MATH 303 | Matrix Multiplication & Transpose

Fall 2024 | Week 2, Session 2

1 Opener: Pop Quiz on Content so Far

1.1 A 3-variable linear homogeneous system has rank 2. How many free variables does it have? How many solutions does it have?

1.2 **Challenge question:** A 3-variable linear homogeneous system has rank 2. How many solutions does it have?

1.3 How many basic solutions does a system with two free variables have?

1.4 Sum the following two matrices:

$$\begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

1.5 Sum the following two matrices:

$$\begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 3 & 2 \\ 4 & 5 \end{bmatrix}$$

2 Vectors

2.1 Row and Column Vectors

2.1 Determine whether the following matrices are *row vectors*, *column vectors*, or neither.

a. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 2 \end{bmatrix}$

2.2 Vector Form of a System of Linear Equations

Recall that linear systems of m equations in n variables have the form:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_3$$

It turns out that we can write this system in a more convenient **vector form**:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

See that each vector would be one column from the augmented matrix, and the column on the right-hand side of the equation would be the "augmented" column on the far right.

2.3 Multiplication of Vector by Matrix

Consider an $m \times n$ matrix $A = \begin{bmatrix} A_1 \dots A_n \end{bmatrix}$ and an n-dimensional column vector $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

The product $A\vec{x}$ is an $m \times 1$ column vector which can be written as a linear combination of the columns of A, similar to the vector form above!

$$A\vec{x} = x_1A_1 + x_2A_2 + \cdots + x_nA_n$$

2.1 Perform the following matrix-vector multiplications by decomposing the problem into vector form:

1. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} =$
2. $\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} =$
3. $\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} =$
4. $\begin{bmatrix} 0 & 2 \\ 1 & 0 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} =$

2.4 Matrix Form of a System of Linear Equations

We can also write a linear system in **matrix form**, taking advantage of the matrix-vector multiplication we just learned!

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

When you decompose the vector multiplication like we did before, you get the vector form:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

So these two forms are indeed interchangeable!

The vector $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ will be a solution to the system if and only if $x_1, x_2, \dots, x_n$ are solutions to the linear system represented in either form.

3 Matrix Multiplication

Similarly, we can multipliy two matrices together. Consider the following example:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix}$$

First, we need to check the size of the matrices to make sure multiplication is possible. In this case, A is 2×3, and B is 3×3. The trick to remember is that when you write the sizes side-by-side *in the same order as the multiplication*, like $(2 \times \boxed{3})(\boxed{3} \times 3)$, you need the "inner" (boxed) entries to be equal. If they're not, the multiplication isn't possible.

3.1 Determine whether the following matrix sizes are compatible for multiplication:

1. 1×2 and 2×1
2. 3×3 and 3×4
3. 2×2 and 4×2
4. 1×1 and 1×2

3.2 Does the fact that a multiplication works in one direction necessarily imply that it works in the other direction?

Once you've determined two matrices are compatible, we can decompose the problem further into matrix-vector form like so:

$$\left[\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \right]$$

That is, the nth column of the resulting matrix will be the column vector created by multiplying A with the nth column of B.

3.1 Perform the following matrix multiplication using the column decomposition method described above:

$$\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

3.2 Perform the matrix multiplication from the example above.

We can also consider matrix multiplication on an entry-by-entry basis (live demo to explain).

3.1 Perform the following matrix multiplication using element-by-element multiplication:

$$\begin{bmatrix} 1 & 3 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$$

4 Transpose

4.1 Vector/Matrix Transposition

The transpose of a vector is the same vector, but "flipped" in direction. That is, a row vector becomes a column vector with the same entries, and vice versa. For example:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

4.1 Find the following vector tranpositions:

1. $\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}^T$

2. $\begin{bmatrix} 1 & 3 & 3 & 2 \end{bmatrix}^T$

3. $\begin{bmatrix} 2 \\ 2 \end{bmatrix}^T$

For a matrix, the (i, j)th entry of A becomes the (j, i)th entry of A^T. In practice, this means the 1st row becomes the 1st column, the 2nd row becomes the 2nd column, etc. See the following example (first row highlighed):

$$\begin{bmatrix} \boxed{1} & \boxed{2} & \boxed{3} \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} \boxed{1} & 4 \\ \boxed{2} & 5 \\ \boxed{3} & 6 \end{bmatrix}$$

4.1 I think the easiest way to understand this is to drill some practice problems, so try the following:

1. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T =$

2. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T =$

3. $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -3 \\ 1 & 5 & 6 \end{bmatrix}^T =$

4.2 Properties of the Transpose

Let A be an $m \times n$ matrix, B an $n \times p$ matrix, and r, s scalars. Then:

a. $(A^T)^T = A$

b. $(AB)^T = B^T A^T$

c. $(rA + sB)^T = rA^T + sB^T$

Additionally, a matrix is said to be **symmetric** if $A = A^T$. It is said to be **skew symmetric** if $A = -A^T$.

4.1 Is the following matrix symmetric?

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

4.3 Closing

Let's finish off the session with some summarizing questions!

4.1 Perform the following matrix multiplication:

$$\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$$

4.2 When writing a linear system in $A\vec{x} = \vec{b}$ form, are the $\vec{x}$ and $\vec{b}$ vectors *row* or *column* vectors?

4.3 Write the following linear system in equation form:

$$\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

4.4 Without doing any calculations, what is the solution to the following system?

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$