MEEN 364 - Recitation 07 Handout

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26 October 2016

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References
-Feedback Control of Dyanamic Systems - Franklin, Powell, Naeini
- Bode Plot Examples - Swarthmore College

Outline

- Superposition and time-invariance
- Frequency response methods
- Bode plots
  - Drawing
  - MATLAB
  - Characteristics

Superposition and Time-Invariance

A system is called linear when it obeys the **principle of superposition**: *If a system has an input that can be expressed as a sum of signals, then the response of the system can be expressed as the sum of individual responses to the respective signals.* I.e., if:

principle of superposition

$$y_1(s) = G(s)u_1(s) \quad (1)$$

$$y_2(s) = G(s)u_2(s) \quad (2)$$

Then:

$$\alpha_1 y_1 + \alpha_2 y_2 = G(s)[\alpha_1 u_1 + \alpha_2 u_2] \quad (3)$$

Superposition can also be expressed as the following two properties:

- **additivity:** $G(s)(u_1 + u_2) = y_1 + y_2$
- **scalability:** $G(s)(\alpha u_1) = \alpha y_1$

A system is **time-invariant** when the system's response to a certain input is independent of the time which the input is applied, i.e., if (note time-domain convolution)

time-invariance

$$y_1(t) = (G * u_1)(t) \quad (4)$$

then,

$$y_1(t-\alpha) = (G * u_1)(t-\alpha) \quad (5)$$

If a system obeys both of these two conditions, linearity and time-invariance, we often call it an LTI system. The LTI properties are important when we talk about frequency responses of systems.

LTI systems

Frequency Response

A stable linear system's steady state response to a sinusoidal input will be sinusoidal at steady-state. This is intuitive when we consider the list of operations that a linear system can perform:

Any composition of these four operations can be written as a transfer function in the Laplace domain, in the form of a fractional polynomial.

- summing signals
- scaling signals
- differentiation
- integration

So, if we apply a sinusoid with a certain frequency, ω_o, and an amplitude of 1:

$$u(t) = sin(\omega_o t) \tag{6}$$

The steady-state response to this input will be another sinusoid of the form:

$$y(t) = Msin(\omega_o t + \phi) \tag{7}$$

This is shown in Section 6.1 of Franklin and Powell's book. The amplitude ratio of the two signals is given by

amplitude ratio

$$M = |G(j\omega_o)| = \sqrt{\{Re[G(j\omega_o)]\}^2 + \{Im[G(j\omega_o)]\}^2} \tag{8}$$

and the phase is given by

phase

$$\phi = \tan^{-1}\left[\frac{Im[G(j\omega_o)]}{Re[G(j\omega_o])}\right] = \angle G(j\omega_o) \tag{9}$$

Keep in mind that the range of the arctan function is only from $[-\frac{\pi}{2}, \frac{\pi}{2}]$. If the denominator is negative, we must add 180 deg to the result. Alternatively, a two input arctan like MATLAB's `atan2` would be sufficient.

Because the system is an LTI system, the steady-state **amplitude ratio,** M**,** and **phase,** ϕ**,** in response to a sinusoidal input are functions only of the input frequency, ω_o.

Bode Plots

Knowing the frequency response of a system over a range of input frequencies is very useful for system analysis and controller design, as we will see later. If we wanted to plot the magnitude and phase of a system in response to a range of frequency inputs, we might use a stacked graph, sharing the input frequency, ω, on the abscissa, as shown below.

These stacked plots are commonly referred to as Bode plots , and are plotted against $\log_{10}(\omega)$. Magnitude is measured in decibels, defined below:

Developed by H.W. Bode at Bell Laboratories from 1932 to 1942

$$\text{Gain (dB)} = 20\log(|G(j\omega)|) \tag{10}$$

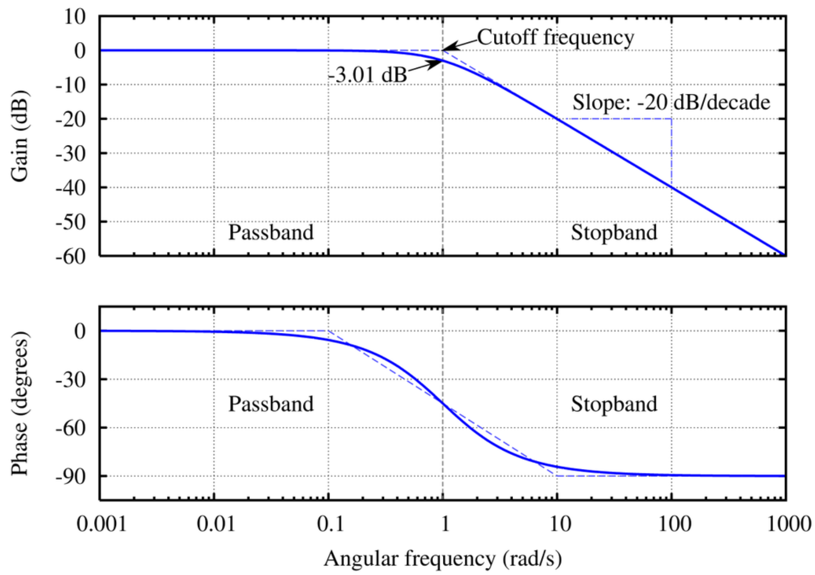

Figure 1: A Bode plot for a first order system. Source: Wikimedia Commons

These conventions serve several purposes:

- The unit decibel (dB) can be used.
- The plot can show a wide range of gains and frequencies.
- The system can be decomposed and drawn by hand as the sum of individual parts, as shown below.

The top plot is magnitude and the bottom plot is phase, both are semi-log plots, although the magnitude is log-log in terms of amplitude ratio, $|G(j\omega)|$.

Consider the generic system:

$$G(s) = \frac{(s - z_1)(s - z_2)...}{(s - p_1)(s - p_2)...} \tag{11}$$

If we wanted to plot the Bode response, we might evaluate the magnitude at frequency ω_o as such:

$$\text{Gain (dB)} = 20\log_{10}|G(j\omega_o)| = 20\log_{10}\left|\frac{(j\omega_o - z_1)(j\omega_o - z_2)...}{(j\omega_o - p_1)(j\omega_o - p_2)...}\right| \tag{12}$$

We could evaluate the above magnitude at many different frequencies, however because of properties of logarithms, we can simplify our task. The logarithm of our transfer function can be expanded as:

$$\log_{10}|G(j\omega_o)| = \log_{10}|j\omega_o - z_1| + \log_{10}|j\omega_o - z_2| \; ... \; - \log_{10}|j\omega_o - p_1| - \log_{10}|j\omega_o - p_2| \; ...$$

By learning to draw just one of these components, we can draw and sum all of them on the magnitude plot and get the magnitude response of our entire system.

The **decibel** was created by Bell Telephone Labs to quantify losses in telephone transmission lines. Since the ear responds to changes in audio power logarithmically, the decibel was originally defined as:
$dB = 10\log_{10}(\frac{P_o}{P_i})$
Power in a signal is proportional to amplitude squared, which yields the common definition:
$dB = 10\log_{10}(\frac{A_o}{A_i})^2 = 20\log_{10}(\frac{A_o}{A_i})$
So when we use decibels to discuss sound levels, we are really referencing a baseline sound level, which we call 0 dB, and is defined as the sound pressure, 20 μPa.

In fact, we will learn to draw plots for the large majority of linear systems, whose numerators and denominators can be decomposed into the following terms:

- k — *a constant gain*
- s — *a pole or zero at* $(s = 0)$
- $(\tau s \pm 1)$ — *a real pole or zero at* $(s = \pm\frac{1}{\tau})$
- $(s^2 + 2\zeta\omega_n s + \omega_n^2)$ — *a pair of complex poles or zeros*

Example 1

As an example, consider the system:

$$H(s) = \frac{100}{s + 30} \tag{13}$$

We can put into Bode form as such:

$$H(s) = \frac{100}{30}\frac{1}{\frac{s}{30} + 1} \tag{14}$$

We have two components from the list above: a constant gain and a pole at $s = -30$. We will plot their magnitudes individually and superimpose the plots.

A constant gain will simply correspond to a constant magnitude line and a phase of zero degrees (no imaginary component), shown in blue. The pole is at 30 rad/s, so we draw an asymptote which breaks at 30 rad/s, and decreases by 20 dB/decade, shown in orange.

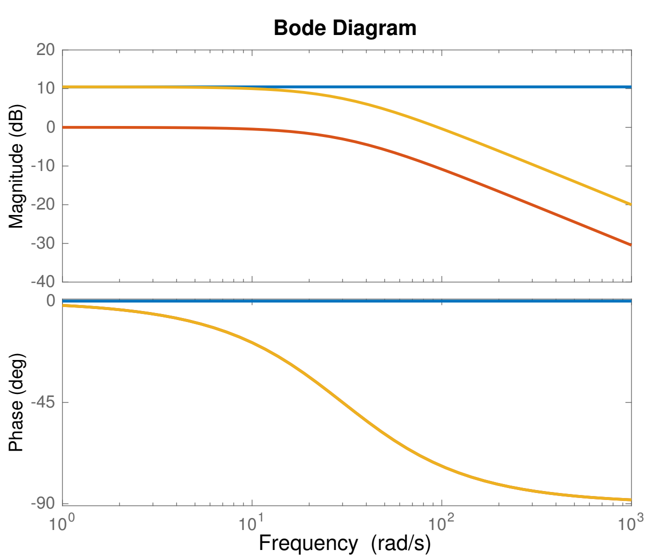

Figure 2: Bode plot for Example 1

Example 1 in MATLAB

```
% Let's plot each component and the entire system, like in Figure 2
sys_gain = tf([100/30], [1]);
sys_pole = tf([1/30], [1]);
sys = tf([100], [1 30]);
bode(sys_gain)

% Use 'hold on' to plot over same figure
hold on

bode(sys_pole)
bode(sys)
```

Example 2

Consider the system:

$$H(s) = 4\frac{s^2 + s + 25}{s^3 + 100s^2} \tag{15}$$

After some factoring, we can put the system into Bode form:

We can quickly evaluate whether a second order polynomial has real or complex roots by checking the condition
$$b^2 - 4ac > 0$$
from the quadratic equation.

$$H(s) = 4\frac{25}{100}\frac{(\frac{s}{5})^2 + \frac{1}{5}(\frac{s}{5}) + 1}{s^2\left(\frac{s}{100} + 1\right)} \tag{16}$$

We can see that the Bode form is composed of:

- two poles at the origin
- a real pole at $s = -100$
- a pair of complex zeros with $\omega_o = 5$, and $\zeta = 0.1$

Our constant gain of 1 corresponds to a 0 dB magnitude and 0 degree phase, and can be essentially ignored. The pole at 100 rad/s is shown in blue, and the repeated poles are shown in orange. Combined, they decay at 40 dB/decade. The complex zeros are shown by the yellow lines, and yield a dip in magnitude.

Example 2 in MATLAB

```
% Let's plot each component and the entire system, like in Figure 2
sys1 = tf([1], [1 0 0]);
sys2 = tf([100], [1 100]);
```

```
sys3 = tf([1 1 25], [25]);

% Multiplying system objects together will cascade their
    inputs and outputs and create a new system object
sys = sys1*sys2*sys3;

bode(sys1)
hold on
bode(sys2)
bode(sys3)
bode(sys)
```

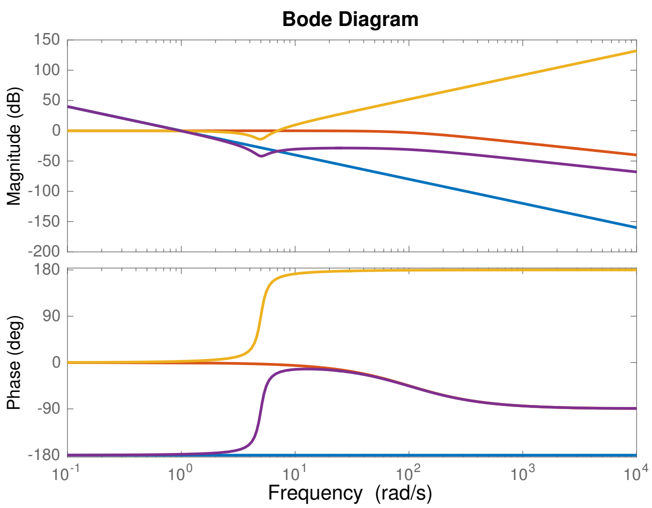

Figure 3: Bode plot for Example 2