Metodo Diferencia progresiva y regresiva

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$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} + u \qquad 0 \leq x \leq 1$$

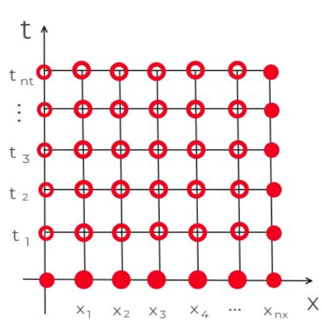
$$\frac{\partial u}{\partial x}(0,t) = 2t \qquad u(1,t) = \frac{t^2}{2} \qquad t > 0$$

$$u(x,0) = sin(x) + cos(x)$$

Limitamos $0 \leq t \leq Tmax$ Discretizamos los intervalos mediante una serie de nodos equiespaciados:

$$nx = ih, \quad i = 0, 1, 2, ..., nx$$
$$nt = jk \quad j = 0, 1, 2, ..., nt$$

donde $\quad h = \frac{1-0}{nx}, \quad k = \frac{Tmax}{nt}$. Generando asi la discretizacion.



1 Método Explícito

Utilizamos formulas de aproximacion en diferencias para u_{xx}, u_t y u_x.

$$u_t = \frac{u_{i,j+1} - u_{i,j}}{k}$$
$$u_x = \frac{u_{i+1,j} - u_{i,j}}{h}$$
$$u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

remplazando en la ecuación original tenemos:

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} - \frac{u_{i+1,j} - u_{i,j}}{h} + u_{i,j}$$

despejamos la variable con el tiempo mayor

$$u_{i,j+1} = \frac{k}{h^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) - \frac{k}{h}(u_{i+1,j} - u_{i,j}) + (1+k)u_{i,j}$$
$$u_{i,j+1} = (\frac{k}{h^2} - \frac{k}{h})u_{i+1,j} + (1 + k + \frac{k}{h} - 2\frac{k}{h^2})u_{i,j} + \frac{k}{h^2}u_{i-1,j}$$

para $i = 0$ tenemos

$$u_{0,j+1} = (\frac{k}{h^2} - \frac{k}{h})u_{1,j} + (1 + k + \frac{k}{h} - 2\frac{k}{h^2})u_{0,j} + \frac{k}{h^2}u_{-1,j} \quad (*)$$
$$\forall j = 1, 2, ..., nt - 1$$

para $u_{-1,j}$ hacemos diferencia regresiva para u_x

$$u_x(0,t) = \frac{u_{0,j} - u_{-1,j}}{h} = 2t_j$$
$$u_{-1,j} = u_{0,j} - 2ht_j$$

remplpazando $u_{-1,j}$ en $(*)$ obtenemos la siguiente ecuacion para $i = 0$

$$u_{0,j+1} = (\frac{k}{h^2} - \frac{k}{h})u_{1,j} + (1 + k + \frac{k}{h} - 2\frac{k}{h^2})u_{0,j} + \frac{k}{h^2}(u_{0,j} - 2ht_j)$$

$$u_{0,j+1} = (\frac{k}{h^2} - \frac{k}{h})u_{1,j} + (1 + k + \frac{k}{h} - \frac{k}{h^2})u_{0,j} - \frac{k}{h^2}2ht_j$$

para $i = nx$ **tenemos**

$$u_{nx,j+1} = \frac{t^2}{2} \qquad \forall j = 0, 1, 2, ..., nt - 1$$

para todo $j = 0, 1, 2, ..., nt - 1$ tenemos

$$\begin{cases} u_{0,j+1} = (\frac{k}{h^2} - \frac{k}{h})u_{1,j} + (1 + k + \frac{k}{h} - \frac{k}{h^2})u_{0,j} - \frac{k}{h^2}2ht_j \\ u_{i,j+1} = (\frac{k}{h^2} - \frac{k}{h})u_{i+1,j} + (1 + k + \frac{k}{h} - 2\frac{k}{h^2})u_{i,j} + \frac{k}{h^2}u_{i-1,j} \\ u_{nx,j+1} = \frac{t^2}{2} \end{cases}$$

Por condicion inicial tenemos

$$u(x, 0) = sin(x) + cos(x) = f(x)$$

$$u_{i,0} = f(ih) \quad \forall i = 0, 1, 2, ..., nx$$

para $j = 0$

$$\begin{aligned} u_{0,1} &= (\frac{k}{h^2} - \frac{k}{h})u_{1,0} + (1 + k + \frac{k}{h} - \frac{k}{h^2})u_{0,0} - \frac{k}{h^2}2ht_0 \\ u_{1,1} &= (\frac{k}{h^2} - \frac{k}{h})u_{2,0} + (1 + k + \frac{k}{h} - 2\frac{k}{h^2})u_{1,0} + \frac{k}{h^2}u_{0,0} \\ u_{2,1} &= (\frac{k}{h^2} - \frac{k}{h})u_{3,0} + (1 + k + \frac{k}{h} - 2\frac{k}{h^2})u_{2,0} + \frac{k}{h^2}u_{1,0} \\ &\vdots \\ u_{nx-1,1} &= (\frac{k}{h^2} - \frac{k}{h})u_{nx,0} + (1 + k + \frac{k}{h} - 2\frac{k}{h^2})u_{nx-1,0} + \frac{k}{h^2}u_{nx-2,0} \\ u_{nx,1} &= \frac{t_0^2}{2} \end{aligned}$$

la forma matricial seria:

$$\lambda = \frac{k}{h^2}$$

$$\begin{pmatrix} u_{0,1} \\ u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ \vdots \\ u_{nx-1,1} \end{pmatrix} = A \begin{pmatrix} u_{0,0} \\ u_{1,0} \\ u_{2,0} \\ u_{3,0} \\ \vdots \\ u_{nx-1,0} \end{pmatrix} + \begin{pmatrix} -\lambda 2ht_0 \\ 0 \\ 0 \\ \vdots \\ (\frac{k}{h^2} - \frac{k}{h})u_{nx,0} \end{pmatrix} \tag{1}$$

$$A = \begin{pmatrix} 1+k+\frac{k}{h}-\lambda & \lambda-\frac{k}{h} & 0 & 0 & \cdots & 0 & 0 \\ \lambda & 1+k+\frac{k}{h}-2\lambda & \lambda-\frac{k}{h} & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1+k+\frac{k}{h}-2\lambda & \lambda-\frac{k}{h} & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda & 1+k+\frac{k}{h}-2\lambda \end{pmatrix}$$

la forma matricial general:

$$\forall j = 0, 1, 2, ..., nt-1$$

$$\begin{pmatrix} u_{0,j+1} \\ u_{1,j+1} \\ u_{2,j+1} \\ \vdots \\ u_{nx-1,j+1} \end{pmatrix} = A \begin{pmatrix} u_{0,j} \\ u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{nx-1,j} \end{pmatrix} + \begin{pmatrix} -\lambda 2ht_j \\ 0 \\ 0 \\ \vdots \\ (\frac{k}{h^2} - \frac{k}{h})u_{nx,j} \end{pmatrix} \tag{2}$$

2 Método Implicito

Utilizamos formulas de aproximacion en diferencias para u_{xx}, u_t y u_x.

$$u_t = \frac{u_{i,j} - u_{i,j-1}}{k}$$
$$u_x = \frac{u_{i,j} - u_{i-1,j}}{h}$$
$$u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

remplazando en la ecuación original tenemos:

$$\frac{u_{i,j} - u_{i,j-1}}{k} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} - \frac{u_{i,j} - u_{i-1,j}}{h} + u_{i,j}$$

despejamos la variable con el tiempo menor

$$u_{i,j} - u_{i,j-1} = \frac{k}{h^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})h^2 - \frac{k}{h}(u_{i,j} - u_{i-1,j}) + ku_{i,j}$$

$$(1 - k + 2\frac{k}{h^2} + \frac{k}{h})u_{i,j} - (\frac{k}{h^2} + \frac{k}{h})u_{i-1,j} - \frac{k}{h^2}u_{i+1,j} = u_{i,j-1}$$

Para $i = 0$ tenemos

$$(1 - k + 2\frac{k}{h^2} + \frac{k}{h})u_{0,j} - (\frac{k}{h^2} + \frac{k}{h})u_{-1,j} - \frac{k}{h^2}u_{1,j} = u_{0,j-1} \quad (*)$$
$$\forall j = 1, 2, ..., nt$$

para $u_{-1,j}$ hacemos diferencia regresiva en u_x

$$u_x(0,t) = \frac{u_{0,j} - u_{-1,j}}{h} = 2t_j$$
$$u_{-1,j} = u_{0,j} - 2ht_j$$

remplazando $u_{-1,j}$ en $(*)$ obtenemos la siguiente ecuacion para $i = 0$

$$(1 - k + 2\frac{k}{h^2} + \frac{k}{h})u_{0,j} - (\frac{k}{h^2} + \frac{k}{h})(u_{0,j} - 2ht_j) - \frac{k}{h^2}u_{1,j} = u_{0,j-1}$$

$$(1 - k + \frac{k}{h^2})u_{0,j} + (\frac{k}{h^2} + \frac{k}{h})2ht_j - \frac{k}{h^2}u_{1,j} = u_{0,j-1}$$

para todo $j = 1, 2, ..., nt$ tenemos

$$\begin{cases} (1 - k + \frac{k}{h^2})u_{0,j} + (\frac{k}{h^2} + \frac{k}{h})2ht_j - \frac{k}{h^2}u_{1,j} = u_{0,j-1} \\ (1 - k + 2\frac{k}{h^2} + \frac{k}{h})u_{i,j} - (\frac{k}{h^2} + \frac{k}{h})u_{i-1,j} - \frac{k}{h^2}u_{i+1,j} = u_{i,j-1} \\ u_{nx,j} = \frac{t_j^2}{2} \end{cases}$$

Por condicion inicial tenemos

$$u(x, 0) = sin(x) + cos(x) = f(x)$$

$$u_{i,0} = f(ih) \quad \forall i = 0, 1, 2, ..., nx$$

para $j = 1$

$$(1 - k + \frac{k}{h^2})u_{0,1} + (\frac{k}{h^2} + \frac{k}{h})2ht_1 - \frac{k}{h^2}u_{1,1} = u_{0,0}$$

$$(1 - k + 2\frac{k}{h^2} + \frac{k}{h})u_{1,1} - (\frac{k}{h^2} + \frac{k}{h})u_{0,1} - \frac{k}{h^2}u_{2,1} = u_{1,0}$$

$$(1 - k + 2\frac{k}{h^2} + \frac{k}{h})u_{2,1} - (\frac{k}{h^2} + \frac{k}{h})u_{1,1} - \frac{k}{h^2}u_{3,1} = u_{2,0}$$

$$\vdots$$

$$(1 - k + 2\frac{k}{h^2} + \frac{k}{h})u_{nx-1,1} - (\frac{k}{h^2} + \frac{k}{h})u_{nx-2,1} - \frac{k}{h^2}u_{nx,1} = u_{nx-1,0}$$

El sistema de arriba puede escribirse en forma de matriz

$$A \begin{pmatrix} u_{0,1} \\ u_{1,1} \\ u_{2,1} \\ \vdots \\ u_{nx-1,1} \end{pmatrix} = \begin{pmatrix} u_{0,0} \\ u_{1,0} \\ u_{2,0} \\ \vdots \\ u_{nx-1,0} \end{pmatrix} + \begin{pmatrix} -(\frac{k}{h^2} + \frac{k}{h})2ht_1 \\ 0 \\ 0 \\ \vdots \\ -\frac{k}{h^2}u_{nx,1} \end{pmatrix} \tag{3}$$

$$A = \begin{pmatrix} 1-k+\frac{k}{h^2} & \frac{k}{h^2}+\frac{k}{h} & 0 & 0 & \cdots & 0 & 0 \\ -(\frac{k}{h^2}+\frac{k}{h}) & 1-k+2\frac{k}{h^2}+\frac{k}{h} & -\frac{k}{h^2} & 0 & \cdots & 0 & 0 \\ 0 & -(\frac{k}{h^2}+\frac{k}{h}) & 1-k+2\frac{k}{h^2}+\frac{k}{h} & -\frac{k}{h^2} & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -(\frac{k}{h^2}+\frac{k}{h}) & 1-k+2\frac{k}{h^2}+\frac{k}{h} \end{pmatrix}$$