MATH 303 | Determinants

Fall 2024 | Week 3, Session 2

1 Pop quiz: Content so Far

1.1 Find the inverse and an LU factorization of the following matrix:

$$\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$$

1.2 Find the inverse and an LU factorization of the following matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

2 Determinants

2.1 Determinant of a 2×2 Matrix

For a 2×2 matrix A, the determinant $det(A)$ can be easily found:

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $det(A) = ad - bc$.

2.1 Find the determinant of the following 2×2 matrices:

$$\begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix}, \begin{bmatrix} 12 & 2 \\ 4 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$$

2.2 When will the determinant of a 2×2 matrix be 0? That is, which values of a, b, c, d would cause the determinant to be 0?

2.2 Matrix Minors

The ijth minor of a matrix A, denoted $minor(A)_{ij}$, is the determinant of the resulting matrix when you delete the ith row and jth column from the matrix. For example:

Let $A = \begin{bmatrix} 1 & \boxed{2} & \boxed{3} \\ 4 & 3 & 2 \\ 3 & \boxed{2} & \boxed{1} \end{bmatrix}$.

Then $minor(A)_{21}$ is the determinant of the 2×2 matrix left over when you delete the 2nd row and 1st column from A. This leaves the 2×2 matrix consisting of the boxed elements above. We can find $minor(A)_{21}$ by taking the determinant of this leftover 2×2 matrix:

$$minor(A)_{21} = det(\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}) = (2)(1) - (3)(2) = -4$$

2.1 Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$. Find the following:

a. $minor(A)_{11}$

b. $minor(A)_{12}$

2.3 Cofactors

Similar to minors, the ijth cofactor of a matrix A is denoted $cof(A)_{ij}$, and is defined as:

$$cof(A)_{ij} = (-1)^{i+j} minor(A)_{ij}$$

First, notice that $(-1)^{i+j}$ can only be two values, 1 or -1. Additionally, it'll be 1 when $i + j$ is even, and -1 when $i + j$ is odd.

So the cofactor of A at the ijth position is just the minor at that position, with the sign flipped if $i + j$ are odd.

2.1 Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$. Find the following:

a. $cof(A)_{31}$

b. $cof(A)_{22}$

2.4 Determinant of a 3×3 Matrix

Now that we've learned cofactors, we can find the determinant of a 3×3 matrix using a process called **Laplace Expansion**, or **Cofactor Expansion**. I think the easiest way to learn this is through an example.

Consider the same matrix we've used so far, $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$.

First, choose a row or column to *expand along*. It doesn't matter which row or column you expand along, you'll always get the same answer! For this example, let's choose the first row.

$$A = \begin{bmatrix} \boxed{1} & \boxed{2} & \boxed{3} \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

$det(A)$ is equal to the sum of $a_{ij} cof(A)_{ij}$ for all elements in the row/column you expand along. For this specific example, this would be:

$$det(A) = 1 \cdot cof(A)_{11} + 2 \cdot cof(A)_{12} + 3 \cdot cof(A)_{13}$$

Now that we have an expression for $det(A)$, let's solve the problem as a group!

It turns out that many matrices have certain rows or columns that are "easier" to expand along. For example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & -4 \\ 2 & 5 & 1 \end{bmatrix}$$

In this example, you could choose to expand along the 1st row and only need to calculate one 2×2 determinant. This is completely fine to do, because you will *always get the same determinant regardless of which row/column you choose.*

We can expand this same proces to arbitrary $n \times n$ square matrices. We can simply choose a row/column to expand along, then find the cofactors of each element. In the case of a 4×4 matrix, our "minor" matrices that we get from excluding a row and column would be 3×3. There's nothing wrong with that; it just means that for each cofactor we'd also need to use Laplace Expansion on each created 3×3 matrix.

2.1 Find $det(\begin{bmatrix} 1 & 0 & 2 \\ 3 & 4 & -1 \\ 2 & 5 & 2 \end{bmatrix})$

2.2 Find $det(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix})$

2.3 Find $det(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 4 & 5 \\ 0 & 2 & -1 & 3 \\ 0 & 5 & 1 & 1 \end{bmatrix})$

2.5 Determinants of Triangular Matrices

For upper or lower triangular matrices, we can use a shortcut to find determinants.

Let A be an upper or lower diagonal matrix. Then $det(A)$ is the product of the elements along the main diagonal.

For example, $det(\begin{bmatrix} 2 & 3 & 4 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix}) = (2)(-1)(3) = -6$

2.1 Find determinants of the following matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} 2 & 1 & -98 \\ 0 & 24 & 56 \\ 0 & 0 & -1 \end{bmatrix}$$

3 Closing

Let's finish off with some practice questions!

3.1 Using the methods we've learned so far, find the determinants of the following matrices:

$$\begin{bmatrix} 2 & 4 \\ 0 & 6 \end{bmatrix}, \begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 4 & 5 & 6 \\ -1 & -2 & 4 \end{bmatrix}$$

3.2 **Challenge question:** Suppose A is a 3×3 matrix in RREF with $rank(A) = 3$. What is its determinant?

3.3 **Challenge question:** Suppose A is a 3×3 matrix in RREF with $rank(A) = 2$. What is its determinant?