PROGRAMMING ASSIGNMENT 3

Due: Thursday, October 1 by 11am. Upload *one* Jupyter notebook to Gradescope.

1. HARMONIC AND ANHARMONIC OSCILLATORS

This is a modified version of chapter 3 problem 4 in your textbook. For simple harmonic motion, the equation of motion is given by

$$\frac{d^2x}{dt^2} = -kx^\alpha$$

with $\alpha = 1$.

- Write a program that uses the Euler-Cromer method to solve for $x(t)$. For convenience, let $\alpha = k = 1$. Show that the period of the oscillations is *independent* of the amplitude of the motion (this is a key feature of simple harmonic motion!). Use several different regularly-spaced values of amplitude and show that the amplitude and period are independent of each other.
- Write a program that uses the Euler-Cromer method to solve for $x(t)$ when $k = 1$ but $\alpha = 3$ and plot a representative solution for $x(t)$. Calculate the period of the oscillation for a variety of amplitudes. Show that the period of oscillation is now *dependent* on the amplitude of the motion (this is called *anharmonic* motion).
- Describe the anharmonic behavior that you observe. Provide an intuitive, qualitative discussion of why this behavior is observed.

Grading Rubric: Program has required features described in prompt (25%), graphs show correct behavior and are easy to interpret (25%), written response is factually correct (25%) and uses proper grammar and spelling (15%), code compiles without errors and reproduces all output (10%)

2. ENERGY FLUCTUATIONS IN A NONLINEAR PENDULUM

Consider the following equation, which describes the (nonlinear) motion of a pendulum:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\sin\theta$$

where g is the gravitational acceleration ($9.81\ \mathrm{m/s^2}$) and ℓ is the length of the pendulum. Since there is no friction or driving force, there is no way to add or remove energy from the system: the total mechanical energy must be conserved.

- Use the Euler-Cromer method to numerically solve for $\theta(t)$, and plot $\theta(t)$ vs. t. Use a time step $\Delta t = 0.04$ and assume $\ell = 1$ for simplicity.
- Compute the total mechanical energy of the system at each time step.

If our programs were solving these equations *exactly*, the total mechanical energy in the system would be exactly the same at all times. However, because our solutions are *not* exact, the energy oscillates over time. One important thing that researchers want to characterize is the accuracy of their program for a given time step: in other words, as the time step $\Delta t \to 0$ the energy oscillations should also $\to 0$.

- Measure the size of the energy fluctuation ΔE from the previous plot (either the amplitude of the fluctuation or the magnitude of the fluctuation about the expected value), which used a time step of $\Delta t = 0.04$.
- Cut the time step in half and measure ΔE again. Keep cutting the time step in half and recording the energy fluctuations until you reach $\Delta t = 10^{-3}$. Create a log-log plot of ΔE vs. Δt. You should find that $\Delta E \propto (\Delta \mathrm{t})^n$ – find the value of n.

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3. An Anharmonic Oscillator with an Asymmetric Force Law

Consider a particle of mass $m = 1$ moving in the following potential:

$$U(x) = \frac{\pi^2}{18}\left(\frac{1}{x^{12}} - \frac{2}{x^6} + 1\right)$$

where we have chosen our units of time so that the period of a small oscillation is 1 and our units of length so the equilibrium is at $x = 1$.

Simulate the motion of the particle using the second-order Runge-Kutta algorithm using a time step $\Delta t = 0.005$. Plot $x(t)$ for at least one full period of oscillation, and also compute the time-averaged value of x:

$$\langle x \rangle = \frac{1}{\text{number of time steps}} \sum_{\text{time}} x(t)$$

- Start at different values of $x > 1$. How big does the starting value have to be before $\langle x \rangle - 1$ is 0.001? How about 0.004? Do your results support the proposition that the average distance from equilibrium is proportional to the energy of the oscillator? (*Hint*: for small x, the linear term of the force dominates, and thus the quadratic term in the potential also dominates)
- Explain, in a few grammatically correct sentences, why it is that increasing the starting distance causes the average position to deviate more and more from the equilibrium point. If this behavior evident in the shape of the graph of $x(t)$? (*Hint*: Is the force equal on both sides? Does it bounce back more strongly on one side than the other?)

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4. Harmonic Generation

Now consider an even more complicated force law:

$$\frac{dv}{dt} = -\frac{2\pi^2}{3}\left(\frac{1}{x^7} - \frac{1}{x^{13}}\right) - \gamma v + F_0 \cos \omega\, t$$

Before you totally panic, let's dissect what's going on in this equation:

- $-\gamma v$: This is a viscous friction term. How do we know? The faster the system moves, the more friction is feels.
- $F_0 \cos \omega t$: This is an externally applied driving force. We know it's an external force because it doesn't depend on *where the particle is right now* (e.g., there's no x or v component); it *only* depends on time. F_0 is the amplitude of the force and ω is the frequency of the driving force. Note that the frequency ω is *not* necessarily the frequency that this system would oscillate at if left to its own devices – *that* frequency should be left as a user-adjusted parameter. For small oscillations, it is 2π – you can see what happens when you drive your system at a frequency completely different from 2π, or at frequency very close to 2π.

Write a program that uses the second-order Runge Kutta method to verify the following statements about this system:

- **When no external driving force is applied to the system**, (1) the amplitude of the oscillation decays exponentially, and (2) the rate of decay proportional to γ. Start the system very close to equilibrium ($x_0 = 0.95$) and try very small values of γ (e.g., 0.2 or so).
- **When a driving force is applied**, the system initially undergoes complicated motion, but at times significantly longer than $1/\gamma$ the system settles down and oscillates with a frequency equal to ω and an amplitude proportional to F_0. Only use small values of F_0, in the range of 0.05 to 0.2. You can use any value of ω you want, and set $\gamma = 1$.

You must submit several graphs to confirm these statements, along with a few grammatically correct sentences explaining why these graphs are sufficient to confirm these statements.

Hint: If you want to show that a quantity y is proportional to x, you need to show that when x doubles y also doubles. If you want to show that y is proportional to x^n, then you need to show that when x doubles y gets multiplied by 2^n. If you want to demonstrate a proportionality, you may need to run more than one simulation.

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5. A Damped, Driven Pendulum

Consider the nonlinear pendulum described in Chapter 3, Section 3.3 of your book:

$$\frac{d^2\theta}{dt^2} = -\left(\frac{g}{\ell}\right)\sin\theta - q\frac{d\theta}{dt} + F_D \sin(\Omega_D t)$$

where $q = 1/2$, $\ell = g = 9.8$, and $\Omega_D = 2/3$.

Write an algorithm (using either the Euler-Cromer method or Runge Kutta) that compares the behavior of two nearly identical chaotic pendulums. Start with $F_D = 1.2$, and start the pendulums with the initial conditions $\theta_1(0) = 4°$, $\theta_2(0) = 4.2°$, and $\omega_1(0) = \omega_2(0) = 0$.

Produce the following plots:

- $\theta(t)$ vs. t for both pendulums
- $\omega(t)$ vs. $\theta(t)$ for both pendulums, similar to those shown in Figure 3.8 of your textbook
- $\Delta\theta(t)$ vs. t, similar to those shown in Figure 3.7 of your textbook.

Determine the Lyapunov exponent (note: it's okay to approximate this "by eye" – no fancy, formal fitting is required). On your graph of $\Delta\theta$ vs. time, draw approximate a best-fit line, as in Figure 3.7 of your textbook. Is the Lyapunov exponent consistent with chaotic motion? Include a few grammatically correct statements answering this question and discussing the plots you've generated.

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