MATH 303 — Vector Span, Linearly Independent Vectors

Fall 2024 — Week 5, Session 1

1 Opening

1.1 What is the determinant of the following matrix, A?

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 4 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

1.2 For the same matrix, what is $det(\frac{1}{2}A)$?

1.3 Is A invertible?

2 Span of a Vector Set

The span of a set of vectors $\{\vec{u_1}, \ldots, \vec{u_k}\}$ in $\mathbb{R}^n$ is known as the span of these vectors, and is written as $span\{\vec{u_1}, \ldots, \vec{u_k}\}$.

In words, if we have a set of vectors $\{\vec{u_1}, \ldots, \vec{u_k}\}$, any vector $\vec{v}$ for which there exists scalars $k_1, k_2, \ldots, k_k$ such that $\vec{v} = k_1\vec{u_1} + k_2\vec{u_2} + \cdots + k_k\vec{u_k}$, then $v \in span\{\vec{u_1}, \ldots, \vec{u_k}\}$.

2.1 Describe the span of the following sets of vectors in $\mathbb{R}^2$:

1. $\{[1, 0]^T\}$
2. $\{[1, 1]^T\}$
3. $\{[1, 0]^T, [0, 1]^T\}$
4. $\{[-1, 0]^T, [1, 0]^T\}$
5. $\{[2, 2]^T, [-1, -1]^T\}$

2.2 Suppose we have a set of vectors $\{\vec{u}, \vec{v}, \vec{w}\}$ such that $\vec{u}$ and $\vec{v}$ span the entire XY-plane, and that $\vec{w} = [2, -5, 0]$. What is the span of this set in $\mathbb{R}^3$, and why?

2.3 Suppose we have a set of vectors $\{\vec{u}, \vec{v}, \vec{w}\}$ such that $\vec{u}$ and $\vec{v}$ span the entire XY-plane, and that $\vec{w} = [2, 2, 1]$. What is the span of this set in $\mathbb{R}^3$, and why?

So how do we show whether a given vector is in the span of a set of vectors? Let's consider a set of two vectors consisting of $\vec{u} = [1, 1, 0]^T$ and $\vec{v} = [3, 2, 0]^T$, and a third vector $\vec{w} = [4, 5, 0]^T$.

To have $\vec{w} \in span\{\vec{u}, \vec{v}\}$, there must exist some scalars a, b such that:

$$\vec{w} = a\vec{u} + b\vec{v}$$

You might realize that this generates a new system for us to solve:

$$\begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

$$a + 3b = 4; a + 2b = 5$$

We can solve this system like usual:

$$\left[\begin{array}{cc|c} 1 & 3 & 4 \\ 1 & 2 & 5 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & -1 \end{array}\right]$$

So, the solution is $a = 7, b = -1$. This means that $\vec{w} = 7\vec{u} - \vec{v}$.

Since we found a solution a, b for this new system, we know that there exists some linear combination of $\vec{u}, \vec{v}$ that equals $\vec{w}$.

2.1 In the example above, how would we be able to tell if $\vec{w} \notin span\{\vec{u}, \vec{v}\}$? Try to come up with a choice of $\vec{w}$ that isn't in the span, and show that it isn't in the span using the augmented matrix.

3 Linear Independence

3.1 Conditions for Linear Dependence/Independence

A set of non-zero vectors is linearly dependent if there exists scalars $k_1, k_2, \ldots k_n$ (where at least one k_i is nonzero), such that the linear combination of said vectors using those scalars equals the zero vector.

You may notice that this is similar to homogeneous systems!

There are three equivalent conditions for linear independence:

1. The inverse of the statement above. That is, the system is linearly *independent* if there doesn't exist such a choice of scalars $k_1, k_2, \ldots k_n$.
2. No vector in the set is in the span of the others.
3. The system of linear equations $A\vec{x} = \vec{0}$ has only the trivial solution, where A is the matrix made up by using the set of vectors as columns.

3.1 Is the following set of vectors linearly independent?

$$\{[1, 1, 0]^T, [2, 3, 1]^T, [-1, 1, 2]^T\}$$

3.2 Let's consider the second condition from above. Suppose we have a set of vectors $\{\vec{u}, \vec{v}, \vec{w}\}$ where $\vec{w} \in span\{\vec{u}, \vec{v}\}$. Prove that there exists a set of scalars k_1, k_2, k_3 such that $\vec{0} = k_1\vec{u} + k_2\vec{v} + k_3\vec{w}$.

3.3 Suppose we have a set of 4 vectors in $\mathbb{R}^3$. Can this set be linearly independent? Why or why not?

Let $U \subseteq \mathbb{R}^n$ be a linearly independent set of vectors. Then any vector in $span(U)$ can be written *uniquely* as a linear combination of vectors of U.

When an n-element set of vectors in $\mathbb{R}^n$ makes up an invertible $n \times n$ matrix A, the vectors are independent and span $\mathbb{R}^n$. The rows of A are thus also independent.

3.1 Why is it true that the columns of an invertible matrix are linearly independent?

4 Closing

4.1 What is the largest n for which a set of 4 vectors can span $\mathbb{R}^n$?

4.2 Are the columns of the following matrix linearly independent? What is the span of the columns?

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 4 \\ 2 & 2 & 2 \end{bmatrix}$$