PROGRAMMING ASSIGNMENT 2

<u>Due:</u> Thursday, September 17 by 11am. Upload *one* Jupyter notebook to Gradescope.

1. POPULATION GROWTH

This is a modified version of chapter 1 problem 6 in your textbook. Population growth problems often give rise to rate equations that are first-order; for example,

$$\frac{dN}{dt} = aN - bN^2$$

might describe how the number of individuals in a population (N) evolves with time. The first term (aN) corresponds to the birth of new members (and is therefore positive) while the second term (bN^2) corresponds to deaths (hence the negative sign). In many cases, the death term is proportional to N^2 (or some other power) to allow for the fact that resources (like food) become increasingly difficult to find when the population gets very large.

- Solve the equation above with $a = 1$ and $b = 0$ using the Euler method and compare your numerical result to the exact solution (plot the number of individuals with time, and *clearly* indicate the numerical and exact results). Assume the initial population contains one hundred individuals at $t = 0$ (e.g., $N(t = 0) = 100$). Compute the results up to $t = 10$ with a step size of 0.1. What happens when you change the time step size?
- Solve the equation above with $a = 1$ and $b = 1$, again using the Euler method, under two initial conditions: one in which $N(0) = 0.1$ and one in which $N(0) = 5$. Use the same time range. Show both results, clearly labeled, on the same plot and describe what causes the qualitative difference between the two results, and justify your choice of time step size.

Grading Rubric: Program has required features described in prompt (25%), graphs show correct behavior and are easy to interpret (25%), written response is factually correct (25%) and uses proper grammar and spelling (15%), code compiles without errors and reproduces all output (10%)

2. RADIOACTIVE DECAY, REVISITED

This is a modified version of chapter 1 problem 4 in your textbook. Consider the radioactive decay of nuclei A into nuclei B, which can then *also* decay. The numbers of each species as a function of time are given by $N_A(t)$ and $N_B(t)$, and the decay of each species is governed by[1]:

$$\frac{dN_A}{dt} = -\frac{N_A}{\tau_A}$$

$$\frac{dN_B}{dt} = \frac{N_A}{\tau_A} - \frac{N_B}{\tau_B}$$

where τ_A and τ_B are the decay time constants for each type of nucleus.

[1] You may find the notes at http://www-naweb.iaea.org/napc/ih/documents/global_cycle/vol%20I/cht_i_06.pdf helpful for this problem.

- Solve both these coupled equations using the Euler method and compare your numerical result to the exact analytical solution. Plot the results, *clearly* indicating the numerical and exact results (similar to what you did in problem 1). Use a time step of 0.01, with a time range spanning from 0 to 8. Take as an initial condition $N_A(t=0) = 100$ and $N_B(t=0) = 0$, and let $\tau_A/\tau_B = 2$. Describe the overall appearance of the curves, and explain (physically) why this behavior makes sense.
- Using the Euler method, show how the behavior of $N_A(t)$ and $N_B(t)$ differs when $\tau_A/\tau_B = 0.1, 2, 10$, and qualitatively explain why you observe this behavior.

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3. Atomic Energy Levels

Let's consider a system of atoms which all have two energy levels: a ground state and an excited state. The fraction of atoms that are in the ground state is given by N_1 and the fraction of atoms in the excited state is N_2. The atoms are exposed to an incident radiation field, which can cause (1) atoms in the ground state to become excited or (2) excited atoms to decay via stimulated emission. Atoms in the excited state can additionally undergo spontaneous decay to the ground state.

The time rate of change of the fraction of atoms in either state is therefore dependent on the rate of excitation from the ground to the excited state (pN_1), the rate of decay due to stimulated emission (pN_2), and the rate of spontaneous decay (N_2/τ). Note that p is proportional to the power of the ambient radiation field (and that the terms pN_1 and pN_2 *must* have opposite signs), and τ is related to the "lifetime" of each excited state (look up "Einstein coefficients" if you're interested). We therefore have a set of coupled equations with two variables (N_1 and N_2) and two parameters (p and τ):

$$\frac{dN_1}{dt} = -pN_1 + pN_2 + \frac{N_2}{\tau}$$

$$\frac{dN_2}{dt} = pN_1 - pN_2 - \frac{N_2}{\tau}$$

To make life easy, let's set $\tau = 1$ and assume that all atoms are in the ground state at $t = 0$.

- Use the Euler method to solve the above equations from $t = 0$ to $t = 3$ assuming a time step of 0.01 for $p = 0.1, 0.3, 1, 3$. Show how the number of atoms in each state evolves with time for each value of p. Note that you are *not* being asked to solve the equations analytically!
- Does increasing the power of the radiation field ever lead to a population inversion, in which *more* atoms occupy the excited state than the ground state? Present a graph(s) to support your answer, and describe your findings. *Don't* just say "The graphs show…" – phrase your answer in terms of the physical system being evaluated! For example: "As we increase the power of the ambient radiation field, we see that…"

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4. THREE-LEVEL SYSTEMS

Now we'll consider a system of atoms with *three* energy levels. Let's assume there's an external pump beam that raises atoms from the ground state (which we will label "1") to an upper excited state (call it "3") at a rate pN_1. Some of the atoms in level 3 will immediately decay to an intermediate excited state (let's be super obvious and call it "2") at rate N_3/τ_{32}, while the other atoms will return to the ground state via stimulated emission at a rate pN_3. The atoms that are in the intermediate state will return to the ground state at rate N_2/τ_{21}.

The result of all these considerations is the following *trio* of rate equations:

$$\frac{dN_1}{dt} = -pN_1 + pN_3 + \frac{N_2}{\tau_{21}}$$

$$\frac{dN_2}{dt} = -\frac{N_2}{\tau_{21}} + \frac{N_3}{\tau_{32}}$$

$$\frac{dN_3}{dt} = pN_1 - pN_3 - \frac{N_3}{\tau_{32}}$$

Let's assume that $\tau_{21} = 1$ and all atoms are initially in the ground state. Let's further assume that the decay from level 3 to level 2 is fast: $\tau_{32} = 0.01$. Solve these equations using the Euler method again – you will have to make a few modifications to your code from the previous problem. Again, you may assume that all atoms are in the ground state at $t = 0$.

- Does increasing the pump power now ever lead to a population inversion, in which *more* atoms occupy the excited state than the ground state? If so, what value of *p* results in an equal number of atoms in level 2 and level 1? Present a graph(s) to support your answer, and describe your findings. *Hint*: If you find $p > 20$ or $p < 0.01$, something has gone wrong.

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5. PROJECTILE MOTION WITH AIR RESISTANCE

Hopefully you have had the following result hammered in to you from your lower division mechanics classes: the range of a projectile is proportional to v_0^2 when there is no air resistance. Fix the launch angle at $\theta_0 = 45°$, vary v_0, and model projectile trajectories *with air resistance*. Is the range still proportional to v_0^2, or is it proportional to some *other* power of v_0 (for example, $v_0^{1.5}$), or is it not a power law at all?

- Present evidence to either support or reject the hypothesis that the range has a power law dependence on v_0 when air resistance is present and explain in a few grammatically correct sentences what this evidence means. If you conclude that a power law relationship does exist, you should also report the value of the index (in other words, if range is proportional to v_0^n, you must state what n is).

I recommend trying a regular pattern of increasing values of v_0. Something like:

$$v_0 = 0.02, 0.04, 0.08, 0.2, 0.4, 0.8, 2, 4, 8$$

You don't have to use these exact values, but please follow some reasonable pattern and make sure to test several values with $v_0 > 1$.

If you are unsure about how to implement air resistance into your code, I suggest looking at Equation 2.20 in your textbook.

Hint: If the range $R \propto v_0^n$, then $\log R$ is a linear function of $\log v_0$.

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