hyperbolic and circular functions.**

Since for all values of θ, $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$ and $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

$\therefore$ Putting $\theta = ix$, we have $\sin ix = \dfrac{e^{-x} - e^x}{2i} = -\dfrac{e^x - e^{-x}}{2i}$ [$\because$ $e^{i\theta} = e^{i \cdot ix} = e^{-x}$]

$$= i^2 \frac{e^x - e^{-x}}{2i} = i \cdot \frac{e^x - e^{-x}}{2} = i\sinh x$$

and, therefore, $\cos ix = \dfrac{e^{-x} + e^x}{2} = \cosh x$

Thus **sin ix = i sinh x** ...(*i*)

cos ix = cosh x ...(*ii*)

and $\therefore$ **tan ix = i tanh x** ...(*iii*)

Cor. **sinh ix = i sin x** ...(*iv*)

cosh ix = cos x ...(*v*)

tanh ix = i tan x ...(*vi*)

V. **Formulae of hyperbolic functions**

(a) Fundamental formulae

(1) $\cosh^2 x - \sinh^2 x = 1$ (2) $\text{sech}^2 x + \tanh^2 x = 1$ (3) $\coth^2 x - \text{cosech}^2 x = 1$.

(b) Addition formulae

(4) $\sinh (x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$ (5) $\cosh (x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

(6) $\tanh (x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$

(c) Functions of 2x.

(7) $\sinh 2x = 2 \sinh x \cosh x$

(8) $\cosh 2x = \cosh^2 x + \sinh^2 x = 2 \cosh^2 x - 1 = 1 + 2 \sinh^2 x$

(9) $\tanh 2x = \dfrac{2 \tanh x}{1 + \tanh^2 x}$

(d) Functions of 3x

(10) $\sinh 3x = 3 \sinh x + 4 \sinh^3 x$

(11) $\cosh 3x = 4 \cosh^3 x - 3 \cosh x$ (12) $\tanh 3x = \dfrac{3 \tanh x + \tanh^3 x}{1 + 3 \tanh^2 x}$

(e) (13) $\sinh x + \sinh y = 2 \sinh \dfrac{x+y}{2} \cosh \dfrac{x-y}{2}$ (14) $\sinh x - \sinh y = 2 \cosh \dfrac{x+y}{2} \sinh \dfrac{x-y}{2}$

(15) $\cosh x + \cosh y = 2 \cosh \dfrac{x+y}{2} \cosh \dfrac{x-y}{2}$ (16) $\cosh x - \cosh y = 2 \sinh \dfrac{x+y}{2} \sinh \dfrac{x-y}{2}$.

Proofs. (1) Since, for all values of θ, we have $\cos^2 \theta + \sin^2 \theta = 1$.

$\therefore$ putting $\theta = ix$, we get $\cos^2 ix + \sin^2 ix = 1$ or $\cosh^2 x - \sinh^2 x = 1$

Otherwise : $\cosh^2 x - \sinh^2 x = \left(\dfrac{e^x + e^{-x}}{2}\right)^2 - \left(\dfrac{e^x - e^{-x}}{2}\right)^2 = \dfrac{1}{4} [e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2] = 1.$

Similarly we can establish the formulae (2) and (3).

(4) $\sinh (x + y) = (1/i) \sin i(x + y) = - i[\sin ix \cos iy + \cos ix \sin iy]$

$= - i[i \sinh x \,.\, \cosh y + \cosh x \,.\, i \sinh y] = \sinh x \cosh y + \cosh x \sinh y.$

Otherwise : $\sinh x \cosh y + \cosh x \sinh y$

$$= \frac{e^x - e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2} = \frac{e^{x+y} - e^{-(x+y)}}{2} = \sinh (x + y)$$

Similarly we can establish the formulae (5) and (6).

(12) $\tan 3A = \dfrac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$

Putting $A = ix$, $\tan 3ix = \dfrac{3 \tan ix - \tan^3 ix}{1 - 3 \tan^2 ix}$ or $i \tanh 3x = \dfrac{3(i \tanh x) - (i \tanh x)^3}{1 - 3(i \tanh x)^2}$

$$\tanh 3x = \frac{3 \tanh x + \tanh^3 x}{1 + 3 \tanh^2 x}$$

Similarly, we can establish the formulae (7) to (11).

(16) $\cos C - \cos D = - 2 \sin \dfrac{C+D}{2} \sin \dfrac{C-D}{2}$

Putting $C = ix$, and $D = iy$, $\cos ix - \cos iy = - 2 \sin i \dfrac{x+y}{2} \sin i \dfrac{x-y}{2}$

or $$\cosh x - \cosh y = - 2\left(i \sinh \frac{x+y}{2}\right)\left(i \sinh \frac{x-y}{2}\right) = 2 \sinh \frac{x+y}{2} \sinh \frac{x-y}{2}$$

Similarly, we can establish the formulae (13) to (15).

19.11 INVERSE HYPERBOLIC FUNCTIONS

(1) Definitions: If *sinh* $u = z$, then u is called the hyperbolic sine inverse of z and is written as $u = \sinh^{-1} z$. Similarly we define $\cosh^{-1} z$, $\tanh^{-1} z$, etc.

The inverse hyperbolic functions like other inverse functions are many-valued, but we shall consider only their principal values.

(2) *To show that (i)* $\sinh^{-1} z = \log [z + \sqrt{(z^2 + 1)}]$ *(Mumbai, 2009)*

(ii) $\cosh^{-1} z = \log [z + \sqrt{(z^2 - 1)}]$, *(iii)* $\tanh^{-1} z = \frac{1}{2} \log \frac{1+z}{1-z}$.

(i) Let $\sinh^{-1} z = u$, then $z = \sinh u = \frac{1}{2}(e^u - e^{-u})$

or $$2z = e^u - 1/e^u \quad \text{or} \quad e^{2u} - 2ze^u - 1 = 0$$

This being a quadratic in e^u, we have

$$e^u = \frac{2z \pm \sqrt{(4z^2 + 4)}}{2} = z \pm \sqrt{(z^2 + 1)}$$

$\therefore$ Taking the positive sign only, we have

$$e^u = z + \sqrt{(z^2 + 1)} \quad \text{or} \quad u = \log [z + \sqrt{(z^2 + 1)}]$$

Similarly we can establish *(ii)*

(iii) Let $\tanh^{-1} z = u$, then $z = \tanh u$

i.e., $$z = \frac{e^u - e^{-u}}{e^u + e^{-u}}.$$

Applying componendo and dividendo, we get $\frac{1+z}{1-z} = e^u / e^{-u} = e^{2u}$

or $$2u = \log\left(\frac{1+z}{1-z}\right) \text{ whence follows the result.}$$ *(P.T.U., 2005)*

Example 19.31. *If* $u = \log \tan (\pi/4 + \theta/2)$, *prove that*

(i) $\tanh u/2 = \tan \theta/2$ *(Mumbai, 2008 ; P.T.U., 2006 ; Madras, 2003)*

(ii) $\theta = -i \log \tan\left(\frac{\pi}{4} + \frac{iu}{2}\right)$. *(Kurukshetra, 2006)*

Solution. We have $$e^u = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) \quad \text{or} \quad \frac{e^{u/2}}{e^{-u/2}} = \frac{1 + \tan \theta/2}{1 - \tan \theta/2}$$

By componendo and dividendo, we get

$$\frac{e^{u/2} - e^{-u/2}}{e^{u/2} + e^{-u/2}} = \tan \theta/2 \quad i.e., \quad \tanh \frac{u}{2} = \tan \frac{\theta}{2} \qquad ...(i)$$

or $$\frac{1}{i} \tan \frac{iu}{2} = \frac{1}{i} \tanh \frac{i\theta}{2} \quad \text{or} \quad \frac{i\theta}{2} = \tanh^{-1}\left(\tan \frac{iu}{2}\right) = \frac{1}{2} \log \frac{1 + \tan iu/2}{1 - \tan iu/2}$$

or $$\theta = \frac{1}{i} \log \tan\left(\frac{\pi}{4} + \frac{iu}{2}\right) = -i \log \tan\left(\frac{\pi}{4} + \frac{iu}{2}\right). \qquad ...(ii)$$

Example 19.32. *Show that* $\tanh^{-1} (\cos \theta) = \cosh^{-1} (\text{cosec } \theta)$. *(Kurukshetra, 2005)*

Solution. Let $\tanh^{-1} (\cos \theta) = \phi$ so that $\cos \theta = \tanh \phi$

or $\tanh^2 \phi = \cos^2 \theta$ or $1 - \text{sech}^2 \phi = \cos^2 \theta$

or $\text{sech}^2 \phi = 1 - \cos^2 \theta = \sin^2 \theta$ or $\text{sech } \phi = \sin \theta$

or $\cosh \phi = \text{cosec } \theta$ or $\phi = \cosh^{-1} (\text{cosec } \theta)$.

Example 19.33. *Find tanh x, if 5 sinh x – cosh x = 5.* (*Mumbai, 2004*)

Solution. We have $5(\sinh x - 1) = \cosh x$

or $$25(\sinh x - 1)^2 = \cosh^2 x = 1 + \sinh^2 x$$

or $$24 \sinh^2 x - 50 \sinh x + 24 = 0 \quad \text{or} \quad 12 \sinh^2 x - 25 \sinh x + 12 = 0$$

or $$(3 \sinh x - 4)(4 \sinh x - 3) = 0 \quad \text{whence } \sinh x = 4/3 \quad \text{or} \quad 3/4.$$

$\therefore$ $$\cosh x = \sqrt{(1 + \sinh^2 x)} = 5/3 \quad \text{or} \quad -5/4 \qquad [\because \cosh x = 5/4 \text{ doesn't satisfy } (i)]$$

Hence $$\tanh x = \frac{4}{5} \quad \text{or} \quad -\frac{3}{5}.$$

PROBLEMS 19.5

1. Separate into real and imaginary parts
   (*i*) $\exp(z^2)$ where $z = x + iy$ (*ii*) $\exp(5 + i\pi/2)$ (*iii*) $\exp(5 + 3i)^2$.
2. From the definitions of $\sin z$ and $\cos z$, prove that
   (*i*) $\cos 2z = 2\cos^2 z - 1$ (*ii*) $\dfrac{\sin 2z}{1 - \cos 2z} = \cot z$ (*iii*) $\sin 3z = 3 \sin z - 4 \sin^3 z$.
3. Prove that $[\sin(\alpha - \theta) + e^{-i\alpha} \sin \theta]^n = \sin^{n-1} \alpha \{\sin(\alpha - n\theta) + e^{-i\alpha} \sin n\theta\}$
4. If $z = e^{i\theta}$, show that $\dfrac{z^2 - 1}{z^2 + 1} = i \tan \theta$.
5. Eliminate z from $p \operatorname{cosech} z + q \operatorname{sech} z + r = 0$, $p' \operatorname{cosech} z + q' \operatorname{sech} z + r' = 0$.
6. If $y = \log \tan x$, show that $\sinh ny = \dfrac{1}{2}(\tan^n x - \cot^n x)$.
7. If $\tan y = \tan \alpha \tanh \beta$ and $\tan z = \cot \alpha \tanh \beta$, prove that $\tan(y + z) = \sinh 2\beta \operatorname{cosec} 2\alpha$.
8. Prove that
   (*i*) $\cosh(\alpha + \beta) - \cosh(\alpha - \beta) = 2 \sinh \alpha \sinh \beta$
   (*ii*) $\sinh(\alpha + \beta) \cosh(\alpha - \beta) = \dfrac{1}{2}(\sinh 2\alpha + \sinh 2\beta)$.
9. Prove that (*i*) $(\cosh \theta \pm \sinh \theta)^n = \cosh n\theta + \sinh n\theta$; (*ii*) $\left(\dfrac{1 + \tanh \theta}{1 - \tanh \theta}\right)^3 = \cosh 6\theta + \sinh 6\theta$.
10. Express $\cosh^7 \theta$ in terms of hyperbolic cosines of multiples of θ.
11. If $\sin \theta = \tanh x$, prove that $\tan \theta = \sinh x$.
12. If $\tan x/2 = \tanh u/2$, prove that
    (*i*) $\tan x = \sinh u$ and $\cos x \cosh u = 1$; (*ii*) $u = \log_e \tan(\pi/4 + x/2)$.
13. If $\cosh x = \sec \theta$, prove that
    (*i*) $\tanh^2 x/2 = \tan^2 \theta/2$ (*ii*) $x = \log_e \tan(\pi/4 + \theta/2)$.
14. Show that $\tan^{-1} z = \dfrac{i}{2} \log \dfrac{i + z}{i - z}$.
15. Prove that
    (*i*) $\sinh^{-1} x = \cosh^{-1} \sqrt{(1 + x^2)} = \tanh^{-1} \dfrac{x}{\sqrt{(1 - x^2)}} = \dfrac{1}{2} \operatorname{cosech}^{-1} \dfrac{1}{2x\sqrt{(1 + x^2)}}$
    (*ii*) $\tanh^{-1} x = \sinh^{-1} \dfrac{x}{\sqrt{(1 - x^2)}}$.
16. Show that
    (*i*) $\sinh^{-1}(\tan \theta) = \log \tan(\pi/4 + \theta/2)$ (*ii*) $\operatorname{sech}^{-1}(\sin \theta) = \log \cot \theta/2$.
17. Solve the equation $7 \cosh x + 8 \sinh x = 1$ for real values of x. (*Mumbai, 2008*)
18. Find $\tanh x$ if $\sinh x - \cosh x = 5$.

19.12 REAL AND IMAGINARY PARTS OF CIRCULAR AND HYPERBOLIC FUNCTIONS

(1) *To separate the real and imaginary parts of*

(i) sin $(x + iy)$; (ii) cos $(x + iy)$; (iii) tan $(x + iy)$; (iv) cot $(x + iy)$; (v) sec $(x + iy)$; (vi) cosec $(x + iy)$.

Proofs. (*i*) $\sin(x + iy) = \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i \cos x \sinh y$.

Similarly, $\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$

(*iii*) Let $\alpha + i\beta = \tan(x + iy)$ then $\alpha - i\beta = \tan(x - iy)$

Adding, $2\alpha = \tan(x + iy) + \tan(x - iy)$

i.e.,
$$\alpha = \frac{\sin(x + iy + x - iy)}{2\cos(x + iy)\cos(x - iy)} = \frac{\sin 2x}{\cos 2x + \cos 2iy} = \frac{\sin 2x}{\cos 2x + \cosh 2y}$$

Subtracting, $2i\beta = \tan(x + iy) - \tan(x - iy)$

i.e.,
$$i\beta = \frac{\sin 2iy}{2\cos(x + iy)\cos(x - iy)} = \frac{i \sinh 2y}{\cos 2x + \cosh 2y}$$

$$\therefore \quad \beta = \frac{\sinh 2y}{\cos 2x + \cosh 2y}$$

Similarly, $\cot(x + iy) = \dfrac{\sin 2x - i \sinh 2y}{\cosh 2y - \cos 2x}$.

(*v*) Let $\alpha + i\beta = \sec(x + iy)$ then $\alpha - i\beta = \sec(x - iy)$

Adding, $2\alpha = \sec(x + iy) + \sec(x - iy)$

i.e.,
$$\alpha = \frac{\cos(x - iy) + \cos(x + iy)}{2\cos(x + iy)\cos(x - iy)} = \frac{2\cos x \cos iy}{\cos 2x + \cos 2iy} = \frac{2\cos x \cosh y}{\cos 2x + \cosh 2y}$$

Subtracting, $2i\beta = \sec(x + iy) - \sec(x - iy)$

i.e.,
$$i\beta = \frac{\cos(x - iy) - \cos(x + iy)}{2\cos(x + iy)\cos(x - iy)} = \frac{2\sin x \sin iy}{\cos 2x + \cos 2iy} = \frac{2i \sin x \sinh y}{\cos 2x + \cosh 2y}$$

$$\therefore \quad \beta = \frac{2\sin x \sinh y}{\cos 2x + \cosh 2y}$$

Similarly, $\text{cosec}(x + iy) = 2\,\dfrac{\sin x \cosh y - i \cos x \sinh y}{\cosh 2y - \cos 2x}$.

(2) *To separate the real and imaginary parts of*

(i) sinh $(x + iy)$; (ii) cosh $(x + iy)$; (iii) tanh $(x + iy)$.

Proofs. (*i*) $\sinh(x + iy) = (1/i)\sin i(x + iy) = (1/i)\sin(ix - y)$

$= (1/i)[\sin ix \cos y - \cos ix \sin y] = (1/i)[i \sinh x \cos y - \cosh x \sin y]$

$= \sinh x \cos y + i \cosh x \sin y$

Similarly, $\cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y$.

(*iii*) If $\alpha + i\beta = \tanh(x + iy) = (1/i)\tan(ix - y)$

then $\alpha - i\beta = \tanh(x - iy) = (1/i)\tan(ix + y)$

Adding, $2\alpha = (1/i)[\tan(ix - y) + \tan(ix + y)]$

$$\alpha = \frac{\sin(ix - y + ix + y)}{i \,.\, 2\cos(ix - y)\cos(ix + y)} = \frac{(1/i)\sin 2ix}{\cos 2ix + \cos 2y} = \frac{\sinh 2x}{\cosh 2x + \cos 2y}.$$

Subtracting, $2i\beta = (1/i)[\tan(ix - y) - \tan(ix + y)]$

i.e.,
$$i\beta = -\frac{\sin[(ix + y) - (ix - y)]}{i \,.\, 2\cos(ix + y)\cos(ix - y)}$$

$$\therefore \quad \beta = \frac{\sin 2y}{\cos 2ix + \cos 2y} = \frac{\sin 2y}{\cosh 2x + \cos 2y}.$$

Example 19.34. *If cosh $(u + iv) = x + iy$, prove that*

$$\frac{x^2}{\cosh^2 u} + \frac{y^2}{\sinh^2 u} = 1 \quad \textit{(P.T.U., 2009 S)} \qquad \frac{x^2}{\cos^2 v} - \frac{y^2}{\sin^2 v} = 1. \qquad \textit{(Madras, 2000)}$$

Solution. Since $x + iy = \cosh(u + iv) = \cos(iu - v)$

$$= \cos iu \cos v + \sin iu \sin v = \cosh u \cos v + i \sinh u \sin v.$$

$\therefore$ equating real and imaginary parts, we get $x = \cosh u \cos v$; $y = \sinh u \sin v$

i.e., $$\frac{x}{\cosh u} = \cos v \quad \text{and} \quad \frac{y}{\sinh u} = \sin v$$

Squaring and adding, we get the first result.

Again $$\frac{x}{\cos v} = \cosh u \quad \text{and} \quad \frac{v}{\sin v} = \sinh u.$$

$\therefore$ squaring and subtracting, we get the second result.

Example 19.35. *If* $\tan(\theta + i\phi) = e^{i\alpha}$, *show that*

$$\theta = (n + 1/2)\pi/2 \quad and \quad \phi = \frac{1}{2}\log\tan(\pi/4 + \alpha/2).$$ *(S.V.T.U., 2007 ; Rohtak, 2005)*

Solution. Since $\tan(\theta + i\phi) = \cos\alpha + i\sin\alpha \quad \therefore \quad \tan(\theta - i\phi) = \cos\alpha - i\sin\alpha$

$\therefore$ $$\tan 2\theta = \tan[(\theta + i\phi) + (\theta - i\phi)]$$

$$= \frac{\tan(\theta + i\phi) + \tan(\theta - i\phi)}{1 - \tan(\theta + i\phi)\tan(\theta - i\phi)} = \frac{2\cos\alpha}{1 - (\cos^2\alpha + \sin^2\alpha)} = \frac{2\cos\alpha}{0} \to \infty$$

i.e., $$2\theta = n\pi + \pi/2 \quad \text{or} \quad \theta = (n + 1/2)\pi/2$$

Also $$\tan 2i\phi = \tan[(\theta + i\phi) - (\theta - i\phi)] = \frac{\tan(\theta + i\phi) - \tan(\theta - i\phi)}{1 + \tan(\theta + i\phi)\tan(\theta - i\phi)}$$

or $$i\tanh 2\phi = \frac{2i\sin\alpha}{1 + (\cos^2\alpha + \sin^2\alpha)} = i\sin\alpha \quad \text{or} \quad \frac{e^{2\phi} - e^{-2\phi}}{e^{2\phi} + e^{-2\phi}} = \frac{\sin\alpha}{1}$$

By componendo and dividendo, we get

$$\frac{e^{2\phi}}{e^{-2\phi}} = \frac{1 + \sin\alpha}{1 - \sin\alpha} = \frac{\cos^2\alpha/2 + \sin^2\alpha/2 + 2\sin\alpha/2\cos\alpha/2}{\cos^2\alpha/2 + \sin^2\alpha/2 - 2\sin\alpha/2\cos\alpha/2}$$

or $$e^{4\phi} = \frac{(\cos\alpha/2 + \sin\alpha/2)^2}{(\cos\alpha/2 - \sin\alpha/2)^2} = \left(\frac{1 + \tan\alpha/2}{1 - \tan\alpha/2}\right)^2$$

or $$e^{2\phi} = \frac{1 + \tan\alpha/2}{1 - \tan\alpha/2} = \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right). \quad \text{Hence } \phi = \frac{1}{2}\log\tan(\pi/4 + \alpha/2).$$

Example 19.36. *Separate* $\tan^{-1}(x + iy)$ *into real and imaginary parts.* *(S.V.T.U., 2009)*

Solution. Let $\alpha + i\beta = \tan^{-1}(x + iy)$. Then $\alpha - i\beta = \tan^{-1}(x - iy)$

Adding, $$2\alpha = \tan^{-1}(x + iy) + \tan^{-1}(x - iy)^* = \tan^{-1}\frac{(x + iy) + (x - iy)}{1 - (x + iy)(x - iy)}$$

$\therefore$ $$\alpha = \frac{1}{2}\tan^{-1}\frac{2x}{1 - x^2 - y^2}$$

Subtracting, $$2i\beta = \tan^{-1}(x + iy) - \tan^{-1}(x - iy) = \tan^{-1}\frac{(x + iy) - (x - iy)}{1 + (x + iy)(x - iy)}$$

$$= \tan^{-1} i\frac{2y}{1 + x^2 + y^2} = i\tanh^{-1}\frac{2y}{1 + x^2 + y^2} \qquad [\because \tan^{-1} iz = i\tanh^{-1} z]$$

$\therefore$ $$\beta = \frac{1}{2}\tanh^{-1}\frac{2y}{1 + x^2 + y^2}.$$

Example 19.37. *Separate* $\sin^{-1}(\cos\theta + i\sin\theta)$ *into real and imaginary parts, where* θ *is a positive acute angle.*

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* $\tan^{-1}A \pm \tan^{-1}B = \tan^{-1}\dfrac{A \pm B}{1 \mp AB}$

Solution. Let $\sin^{-1}(\cos\theta + i\sin\theta) = x + iy$

Then $$\cos\theta + i\sin\theta = \sin(x + iy) = \sin x\cosh y + i\cos x\sinh y$$

$\therefore$ $$\cos\theta = \sin x\cosh y \quad ...(i) \qquad \text{and} \quad \sin\theta = \cos x\sinh y \quad ...(ii)$$

Squaring and adding, we have

$$1 = \sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y = \sin^2 x(1 + \sinh^2 y) + \cos^2 x\sinh^2 y$$
$$= \sin^2 x + \sinh^2 y(\sin^2 x + \cos^2 x)$$

or $$1 - \sin^2 x = \sinh^2 y, \quad i.e. \quad \cos^2 x = \sinh^2 y.$$

Hence from (*ii*), we have $\sin^2\theta = \cos^4 x$, *i.e.*, $\cos^2 x = \sin\theta$ because θ being a positive acute angle, $\sin\theta$ is positive.

As x is to be between $-\pi/2$ and $\pi/2$, therefore, we have

$$\cos x = +\sqrt{(\sin\theta)} \quad \text{or} \quad x = \cos^{-1}\sqrt{(\sin\theta)}$$

The relation (*ii*), then, gives $\sinh y = \sqrt{(\sin\theta)}$ so that $y = \log[\sqrt{(\sin\theta)} + \sqrt{(1 + \sin\theta)}]$.

PROBLEMS 19.6

1. If $\sin(A + iB) = x + iy$, prove that

(*i*) $\dfrac{x^2}{\cosh^2 B} + \dfrac{y^2}{\sinh^2 B} = 1$ (*ii*) $\dfrac{x^2}{\sin^2 A} - \dfrac{y^2}{\cos^2 A} = 1.$ (*P.T.U., 2010*)

2. If $\cos(\alpha + i\beta) = r(\cos\theta + i\sin\theta)$, prove that (*i*) $e^{2\beta} = \dfrac{\sin(\alpha - \theta)}{\sin(\alpha + \theta)}$. (*Kurukshetra, 2005 ; Madras, 2003*)

(*ii*) $\beta = \dfrac{1}{2}\log\dfrac{\sin(\alpha - \theta)}{\sin(\alpha + \theta)}$. (*V.T.U., 2006*)

3. If $\cos(\theta + i\phi) = \cos\alpha + i\sin\alpha$, prove that
(*i*) $\sin^2\theta = \pm\sin\alpha$ (*Madras, 2003*) (*ii*) $\cos 2\theta + \cosh 2\phi = 2$.
4. If $\tan(A + iB) = x + iy$, prove that
(*i*) $x^2 + y^2 + 2x\cot 2A = 1$. (*ii*) $x^2 + y^2 - 2y\coth 2B + 1 = 0$. (*iii*) $x\sinh 2B = y\sin 2A$.
5. If $\tan(\theta + i\phi) = \tan\alpha + i\sec\alpha$, prove that $e^{2\phi} = \pm\cot\alpha/2$ and $2\theta = \left(n + \dfrac{1}{2}\right)\pi + \alpha$. (*Nagpur, 2009 ; S.V.T.U., 2008*)
6. If $\tan(x + iy) = \sin(u + iv)$, prove that $\dfrac{\sin 2x}{\sinh 2y} = \dfrac{\tan u}{\tan v}$. (*S.V.T.U., 2006*)
7. If $\operatorname{cosec}(\pi/4 + ix) = u + iv$, prove that $(u^2 + v^2) = 2(u^2 - v^2)$. (*Mumbai, 2009*)
8. If $x = 2\cos\alpha\cosh\beta$, $y = 2\sin\alpha\sinh\beta$, prove that $\sec(\alpha + i\beta) + \sec(\alpha - i\beta) = \dfrac{4x}{x^2 + y^2}$.
9. If $a + ib = \tanh(v + i\pi/4)$, prove that $a^2 + b^2 = 1$.
10. Reduce $\tan^{-1}(\cos\theta + i\sin\theta)$ to the form $a + ib$. (*Mumbai, 2009*)

Hence show that $\tan^{-1}(e^{i\theta}) = \dfrac{n\pi}{2} + \dfrac{\pi}{4} - \dfrac{i}{2}\log\tan\left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right)$.

11. Separate $\cos^{-1}(\cos\theta + i\sin\theta)$ into real and imaginary parts, where θ is a positive acute angle.
12. If $\sin^{-1}(u + iv) = \alpha + i\beta$, prove that $\sin^2\alpha$ and $\cosh^2\beta$ are the roots of the equation
$$x^2 - x(1 + u^2 + v^2) + u^2 = 0.$$
13. If $\cos^{-1}(x + iy) = \alpha + i\beta$, show that
(*i*) $x^2\sec^2\alpha - y^2\operatorname{cosec}^2\alpha = 1$, (*ii*) $x^2\operatorname{sech}^2\beta + y^2\operatorname{cosech}^2\beta = 1$.
14. Prove that (*i*) $\sin^{-1}(ix) = 2n\pi + i\log(\sqrt{1 + x^2} + x)$ (*ii*) $\sin^{-1}(\operatorname{cosec}\theta) = \pi/2 + i\log\cot\theta/2$.

19.13 LOGARITHMIC FUNCTION OF A COMPLEX VARIABLE

(1) *Definition. If $z(= x + iy)$ and $w(= u + iv)$ be so related that $e^w = z$, then w is said to be a logarithm of z to the base e and is written as $w = \log_e z$.* ...(*i*)

Also $$e^{w + 2in\pi} = e^w \cdot e^{2in\pi} = z \qquad [\because \ e^{2in\pi} = 1]$$

$\therefore$ $$\log z = w + 2in\pi \qquad ...(ii)$$

i.e., the logarithm of a complex number has an infinite number of values and is, therefore, a multi-valued function.

The general value of the logarithm of z is written as $Log\, z$ (beginning with capital L) so as to distinguish it from its principal value which is written as $\log z$. This principal value is obtained by taking $n = 0$ in $Log\, z$.

Thus from (*i*) and (*ii*), $Log\,(x + iy) = 2in\pi + \log (x + iy)$.

Obs. 1. If $y = 0$, then $Log\, x = 2in\pi + \log x$.

This shows that the logarithm of a real quantity is also multi-valued. Its principal value is real while all other values are imaginary.

2. *We know that the logarithm of a negative quantity has no real value. But we can now evaluate this.*

e.g. $$\log_e (-2) = \log_e 2(-1) = \log_e 2 + \log_e (-1) = \log_e 2 + i\pi \quad [\because -1 = \cos\pi + i\sin\pi = e^{i\pi}]$$
$$= 0.6931 + i\,(3.1416).$$

(2) *Real and imaginary parts of Log* $(x + iy)$.

$$Log\,(x + iy) = 2in\pi + \log (x + iy)$$
$$= 2in\pi + \log [r(\cos\theta + i\sin\theta)] \quad \left\{\begin{array}{l}\text{Put } x = r\cos\theta,\ y = r\sin\theta \text{ so that} \\ r = \sqrt{(x^2 + y^2)} \text{ and } \theta = \tan^{-1}(y/x)\end{array}\right.$$
$$= 2in\pi + \log (re^{i\theta})$$
$$= 2in\pi + \log r + i\theta = \log \sqrt{(x^2 + y^2)} + i\,\{2n\pi + \tan^{-1}(y/x)\}$$

(3) *Real and imaginary parts of* $(\alpha + i\beta)^{x+iy}$

$$(\alpha + i\beta)^{x+iy} = e^{(x+iy)\,Log\,(\alpha+i\beta)} = e^{(x+iy)\,[2in\pi + \log(\alpha+i\beta)]} \quad \left\{\begin{array}{l}\text{Put } \alpha = r\cos\theta, \beta = r\sin\theta \text{ so that} \\ r = \sqrt{(\alpha^2 + \beta^2)} \text{ and } \theta = \tan^{-1}\beta/\alpha\end{array}\right.$$
$$= e^{(x+iy)\,[2in\pi + \log re^{i\theta}]} = e^{(x+iy)\,[\log r + i\,(2n\pi+\theta)]}$$
$$= e^{A+iB} = e^A(\cos B + i\sin B).$$

where $A = x\log r - y\,(2n\pi + \theta)$ and $B = y\log r + x\,(2n\pi + \theta)$.

$\therefore$ the required real part $= e^A \cos B$ and the imaginary part $= e^A \sin B$.

Example 19.38. *Find the general value of* $\log(-i)$.

Solution. $$\text{Log}\,(-i) = 2in\pi + \log [0 + i(-1)] \quad \left\{\begin{array}{l}\text{Put } 0 = r\cos\theta, -1 = r\sin\theta 2 \\ \text{so that } r = 1 \text{ and } \theta = -\pi/2\end{array}\right.$$
$$= 2in\pi + \log [r(\cos\theta + i\sin\theta)] = 2in\pi + \log (re^{i\theta})$$
$$= 2in\pi + \log r + i\theta = 2in\pi + \log 1 + i\,(-\pi/2) = i\left(2n - \frac{1}{2}\right)\pi.$$

Example 19.39. *Prove that* (*i*) $i^i = e^{-(4n+1)\pi/2}$ *and* $Log\, i^i = -\left(2n + \frac{1}{2}\right)\pi$.

(*ii*) $(\sqrt{i})^{\sqrt{i}} = e^{-\alpha}\, cis\, \alpha$ *where* $\alpha = \pi/4\sqrt{2}$. *(Mumbai, 2008)*

Solution. (*i*) By definition, we have

$$i^i = e^{i\,Log\,i} = e^{i\,(2in\pi + \log i)} = e^{-2n\pi + i\log[\exp(i\pi/2)]} \quad [\because i = \text{cis}\,\pi/2 = \exp(i\pi/2)]$$
$$= e^{-2n\pi + i(i\pi/2)} = e^{-(2n+1/2)\pi}$$

Taking logarithms, we get (*ii*)

(*ii*) $$(\sqrt{i})^{\sqrt{i}} = e^{\sqrt{i}\log\sqrt{i}}$$

Now $$\sqrt{i}\,\log\sqrt{i} = \frac{1}{2}\left(\cos\frac{\pi}{2} + i\sin\pi/2\right)^{1/2}\log\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$
$$= \frac{1}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\log(e^{i\pi/2}) = \frac{1}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\frac{i\pi}{2}$$
$$= \frac{i\pi}{4}\left(\frac{i}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = -\frac{\pi}{4\sqrt{2}} + i\,\frac{\pi}{4\sqrt{2}}$$

Hence $(\sqrt{i})^{\sqrt{i}} = e^{-\alpha + i\alpha}$ where $\alpha = \pi/4\sqrt{2}$

$$= e^{-\alpha} . e^{i\alpha} = e^{-\alpha} (\cos \alpha + i \sin \alpha).$$

Example 19.40. *If $(a + ib)^p = m^{x + iy}$, prove that one of the values of y/x is $2 \tan^{-1} (b/a) \div \log (a^2 + b^2)$.*

Solution. Taking logarithms, $(a + ib)^p = m^{x+iy}$ gives $p \log (a + ib) = (x + iy) \log m$

or
$$p\left(\frac{1}{2} \log (a^2 + b^2) + i \tan^{-1} \frac{b}{a}\right) = x \log m + iy \log m$$

Equating real and imaginary parts from both sides, we get

$$\frac{p}{2} \log (a^2 + b^2) = x \log m \quad ...(i), \qquad p \tan^{-1} \frac{b}{a} = y \log m \quad ...(ii)$$

Division of (*ii*) by (*i*) gives

$$y/x = 2 \tan^{-1} (b/a)/\log (a^2 + b^2).$$

Example 19.41. *If $i^{i^{i \dots \infty}} = A + iB$, prove that $\tan \pi A/2 = B/A$ and $A^2 + B^2 = e^{-\pi B}$.* (S.V.T.U., 2006 S)

Solution. $i^{i^{i \dots \infty}} = A + iB$ i.e. $i^{A+iB} = A + iB$

or
$$A + iB = e^{(A+iB)\log i} = e^{(A+iB)\log(\cos \pi/2 + i \sin \pi/2)}$$
$$= \exp [(A + iB) \log (e^{i\pi/2})] = e^{(A+iB)(i\pi/2)}$$
$$= e^{-B\pi/2} . e^{i\pi A/2} = e^{-B\pi/2} \left(\cos \frac{\pi A}{2} + i \sin \frac{\pi A}{2}\right)$$

Equating real and imaginary parts, we get

$$A = e^{-B\pi/2} \cos \frac{\pi A}{2} \quad ...(i) \qquad B = e^{-B\pi/2} \sin \frac{\pi A}{2} \quad ...(ii)$$

Division of (*ii*) by (*i*) gives $B/A = \tan \pi A/2$

Squaring and adding (*i*) and (*ii*), $A^2 + B^2 = e^{-B\pi}$.

Example 19.42. *Prove that $\log \left(\frac{a+ib}{a-ib}\right) = 2i \tan^{-1} \left(\frac{b}{a}\right)$. Hence evaluate $\cos \left[i \log \left(\frac{a+ib}{a-ib}\right)\right]$.* (P.T.U., 2006)

Solution. Putting $a = r \cos \theta$, $b = r \sin \theta$ so that $\theta = \tan^{-1} b/a$, we have

$$\log \left(\frac{a+ib}{a-ib}\right) = \log \frac{r (\cos \theta + i \sin \theta)}{r (\cos \theta - i \sin \theta)} = \log (e^{i\theta} \div e^{-i\theta})$$
$$= \log e^{2i\theta} = 2i\theta = 2i \tan^{-1} b/a.$$

Thus
$$\cos\left[i \log \left(\frac{a+ib}{a-ib}\right)\right] = \cos [i (2i\theta)] = \cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{1 - (b/a)^2}{1 + (b/a)^2} = \frac{a^2 - b^2}{a^2 + b^2}.$$

Example 19.43. *Separate into real and imaginary parts $\log \sin (x + iy)$.*

Solution.
$$\log \sin (x + iy) = \log (\sin x \cos iy + \cos x \sin iy)$$
$$= \log (\sin x \cosh y + i \cos x \sinh y) = \log r (\cos \theta + i \sin \theta),$$

where $r \cos \theta = \sin x \cosh y$ and $r \sin \theta = \cos x \sinh y$,

so that
$$r = \sqrt{(\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y)}$$
$$= \sqrt{\frac{1 - \cos 2x}{2} . \frac{1 + \cosh 2y}{2} + \frac{1 + \cos 2x}{2} . \frac{\cosh 2y - 1}{2}} = \sqrt{\left[\frac{1}{2} (\cosh 2y - \cos 2x)\right]}$$

and $\theta = \tan^{-1} (\cot x \tanh y)$.

Thus $\log \sin (x + iy) = \log (re^{i\theta}) = \log r + i\theta$

$$= \frac{1}{2} \log \left[\frac{1}{2} (\cosh 2y - \cos 2x)\right] + i \tan^{-1} (\cot x \tanh y).$$

Example 19.44. *Find all the roots of the equation*
(i) $\sin z = \cosh 4$ *(ii)* $\sinh z = i$.

Solution. (*i*) $\sin z = \cosh 4 = \cos 4i = \sin(\pi/2 - 4i)$

$\therefore$ $z = n\pi + (-1)^n (\pi/2 - 4i)$ $\left\{\because \text{ If } \sin\theta = \sin\alpha \text{ then } \theta = n\pi + (-1)^n \alpha\right.$

(*ii*) $$i = \sinh z = \frac{e^z - e^{-z}}{2}$$

or $e^{2z} - 2ie^z - 1 = 0$, *i.e.* $(e^z - i)^2 = 0$ *i.e.*, $e^z = i$

or $$z = Log\, i = 2in\pi + \log i = 2in\pi + \log e^{i\pi/2} = 2in\pi + i\pi/2 = i\left(2n + \frac{1}{2}\right)\pi.$$

PROBLEMS 19.7

1. Find the general value of
   (*i*) $\log(6 + 8i)$ (*Rohtak, 2006*) (*ii*) $\log(-1)$. (*J.N.T.U., 2003*)
2. Show that (*i*) $\log(1 + i\tan\alpha) = \log(\sec\alpha) + i\alpha$, where α is an acute angle.
   (*ii*) $\text{Log}_e \dfrac{3-i}{3+i} = 2i\left(n\pi - \tan^{-1}\dfrac{1}{3}\right)$.
3. If $(a_1 + ib_1)(a_2 + ib_2)......(a_n + ib_n) = A + iB$, prove that
   (*i*) $(a_1^2 + b_1^2)(a_2^2 + b_2^2)......(a_n^2 + b_n^2) = A^2 + B^2$
   (*ii*) $\tan^{-1}\dfrac{b_1}{a_1} + \tan^{-1}\dfrac{b_2}{a_2} + ... + \tan^{-1}\dfrac{b_n}{a_n} = \tan^{-1}\dfrac{B}{A}$.
4. Find the modulus and argument of (*i*) $(1 - i)^{1+i}$. (*P.T.U., 2010*) (*ii*) $i^{\log(1+i)}$
5. If $i^{\alpha + i\beta} = \alpha + i\beta$, prove that $\alpha^2 + \beta^2 = e^{-(4n+1)\pi\beta}$. (*Kurukshetra, 2005*)
6. Prove that $\log\left\{\dfrac{\sin(x + iy)}{\sin(x - iy)}\right\} = 2i\tan^{-1}(\cot x \tanh y)$. (*Mumbai, 2007*)
7. Prove that $\tan\left[i\log\left(\dfrac{a - ib}{a + ib}\right)\right] = \dfrac{2ab}{a^2 - b^2}$.
8. If $\tan\log(x + iy) = a + ib$ where $a^2 + b^2 \neq 1$, show that $\tan\log(x^2 + y^2) = \dfrac{2a}{1 - a^2 - b^2}$.
9. If $\sin^{-1}(x + iy) = \log(A + iB)$, show that $\dfrac{x^2}{\sin^2 u} - \dfrac{y^2}{\cos^2 u} = 1$, where $A^2 + B^2 = e^{2u}$.
10. Separate into real and imaginary parts $\log\cos(x + iy)$.
11. Find all the roots of the equation, (*i*) $\cos z = 2$, (*ii*) $\tanh z + 2 = 0$.

19.14 SUMMATION OF SERIES – 'C + iS' METHOD

This is the most general method and is applied to find the sum of a series of the form
$$a_0 \sin\alpha + a_1\sin(\alpha + \beta) + a_2\sin(\alpha + 2\beta) + ...$$
or
$$a_0\cos\alpha + a_1\cos(\alpha + \beta) + a_2\cos(\alpha + 2\beta) + ...$$

Procedure. (*i*) Put the given series = S (or C) according as it is a series of sines (or cosines).

Then write C (or S) = a similar series of cosines (or sines).

e.g., If $S = a_0\sin\alpha + a_1\sin(\alpha + \beta) + a_2\sin(\alpha + 2\beta) + ...$

then $C = a_0\cos\alpha + a_1\cos(\alpha + \beta) + a_2\cos(\alpha + 2\beta)...$

(*ii*) Multiply the series of sines by i and add to the series of cosines, so that
$$C + iS = a_0[\cos\alpha + i\sin\alpha] + a_1[\cos(\alpha + \beta) + i\sin(\alpha + \beta)] + ...$$
$$= a_0e^{i\alpha} + a_1e^{i(\alpha+\beta)} + a_2e^{i(\alpha+2\beta)} + ...$$

(*iii*) Sum up this last series using any of the following standard series :

(1) Exponential series *i.e.*, $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \infty = e^x$

(2) Sine, cosine, sinh or cosh series

i.e., $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \infty = \sin x,\quad 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \infty = \cos x$

$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \infty = \sinh x,\quad 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \infty = \cosh x$

(3) Logarithmic series

i.e., $x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \infty = \log(1+x),\quad -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots \infty\right) = \log(1-x)$

(4) Gregory's series

i.e., $x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots \infty = \tan^{-1} x,\quad x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots \infty = \tanh^{-1} x = \frac{1}{2} \log \frac{1+x}{1-x}$

(5) Binomial series

i.e., $1 + nx + \frac{n(n-1)}{1.2} x^2 + \frac{n(n-1)(n-2)}{1.2.3} x^3 + \ldots \infty = (1+x)^n$

$1 - nx + \frac{n(n+1)}{2!} x^2 - \frac{n(n+1)(n+2)}{3!} x^3 + \ldots \infty = (1+x)^{-n}$

$1 + nx + \frac{n(n+1)}{2!} x^2 + \frac{n(n+1)(n+2)}{3!} x^3 + \ldots \infty = (1-x)^{-n}$

(6) Geometric series

i.e., $a + ar + ar^2 + \ldots$ to n terms $= a\,\frac{1-r^n}{1-r}$, $a + ar + ar^2 + \ldots \infty = \frac{a}{1-r}, |r| < 1.$

(*iv*) Finally express the sum thus obtained in the form $A + iB$ so that by equating the real and imaginary parts, we get $C = A$ and $S = B$.

Series depending on exponential series

Example 19.45. *Sum the series* $\sin \alpha + x \sin(\alpha + \beta) + \frac{x^2}{2!} \sin(\alpha + 2\beta) + \ldots \infty$.

Solution. Let $S = \sin \alpha + x \sin(\alpha + \beta) + \frac{x^2}{2!} \sin(\alpha + 2\beta) + \ldots \infty$

and $C = \cos \alpha + x \cos(\alpha + \beta) + \frac{x^2}{2!} \cos(\alpha + 2\beta) + \ldots \infty$

$$\therefore \quad C + iS = [\cos \alpha + i \sin \alpha] + x[\cos(\alpha + \beta) + i \sin(\alpha + \beta)] + \frac{x^2}{2!}[\cos(\alpha + 2\beta) + i \sin(\alpha + 2\beta)] + \ldots\ldots \infty$$

$$= e^{i\alpha} + xe^{i(\alpha+\beta)} + \frac{x^2}{2!} \cdot e^{i(\alpha+2\beta)} + \ldots \infty = e^{i\alpha}\left[1 + \frac{xe^{i\beta}}{1!} + \frac{x^2 e^{2i\beta}}{2!} + \ldots \infty\right]$$

$$= e^{i\alpha} \cdot e^{xe^{i\beta}} = e^{i\alpha} e^{x(\cos\beta + i \sin\beta)} = e^{x\cos\beta + i(\alpha + x\sin\beta)} = e^{x\cos\beta} e^{i(\alpha + x\sin\beta)}$$

$$= e^{x\cos\beta}[\cos(\alpha + x\sin\beta) + i \sin(\alpha + x\sin\beta)]$$

Equating imaginary parts from both sides, we have $S = e^{x\cos\beta} \sin(\alpha + x\sin\beta)$.

Series depending on logarithmic series

Example 19.46. *Sum the series*

$$\sin^2\theta - \frac{1}{2}\sin 2\theta \sin^2\theta + \frac{1}{3}\sin 3\theta \sin^3\theta - \frac{1}{4}\sin 4\theta \sin^4\theta + \ldots \infty.$$

(P.T.U., 2010 ; V.T.U., 2006 S)

Solution. Let $S = \sin\theta \,.\, \sin\theta - \frac{1}{2}\sin 2\theta \,.\, \sin^2\theta + \frac{1}{3}\sin 3\theta \,.\, \sin^3\theta - \dots \infty$

and $C = \cos\theta \,.\, \sin\theta - \frac{1}{2}\cos 2\theta \,.\, \sin^2\theta + \frac{1}{3}\cos 3\theta \,.\, \sin^3\theta - \dots \infty$

$$\therefore \quad C + iS = e^{i\theta}\sin\theta - \frac{e^{2i\theta}\sin^2\theta}{2} + \frac{e^{3i\theta}\sin^3\theta}{3} - \dots \infty$$
$$= \log(1 + e^{i\theta}\sin\theta) = \log[1 + (\cos\theta + i\sin\theta)\sin\theta]$$
$$= \log[1 + \cos\theta\sin\theta + i\sin^2\theta] \text{ [Put } 1 + \cos\theta\sin\theta = r\cos\alpha \,;\, \sin^2\theta = r\sin\alpha] \quad \dots(i)$$
$$= \log r(\cos\alpha + i\sin\alpha) = \log re^{i\alpha} = \log r + i\alpha$$

Equating imaginary parts, we have $S = \alpha = \tan^{-1}\left(\frac{\sin^2\theta}{1 + \cos\theta\sin\theta}\right)$. [from (i)]

Series depending on binomial series

Example 19.47. *Find the sum to infinity of the series*

$$1 - \frac{1}{2}\cos\theta + \frac{1.3}{2.4}\cos 2\theta - \frac{1.3.5}{2.4.6}\cos 3\theta + \dots (-\pi < \theta < \pi).$$ *(S.V.T.U., 2009)*

Solution. Let $C = 1 - \frac{1}{2}\cos\theta + \frac{1.3}{2.4}\cos 2\theta - \frac{1.3.5}{2.4.6}\cos 3\theta + \dots \infty$

and $S = 0 - \frac{1}{2}\sin\theta + \frac{1.3}{2.4}\sin 2\theta - \frac{1.3.5}{2.4.6}\sin 3\theta + \dots \infty$

$$\therefore \quad C + iS = 1 - \frac{1}{2}e^{i\theta} + \frac{1.3}{2.4}e^{2i\theta} - \frac{1.3.5}{2.4.6}e^{3i\theta} - \dots$$
$$= 1 + \left(-\frac{1}{2}\right)e^{i\theta} + \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{1.2}e^{2i\theta} + \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{1.2.3}e^{3i\theta} + \dots$$
$$= (1 + e^{i\theta})^{-1/2} = (1 + \cos\theta + i\sin\theta)^{-1/2} = \left(2\cos^2\frac{\theta}{2} + i\,.\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{-1/2}$$
$$= \left(2\cos\frac{\theta}{2}\right)^{-1/2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^{-1/2} = \left(2\cos\frac{\theta}{2}\right)^{-1/2}\left(\cos\frac{\theta}{4} - i\sin\frac{\theta}{4}\right).$$

Equating real parts, we have $C = (2\cos\theta/2)^{-1/2}\cos\theta/4$.

PROBLEMS 19.8

Sum the following series :

1. $\cos\theta + \sin\theta\cos 2\theta + \frac{\sin^2\theta}{1.2}\cos 3\theta + \dots \infty.$ *(P.T.U., 2005)*
2. $\sin\alpha - \frac{\sin(\alpha + 2\beta)}{2!} + \frac{\sin(\alpha + 4\beta)}{4!} - \dots \infty.$
3. $x\sin\theta - \frac{1}{2}x^2\sin 2\theta + \frac{1}{3}x^3\sin 3\theta - \dots \infty.$ *(Kurukshetra, 2005)*
4. $\cos\theta - \frac{1}{2}\cos 2\theta + \frac{1}{3}\cos 3\theta \dots \infty.$ *(S.V.T.U., 2006)*
5. $e^{\alpha}\cos\beta - \frac{e^{3\alpha}}{3}\cos 3\beta + \frac{e^{5\alpha}}{5}\cos 5\beta - \dots \infty.$
6. $c\sin\alpha + \frac{c^3}{3}\sin 3\alpha + \frac{c^5}{5}\sin 5\alpha + \dots \infty.$
7. $1 - \frac{1}{2}\cos 2\theta + \frac{1.3}{2.4}\cos 4\theta - \frac{1.3.5}{2.4.6}\cos 6\theta + \dots \infty.$ *(Kurukshetra, 2006)*
8. $n\sin\alpha + \frac{n(n+1)}{1.2}\sin 2\alpha + \frac{n(n+1)(n+2)}{1.2.3}\sin 3\alpha + \dots \infty.$
9. $\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \dots \sin(\alpha + \overline{n-1}\,\beta)$ *(P.T.U., 2009 S)*
10. $\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \dots$ to n terms. *(Kurukshetra, 2006)*
11. $\sin\alpha\cos\alpha + \sin^2\alpha\cos 2\alpha + \sin^3\alpha\cos 3\alpha + \dots\infty.$
12. $1 + x\cos\theta + x^2\cos 2\theta + \dots + x^{n-1}\cos(n-1)\theta.$

19.15 APPROXIMATIONS AND LIMITS

Example 19.48. *If* $\frac{\sin\theta}{\theta} = \frac{599}{600}$, *find an approximate value of* θ *in radians.*

Solution. Since $\frac{\sin\theta}{\theta} = 1 - \frac{1}{600}$ which is nearly equal to 1. $\therefore$ θ must be very small.

We know that $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots$

$$\therefore \quad \frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{6} + \frac{\theta^4}{5!}$$

Omitting θ^4 and higher powers, we have

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{6} = 1 - \frac{1}{600} \quad \text{or} \quad \theta^2 = \frac{1}{100}. \text{ Hence } \theta = 0.1 \text{ radians.}$$

Example 19.49. *Solve approximately* $\sin\left(\frac{\pi}{6} + \theta\right) = 0.51$.

Solution. Since 0.51 is nearly equal to 1/2, which is the value of $\sin \pi/6$, so θ must be very small.

$$\therefore \quad \sin\left(\frac{\pi}{6} + \theta\right) = \sin\frac{\pi}{6}\cos\theta + \cos\frac{\pi}{6}\sin\theta = \frac{1}{2}\left(1 - \frac{\theta^2}{2!} + \ldots\right) + \frac{\sqrt{3}}{2}\left(\theta - \frac{\theta^3}{3!} + \ldots\right)$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}\theta, \text{ omitting } \theta^2 \text{ and higher powers of } \theta.$$

Hence the given equation becomes,

$$\frac{1}{2} + \frac{\sqrt{3}}{2}\theta = 0.51 \quad \text{or} \quad \theta = \frac{1}{50\sqrt{3}}$$

or $$\theta = \frac{1}{50\sqrt{3}} \text{ radian} = \frac{\sqrt{3}}{150} \times 57.29 \text{ degrees nearly} = 39.7'.$$

PROBLEMS 19.9

1. Given $\frac{\sin\theta}{\theta} = \frac{5045}{5046}$, show that θ is $1°58'$ nearly.
2. If $\frac{\sin\theta}{\theta} = \frac{2165}{2166}$, find an approximate value of θ in radians. *(Madras, 2003)*
3. If $\cos\theta = \frac{1681}{1682}$, find θ approximately.
4. Solve approximately the equation $\cos\left(\frac{\pi}{3} + \theta\right) = 0.49$.

19.16 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 19.10

Choose the correct answer or fill up the blanks in each of the following problems :

1. If $x + iy = \sqrt{2} + 3i$, then $x^2 + y$ is

   (*a*) 7 (*b*) 5 (*c*) 13 (*d*) $\sqrt{2} + 3$.

2. The real part of $(\sin x + i\cos x)^5$ is

   (*a*) $-\cos 5x$ (*b*) $-\sin 5x$ (*c*) $\sin 5x$ (*d*) $\cos 5x$.

3. The number $(i)^i$ is
(a) a purely imaginary number (b) an irrational number
(c) a rational number (d) an integer.

4. The relation $|3 - z| + |3 + z| = 5$ represents
(a) a circle (b) a parabola (c) an ellipse (d) a hyperbola.

5. z is a complex number with $|z| = 1$ and arg $(z) = 3\pi/4$. The value of z is
(a) $(1 + i)/\sqrt{2}$ (b) $(-1 + i)/\sqrt{2}$ (c) $(1 - i)/\sqrt{2}$ (d) $(-1 - i)/\sqrt{2}$.

6. If $f(z) = e^{2z}$, then the imaginary part of $f(z)$ is
(a) $e^y \sin x$ (b) $e^x \cos y$ (c) $e^{2x} \cos 2y$ (d) $e^{2x} \sin 2y$.

7. Expansion of $\sin^m \theta \cos^n \theta$ is a series of sines of multiples of θ when m is ......

8. Expansion of $\cos 6\theta$ in terms of $\cos \theta$ is ......

9. If $f(z) = 3\bar{z}$, then the value of $f(z)$ at $z = 2 + 4i$ is ......

10. If $x = \cos \theta + i \sin \theta$, then $x^n - 1/x^n =$ ......

11. Imaginary part of $(2 + i3)/(3 - i4)$ is ......

12. Real part of $\cosh (x + iy)$ is ......

13. If $\dfrac{\sin \theta}{\theta} = \dfrac{2165}{2166}$, then $\theta =$ ...... approximately.

14. If $\tan x/2 = \tanh y/2$, then $\cos x \cosh y =$ ......

15. Imaginary part of $\sin \bar{z}$ is ......

16. Modulus of $(\sqrt{i})^{\sqrt{i}} =$ ....

17. If $\sin \alpha + \sin \beta + \sin \gamma = 0 = \cos \alpha + \cos \beta + \cos \gamma$, then $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3 \cos$ (.............)

18. $\log(-1) =$ ....

19. $(i)^i$ is purely real or imaginary

20. If $\sin \theta = \tanh \phi$, then $\tan \theta =$ ....

21. Imaginary part of $\tan (\theta + i\phi) =$ ....

22. $\cos 5\alpha =$ (.........) $\cos^5 \alpha +$ (.....) $\cos^3 \alpha +$ (.....) $\cos \alpha$.

23. Cube roots of unity form .... triangle.

24. If $|z_1 + z_2| = |z_1 - z_2|$ then amp $(z_1) -$ amp (z_2) is ....

25. If $-3 + ix^2y$ and $x^2 + y + 4i$ represent conjugate complex number then $x =$ .... and $y =$ ....

26. If $\left|\dfrac{z - a}{z - b}\right| = k\ (\neq 1)$, then the locus of z is ....

27. $(-i)^{-i}$ is purely real. (True or False)

28. The statements Re $z > 0$ and $|z - 1| < |z + 1|$ are equivalent. (*Mumbai, 2007*) (True or False)

29. Hyperbolic functions are periodic. (True or False)

30. nth roots of unity form a G.P. (True or False)

31. $\sin ix = -i \sinh x$. (*Mumbai, 2008*) (True or False)

32. If the sum and product of two complex numbers are real, then the two numbers must be either real or conjugate. (*Mumbai, 2008*) (True or False)

33. The modulus of the sum of two complex numbers $\geq$ to the sum of their moduli. (True or False)

CHAPTER 20

Calculus of Complex Functions

20.1 INTRODUCTION

In the previous chapter, we have dealt with some elementary complex functions—the exponential, logarithmic, circular and hyperbolic functions, evaluated at specific complex values. These functions are useful in the study of fluid mechanics, thermodynamics and electric fields. It, therefore, seems desirable to study the calculus of such functions.

20.2 (1) LIMIT OF A COMPLEX FUNCTION

A *function* $w = f(z)$ *is said to tend to* **limit** l *as* z *approaches a point* z_0, *if for every real* $\in$, *we can find a positive real* δ *such that*

$$| f(z) - l | < \in \quad \text{for} \quad | z - z_0 | < \delta$$

i.e., for every $z \neq z_0$ in the δ–disc (dotted) of z-plane, $f(z)$ has a value lying in the $\in$-disc of w-plane (Fig. 20.1). In symbols, we write $\underset{z \to z_0}{\text{Lt}} f(z) = l$.

This definition of limit though similar to that in ordinary calculus, is quite different for in real calculus x approaches x_0 only along the line whereas here *z approaches* z_0 *from any direction in the z-plane.*

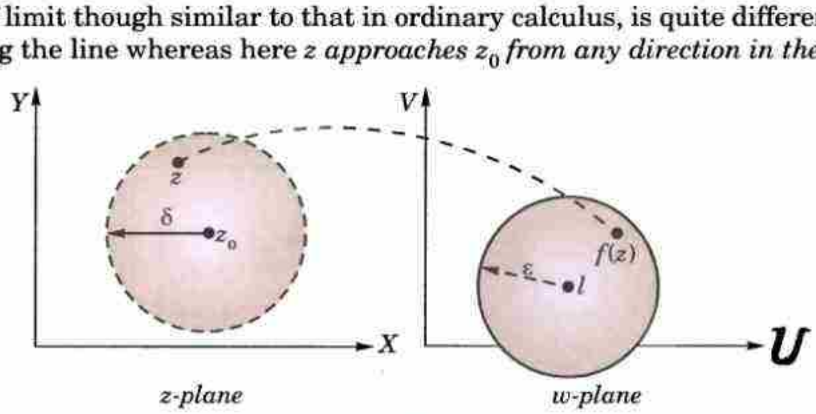

Fig. 20.1

(2) Continuity of f(z). *A function $w = f(z)$ is said to be* **continuous** *at $z = z_0$, if*

$$\underset{z \to z_0}{Lt} f(z) = f(z_0).$$

Further $f(z)$ is said to be continuous in any region R of the z-plane, if it is continuous at every point of that region.

Also if $w = f(z) = u(x, y) + iv(x, y)$ is continuous at $z = z_0$, then $u(x, y)$ and $v(x, y)$ are also continuous at $z = z_0$, *i.e.*, at $x = x_0$ and $y = y_0$. Conversely if $u(x, y)$ and $v(x, y)$ are continuous at (x_0, y_0), then $f(z)$ will be continuous at $z = z_0$. [cf. § 5.1 (3)].

20.3 (1) DERIVATIVE OF f(z)

Let $w = f(z)$ be a single-valued function of the variable $z = x + iy$. Then *the derivative of $w = f(z)$ is defined to be*

$$\frac{dw}{dz} = f'(z) = \underset{\delta z \to 0}{Lt} \frac{f(z + \delta z) - f(z)}{\delta z},$$

provided the limit exists and has the same value for all the different ways in which δz approaches zero.

Suppose $P(z)$ is fixed and $Q(z + \delta z)$ is a neighbouring point (Fig. 20.2). The point Q may approach P along any straight or curved path in the given region, *i.e.*, δz may tend to zero in any manner and dw/dz may not exist. It, therefore, becomes a fundamental problem to determine the necessary and sufficient conditions for dw/dz to exist. The fact is settled by the following theorem.

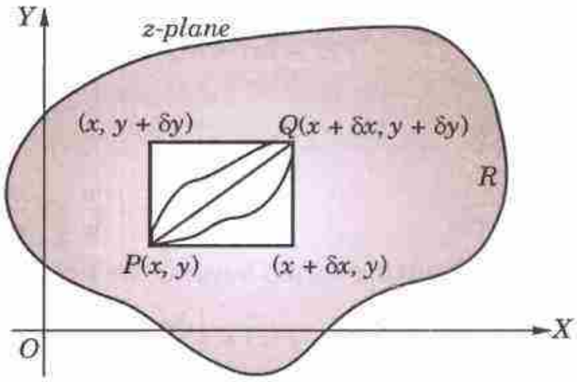

Fig. 20.2

(2) Theorem. *The necessary and sufficient conditions for the derivative of the function $w = u(x, y) + iv(x, y) = f(z)$ to exist for all values of z in a region R, are*

(*i*) $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ *are continuous functions of x and y in R;*

(*ii*) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$.

The relations (ii) are known as **Cauchy-Riemann* equations** or briefly *C-R* equations.

(*a*) *Condition is necessary.*

If $f(z)$ possesses a unique derivative at $P(z)$, then

$$f'(z) = \underset{\delta z \to 0}{\text{Lt}} \frac{f(z + \delta z) - f(z)}{\delta z}$$

$$= \underset{\substack{\delta x \to 0 \\ \delta y \to 0}}{\text{Lt}} \frac{\{u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y)\} - \{u(x, y) + iv(x, y)\}}{\delta x + i\delta y}$$

Since δz can approach zero in any manner, we can first assume δz to be wholly real and then wholly imaginary. When δz is wholly real, then $\delta y = 0$ and $\delta z = \delta x$.

$$\therefore \quad f'(z) = \underset{\delta x \to 0}{\text{Lt}} \left(\frac{u(x + \delta x, y) - u(x, y)}{\delta x} + i\,\frac{v(x + \delta x, y) - v(x, y)}{\delta x} \right) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \quad \text{...(1)}$$

When δz is wholly imaginary, then $\delta x = 0$ and $\delta z = i\delta y$.

$$\therefore \quad f'(z) = \underset{\delta y \to 0}{\text{Lt}} \left(\frac{u(x, y + \delta y) - u(x, y)}{i\delta y} + i\,\frac{v(x, y + \delta y) - v(x, y)}{i\delta y} \right) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\delta y} - i\,\frac{\partial u}{\partial y} \quad \text{...(2)}$$

Now the existence of $f'(z)$ requires the equality of (1) and (2).

* Named after *Cauchy* (p. 144) and the German mathematician *Bernhard Riemann* (1826—1866) who along with *Weierstrass* (p. 390) laid the foundations of complex analysis. Riemann introduced the concept of integration and made basic contributions to number theory and mathematical analysis. He developed the Riemannian geometry which formed the mathematical base for Einstein's relativity theory.

$$\therefore \qquad \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

On equating the real and imaginary parts from both sides, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad ...(3)$$

Thus the necessary conditions for the existence of the derivative of f (z) is that the C-R equations should be satisfied. (*V.T.U., 2011 S*)

(*b*) *Condition is sufficient.* Suppose $f(z)$ is a single-valued function possessing partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ at each point of the region and the C-R equations (3) are satisfied.

Then by Taylor's theorem for functions of two variables (p. 220)

$$f(z + \delta z) = u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y)$$

$$= u(x, y) + \left(\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y\right) + ... + i\left[v(x, y) + \left(\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y\right) + ...\right]$$

$$= f(z) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\partial x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

[Omitting terms beyond the first powers of δx and δy]

or $$f(z + \delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\partial x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y.$$

Now using the C-R equation (3), replace $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ by $-\frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial x}$ respectively.

Then $$f(z + \delta z) - f(z) = \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\delta x + \left[-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right]\delta y = \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\delta x + \left[i\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\right]i\delta y$$

$$= \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right](\delta x + i\delta y) = \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\delta z$$

$$\therefore \qquad f'(z) = \underset{\delta z \to 0}{\text{Lt}} \frac{f(z + \delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{or} \quad \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

which by (1) or (2) proves the sufficiency of conditions.

20.4 ANALYTIC FUNCTIONS

A function f (z) which is single-valued and possesses a unique derivative with respect to z at all points of a region R, is called an **analytic function** *of z in that region. An analytic function is also called a regular function* or an *holomorphic function.*

A function which is analytic everywhere in the complex plane, is known as an ***entire function.*** As derivative of a polynomial exists at every point, a polynomial of any degree is an *entire function.*

A point at which an analytic function ceases to possess a derivative is called a **singular point** *of the function.*

Thus if u and v are real single-valued functions of x and y such that $\delta u/\delta x$, $\delta u/\delta y$, $\delta v/\delta x$, $\delta v/\delta y$ are continuous throughout a region R, then the Cauchy-Riemann equations

$$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \frac{\partial \mathbf{v}}{\partial \mathbf{y}} \text{ and } \frac{\partial \mathbf{u}}{\partial \mathbf{y}} = -\frac{\partial \mathbf{v}}{\partial \mathbf{x}} \qquad ...(1)$$

are both necessary and sufficient conditions for the function $f(z) = u + iv$ to be analytic in R. The derivative of $f(z)$ is then, given by (1) of p. 664 or (2) of p. 665.

The real and imaginary parts of an analytic function are called *conjugate functions*. The relation between two conjugate functions is given by C-R equation (1).

Example 20.1. *If $w = \log z$, find dw/dz and determine where w is non-analytic.* (*U.P.T.U., 2005 ; J.N.T.U., 2005*)

Solution. We have $w = u + iv = \log(x + iy) = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1} y/x$ [By **(2)**, p. 665]

so that $u = \frac{1}{2}\log(x^2 + y^2),\ v = \tan^{-1} y/x.$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} = \frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2} = -\frac{\partial v}{\partial x}.$$

Since the Cauchy-Riemann equations are satisfied and the partial derivatives are continuous except at (0, 0). Hence w is analytic everywhere except at $z = 0$.

$$\therefore \quad \frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{1}{x+iy} = \frac{1}{z}\ (z \neq 0).$$

Obs. The definition of the derivative of a function of complex variable is identical in form to that of the derivative of a function of real variable. Hence the rules of differentiation for complex functions are the same as those of real calculus. *Thus* **if, a complex function is once known to be analytic, it can be differentiated just in the ordinary way.**

Example 20.2. *If $f(z)$ is an analytic function with constant modulus, show that $f(z)$ is constant.*
(U.P.T.U., 2008 ; Mumbai, 2005 S ; Madras 2003 ; Bhopal, 2002 S)

Solution. If $f(z) = u + iv$ is an analytic function, then

$$|f(z)| = \sqrt{(u^2 + v^2)} \text{ is constant } = c \text{ (say) or } u^2 + v^2 = c^2 \qquad ...(i)$$

Differentiating (i) partially w.r.t. x and y, we get

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0\ ; \qquad 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

or $$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \quad ...(ii) \qquad u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0 \qquad ...(iii)$$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y};\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ by C-R equations,

$$\therefore \quad (iii) \text{ becomes } -u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} = 0 \qquad ...(iv)$$

Squaring and adding (ii) and (iv), we obtain

$$u^2\left(\frac{\partial u}{\partial x}\right)^2 + v^2\left(\frac{\partial v}{\partial x}\right)^2 + u^2\left(\frac{\partial v}{\partial x}\right)^2 + v^2\left(\frac{\partial u}{\partial x}\right)^2 = 0$$

or $$(u^2+v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] = 0 \text{ or } \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = 0 \quad [\because\ u^2 + v^2 = c^2 \neq 0] \qquad ...(v)$$

Now $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\therefore \quad |f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = 0 \qquad \text{[By } (v)]$$

or $f'(z) = 0.$ or $f(z) = \text{constant}.$

Example 20.3. *Show that the function $f(z) = \sqrt{|xy|}$ is not analytic at the origin even though C.R. equations are satisfied thereof.* *(A.M.I.E.T.E., 2005 S ; Osmania, 2003)*

Solution. If $f(z) = \sqrt{|xy|} = u(x, y) + iv(x, y)$, then $u(x, y) = \sqrt{|xy|}$, $v(x, y) = 0$

At the origin, we have

$$\frac{\partial u}{\partial x} = \underset{x \to 0}{\text{Lt}}\ \frac{u(x,0) - u(0,0)}{x} = \underset{x \to 0}{\text{Lt}}\ \frac{0-0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \underset{y \to 0}{\text{Lt}}\ \frac{u(0,y) - u(0,0)}{y} = \underset{x \to 0}{\text{Lt}}\ \frac{0-0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \underset{x \to 0}{\text{Lt}}\ \frac{v(x,0) - v(0,0)}{x} = \underset{x \to 0}{\text{Lt}}\ \frac{0-0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \underset{y \to 0}{\text{Lt}} \frac{v(0, y) - v(0, 0)}{y} = \underset{y \to 0}{\text{Lt}} \frac{0 - 0}{y} = 0$$

$\therefore$ $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

i.e., C.R. equations are satisfied at the origin.

However $$f'(0) = \underset{z \to 0}{\text{Lt}} \frac{f(z) - f(0)}{z - 0} = \underset{z \to 0}{\text{Lt}} \frac{\sqrt{|xy|} - 0}{x + iy}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\sqrt{|mx^2|} - 0}{x(1 + im)}, \text{ when } z \to 0 \text{ along the line } y = mx$$

$$= \frac{\sqrt{|m|}}{1 + im} \text{ which is not unique.}$$

$\therefore$ $f'(0)$ does not exist. Hence $f(z)$ is not analytic at the origin.

Example 20.4. *Prove that the function $f(z)$ defined by*

$$f(z) = \frac{x^3(1 + i) - y^3(1 - i)}{x^2 + y^2} (z \neq 0), f(0) = 0$$

is continuous and the Cauchy-Riemann equations are satisfied at the origin, yet $f'(0)$ does not exist.

(S.V.T.U., 2009 ; V.T.U., 2001)

Solution. $$\underset{z \to 0}{\text{Lt}} f(z) = \underset{\substack{x \to 0 \\ y \to 0}}{\text{Lt}} \frac{x^3(1 + i) - y^3(1 - i)}{x^2 + y^2} = \underset{y \to 0}{\text{Lt}} \frac{-y^3(1 - i)}{y^2} = \underset{y \to 0}{\text{Lt}} [-y(1 - i)] = 0$$

$$\underset{z \to 0}{\text{Lt}} f(z) = \underset{\substack{y \to 0 \\ x \to 0}}{\text{Lt}} \frac{x^3(1 + i) - y^3(1 - i)}{x^2 + y^2} = \underset{x \to 0}{\text{Lt}} \frac{x^3(1 + i)}{x^2} = \underset{x \to 0}{\text{Lt}} [x(1 + i)] = 0$$

Also $f(0) = 0$ (given).

Thus $\underset{z \to 0}{\text{Lt}} f(z) = f(0)$ when $x \to 0$ first and then $y \to 0$ and also vice-versa. Now let both x and y tend to zero simultaneously along the path $y = mx$. Then

$$\underset{z \to 0}{\text{Lt}} f(z) = \underset{\substack{y \to mx \\ x \to 0}}{\text{Lt}} \frac{x^3(1 + i) - y^3(1 - i)}{x^2 + y^2}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x^3(1 + i) - m^3x^3(1 - i)}{(1 + m^2)x^2} = \underset{x \to 0}{\text{Lt}} \frac{x[1 + i - m^3(1 - i)]}{1 + m^2} = 0$$

Hence $\underset{z \to 0}{\text{Lt}} f(z) = f(0)$, in whatever manner $z \to 0$. $\therefore$ $f(z)$ is continuous at the origin.

Now $$f(z) = \frac{x^3 - y^3}{x^2 + y^2} + i \frac{x^3 + y^3}{x^2 + y^2} = u(x, y) + iv(x, y).$$

Also $u(0, 0) = 0$, and $v(0, 0) = 0$ $[\because \ f(0) = 0]$

$\therefore$ $$\left(\frac{\partial u}{\partial x}\right)_{0,0} = \underset{x \to 0}{\text{Lt}} \frac{u(x, 0) - u(0, 0)}{x} = \underset{x \to 0}{\text{Lt}} \frac{x}{x} = 1$$

$$\left(\frac{\partial u}{\partial y}\right)_{0,0} = \underset{y \to 0}{\text{Lt}} \frac{u(0, y) - u(0, 0)}{y} = \underset{y \to 0}{\text{Lt}} \frac{-y}{y} = -1$$

$$\left(\frac{\partial v}{\partial x}\right)_{0,0} = \underset{x \to 0}{\text{Lt}} \frac{v(x, 0) - v(0, 0)}{x} = \underset{x \to 0}{\text{Lt}} \frac{x}{x} = 1$$

and $$\left(\frac{\partial v}{\partial y}\right)_{0,0} = \underset{y \to 0}{\text{Lt}} \frac{v(0, y) - v(0, 0)}{y} = \underset{y \to 0}{\text{Lt}} \frac{y}{y} = 1.$$

Hence at (0, 0), $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Thus the C-R equations are satisfied at the origin.

But $$f'(0) = \underset{z \to 0}{\text{Lt}} \frac{f(z) - f(0)}{z} = \underset{z \to 0}{\text{Lt}} \frac{f(z)}{z} = \underset{z \to 0}{\text{Lt}} \frac{x^3 - y^3 + i(x^3 + y^3)}{(x^2 + y^2)(x + iy)}.$$

If $z \to 0$ along the path $y = mx$, then $f'(0) = \dfrac{1 - m^3 + i(1 + m^3)}{(1 + m^2)(1 + im)}$

which assumes different values as m varies. So $f'(z)$ is not unique at (0, 0) *i.e.*, $f'(0)$ does not exist. Thus $f(z)$ is not analytic at the origin even though it is continuous and satisfies the C-R equations thereat.

Example 20.5. *Show that polar form of Cauchy-Riemann equations are*

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta},\ \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$ *(U.P.T.U., 2008 ; V.T.U., 2006)*

Deduce that $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$ *(Bhopal, 2009 ; Kurukshetra, 2005)*

Solution. If (r, θ) be the coordinates of a point whose cartesian coordinates are (x, y), then $z = x + iy = re^{i\theta}$.

$\therefore$ $$u + iv = f(z) = f(re^{i\theta})$$

where u and v are now expressed in terms of r and θ.

Differentiating it partially w.r.t. r and θ, we have

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta}) \cdot e^{i\theta}$$

and $$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta}) \cdot ire^{i\theta} = ir\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

Equating real and imaginary parts, we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad ...(i) \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \quad ...(ii)$$

Differentiating (i) partially w.r.t. r, we get

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial \theta \partial r} \quad ...(iii)$$

Differentiating (ii) partially w.r.t. θ, we have

$$\frac{\partial^2 u}{\partial \theta^2} = -r\frac{\partial^2 v}{\partial r \partial \theta} \quad ...(iv)$$

Thus using (i), (ii) and (iv)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial \theta \partial r} + \frac{1}{r}\left(\frac{1}{r}\frac{\partial v}{\partial \theta}\right) + \frac{1}{r^2}\left(-r\frac{\partial^2 v}{\partial r \partial \theta}\right) = 0 \qquad \left[\because \frac{\partial^2 v}{\partial \theta \partial r} = \frac{\partial^2 v}{\partial r \partial \theta}\right]$$

20.5 (1) HARMONIC FUNCTIONS

If $f(z) = u + iv$ be an analytic function in some region of the z-plane, then the Cauchy-Riemann equations are satisfied.

i.e., $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad ...(1) \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \quad ...(2)$$

Differentiating (1) with respect to x and (2) with respect to y, we obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad ...(3) \qquad \text{and} \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}. \quad ...(4)$$

Adding (3) and (4) and assuming that $\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$, we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \qquad ...(5)$$

Similarly, by differentiating (1) with respect to y and (2) with respect to x and subtracting, we obtain

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \qquad ...(6)$$

Thus both the functions u and v satisfy the Laplace's equation in two variables. For this reason, they are known as **harmonic functions** and their theory is called **potential theory.** *(Rohtak, 2005)*

(2) Orthogonal system. Consider the two families of curves

$$u(x, y) = c_1 \quad ...(7) \qquad \text{and} \qquad v(x, y) = c_2 \qquad ...(8)$$

Differentiating (7), we get $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0$

or $$\frac{dy}{dx} = -\frac{\partial u / \partial x}{\partial u / \partial y} = \frac{\partial v / \partial y}{\partial v / \partial x} = m_1 \text{ (say)} \qquad \text{[By (1) and (2)]}$$

Similarly (8) gives $$\frac{dy}{dx} = -\frac{\partial v / \partial x}{\partial v / \partial y} = m_2 \text{ (say)}$$

$\therefore$ $m_1 m_2 = -1$, *i.e.*, (7) and (8) form an orthogonal system.

Hence *every analytic function $f(z) = u + iv$ defines two families of curves $u(x, y) = c_1$ and $v(x, y) = c_2$, which form an orthogonal system.* *(U.P.T.U., 2009)*

20.6 APPLICATIONS TO FLOW PROBLEMS

As the real and imaginary parts of an analytic function are the solutions of the Laplace's equation in two variables, the conjugate functions provide solutions to a number of field and flow problems.

As an illustration, consider the irrotational motion of an incompressible fluid in two dimensions. Assuming the flow to be in planes parallel to the xy-plane, the velocity $\mathbf{V}$ of a fluid particle can be expressed as

$$\mathbf{V} = v_x \mathbf{I} + v_y \mathbf{J} \qquad ...(1)$$

Since the motion is irrotational, therefore, by § 6.18 (1), there exist a scalar function $\phi(x, y)$ such that

$$\mathbf{V} = \nabla \phi(x, y) = \frac{\partial \phi}{\partial x} \mathbf{I} + \frac{\partial \phi}{\partial y} \mathbf{J} \qquad ...(2)$$

[The function $\phi(x, y)$ is called the *velocity potential* and the curves $\phi(x, y) = c$ are known as *equipotential lines*.]

Thus from (1) and (2), $v_x = \frac{\partial \phi}{\partial x}$ and $v_y = \frac{\partial \phi}{\partial y}$ ...(3)

Also the fluid being incompressible div $\mathbf{V} = 0$ [by § 8.7 (1)] *i.e.*, $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0.$

Substituting the values of v_x and v_y from (3), we get $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$

which shows that the velocity potential ϕ is *harmonic*. It follows that there must exist a conjugate harmonic function $\psi(x, y)$ such that $w(z) = \phi(x, y) + i\psi(x, y)$ ...(4)
is analytic.

Also the slope at any point of the curve $\psi(x, y) = c'$ is given by

$$\frac{dy}{dx} = -\frac{\partial \psi / \partial x}{\partial \psi / \partial y} = \frac{\partial \phi / \partial y}{\partial \phi / \partial x} \qquad \text{[By } C\text{-}R \text{ equations]}$$

$$= v_y / v_x \qquad \text{[By (3)]}$$

This shows that the velocity of the fluid particle is along the tangent to the curve $\psi(x, y) = c'$, *i.e.* the particle moves along this curve. Such curves are known as *stream lines* and $\psi(x, y)$ is called the *stream function*. Also the equipotential lines $\phi(x, y) = c$ and the stream lines $\psi(x, y) = c'$ cut orthogonally.

From (4), $$\frac{\partial w}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}$$ [By *C-R* equations]

$$= v_x - iv_y$$ [By (3)]

$\therefore$ The magnitude of the fluid velocity $= \sqrt{(v_x^2 + v_y^2)} = | dw/dz |$.

Thus the flow pattern is fully represented by the function $w(z)$ which is known as the **complex potential.**

Similarly the complex potential $w(z)$ can be taken to represent any other type of 2-dimensional steady flow. In electrostatics and gravitational fields, the curves $\phi(x, y) = c$ and $\psi(x, y) = c'$ are *equipotential lines* and *lines of force*. In heat flow problems, the curves $\phi(x, y) = c$ and $\psi(x, y) = c'$ are known as *isothermals* and *heat flow lines* respectively.

Given $\phi(x, y)$, we can find $\psi(x, y)$ and *vice-versa.*

Example 20.6. *If $w = \phi + i\psi$ respresents the complex potential for an electric field and $\psi = x^2 - y^2 + \frac{x}{x^2+y^2}$, determine the function ϕ.* (V.T.U., 2011 ; Mumbai, 2008 ; Bhopal, 2002 S)

Solution. It is readily verified that ψ satisfies the Laplace's equation.

$\therefore$ ϕ and ψ must satisfy the Cauchy-Riemann equations :

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad ...(i) \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \quad ...(ii)$$

$\therefore$ by (i), $$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial y}\left[x^2 - y^2 + \frac{x}{x^2+y^2}\right] = -2y - \frac{2xy}{(x^2+y^2)^2}$$

Integrating w.r.t. x, we get $\phi = -2xy + \frac{y}{x^2+y^2} + \eta(y)$ where $\eta(y)$ is an arbitrary function of y.

$\therefore$ (ii) gives $-2x + \frac{x^2-y^2}{(x^2+y^2)^2} + \eta'(y) = -2x + \frac{x^2-y^2}{(x^2+y^2)^2}$

whence $\eta'(y) = 0$, *i.e.*, $\eta(y) = c$, an arbitrary constant.

Thus $$\phi = -2xy + \frac{y}{x^2+y^2} + c$$

Otherwise (Milne-Thomson's method*) :

We have $$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} + i\frac{\partial \psi}{\partial x} = \left[-2y - \frac{2xy}{(x^2+y^2)^2}\right] + i\left[2x + \frac{y^2-x^2}{(x^2+y^2)^2}\right]$$

By Milne-Thomson's method, we express dw/dz in terms of z, on replacing x by z and y by 0.

$\therefore$ $$\frac{dw}{dz} = i\left(2z - \frac{1}{z^2}\right)$$

Integrating w.r.t. z, we get $w = i\,(z^2 + 1/z) + A$ where A is a complex constant.

---

* Since $z = x + iy$ and $\bar{z} = x - iy$, we have

$$x = \frac{1}{2}(z + \bar{z}), \qquad y = \frac{1}{2i}(z + \bar{z})$$

$\therefore$ $$f(z) = \phi(x, y) + i\psi(x, y) \quad ...(1)$$

$$= \phi\left[\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right] + i\psi\left[\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right]$$

Now considering this as a formal identity in the two independent variables z, $\bar{z}$ and putting $\bar{z} = z$, we get

$$f(z) = \phi(z, 0) + i\psi(z, 0) \quad ...(2)$$

$\therefore$ (2) is the same as (1), if we replace x by z and y by 0.

Thus **to express any function in terms of z, replace x by z and y by 0.** *This provides an elegant method of finding $f(z)$ when its real part or the imaginary part is given. It is due to Milne-Thomson.*

Hence $$\phi = R\left[i\left(z^2 + \frac{1}{z}\right) + A\right] = -2xy + \frac{y}{x^2 + y^2} + c.$$

Example 20.7. *Find the analytic function, whose real part is sin 2x / (cosh 2y – cos 2x).*

(J.N.T.U., 2005 ; Anna, 2003)

Solution. Let $f(z) = u + iv$, where $u = \sin 2x/(\cosh 2y - \cos 2x)$

$$\therefore \quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{[By } C\text{-}R \text{ equations]}$$

$$= \frac{(\cosh 2y - \cos 2x)\, 2\cos 2x - \sin 2x\,(2\sin 2x)}{(\cosh 2y - \cos 2x)^2} - i\,\frac{\sin 2x(-2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2} + i\,\frac{2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

By Milne-Thomson's method, we express $f'(z)$ in terms of z by putting $x = z$ and $y = 0$.

$$\therefore \quad f'(z) = \frac{2\cos 2z - 2}{(1 - \cos 2z)^2} + i(0) = \frac{-2}{1 - \cos 2z} = \frac{-2}{2\sin^2 z} = -\operatorname{cosec}^2 z$$

Integrating w.r.t. z, we get $f(z) = \cot z + ic$, taking the constant of integration as imaginary since u does not contain any constant.

Example 20.8. *Determine the analytic function* $f(z) = u + iv$, *if* $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$ *and* $f(\pi/2) = 0$.

(A.M.I.E.T.E., 2005 ; Osmania, 2003)

Solution. We have $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$

$$\therefore \quad \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = \frac{(\sin x - \cos x)\cosh y + 1 - e^{-y}\sin x}{2(\cos x - \cosh y)^2} \qquad ...(i)$$

and $$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = \frac{(\cos x - \cosh y)\, e^{-y} + (\cos x + \sin x - e^{-y})\sinh y}{2(\cos x - \cosh y)^2}$$

or $$-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} = \frac{(\sin x + \cos x)\sinh y + e^{-y}(\cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2} \qquad ...(ii)$$

Subtracting (ii) from (i), we get

$$2\frac{\partial u}{\partial x} = \frac{(\sin x - \cos x)\cosh y - (\sin x + \cos x)\sinh y + 1 - e^{-y}(\sin x + \cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2}$$

Adding (i) and (ii), we have

$$-2\frac{\partial v}{\partial x} = \frac{(\sin x - \cos x)\cosh y + (\sin x + \cos x)\sinh y + 1 + e^{-y}(-\sin x + \cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2}$$

Thus $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{1 - \cos z}{2(1 - \cos z)^2} \qquad [\text{Putting } x = z \text{ and } y = 0]$$

$$= \frac{1}{2(1 - \cos z)} = \frac{1}{4\sin^2 z/2} = \frac{1}{4}\operatorname{cosec}^2 \frac{z}{2} \quad \text{or} \quad f(z) = -\frac{1}{2}\cot\frac{z}{2} + c$$

Since $f(\pi/2) = 0$, $\quad 0 = -\dfrac{1}{2}\cot \pi/4 + c$, whence $c = \dfrac{1}{2}$

Hence $$f(z) = \frac{1}{2}\left(1 - \cot\frac{z}{2}\right).$$

Example 20.9. *Find the conjugate harmonic of $v(r, \theta) = r^2 \cos 2\theta - r \cos \theta + 2$. Show that v is harmonic.* (*Marathwada, 2008*)

Solution. Let $f(z) = u + v$. Using C-R equations in polar coordinates (Ex. 20.5),

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} = -2r^2 \sin 2\theta + r \sin \theta \quad ...(i)$$

$$-\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r} = 2r \cos 2\theta - \cos \theta \quad ...(ii)$$

$\therefore$ (*i*) gives, $\dfrac{\partial u}{\partial r} = -2r \sin 2\theta + \sin \theta$

Integrating w.r.t., r

$$u = -r^2 \sin 2\theta + r \sin \theta + \phi(\theta)$$ where $\phi(\theta)$ is an arbitrary function.

$$\therefore \quad \frac{\partial u}{\partial \theta} = -2r^2 \cos 2\theta + r \cos \theta + \phi'(\theta) \quad ...(iii)$$

From (*ii*) and (*iii*), we get

$$-2r^2 \cos 2\theta + r \cos \theta = \frac{\partial u}{\partial \theta} = -2r^2 \cos 2\theta + r \cos \theta + \phi'(\theta)$$

$$\therefore \quad \phi'(\theta) = 0 \quad \text{or} \quad \phi(\theta) = c$$

Thus $u = -r^2 \sin 2\theta + r \sin \theta + c$ is the conjugate harmonic of v.

Now v will be harmonic if it satisfies the Laplace equation $\dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r} + \dfrac{\partial v}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial \theta^2} = 0$

From (*i*), $\dfrac{\partial^2 v}{\partial \theta^2} = -4r^2 \cos 2\theta + r \cos \theta$. From (*ii*), $\dfrac{\partial^2 v}{\partial r^2} = 2 \cos 2\theta$

$$\therefore \quad \frac{\partial t^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 2 \cos 2\theta + \frac{1}{r}(2r \cos 2\theta - \cos \theta) + \frac{1}{r^2}(-4r^2 \cos 2\theta + r \cos \theta)$$

$$= 4 \cos 2\theta - \frac{1}{r}\cos \theta - 4 \cos 2\theta + \frac{1}{r}\cos \theta = 0$$

Hence v is harmonic.

Example 20.10. (*a*) *Find the orthogonal trajectories of the family of curves*

$$x^4 + y^4 - 6x^2y^2 = constant$$

(*b*) *Show that the curves $r^n = \alpha \sec n\theta$ and $r^n = \beta \operatorname{cosec} n\theta$ cut orthogonally.*

(*Mumbai, 2005 ; J.N.T.U., 2003*)

Solution. (*a*) Take $u(x, y) = x^4 + y^4 - 6x^2y^2$. Then the family of curves $v(x, y)$ = constant will be the required trajectories if $f(z) = u + iv$ is analytic.

Now $\dfrac{\partial u}{\partial x} = 4x^3 - 12xy^2$, $\dfrac{\partial u}{\partial y} = 4y^3 - 12x^2y$

$$\therefore \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 4x^3 - 12xy^2$$

Integrating, $v = 4x^3y - 4xy^3 + c(x)$

Differentiating partially w.r.t. x

$$12x^2y - 4y^3 + \frac{dc(x)}{dx} = \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -4y^3 + 12x^2y$$

$$\therefore \quad \frac{dc(x)}{dx} = 0 \quad \text{or} \quad c = \text{constant}$$

Thus the required orthogonal trajectories are v = constant or $x^3y - xy^3$ = constant.

(*b*) Writing $u(r, \theta) = r^n \cos n\theta = \alpha$ and $v(r, \theta) = r^n \sin n\theta = \beta$,
we have $u(r, \theta) + iv(r, \theta) = \alpha + i\beta = r^n (\cos n\theta + i \sin n\theta) = r^n . e^{in\theta} = (re^{i\theta})^n = z^n$