MATH 303 Fall 2024

Midterm Exam Review

Week 4, Session 1

1 Exam Practice

1.1 Linear Systems and Matrices

1.1 Suppose a linear system has one solution. Is it consistent or inconsistent?

1.2 Are the linear systems represented by the following augmented matrices consistent or inconsistent?

$$\left[\begin{array}{cc|c} 2 & 1 & 3 \\ 0 & 1 & 3 \end{array}\right], \left[\begin{array}{ccc|c} 2 & -1 & 5 & 5 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

1.3 For what value of k will the linear system below be consistent?

$$\begin{cases} x - 6y = -5 \\ x - 4y = k \\ 3x - 9y = -3 \end{cases}$$

1.4 If $A = \begin{bmatrix} x & 4 \\ -5 & y \end{bmatrix}$ and $A^2 = \begin{bmatrix} 29 & 24 \\ -30 & -19 \end{bmatrix}$, what are the values of x and y?

1.5 Reduce the following matrices to row-echelon form:

$$\begin{bmatrix} 1 & -5 & 3 \\ -4 & 2 & -3 \end{bmatrix}, \begin{bmatrix} 3 & 4 & 2 \\ 6 & 8 & 4 \\ 6 & 3 & 2 \end{bmatrix}$$

1.2 Homoegeneous Systems

1.1 Build a 2×2, non-zero coefficient matrix such that the vector $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ is a solution to the corresponding homogeneous system.

1.2 Suppose the coefficient matrix for an 11-variable homogeneous system has rank 10. How many solutions will the homogeneous system have?

1.3 Inverses, Transposes, and LU Decomposition

1.1 Find the inverse (if one exists) for the following matrices:

$$\begin{bmatrix} 1 & -3 \\ 5 & -14 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 5 & 0 \end{bmatrix}, \begin{bmatrix} 7 & 2 \\ -11 & -3 \end{bmatrix}$$

1.2 Find the tranpositions of the following matrices:

$$\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} -2 & 0 & -3 \\ 4 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

1.3 Are the following matrices symmetric? Skew symmetric?

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 3 \end{bmatrix}$$

1.4 Find an LU decomposition for the following matrices:

$$\begin{bmatrix} -6 & 9 \\ -7 & 9 \end{bmatrix}, \begin{bmatrix} 10 & -4 \\ -4 & 6 \end{bmatrix}$$

1.4 Determinants

1.1 Find the determinants of the following 2×2 matrices:

$$\begin{bmatrix} 4 & 3 \\ -3 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 3 & -3 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$$

1.2 Find the determinants of the following 3×3 matrices:

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 5 & -2 \\ 4 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 0 & 5 & 4 \\ 0 & 0 & -2 \end{bmatrix}$$

1.5 Cramer's Rule

We'll practice this more in the second session of this week, but it's something that may show up on your exam, so let's practice it!

Suppose we have a linear system of the form $A\vec{x} = \vec{b}$, where A is a square invertible matrix. Then $\vec{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T$ and $x_i = \dfrac{det(A_i)}{det(A)}$, where A_i is the matrix formed by replacing the ith column of A with $\vec{b}$.

As an example, consider the system defined by $A = \begin{bmatrix} -1 & -3 \\ 3 & 4 \end{bmatrix}, \vec{b} = \begin{bmatrix} 11 \\ -13 \end{bmatrix}$.

Then $A_1 = \begin{bmatrix} 11 & -3 \\ -13 & 4 \end{bmatrix}$ (A but with its first column replaced by $\vec{b}$),
and $A_2 = \begin{bmatrix} -1 & 11 \\ 3 & -13 \end{bmatrix}$ (A but with its second column replaced by $\vec{b}$).

Then $x_1 = \dfrac{det(A_1)}{det(A)}$ and $x_2 = \dfrac{det(A_2)}{det(A)}$. What are these determinants, and subsequently the solution to the system, $\vec{x}$?

1.1 Use Cramer's rule to solve the following system:

$$\begin{cases} 4x + 5y = 23 \\ -x + 2y = 4 \end{cases}$$

1.6 Properties of Determinants

1.1 Suppose A and B are invertible 2×2 matrices such that $det(A) = 2$ and $det(B) = 3$. Then find the following determinants:

a. $det(AB)$

b. $det(A^T)$

c. $det(A^{-1})$

d. $det(2A)$

e. $det(B^{-1}A^{-1})$