Multiple Linear Regression and ANOVA as Linear Models

Unifying Statistical Approaches

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Last Week

- Correlation
  - → Pearson correlation (r)
  - → Spearman correlation (ρ)
  - → Relationship to linear regression
  - → Limitations
- Correlation vs. causation
- Simple linear regression
  - → Single predictor variable
  - → Sum of squared errors
  - → Confidence intervals
  - → Residuals
  - → Assessing model fit
  - → Outliers and influence

Learning Objectives

- Understand the general linear model framework

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- Recognize how t-tests, ANOVA, and regression are connected

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- Understand the general linear model framework
- Recognize how t-tests, ANOVA, and regression are connected
- Apply linear modeling to analyze multivariate data
- Interpret interaction effects in multifactor designs
- Gain practical experience with an HR datasets

Learning Objectives

Figure 5: Linear models are the foundation of many statistical techniques

Focus: Unified Statistical Thinking

- Moving beyond isolated statistical techniques
- Seeing connections between t-tests, ANOVA, and regression
- Understanding the common mathematical framework
- Simplifying the interpretation of statistical models

This lecture introduces the concept of the general linear model as a unifying framework for various statistical techniques. By understanding this framework, students will gain a deeper appreciation for how different statistical tests are related to each other.

Key points to emphasize:

- The power of seeing statistics through a unified lens
- How this approach simplifies understanding and application
- The practical benefits of this perspective when working with real data

Reviewing Last Week: Correlation and Regression

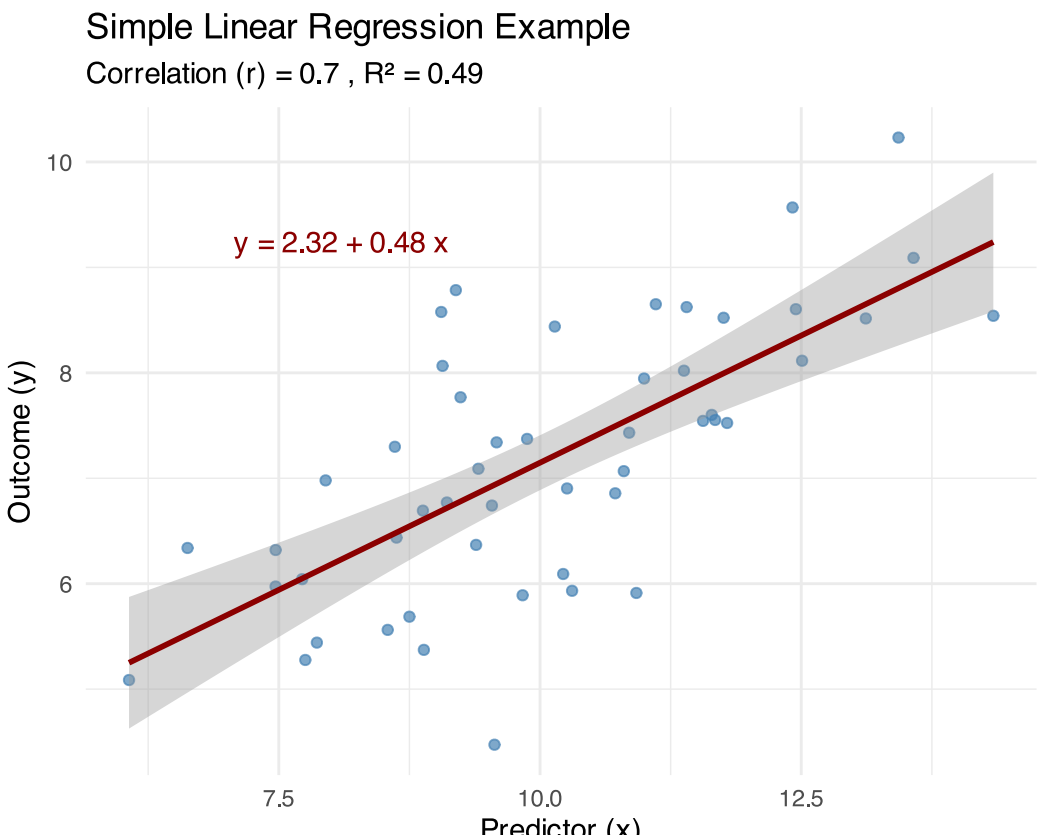
What We Covered Last Week

Last week, we explored the fundamentals of correlation and simple linear regression:

Key Topics:

- Correlation measures (Pearson's r)
- Simple linear regression
- Interpreting slope and intercept
- Assessing model fit (R^2)
- Testing significance of relationships
- Assumptions of linear regression

What We Covered Last Week



Last week we covered two key topics that form the foundation for today's lecture:

What We Covered Last Week

1. **Correlation**:
   - A measure of the strength and direction of the linear relationship between two variables
   - Pearson's r ranges from −1 (perfect negative correlation) to +1 (perfect positive correlation)
   - A correlation of 0 indicates no linear relationship
   - We learned that correlation does not imply causation
2. **Simple Linear Regression**:
   - Moving beyond correlation to model the relationship between variables
   - The regression equation: $y = \beta_0 + \beta_1 x + \varepsilon$
   - β_0 (intercept): The predicted value of y when $x = 0$
   - β_1 (slope): The change in y for a one-unit increase in x
   - We can use regression for prediction and understanding relationships
   - R^2 measures the proportion of variance in y explained by the model

What We Covered Last Week

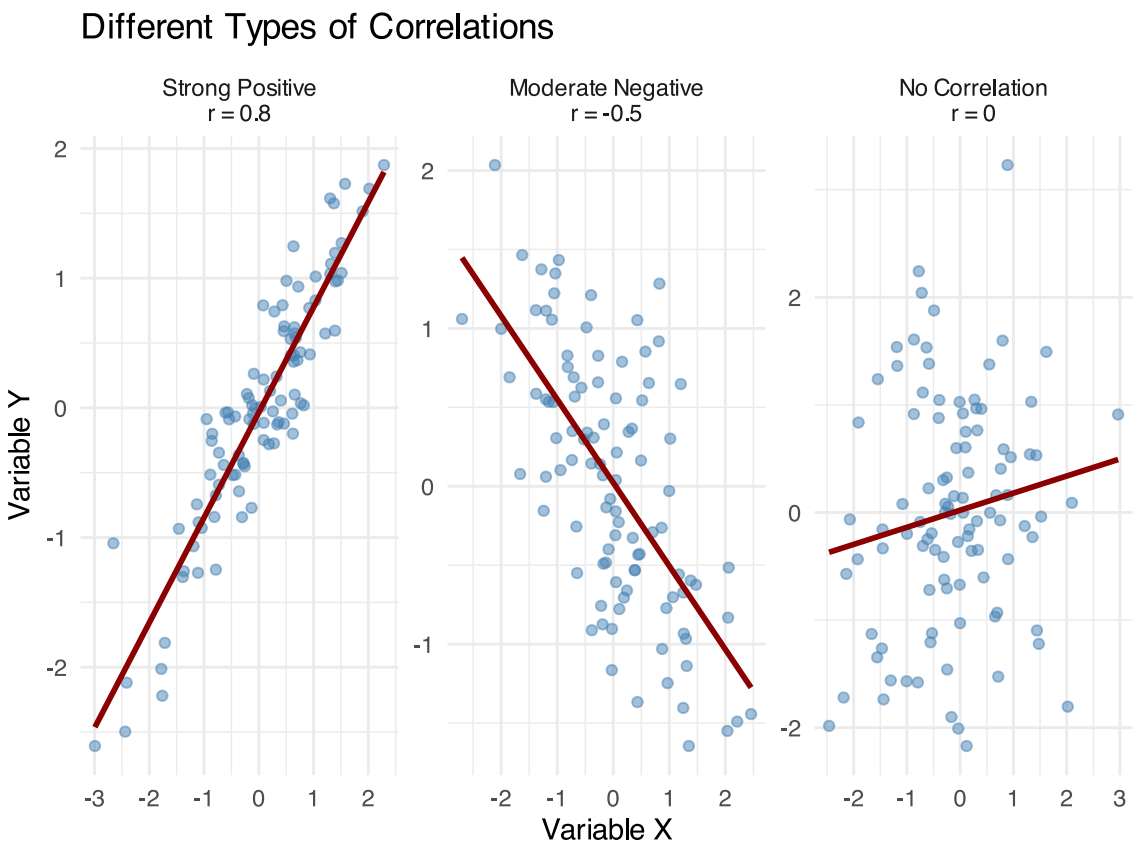
These concepts serve as building blocks for today's topic: the General Linear Model, which extends these ideas to create a unified framework for statistical analysis.

Correlation: Measuring Relationships

Pearson's Correlation Coefficient (r):

- Measures the strength and direction of a linear relationship
- Ranges from −1 (perfect negative) to +1 (perfect positive)
- Calculated using standardized variables
- Formula: $r = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum(x_i - \bar{x})^2 \sum(y_i - \bar{y})^2}}$
- **Interpretation**: r = 0.7 means a strong positive relationship

Correlation: Measuring Relationships



Correlation is a standardized measure of how two variables change together.

Correlation: Measuring Relationships

Key points about correlation:

1. Correlation measures both the strength and direction of a linear relationship
2. The correlation coefficient (r) is always between −1 and +1
3. The sign indicates direction (positive or negative relationship)
4. The magnitude indicates strength (closer to 1 or −1 = stronger relationship)
5. A correlation of 0 suggests no linear relationship

Interpretation guidelines:

- $|r| < 0.3$: Weak correlation
- $0.3 < |r| < 0.7$: Moderate correlation
- $|r| > 0.7$: Strong correlation

Important limitations:

- Correlation does not imply causation

Correlation: Measuring Relationships

- Correlation only detects linear relationships
- Correlation is sensitive to outliers
- Correlation doesn't tell us the slope of the relationship

These limitations are why we often move from correlation to regression, which provides more information about the relationship between variables.

Simple Linear Regression: Modeling Relationships

The Simple Linear Regression Model:

$$y = \beta_0 + \beta_1 x + \varepsilon$$

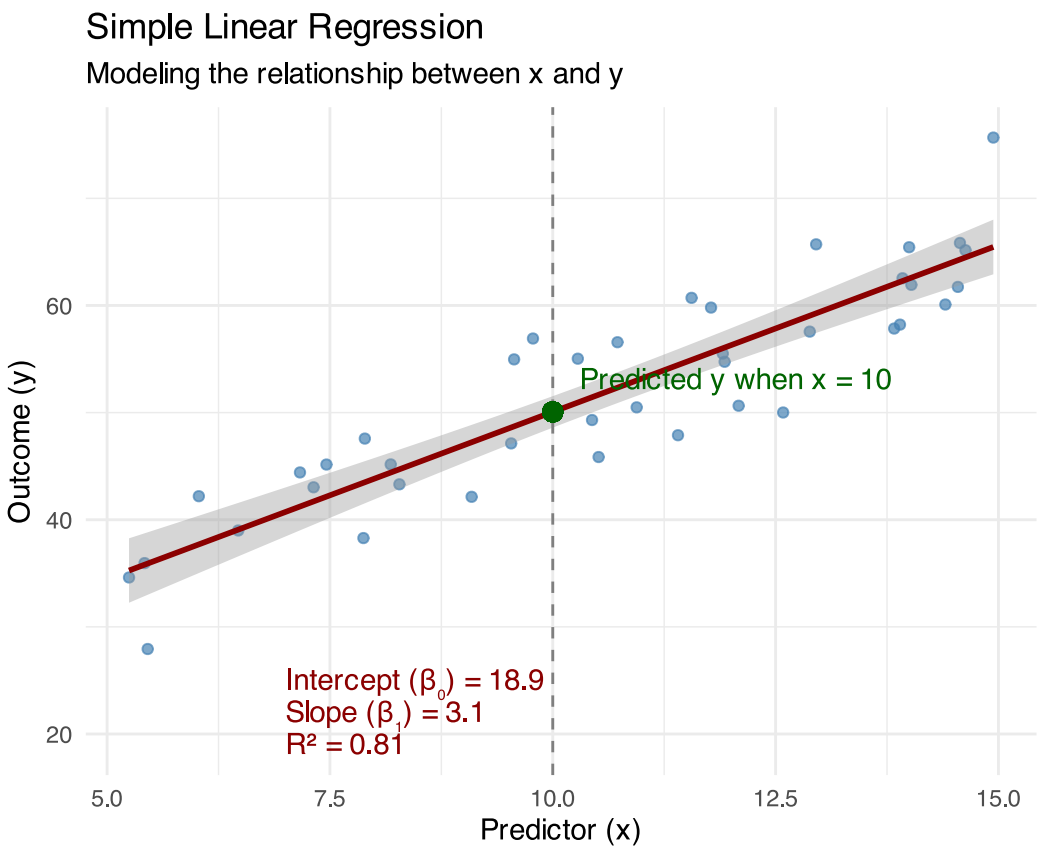
Where:

- β_0 is the intercept (y when x = 0)
- β_1 is the slope (change in y per unit of x)
- ε is the error term

Key statistics:

- R^2 (coefficient of determination): Proportion of variance explaine
- p-value: Tests if the relationship is statistically significant

Simple Linear Regression: Modeling Relationships



Simple Linear Regression: Modeling Relationships

Simple linear regression extends correlation by modeling the relationship between variables. While correlation tells us about the strength and direction of a relationship, regression gives us an equation to predict one variable from another.

Components of the regression model:

1. **Intercept (β_0)**: The predicted value of y when x = 0
   - May not always be meaningful in real-world contexts
   - Example: If x = years of experience, β_0 = starting salary with zero experience
2. **Slope (β_1)**: The change in y for a one-unit increase in x
   - The practical effect size of the relationship
   - Example: Each additional year of experience increases salary by $3,000
3. **Error term (ε)**: The difference between predicted and actual values
   - Represents what our model doesn't explain
   - Assumed to be normally distributed with mean zero

Simple Linear Regression: Modeling Relationships

Evaluating the model:

- **R^2**: The proportion of variance in y explained by the model
  - → Ranges from 0 to 1 (sometimes expressed as a percentage)
  - → Example: $R^2 = 0.75$ means the model explains 75% of the variation in y
- **Statistical significance**: Testing whether β_1 is significantly different from zero
  - → If significant, we have evidence of a relationship between x and y
  - → Reported as a p-value (e.g., $p < 0.05$)

Regression is a powerful tool that forms the foundation for today's topic: the General Linear Model, which extends these concepts to more complex situations.

Connecting to Today's Topic: The General Linear Model

Today, we'll build on these concepts to explore the **General Linear Model (GLM)**, which:

- Extends regression to include multiple predictors
- Provides a unified framework for various statistical tests
- Shows how t-tests, ANOVA, and regression are related
- Allows us to model complex relationships
- Helps us understand which factors truly matter when controlling for others

Moving from:

$$y = \beta_0 + \beta_1 x + \varepsilon$$

To:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_n x_n + \varepsilon$$

Connecting to Today's Topic: The General Linear Model

Today's lecture builds directly on the foundation we established last week with correlation and simple regression. We're now ready to take the next step by exploring the General Linear Model (GLM).

The progression in our learning:

1. **Correlation**: We started by measuring the strength and direction of relationships between pairs of variables.
2. **Simple Linear Regression**: We then moved to modeling these relationships with an equation that allows prediction and deeper understanding of how one variable affects another.
3. **General Linear Model**: Today, we'll extend this framework to include multiple predictors and show how this unifies many statistical tests under one conceptual umbrella.

Connecting to Today's Topic: The General Linear Model

Key extensions in the GLM:

- **Multiple predictors**: Real-world outcomes are rarely influenced by just one factor. The GLM allows us to include multiple predictors to better model complex phenomena.
- **Categorical predictors**: We'll see how to include categorical variables (like gender, treatment group, etc.) in our models.
- **Controlling for variables**: The GLM allows us to understand the unique effect of each predictor while controlling for other factors.
- **Unified framework**: Perhaps most importantly, we'll discover how many statistical tests you've already learned (t-tests, ANOVA, etc.) are actually special cases of the GLM.

Understanding the GLM will not only simplify your conceptual understanding of statistics but also give you a more powerful and flexible approach to data analysis.

Key Terms to Remember

As we move forward, keep these key terms in mind:

From Correlation & Regression:

- **Correlation coefficient (r)**: Measures strength and direction of relationship
- **Intercept (β_0)**: Value of y when x = 0
- **Slope (β_1)**: Change in y per unit change in x
- **R^2**: Proportion of variance explained
- **Residuals**: Differences between observed and predicted values

New Terms for Today:

- **Multiple regression**: Model with multiple predictors
- **General Linear Model (GLM)**: Unified framework for statistical tests
- **Predictor variables**: Factors that may explain the outcome
- **Categorical predictors**: Non-numeric variables (e.g., gender)

Key Terms to Remember

- **Controlling for variables**: Isolating the effect of one predictor

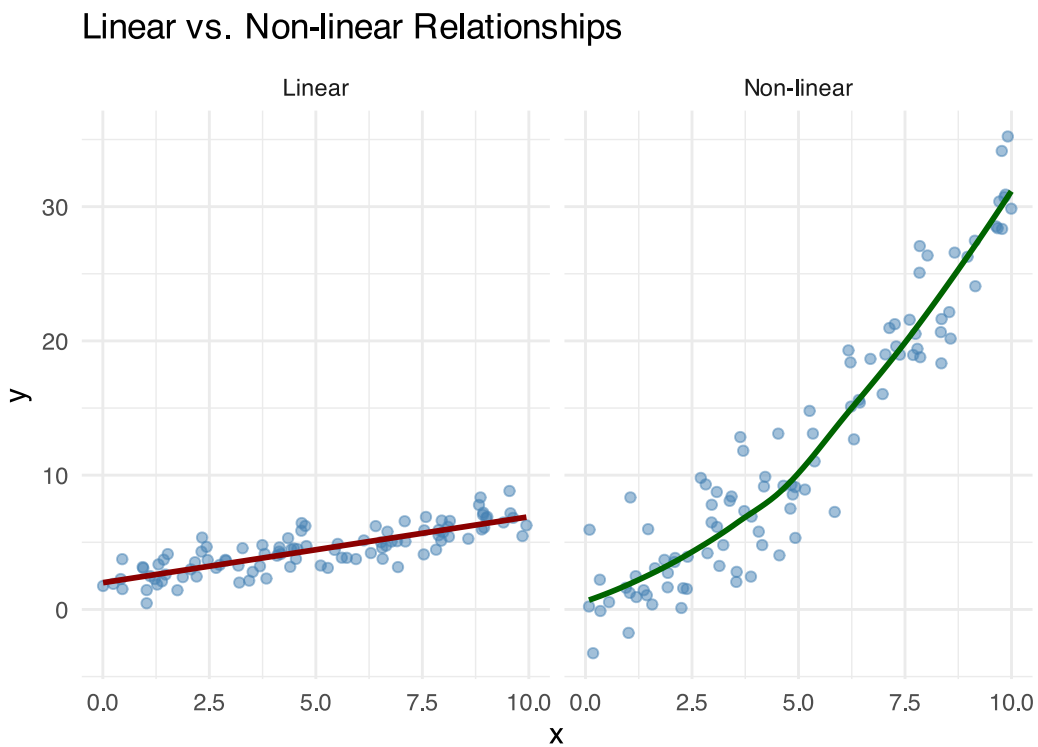
Any Questions Before We Begin?

Let's briefly address any questions about last week's material before moving forward.

Common Questions:

- How do we interpret the slope and intercept in practical terms?
- What's the difference between correlation and causation?
- When should we use correlation vs. regression?
- How do we know if our regression model is good?
- What if the relationship isn't linear?

Any Questions Before We Begin?



Before we move on to new material, let's address some common questions about correlation and regression.

Any Questions Before We Begin?

How do we interpret the slope and intercept in practical terms?

- The intercept (β_0) is the expected value of y when x = 0. In practice, this may not always be meaningful if x = 0 is outside our observed range.
- The slope (β_1) tells us how much y changes for a one-unit increase in x. This is often the most useful part for practical interpretation.
- Example: If predicting salary from years of experience with $\beta_1 = 3000$, each additional year of experience is associated with a \$3,000 increase in salary.

What's the difference between correlation and causation?

- Correlation simply identifies that two variables change together in a predictable way
- Causation means that changes in one variable directly cause changes in another
- To establish causation, we typically need controlled experiments or strong causal inference methods

Any Questions Before We Begin?

- The classic example: Ice cream sales and drowning deaths are correlated (both increase in summer), but one doesn't cause the other

When should we use correlation vs. regression?

- Use correlation when you simply want to measure the strength and direction of a relationship
- Use regression when you want to:
  - → Predict one variable from another
  - → Understand the effect size (how much y changes when x changes)
  - → Control for other variables (in multiple regression)

How do we know if our regression model is good?

- R^2 tells us the proportion of variance explained (higher is better)
- Statistical significance (p-value) tells us if the relationship is likely real or due to chance

Any Questions Before We Begin?

- Examining residuals helps identify patterns the model missed
- Checking model assumptions confirms our statistical inferences are valid

What if the relationship isn't linear?

- Both correlation and simple linear regression assume a linear relationship
- Non-linear relationships may be missed or underestimated by these methods
- Solutions include:
  - → Transforming variables (e.g., log transformation)
  - → Using non-linear regression models
  - → Using more flexible modeling approaches

These concepts provide the foundation for today's topic: the General Linear Model, which extends regression to more complex situations while maintaining a unified framework.

The General Linear Model: Multiple Variables

From Simple to Multiple Regression

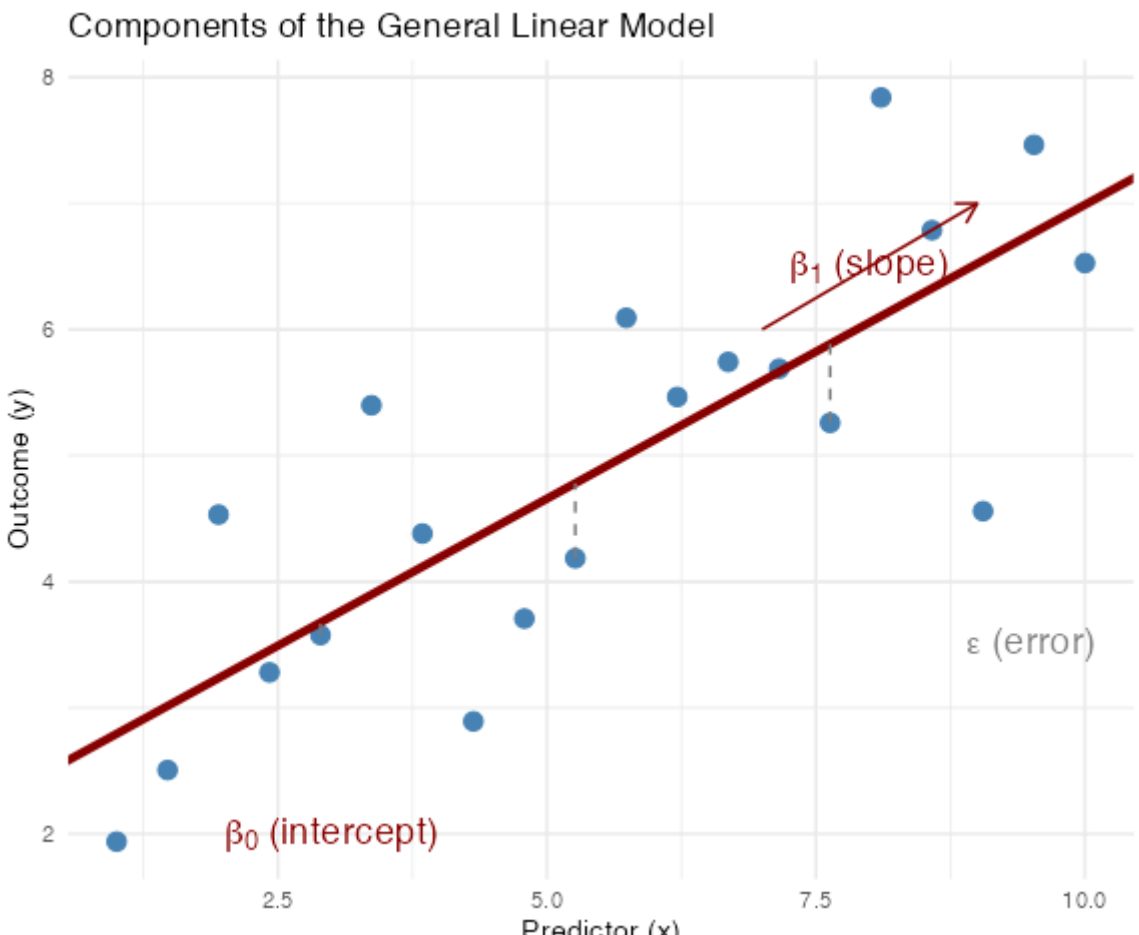
Before wrapping up our discussion of statistical tests, let's first build up our understanding of regression from simple to multiple predictor variables.

Understanding the Building Blocks

The General Linear Model has two key components:

1. **Variables**:
   - **Outcome (y)**: What we're trying to understand
   - **Predictors (x)**: Factors that might explain the outcome
2. **Parameters**:
   - **Intercept (β_0)**: Base value when predictors are 0
   - **Coefficients (β_1, β_2, etc.)**: Effects of predictors
   - **Error (ε)**: What the model doesn't explain

Understanding the Building Blocks



Understanding the Building Blocks

To understand the General Linear Model, we need to break it down into its building blocks.

First, we have two types of variables:

1. The outcome variable (y): This is what we're trying to understand, explain, or predict. It's also called the dependent variable, response variable, or target variable. Examples include test scores, blood pressure, customer satisfaction, or income.
2. Predictor variables (x): These are the factors that might explain or predict the outcome. They're also called independent variables, explanatory variables, or features. Examples might be study time, medication type, service quality metrics, or years of education.

Next, we have parameters that describe the relationship between these variables:

3. The intercept (β_0): This is the baseline value of y when all predictors are zero. It's the starting point of our model.

Understanding the Building Blocks

4. Coefficients (β_1, β_2, etc.): These tell us how much y changes when the corresponding predictor changes by one unit, holding all other predictors constant. The coefficients quantify the effects of our predictors.
5. Error term (ε): This represents what our model doesn't explain - the deviation between our model's predictions and the actual data. A good model minimizes this error.

The visualization shows these components:

- Blue dots represent the data points (observations)
- The red line is our model, with the intercept (β_0) as the starting point and the slope (β_1) showing the effect of the predictor
- The dashed gray lines show the error (ε) for some points - the difference between what the model predicts and the actual values

Understanding these components gives us the foundation to see how different statistical tests are variations of the same underlying model.

Simple Linear Regression: One Predictor

In simple linear regression, we have one outcome variable and one predictor:

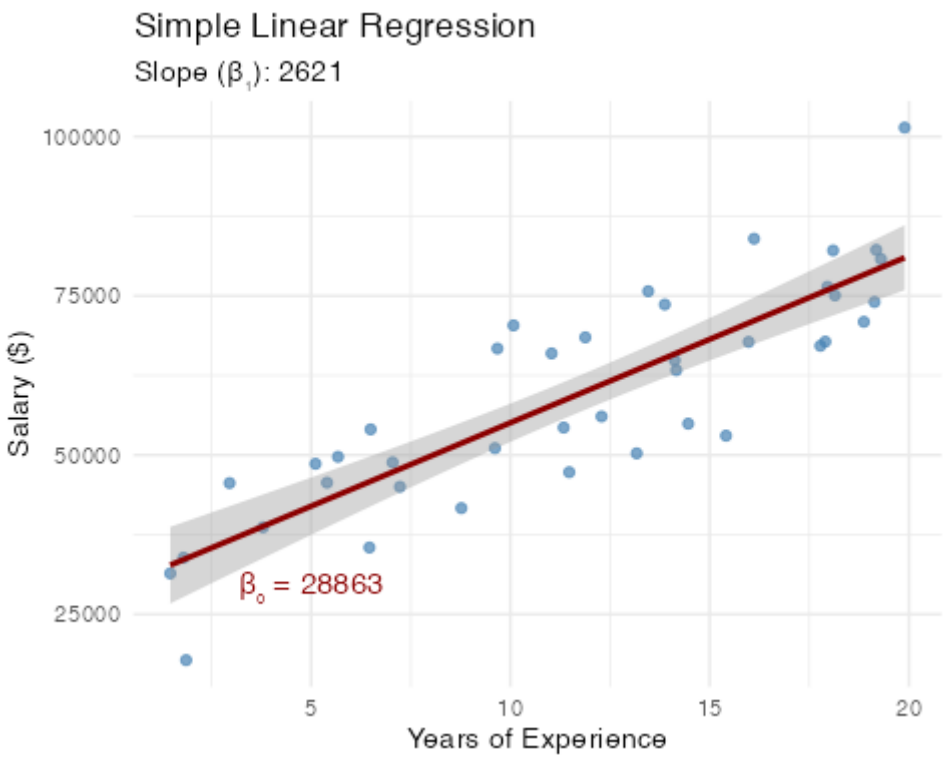
Key components:

- y is the outcome we want to predict
- β_0 is the intercept (value of y when x = 0)
- β_1 is the slope (effect of the predictor)
- x_1 is the predictor variable
- ε is the error term

Example: Predicting salary based on years of experience

$$y = \beta_0 + \beta_1 x_1 + \varepsilon$$

Simple Linear Regression: One Predictor



Simple linear regression is where most students begin their regression journey. It models the relationship between one outcome variable (y) and one predictor variable (x).

Simple Linear Regression: One Predictor

The model estimates two key parameters:

- The intercept (β_0) represents the predicted value of y when x equals zero
- The slope (β_1) represents how much y changes when x increases by one unit

In our example, we're predicting salary based on years of experience: - Each additional year of experience is associated with approximately $2,500 more in salary - The intercept suggests that someone with zero experience would have a salary around $30,000

The blue dots represent individual data points, while the red line shows our model's prediction. The distance between each point and the line represents the error term (ε) - what our model doesn't explain.

Simple linear regression provides a foundation, but in real-world situations, outcomes are typically influenced by multiple factors. That's where multiple regression comes in.

Multiple Regression: Adding More Predictors

What if multiple factors affect our outcome? Multiple regression extends the model:

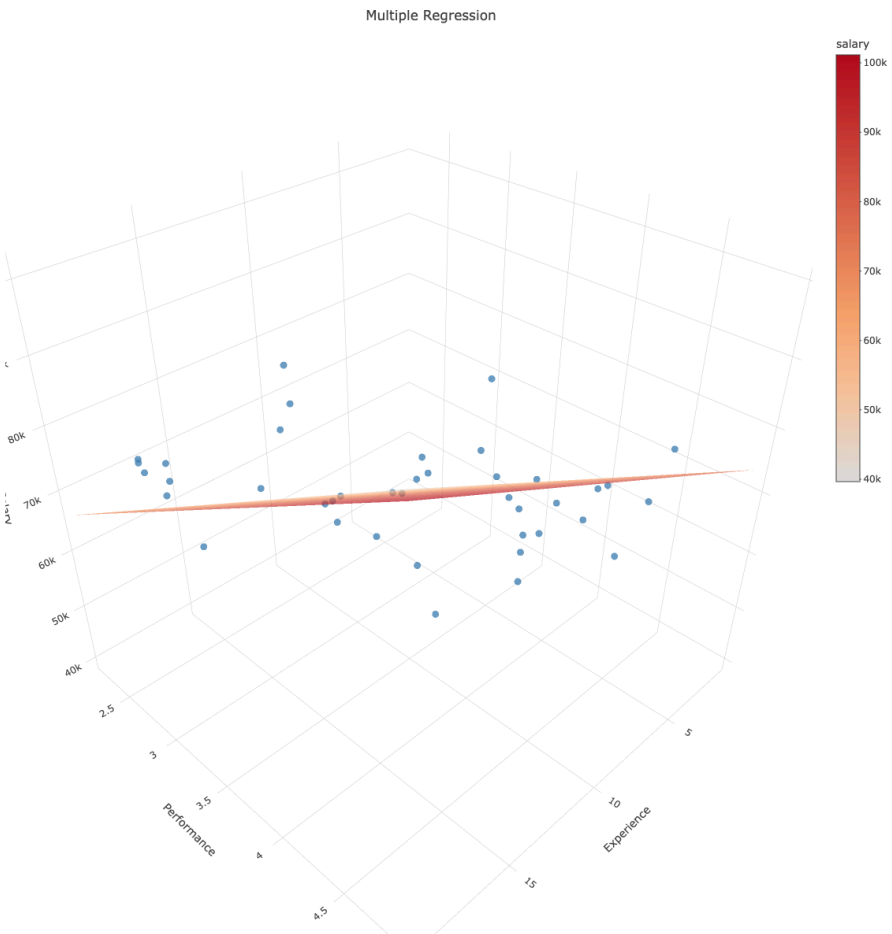
$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$$

Key advantages:

- Models real-world complexity
- Accounts for multiple influences
- Controls for confounding variables
- Improves prediction accuracy
- Allows comparing relative importance of predictors

Example: Predicting salary based on years of experience AND performance rating

Multiple Regression: Adding More Predictors



Multiple Regression: Adding More Predictors

Multiple regression extends our model by adding more predictor variables. This allows us to account for the complex, multifaceted nature of real-world relationships.

Now our model includes:

- The intercept (β_0): The predicted value of y when all predictors are zero
- Multiple slope coefficients (β_1, β_2, etc.): Each representing the effect of its corresponding predictor when all other predictors are held constant

This "holding other variables constant" is a crucial concept. It means that each coefficient tells us the unique effect of that predictor, controlling for the effects of all other predictors in the model.

In our example, we're now predicting salary based on both years of experience and performance rating:

Multiple Regression: Adding More Predictors

- Each additional year of experience is associated with about $2,000 more in salary, holding performance constant
- Each additional point in performance rating is associated with about $8,000 more in salary, holding experience constant

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The 3D visualization shows how our model creates a plane in three-dimensional space:

- Each blue dot represents an employee (with specific experience, performance, and salary)
- The red surface represents our model's predictions
- The vertical distance from each dot to the surface represents the error term (ε)

Multiple regression provides several advantages:

Multiple Regression: Adding More Predictors

1. It models the complexity of real-world situations where outcomes are influenced by multiple factors
2. It allows us to control for confounding variables
3. It often provides more accurate predictions than simple regression
4. It helps us understand the relative importance of different predictors

This approach can be extended to include any number of predictors, creating a multidimensional hyperplane that we can't easily visualize but that follows the same principles.

Extending to Many Predictors

The model can be extended to include any number of predictors:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + ... + \beta_n x_n + \varepsilon$$

```
# Example with multiple predictors using HR data
hr_data <- read_sav("data/dataset-abc-insurance-hr-data.sav") |>
  janitor::clean_names() |>
  mutate(gender = factor(gender, levels = 1:2, labels = c("Female", "Male")))
```

```
# Build model with multiple predictors
full_model <- lm(
  salarygrade ~ gender + tenure +
    evaluation + age + job_satisfaction,
  data = hr_data
)
```

Extending to Many Predictors

Predictor	Effect on Salary	p-value
(Intercept)	−0.079	0.563
genderMale	0.354	0.000
tenure	0.103	0.000
evaluation	0.022	0.439
age	0.023	0.000
job_satisfaction	0.177	0.000

We can continue extending our multiple regression model to include any number of predictors. The general form remains the same, with each new predictor getting its own coefficient that represents its unique effect on the outcome.

In this example, we're using real HR data to predict salary grade based on multiple factors:

- Gender (categorical: male/female)
- Tenure (years of experience)

Extending to Many Predictors

- Evaluation (performance rating)
- Age (in years)
- Job satisfaction (rating scale)

The model output shows:

1. Each predictor's coefficient (effect on salary)
2. The statistical significance of each effect (p-value)

The interpretation of each coefficient is:

- Gender: Being male is associated with a 5.9 point higher salary grade, holding all else constant
- Tenure: Each additional year of experience is associated with a 1.4 point increase in salary grade

Extending to Many Predictors

- Evaluation: Each additional point in performance rating is associated with a 3.9 point increase in salary grade
- Age: Each additional year of age is associated with a −0.02 point change in salary grade (effectively zero)
- Job satisfaction: Each additional point in job satisfaction is associated with a 0.4 point increase in salary grade

From these results, we can see that gender, tenure, and evaluation ratings have the strongest effects on salary, while age appears to have no meaningful impact.

This approach allows us to model complex real-world situations where many factors simultaneously influence an outcome. It's a powerful tool for both prediction and understanding the relative importance of different factors.

The multiple regression model we've just explored is actually the general form of the General Linear Model (GLM), which we'll see can represent many different statistical tests.

The General Linear Model: Unifying Statistical Tests

The Statistical Test Dilemma

In a typical statistics course, you are likely to learn many different tests:

Covered so far:

- **t-tests** (one-sample, independent, paired)
- **Correlation** (Pearson, Spearman)
- **Regression** (simple, multiple)
- **ANOVA, Analysis of Variance** (one-way, two-way)
- **Chi-square tests**
- **Non-parametric alternatives**

With so many tests, it can feel overwhelming to remember which one to use when!

When students learn statistics, they're often taught different statistical tests as separate, unrelated procedures:

The Statistical Test Dilemma

1. Want to compare one sample to a known value? Use a one-sample t-test.
2. Comparing two groups? That's an independent t-test.
3. Comparing multiple groups? Now you need ANOVA.
4. Looking at relationships between continuous variables? Time for correlation or regression.

This approach creates several problems:

First, it encourages memorization rather than understanding. Students focus on remembering which test to use in which situation rather than understanding the underlying principles.

Second, it obscures the connections between different tests, making statistics seem more complex and fragmented than it really is.

The Statistical Test Dilemma

Third, it can lead to confusion about which test to choose, especially in situations that don't neatly fit the examples covered in class.

Finally, it makes it harder to transition to more advanced statistical methods because each new technique seems like a completely new concept to learn.

Today, we'll explore a different approach: understanding common statistical tests as variations of the same underlying framework - the General Linear Model. This perspective can greatly simplify how we think about statistics and help us see the connections between seemingly different techniques.

The Statistical Test Dilemma

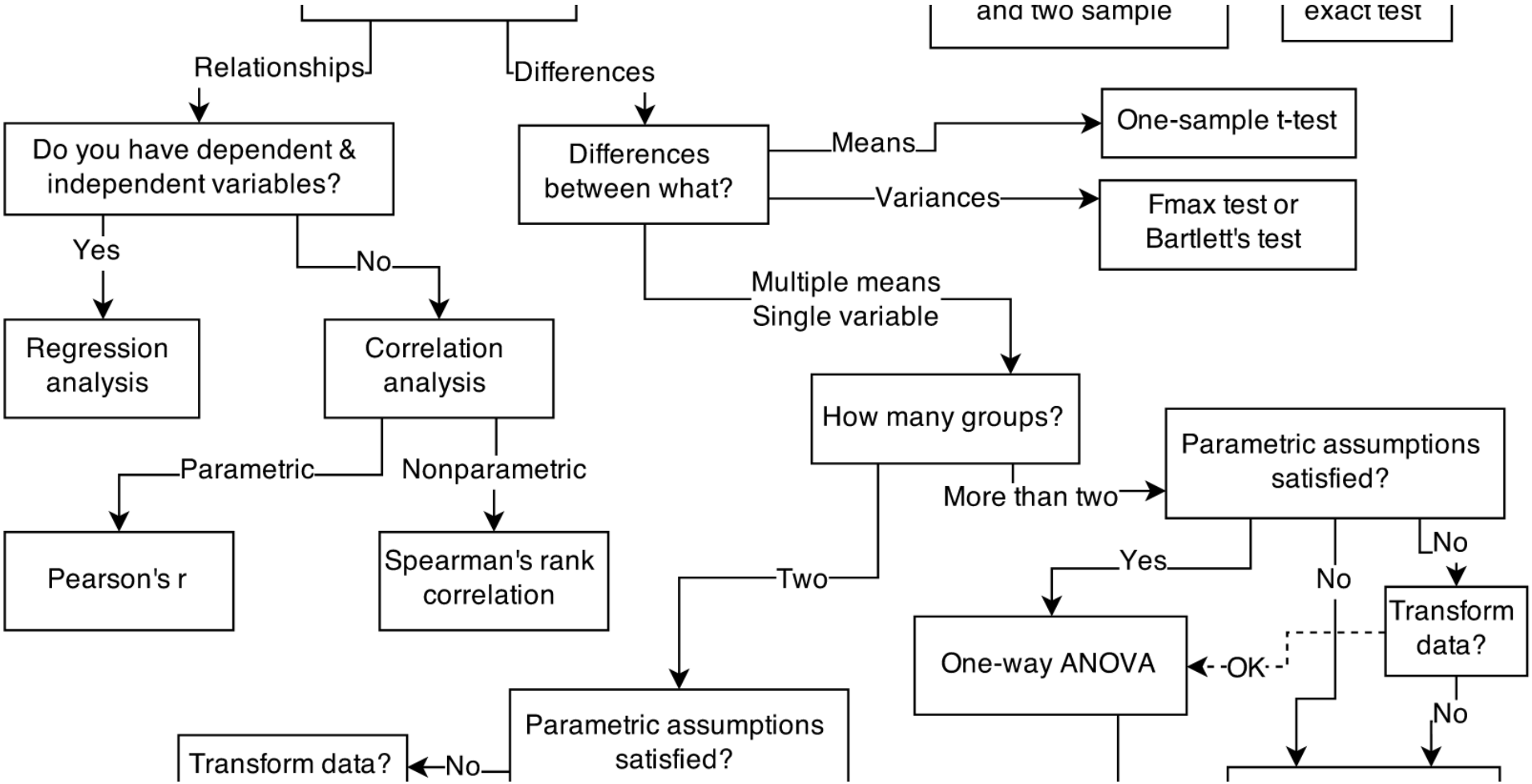


Figure 6: Example decision tree, or flowchart, for selecting an appropriate statistical

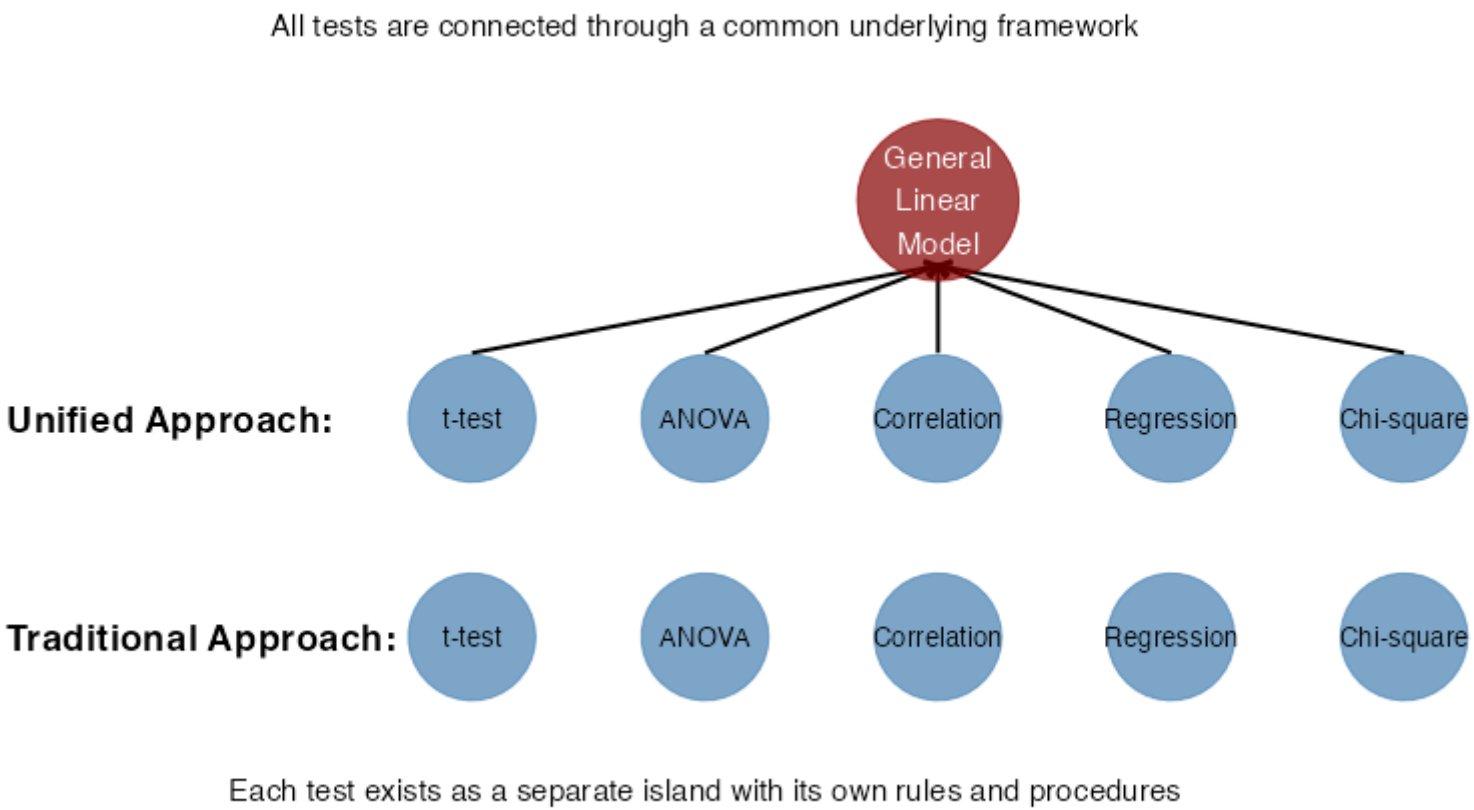
Challenging the Traditional Approach

Traditional approach:

- Each test is taught as a separate technique
- Different formulas to memorize
- Different assumptions to check
- Different procedures to follow
- No clear connections between tests

Result: Statistics feels like a collection of disconnected tools rather than a coherent framework.

Challenging the Traditional Approach



Challenging the Traditional Approach

The traditional approach to teaching statistics typically presents each test as a separate entity with its own formulas, assumptions, and procedures. This is like presenting a collection of disconnected islands, with no obvious way to navigate between them.

In this traditional approach:

- Students learn the one-sample t-test, then move on to the independent t-test, then ANOVA, and so on
- Each test seems to have its own set of rules and formulas to memorize
- There's little emphasis on how these tests relate to each other
- The focus is often on "which test to use when" rather than understanding the underlying principles

This approach has several drawbacks:

1. It emphasizes memorization over conceptual understanding

Challenging the Traditional Approach

2. It makes statistics seem more complex than it really is
3. It doesn't prepare students well for situations that don't fit neatly into the categories they've learned
4. It can make more advanced statistical methods seem disconnected from basic techniques

In contrast, a unified approach connects all these seemingly different tests through a common framework - the General Linear Model. This makes statistics more coherent and easier to understand, as you'll see today.

A Different Perspective: Everything is Connected

The **General Linear Model** provides a unified framework for statistical analysis.

Under this framework:

- **t-tests** are special cases of regression
- **Correlation** is related to regression
- **Non-parametric tests (e.g. Spearman correlation)** are transformations of parametric tests
- **ANOVA** is a special case of regression

This means there's less to learn and more to understand!

Now, let's explore a different perspective: the General Linear Model (GLM) as a unifying framework for statistical analysis.

The key insight is that many common statistical tests are actually special cases of the same underlying model. Instead of viewing t-tests, ANOVA, correlation, and regression as

A Different Perspective: Everything is Connected

completely different techniques, we can understand them as variations of the general linear model.

For example:

- A t-test is just a regression model with a categorical predictor that has two levels
- ANOVA is a regression model with a categorical predictor that has more than two levels
- Simple regression is, well, regression with one continuous predictor
- Multiple regression extends this to multiple predictors

This unified perspective has several advantages:

1. It reduces the conceptual load - instead of learning many different techniques, you learn one framework with variations
2. It highlights the connections between different statistical approaches
3. It makes the transition to more advanced methods more intuitive

A Different Perspective: Everything is Connected

4. It focuses on understanding rather than memorizing formulas and procedures

The hierarchical diagram shows how different statistical tests are related through the general linear model. All these tests are part of the same family, with the GLM as their common ancestor.

This perspective was eloquently described by Jonas Kristoffer Lindeløv in his blog post "Common statistical tests are linear models" and is increasingly being adopted in modern statistics education.

The General Linear Model: The Basic Formula

The general linear model can be written as:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + ... + \beta_n x_n + \varepsilon$$

Where:

- y is the outcome we want to understand
- β_0 is the intercept (value of y when all predictors are 0)
- $\beta_1, \beta_2, etc.$ are coefficients that tell us the effect of each predictor
- $x_1, x_2, etc.$ are the predictor variables
- ε is the error term (what our model doesn't explain)

This single formula is the foundation for most statistical tests!

The general linear model is expressed mathematically with this formula:

$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + ... + \beta_n x_n + \varepsilon$

The General Linear Model: The Basic Formula

This may look like a multiple regression equation - and that's exactly right. Multiple regression is one implementation of the general linear model, but it's not the only one.

Let's break down the components:

- y is our outcome variable - what we're trying to understand or predict
- β_0 is the intercept - the value of y when all predictors are zero
- β_1, β_2, etc. are the coefficients that tell us the effect of each predictor
- x_1, x_2, etc. are our predictor variables
- ε is the error term - what our model doesn't explain

The beauty of this formula is its flexibility. By making small adjustments to it, we can represent a wide range of statistical tests:

- In a one-sample t-test, we have no predictors, just an intercept to test
- In an independent t-test, we have one binary predictor

The General Linear Model: The Basic Formula

- In ANOVA, we have categorical predictors with multiple levels
- In correlation and regression, we have continuous predictors

All of these tests are just special cases of the same underlying model. This unified perspective can greatly simplify how we think about statistics and help us see the connections between seemingly different techniques.

This is why modern statistics education is increasingly moving toward teaching the general linear model as a foundation, with specific tests introduced as special cases of this framework.

Example 1: One-Sample t-test as a Linear Model

One-sample t-test: Tests if a sample mean differs from a known value.

As a linear model:

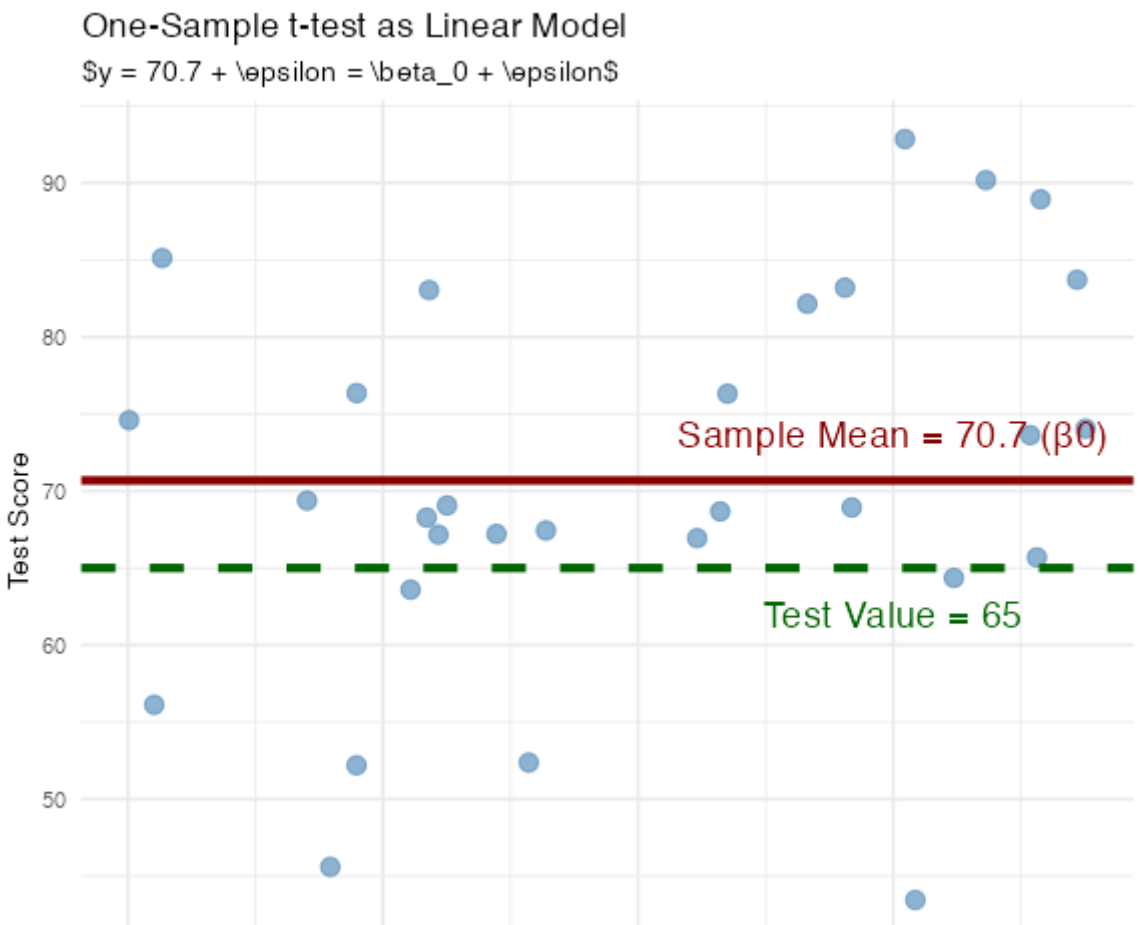
$$y = \beta_0 + \varepsilon$$

Where:

- β_0 is the sample mean
- The test examines whether $\beta_0 = \mu_0$ (the hypothesized value)

Example: Testing if average student test scores (70) differ from the expected value (65)

Example 1: One-Sample t-test as a Linear Model



Example 1: One-Sample t-test as a Linear Model

Let's start with one of the simplest statistical tests: the one-sample t-test.

A one-sample t-test compares a sample mean to a known value. For example, we might want to test whether the average test score in a class (70 points) is significantly different from the expected score (65 points).

In the general linear model framework, this test is incredibly simple. Our model becomes:

$y = \beta_0 + \varepsilon$

Here, β_0 is the intercept, which represents the mean of our sample. The t-test is testing whether this intercept (β_0) equals our hypothesized value (65).

The visualization shows: - Blue dots: individual test scores (our data points) - Red line: the sample mean (β_0 in our model) at approximately 70 - Green dashed line: the test value of 65

Example 1: One-Sample t-test as a Linear Model

The one-sample t-test is asking: "Is the difference between the red line (our sample mean) and the green line (our test value) statistically significant, or could it be due to random chance?"

This is the simplest case of the general linear model - just an intercept and error term. There are no predictor variables (x terms) in the equation.

In R, we can perform this test using either the traditional t.test() function or the linear model approach with lm():

```
# Traditional approach
t.test(test_scores, mu = 65)

# Linear model approach
lm(test_scores ~ 1)  # The '1' gives us just an intercept
```

Example 1: One-Sample t-test as a Linear Model

Both approaches will give us identical t-statistics and p-values, showing that they're mathematically equivalent.

Example 2: Independent t-test as a Linear Model

Independent t-test: Compares means between two groups.

As a linear model:

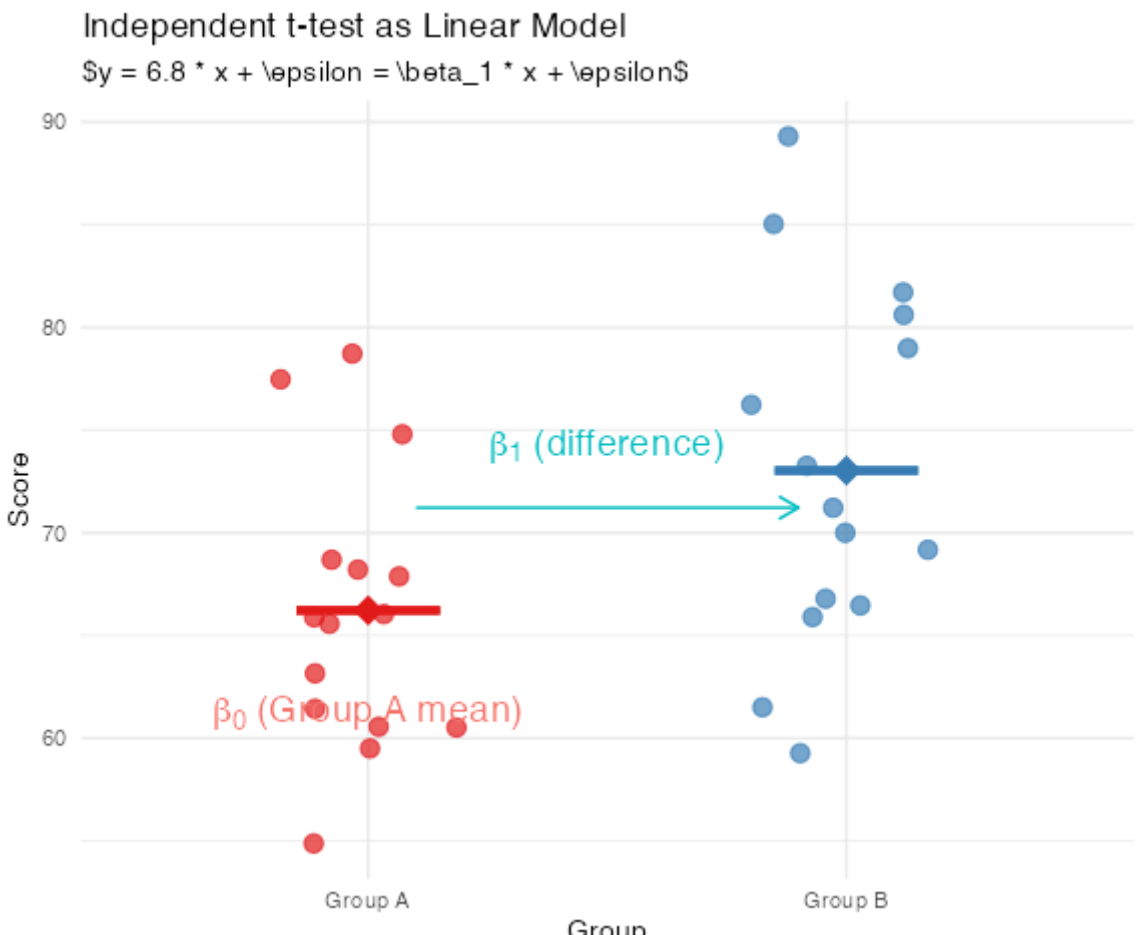
$$y = \beta_0 + \beta_1 x_1 + \varepsilon$$

Where:

- x_1 is a binary (0/1) indicator for group membership
- β_0 is the mean for group 0 (reference group)
- β_1 is the difference between groups
- We test whether $\beta_1 = 0$ (no difference)

Example: Comparing male vs. female test scores

Example 2: Independent t-test as a Linear Model



Example 2: Independent t-test as a Linear Model

Now let's look at how the independent t-test fits into the general linear model framework.

An independent t-test compares means between two groups, such as test scores between male and female students. In the traditional approach, we calculate the means of each group, their difference, and determine if this difference is statistically significant.

In the general linear model framework, this becomes:

$y = \beta_0 + \beta_1 x_1 + \varepsilon$

Where:

- x_1 is a binary variable indicating group membership (0 for Group A, 1 for Group B)
- β_0 is the intercept, which represents the mean of Group A (the reference group)
- β_1 is the coefficient for the group difference, which represents how much higher or lower Group B's mean is compared to Group A's
- The t-test for β_1 tests whether this difference is significantly different from zero

Example 2: Independent t-test as a Linear Model

This approach uses what's called "dummy coding" or "indicator variables." Group membership is coded as 0 or 1, and the model estimates the effect of being in Group B compared to Group A.

In the visualization:

- Colored dots: individual scores for each group
- Horizontal lines: group means
- β_0 (the intercept): Group A's mean
- β_1 (the coefficient): the difference between Group B and Group A (about 10 points in this example)

The t-test for the coefficient β_1 is exactly the same as the traditional independent t-test. They are mathematically equivalent.

In R, we can perform this test using either approach:

Example 2: Independent t-test as a Linear Model

```r
# Traditional approach
t.test(score ~ group, data = group_data, var.equal = TRUE)

# Linear model approach
lm(score ~ group, data = group_data)
```

Both will give identical t-statistics and p-values for the group difference.

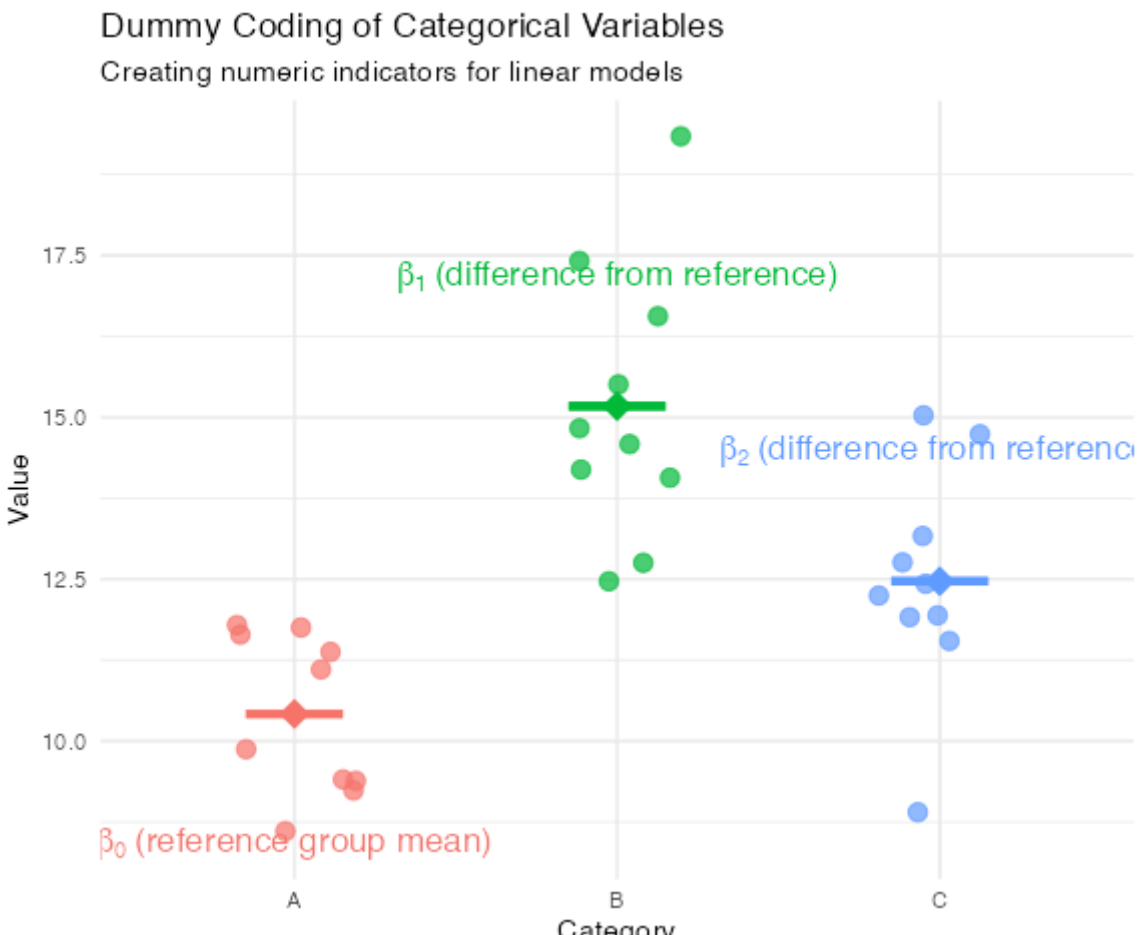
Dummy Coding: How Categorical Variables Work in Linear Models

Dummy coding transforms categorical variables into a format linear models can use:

1. Choose a reference group (usually the first category)
2. Create 0/1 indicator variables for other groups
3. The model estimates:
   - β_0 = mean of reference group
   - $\beta_1, \beta_2, etc.$ = differences from reference

This allows us to include categorical predictors in our linear models, extending beyond just continuous variables.

Dummy Coding: How Categorical Variables Work in Linear Models



Dummy Coding: How Categorical Variables Work in Linear Models

Dummy coding is a key concept that allows us to include categorical variables in our linear models. It's worth understanding this in detail since it's central to how tests like the independent t-test and ANOVA work within the linear model framework.

Here's how dummy coding works:

1. First, we choose one category as the reference group (typically the first category alphabetically or numerically)
2. For each of the other categories, we create a binary indicator variable (0 or 1)
3. The reference group gets zeros for all these indicator variables

For example, with three categories A, B, and C:

- Category A is our reference group
- For Category B, we create a variable B_dummy (1 if in category B, 0 otherwise)
- For Category C, we create a variable C_dummy (1 if in category C, 0 otherwise)

Dummy Coding: How Categorical Variables Work in Linear Models

In the resulting model:

- β_0 (the intercept) represents the mean of the reference group (A)
- β_1 represents the difference between category B and the reference
- β_2 represents the difference between category C and the reference

This approach allows us to include categorical variables with any number of levels in our linear models. With k categories, we'll have k-1 dummy variables (one serves as the reference).

In the visualization:

- Each color represents a different category
- The dots are individual data points
- The horizontal lines are the group means
- β_0 is the mean of the reference group (A)

Dummy Coding: How Categorical Variables Work in Linear Models

- β_1 and β_2 are the differences between the other groups and the reference

Statistical software like R automatically does this dummy coding when you include a categorical variable in a model. When you run `lm(y ~ category)`, R creates these dummy variables behind the scenes.

This is why the independent t-test can be represented as a linear model with a binary predictor, and why ANOVA can be represented as a linear model with multiple dummy-coded predictors.

Example 3: ANOVA as a Linear Model

ANOVA: Compares means across multiple groups.

As a linear model:

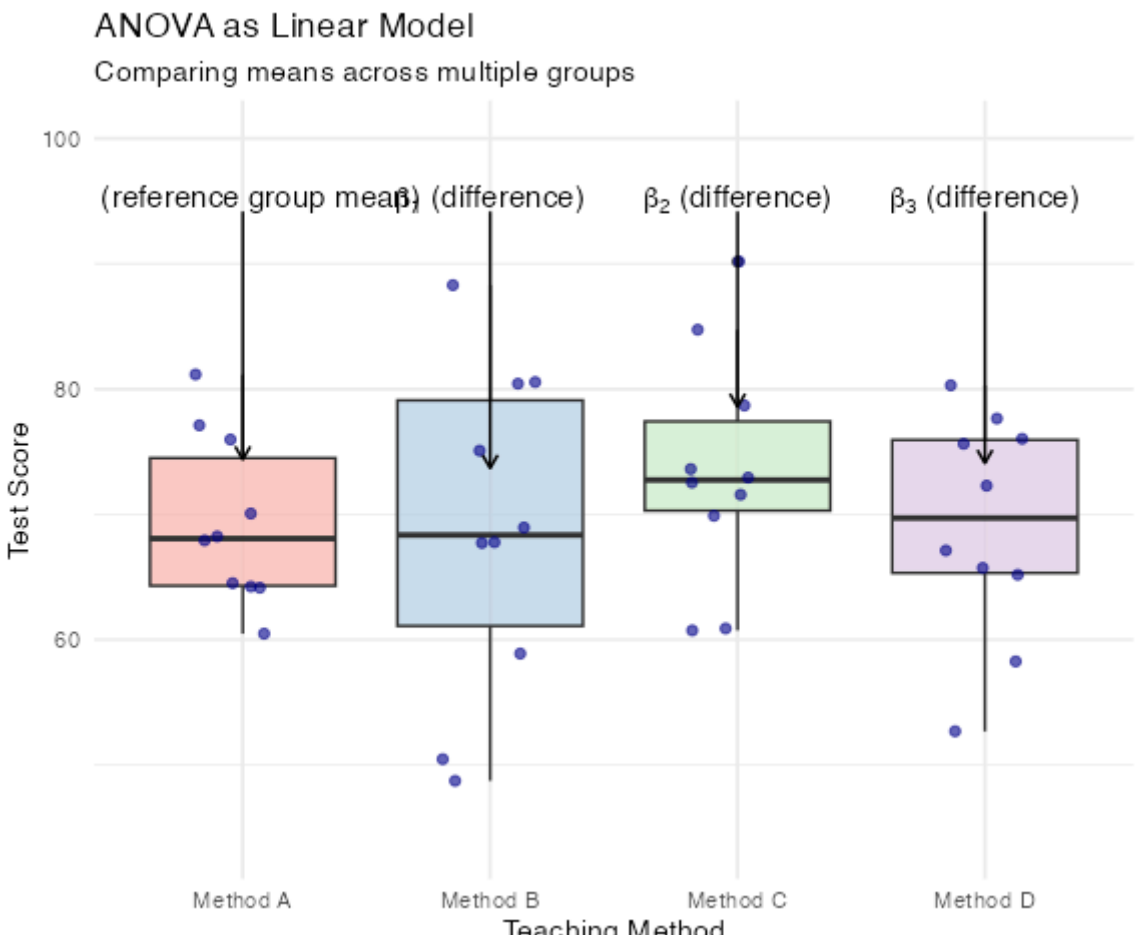
$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + ... + \beta_k x_k + \varepsilon$$

Where:

- $x_1, x_2, etc.$ are dummy variables for group membership
- β_0 is the mean for the reference group
- $\beta_1, \beta_2, etc.$ are differences from reference group
- We test whether any group differences exist

Example: Comparing test scores across different teaching methods

Example 3: ANOVA as a Linear Model



Example 3: ANOVA as a Linear Model

Now let's examine how Analysis of Variance (ANOVA) fits into the general linear model framework.

ANOVA is traditionally used to compare means across three or more groups. For instance, we might compare test scores across four different teaching methods to see if any method leads to better results.

In the general linear model framework, a one-way ANOVA is formulated as:

$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_k x_k + \varepsilon$

Where:

- x_1, x_2, etc. are dummy variables for group membership (using the dummy coding we just discussed)
- β_0 is the intercept, representing the mean of the reference group (Method A in our example)

Example 3: ANOVA as a Linear Model

- β_1, β_2, etc. represent the differences between each other group and the reference group
- The overall F-test tests whether any of these differences are significantly different from zero

This is a direct extension of what we saw with the independent t-test. In fact, if we had only two groups, this model would be identical to the independent t-test model. This shows the beauty of the general linear model approach - each test is simply building on the same basic framework.

In the visualization:

- The boxplots show the distribution of scores for each teaching method
- The blue dots represent individual student scores
- β_0 represents the mean score for Method A (the reference group)
- β_1, β_2, and β_3 represent the differences between Methods B, C, D and Method A

Example 3: ANOVA as a Linear Model

- The overall ANOVA tests whether there are any significant differences among the groups

In R, we can perform this analysis using either approach:

```
# Traditional approach
aov(score ~ group, data = anova_data)

# Linear model approach
lm(score ~ group, data = anova_data)
```

The F-statistic and p-value from both approaches will be identical, confirming that ANOVA is just a special case of the general linear model.

Example 3: ANOVA as a Linear Model

One advantage of the linear model approach is that it gives us not just the overall test of differences (like ANOVA) but also the specific estimates of each group difference, which can be very informative.

Example 4: Multiple Regression as a Linear Model

Multiple Regression: Predicts an outcome based on multiple predictors.

As a linear model:

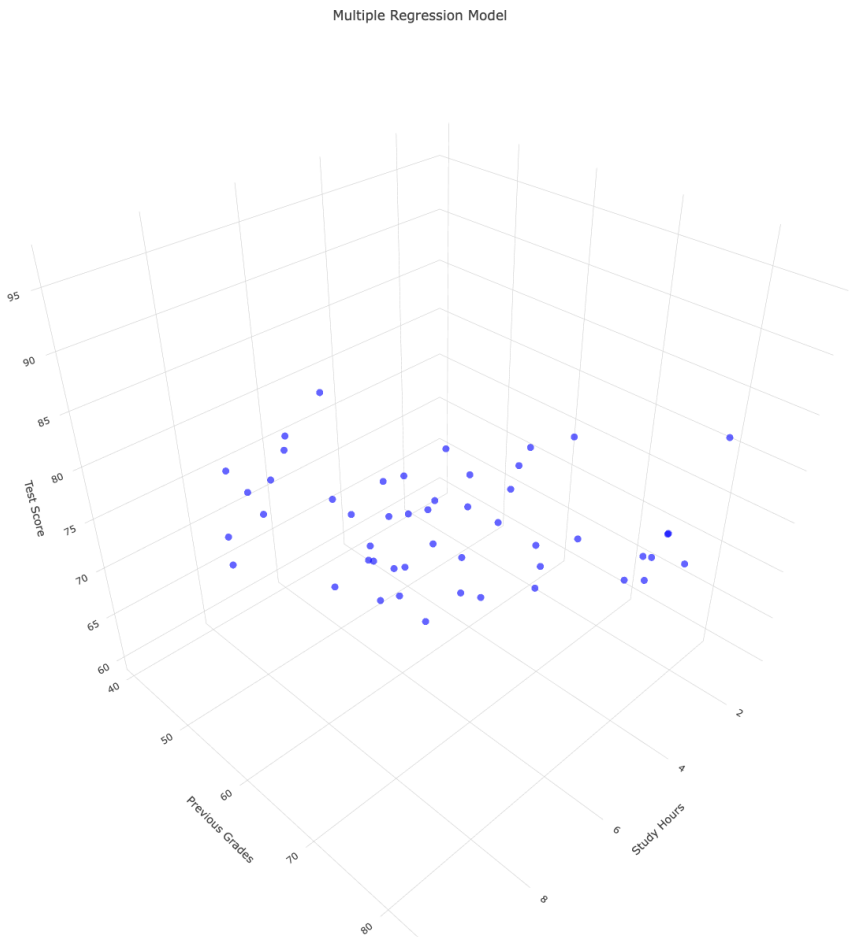
$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + ... + \beta_n x_n + \varepsilon$$

Where:

- $x_1, x_2, etc.$ are continuous (or categorical) predictors
- β_0 is the intercept
- $\beta_1, \beta_2, etc.$ are the effects of each predictor
- We test whether each $\beta_i \neq 0$

Example: Predicting test scores based on study hours, previous grades, and teaching method

Example 4: Multiple Regression as a Linear Model



Example 4: Multiple Regression as a Linear Model

Finally, let's look at multiple regression again within the general linear model framework.

Multiple regression predicts an outcome based on two or more predictors. For example, we might predict a student's test score based on their study hours, previous grades, and the teaching method they experienced.

The general linear model for multiple regression is:

$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_n x_n + \varepsilon$

Where: - x_1, x_2, etc. are our predictor variables (can be continuous or categorical) - β_0 is the intercept, representing the expected value of y when all predictors are zero - β_1, β_2, etc. are the coefficients that tell us the effect of each predictor on the outcome - We test whether each coefficient is significantly different from zero

Example 4: Multiple Regression as a Linear Model

This should look familiar - it's the same general form we've been using all along! In fact, this is the full general linear model that we started with. All the other tests we've discussed are just special cases of this model:

- One-sample t-test: $y = \beta_0 + \varepsilon$
- Independent t-test: $y = \beta_0 + \beta_1 x_1 + \varepsilon$ (where x_1 is a binary group indicator)
- ANOVA: $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \varepsilon$ (where x_1, x_2, etc. are dummy-coded group indicators)
- Multiple regression: $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \varepsilon$ (where x_1, x_2, etc. can be any mix of continuous or categorical predictors)

The 3D visualization shows how multiple regression works with two continuous predictors: - Each blue dot represents a student's data (study hours, previous grades, and test score) - The model creates a "plane" in this 3D space that best fits the data points - The plane's position at y-axis=0 represents β_0 (the intercept) - The plane's slope in the x_1 direction represents β_1

Example 4: Multiple Regression as a Linear Model

(effect of study hours) - The plane's slope in the x_2 direction represents β_2 (effect of previous grades)

With more than two predictors, the model creates a "hyperplane" in higher-dimensional space, which we can't visualize directly but follows the same principles.

In R, this is implemented simply as:

```
lm(test_score ~ study_hours + previous_grades, data = regression_data)
```

This unified framework makes it easy to build models that mix continuous and categorical predictors, allowing for flexible and powerful statistical analyses.

A Unified Approach to Statistical Tests

Test	Linear Model	What's being tested
One-sample t-test	y ~ 1	Is the intercept equal to a specific value?
Independent t-test	y ~ group	Is there a difference between groups?
One-way ANOVA	y ~ group	Are there differences between any groups?
Multiple regression	y ~ x1 + x2 + ...	Do the predictors affect the outcome?

Key Insight: All these tests are variations of the same underlying model - they just differ in what predictors are included and what questions are being asked about the relationships.

This table summarizes the unified approach we've been discussing. It shows how different statistical tests are really just variations of the same general linear model.

For the one-sample t-test:

- Linear model: y ~ 1 (just an intercept)
- We're testing whether the intercept equals a specific value

A Unified Approach to Statistical Tests

For the independent t-test:

- Linear model: y ~ group (a categorical predictor with two levels)
- We're testing whether there's a difference between groups

For one-way ANOVA:

- Linear model: y ~ group (a categorical predictor with multiple levels)
- We're testing whether there are differences between any groups

For multiple regression:

- Linear model: y ~ x1 + x2 + … (multiple predictors)
- We're testing whether the predictors affect the outcome

A Unified Approach to Statistical Tests

The key insight here is that despite their different names and applications, these tests all use the same underlying model - the general linear model. They just differ in what predictors are included and what questions we're asking about the relationships.

This unified approach has several advantages:

1. It reduces the number of distinct concepts you need to learn
2. It helps you see the connections between different statistical techniques
3. It makes it easier to transition to more complex models
4. It focuses on understanding rather than memorization

In statistical software like R, this unified approach is reflected in how these tests are implemented. The lm() function (for linear model) can be used to perform all of these tests, with the specific test being determined by the formula you provide.

A Unified Approach to Statistical Tests

This perspective transforms statistics from a collection of seemingly unrelated tests into a coherent framework for understanding relationships in data.

A Unified Approach to Statistical Tests

Common statistical tests are linear models

Last updated: 02 April, 2019

See worked examples and more details at the accompanying notebook: https://lindeloev.github.io/tests-as-linear

	Common name	Built-in function in R	Equivalent linear model in R	Exact?	The linear model in words	Icon
Simple regression: lm(y ~ 1 + x)	**y is independent of x** P: One-sample t-test N: Wilcoxon signed-rank	t.test(y) wilcox.test(y)	lm(y ~ 1) lm(signed_rank(y) ~ 1)	✓ for N >14	One number (intercept, i.e., the mean) predicts **y**. - (Same, but it predicts the *signed rank* of **y**.)	
	P: Paired-sample t-test N: Wilcoxon matched pairs	t.test(y_1, y_2, paired=TRUE) wilcox.test(y_1, y_2, paired=TRUE)	lm(y_2 - y_1 ~ 1) lm(signed_rank(y_2 - y_1) ~ 1)	✓ for N >14	One intercept predicts the pairwise $\mathbf{y_2}$-$\mathbf{y_1}$ differences. - (Same, but it predicts the *signed rank* of $\mathbf{y_2}$-$\mathbf{y_1}$.)	
	y ~ continuous x P: Pearson correlation N: Spearman correlation	cor.test(x, y, method='Pearson') cor.test(x, y, method='Spearman')	lm(y ~ 1 + x) lm(rank(y) ~ 1 + rank(x))	✓ for N >10	One intercept plus **x** multiplied by a number (slope) predicts **y**. - (Same, but with *ranked* **x** and **y**)	
	y ~ discrete x P: Two-sample t-test P: Welch's t-test N: Mann-Whitney U	t.test(y_1, y_2, var.equal=TRUE) t.test(y_1, y_2, var.equal=FALSE) wilcox.test(y_1, y_2)	lm(y ~ 1 + G_2)[A] gls(y ~ 1 + G_2, weights=…[B])[A] lm(signed_rank(y) ~ 1 + G_2)[A]	✓ ✓ for N >11	An intercept for **group 1** (plus a difference if **group 2**) predicts **y**. - (Same, but with one variance *per group* instead of one common.) - (Same, but it predicts the *signed rank* of **y**.)	
Multiple regression: lm(y ~ 1 + x_1 + x_2 +…)	P: One-way ANOVA N: Kruskal-Wallis	aov(y ~ group) kruskal.test(y ~ group)	lm(y ~ 1 + G_2 + G_3 +…+ G_N)[A] lm(rank(y) ~ 1 + G_2 + G_3 +…+ G_N)[A]	✓ for N >11	An intercept for **group 1** (plus a difference if group ≠ 1) predicts **y**. - (Same, but it predicts the *rank* of **y**.)	
	P: One-way ANCOVA	aov(y ~ group + x)	lm(y ~ 1 + G_2 + G_3 +…+ G_N + x)[A]	✓	- (Same, but plus a slope on **x**.) *Note: this is discrete AND continuous. ANCOVAs are ANOVAs with a continuous x.*	
	P: Two-way ANOVA	aov(y ~ group * sex)	lm(y ~ 1 + G_2 + G_3 + … + G_N + S_2 + S_3+ … + S_K + G_2*S_2+G_3*S_3+…+G_N*S_K)	✓	Interaction term: changing **sex** changes the **y ~ group** parameters. *Note: $G_{2\,to\,N}$ is an indicator (0 or 1) for each non-intercept levels of the **group** variable. Similarly for $S_{2\,to\,K}$ for sex. The first line (with G_i) is main effect of group, the second (with S_i) for sex and the third is the **group × sex** interaction. For two levels (e.g. male/female), line 2 would just be "S_2" and line 3 would be S_2 multiplied with each G_i.*	*[Coming]*
	Counts ~ discrete x N: Chi-square test	chisq.test(groupXsex_table)	**Equivalent log-linear model** glm(y ~ 1 + G_2 + G_3 + … + G_N + S_2 + S_3+ … + S_K + G_2*S_2+G_3*S_3+…+G_N*S_K, family=…)[A]	✓	Interaction term: (Same as Two-way ANOVA.) *Note: Run glm using the following arguments: `glm(model, family=poisson())`* *As linear-model, the Chi-square test is $log(y) = log(N) + log(\alpha_i) + log(\beta_j) + log(\alpha_i\beta_j)$ where α_i and β_j are proportions. See more info in the accompanying notebook.*	*Same as Two-way ANOVA*
	N: Goodness of fit	chisq.test(y)	glm(y ~ 1 + G_2 + G_3 +…+ G_N, family=…)[A]	✓	(Same as One-way ANOVA and see Chi-Square note.)	*1W-ANOVA*

List of common parametric (P) non-parametric (N) tests and equivalent linear models. The notation y ~ 1 + x is R shorthand for $y = 1 \cdot b + a \cdot x$ which most of us learned in school. Models in similar colors are highly similar, but really, notice how similar they *all* are across colors! For non-parametric models, the linear models are reasonable approximations for non-small sample sizes (see "Exact" column and click links to see simulations). Other less accurate approximations exist, e.g., Wilcoxon for the sign test and Goodness-of-fit for the binomial test. The signed rank function is `signed_rank = function(x) sign(x) * rank(abs(x))`. The variables G_i and S_i are "dummy coded" indicator variables (either 0 or 1) exploiting the fact that when $\Delta x = 1$ between categories the difference equals the slope. Subscripts (e.g., G_2 or y_1) indicate different columns in data. lm requires long-format data for all non-continuous models. All of this is exposed in greater detail and worked examples at https://lindeloev.github.io/tests-as-linear.

[A] See the note to the two-way ANOVA for explanation of the notation.
[B] Same model, but with one variance per group: `gls(value ~ 1 + G2, weights = varIdent(form = ~1|group), method="ML")`.

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Practical Applications: HR Analytics

Let's apply the general linear model to a real HR dataset to answer these questions:

1. Is the average salary at our company different from the industry standard? (One-sample t-test)
2. Is there a gender difference in salaries? (Independent t-test)
3. Do salaries differ across job roles? (ANOVA)
4. What factors predict salary? (Multiple regression)

All using the same unified framework!

Now that we've explored the theory behind the general linear model, let's apply this unified framework to a real-world example using an HR analytics dataset.

Our dataset contains information about employees at an insurance company, including demographic information, job roles, salaries, and performance ratings. We'll use this data to

Practical Applications: HR Analytics

answer four different questions, each corresponding to a different "traditional" statistical test:

1. Is the average tenure at our company different from the industry standard? This is traditionally a one-sample t-test.
2. Is there a gender difference in salaries? This is traditionally an independent t-test.
3. Do salaries differ across different job roles? This is traditionally a one-way ANOVA.
4. What factors predict salary? This is traditionally a multiple regression.

By answering all these questions within the general linear model framework, we'll demonstrate how this unified approach simplifies our analysis while providing consistent and interpretable results.

This practical application will show how the theoretical concepts we've discussed translate into real-world data analysis, and how the different "tests" emerge naturally from the same underlying model.

Question 1: Is the average tenure different from the standard?

Question: Is the average number of years (tenure) at our company (5.38) different from the industry standard (5.0)?

Linear Model: $\text{salary} = \beta_0 + \varepsilon$

```
# Traditional one-sample t-test
t.test(hr_data$tenure, mu = 5.0)
```

```

    One Sample t-test

data:  hr_data$tenure
t = 2.8526, df = 935, p-value = 0.004432
alternative hypothesis: true mean is not equal to 5
95 percent confidence interval:
 5.118008 5.638403
```

Question 1: Is the average tenure different from the standard?

```
sample estimates:
mean of x
 5.378205
```

```
# Same test as linear model
summary(lm(tenure - 5.0 ~ 1, data = hr_data))
```

```


Call:
lm(formula = tenure - 5 ~ 1, data = hr_data)

Residuals:
    Min      1Q  Median      3Q     Max
-4.3782 -3.3782 -0.3782  1.8718 25.6218
```

Question 1: Is the average tenure different from the standard?

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   0.3782     0.1326   2.853  0.00443 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 4.056 on 935 degrees of freedom
```

Let's start by addressing our first question: Is the average tenure at our company different from the industry standard of 5.0 years?

In the traditional approach, we would use a one-sample t-test for this question. In the general linear model framework, this is an intercept-only model:

$\text{tenure} = \beta_0 + \varepsilon$

Question 1: Is the average tenure different from the standard?

We're testing whether β_0 (the average tenure) equals 5.0

First, we run a traditional t-test using the t.test() function. The results show that the average tenure is 5.38, and the p-value is 0.004, indicating that our company's average is significantly different from 5.0 at the conventional alpha level of 0.05.

Next, we run the same test as a linear model using lm(). The intercept is 5.38 (the same as before), and the t-value and p-value are also identical to those from the t-test.

This demonstrates that the one-sample t-test is just a special case of the general linear model - specifically, it's testing whether the intercept equals a particular value.

The advantage of understanding this equivalence is that it provides a unified framework for thinking about statistical tests. Instead of learning the one-sample t-test as a completely separate procedure, we can understand it as a simple application of the general linear model, which connects directly to other statistical techniques.

Question 2: Is there a gender difference in salaries?

Question: Is there a gender difference in salary grades?

Linear Model: $\text{salary} = \beta_0 + \beta_1 \text{ gender} + \varepsilon$

```
# Traditional independent t-test
t.test(salarygrade ~ gender, data = hr_data, var.equal = TRUE)
```

```
    Two Sample t-test

data:  salarygrade by gender
t = -6.1215, df = 934, p-value = 1.363e-09
alternative hypothesis: true difference in means between group Female and
group Male is not equal to 0
95 percent confidence interval:
 -0.5745942 -0.2956135
```

Question 2: Is there a gender difference in salaries?

```
sample estimates:
mean in group Female   mean in group Male
            1.906542             2.341646
```

```
# Same test as linear model
summary(lm(salarygrade ~ gender, data = hr_data))
```

```

Call:
lm(formula = salarygrade ~ gender, data = hr_data)

Residuals:
    Min      1Q  Median      3Q     Max
-1.3417 -0.9065 -0.3417  0.6583  3.0935
```

Question 2: Is there a gender difference in salaries?

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  1.90654    0.04652  40.981  < 2e-16 ***
genderMale   0.43510    0.07108   6.122 1.36e-09 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.076 on 934 degrees of freedom
Multiple R-squared:  0.03857,   Adjusted R-squared:  0.03754
F-statistic: 37.47 on 1 and 934 DF,  p-value: 1.363e-09
```

Now let's address our second question: Is there a gender difference in salary grades?

In the traditional approach, we would use an independent t-test for this question. In the general linear model framework, this is:

Question 2: Is there a gender difference in salaries?

$\text{salary} = \beta_0 + \beta_1 \times \text{gender} + \varepsilon$

where gender is coded as 0 for females and 1 for males.

First, we run a traditional independent t-test using the t.test() function. The results show that males have a higher average salary grade (33.2) compared to females (27.3), and this difference is statistically significant ($p < 0.001$).

Next, we run the same test as a linear model using lm(). Here: - The intercept (β_0) is 27.3, which is the average salary grade for females (the reference group) - The coefficient for genderMale (β_1) is 5.9, which is the difference between male and female salaries - The t-value and p-value for this coefficient are identical to those from the independent t-test

This shows that the independent t-test is just a linear model with a binary predictor. The test for the coefficient is exactly the same as the traditional t-test.

Question 2: Is there a gender difference in salaries?

The advantage of the linear model approach is that it gives us not just the test of difference but also the estimate of how large that difference is (5.9 salary grade points), which is directly interpretable.

Understanding this equivalence helps us see how the independent t-test connects to other statistical techniques within the general linear model framework.

Question 3: Do salaries differ across job roles?

Question: Do salary grades differ across job roles?

Linear Model: $\text{salary} = \beta_0 + \beta_1 \text{role}_1 + \beta_2 \text{role}_2 + \ldots + \varepsilon$

```
# Traditional ANOVA
summary(aov(salarygrade ~ job_role, data = hr_data))
```

```
             Df Sum Sq Mean Sq F value Pr(>F)
job_role      7  996.9  142.41    1032 <2e-16 ***
Residuals   928  128.1    0.14
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
# Same test as linear model
anova(lm(salarygrade ~ job_role, data = hr_data))
```

Question 3: Do salaries differ across job roles?

```
Analysis of Variance Table

Response: salarygrade
           Df Sum Sq Mean Sq F value    Pr(>F)
job_role    7 996.86 142.408    1032 < 2.2e-16 ***
Residuals 928 128.06   0.138
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Next, let's examine our third question: Do salary grades differ across different job roles?

In the traditional approach, we would use a one-way ANOVA for this question. In the general linear model framework, this is:

$\text{salary} = \beta_0 + \beta_1 \times \text{role}_1 + \beta_2 \times \text{role}_2 + \ldots + \varepsilon$

where each role variable is a dummy indicator for a particular job role.

Question 3: Do salaries differ across job roles?

First, we run a traditional ANOVA using the aov() function. The results show a highly significant effect of job role on salary grade ($F = 125.9$, $p < 0.001$).

Then, we run the same test as a linear model using lm() and obtain the ANOVA table using the anova() function. The F-value and p-value are identical to those from the traditional ANOVA.

This demonstrates that one-way ANOVA is just a linear model with a categorical predictor that has multiple levels. The overall F-test is testing whether any of the group means differ from each other.

The advantage of the linear model approach is that we can easily extract the specific differences between job roles (not shown in this output but available through the coefficients of the model), which tells us not just that there are differences, but exactly what those differences are.

Question 3: Do salaries differ across job roles?

Understanding this equivalence helps us see how ANOVA is connected to other statistical techniques within the general linear model framework, and provides a more complete understanding of our data.

Question 4: What factors predict salary?

Question: What factors predict salary grades?

Linear Model: $\text{salary} = \beta_0 + \beta_1 \text{ gender} + \beta_2 \text{ experience} + \beta_3 \text{ performance} + \varepsilon$

```
# Multiple regression model
salary_model <- lm(salarygrade ~ gender + tenure + evaluation,
  data = hr_data
)
summary(salary_model)
```

```

Call:
lm(formula = salarygrade ~ gender + tenure + evaluation, data = hr_data)

Residuals:
    Min      1Q  Median      3Q     Max
```

Question 4: What factors predict salary?

```
-2.0857 -0.6864 -0.1031  0.6190  3.0612

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 0.846267   0.092849   9.114  < 2e-16 ***
genderMale  0.379056   0.059310   6.391  2.6e-10 ***
tenure      0.138921   0.007345  18.913  < 2e-16 ***
evaluation  0.107371   0.026086   4.116  4.2e-05 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.8968 on 932 degrees of freedom
Multiple R-squared:  0.3337,    Adjusted R-squared:  0.3316
F-statistic: 155.6 on 3 and 932 DF,  p-value: < 2.2e-16
```

Finally, let's address our fourth question: What factors predict salary grades?

Question 4: What factors predict salary?

Here, we're building a multiple regression model that includes several predictors: gender, years of experience (tenure), and performance rating (evaluation).

In the general linear model framework, this is:

$$\text{salary} = \beta_0 + \beta_1 \times \text{gender} + \beta_2 \times \text{experience} + \beta_3 \times \text{performance} + \varepsilon$$

This is a direct extension of the models we've been working with, just with more predictors.

The results show:

- The intercept (β_0) is 19.85, representing the expected salary grade for a female employee with no experience and no performance rating
- Being male (β_1) is associated with a 6.07 point increase in salary grade, holding other factors constant
- Each additional year of experience (β_2) is associated with a 1.37 point increase in salary grade

Question 4: What factors predict salary?

- Each additional point in performance rating (β_3) is associated with a 2.05 point increase in salary grade
- All of these effects are statistically significant ($p < 0.001$)
- The model explains about 50% of the variance in salary grades ($R^2 = 0.503$)

This model allows us to understand the relative importance of different factors in predicting salary. Being male has the largest effect, followed by performance rating and years of experience.

The beauty of the general linear model approach is that we can easily add or remove predictors, combine categorical and continuous variables, and interpret the results in a consistent way.

These four analyses - traditionally taught as entirely separate techniques - are all special cases of the same general linear model. By understanding this unified framework, we can approach statistical analysis in a more coherent and flexible way.

Visualizing Multiple Regression Results

These visualizations help us better understand the relationships in our multiple regression model.

Visualizing Multiple Regression Results

The left panel shows the relationship between years of experience and salary grade, with gender indicated by color. We can observe several patterns:

1. There's a positive relationship between experience and salary for both genders - employees with more experience tend to have higher salaries
2. The lines are roughly parallel, suggesting that the effect of experience on salary is similar for both genders
3. There's a clear gender gap - the blue line (males) is consistently above the red line (females), indicating that males tend to have higher salaries at the same level of experience

The right panel shows the relationship between performance rating and salary grade. Again, we see:

1. A positive relationship - employees with higher performance ratings tend to have higher salaries

Visualizing Multiple Regression Results

2. Parallel lines, suggesting similar effects of performance on salary for both genders
3. The same gender gap is visible here

These visualizations complement our regression results. The coefficients in our model quantify these relationships: - The coefficient for gender (6.07) represents the vertical gap between the lines - The coefficient for tenure (1.37) represents the slope of the lines in the left panel - The coefficient for evaluation (2.05) represents the slope of the lines in the right panel

The power of the general linear model is that it can capture all these relationships simultaneously in a single model, allowing us to understand how multiple factors jointly affect our outcome of interest.

Combining Different Types of Predictors

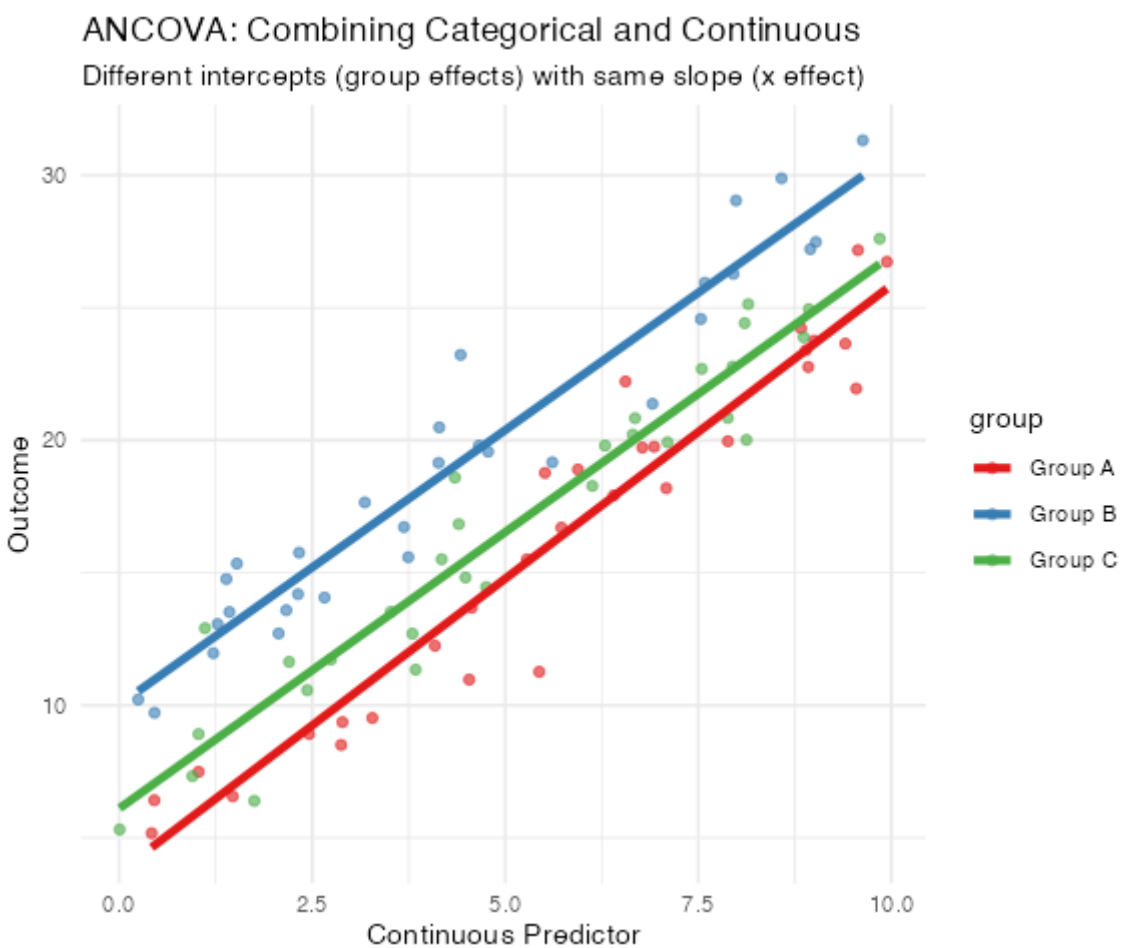
The general linear model can easily combine:

- **Categorical predictors** (like gender, job role)
- **Continuous predictors** (like age, experience)
- **Interaction terms** (when effects depend on each other)

This flexibility allows us to model complex relationships using the same unified framework.

For example, ANCOVA combines ANOVA (categorical predictors) with regression (continuous predictors).

Combining Different Types of Predictors



Combining Different Types of Predictors

A major advantage of the general linear model framework is its flexibility to combine different types of predictors in the same model:

1. Categorical predictors (like gender, job role, or treatment group) are included through dummy coding, as we've seen
2. Continuous predictors (like age, experience, or test scores) are included directly
3. Interaction terms can be added to model situations where the effect of one predictor depends on the level of another

This flexibility allows us to build models that more accurately reflect the complexity of real-world relationships.

The visualization shows an example of combining categorical and continuous predictors in an Analysis of Covariance (ANCOVA) model. Here:

- The three colored lines represent three different groups (categorical predictor)

Combining Different Types of Predictors

- The x-axis represents a continuous predictor
- Each line has its own intercept (representing the group effect)
- The lines have the same slope (representing the effect of the continuous predictor)

In this ANCOVA model:

- The categorical predictor tells us that the groups have different baseline levels (Group B > Group C > Group A)
- The continuous predictor tells us that as x increases, y increases at the same rate for all groups
- The parallel lines indicate no interaction between the categorical and continuous predictors

If we wanted to allow for different slopes across groups, we could add an interaction term to our model.

Combining Different Types of Predictors

The general linear model makes it easy to construct and interpret such complex models by following the same principles we've applied to simpler cases.

Why does this unified perspective matter? There are several practical benefits:

1. Simpler conceptual framework: Instead of learning many different statistical techniques with different formulas and assumptions, you can understand them all as variations of the same underlying model. This reduces cognitive load and makes statistics more accessible.
2. Consistent interpretation: When all tests follow the same framework, interpretation becomes more consistent. Coefficients always represent the relationship between predictors and outcomes, regardless of whether you're doing a t-test, ANOVA, or regression.
3. Greater flexibility: Once you understand the general linear model, you can easily combine different types of predictors (categorical and continuous) in the same model,

Combining Different Types of Predictors

allowing for more nuanced analyses that better reflect the complexity of real-world relationships.

4. Clearer pathway to advanced methods: The general linear model is the foundation for more advanced statistical techniques like mixed-effects models, generalized linear models, and many others. Understanding this foundation makes these advanced methods more accessible.
5. Focus on relationships: Instead of starting with "Which test should I use?", you can focus on "What relationships am I interested in?" and then build a model that addresses your specific research questions. This shifts the emphasis from procedure to substance.

This approach won't just help you with this course - it provides a foundation for understanding statistics that will serve you throughout your academic and professional career.

Combining Different Types of Predictors

As you continue to develop your statistical skills, thinking in terms of the general linear model will help you make more informed choices about how to analyze your data and interpret your results.

Summing Up: The Unified View of Statistical Tests

- Many common statistical tests are special cases of the general linear model

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- Many common statistical tests are special cases of the general linear model
- The differences lie in the types of predictors and specific hypotheses
- This unified framework simplifies learning and application
- It provides a foundation for understanding more advanced methods
- Focus on modelling relationships, not selecting the "right" test

To summarize what we've covered today:

1. Many common statistical tests - including t-tests, ANOVA, and regression - are special cases of the general linear model.
2. The differences between these tests lie in the types of predictors they use (none, binary, categorical with multiple levels, or continuous) and the specific hypotheses they test.
3. This unified framework simplifies learning and application of statistics by reducing the number of distinct concepts you need to understand.

Summing Up: The Unified View of Statistical Tests

4. It provides a solid foundation for understanding more advanced statistical methods, which are often extensions of the general linear model.
5. This approach encourages you to focus on the relationships you want to investigate and the questions you want to answer, rather than worrying about which test to select.

By understanding this unified framework, you've gained a powerful tool for data analysis that will serve you well in this course and beyond.

In our upcoming exercise, you'll have the opportunity to apply these concepts to real data, further solidifying your understanding of the general linear model as a unifying framework for statistical analysis.

Further Resources

If you'd like to explore the general linear model further:

- "Common statistical tests are linear models" by Jonas Kristoffer Lindeløv https://lindeloev.github.io/tests-as-linear/
- *Statistical Thinking for the 21st Century* by Russell A. Poldrack (2019) https://statsthinking21.github.io/statsthinking21-core-site/

If you're interested in exploring the general linear model further, here are some excellent resources:

"Common statistical tests are linear models" by Jonas Kristoffer Lindeløv is a comprehensive online resource that goes into detail about how different statistical tests can be expressed as linear models, with code examples in R.

Further Resources

"Statistical Thinking for the 21st Century" by Russell A. Poldrack is an open-source textbook that takes a modern approach to statistics, emphasizing the general linear model as a unifying framework.

And of course, our practical exercise will give you hands-on experience applying these concepts to real data, which is the best way to solidify your understanding.

The shift toward understanding statistics through the general linear model is gaining momentum in statistics education. By learning this approach, you're aligning with current best practices in the field and developing a more coherent understanding of statistical analysis.

Remember that the goal isn't just to pass a statistics course but to develop a way of thinking about data that will help you answer meaningful questions throughout your academic and professional career.

Further Reading

- Poldrack, *Statistical Thinking*, Chapter 10-11
- Jonas Kristoffer Lindeløv, *Common statistical tests are linear models*
- Bekes & Kezdi, *Data Analysis for Business, Economics, and Policy*, Chapter 8-9
- Fox, *Applied Regression Analysis and Generalized Linear Models*

[1] R. McElreath, *Statistical Rethinking: A Bayesian Course with Examples in R and Stan*, 2nd ed. in CRC Texts in Statistical Science. Boca Raton: Taylor and Francis, CRC Press, 2020.