MATH 303 — Identity Matrices, Inverses, LU Factorization

Fall 2024 — Week 3, Session 1

1 Pop Quiz: Content so far...

1.1 Perform the following matrix multiplication:

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} -2 & -4 \\ -4 & 3 \end{bmatrix}$$

1.2 Are the following matrices symmetric? Skew symmetric?

$$\begin{bmatrix} 0 & 2 & -1 \\ 2 & 0 & 4 \\ 1 & -4 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 & -1 \\ 2 & 0 & 4 \\ 1 & -4 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 4 & 11 \\ 4 & 2 & 4 \\ 11 & 4 & 2 \end{bmatrix}$$

1.3 What are the ranks of the following linear systems (excluding the augmented column)? Number of solutions?

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -34 \\ 0 & 0 & 1 & 15 \end{array}\right], \left[\begin{array}{ccc|c} 2 & 0 & 5 & 6 \\ 0 & 4 & 8 & 17 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

2 The Identity Matrix and Inverses

2.1 Identity Matrices

The identity matrix I_n is the $n \times n$ *square* matrix consisting of ones down the main diagonal, and zeroes elsewhere.

We say that I_n is defined by $I_{ij} = \delta_{ij}$, where:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

In words, the ij-th element of the identity matrix (the element in the ith row and jth column) is 1 if it is along the main diagonal, where $i = j$, and 0 otherwise.

2.1 Identify which of the following matrices are identity matrices:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}$$

The identity matrix has the special property that for any $m \times n$ matrix A, $AI_n = A$ and $I_m A = A$.

Note that we use I_n in one direction and I_m in the other; why might this be?

2.2 Matrix Inverse

The *inverse* A^{-1} of an $n \times n$ *square* matrix is a matric such that:

$$AA^{-1} = A^{-1}A = I_n$$

Note: It is not necessarily true that an inverse matrix exists at all for any particular choice of A! However, if such an inverse does exist, it is *unique*. That is, there cannot be multiple distinct matrices which satisfy the above definition of an inverse matrix.

2.1 Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. Show that $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ is the inverse of A.

2.3 Finding an Inverse

Suppose we have a 2×2 matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. When searching for A^{-1}, what we're trying to do is find another matrix satisfying the following:

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x & z \\ y & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Live demo: solve for the above inverse via multiplication.

It turns out that there's a simple algorithm for finding the inverse. Simply construct the augmented matrix $\left[A \mid I \right]$ and do row operations to reduce the A side to I, such that the augmented matrix looks like $\left[I \mid B \right]$. If it's impossible to reduce A to I via row operations, then there is no inverse.

Live demo: solve for the same inverse using the augmented matrix method.

2.1 Using the augmented matrix method, find the inverses of the following matrices, or state that one does not exist:

$$\begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 4 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

2.4 Solving Systems Using the Inverse

Suppose we have a linear system of the form $A\vec{x} = \vec{b}$, and suppose A has an inverse.

Then we can obtain the following expression for the solution of the system, $\vec{x}$:

$$(A^{-1})A\vec{x} = A^{-1}\vec{b} \text{ (multiply both sides on the left by } A^{-1})$$

$$(A^{-1}A)\vec{x} = A^{-1}\vec{b} \text{ (use the associate property of matrices } A^{-1})$$

$$I\vec{x} = A^{-1}\vec{b} \text{ (replace the product with the identity matrix)}$$

$$\boxed{\vec{x} = A^{-1}\vec{b}} \text{ (the identity matrix vanishes)}$$

2.1 Solve the following system of equations using the inverse method above:

$$x + z = 1$$

$$x - y + z = 3$$

$$x + y - z = 2$$

2.5 Miscellaneous Properties of the Inverse

Let A, B, and A_i for $i \in 1 \dots k$ be $n \times n$ matrices. Then:

1. If A is invertible, then $(A^T)^{-1} = (A^{-1})^T$.
2. If A and B are invertible, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
3. If $A_1, A_2, \dots A_k$ are invertible, then the product of such matrices is invertible, and $(A_1A_2 \dots A_k)^{-1} = A_k^{-1}A_{k-1}^{-1} \dots A_2^{-1}A_1^{-1}$. That is, the inverse of a product sequence of matrices is the same as the reversed-order product sequence of those matrices' inverses.

Let A be an $n \times n$ matrix and I_n the identity matrix. Then:

1. I is invertible and $I^{-1} = I$.
2. If A is invertible then so is A^{-1}, and $(A^{-1})^{-1} = A$.
3. Of A is invertible then so is A^k, and $(A^k)^{-1} = (A^{-1})^k$. That is, the inverse of some power of a matrix is the same as the matrix's inverse raised to the same power.
4. If A is invertible and p is a nonzero real number, then pA is inveritble and $(pA)^{-1} = \frac{1}{p}A^{-1}$.

3 LU Factorization

3.1 Finding LU Factorizations

Sometimes, we can write a matrix in its *LU Factorization* form. For a matrix A, the LU Factorization is the product of two matrices L, U such that $A = LU$.

L is a *lower triangular* matrix where the main diagonal consists of all 1s and everything above the main diagonal is 0. (Live demo: show an example)

U is a *upper triangular* matrix which only needs to have 0s below the main diagonal. The main diagonal and all entries above it can be any number. (Live demo: show an example)

The easiest method for finding an LU Factorization is the *multiplier method.* We'll demonstrate this with the following example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ -2 & 3 & -2 \end{bmatrix}$$

First, write A as the product of the identity matrix and itself:

$$A = I_n A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ -2 & 3 & -2 \end{bmatrix}$$

Next, we'll row-reduce the right matrix until it is in *upper triangular* form. This is similar to reducing it to RREF, except we don't need to cancel the entries above the main diagonal! **Note: You cannot use the row interchange or row-multiplication operations during this process, only the row-multiple addition operation**.

Each time we perform a row-multiple addition operation, we'll update the left matrix with the *negative* of the multiple we used. For example, consider wanting to eliminate the boxed entry below:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ \boxed{2} & 3 & 1 \\ -2 & 3 & -2 \end{bmatrix}$$

We'll need to perform the row-multiple addition operation $R_2 = R_2 + (-2)R_1$ to cancel out the 2. As we perform that operation, we'll update the corresponding entry in the left matrix with $-(-2) = 2$:

$$\begin{bmatrix} 1 & 0 & 0 \\ \boxed{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ -2 & 3 & -2 \end{bmatrix}$$

Don't forget to also update the other elements in the row (besides the one we're cancelling)!

We'll repeat this process for any remaining entries under the main diagonal (boxed entries below):

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ \boxed{-2} & \boxed{3} & -2 \end{bmatrix}$$

Using the same method, we'd eventually arrive at the LU Factorization, where L is the left matrix and U is the right matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & -7 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & -31 \end{bmatrix}$$

3.1 Find an LU Factorization for $A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix}$

3.2 Using LU Factorizations to Solve Systems

Suppose we have a linear system $A\vec{x} = \vec{b}$ where $A = LU$. Then we can rewrite the system as $(LU)\vec{x} = \vec{b}$. Since matrix multiplication is associative, it is also valid to say $L(U\vec{x}) = \vec{b}$. If we set a new variable $y = U\vec{x}$, then we can solve two systems consectuviely to solve for $\vec{x}$:

$$L\vec{y} = \vec{b}$$

$$U\vec{x} = \vec{y}$$

This might seem like more work at first, but it turns out that solving systems where the matrix is *triangular* (as L and U are) tends to be pretty convenient!

3.1 Solve the following system by LU Factorization:

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 4 & 3 & 1 & 1 \\ 1 & 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

4 Closing

4.1 Find the inverse of the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

4.2 Suppose we have a system $A\vec{x} = \vec{b}$, where A is invertible. Solve for $\vec{x}$ in terms of A^{-1}.

4.3 Can a 4×3 matrix have an inverse?