MATH 303 | One-to-One and Onto Transformations
Fall 2024 | Week 7, Session 1

1 Opening

1.1 Let $T : \mathbb{R}^2 \to \mathbb{R}^{\nvDash}$ be a linear transformation induced by the matrix $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$. How would we find T's inverse? What would T^{-1}'s matrix be?

1.2 Suppose $T[1,0]^T = [2,3]^T$ and $T[0,1]^T = [3,-1]^T$. What is the matrix associated with T?

2 One-to-One and Onto Transformations

Recall that a linear transformation maps vectors from $\mathbb{R}^n$ to $\mathbb{R}^m$, where the number of rows (m) and the number of columns (n) of the matrix of the transformation defines the dimensions of the pre and post-transformation spaces.

Let's think conceptually:

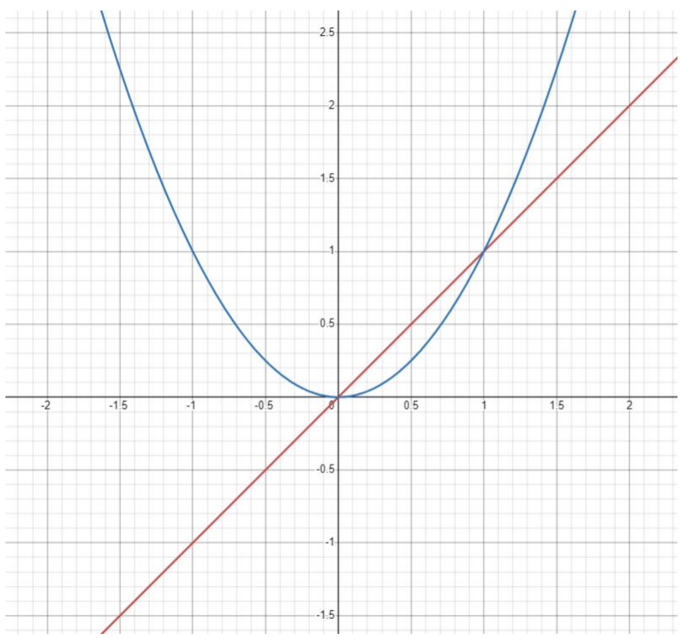
2.1 Suppose $m > n$. Is it possible for *every* vector in $\mathbb{R}^m$ to be the image of some vector in $\mathbb{R}^n$?

2.2 Suppose $n > m$. Is it possible for *every* vector in $\mathbb{R}^n$ to be the image of *at most* one vector in $\mathbb{R}^n$?

2.1 One-to-One Transformations

A **one-to-one** transformation (or, an **injection**) is a transformation with the property that no vector in the post-transformation space is mapped to by more than one vector in the pre-transformation space.

Transformations can be thought of as functions. If we consider an analagous function $f(x)$ like we're familiar with, a one-to-one function maps each x value into a different y value. In the graph below, $f(x) = x$ is one-to-one, but $f(x) = x^2$ isn't one-to-one, since it carries two x values to $f(x) = 1$, for example.



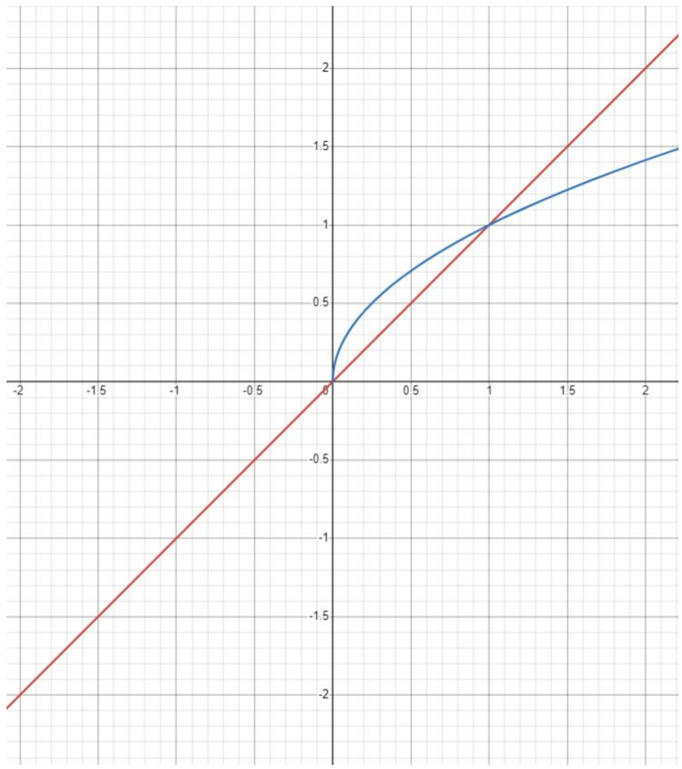
It turns out that we have a couple ways to immediately tell if a transformation is one-to-one:

1. If A is the matrix of the transformation, $\text{rank}(A) = n$ (the dimension of the pre-transformation space)
2. If the only vector which is mapped to $\vec{0}$ is the zero vector itself. This is the same as solving the homogeneous system $A\vec{x} = \vec{0}$ and verifying that it only has the trivial solution!

2.2 Onto Transformations

An **onto** transformation (or, a **surjection**) is a transformation with the property that every vector in the post-transformation space is mapped to by *some* vector in the pre-transformation space.

Using the same function analogy, an onto function is one where *every* y value is "touched" by the graph somewhere - even if it's more than once! $f(x) = x$ is onto, but $f(x) = \sqrt{x}$ is not (because negative values aren't mapped to by any x).



Similarly, we can verify that a transformation is onto by checking if $\text{rank}(A) = m$ (the dimension of the post-transformation space)

3 Practice/Conceptual Questions

3.1 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$, and suppose T is one-to-one. What is the rank of the transformation's corresponding matrix? Is T onto?

3.2 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$, and suppose that T is onto. Is it one-to-one?

3.3 Let $T : \mathbb{R}^2 \to \mathbb{R}^3$. Is it possible for T to be one-to-one? Onto?

3.4 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$, and suppose $T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = T \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Is T one-to-one? Onto?