AES 561 Homework 4

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1 Problem 7.4

k_e	3.89×10^2	1002.8	0.45	149.9
N	6.57×10^21	9.97×10^9	80	10^9
A	2.8×10^{-19}	7.07×10^{-14}	3×10^{-6}	3.14×10^{-10}
Q_e	1.01×10^{-4}	0.2	0.6	2
$\tilde{\omega}$	0	0.1	0.399	0.9
m	7.3×10^{-26}	1.41×10^{-17}	4.2×10^{-5}	4.19×10^{-12}
ρ	4.8×10^{-4}	1.41×10^{-7}	3.35×10^{-4}	4.19×10^{-3}
σ_e	2.84×10^{-23}	1.414×10^{-14}	1.89×10^{-6}	6.28×10^{-10}
β_e	0.1867	1.41×10^{-4}	1.51×10^{-4}	0.628
β_s	0.1867	1.41×10^{-5}	6.03×10^{-5}	0.565

2 Problem 7.6

Given water density $\rho_w = 1 \times 10^{-4} kg\, m^{-3}$, cloud mass extinction coefficient $k_{ew} = 150 m^2\, kg^{-1}$, and cloud single-scatter albedo $\tilde{\omega}_w = 1.0$, the extinction, scattering, and absorption coefficients for the cloud water are calculated as follows:

$$\begin{aligned}
\beta_{ew} &= \rho_w k_{ew} = 1 \times 10^{-4} \frac{kg}{m^3} \cdot 150 \frac{m^2}{kg} = 0.015 m^{-1} \\
\beta_{sw} &= \beta_{ew} \tilde{\omega}_w = 0.015 m^{-1} \cdot 1.0 = 0.015 m^{-1} \\
\beta_{aw} &= \beta_{ew} - \beta_{sw} = 0.015 m^{-1} - 0.015 m^{-1} = 0
\end{aligned}$$

Now for the atmospheric medium, given atmospheric absorption coefficient $\beta_{av} = 0.01 m^{-1}$ and single-scatter albedo $\tilde{\omega}_v = 0$, we can conclude that the scattering coefficient $\beta_{sw} = 0$ since $\tilde{\omega}_v = \frac{\beta_s}{\beta_a + \beta_s} = 0$. Furthermore for the extinction coefficient, $\beta_{ev} = \beta_{av} + \beta_{sv} = .01 m^{-1} + 0 = .01 m^{-1}$.

The coefficients of the cloud water mixed with with the atmospheric medium are just the sum of each component's respective coefficients. With the subscript 'm' denoting the mixture, we find that...

$$\begin{aligned}
\beta_{sm} &= \beta_{sv} + \beta_{sw} = 0 + 0.015 m^{-1} \\
\beta_{am} &= \beta_{av} + \beta_{aw} = 0.01 m^{-1} + 0 \\
\beta_{em} &= \beta_{ev} + \beta_{ew} = 0.01 m^{-1} + 0.015 m^{-1} = 0.025 m^{-1} \\
\tilde{\omega}_m &= \beta_{sm} \beta_{em}^{-1} = \frac{0.015 m^{-1}}{0.025 m^{-1}} = 0.6
\end{aligned}$$

Assuming β_{em} is constant throughout the layer, the optical thickness and transmittance through the cloud of light incident at $\theta = 60°$ are found such that...

$$\tau = \int_{z_0}^{z_f} \beta_{em} dz = \beta_{em} \Delta z = 0.025 m^{-1}\, 100 m = 2.5$$

$$t = e^{-\tau \mu^{-1}} = e^{-\frac{\tau}{|\cos(\theta)|}} = e^{-\frac{2.5}{0.5}} \approx 6.74 \times 10^{-4}$$

Thus $I_{\lambda,bot}$, the intensity transmitted from $I_{\lambda,top}$ at a zenith angle $\theta = 60°$, is...

$$I_{\lambda,bot} = .000674\, I_{\lambda,top}$$

3 Problem 7.7

To find the mass and quantity of oxygen molecules per unit area in the atmospheric column, we are given pressure $P = 1.01 \times 10^5 Pa$, diatomic oxygen's molar mass $M_{mol} = 0.029\, kg\, mol^{-1}$, oxygen molar fraction $f_{O_2} = 0.21$, and gravitational constant $g = 9.81\, m\, s^{-1}$. With the hydrostatic assumption, the mass of the unit atmospheric column M_c is...

$$M_c = \frac{P}{g} = \frac{1.01 \times 10^5 Pa}{9.81\, ms^{-2}} = 10{,}295.62\, \frac{kg}{m^2}$$

And with Avogadro's number $\mathcal{N}$, the number of particles per unit area N_c is...

$$N_c = f_{O_2} \mathcal{N} \frac{M_c}{M_{mol}} = 0.21 \cdot \frac{10{,}295.62\, kg}{0.029\, kg\, mol^{-1}} = 4.488 \times 10^{28}\, m^{-2}$$

With absorption cross-section $\sigma_a = 7 \times 10^{-29} m^2$, the absorption coefficient β_a of the medium is shown to be...

$$\beta_a = \sigma_a N_c = 7 \times 10^{-29} m^2 \cdot 4.488 \times 10^{28} = 3.142$$

In order to solve the integral for optical depth, I contrive an undetermined function $N(z)$ that models the total number of particles encountered up to an altitude z along a vertical path. This function can be substituted as the vertical coordinate since it is monotonically related to z. The derivative $\frac{dN(z)}{dz}$ describes the number of particles encountered per increment altitude.

At the top of the atmosphere ($z = \infty$), $N(\infty) = N_c$, and at ground level ($z = 0$), $N(0) = 0$. With this function, $z_0 = 0$, and $z_f = \infty$, we can show that the optical depth $\tau = \sigma_a N_c = \beta_a$...

$$\tau(0, \infty) = \int_{z_0}^{z_f} \beta_a dz = \int_0^{\infty} \sigma_a \frac{dN(z)}{dz} dz = \sigma_a \int_0^{N_c} dN(z) = \sigma_a (N_c - 0) = \sigma_a N_c = \beta_a$$

So the optical depth of the full atmospheric column $\tau = \beta_a = 3.142$, and the vertical transmittance of the column is...

$$t(0, \infty) = e^{-\tau} = e^{-3.142} = .043$$