MEEN 364 - Recitation 01 Handout

Mitchell Allain

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Email
mitchallain@tamu.edu
Web
people.tamu.edu/~mitchallain/meen364

Outline

- Translational dynamics
- State-space equations of motion
- Rotational dynamics
- Kinematic constraints

Translational Dynamics

Example 1.1

Derive the equations of motion and put them in state space matrix form with the output taken as x_3 and the input as F(t).

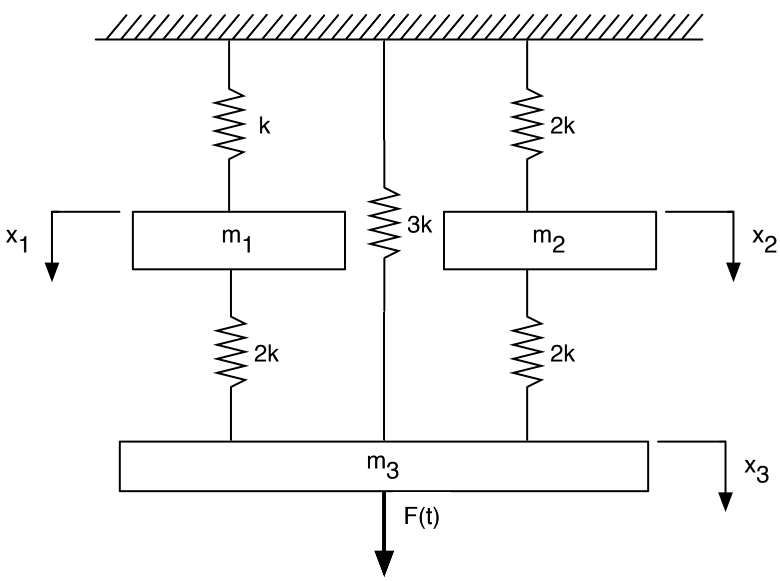


Figure 1: A hanging mass-spring system with 3 DOF.

In order to match the direction of forces in springs with Hooke's law for each spring, we will assume[1]:

$$x_1 > x_3 \text{ and } x_2 > x_3$$

Now we must draw the free body diagrams for each mass.

[1] This assumption is arbitrary, and determines whether we assume the springs are in tension or compression. Try making a different assumption and see that the equations of motion are equivalent.

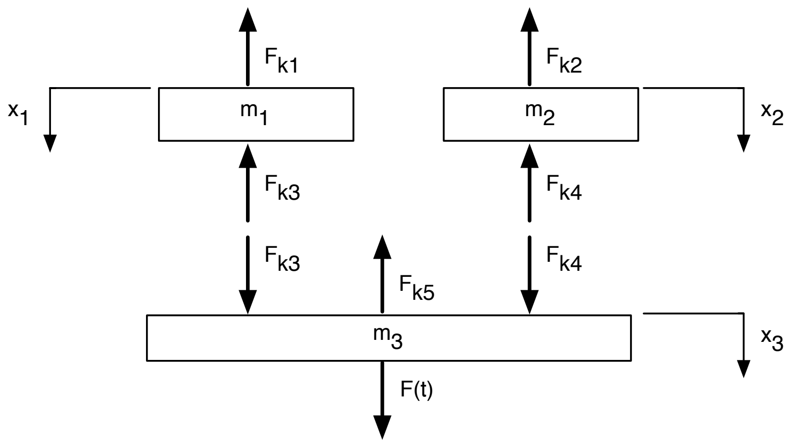


Figure 2: The FBD diagrams of each mass, arranged similar to original diagram.

Now let us apply Newton's law for each mass.

At first, you may want to also draw each spring and damper element with the displacements on each. This may help you avoid errors regarding the state of the springs (compression or tension).

$$\sum F(t) = ma$$

$$\begin{aligned}
m_1\ddot{x}_1 &= -F_{k1} - F_{k3} && (1)\\
&= -kx_1 - 2k(x_1 - x_2) && (2)\\
m_2\ddot{x}_2 &= -F_{k2} - F_{k4} && (3)\\
&= -2kx_2 - 2k(x_2 - x_3) && (4)\\
m_3\ddot{x}_3 &= F_{k3} + F_{k4} - F_{k5} + F(t) && (5)\\
&= 2k(x_1 - x_3) + 2k(x_2 - x_3) - 3kx_3 + F(t) && (6)\\
& && (7)
\end{aligned}$$

Let's define our state vector as q = $[x_1\ \dot{x}_1\ x_2\ \dot{x}_2\ x_3\ \dot{x}_3]^T$. Our state space equations of motion are as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \ddot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_2 \\ \dot{x}_3 \\ \ddot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{-3k_1}{m_1} & 0 & 0 & 0 & \frac{2k_1}{m_1} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-4k_1}{m_2} & 0 & \frac{2k_1}{m_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{2k_1}{m_3} & 0 & \frac{2k_1}{m_3} & 0 & \frac{-7k_1}{m_3} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \\ x_3 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{m_3} \end{bmatrix} F(t) \tag{8}$$

$$y = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \\ x_3 \\ \dot{x}_3 \end{bmatrix} + (0)F(t) \tag{9}$$

Q: What if we had assumed $x_3 > x_1$ and $x_3 > x_2$?

A: For example, spring "k3" would now be in compression, but Hooke's law for that spring would be (note change of signs on x_3 and x_1):

$$F_{k3} = 2k(x_3 - x_1)$$

effectively cancelling the change in assumption.

Q: Why don't we need to consider the weight of each mass in the FBDs?

A: The coordinates are taken for the system static equilibrium (after the springs have been stretched by the weight of the masses).

Q: In state-space representations, can we only have one input and one output?

A: No, the general form of state-space systems is as follows: **incomplete** from F&P If y and u are scalars ($n = m = 1$), the system is called a single-input, single-output (SISO) system. Control of multi-input, multi-output (MIMO) systems is one of the major challenges in control systems research today.

Example 1.2 - A piezoelectric accelerometer.

Consider a piezoelectric accelerometer model like the one shown to the right. The inputs to the system are the motion of the measurand, z and $\dot{z}$. The accelerometer has flexibility and damping in its attachment, k_1 and c_1, and a small mass moves against the piezoelectric crystal. The mass is suspended by a damper and spring, k_2 and c_2, and the piezoelectric crystal has an inherent stiffness, k_3.

Piezoelectric elements sense displacement, and therefore the output will be the relative distance between m_2 and m_1, or $y = x_2 - x_1$.

Draw the free body diagrams and derive the governing differential equations of motion. Write the equations of motion in state-space representation. Assume coordinates referenced from static equilibrium, and use $q = [x_1\ \dot{x}_1\ x_2\ \dot{x}_2]^T$ *as the state vector.*

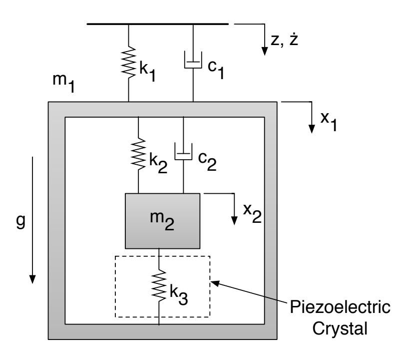
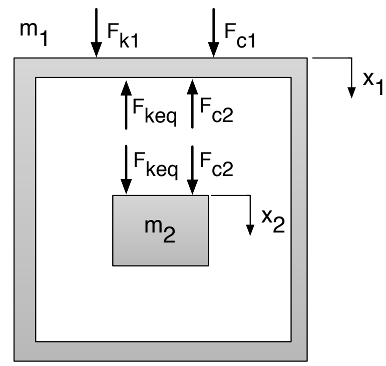


Figure 3: An accelerometer model with flexible attachment.

Our assumptions will be that:

$$z > x_1 > x_2 \text{ and } \dot{z} > \dot{x}_1 > \dot{x}_2$$

Before drawing the FBD, observe that there are parallel springs, which can be combined to form $k_{eq} = k_2 + k_3$. Using this, the FBD diagram for the system is shown to the right.



The spring and damper component forces, using our assumptions above, are:

$$F_{k1} = k_1(z - x_1) \tag{10}$$
$$F_{keq} = k_{eq}(x_1 - x_2) \tag{11}$$
$$F_{c1} = c_1(\dot{z} - \dot{x}_1) \tag{12}$$
$$F_{c2} = c_2(\dot{x}_1 - \dot{x}_2) \tag{13}$$

Therefore, our equations of motion are:

It is a good habit to gather terms for each state variable in the equations of motion to prepare for state-space formulation. This will make it easier to move directly to a matrix equation.

$$\sum F = ma$$

$$\begin{aligned}
(\downarrow +)\quad m_1\ddot{x}_1 &= F_{k1} + F_{c1} - F_{keq} - F_{c2} \\
&= k_1(z - x_1) + c_1(\dot{z} - \dot{x}_1) - k_{eq}(x_1 - x_2) - c_2(\dot{x}_1 - \dot{x}_2) \\
&= (-k_1 - k_{eq})x_1 + k_{eq}x_2 + (-c_1 - c_2)\dot{x}_1 + c_2\dot{x}_2 + k_1 z + c_1\dot{z} \\
(\downarrow +)\quad m_2\ddot{x}_2 &= F_{keq} + F_{c2} \\
&= k_{eq}(x_1 - x_2) + c_2(\dot{x}_1 - \dot{x}_2) \\
&= k_{eq}x_1 - k_{eq}x_2 + c_2\dot{x}_1 - c_2\dot{x}_2
\end{aligned}$$

Using our state vector, $q = [x_1\ \dot{x}_1\ x_2\ \dot{x}_2]$, and noting our two inputs, z and $\dot{z}$, and output, $x_2 - x_1$, the state-space equations of motion are:

$$\begin{bmatrix} \dot{x}_1 \\ \ddot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ \frac{-k_1-k_{eq}}{m_1} & \frac{-c_1-c_2}{m_1} & \frac{k_{eq}}{m_1} & \frac{c_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_{eq}}{m_2} & \frac{c_2}{m_2} & \frac{-k_{eq}}{m_2} & \frac{-c_2}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{k_1}{m_1} & \frac{c_1}{m_1} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z \\ \dot{z} \end{bmatrix} \tag{14}$$

$$y = \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} z \\ \dot{z} \end{bmatrix} \tag{15}$$

Rotational Dynamics

Example 1.3

Determine the equations of motion for the system given to the right. The displacement of the body is considered from the static equilibrium position. The disk is fixed at its center and can freely rotate about it.

The system has two degrees of freedom, represented by the angular displacement of the pulley, θ, and the translation displacement of the block, x. The displacement of the string around the pulley is θr, and we will assume:

$$\theta r > x$$

This completes the kinematics stage.

An FBD for the two masses is shown to the right. The forces in each spring and damper are:

$$F_{k1} = k_1(\theta r - x)$$
$$F_{k2} = k_2\theta r$$
$$F_{k3} = k_3 x$$
$$F_b = b\dot{x}$$

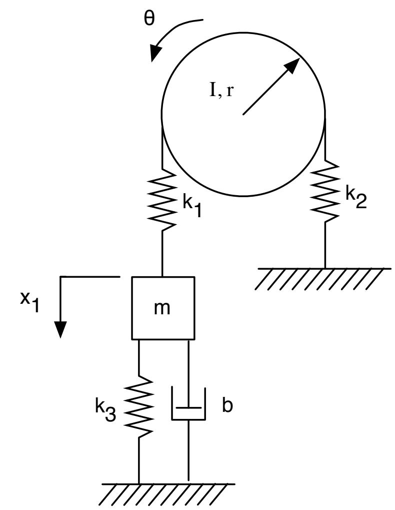


Figure 4: A mixed translational and rotational system.

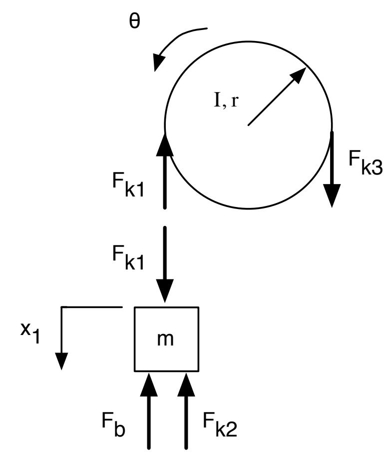


Figure 5: A FBD diagram showing the direction of spring and damper forces.

Given these, we can construct our equations of motion for each mass:

$$\sum F = ma$$
$$\sum M_o = I_o\alpha$$

$$(\circlearrowleft +)\ I_o\ddot{\theta} = F_{k2}r - F_{k1}r \tag{16}$$
$$= k_2\theta r^2 - k_1 r(\theta r - x) \tag{17}$$
$$= (k_2 - k_1)\theta r^2 - k_1 r x \tag{18}$$
$$(\downarrow +)\ m\ddot{x} = F_{k1} - F_b - F_{k2} \tag{19}$$
$$= (\theta r - x)k_1 - b\dot{x} - \theta r k_2 \tag{20}$$
$$= (k_1 - k_2)\theta r - k_1 x - b\dot{x} \tag{21}$$

Q: What happens if we assume the pulley in Example 1.3 has no inertia?

A: The degrees of freedom will change from 2 to 1. The springs k_1 and k_2 are effectively in series.