MATH 303 Diagonalization

Fall 2024 Week 8, Session 1

1 Opening: Eigenvalues and Eigenvectors

1.1 Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} -1 & -4 \\ -3 & -2 \end{bmatrix}$.

1.2 Suppose a matrix A is known to have an eigenvalue $\lambda = 3$ with associated eigenvector $\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Solve for $\vec{y} = A \begin{bmatrix} -6 \\ -4 \end{bmatrix}$.

2 Similarity

Two matrices A, B are similar if and only if we can find some invertible matrix P such that:

$$A = P^{-1}BP$$

Similar matrices have the same determinants, rank, trace, and eigenvalues.

When a matrix is similar to a diagonal matrix, we say that the matrix is *diagonalizable*.

3 Diagonalization

When diagonalizing a matrix A, we're trying to find three other matrices P, D, and P^{-1} such that $A = PDP^{-1}$, and D is a matrix with nonzero entries *only* on its diagonal.

A matrix A is diagonalizable if and only if there is an invertible matrix P given by:

$$P = \begin{bmatrix} X_1 X_2 \ldots X_n \end{bmatrix}$$

where the X_k are the eigenvectors of A. Additionally, the matrix D constructed by $A = PDP^{-1}$ will have diagonal entries consisting of the corresponding eigenvalues of A.

3.1 Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix}$. Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

3.1 Conditions for Diagonalization

It turns out that we can always diagonalize an $n \times n$ matrix if it has n distinct eigenvectors. In practice, this means that if we have n different eigenvalues, we can diagonalize. If the number of eigenvalues if less than n (implying that some eigenvalue(s) have multiplicity > 1), we may still be able to diagonalize if each eigenvalue of multiplicity m corresponds to m different eigenvectors.

In other words, if A has eigenvalues with multiplicity > 1, that eigenvalue needs to have as many eigenvectors associated with it as its multiplicity. If this is the case for all eigenvalues, the matrix will still be diagonalizable.

3.2 A Useful Property of Diagonal Matrices

One useful case for diagonal matrices is when multiplying a matrix by itself some number of times (i.e. raising it to a power).

Consider trying to find A^{100}. This would be very annoying to calculate by hand, since you'd be perfoming the matrix multiplication 100 times!

Instead, you could diagonalize A into $A = PDP^{-1}$ and raise this representation to the 100th power. This results in $A^{100} = (PDP^{-1})^{100} = PD^{100}P^{-1}$. Since D is a matrix with only entries on the diagonal, D^{100} is very easy to find; just raise each nonzero entry to the 100th power!

4 Closing/Practice Problems

4.1 Diagonalize (find P, D, P^{-1}) for the following matrix:

$$A = \begin{bmatrix} 3 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix}$$

4.2 Use diagonalization to find A^4 for $A = \begin{bmatrix} 1 & -6 \\ 2 & -6 \end{bmatrix}$.