MATH 303 — Linear Transformations

Fall 2024 — Week 6, Session 2

1 Refresher

1.1 Is the following set of vectors orthogonal? Orthonormal?

$$\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}$$

1.2 Let U be an orthogonal matrix, and consider the set of vectors created using the rows of U. Which property does this set of vectors have?

2 Linear Transformations

An $m \times n$ matrix transforms an $n \times 1$ column vector into an $m \times 1$ column vector.

A transformation T is linear if for all scalars k_1, k_2 and for all vectors $\vec{x_1}, \vec{x_2} \in \mathbb{R}^n$, the following is true:

$$T(k_1\vec{x_1} + k_2\vec{x_2}) = k_1 T(\vec{x_1}) + k_2 T(\vec{x_2})$$

If you define T as $T(\vec{x}) = A\vec{x}$ (that is, $\vec{x}$ multiplied by some matrix), then T is linear.

With that said, how can we find A given only examples of T? (Live demo)

2.1 Find A for the following transformation T:

$$T([1,0,0]^T) = [1,2]^T, T([0,1,0]^T) = [9,-3]^T, T([0,0,1]^T) = [1,1]^T$$

2.2 Find A for the following transformation T:

$$T([1,1]^T) = [1,2]^T, T([0,-1]^T) = [3,2]^T$$

3 Properties of Linear Transformations

1. T preserves the zero vector. That is, $T(\vec{0}) = \vec{0}$.
2. T preserves the negative of a vector. That is, $T(-\vec{x}) = -T(\vec{x})$.
3. T preserves linear combinations. That is, $T(a_1\vec{x_1} + a_2\vec{x_2}) = a_1T(\vec{x_1}) + a_2T(\vec{x_2})$.

3.1 Suppose T is defined as follows:

$$T[1,3,1]^T = [4,4,0,-2]^T, T[4,0,5]^T = [4,5,-1,5]^T$$

Find $T[-7,3,-9]^T$.

3.1 Compositions of Transformations

Performing two transformations back-to-back is called a composition of transformations:

$$(S \circ T)(\vec{x}) = S(T(\vec{x}))$$

If A is the matrix for T and B is the matrix for S, then the composed transformation $S \circ T$ has matrix BA.

3.2 Inverses of Transformations

Similarly, if a transformation S serves to reverse the effect of transformation T such that $(S \circ T)(\vec{x}) = (T \circ S)(\vec{x}) = \vec{x}$, then the transformations are *inverses of each other*. Since the transformations are defined by multiplying a matrix by vectors, this is the exact same as matrix inverses!

4 Closing

4.1 Let $T : \mathbb{R}^2 \to \mathbb{R}^{\nvDash}$ be a linear transformation induced by the matrix $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$. How would we find T's inverse? What would T^{-1}'s matrix be?

4.2 Suppose $T[1,0]^T = [2,3]^T$ and $T[0,1]^T = [3,-1]^T$. What is the matrix associated with T?