Metodo Diferencia progresiva y regresiva

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$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} + u \qquad 0 \leq x \leq 1$$

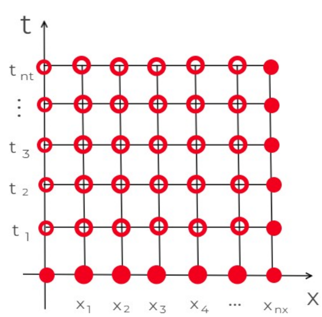
$$\frac{\partial u}{\partial x}(0,t) = 2t \qquad u(1,t) = \frac{t^2}{2} \qquad t > 0$$

$$u(x,0) = sin(x) + cos(x)$$

Limitamos $0 \leq t \leq Tmax$ Discretizamos los intervalos mediante una serie de nodos equiespaciados:

$$nx = ih, \quad i = 0, 1, 2, ..., nx$$
$$nt = jk \quad j = 0, 1, 2, ..., nt$$

donde $\quad h = \frac{1-0}{nx}, \quad k = \frac{Tmax}{nt}$. Generando asi la discretizacion.



1 Método Explícito

Utilizamos formulas de aproximacion en diferencias para u_{xx}, u_t y u_x.

$$u_t = \frac{u_{i,j+1} - u_{i,j}}{k}$$
$$u_x = \frac{u_{i+1,j} - u_{i,j}}{h}$$
$$u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

remplazando en la ecuación original tenemos:

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} - \frac{u_{i+1,j} - u_{i,j}}{h} + u_{i,j}$$

despejamos la variable con el tiempo mayor

$$u_{i,j+1} = \frac{k}{h^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) - \frac{k}{h}(u_{i+1,j} - u_{i,j}) + (1+k)u_{i,j}$$
$$u_{i,j+1} = (\frac{k}{h^2} - \frac{k}{h})u_{i+1,j} + (1 + k + \frac{k}{h} - 2\frac{k}{h^2})u_{i,j} + \frac{k}{h^2}u_{i-1,j}$$

para $i = 0$ tenemos

$$u_{0,j+1} = (\frac{k}{h^2} - \frac{k}{h})u_{1,j} + (1 + k + \frac{k}{h} - 2\frac{k}{h^2})u_{0,j} + \frac{k}{h^2}u_{-1,j} \quad (*)$$
$$\forall j = 1, 2, ..., nt - 1$$

para $u_{-1,j}$ hacemos diferencia regresiva para u_x

$$u_x(0,t) = \frac{u_{0,j} - u_{-1,j}}{h} = 2t_j$$
$$u_{-1,j} = u_{0,j} - 2ht_j$$

remplpazando $u_{-1,j}$ en $(*)$ obtenemos la siguiente ecuacion para $i = 0$

$$u_{0,j+1} = (\frac{k}{h^2} - \frac{k}{h})u_{1,j} + (1 + k + \frac{k}{h} - 2\frac{k}{h^2})u_{0,j} + \frac{k}{h^2}(u_{0,j} - 2ht_j)$$

$$u_{0,j+1} = (\frac{k}{h^2} - \frac{k}{h})u_{1,j} + (1 + k + \frac{k}{h} - \frac{k}{h^2})u_{0,j} - \frac{k}{h^2}2ht_j$$

para $i = nx$ **tenemos**

$$u_{nx,j+1} = \frac{t^2}{2} \qquad \forall j = 0, 1, 2, ..., nt - 1$$

para todo $j = 0, 1, 2, ..., nt - 1$ tenemos

$$\begin{cases} u_{0,j+1} = (\frac{k}{h^2} - \frac{k}{h})u_{1,j} + (1 + k + \frac{k}{h} - \frac{k}{h^2})u_{0,j} - \frac{k}{h^2}2ht_j \\ u_{i,j+1} = (\frac{k}{h^2} - \frac{k}{h})u_{i+1,j} + (1 + k + \frac{k}{h} - 2\frac{k}{h^2})u_{i,j} + \frac{k}{h^2}u_{i-1,j} \\ u_{nx,j+1} = \frac{t_{j+1}^2}{2} \end{cases}$$

i,

Por condicion inicial tenemos

$$u(x, 0) = sin(x) + cos(x) = f(x)$$

$$u_{i,0} = f(ih) \quad \forall i = 0, 1, 2, ..., nx$$

para $j = 0$

$$\begin{aligned} u_{0,1} &= (\frac{k}{h^2} - \frac{k}{h})u_{1,0} + (1 + k + \frac{k}{h} - \frac{k}{h^2})u_{0,0} - \frac{k}{h^2}2ht_0 \\ u_{1,1} &= (\frac{k}{h^2} - \frac{k}{h})u_{2,0} + (1 + k + \frac{k}{h} - 2\frac{k}{h^2})u_{1,0} + \frac{k}{h^2}u_{0,0} \\ u_{2,1} &= (\frac{k}{h^2} - \frac{k}{h})u_{3,0} + (1 + k + \frac{k}{h} - 2\frac{k}{h^2})u_{2,0} + \frac{k}{h^2}u_{1,0} \\ &\vdots \\ u_{nx-1,1} &= (\frac{k}{h^2} - \frac{k}{h})u_{nx,0} + (1 + k + \frac{k}{h} - 2\frac{k}{h^2})u_{nx-1,0} + \frac{k}{h^2}u_{nx-2,0} \end{aligned}$$

la forma matricial seria:

$$\lambda = \frac{k}{h^2}$$

$$\begin{pmatrix} u_{0,1} \\ u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ \vdots \\ u_{nx-1,1} \end{pmatrix} = A \begin{pmatrix} u_{0,0} \\ u_{1,0} \\ u_{2,0} \\ u_{3,0} \\ \vdots \\ u_{nx-1,0} \end{pmatrix} + \begin{pmatrix} -\lambda 2ht_0 \\ 0 \\ 0 \\ \vdots \\ (\frac{k}{h^2} - \frac{k}{h})u_{nx,0} \end{pmatrix} \tag{1}$$

$$A = \begin{pmatrix} 1+k+\frac{k}{h}-\lambda & \lambda - \frac{k}{h} & 0 & 0 & \cdots & 0 & 0 \\ \lambda & 1+k+\frac{k}{h}-2\lambda & \lambda - \frac{k}{h} & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1+k+\frac{k}{h}-2\lambda & \lambda - \frac{k}{h} & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda & 1+k+\frac{k}{h}-2\lambda \end{pmatrix}$$

la forma matricial general:

$$\forall j = 0, 1, 2, ..., nt-1$$

$$\begin{pmatrix} u_{0,j+1} \\ u_{1,j+1} \\ u_{2,j+1} \\ \vdots \\ u_{nx-1,j+1} \end{pmatrix} = A \begin{pmatrix} u_{0,j} \\ u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{nx-1,j} \end{pmatrix} + \begin{pmatrix} -\lambda 2ht_j \\ 0 \\ 0 \\ \vdots \\ (\frac{k}{h^2} - \frac{k}{h})u_{nx,j} \end{pmatrix} \tag{2}$$

2 Método Implicito

Utilizamos formulas de aproximacion en diferencias para u_{xx}, u_t y u_x.

$$u_t = \frac{u_{i,j} - u_{i,j-1}}{k}$$
$$u_x = \frac{u_{i+1,j} - u_{i-1,j}}{2h}$$
$$u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

remplazando en la ecuación original tenemos:

$$\frac{u_{i,j} - u_{i,j-1}}{k} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} - \frac{u_{i+1,j} - u_{i-1,j}}{2h} + u_{i,j}$$

despejamos la variable con el tiempo menor

$$u_{i,j} - u_{i,j-1} = \frac{k}{h^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) - \frac{k}{2h}(u_{i+1,j} - u_{i-1,j}) + ku_{i,j}$$

$$-(\frac{k}{2h} + \frac{k}{h^2})u_{i-1,j} + (1 - k + 2\frac{k}{h^2})u_{i,j} + (\frac{k}{2h} - \frac{k}{h^2})u_{i+1,j} = u_{i,j-1}$$

Para $i = 0$ y $\lambda = k/h^2$ tenemos

$$-(\frac{k}{2h} + \lambda)u_{-1,j} + (1 - k + 2\lambda)u_{0,j} + (\frac{k}{2h} - \lambda)u_{1,j} = u_{0,j-1} \quad (*)$$
$$\forall j = 1, 2, ..., nt$$

para $u_{-1,j}$ hacemos diferencia regresiva en u_x

$$u_x(0,t) = \frac{u_{1,j} - u_{-1,j}}{2h} = 2t_j$$
$$u_{-1,j} = u_{1,j} - 4ht_j$$

remplazando $u_{-1,j}$ en $(*)$ obtenemos la siguiente ecuacion para $i = 0$

$$-(\frac{k}{2h} + \lambda)(u_{1,j} - 4ht_j) + (1 - k + 2\lambda)u_{0,j} + (\frac{k}{2h} - \lambda)u_{1,j} = u_{0,j-1}$$

$$(\frac{k}{2h}+\lambda)4ht_j+(1-k+2\lambda)u_{0,j}-2\lambda u_{1,j}=u_{0,j-1}$$

para todo $j=1,2,...,nt$ tenemos

$$\begin{cases} (\frac{k}{2h}+\lambda)4ht_j+(1-k+2\lambda)u_{0,j}-2\lambda u_{1,j}=u_{0,j-1} \\ -(\frac{k}{2h}+\lambda)u_{i-1,j}+(1-k+2\lambda)u_{i,j}+(\frac{k}{2h}-\lambda)u_{i+1,j}=u_{i,j-1} \\ u_{nx,j}=\frac{t_j^2}{2} \end{cases}$$

Por condicion inicial tenemos

$$u(x,0)=sin(x)+cos(x)=f(x)$$

$$u_{i,0}=f(ih) \quad \forall i=0,1,2,...,nx$$

para $j=1$

$$(\frac{k}{2h}+\lambda)4ht_1+(1-k+2\lambda)u_{0,1}-2\lambda u_{1,1}=u_{0,0}$$

$$-(\frac{k}{2h}+\lambda)u_{0,1}+(1-k+2\lambda)u_{1,1}+(\frac{k}{2h}-\lambda)u_{2,1}=u_{1,0}$$

$$-(\frac{k}{2h}+\lambda)u_{1,1}+(1-k+2\lambda)u_{2,1}+(\frac{k}{2h}-\lambda)u_{3,1}=u_{2,0}$$

$$\vdots$$

$$-(\frac{k}{2h}+\lambda)u_{nx-2,1}+(1-k+2\lambda)u_{nx-1,1}+(\frac{k}{2h}-\lambda)u_{nx,1}=u_{nx-1,0}$$

El sistema de arriba puede escribirse en forma de matriz

$$A\begin{pmatrix} u_{0,1} \\ u_{1,1} \\ u_{2,1} \\ \vdots \\ u_{nx-1,1} \end{pmatrix}=\begin{pmatrix} u_{0,0} \\ u_{1,0} \\ u_{2,0} \\ \vdots \\ u_{nx-1,0} \end{pmatrix}+\begin{pmatrix} -(\frac{k}{2h}+\lambda)4ht_j \\ 0 \\ 0 \\ \vdots \\ -(\frac{k}{2h}-\lambda)u_{nx,1} \end{pmatrix} \tag{3}$$

$$A = \begin{pmatrix} (1-k+2\lambda) & -2\lambda) & 0 & 0 & \cdots & 0 & 0 \\ -(\frac{k}{2h}+\lambda) & (1-k+2\lambda) & (\frac{k}{2h}-\lambda) & 0 & \cdots & 0 & 0 \\ 0 & -(\frac{k}{2h}+\lambda) & (1-k+2\lambda) & (\frac{k}{2h}-\lambda) & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -(\frac{k}{2h}+\lambda) & (1-k+2\lambda) \end{pmatrix}$$

3 Metodo de Crank-nicholson

Método de Diferencia Progresiva:

$$\frac{u_{i,j+1}-u_{i,j}}{k}=\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}-\frac{u_{i+1,j}-u_{i-1,j}}{2h}+u_{i,j}$$

Método de Diferencia Regresiva:

$$\frac{u_{i,j+1}-u_{i,j}}{k}=\frac{u_{i+1,j+1}-2u_{i,j+1}+u_{i-1,j+1}}{h^2}-\frac{u_{i+1,j+1}-u_{i-1,j+1}}{2h}+u_{i,j+1}$$

Método de Diferencia Promedio:

$$\left(\frac{u_{i,j+1}-u_{i,j}}{k}\right)=\frac{1}{2}\left(\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}+\frac{u_{i+1,j+1}-2u_{i,j+1}+u_{i-1,j+1}}{h^2}\right)$$
$$-\frac{1}{2}\left(\frac{u_{i+1,j}-u_{i-,j}}{2h}+\frac{u_{i+1,j+1}-u_{i-1,j+1}}{2h}\right)+\frac{1}{2}(u_{i,j}+u_{i,j+1})$$

$$(u_{i,j+1}-u_{i,j})=\frac{k}{2h^2}(u_{i+1,j}-2u_{i,j}+u_{i-1,j}+u_{i+1,j+1}-2u_{i,j+1}+u_{i-1,j+1})$$
$$-\frac{k}{4h}(u_{i+1,j}-u_{i-1,j}+u_{i+1,j+1}-u_{i-1,j+1})+\frac{k}{2}(u_{i,j}+u_{i,j+1})$$

sea $\lambda = k/h^2$

$$-\left(\frac{\lambda}{2}+\frac{k}{4h}\right)u_{i-1,j+1}+\left(1+\lambda-\frac{k}{2}\right)u_{i,j+1}+\left(\frac{k}{4h}-\frac{\lambda}{2}\right)u_{i+1,j+1}=\left(\frac{\lambda}{2}+\frac{k}{4h}\right)u_{i-1,j}+\left(1+\frac{k}{2}-\lambda\right)u_{i,j}+\left(\frac{\lambda}{2}-\frac{k}{4h}\right)u_{i+1,j}$$

para $i=0$ tenemos:

$$-\left(\frac{\lambda}{2}+\frac{k}{4h}\right)u_{-1,j+1}+\left(1+\lambda-\frac{k}{2}\right)u_{0,j+1}+\left(\frac{k}{4h}-\frac{\lambda}{2}\right)u_{1,j+1}=\left(\frac{\lambda}{2}+\frac{k}{4h}\right)u_{-1,j}+\left(1+\frac{k}{2}-\lambda\right)u_{0,j}+\left(\frac{\lambda}{2}-\frac{k}{4h}\right)u_{1,j} \quad (*)$$

$$sea \quad \frac{u_{1,j}-u_{-1,j}}{2h}=2t_j \quad \rightarrow \quad u_{-1,j}=u_{1,j}-4ht_j$$
$$\frac{u_{1,j+1}-u_{-1,j+1}}{2h}=2t_{j+1} \quad \rightarrow \quad u_{-1,j+1}=u_{1,j+1}-4ht_{j+1}$$

remplazando $u_{-1,j}$ y $u_{-1,j+1}$ en $(*)$

$$-(\frac{\lambda}{2}+\frac{k}{4h})(u_{1,j+1}-4ht_{j+1})+(1+\lambda-\frac{k}{2})u_{0,j+1}+(\frac{k}{4h}-\frac{\lambda}{2})u_{1,j+1}=(\frac{\lambda}{2}+\frac{k}{4h})(u_{1,j}-4ht_j)$$
$$+(1+\frac{k}{2}-\lambda)u_{0,j}+(\frac{\lambda}{2}-\frac{k}{4h})u_{1,j}$$

$$(\frac{\lambda}{2}+\frac{k}{4h})4ht_{j+1}+(1+\lambda-\frac{k}{2})u_{0,j+1}-\lambda u_{1,j+1}=-(\frac{\lambda}{2}+\frac{k}{4h})4ht_j+(1+\frac{k}{2}-\lambda)u_{0,j}+\lambda u_{1,j}$$

Por condicion inicial tenemos

$$u(x,0)=sin(x)+cos(x)=f(x)$$
$$u_{i,0}=f(ih)\quad \forall i=0,1,2,...,nx$$

para todo $j=0,1,2,...,nt-1$ tenemos

$$\left\{\begin{array}{r}(\frac{\lambda}{2}+\frac{k}{4h})4ht_{j+1}+(1+\lambda-\frac{k}{2})u_{0,j+1}-\lambda u_{1,j+1}=-(\frac{\lambda}{2}+\frac{k}{4h})4ht_j+(1+\frac{k}{2}-\lambda)u_{0,j}+\lambda u_{1,j}\\ -(\frac{\lambda}{2}+\frac{k}{4h})u_{i-1,j+1}+(1+\lambda-\frac{k}{2})u_{i,j+1}+(\frac{k}{4h}-\frac{\lambda}{2})u_{i+1,j+1}=(\frac{\lambda}{2}+\frac{k}{4h})u_{i-1,j}\\ +(1+\frac{k}{2}-\lambda)u_{i,j}+(\frac{\lambda}{2}-\frac{k}{4h})u_{i+1,j}\\ u_{nx,j+1}=\frac{t^2}{2}\end{array}\right.$$

El sistema de arriba puede escribirse en forma de matriz

$$A\begin{pmatrix}u_{0,1}\\u_{1,1}\\u_{2,1}\\\vdots\\u_{nx-1,1}\end{pmatrix}=B\begin{pmatrix}u_{0,0}\\u_{1,0}\\u_{2,0}\\\vdots\\u_{nx-1,0}\end{pmatrix}+\begin{pmatrix}-(\frac{\lambda}{2}+\frac{k}{4h})4h(t_j+t_{j+1})\\0\\0\\\vdots\\(\frac{\lambda}{2}-\frac{k}{4h})(u_{i+1,j}+u_{i+1,j+1})\end{pmatrix}\qquad (4)$$

$$A = \begin{pmatrix} (1+\lambda-\frac{k}{2}) & -\lambda & 0 & 0 & \cdots & 0 & 0 \\ -(\frac{\lambda}{2}+\frac{k}{4h}) & (1+\lambda-\frac{k}{2}) & (\frac{k}{4h}-\frac{\lambda}{2}) & 0 & \cdots & 0 & 0 \\ 0 & -(\frac{\lambda}{2}+\frac{k}{4h}) & (1+\lambda-\frac{k}{2}) & (\frac{k}{4h}-\frac{\lambda}{2}) & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -(\frac{\lambda}{2}+\frac{k}{4h}) & (1+\lambda-\frac{k}{2}) \end{pmatrix}$$

$$B = \begin{pmatrix} (1+\frac{k}{2}-\lambda) & \lambda & 0 & 0 & \cdots & 0 & 0 \\ (\frac{\lambda}{2}+\frac{k}{4h}) & (1+\frac{k}{2}-\lambda) & (\frac{\lambda}{2}-\frac{k}{4h}) & 0 & \cdots & 0 & 0 \\ 0 & (\frac{\lambda}{2}+\frac{k}{4h}) & (1+\frac{k}{2}-\lambda) & (\frac{\lambda}{2}-\frac{k}{4h}) & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (\frac{\lambda}{2}+\frac{k}{4h}) & (1+\frac{k}{2}-\lambda) \end{pmatrix}$$