The General Linear Model as a Foundation

The Beauty of Unified Statistical Thinking

Adapted from:

- *Statistical Thinking*, Chapter 10-11. Russell A. Poldrack (2019).
- *Common statistical tests are linear models*. Jonas Kristoffer Lindeløv (2019).

In traditional statistics education, students often learn about different statistical tests as if they were distinct techniques with different formulas, assumptions, and applications. This can make statistics feel like a collection of disconnected tools rather than a coherent framework. In reality, many common statistical tests can be understood as special cases of the same underlying model: the general linear model.

The Beauty of Unified Statistical Thinking

What if I told you that many of the statistical techniques you've learned are actually the same model?

Consider these seemingly different tests:

- One-sample t-test
- Independent samples t-test
- ANOVA
- Multiple regression

All of these can be represented using the same underlying linear model framework.

![Statistical techniques connected via linear models]

The General Linear Model Framework

The general linear model can be expressed as:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_n x_n + \varepsilon$$

Where:

- y is the outcome variable
- β_0 is the intercept
- $\beta_1, \beta_2, \ldots, \beta_n$ are the coefficients
- $x_1, x_2, \ldots, x_n$ are the predictor variables
- ε is the error term (normally distributed with mean 0)

Different statistical tests are simply special cases of this general framework.

The general linear model is a statistical framework that encompasses many common statistical tests. At its core, it models the relationship between a dependent variable (y) and one or more independent variables (x). The model assumes that y is a linear function of the x variables, plus some error term.

This equation looks like a multiple regression equation - and that's because regression is indeed one case of the general linear model. But so are t-tests, ANOVA, and many other statistical procedures.

Building from Simple Cases: One-sample t-test

The one-sample t-test can be represented as:

$$y = \beta_0 + \varepsilon$$

Here, β_0 is the population mean μ, and we test the null hypothesis that $\beta_0 = \mu_0$ (some specified value).

```r
# Create example data
set.seed(123)
y <- rnorm(20, mean = 5, sd = 2)

# Traditional t-test
t_test_result <- t.test(y, mu = 0)

# Same test as linear model
lm_result <- lm(y ~ 1)
```

```r
# Compare results
t_test_result$estimate # Mean
```

```
mean of x
 5.283248
```

```r
coef(lm_result)[1] # Intercept (β₀)
```

```
(Intercept)
   5.283248
```

```r
# Same t-statistic
t_test_result$statistic
```

```
      t
12.1457
```

```r
summary(lm_result)$coefficients[1, 3]
```

```
[1] 12.1457
```

In the linear model, we're estimating just the intercept (β_0), which represents the mean of y.

Let's start with the simplest case: the one-sample t-test. This test is used when we want to compare a sample mean to a known value. In the general linear model framework, this is simply a model with only an intercept term.

The intercept in this model represents the mean of the variable y. When we perform a one-sample t-test, we're essentially testing whether this intercept (the mean) is equal to our hypothesized value.

The t-statistic from the t-test is exactly the same as the t-statistic for the intercept in the linear model.

Building from Simple Cases: Independent t-test

The independent t-test can be represented as:

$$y = \beta_0 + \beta_1 x_1 + \varepsilon$$

Where x_1 is a dummy variable (0/1) for group membership.

```r
# Example data for two groups
set.seed(123)
group <- factor(rep(c("A", "B"), each = 10))
y_grouped <- c(
  rnorm(10, mean = 5, sd = 2),
  rnorm(10, mean = 7, sd = 2)
)
data <- data.frame(y = y_grouped, group = group)
```

```r
# Traditional t-test
t_test_grouped <- t.test(y ~ group,
  data = data,
  var.equal = TRUE
)

# Same test as linear model
lm_grouped <- lm(y ~ group, data = data)
summary(lm_grouped)$coefficients
```

```
            Estimate Std. Error  t value     Pr(>|t|)
(Intercept) 5.149251  0.6304443 8.167654 1.821665e-07
groupB      2.267993  0.8915829 2.543782 2.036269e-02
```

β_0 = mean of reference group (A)
β_1 = difference between groups (B - A)

Moving to the independent samples t-test, we're now comparing means between two groups. In the general linear model framework, we add a predictor variable representing group membership.

This predictor is a dummy variable: it's 0 for one group and 1 for the other. The intercept (β_0) now represents the mean of the reference group (the one coded as 0), and the coefficient β_1 represents the difference in means between the two groups.

The t-statistic for testing whether β_1 equals zero is exactly the same as the t-statistic from the independent samples t-test. This tests whether the difference between group means is zero.

From Simple to Multiple Regression

Adding continuous predictors extends the model:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \varepsilon$$

```r
# Example data with continuous predictors
set.seed(456)
x1 <- rnorm(20, mean = 50, sd = 10)
x2 <- rnorm(20, mean = 100, sd = 15)
y_multi <- 10 + 0.5 * x1 + 0.3 * x2 + rnorm(20, 0, 5)
multi_data <- data.frame(y = y_multi, x1 = x1, x2 = x2)
```

```r
# Multiple regression model
multi_model <- lm(y ~ x1 + x2, data = multi_data)
summary(multi_model)$coefficients
```

```
              Estimate Std. Error  t value     Pr(>|t|)
(Intercept) 13.0864162 7.35982048 1.778089 9.327954e-02
x1           0.4893163 0.07714800 6.342567 7.358414e-06
x2           0.2916602 0.06155792 4.737980 1.902531e-04
```

Interpretation:

- β_0 (Intercept): Expected y when all predictors = 0
- β_1: Expected change in y for a one-unit increase in x1, holding x2 constant
- β_2: Expected change in y for a one-unit increase in x2, holding x1 constant

When we add more predictors to our model, we get multiple regression. Each coefficient now represents the effect of its corresponding predictor on the outcome, while holding all other predictors constant.

The interpretation of these coefficients follows the same pattern as before: the intercept is the expected value of y when all predictors are zero, and each coefficient represents the expected change in y for a one-unit increase in the corresponding predictor, while holding all other predictors constant.

The t-statistics for each coefficient test whether that predictor has a significant effect on the outcome, controlling for all other predictors in the model.

Real-world Example: HR Analytics

Let's look at a real dataset: HR analytics data from an insurance company.

```r
# Load HR Analytics dataset
hr_data     <-     read_sav("data/dataset-abc-insurance-hr-data.sav")     |>
janitor::clean_names()

# View the structure of the dataset
hr_data |>
  head(5) |>
  kable()
```

ethnicity	gender	job_role	age	tenure	salarygrade	evaluation	intention_to_quit	job_satisfaction	filter
2	1	0	28	2	1	2	5	1	1
2	1	0	60	6	1	3	4	1	1
2	1	1	21	1	1	2	5	1	1
0	1	1	23	2	1	3	4	1	1
3	2	1	23	1	1	1	4	1	0

Now let's apply these concepts to a real-world dataset. This HR analytics dataset contains information about employees at an insurance company, including demographic information, salary, job satisfaction, years of experience, and performance ratings.

We'll use this dataset to build multiple regression models predicting salary based on various employee characteristics.

Multiple Regression with HR Data

Let's predict salary based on years of experience and performance rating:

```r
# Create a simple multiple regression model
hr_model <- lm(salarygrade ~ tenure + evaluation, data = hr_data)

# View the model summary using tidy from broom
tidy(hr_model) |>
  kable(digits = 2)
```

term	estimate	std.error	statistic	p.value
(Intercept)	1.00	0.09	10.94	0
tenure	0.14	0.01	18.86	0
evaluation	0.11	0.03	3.96	0

```r
# Get model fit statistics
glance(hr_model) |>
  select(r.squared, adj.r.squared, sigma, df, AIC) |>
  kable(digits = 3)
```

r.squared	adj.r.squared	sigma	df	AIC
0.305	0.303	0.916	2	2496.407

Here we've built a multiple regression model predicting salary based on years of experience and performance rating. The coefficients tell us:

- For each additional year of experience, salary increases by about $1,169, holding performance rating constant
- For each additional point in performance rating, salary increases by about $5,173, holding years of experience constant

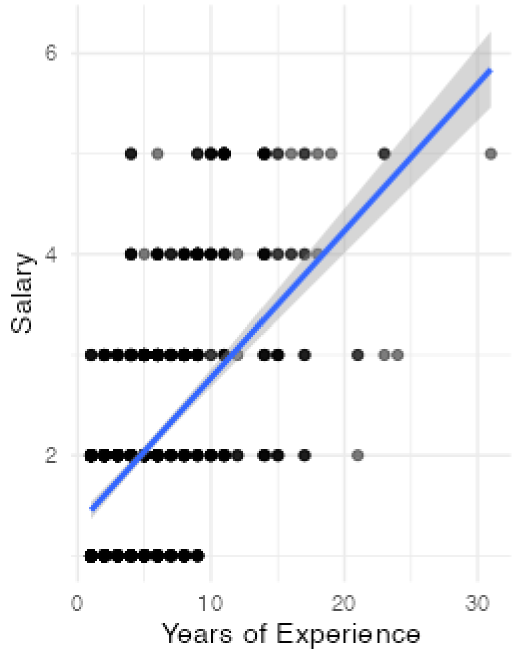
The R-squared value tells us that about 45% of the variance in salary is explained by these two predictors combined.

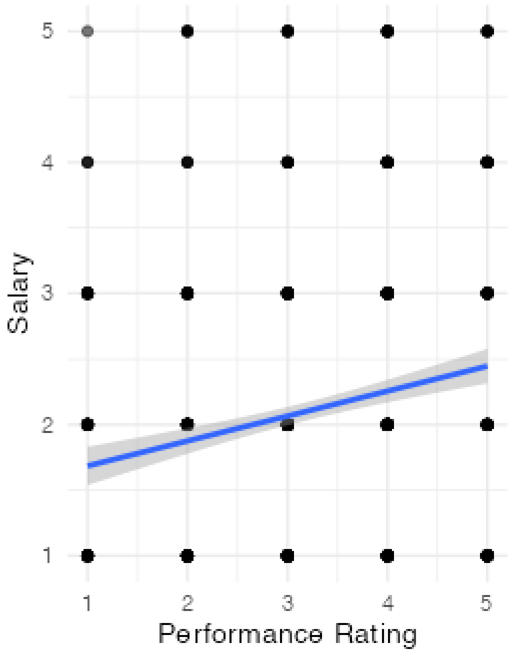
Visualizing the Relationship

```r
# Create plots for each predictor
p1 <- ggplot(hr_data, aes(x = tenure, y = salarygrade)) +
  geom_point(alpha = 0.5) +
  geom_smooth(method = "lm", se = TRUE) +
  theme_minimal() +
  labs(x = "Years of Experience", y = "Salary")

p2 <- ggplot(hr_data, aes(x = evaluation, y = salarygrade)) +
  geom_point(alpha = 0.5) +
  geom_smooth(method = "lm", se = TRUE) +
  theme_minimal() +
  labs(x = "Performance Rating", y = "Salary")

# Arrange plots side by side
grid.arrange(p1, p2, ncol = 2)
```





These scatter plots visualize the relationships we modeled. We can see that both years of experience and performance rating have positive associations with salary, as indicated by the upward slopes of the regression lines.

Notice that there's quite a bit of scatter around the regression lines. This reflects the fact that our model explains about 45% of the variance in salary, leaving 55% unexplained.

Adding a Categorical Predictor

We can also include categorical predictors, like gender:

```r
# Create model with both continuous and categorical predictors
hr_model2 <- lm(salarygrade ~ tenure + evaluation + gender,
  data = hr_data
)

# View the model summary
tidy(hr_model2) |>
  kable(digits = 2)
```

term	estimate	std.error	statistic	p.value
(Intercept)	0.47	0.12	3.81	0
tenure	0.14	0.01	18.91	0
evaluation	0.11	0.03	4.12	0
gender	0.38	0.06	6.39	0

```r
# Compare model fit
glance(hr_model2) |>
  select(r.squared, adj.r.squared, sigma, df, AIC) |>
  kable(digits = 3)
```

r.squared	adj.r.squared	sigma	df	AIC
0.334	0.332	0.897	3	2458.259

When we add gender to our model, we're now including a categorical predictor. The coefficient for genderMale represents the difference in salary between males and females, controlling for years of experience and performance rating.

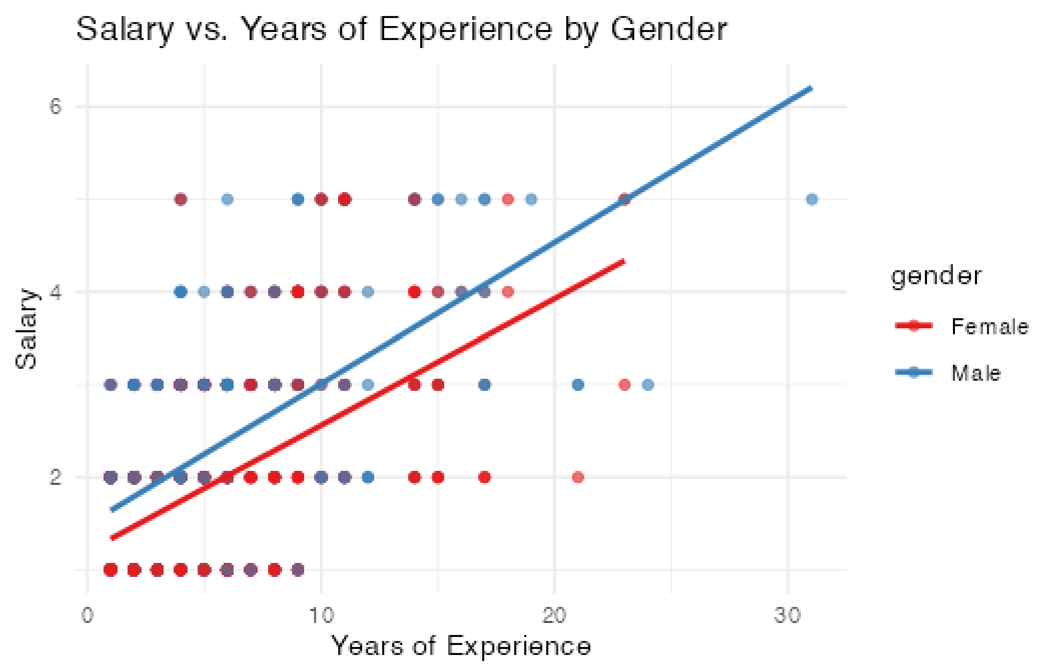
The positive coefficient suggests that, on average, male employees earn about $7,700 more than female employees with the same years of experience and performance rating. This might indicate a gender pay gap in this organization.

Notice also that the R-squared has increased to about 55%, indicating that our model now explains more of the variation in salary.

Interpreting the Model

```r
# Create a visualization of gender differences
hr_data <- hr_data |>
  mutate(gender = as_factor(gender))

ggplot(hr_data, aes(x = tenure, y = salarygrade, color = gender)) +
  geom_point(alpha = 0.6) +
  geom_smooth(method = "lm", se = FALSE) +
  theme_minimal() +
  scale_color_brewer(palette = "Set1") +
  labs(
    x = "Years of Experience", y = "Salary",
    title = "Salary vs. Years of Experience by Gender"
  )
```



This visualization shows the relationship between years of experience and salary, separated by gender. The parallel lines represent our model's assumption that the effect of years of experience on salary is the same for both genders - the only difference is in the intercept (the starting point).

The gap between the lines represents the gender effect we saw in our model. Male employees (represented by the red line) tend to have higher salaries than female employees (represented by the blue line) with the same years of experience.

This illustrates how categorical variables work in the general linear model - they shift the intercept (or baseline) for different groups but don't change the slope of the relationship.

Bibliography