ANOVA as a Linear Model

ANOVA: Comparing Multiple Groups

ANOVA (Analysis of Variance) is traditionally taught as a distinct statistical test for comparing means across multiple groups.

- Null hypothesis: All group means are equal ($\mu_1 = \mu_2 = \ldots = \mu_k$)
- Alternative hypothesis: At least one group mean differs from the others
- Test statistic: F-ratio (ratio of between-group to within-group variance)

ANOVA is traditionally taught as a distinct test from regression, with its own set of formulas and concepts like "sums of squares" and "F-ratios." However, ANOVA is actually just another manifestation of the general linear model.

The key insight is that when we compare means across groups, we're essentially predicting an outcome (y) based on group membership (a categorical variable). This can be seamlessly represented within the linear model framework.

Fuel Consumption Dataset

Let's use a real dataset on fuel consumption in Canada to demonstrate ANOVA as a linear model.

```r
# View the structure of the fuel consumption dataset
fuel_data |>
  select(make, model, class, enginesize, cylinders468, fueluseboth) |>
  head(5) |>
  kable()
```

make	model	class	enginesize	cylinders468	fueluseboth
ACURA	ILX	COMPACT	2.0	4	8.3
ACURA	ILX	COMPACT	2.4	4	9.3
ACURA	ILX HYBRID	COMPACT	1.5	4	6.1
ACURA	MDX SH-AWD	SUV - SMALL	3.5	6	11.1
ACURA	RDX AWD	SUV - SMALL	3.5	6	10.6

This dataset contains information about vehicles sold in Canada, including their fuel consumption (measured in liters per 100 kilometers), engine characteristics, and vehicle class.

We'll use this data to compare average fuel consumption across different vehicle classes, first using traditional ANOVA and then showing the equivalent linear model approach.

One-way ANOVA: Traditional Approach

Let's compare fuel consumption across vehicle classes:

```r
# Run traditional ANOVA
anova_result <- aov(fueluseboth ~ class, data = fuel_data)
summary(anova_result)
```

```
              Df Sum Sq Mean Sq F value Pr(>F)    
class         14   4099  292.78   62.77 <2e-16 ***
Residuals   1067   4977    4.66                   
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

The significant p-value (< 0.05) indicates that average fuel consumption differs significantly across vehicle classes.

The traditional ANOVA output shows us the familiar ANOVA table with sums of squares, degrees of freedom, mean squares, and the F-statistic. The very small p-value tells us that there are significant differences in fuel consumption between vehicle classes.

But how does this relate to the linear model? Let's see.

One-way ANOVA as Linear Model

The same analysis using the linear model approach:

```r
# Run equivalent linear model
lm_result <- lm(fueluseboth ~ class, data = fuel_data)
anova(lm_result)
```

```
Analysis of Variance Table

Response: fueluseboth
            Df Sum Sq Mean Sq F value    Pr(>F)    
class       14 4098.9 292.775  62.772 < 2.2e-16 ***
Residuals 1067 4976.6   4.664                      
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Notice the identical F-value and p-value as the traditional ANOVA!

When we run the same analysis using lm() instead of aov(), and then use anova() on the result, we get the exact same F-value and p-value as the traditional ANOVA. That's because they're mathematically equivalent - ANOVA is just a linear model with categorical predictors.

In this linear model, we're predicting fuel consumption based on vehicle class. The model creates dummy variables for each vehicle class (except one, which serves as the reference group).

Understanding the Linear Model Coefficients

```r
# View coefficients from the linear model
tidy(lm_result) |>
  kable(digits = 3)
```

term	estimate	std.error	statistic	p.value
(Intercept)	9.355	0.162	57.792	0.000
classFULL-SIZE	2.504	0.285	8.793	0.000
classMID-SIZE	0.279	0.229	1.220	0.223
classMINICOMPACT	0.747	0.329	2.272	0.023
classMINIVAN	2.925	0.581	5.037	0.000
classPICKUP TRUCK - SMALL	2.531	0.488	5.186	0.000
classPICKUP TRUCK - STANDARD	4.905	0.340	14.407	0.000
classSPECIAL PURPOSE VEHICLE	0.734	0.738	0.995	0.320
classSTATION WAGON - MID-SIZE	0.359	0.832	0.432	0.666
classSTATION WAGON - SMALL	−0.774	0.415	−1.866	0.062
classSUBCOMPACT	0.951	0.284	3.352	0.001
classSUV - SMALL	1.038	0.231	4.495	0.000
classSUV - STANDARD	4.497	0.271	16.610	0.000
classTWO-SEATER	1.575	0.303	5.194	0.000
classVAN - PASSENGER	10.466	0.521	20.079	0.000

Interpretation: - The intercept (9.171) is the mean fuel consumption for the reference class (COMPACT) - Each coefficient represents the difference between that class and the reference class - E.g., FULL-SIZE vehicles consume 3.514 L/100km more fuel than COMPACT vehicles, on average

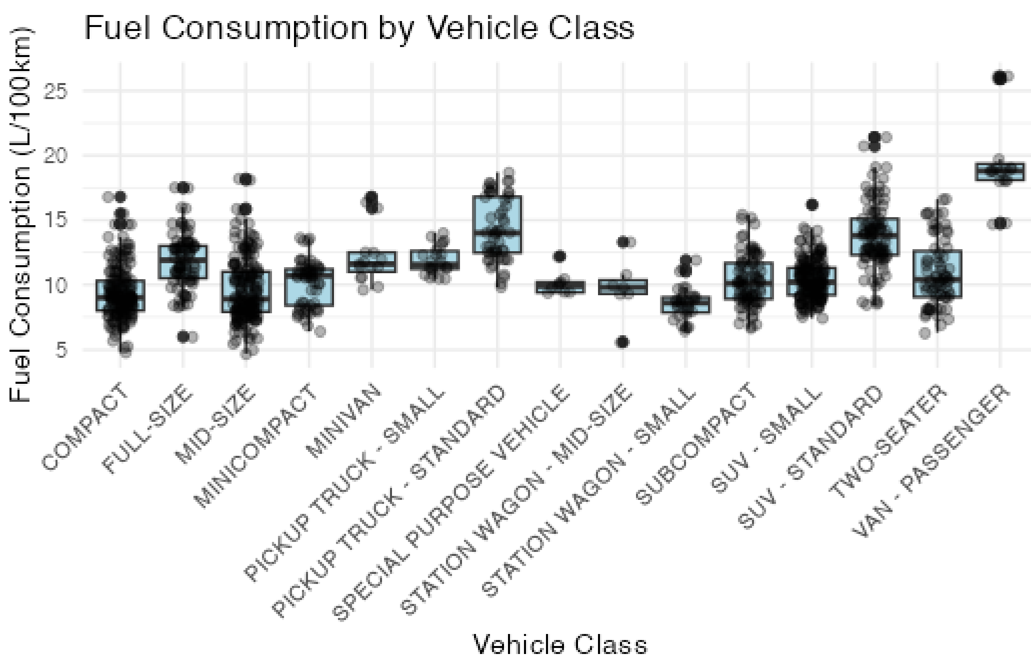
Looking at the coefficients from the linear model gives us more detailed information than the ANOVA table alone. The intercept represents the mean fuel consumption for the reference group (in this case, COMPACT vehicles).

Each other coefficient represents the difference in mean fuel consumption between that vehicle class and the reference class. For example, the coefficient for classFULL-SIZE is 3.514, which means that, on average, full-size vehicles consume 3.514 liters per 100km more fuel than compact vehicles.

This is a much more detailed result than the overall ANOVA, which only tells us that there are differences somewhere. The linear model pinpoints exactly where those differences are.

Visualizing the ANOVA Results

```r
# Create a visualization of group means
ggplot(fuel_data, aes(x = class, y = fueluseboth)) +
  geom_boxplot(fill = "lightblue") +
  geom_point(position = position_jitter(width = 0.2), alpha = 0.3) +
  theme_minimal() +
  theme(axis.text.x = element_text(angle = 45, hjust = 1)) +
  labs(
    x = "Vehicle Class", y = "Fuel Consumption (L/100km)",
    title = "Fuel Consumption by Vehicle Class"
  )
```



This visualization helps us see the differences in fuel consumption across vehicle classes. We can visually confirm that larger vehicle classes like full-size, SUV, and pickup trucks tend to have higher fuel consumption than compact and subcompact vehicles.

The boxplots show the median (middle line), quartiles (box), and range (whiskers) of fuel consumption for each class, while the individual points represent actual vehicles in the dataset.

Two-way ANOVA: Adding Another Factor

Let's extend our model to include the number of cylinders468:

```r
# Create a simplified cylinder factor
fuel_data <- fuel_data |>
  mutate(cyl_factor = factor(case_when(
    cylinders468 <= 4 ~ "4 or fewer",
    cylinders468 == 6 ~ "6",
```

```r
    cylinders468 >= 8 ~ "8 or more"
  )))

# Run two-way ANOVA
two_way_model <- lm(fueluseboth ~ class + cyl_factor, data = fuel_data)
anova(two_way_model)
```

```
Analysis of Variance Table

Response: fueluseboth
             Df Sum Sq Mean Sq F value    Pr(>F)    
class        14 4098.9  292.78  131.26 < 2.2e-16 ***
cyl_factor    2 2601.1 1300.53  583.05 < 2.2e-16 ***
Residuals  1065 2375.6    2.23                      
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Both vehicle class and number of cylinders468 significantly affect fuel consumption.

Here we've extended our model to include two factors: vehicle class and number of cylinders468. This is called a two-way ANOVA in traditional statistics.

The ANOVA table shows that both factors have significant effects on fuel consumption. In other words, fuel consumption varies significantly based on both vehicle class and number of cylinders468.

But this model only looks at the main effects - it doesn't consider interactions between the factors.

Adding Interaction Effects

In the linear model framework, interactions are easy to add:

```r
# Run two-way ANOVA with interaction
interaction_model <- lm(fueluseboth ~ class * cyl_factor, data = fuel_data)
anova(interaction_model)
```

```
Analysis of Variance Table

Response: fueluseboth
                   Df Sum Sq Mean Sq  F value  Pr(>F)    
class              14 4098.9  292.78 133.0732 < 2e-16 ***
cyl_factor          2 2601.1 1300.53 591.1207 < 2e-16 ***
class:cyl_factor   21   78.7    3.75   1.7024 0.02506 *  
Residuals        1044 2296.9    2.20                     
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

The interaction term tests whether the effect of one factor depends on the level of the other factor.

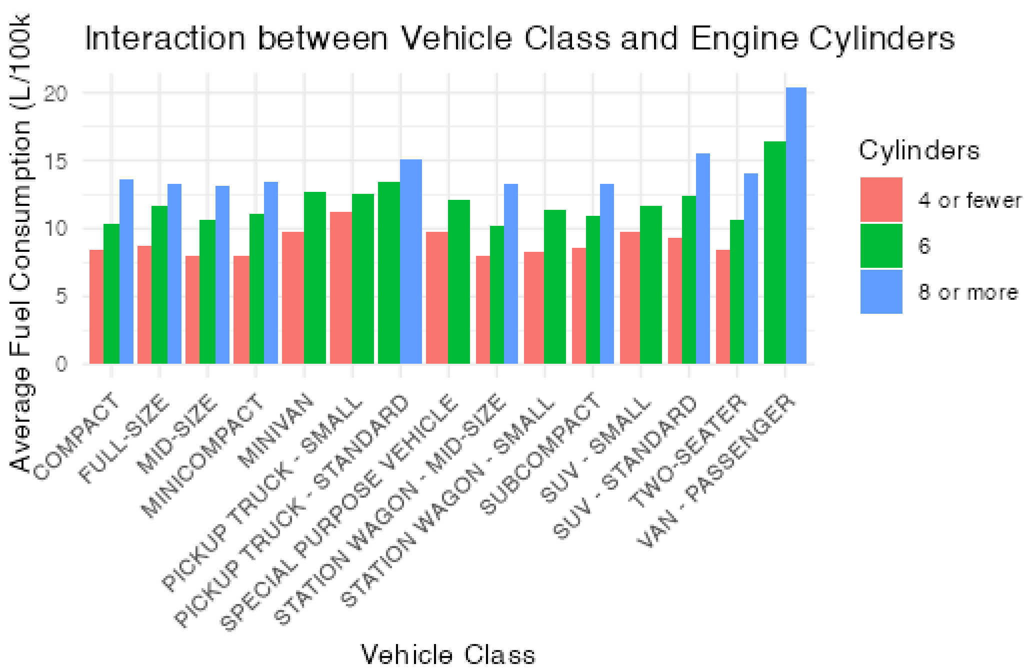
An interaction effect occurs when the effect of one factor depends on the level of another factor. For example, the difference in fuel consumption between 4-cylinder and 8-cylinder engines might be larger for SUVs than for compact cars.

In the linear model, we can easily test for interactions by using the * operator instead of +. This adds both main effects and their interaction.

The ANOVA table shows a significant interaction effect, indicating that the effect of cylinders468 on fuel consumption differs across vehicle classes (or equivalently, the effect of vehicle class differs depending on the number of cylinders468).

Visualizing the Interaction

```r
# Create an interaction plot
ggplot(fuel_data, aes(x = class, y = fueluseboth, fill = cyl_factor)) +
  stat_summary(fun = mean, geom = "bar", position = "dodge") +
  theme_minimal() +
  theme(axis.text.x = element_text(angle = 45, hjust = 1)) +
  labs(
    x = "Vehicle Class", y = "Average Fuel Consumption (L/100km)",
    fill = "Cylinders",
    title = "Interaction between Vehicle Class and Engine Cylinders"
  )
```



This bar chart helps visualize the interaction effect. Each group of bars represents a vehicle class, and the different colored bars within each group represent different cylinder categories.

If there were no interaction, the pattern of differences between cylinder categories would be consistent across all vehicle classes. The fact that the pattern varies - for example, the difference between 4-cylinder and 8-cylinder engines seems larger for some vehicle classes than others - illustrates the interaction effect.

This is a powerful aspect of the general linear model: it allows us to model and interpret complex relationships between variables, including interactions.

ANCOVA: Mixing Categorical and Continuous Predictors

ANCOVA (Analysis of Covariance) combines ANOVA with regression by including both categorical and continuous predictors:

```r
# Run ANCOVA
ancova_model <- lm(fueluseboth ~ class + enginesize, data = fuel_data)
summary(ancova_model)
```

```

Call:
lm(formula = fueluseboth ~ class + enginesize, data = fuel_data)

Residuals:
    Min      1Q  Median      3Q     Max 
-4.2020 -0.7662 -0.1031  0.5934  6.5633 

Coefficients:
                              Estimate Std. Error t value Pr(>|t|)    
(Intercept)                    5.70095    0.15343  37.156  < 2e-16 ***
classFULL-SIZE                 0.53708    0.20372   2.636 0.008500 ** 
classMID-SIZE                 -0.45814    0.15871  -2.887 0.003972 ** 
classMINICOMPACT              -0.05542    0.22699  -0.244 0.807157    
classMINIVAN                   1.56351    0.40083   3.901 0.000102 ***
classPICKUP TRUCK - SMALL      1.49943    0.33662   4.454 9.30e-06 ***
classPICKUP TRUCK - STANDARD   1.77451    0.25079   7.076 2.69e-12 ***
classSPECIAL PURPOSE VEHICLE   0.87872    0.50691   1.733 0.083302 .  
classSTATION WAGON - MID-SIZE -0.62622    0.57240  -1.094 0.274190    
classSTATION WAGON - SMALL    -0.02474    0.28570  -0.087 0.931013    
classSUBCOMPACT                0.25683    0.19587   1.311 0.190059    
classSUV - SMALL               0.79530    0.15879   5.009 6.41e-07 ***
classSUV - STANDARD            1.68690    0.20301   8.310 2.89e-16 ***
classTWO-SEATER                0.30611    0.21146   1.448 0.148023    
classVAN - PASSENGER           6.11458    0.37956  16.110  < 2e-16 ***
enginesize                     1.48977    0.04310  34.567  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.484 on 1066 degrees of freedom
```

```
Multiple R-squared:  0.7414,	Adjusted R-squared:  0.7378 
F-statistic: 203.8 on 15 and 1066 DF,  p-value: < 2.2e-16
```

This model predicts fuel consumption based on both vehicle class (categorical) and engine size (continuous).

ANCOVA is traditionally taught as yet another distinct technique, but in the general linear model framework, it's simply a model that includes both categorical and continuous predictors.

In this model, we're predicting fuel consumption based on vehicle class and engine size. The coefficients for vehicle class represent the differences between classes after controlling for engine size. The coefficient for engine size represents the effect of engine size on fuel consumption, controlling for vehicle class.

This is another example of how the general linear model provides a unified framework for various statistical techniques.

Effect Sizes: Understanding Practical Significance

Statistical significance (p-values) tells us if effects are likely real, but effect sizes tell us if they're practically important:

```
# Calculate effect sizes
eta_squared(anova_result)
```

```
# Effect Size for ANOVA

Parameter | Eta2 |       95% CI
-------------------------------
class     | 0.45 | [0.41, 1.00]

- One-sided CIs: upper bound fixed at [1.00].
```

Interpretation:

- η^2 = proportion of variance explained by each factor
- Vehicle class explains about 43% of the variance in fuel consumption
- Values of 0.01, 0.06, and 0.14 are considered small, medium, and large effects

While p-values tell us whether an effect is statistically significant (unlikely to be due to chance), effect sizes tell us about the practical significance or magnitude of the effect.

For ANOVA, a common effect size is eta-squared (η^2), which represents the proportion of variance explained by each factor. Values around 0.01 are considered small, 0.06 medium, and 0.14 large.

The eta-squared value of 0.43 for vehicle class indicates that about 43% of the variance in fuel consumption is explained by vehicle class, which is a very large effect.

Effect sizes are important because with large enough sample sizes, even tiny, practically meaningless effects can become statistically significant.

Post-hoc Tests: Which Groups Differ?

When ANOVA finds significant differences, post-hoc tests help identify which specific groups differ:

```r
# Calculate estimated marginal means
emm <- emmeans(lm_result, ~class)

# Pairwise comparisons with Tukey adjustment
pairs(emm) |>
  as_tibble() |>
  filter(p.value < 0.05) |>
  arrange(p.value) |>
  head(5) |>
  kable(digits = 3)
```

contrast	estimate	SE	df	t.ratio	p.value
COMPACT - (PICKUP TRUCK - STANDARD)	−4.905	0.340	1067	−14.407	0
COMPACT - (SUV - STANDARD)	−4.497	0.271	1067	−16.610	0
COMPACT - (VAN - PASSENGER)	−10.466	0.521	1067	−20.079	0
(FULL-SIZE) - (VAN - PASSENGER)	−7.962	0.548	1067	−14.528	0
(MID-SIZE) - (PICKUP TRUCK - STANDARD)	−4.625	0.340	1067	−13.586	0

When ANOVA indicates significant differences between groups, we often want to know which specific groups differ from each other. Post-hoc tests help answer this question.

Here we're using estimated marginal means and pairwise comparisons with Tukey's adjustment for multiple comparisons. The results show the estimated difference between each pair of vehicle classes, along with confidence intervals and adjusted p-values.

The table shows the 5 most significant pairwise differences. For example, fuel consumption differs significantly between SUV-UTILITY and COMPACT-SUV vehicle classes.

This is another example of how the linear model framework provides a comprehensive approach to statistical analysis, from overall tests to detailed comparisons.

Integrated Example: HR Analytics with ANOVA

Let's return to our HR dataset and use ANOVA to compare job satisfaction across job roles:

```r
# Load HR Analytics dataset if not already loaded
if (!exists("hr_data")) {
       hr_data    <-    read_sav("data/dataset-abc-insurance-hr-data.sav")    |>
```

```r
janitor::clean_names()
}

# Run ANOVA for job satisfaction by department
hr_anova <- lm(job_satisfaction ~ job_role, data = hr_data)
summary(hr_anova)
```

```

Call:
lm(formula = job_satisfaction ~ job_role, data = hr_data)

Residuals:
     Min       1Q   Median       3Q      Max 
-2.67123 -0.82659  0.00448  0.83555  2.17341 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  2.65766    0.06886  38.597  < 2e-16 ***
job_role     0.16893    0.02155   7.839 1.23e-14 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.103 on 934 degrees of freedom
Multiple R-squared:  0.06173,	Adjusted R-squared:  0.06073 
F-statistic: 61.45 on 1 and 934 DF,  p-value: 1.235e-14
```

Now let's apply what we've learned to our HR analytics dataset. Here we're comparing job satisfaction across different job roles using a linear model (which is equivalent to ANOVA).

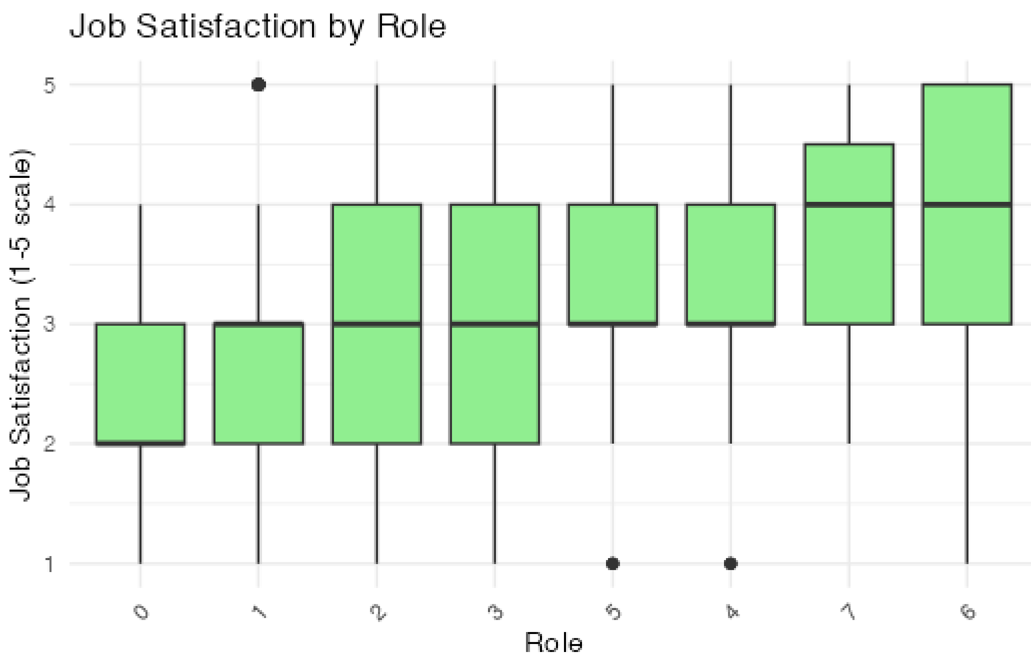
The results show the mean job satisfaction for the reference role (the intercept) and the differences between each other role and the reference role Some departments appear to have significantly higher or lower job satisfaction than others.

This is a practical application of ANOVA as a linear model in a human resources context.

Visualizing HR Department Differences

```r
# Create a visualization of satisfaction by department
ggplot(hr_data, aes(
  x = reorder(job_role, job_satisfaction, FUN = mean),
  y = job_satisfaction
)) +
  geom_boxplot(fill = "lightgreen") +
  theme_minimal() +
  theme(axis.text.x = element_text(angle = 45, hjust = 1)) +
  labs(
    x = "Role", y = "Job Satisfaction (1-5 scale)",
```

```r
    title = "Job Satisfaction by Role"
  )
```



This visualization helps us see the differences in job satisfaction across departments. The departments are ordered by their mean job satisfaction, with departments having higher average satisfaction appearing towards the right.

We can see variations in both the central tendency (median, indicated by the line in the middle of each box) and the spread of job satisfaction scores within each department.

This kind of analysis could help HR identify departments that might need intervention to improve employee satisfaction, or departments with particularly high satisfaction that might serve as models for others.

Combining ANOVA and Regression

We can build more complex models that include: - Multiple categorical predictors (multi-way ANOVA) - Continuous predictors alongside categorical ones (ANCOVA) - Interaction terms between predictors

```r
# Build a complex model
complex_model <- lm(job_satisfaction ~ job_role + gender + evaluation +
  job_role:evaluation, data = hr_data)

# View model summary
anova(complex_model) |>
```

```r
  as_tibble() |>
  kable(digits = 3)
```

Df	Sum Sq	Mean Sq	F value	Pr(>F)
1	74.760	74.760	81.951	0.000
1	1.478	1.478	1.620	0.203
1	285.426	285.426	312.880	0.000
1	0.101	0.101	0.111	0.739
931	849.307	0.912	NA	NA

Here we've built a more complex model that includes multiple predictors: department (categorical), gender (categorical), and performance rating (continuous), as well as an interaction between department and performance rating.

This model tests whether job satisfaction varies by department, gender, and performance rating, and whether the relationship between performance rating and job satisfaction differs across departments.

The ANOVA table shows which effects are statistically significant. This demonstrates how the general linear model framework allows us to build and test complex models that would be difficult to conceptualize using traditional statistical procedures taught in isolation.

The Power of the Unified Approach

Benefits of viewing statistical tests as linear models:

1. **Conceptual simplicity**: Learn one framework instead of many isolated techniques
2. **Flexibility**: Easily combine and extend models to suit your research questions
3. **Interpretability**: Consistent approach to understanding and communicating results
4. **Practicality**: Simplifies implementation in statistical software
5. **Extensibility**: Natural pathway to more advanced methods (mixed effects, generalized linear models)

The unified linear model approach offers several benefits over the traditional approach of teaching statistical tests as separate, unrelated techniques.

First, it's conceptually simpler. Instead of learning different formulas and procedures for t-tests, ANOVA, regression, etc., you learn one framework that encompasses all of these.

Second, it's more flexible. You can easily combine different types of predictors and test complex hypotheses within the same framework.

Third, it provides a consistent approach to interpretation. The coefficients in a linear model always have the same basic interpretation, regardless of whether the model is implementing a t-test, ANOVA, or regression.

Fourth, it's practical. In R and many other statistical software packages, the linear model (lm() function in R) is the workhorse for a wide range of analyses.

Finally, it provides a natural pathway to more advanced methods like mixed-effects models and generalized linear models, which extend the linear model framework to handle more complex data structures and non-normal distributions.

Concluding Thoughts

- Statistical tests are not isolated tools but connected members of the same family
- The general linear model provides a unified framework for understanding these connections
- This perspective simplifies learning, application, and interpretation of statistics
- When facing a new analytical problem, think in terms of the linear model: what is my outcome? What are my predictors? What relationships am I testing?

In conclusion, the general linear model provides a powerful, unified framework for statistical analysis. By understanding that many common statistical tests are special cases of the linear model, we gain a deeper and more coherent understanding of statistics.

Rather than memorizing different formulas and procedures for different tests, we can focus on understanding the core principles of the linear model and how to apply them to different research questions.

When approaching a new analytical problem, thinking in terms of the linear model helps clarify the essential components: the outcome variable, the predictor variables, and the relationships we're interested in testing.

This approach not only simplifies learning and application but also enables us to build more sophisticated models that better capture the complexity of real-world phenomena.

Bibliography