MATH 303 Homogeneous Systems, Rank, Matrix Arithmetic

Fall 2024 Week 2, Session 1

1 Homogeneous Systems

Recall that homogeneous systems have the form:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0$$

Equivalently:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & 0 \end{array}\right]$$

1.1 Trivial Solution of Homogeneous Systems

1.1 Fill in the blank: Homogeneous systems always have at least ___ solution(s).

1.2 Fill in the blank: If a homogeneous system has a solution in which not all of the $x_1 \dots x_n$ are equal to zero, this solution is called a ___ solution.

1.3 Consider a three-variable homogeneous system. For its trivial solution, what are the values of x, y, z?

1.2 Basic Solutions of Homogeneous Systems

1.1 Consider a homogeneous system with the following solution:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

What are the basic solution(s) of this system?

1.2 Find the basic solution(s) (if any) of the following homogeneous system:

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 4 & 8 & 0 \end{array}\right]$$

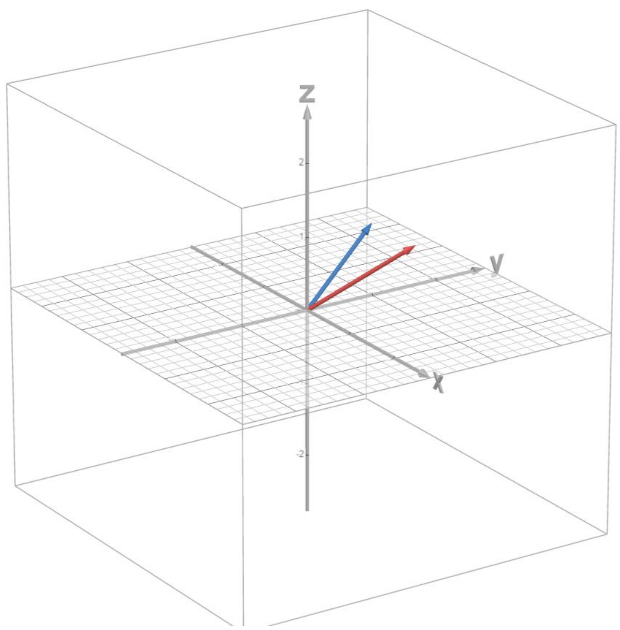
1.3 Find the basic solution(s) (if any) of the following homogeneous system:

$$\left[\begin{array}{ccc|c} 1 & 1 & 6 & 0 \\ 1 & 3 & 14 & 0 \\ 2 & 2 & 12 & 0 \end{array}\right]$$

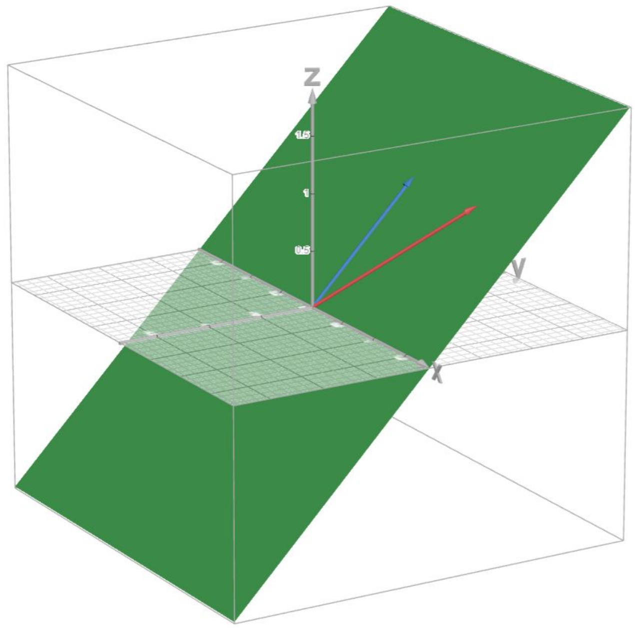
We say that the set of solutions of a homogeneous system is a *linear combination* of the basic solutions.

For example, if a homogeneous system has two basic solutions X_1 and X_2, the set of solutions of the system can be expressed as $X = sX_1 + tX_2$, where s, t are real numbers. In essence, X is a "list" of any points that can be "reached" by combining some "amount" of X_1 and X_2.

To try to build some intuition, consider the following graphical example. Suppose the red/blue arrows are two vectors which are basic solutions of a system.



These two vectors define a plane that both vectors lie within; any point in this plane can be reached by combining these two vectors in some amounts!



1.3 Matrix Sizes and Nontrivial Solutions

When the *coefficient matrix* of the homogeneous system (that is, the matrix without the augmented row) has fewer rows than columns, it turns out that the system will *always* have a nontrivial solution!

Reason: Remember that each row is a separate equation, and each column is a variable. When we have fewer equations than variables, it's impossible to construct equations such that all variables have a single unique value, so we will have infinite solutions. Since we have infinite solutions, there must be at least one parameter.

We normally say a matrix has m rows and n columns. With that in mind, let's try to answer the following conceptual questions:

1.1 When $m = n$ for a homogeneous system, will we always, never, or sometimes have a nontrivial solution?

1.2 When $m > n$ for a homogeneous system, will we always, never, or sometimes have a nontrivial solution?

2 Rank

The *rank* of a matrix is the number of pivot columns in the row-echelon or reduced row-echelon forms of the matrix. For a matrix A, we can write this as $\text{Rank}(A)$.

For an $m \times n$ coefficient matrix with rank r, the solution to the system will have $n - r$ parameters/free variables/basic solutions.

2.1 Consider the following 3×3 homogeneous system, already reduced to RREF:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

1. What is the rank of this matrix?
2. How many parameters will the solution to this system have?

2.1 Rank and Consistent Systems

Now, let's consider how we can use rank for non-homogeneous, consistent systems. Note that non-homogeneous systems *can be* inconsistent; if that's the case, none of this discussion applies.

1. When the rank r of a consistent system is less than the number of variables n, the system will have at least one parameter. Thus it will have infinite solutions.
2. When the rank r of a system is equal to the number of variables n, the system doesn't have any parameters. Thus, there is either exactly one or no solutions. Since we presupposed the system to be consistent, it has exactly one solution.

Again, the rank of a system does not determine the consistency or inconsistency of a system in any way!

2.1 Use the concepts of rank and consistency to classify the number of solutions and parameters (if any) of the following non-homogeneous systems:

1. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$
2. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$
3. $\left[\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 5 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$

3 Matrix Arithmetic

3.1 Introduction

3.1 A square matrix has **3** rows. How many columns does it have?

3.2 A 3×3 zero matrix has **3** rows and **3** columns. What is the value of the top-leftmost entry?

3.3 Two 2×2 matrices, A and B have the same set of entries, but listed in different orders in the matrix. Does $A = B$?

3.2 Matrix Addition

3.1 Perform the following matrix additions:

1. $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$
2. $\begin{bmatrix} 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \end{bmatrix}$
3. $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 2 \end{bmatrix}$

3.3 Scalar Multiplication of Matrices

3.1 Perform the following scalar matrix multiplications:

1. $3 \times \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$
2. $\frac{3}{2} \times \begin{bmatrix} 1 & 5 \end{bmatrix}$
3. $0 \times \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

4 Closing

Let's wrap today's session up with some summarizing questions!

4.1 Suppose a homogeneous system is found to have two basic solutions. How many free variables does the system have, and how many solutions does the system have?

4.2 Suppose a 5×3 homogeneous system has rank 3. How many free variables does the system have, and how many solutions does the system have?

4.3 Fill in the blank: Rank is the number of ___ columns in a system.

4.4 Consider the following system:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

What is the rank of the system? How many solutions does it have?

4.5 Find the value of matrix A:

$$A = 2\begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix} - 3\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$