Distributed Operating Room Scheduling using Lazy Constraints and Networks

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Abstract

We re-implemented existing models[1] for solving the distributed operating room scheduling (DORS) problem. We extended the existing formulation with a new LBBD cut and a network formulation, we extended implementation by employing lazy constraints in Gurobi. Results in line with the original paper were not able to be replicated. It was found that all models were unable to scale to larger than 20 patients when solving to optimality. It was found that all models were able to scale up to 80 patients when solving to a 1% gap. The pure MIP exhibited the best performance in terms of both its ability to solve within time limits and speed when solving to both optimality and a 1% gap.

1 Introduction

Our report is based upon and summarises Roshanaei, Luong, Aleman and Urbach [1] which we will also refer to as *the original paper* throughout. The paper solves the deterministic distributed operating room scheduling (DORS) problem using Logic-Based Benders' Decomposition (LBBD) and a cut propagation method. This formulation has been extended upon in a stochastic setting[2]. We re-implemented the paper and extended the formulation with a new LBBD cut and a network model. We extended implementation by using lazy constraints in Gurobi. Our main goals were to replicate the results found in the original paper and investigate if there was potential improvement to be had by using lazy constraints instead of iteration.

In the DORS problem, we seek to decide which hospital operating suites to open and which of their respective operating rooms to open. We also decide which patients to allocate to each hospital day. We make these decisions in such a way as to minimize the total cost of opening facilities while also trying to maximize the reward for assigning critical patients[1].

The LBBD framework outlined in [1] is structured as a location-allocation master problem with disaggregated bin-packing sub problems simplified by symmetric operating rooms. Both the master problem and sub problem are solved as mixed integer programming (MIP) problems. The results in the original paper show that the proposed LBBD models scale well for up to 3 hospitals, 5 operating rooms and 160 patients.

The original paper contains many variants of models, they provide a pure MIP formulation, and an LBBD formulation with three different types of cuts. For each of these cuts they experiment with a version with and without propagation. For each of these they implement different cut generation schemes which determine how many sub problems to iterate over before re-solving the master problem. We choose some of their model variants to best achieve our aims. LBBD1 and LBBD2 cuts from [1] were chosen to be reimplemented using the maximal cut generation scheme. LBBD1 was to be baseline and LBBD2 was a top performer. Maximal cut generation iterates over all sub problems before re-solving the master problem.

2 Model Formulation

2.1 Master Problem

The following is the LBBD formulation of the problem. For a pure IP formulation refer to [1]. We will first outline the formulation of the master problem. The master problem handles assignment of patients to hospital-days.

SETS

$\mathcal{P}$	Set of patients $p \in \mathcal{P}$
$\mathcal{P}'$	Set of mandatory patients, $\mathcal{P}' = \{p \vert \rho_p(\vert\mathcal{D}\vert - \alpha_p) \leq -\Gamma\}$
$\mathcal{H}$	Set of hospitals, $h \in \mathcal{H}$
$\mathcal{D}$	Set of days in the planning horizon, $d \in \mathcal{D}$
$\mathcal{R}_h$	Set of ORs in each hospital's surgical suite, $r \in \mathcal{R}_h$

DATA

G_{hd}	Cost of opening the surgical suite in hosptial h on day d.
F_{hd}	Cost of opening and OR in hospital h on day d.
B_{hd}	Regular operating hours of each OR on day d. in hospital h.
T_{hp}	Total booked time (preparation time + surgery time + cleaning time) of patient p.
ρ_p	Health status score assigned to patient p.
α_p	Number of days elapsed from the referal date of patient p.
κ_1	Waiting cost for scheduled patients.
κ_2	Waiting cost for unscheduled patients.
Γ	Health status threshold above which patients have to be operated.

VARIABLES

x_{hdp}	1 if patient p is assigned to hospital h on day d, 0 otherwise.
u_{hd}	1 if the surgical suite in hospital h is opened on day d, 0 otherwise.
y_{hd} rooms	$\in \mathbb{Z}^+$, lower bound on number of operating open in hospital h on day d.
w_p	1 if patient p is not scheduled this horizon, 0 otherwise.

Objective The objective function balances the minimisation of costs associated with opening hospitals and ORs and maximising the reward of assigning patients to surgeries.

$$\text{minimize} \Bigg(\sum_{h\in\mathcal{H}} \sum_{d\in\mathcal{D}} G_{hd} U_{hd} + \sum_{h\in\mathcal{H}} \sum_{d\in\mathcal{D}} F_{hd} y_{hd} \quad (1)$$
$$+ \sum_{h\in\mathcal{H}} \sum_{d\in\mathcal{D}} \sum_{p\in\mathcal{P}} \kappa_1 [\rho_p (d - \alpha_p) x_{hdp}]$$
$$+ \sum_{p\in\mathcal{P}\setminus\{\mathcal{P}'\}} \kappa_2 [\rho_p (\mathcal{D} + 1 - \alpha_p) w_p] \Bigg)$$

Constraints The constraints for the MP are formulated as follows.

$$\sum_{h\in\mathcal{H}} \sum_{d\in\mathcal{D}} x_{hdp} = 1 \qquad \forall p \in \mathcal{P}' \tag{2}$$

$$\sum_{h\in mathcalH} \sum_{d\in\mathcal{D}} x_{hdp} + w_p = 1 \qquad \forall p \in \mathcal{P}\setminus\{\mathcal{P}'\} \tag{3}$$

$$x_{hdp} \leq u_{hd} \quad \forall h \in \mathcal{H}, d \in \mathcal{D}, p \in \mathcal{P} \tag{4}$$

$$\sum_{p\in\mathcal{P}} T_p x_{hdp} \leq |\mathcal{R}_h| B_{hd} u_{hd} \qquad \forall h \in \mathcal{H}, d \in \mathcal{D} \tag{5}$$

$$T_p\, x_{hdp} \leq B_{hd} \quad \forall h \in \mathcal{H}, d \in \mathcal{D}, p \in \mathcal{P} \tag{6}$$

$$\sum_{p\in\mathcal{P}} T_p x_{hdp} \leq B_{hd} y_{hd} \qquad \forall h \in \mathcal{H}, d \in \mathcal{D} \tag{7}$$

$$y_{hd} \leq |\mathcal{R}_h| \qquad \forall h \in \mathcal{H}, d \in \mathcal{D} \tag{8}$$

$$u_{hd}, x_{hdp} \in \{0, 1\} \quad \forall h \in \mathcal{H}, d \in \mathcal{D}, p \in \mathcal{P} \tag{9}$$

$$w_p \in \{0, 1\} \qquad \forall p \in \mathcal{P}\setminus\{\mathcal{P}'\} \tag{10}$$

Constraint (2) ensures all mandatory patients are assigned in the planning horizon. Constraint (3) ensures that variables x_{hdp} and u_{hd} are not turned on simultaneously. Constraint (4) ensures that if a patient is assigned a hospital-day then that hospital-day is open. (5) ensures that surgery time of patients assigned to a hospital-day does not exceed the available surgery time in that hospital day. (6) ensures an individual's surgery time does not exceed the hospital-day's available time. (7) ensures y_{hd} gives a lower bound on the number of operating rooms. (8) ensures y_{hd} does not exceed the number of operating rooms available on a hospital-day. Constraints (9) – (10) simply restrict variables to binary.

2.2 Subproblems

Given a solution to the master problem $(\widehat{Y}_{hd}^{(i)}, \widehat{\mathcal{P}}_{hd}^{(i)})$ the sub problem minimises the number of ORs to open for a given hospital day. Each sub problem is formulated as follows.

ADDITIONAL VARIABLES

y_r $\in \mathbb{Z}^+$, number of open operating rooms.
x_{pr} 1 if patient p is assigned to operating room r, 0 otherwise.

$$\text{minimise} \quad \overline{Y}_{hd} = \sum_{r \in \mathcal{R}_h} y_r \tag{11}$$

With constraints given as follows:

$$\sum_{r \in \mathcal{R}_h} x_{pr} = 1 \qquad \forall p \in \widehat{\mathcal{P}}_{hd}^{(i)} \tag{12}$$

$$\sum_{p \in \widehat{\mathcal{P}}_{hd}^{(i)}} T_p x_{pr} \leq B_{hd} y_r \qquad \forall r \in \mathcal{R}_h \tag{13}$$

$$x_{pr} \leq y_r \qquad \forall p \in \widehat{\mathcal{P}}_{hd}^{(i)}, r \in \mathcal{R}_h \tag{14}$$

$$y_r \leq y_{r-1} \qquad \forall r \in \mathcal{R}_h \backslash \{1\} \tag{15}$$

$$x_{pr}, \, y_r \in \{0, 1\} \qquad \forall p \in \widehat{\mathcal{P}}_{hd}^{(i)}, r \in \mathcal{R}_h \tag{16}$$

Constraint (12) ensures that each patient is assigned to only one operating room. Constraint (13) ensures that no OR is overcapacitated. Constraint (14) ensures that patients are assigned to open ORs. Constraint (15) breaks symmetry among ORs.

2.3 Benders Cuts

There are multiple forms of benders cuts outlined in the original paper. We will discuss pertinent forms based on their performance as discussed in the original paper. These are the LBBD1 and LBBD2 benders cuts. In order to describe these cut types, we will first discuss the first-fit decreasing heuristic algorithm (FFD) as this is used to determine optimality of SPs.

First-fit decreasing heuristic algorithm Since the SP packing problem can be difficult to solve, we can first find a feasible solution ($\overline{F}_{hd}^{(i)}$) using the FFD heuristic. This process is faster than other techniques such as integer and constraint programming. Moreover, we have the following relationship between the FFD, MP and SP solutions;

$$\hat{Y}_{hd}^{(i)} \leq \overline{Y}_{hd}^{(i)} \leq \overline{F}_{hd}^{(i)} \tag{17}$$

We can use the FFD solution $\left(\overline{F}_{hd}^{(i)}\right)$ to find an optimal SP solution $\left(\overline{Y}^{(i)hd}\right)$ without explicitly solving the SP. Moreover, if $\overline{F}_{hd}^{(i)} \neq \tilde{Y}_{hd}^{(i)}$ then when solving the SP we can use $\min\{\overline{F}_{hd}^{(i)}, |\mathcal{R}_h|\}$ as an upper bound.

LBBD1 LBBD1 [1] utilises both feasibility and optimality cuts to correct the MP to find a solution. If the SP is infeasible the following "no good" cut is added to the MP which requires at least one patient be removed from $\tilde{\mathcal{P}}_{hd}^{(i)}$.

$$\sum_{p \in \tilde{\mathcal{P}}_{hd}^{(i)}} (1 - x_{hdp}) \geq 1 \qquad \forall (h, d) \in \mathcal{U}_{hd}^{(i)} \tag{18}$$

Where $\mathcal{U}_{hd}^{(i)}$ is the set of infeasible SPs at this stage of solving the MP. If the SP is optimal, that is $\tilde{Y}_{hd}^{(i)} = \overline{Y}_{hd}^{(i)}$, no cuts are required. However, if this is not the case, the following optimality cut is added:

$$y_{hd} \geq \overline{Y}_{hd}^{(i)} - \sum_{p \in \hat{\mathcal{P}}} (1 - x_{hdp}) \qquad \forall (h, d) \in \overline{\mathcal{J}}^{(i)}.$$

Where $\overline{\mathcal{J}}^{(i)}$ is the set of SPs that are not optimal at a particular stage of solving the MP. This cut effectively has two results. It either forces at least one more OR open, or removes one patient from $\tilde{\mathcal{P}}_{hd}^{(i)}$.

LBBD2 LBBD2 differs from LBBD1 in its feasibility cut which is given as follows:

$$y_{hd} \geq (|\mathcal{R}_h| + 1) - \sum_{p \in \hat{\mathcal{P}}_{hd}^{(i)}} (1 - x_{hdp}) \quad (h, d) \in \overline{\mathcal{U}}^{(i)}.$$

If $\overline{Y}_{hd}^{(i)} = |\mathcal{R}_h|$ and the SP is infeasible then this cut simply removes one patient from $\overline{\mathcal{P}}_{hd}^{(i)}$. However, if $\hat{Y}_{hd}^{(i)} \leq |\mathcal{R}_h|$ and the SP is infeasible then the cut either removes two patients from $\overline{\mathcal{P}}_{hd}^{(i)}$, or removes one patient and/or opens at least one more OR.

LBBD4 We implement an improvement upon LBBD2 in addition to the original paper. We refer to this cut as LBBD4 to differentiate it from LBBD3 given in the original paper. This cut is described as follows:

$$\text{maxDur} = \max_{p \in \hat{\mathcal{P}}_{hd}^{(i)}} \{T[p]\},$$

$$y_{hd} \geq (|R_h| + 1) - \sum_{p \in \mathcal{P}} (1 - x_{hdp}) + \sum_{\substack{p \in \mathcal{P} \setminus \hat{\mathcal{P}}_{hd}^{(i)} | \\ T_p \geq \text{maxDur}}} x_{hdp}.$$

This cut has the same function as LBBD2 with the addition that it prescribes that any new patients that take the place of a removed patient must have an operation duration less than the maximum operation duration of patients for this sub problem.

Cut Propagation The original paper utilises cut propagation for LBBDs in order to generate multiple cuts for each infeasible SP. This is done by recognising that an infeasible set of patients for a particular hospital-day cannot be packed into a hospital-day with less or equal OR time.

2.4 Network Problem

In addition to the original paper, we give a network formulation of the problem. The problem contains the sets and data of the master problem previously outlined with the addition of the following sets and data. Nodes represent hospital-day-time, where time is in minute intervals. Arcs represent assignment of patients to operating rooms.

SETS

$\mathcal{N}$	Set of nodes $n \in \mathcal{N}$ for each hospital-day-minute
$\mathcal{N}'$	Set of nodes $n \in \mathcal{N}$ s. t. $\text{minDur} \leq n[time] \leq B_{n[time],n[day]} - \text{minDur}$
$\mathcal{A}$	Set of arcs $a \in \mathcal{A}$ between nodes $n \in N$

DATA

t_n	Arcs that enter node $n \in \mathcal{N}$
f_n	Arcs that leave node $n \in \mathcal{N}$
h_a	Hospital associated with arc $a \in \mathcal{A}$
d_a	Day associated with arc $a \in \mathcal{A}$
patient_a	Patient associated with arc $a \in \mathcal{A}$
start_a	Start time of arc $a \in \mathcal{A}$
end_a	End time of arc $a \in \mathcal{A}$
minDur	minimum surgery duration

The variables are also shared with the master problem with the addition of the following variable.

VARIABLES

z_a	1 if arc a is turned on, 0 otherwise.

Objective The objective function is identical to master problem objective given in 1.

$$\text{minimize} \Bigg(\sum_{h \in \mathcal{H}} \sum_{d \in \mathcal{D}} G_{hd} U_{hd} + \sum_{h \in \mathcal{H}} \sum_{d \in \mathcal{D}} F_{hd} y_{hd} \tag{19}$$

$$+ \sum_{h \in \mathcal{H}} \sum_{d \in \mathcal{D}} \sum_{p \in \mathcal{P}} \kappa_1 [\rho_p (d - \alpha_p) x_{hdp}] \tag{20}$$

$$+ \sum_{p \in \mathcal{P} \setminus \{\mathcal{P}'\}} \kappa_2 [\rho_p (\mathcal{D} + 1 - \alpha_p) w_p] \Bigg)$$

Constraints The constraints for the network formulation are given as follows,

$$\sum_{a \in f_n} z_a = \sum_{a \in t_n} z_a \qquad \forall n \in \mathcal{N}' \tag{21}$$

$$\sum_{\substack{a \in \mathcal{A} | \\ (h_a, d_a, \text{start}_a) \\ = (h,d,0)}} z_a \leq y_{hd} \qquad \forall d \in \mathcal{D}, \forall h \in \mathcal{H} \tag{22}$$

$$\sum_{\substack{a \in \mathcal{A} | \\ (h_a, d_a, \text{patient}_a) \\ = (h,d,p)}} z_a = x_{hdp} \quad \forall h \in \mathcal{H}, \forall d \in \mathcal{D}, \forall p \in \mathcal{P} \tag{23}$$

$$\sum_{h \in \mathcal{H}} \sum_{d \in \mathcal{D}} x_{hdp} = 1 \qquad \forall p \in \mathcal{P}' \tag{24}$$

$$\sum_{h \in \mathcal{H}} \sum_{d \in \mathcal{D}} x_{hdp} + w_p = 1 \qquad \forall p \in \mathcal{P} \setminus \{\mathcal{P}'\} \tag{25}$$

$$x_{hdp} \leq u_{hd} \qquad \forall h \in \mathcal{H}, d \in \mathcal{D}, p \in \mathcal{P} \tag{26}$$

$$y_{hd} \leq |\mathcal{R}_h| \qquad \forall h \in \mathcal{H}, d \in \mathcal{D} \tag{27}$$

$$u_{hd}, x_{hdp} \in \{0, 1\} \qquad \forall h \in \mathcal{H}, d \in \mathcal{D}, p \in \mathcal{P} \tag{28}$$

$$w_p \in \{0, 1\} \qquad \forall p \in \mathcal{P} \setminus \{\mathcal{P}'\} \tag{29}$$

We will only give descriptions of constraints unique to the network problem. Constraint 21 ensures conservation of flow between all nodes. Constraint 22 ensures that if an arc is turned on, then the surgical suite relating to that arc must be open at that time to accommodate it. Constraint 23 assigned patients to arcs.

3 Data

We generate our own data based on the parameters provided in the original paper. This data is derived from 7500 elective and emergency patients from 2011 to 2013 in the General Surgery Departments of the University Health Network (UHN) hospitals (Toronto, Ontario, Canada) by considering the average number of surgical cases in a week[1]. The original instances used were not available. Using examples given by Guo[2] we round surgery times to integer values. We generated data according to the distributions in Table 1.

Table 1: *Distributions used for data generation.*

Data	Distribution
κ_1	50 dollars
κ_2	5 dollars
Γ	500
ρ_p	Discrete uniform distribution [1, 5]
B_{hd}	Discrete uniform distribution [420, 480] minutes in 15-minute intervals
α_p	Discrete uniform distribution [60, 120] days.
F_{hd}	Discrete uniform distribution [4000, 6000] dollars
G_{hd}	Discrete uniform distribution [1500, 2500] dollars
T_p	Truncated normal distribution [45, 480] minutes, $\mu = 160$, $\sigma = 40$

4 Implementation

We implemented two different practical models for LBBD, one following [1] which solves the master problem iteratively in a loop, rebuilding the master problem branch-and-bound tree on every iteration. The other model leveraged lazy constraints in Gurobi to perform a branch and check routine[3], building the branch-and-bound tree once while applying Benders' cuts as lazy constraints at each new incumbent solution.

Both implementations of LBBD first solved each sub problem using the first-fit decreasing (FFD) heuristic[4] and used its results as a warm-start to a MIP model as outlined in [1]. The pure MIP and network models were solved using standard Gurobi routines.

We implemented the caching of sub problems to avoid repeated work. During run-time, we keep a cache of solved sub problems, which we check at each incumbent to see if we are looking at an already solved sub problem, exploratory testing showed this to be beneficial. We also informally tested different Gurobi parameters, such as *MIPFocus*, *Branching Priorities* on important variables such as y_{hd} and u_{hd}, *Heuristics* and *Method*. Some improvements were seen using branching priorities but overall no significance improvement was observed tuning these parameters.

Model formulations were simple enough to interpret, but practical implementation was difficult for a few reasons. Firstly, the original paper did not supply any code. This made it hard to find implementation details that may have drastically improved results, for example some important caching protocol may have been used that was not mentioned in the paper. Further, the paper supplied distribution parameters that were useful in generating data, however it did not supply the data instances that were used to generate their results. This made it impossible to verify if the differences seen in our results were due to an implementation error, random chance or something else. It also left data inadequately described. For example the surgery times were said to be from a normal distribution, this might imply that they should be real-valued. However, upon inspection of another paper's data which worked on a similar problem[2] and the same data, it was noticed that surgery times were rounded to an integer value. Implementing this with our data generation scheme immediately improved the performance of our models. We reached out to the author of the original paper and we were told that they were no longer in possession of either the code or data that was used.

The paper did not supply example objective values found for their instances which made it hard to determine feasibility of our solutions. The original paper also did not specify if a certain MIP gap parameter was set, it mentions "Gurobi optimal solutions" but by default Gurobi treats a MIP gap of 1% as optimal, [2] states that they solve their models to "optimality with a gap of 1%" when they seek to make comparisons with the original paper, meaning that the exact gap used in the original paper was left unclear to us. Ultimately the main verification method of our results was by the consistency of outputted optimal objective values across all models implemented. That is, the pure MIP, Network and Benders' models were implemented in sufficiently different ways, but all provided the same output giving us some level of confidence in our implementation's optimality. Feasibility was verified by manual inspection of patient allocations for instances with small number of patients. We had no way of verifying if the speed of our models would be the same as achieved by the models used in the original paper.

5 Results

Models were run on a Windows 11 computer with an AMD Ryzen 5 5625U 2.3GHz processor with 16 GB of RAM. Gurobi version 10.0.1 was used with Python version 3.9.16.

We aimed to compare 7 models: The pure MIP, the network model, an iterative LBBD1 (iLBBD1), LBBD1 using lazy constraints (cLBBD1), iterative LBBD2 with propagation (iLBBD2p), LBBD2 with lazy constraints and propagation (cLBBD2p) and LBBD4 with lazy constraints and propagation (cLBBD4). The pure MIP was used as a baseline. iLBBD1 and cLBBD1 were chosen to be used as baseline LBBD models. LBBD2 with propagation was chosen as it was one of the best performers in the original paper, we intended to use it as the main comparison to the network model and our new cut. We chose to collect results for LBBD4 only using propagation and lazy constraints as we believed this would give us the best results while adhering to the project time constraints.

We generate 5 seeded instances of data similarly to [1]. The data is generated for 3 hospitals and 5 operating rooms over a 5-day planning horizon for varying number of patients. Unlike [1] who generate a set of data for 3 and 5 operating rooms while only varying the surgery times in each instance, we generate each instance with entirely different data. This was believed to give results more indicative of performance on a wider variation in problem parameters. Also unlike in the original paper, where models were run with a time limit of 7200 seconds, we set our time limit to 900 seconds due to project time constraints. At a time limit of 7200 seconds the worst case time to run all models over all 5 instances was 70 hours for only one patient size. For this same reason we only ran models with 5 available operating rooms, instead of testing both 5 and 3. We chose to use 5 operating rooms, this should have led to harder sub problems and relatively easier master problems compared to using 3 operating rooms[1].

There was uncertainty as to whether the models were solved to true optimality or to a relative MIP gap of 1% as was done by [2], so we report results for both scenarios. We follow [1] by defining the best performing model to be the most robust, that is, the one able to solve the model to optimality within the given time constraints while breaking ties by considering the fastest models averaged over solved instances.

The 7 models were solved to optimality on each of the 5 instances, the average time to solve can be seen in Table 2 and the average gaps can be seen in Table 3.

We can see from Table 2 that all models had difficulty solving to optimality for number of patients greater than only 20. The pure MIP was the most robust across all patient sizes while the network model was the least. We can see that for sizes where LBBD models do solve an instance, they do so on average faster than the pure MIP. For example, for a patient size of 20 cLBBD4p performed the best, being solved in the least time on average, with cLBBD2p being close second. For a patient size of 60, most LBBD models solved faster than the pure MIP. Comparing between LBBD models, all using lazy constraints outperform those using iteration in terms of time to solve for every number of patients. The results shown in Table 2 suggest that none of the 7 models scale well for solving problems with sets of patients as large or larger than 60 when solving to optimality. No models were able to solve instances with 80 patients to optimality.

Table 3 shows that although not many instances solved to optimality, by the end of the prescribed time limit, the optimality gaps were very small. All optimality gaps had reached at least 1% or lower. In fact, increasing the number of patients did not appear to have a large negative effect on the final optimality gaps, giving merit to the scalability of all models in terms of achieving a small gap in under 15 minutes for increasing number of patients. Interestingly, nearly all gaps for number of patients greater than 20 were of the same magnitude, it seems there is some threshold that the model can reach very quickly but closing

Table 2: *Average time (seconds) until solved to optimality over 5 instances. The number of instances not solved to optimality are superscripted. Non-solved instances are not included in average. Asterisks indicate that no instances solved in time.*

$\lvert\mathcal{P}\rvert$	Pure MIP	Network	iLBBD1	cLBBD1	iLBBD2p	cLBBD2p	cLBBD4p
20	16.06	$****^{(5)}$	1.509	0.8829	1.431	0.8800	0.7890
40	$179.3^{(2)}$	$268.5^{(3)}$	$4.429^{(4)}$	$1.911^{(4)}$	$4.503^{(4)}$	$1.911^{(4)}$	$1.959^{(4)}$
60	$30.80^{(4)}$	$****^{(5)}$	$24.46^{(4)}$	$10.73^{(4)}$	$34.61^{(4)}$	$21.46^{(4)}$	$25.80^{(4)}$
80	$****^{(5)}$	$****^{(5)}$	$****^{(5)}$	$****^{(5)}$	$****^{(5)}$	$****^{(5)}$	$****^{(5)}$

Table 3: *Average gap (%) over 5 instances after trying to solve to optimality. MIPGap is reported for pure MIP, Network and callback implementations of LBBD. Gap between master problem lowerbound and best sub problem upperbound is reported for iterative implementations of LBBD.*

$\lvert\mathcal{P}\rvert$	Pure MIP	Network	iLBBD1	cLBBD1	iLBBD2p	cLBBD2p	cLBBD4p
20	0.000	0.4950	0.000	0.000	0.000	0.000	0.000
40	0.04220	0.3382	0.6252	0.3532	0.09500	0.4765	0.4948
60	0.1827	0.4131	0.5514	0.2540	0.1700	0.3513	0.3273
80	0.2630	0.2048	0.9390	0.6000	0.2100	0.4718	0.5218

the last of the gap becomes difficult. These results gave more motivation to investigate solving to a 1% gap.

The 7 models were solved to a 1% gap on each of the 5 instances, the average time to solve can be seen in Table 4.

We can see from Table 4 that now only going to a 1%, most models are able to complete for most instances for number of patients up to 80. The pure MIP and the network model were the most robust across all number of patients. The least robust model across all number of patients was iLBBD2p. For 20 patients iLBBD1 solved the fastest on average. For all other number of patients the pure MIP solved the fastest. Overall, lazy constraint implementations were slightly better than there iterative counterparts. cLBBD2p performed better than iLBBD2p for 20, 40 and 60 patients. cLBBD1 performed better than iLBBD1 for 40 and 60 patients.

The average time in the master problem and sub problem when trying to solve to a 1% gap was plotted for different patient set sizes and can be seen in Figure 1.

From Figure 1a we can see that that iterative variants of LBBD spent a high proportion of their time in the master problem and the lazy constraint implementations spent a high proportion of their time in the sub problems. This is expected for the iterative implementation a large amount of time would be spent re-building the master problem branch-and-bound tree on every iteration. This is expected for the sub problem as we need to solve the sub problems for every incumbent solution, even though it was hoped to be remedied somewhat by caching of sub problems.

Figure 1b shows that for $|P| = 40$ most variants of LBBD including lazy constraint implementations spent large portions of their time in the master problem. This provided some evidence that sub problem implementation was not the main bottleneck for this size of problem, but was in fact the highly optimized Gurobi solver.

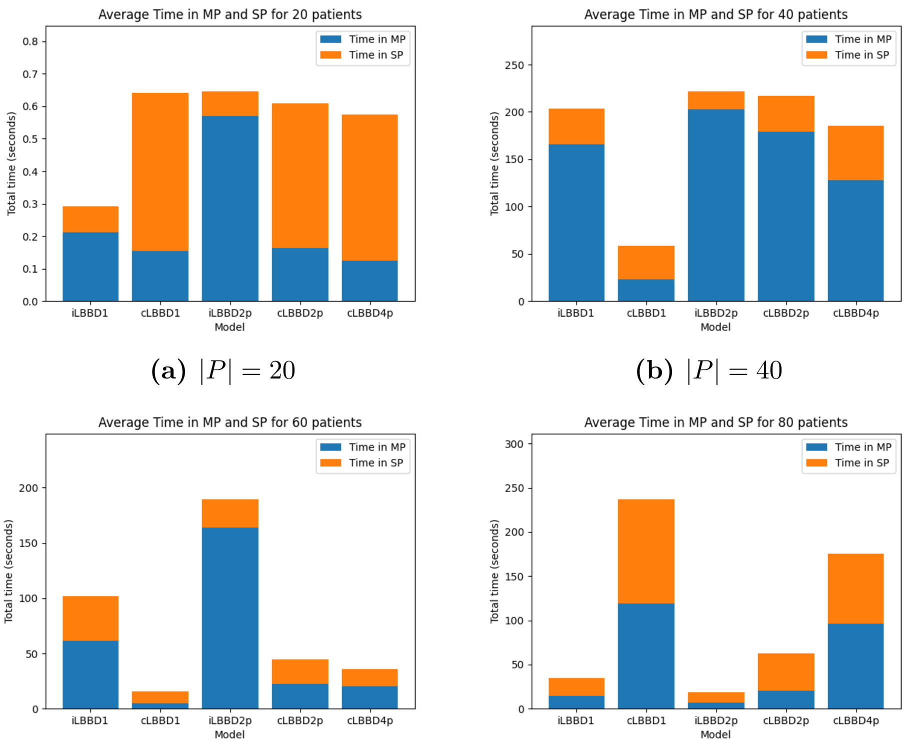
Across all patient set sizes shown in Figure 1 iterative versions consistently spent a high portion of their time in the master problem. Lazy constraint versions seemed to vary with regard to whether they spent more time in master problem or sub problem, this indicates that the relative difficulty of sub problems to master problems was likely highly dependent on the data and not a product of lazy constraints, this was even with using 5 operating rooms which should have influenced the problem to have easier master problems than sub problems[1].

6 Discussion

It should be emphasised that because our time constraints were stricter robustness may have been

Table 4: *Average time (seconds) until solved to optimality with a 1% gap over 5 instances. The gap used to terminate optimisation was the MIPGap for all models except for iterative LBBDs which were terminated by gap between master problem and sub problem. The number of instances not solved to optimality are superscripted. Non-solved instances are not included in average. Asterisks represent that no instances solved in time. Network times do not include node and arc generation time.*

$\|\mathcal{P}\|$	Pure MIP	Network	iLBBD1	cLBBD1	iLBBD2p	cLBBD2p	cLBBD4p
20	0.8406	94.56	0.2942	0.6679	0.6466	0.6335	0.6047
40	5.448	17.52	$28.90^{(1)}$	58.87	$52.35^{(1)}$	$46.21^{(1)}$	$6.077^{(1)}$
60	6.012	29.11	101.8	15.74	$12.01^{(1)}$	45.06	36.22
80	9.716	30.48	34.90	$71.66^{(1)}$	18.79	62.25	175.4



(a) $|P| = 20$ **(b)** $|P| = 40$

(c) $|P| = 60$ **(d)** $|P| = 80$

Figure 1: *Average time in master problem (MP) and sub problem (SP) for different patient set sizes.*

greatly affected giving different results to the original paper, however we found that our results were already disparate without this factor. For example, the original paper was able to solve with patient sizes up to 100 without ever going over an average runtime of 200 seconds. This change in experimental setup should have allowed our models to still attain similar results.

One possible reason the results were not able to replicate the original paper was that undisclosed implementation techniques might have been used and been important for performance. Due to our testing of optimality and feasibility across models, it was thought that the modelling implementation was correct, meaning that our relatively poor results were mainly thought to be due to algorithm optimization. Caching sub-problems was used to get more performance out of our models but was not enough to give us results similar to [1]. Caching was used to avoid repeated calculation for identical sub problems and did give immediate improvements in the prototyping stage. In Figure 1 it was shown that for larger patient sizes, average time in the sub problem was not disproportionately larger than in the master problem. This does give some confidence that poor sub problem implementation was not the main bottleneck of performance, as about half or more of the time was spent in the master problem, handled by Gurobi.

The network formulation was slower and less robust than the MIP formulation, even when not considering node and arc generation time. It was initially thought that this formulation could yield improvements as it had shown promising results as a prototype for a similar hard surgery allocation problem that was thought to have needed the Branch and Price algorithm[5]. After collecting our results it was realised that the key reason that the network formulation performed well for the aforementioned problem was due to importance of surgery ordering. For the DORS problem we only care about packing operations into an OR, order is not important. For this reason, the permutation generation of arcs for different orderings of surgeries was mostly redundant making the network fundamentally inefficient at solving this problem compared to other methods, particularly the pure MIP. Surprisingly, even with this shortcoming the network model was still able to be more robust than some of the Benders' Decomposition implementations when solving to a 1% gap.

By construction, the stronger LBBD cuts were able to leverage the structure of an iterative implementation which may have bottlenecked greater improvement from using the callback. LBBD2 and LBBD4 both try to leverage the fact that the current solution to the master problem is optimal. That is for both cuts, there is dependency on $\hat{Y}_{hd}$ to be an optimal lower bound for the problem. However, Gurobi callbacks are called at each *incumbent* solution, as the lazy constraints must cut off new solutions as they are reached in the branch-and-bound tree and cannot retroactively remove old solutions that violate a lazy constraint. We can still apply cuts in a callback despite this, however it was speculated that some were redundant or that this may have caused sub problems to be needlessly solved.

The simple pure MIP performed much better compared to the other more involved models, and much better than the original paper's pure MIP from the perspective of a 1% gap. This may have been due to the simplicity of the model and furthermore of implementation. Very little model-critical code was implemented for the pure MIP besides the utilization of Gurobi classes and methods, which are all rigidly defined by that package and its documentation. This means that chance of introducing a logic or optimization error was much lower. This very well could have been the main driver of this model's success in our results. Alternatively, the great performance of the pure MIP could be due to the ongoing improvements being made to Gurobi's solver. Gurobi has gone through 3 major versions since the release of [1], in the latest major release alone it reports a 13% speed improvement on its test MIP problems[6]. The hardware difference between the original paper would also be a highly significant factor with [1] running models on a Windows XP computer with an Intel(R) Core(TM)2 Duo 3.00 gigahertz CPU with 8 gigabytes RAM. Other than these reasons the pure MIP may just be the best model out of those tested for the DORS problem and the specific data used for our tests.

7 Conclusion

We reimplemented a pure MIP and Benders' Decomposition models from [1]. Models were extended by constructing a new Benders' Cut LBBD4 and by trialling a network model. Implementation was extended by using lazy constraints in Gurobi. Implementations successfully generated consistent optimal solutions across all models. Implementations generated feasible solutions from observation. Models were able to solve to optimality for only a small patient set size of 20 but were able to scale up to 80 patients when solving to a 1% gap. We were not able to replicate the efficiency of Benders' Decomposition from the original paper, this was believed to be caused by some suboptimal implementation used or the difference in data generation.

From our results, the pure MIP model performed the best overall, being the most robust when solving to optimality and being the fastest to solve across all sizes of patient sets when solving to a 1% gap. The Network model was the least robust when solving to optimality and the slowest when solving to a 1% gap. Overall, callback implementations performed slightly better than iterative variants.

Discretizing time further into periods may remedy some of the scaling issues found with all models when solving to optimality, this would sacrifice accuracy for speed, although this can already be done by simply using a larger gap.

Larger number of patients could be tested to find the limits of the models when solving to a 1% gap as it was seen that all models were still quite robust up to 80 patients.

Further algorithmic optimization could be trialled. Caching of sub problems was used to check for identical sub problems but exploratory tests using a more sophisticated caching protocol showed promise, project time constraints and long run-times constrained the collection of more results. The more sophisticated caching methods depend on looking at the sorted set of patients for a given solved sub problem. And the sorted set of patients for the current sub problem. If patient-wise times for the current sub problem are less than or equal to those of a feasible cached sub problem, we know a feasible solution to the current sub problem. Similarly, if the patient-wise time for the current sub problem are greater than or equal to those of an infeasible cached sub problem, we know the current sub problem must be infeasible. The results for this may show great improvement, especially for lazy constraint implementations where similar sub problems are visited often.

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A Appendix

Table 5: *Time taken for each instance when trying to solve to optimality.*

seed	$\lvert\mathcal{P}\rvert$	Pure MIP	Network	iLBBD1	cLBBD1	iLBBD2p	cLBBD2p	cLBBD4
42	20	55.621249	900	4.123265	0.949113	3.958082	0.605675	0.654821
831	20	18.126045	900	0.296833	0.253265	0.274671	0.416003	0.441198
306	20	2.041458	900	1.361992	2.055692	1.106399	1.808341	1.716485
542	20	3.000340	900	0.580737	0.504805	0.600731	0.514555	0.520827
1	20	1.505760	900	1.181184	0.651583	1.215210	1.055650	0.611608
42	40	39.046759	64.643897	901.969151	900.928267	900.922126	902.530971	901.404782
831	40	900.074868	472.309139	1171.195916	900.966696	1605.561038	900.273473	900.329041
306	40	3.928887	900.159554	4.428572	1.911088	4.503378	1.910761	1.959761
542	40	900.199757	900.153089	1045.090099	901.370528	907.687432	901.559075	900.948533
1	40	494.895451	1394.601143	905.294060	905.809085	935.405877	900.435013	904.002700
42	60	900.136048	900	902.127126	900.441417	905.850299	903.361533	900.170979
831	60	900.159431	900	912.129558	900.112896	1183.786823	900.232333	902.206866
306	60	30.804891	900	24.459900	10.726104	34.606863	21.455653	25.795459
542	60	900.154018	900	901.279073	900.676095	919.253635	901.173662	901.073426
1	60	900.127659	900	1071.186251	900.254857	1084.595362	900.801214	900.983243
42	80	900.069266	900.105599	920.841381	901.018549	949.759938	900.152319	900.167128
831	80	900.084165	900.246634	1491.767385	900.493704	1251.183445	901.669027	901.097375
306	80	900.093483	900.246199	1307.066104	900.037347	1297.656757	900.081336	901.133256
542	80	900.088750	900.201944	903.723640	900.139623	903.938128	900.088399	903.650919
1	80	900.167469	900.200223	900.285928	901.299396	900.273110	900.055926	900.098382

Table 6: *Objective values when trying to solve to optimality.*

seed	$\lvert\mathcal{P}\rvert$	Pure MIP	Network	iLBBD1	cLBBD1	iLBBD2p	cLBBD2p	cLBBD4
42	20	-312419.0	-312419.0	-312419.0	-312419.000000	-312419.0	-312419.0	-312419.0
831	20	-187989.0	-187989.0	-187989.0	-187989.000000	-187989.0	-187989.0	-187989.0
306	20	-220079.0	-220079.0	-220079.0	-220079.000000	-220079.0	-220079.0	-220079.0
542	20	-238297.0	-238030.0	-238297.0	-238297.000000	-238297.0	-238297.0	-238297.0
1	20	-244112.0	-244112.0	-244112.0	-244112.002591	-244112.0	-244112.0	-244112.0
42	40	-503251.0	-503151.0	-503651.0	-501294.0	-503651.0	-498116.0	-498046.0
831	40	-367748.0	-367748.0	-368833.0	-366247.0	-368833.0	-366197.0	-366047.0
306	40	-383537.0	-383487.0	-383537.0	-383537.0	-383537.0	-383537.0	-383537.0
542	40	-412576.0	-413122.0	-413322.0	-412576.0	-413322.0	-412722.0	-412576.0
1	40	-402834.0	-402067.0	-403067.0	-401653.0	-403067.0	-401653.0	-401653.0
42	60	-741357.0	-744057.0	-744307.0	-741357.0	-744307.0	-740007.0	-740542.0
831	60	-654542.0	-654683.0	-655215.0	-653458.0	-655215.0	-653458.0	-652005.0
306	60	-564312.0	-564212.0	-564312.0	-564312.0	-564312.0	-564312.0	-564312.0
542	60	-675220.0	-673723.0	-675400.0	-673288.0	-675400.0	-671849.0	-673117.0
1	60	-602474.0	-602174.0	-604042.0	-602474.0	-604042.0	-602474.0	-602474.0
42	80	-957258.0	-959091.0	-959591.0	-948632.0	-959591.0	-957158.0	-955093.0
831	80	-830774.0	-832624.0	-833293.0	-829544.0	-833293.0	-830028.0	-827186.0
306	80	-801771.0	-803513.0	-803663.0	-800453.0	-803663.0	-798478.0	-801133.0
542	80	-892084.0	-892173.0	-892573.0	-889573.0	-892573.0	-889273.0	-890231.0
1	80	-806619.0	-808385.0	-809624.0	-805178.0	-809624.0	-805027.0	-804203.0