MATH 303 Orthogonality

Fall 2024 Week 6, Session 1

1 Opener

1.1 Suppose a 4×4 matrix U has $dim(row(U)) = 3$. What is its rank?

1.2 Suppose a 4×4 matrix U has $dim(row(U)) = 1$. What is its nullity (the dimension of its null space)?

1.3 Loosely describe the process to find the null space of a matrix.

2 Orthogonal Vectors

2.1 Conditions for Orthogonality

Two vectors $\vec{u}, \vec{v}$ are orthogonal if their dot product is 0:

$$\vec{u} \cdot \vec{v} = 0$$

Subsequently, an *orthogonal set of vectors* is a set $\{\vec{u_1}, \vec{u_2}, \ldots, \vec{u_k}\}$ where each vector is mutually orthogonal:

1. $\vec{u_j} \cdot \vec{u_j} = 0$ if $i \neq j$
2. $\vec{u_1} \neq \vec{0}$ for all i

Additionally, the set is orthonormal if each such dot product is either 0 or 1 (when $i = j$).

Also, orthogonal vectors are definitionally linearly independent!

We can normalize an orthonormal set by dividing each vector by its magnitude. For example:

$$\left\{ \frac{1}{||\vec{u_1}||}\vec{u_1}, \frac{1}{||\vec{u_2}||}\vec{u_2}, \ldots, \frac{1}{||\vec{u_k}||}\vec{u_k} \right\}$$

This is just changing the vector to point in the same direction but with its magnitude coerced to 1.

2.1 If the dot product of a vector times itself is 1, what does this mean in practice?

2.2 Suppose $\{\vec{u_1}, \vec{u_2}\} = \{[1, 1]^T, [-1, 1]^T\}$. Find whether this set is orthogonal and/or orthonormal.

3 Orthogonal Matrices

An $n \times n$ matrix U is orthogonal if $UU^T = U^TU = I$. Since U is assumed to be square, it's enough to check that one of these equalities to I holds.

It turns out that the rows of an $n \times n$ orthogonal matrix form an *orthonormal* basis of $\mathbb{R}^n$! Similarly, you can construct an $n \times n$ orthogonal matrix using an orthonormal basis of $\mathbb{R}^n$.

The determinant of an orthogonal matrix is ± 1.

3.1 Is the matrix $U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$ orthogonal? What is its determinant?

4 Gram-Schmidt Process

(Live walkthrough using textbook)

4.1 Find an orthogonal basis for the space spanned by $[-1,-2,1]^T$ and $[0,1,-2]^T$.

4.2 Find an orthogonal basis for the column space of $\begin{bmatrix} 2 & 1 \\ 2 & 1 \\ 6 & 0 \end{bmatrix}$.

4.3 Find an orthonormal basis for the column space of $\begin{bmatrix} 2 & -1 \\ 8 & -1 \\ 0 & 2 \end{bmatrix}$.

5 Closing

Let's finish up with a few conceptual questions!

5.1 Suppose we have a set of three vectors with magnitudes of 1 in $\mathbb{R}^n$ such that $\vec{u_1}$ lies along the x-axis, $\vec{u_2}$ lies along the z-axis, and $\vec{u_3}$ lies along the y-axis. What space do these vectors form a basis of? Is the basis orthogonal? Orthonormal?

5.2 Suppose we construct a matrix using the same vectors, $U = \begin{bmatrix} \vec{u_1}^T \\ \vec{u_2}^T \\ \vec{u_3}^T \end{bmatrix}$. What is the determinant of this matrix?