# Differentiating Data Structures

Mitch Stevens

# What is an Algebra?

An **algebra** is a pair of monoids over some type.

Examples of Algebras include:

- $\mathbb{B}$: (False, True, or, and)
- $\mathbb{N}, \mathbb{R}, \mathbb{C}$: $(0, 1, +, \times)$
- Sets: $(\varnothing, U, \cup, \cap)$
- Polynomials of Algebras
- Square Matrices: $(0_n, I_n, +, \times)$
- Musical Notes: $\mathbb{Z}_{12}$
- Musical Sequences (Euterpea)
- Pretty Printing
- Image Composition

```haskell
class Alg a where
  zero :: a
  one  :: a
  (+)  :: a -> a -> a
  (*)  :: a -> a -> a
```

Laws:

- Monoid Laws
- $(a + b) * c = (a * c) + (a * c)$

# Types form an Algebra

A **sum type** takes a value from $A$ **or** from $B$

- `Either a b`
- `data Bool = True | False`
- `data Maybe a = Nothing | Just a`

---

A **product type** takes a value from $A$ **and** from $B$

- `(a, b)`
- `data Person = Person Name Age`
- `data Time = Time Hour Minutes Seconds`

```haskell
instance Algebra Type where
  zero = Void
  one  = ()
  (+)  = Either
  (*)  = (,)
```

Does this satisfy the algebra law from earlier?

$$(a+b)*c = a+c*b+c$$

# Type Isomorphism

There is a difference between the set of Types and the equivalence class of types ($\mathbb{T}$)

---

The size of a type is the number of values that can inhabit it. Lets call this size computing funtion

$$\sigma :: \mathbb{T} \to \mathbb{N}$$

| Name | Constructor/Type | Size |
|---|---|---|
| Void | N/A | 0 |
| Unit | () | 1 |
| Boolean | `False | True` | 2 |
| Int | $-2^{31} \ldots 2^{31} - 1$ | $2^{32}$ |
| String | `[Char]` | $\infty$ |

# Calculating the size of a Type

We can calculate $\sigma$ in a mechanical way using these rules:

- $\sigma(\text{Sum a b}) = \sigma(a) + \sigma(b)$
- $\sigma(\text{Prod a, b}) = \sigma(a) \times \sigma(b)$
- $\sigma(a \to b) \equiv \prod_{n=1}^{\sigma(a)} b = \sigma(b)^{\sigma(a)}$

| Name | Constructor/Type | Size |
|---|---|---|
| Identity a | Identity a | $\sigma(a)$ |
| Maybe a | `Nothing | Just a` | $1 + \sigma(a)$ |
| Either a b | `Left a | Right b` | $\sigma(a) + \sigma(b)$ |
| (a, b) | `Tuple a b` | $\sigma(a) \times \sigma(b)$ |

# Traffic light example

- How many values inhabit `Traffic`?
- How many values inhabit `Intersection`?
- How many values inhabit `ComplexInter`?

```
data Traffic = R | Y | G

data Intersection = Inter
    { north :: Traffic
    , east  :: Traffic
    , south :: Traffic
    , west  :: Traffic
    }

type ComplexInter lane =
    lane -> Traffic
```

# How many values inhabit List?

$$
\begin{aligned}
\sigma(\texttt{List a}) &= \sigma(\texttt{Nil | Cons a (List a)}) \\
&= \sigma(\texttt{Nil}) + \sigma(\texttt{Cons a (List a)}) \\
&= 1 + \sigma(a) \times \sigma(\texttt{List a}) \\
&= 1 + \mathsf{a} \times (1 + a \times \sigma(\texttt{List a})) \\
&= 1 + \mathsf{a} + \mathsf{a}^2 \times \sigma(\texttt{List a}) \\
&= 1 + \mathsf{a} + \mathsf{a}^2 \times (1 + a \times \sigma(\texttt{List a})) \\
&= 1 + \mathsf{a} + \mathsf{a}^2 + a^3 + a^4 + \ldots
\end{aligned}
$$

---

So we have:

$$\sigma(\texttt{List a}) = \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

Which strongly resembles the geometric sum ($|r| \geq 1$):

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

# Refactoring

Give that $\sigma$ is a isomorphism, we know that $id_{\mathbb{T}} = \sigma \circ id_{\mathbb{N}} \circ \sigma^{-1}$.

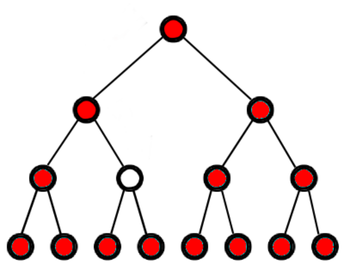
$$
\begin{tikzcd}
\mathbb{T} \arrow[r, "id_{\mathbb{T}}"] \arrow[d, "\sigma"] & \mathbb{T} \\
\mathbb{N} \arrow[r, "id_{\mathbb{N}}"] & \mathbb{N} \arrow[u, "\sigma^{-1}"']
\end{tikzcd}
$$

---

Example: Refactoring the type `Either (b, a, [b]) a`

# One-Hole Contexts and Zippers

A 'One-hole Context' is a data structure with a 'hole' where a value can be inserted.



If you include a value, you can use a one-hole context to recreate the original data-structure. The product of a value and a one-hole context is called a zipper.

# Rules of Differentiation

Usually only differentiation is only defined over functions $\mathbb{R}^m \to \mathbb{R}^n$.
But the rules that come from differential calculus can be applied to any algebra.
This is not usually meaningful, however.

Sum Rule:

$$\partial_x(A + B) = \partial_x A \partial_x B$$

Product Rule:

$$\partial_x(A \times B) = \partial_x A \times B + A \times \partial_x B$$

Chain Rule:

$$\partial_x(A \circ B) = \partial_x A(B) \times \partial_x A$$

# Differentiating Types

The dervivative of a Type is meaningful, it is **exactly** the type of its one-hole context!
It is easier to differentiate a function in $\mathbb{N}$ than a function in $\mathbb{T}$. We apply the same trick as before.

$$\begin{tikzcd}
\mathbb{T} \arrow[r, "\partial_{\mathbb{T}}"] \arrow[d, "\sigma"] & \mathbb{T} \\
\mathbb{N} \arrow[r, "\partial_{\mathbb{N}}"] & \mathbb{N} \arrow[u, "\sigma^{-1}"']
\end{tikzcd}$$

# The one-hole context of a List

# Other Questions

- Can we perform more than one differentiation?
  Yes.
- Is there a method for computing anti-differentiation?
  No idea.
- Is there a reasonable interpretation for a fractional or negative type?
  Kind of.