MEEN 364 - Recitation 03 Handout

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Outline

- State variable methods discussion
- Modeling electrical circuits
- Modeling electromechanical systems
- Q&A

State Variable Selection

Unlike transfer functions, state space representations are not unique. For a given system, there may be many state space forms that relate the input to the output.

However, it is common practice to select state variables which correspond to energy storage elements in the system. In mechanical systems, energy is stored in displacement of springs, i.e., **position** and momentum of masses, i.e., **velocity**. In electrical systems, energy is stored in the **current through inductors** and the **voltage across capacitors**.

You should have an understanding of the inputs, outputs, and the state variable of your system **before** you begin applying Newton's law or circuit analysis.

Modeling Electrical Circuits

After we have designated our state variables, and assumed current directions through each element, we can apply the elemental equations for each component.

Elements of Electric Circuits

Resistor:

$$v = Ri \tag{1}$$

Capacitor

$$i = C\frac{dv}{dt} = C\dot{v} \tag{2}$$

Inductor

$$v = L\frac{di}{dt} = Li' \tag{3}$$

Example 3.1 - Bridge-Tee Circuit

Determine the differential equations for the circuit shown to the right in Figure 1. Then, put the system into state-space form using the energy storage variables. $v_i n$ is the input, and $v_o ut$ is the output.

Before beginning our analysis, let us note the input and output are voltages. The state vector is the voltages across the capacitors:

$$x = \begin{bmatrix} v_{C1} \\ v_{C2} \end{bmatrix} \tag{4}$$

This electrical network is well suited for nodal analysis, so we will use this technique. Nodal analysis is just a slightly different application of KVL and KCL. The nodes 1 through 4 have already been labeled.

We will ground the circuit at node 4. This choice of ground allows us to immediately define each node voltage in terms of the input, output, and state:

$$v_1 = v_{in} \tag{5}$$

$$v_2 = v_{C1} \tag{6}$$

$$v_3 = v_{in} - v_{C2} \tag{7}$$

$$v_4 = 0 \tag{8}$$

Also each capacitor is related by

$$i = C\dot{v} \tag{9}$$

At node 2, KCL gives

$$\frac{v_2 - v_i}{R_1} - \frac{v_2 - v_o}{R_2} - i_{C1} = 0 \tag{10}$$

After substituting in Eqs. 5-9, this becomes

$$\frac{v_{C1} - v_i}{R_1} - \frac{v_{C1} - v_i + v_{C2}}{R_2} - \dot{v}_{C1}C_1 = 0 \tag{11}$$

At node 3, KCL gives

$$i_{C2} + \frac{v_2 - v_o}{R_2} = 0 \tag{12}$$

$$\dot{v}_{C2}C_2 + \frac{v_{C1} - v_i + v_{C2}}{R_2} = 0 \tag{13}$$

Rearranging equations 11 and 13 we have:

$$\dot{v}_{C1} = (\frac{1}{R_1C_1} - \frac{1}{R_2C_1})v_{C1} + (\frac{1}{R_2C_1} - \frac{1}{R_1C_1})v_i - \frac{v_{C2}}{R_2C_1} \tag{14}$$

$$\dot{v}_{C2} = -\frac{v_{C1}}{R_2C_2} - \frac{v_{C2}}{R_2C_2} + \frac{v_i}{R_2C_2} \tag{15}$$

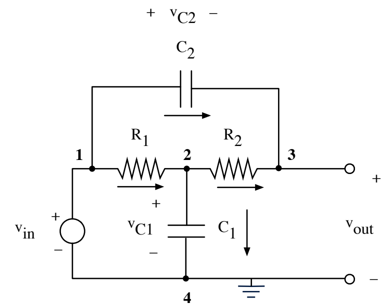


Figure 1: A bridged-tee notch filter. For an example application in guitar effects pedals, see `www.muzique.com/lab/notch.htm`

And finally:

$$\begin{bmatrix} \dot{v}_{C1} \\ \dot{v}_{C2} \end{bmatrix} = \begin{bmatrix} \frac{1}{R_1C_1} - \frac{1}{R_2C_1} & \frac{-1}{R_2C_1} \\ -\frac{1}{R_2C_2} & -\frac{1}{R_2C_2} \end{bmatrix} \begin{bmatrix} v_{C1} \\ v_{C2} \end{bmatrix} + \begin{bmatrix} \frac{1}{R_2C_1} - \frac{1}{R_1C_1} \\ \frac{1}{R_2C_2} \end{bmatrix} v_i \tag{16}$$

$$y = \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} v_{C1} \\ v_{C2} \end{bmatrix} + (1)v_i \tag{17}$$

Frequency Response of Example 3.1

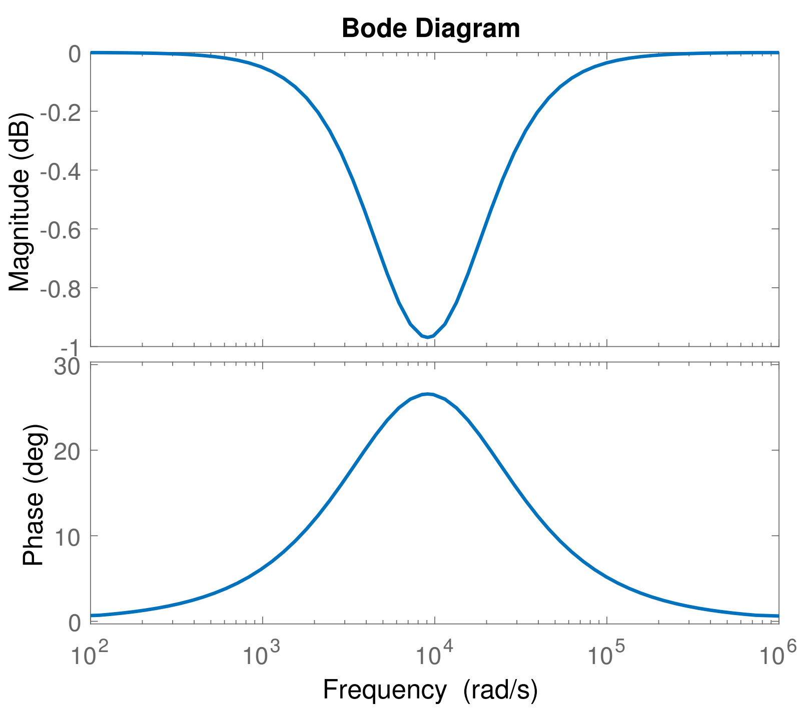
The above bridged-tee circuit was used as a notch filter in Gibson guitar amplifiers, such as the Gibson Hawk Reverb, to cut the mid-range of the guitar and reduce muddiness in the sound. We will plot the frequency response. The components are:

$$R_1 = R_2 = 220k\Omega \tag{18}$$

$$C_1 = 47nF \tag{19}$$

$$C_2 = 500pF \tag{20}$$

Figure 2: Bode plot of the above system



Example 3.2 - Electromechanical System

Find the equations and state-space representation for a DC motor shown below. Assume the rotor has inertia J and viscous friction b. The input is the applied voltage, v, and the output is the position, θ, of the rotor.

The magnetic induction of the rotor coils produces a back EMF proportional to the rotor velocity by:

$$v_m = K_v\dot{\theta} \tag{21}$$

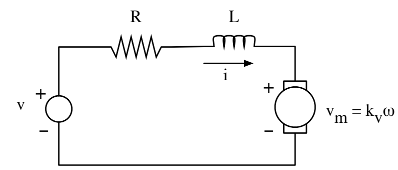


Figure 3: A DC motor circuit; the rotor diagram and FBD have been omitted for brevity

The torque on the rotor is proportional to current by:

$$T = K_t i \tag{22}$$

Our state vector, using the energy storage variable approach, will be the current through the inductor and the position and velocity of the rotor, i.e.:

$$x = \begin{bmatrix} i \\ \theta \\ \dot{\theta} \end{bmatrix} \tag{23}$$

Back EMF is a separate effect from inductance of the motor windings, and is caused by current carrying conductors moving through the field of the stator. For more info, see `www.youtube.com/watch?v=LAtPHANEfQo`

After drawing FBD's Application of Newton's law to the motor rotor gives us:

$$J\ddot{\theta} = K_t i - b\dot{\theta} \tag{24}$$

KVL applied to the single loop in our circuit yields:

$$v - Ri - L\frac{di}{dt} - K_v\dot{\theta} = 0 \tag{25}$$

Now, let us form our state equations:

$$\begin{bmatrix} \frac{di}{dt} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{R}{L} & -\frac{K_v}{L} & 0 \\ 0 & 0 & 1 \\ \frac{K_t}{J} & 0 & -\frac{b}{J} \end{bmatrix} \begin{bmatrix} i \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \\ 0 \end{bmatrix} v \tag{26}$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} i \\ \theta \\ \dot{\theta} \end{bmatrix} + (0)v \tag{27}$$