MATH 303 — Properties of Determinants & Cramer's Rule

Fall 2024 — Week 4, Session 2

1 Opener: Midterm Exam Recap

By now, I think you all would've taken the midterm. With that in mind, let's take a second to reflect on the course so far. Take a few minutes, and think quietly about your responses to the following prompts. We'll take a few minutes to discuss as a group afterwards!

1. What is one concept from the midterm (or course so far) that you wish you understood better?
2. Similarly, what's one concept that you feel particularly strong about; especially if it's something difficult that you feel proud of yourself for having learned?
3. On a scale from 1 to 10, how effective do you think your study strategy has been for the course so far? What's one strategy you've used that you've found to be successful?

2 Properies of Determinants

2.1 Determinants and Row Operations

Recall that there are three row operations for matrices:

1. Switching or interchanging two rows
2. Multiplying a single row by a scalar
3. Adding a scalar multiple of one row to another row

These row operations each have different mutating effects on the determinant of a matrix. Consider a matrix A with determinant $det(A)$. Consider a matrix B which is created by performing each of the above row operations on A. Then:

1. (Switching rows) $det(B) = -det(A)$
2. (Scalar multiple of a row) $det(B) = k \cdot det(A)$, where k is the scalar multiple used
3. (Adding a scalar multiple of a row to another) $det(B) = det(A)$

For the following problems, suppose $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 4 \\ 2 & 4 & 3 \end{bmatrix}$.

2.1 What is $det(A)$?

2.2 Suppose $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 3 \\ 2 & 3 & 4 \end{bmatrix}$. Without manual calculation, what is $det(B)$?

2.3 Suppose $B = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 3 & 4 \\ 2 & 4 & 3 \end{bmatrix}$. Without manual calculation, what is $det(B)$?

2.4 Suppose $B = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 9 & 12 \\ 2 & 4 & 3 \end{bmatrix}$. Without manual calculation, what is $det(B)$?

2.5 Suppose $B = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 6 & 8 \\ 2 & 4 & 3 \end{bmatrix}$. Without manual calculation, what is $det(B)$? *Hint: Multiple row operations were applied to row 2.*

2.6 Suppose $B = 2A$. Without manual calculation, what is $det(B)$?

2.2 Other Properties of Determinants

As we saw above, multiplying a matrix by a scalar k is the same as multiplying each row of the matrix by k. Since each row multiplication multiplies the determinant by k, for an $n \times n$ matrix you will be performing the multiplication n times. As such, the determinant will cumulatively be multiplied by k^n.

In other words for an $n \times n$ matrix A, $det(kA) = k^n det(A)$.

There are several other properties of determinants we'll need to know; suppose we have two matrices A and B. Then:

1. $det(AB) = det(A)det(B)$
2. $det(A^T) = det(A)$
3. $det(A^{-1}) = \dfrac{1}{det(A)}$ (when $det(A) \neq 0$)

3 Cramer's Rule

Suppose we have a linear system of the form $A\vec{x} = \vec{b}$, where A is a square invertible matrix. Then $\vec{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T$ and $x_i = \dfrac{det(A_i)}{det(A)}$, where A_i is the matrix formed by replacing the ith column of A with $\vec{b}$.

As an example, consider the system defined by $A = \begin{bmatrix} -1 & -3 \\ 3 & 4 \end{bmatrix}, \vec{b} = \begin{bmatrix} 11 \\ -13 \end{bmatrix}$.

Then $A_1 = \begin{bmatrix} 11 & -3 \\ -13 & 4 \end{bmatrix}$ (A but with its first column replaced by $\vec{b}$),
and $A_2 = \begin{bmatrix} -1 & 11 \\ 3 & -13 \end{bmatrix}$ (A but with its second column replaced by $\vec{b}$).

Then $x_1 = \dfrac{det(A_1)}{det(A)}$ and $x_2 = \dfrac{det(A_2)}{det(A)}$. What are these determinants, and subsequently the solution to the system, $\vec{x}$?

3.1 Use Cramer's rule to solve the following system:

$$\begin{cases} 4x + 5y = 23 \\ -x + 2y = 4 \end{cases}$$

Live demo: Proof of Cramer's Rule for 2×2 systems.

4 Closing

4.1 Based on the 2×2 proof of Cramer's rule, do you think we use it for non-invertible coefficient matrices? How about non-square coefficient matrices?

4.2 If $det(A) = 3$ and $det(B) = \frac{2}{3}$, what is $det(AB^{-1})$?

4.3 If $det(A) = 5$ and $det(B) = 4$ and both are 3×3 matrices, what is $det(3AB^T)$?